Notes on: *Limits on the Size of the Universe with the Stochastic GW Background*

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1 Introduction

We aim to set a limit on the size of a static Minkowski universe using the nonde-tection of a stochastic background of gravitational waves. We assume that the stochastic background is dominated by gravitational waves (GWs) from compact binary coalescences (CBCs) and assume a constant rate of CBCs throughout the universe.

2 Calculation

We start with Equation 5 in [1]. Namely,

$$\rho_c c^2 \Omega_{\mathrm{gw}}(f) = \int_0^\infty \frac{N(z)}{1+z} (f_r \frac{dE_{\mathrm{gw}}}{df_r})_{f_r=f(1+z)} \mathrm{d}z, \tag{1}$$

where $\frac{dE_{\mathrm{gw}}}{df}$ is the energy spectrum emitted by a CBC source, f_r is the frequency in the source frame, and $N(z)$ is comoving number density of sources. The factor of $(1+z)$ in the denominator is to account for redshifting of the gravitons since emission. $\rho_c = 3H_0^2/(8\pi G)$ is the critical density to close the universe.

We wish to consider the case of a static, Minkowski spacetime and con-strain the size of this universe under the assumption that the number density of stochastic sources is a spatial and temporal constant throughout the universe. In this scenario, redshift has no meaning, so we must convert the integral over redshift in Equation 1 to an integral over distance. To do this, we assert that the number of galaxies in a spherical shell of infintesimal width should be invaraiant under a parametrization in terms of redshift or distance. That is,

$$\#_{\mathrm{shell}} = N(V_C)dV_C, \tag{2}$$

where "C" indicates a comoving value. The volume element of a spherical shell with radius r and width dr is $dV = 4\pi r^2 dr$. Thus, equation 2 written in terms of comoving distance D_C only is

$$\#_{\mathrm{shell}} = N(D_C)4\pi D_C^2 dD_C. \tag{3}$$

The comoving volume element of a spherical shell of radius D_C and width dz is given by Equation 41 of [2]:

$$dV_{\text{com}} = \frac{c4\pi}{H(z)}D_C^2 dz,$$

so the number of sources in this volume element is

$$\#_{\text{shell}} = N(z)\frac{c4\pi}{H(z)}D_C^2 dz. \tag{4}$$

Setting Equation 4 equal to 3 we have

$$N(z)\frac{c4\pi}{H(z)}D_C^2 dz = N(D_C)4\pi D_C^2 dD_C$$

$$N(z)\frac{c}{H(z)}dz = N(D_C)dD_C$$

$$N(z)dz = \frac{H(z)}{c}N(D_C)dD_C. \tag{5}$$

Substituting this into Equation 1 we have

$$\rho_c c^2 \Omega_{\text{gw}}(f) = \int_0^\infty \frac{H(z)}{c}\frac{N(D_C)}{1+z}(f_r \frac{dE_{\text{gw}}}{df_r})_{f_r=f(1+z)}\mathrm{d}D_C.$$

We wish here to use constraints on Ω_{gw} to put an upper limit on the horizon distance r_{hor} of a static Minkowski universe, so we change the upper limit of the integral from ∞ to r_{hor}. Noting that in a non-expanding universe, gravitons do not get cosmologically redshifted, we set z=0. This means that the frequency and energy in the source frame (r) are the same as in the detector frame. Additionally, $H(z)$ becomes the local H_0 . Thus, we have

$$\rho_c c^2 \Omega_{\text{gw}}(f) = \int_0^\infty \frac{H_0}{c}N(D_C)(f\frac{dE_{\text{gw}}}{df})\mathrm{d}D_C. \tag{6}$$

Evaluating with the assumption that $N(r) = N$ is a spatial constant, we have

$$\Omega_{\text{gw}}(f) = \frac{H_0 f N}{\rho_c c^3}\frac{dE_{\text{gw}}}{df}\int_0^{r_{\text{hor}}} dD_C$$

$$= \frac{H_0 f N}{\rho_c c^3}\frac{dE_{\text{gw}}}{df}r_{\text{hor}}.$$

For CBC sources, $\frac{dE_{\text{gw}}}{df} = \frac{\mathcal{M}^{5/3}(G\pi)^{2/3}}{3f^{1/3}}$ so

$$\Omega_{\text{gw}}(f) = \frac{H_0 N f^{2/3}}{3\rho_c c^3}\mathcal{M}^{5/3}(G\pi)^{2/3}r_{\text{hor}}.$$

$\mathcal{R}(\mathrm{yr}^{-1}\mathrm{Gpc}^{-3})$	$\mathcal{M}(M_{\odot})$	Ω_{ref}	$H_0(km/s/Mpc)$
53.2	26	4.8×10^{-8}	67.9

Table 1: List of parameters used

Rearranging and substituting $\rho_c = 3H_0^2/(8\pi G)$,

$$r_{\mathrm{hor}} = \frac{3\Omega_{\mathrm{gw}}(f)\rho_c c^3}{NH_0}(fG\pi)^{-2/3}\mathcal{M}^{-5/3}$$
$$= \frac{9\Omega_{\mathrm{gw}}(f)H_0 c^3}{8N}(f)^{-2/3}(G\pi\mathcal{M})^{-5/3}.$$

In the GW stochastic background literature (e.g. [3, 4]) Ω_{gw} for a CBC source is commonly written as $\Omega_{\mathrm{gw}} = \Omega_{\mathrm{ref}}(\frac{f}{f_{\mathrm{ref}}})^{2/3}$ so that Ω can be calculated at a single frequency. The reference frequency for LIGO searches is $f_{\mathrm{ref}} = 25Hz$. The number density of sources N is related to the rate density of CBCs $\mathcal{R}$ by $N(t) = \int_0^t \mathcal{R}(t')\mathrm{d}t'$. Again using our assumption of the rate as a spatial and temporal constant, we have $N = \mathcal{R}t$. We take t to be the time it takes GWs to reach us from the horizon distance, so $N = \mathcal{R}r_{\mathrm{hor}}/c$.

Making these replacements we have

$$r_{\mathrm{hor}}^2 = \frac{9\Omega_{\mathrm{ref}}H_0 c^4}{8\mathcal{R}}(f_{\mathrm{ref}})^{-2/3}(G\pi\mathcal{M})^{-5/3}. \tag{7}$$

This is our final expression. Now comes the time to compute with characteristic values, which we summarize in Table 1. The rate is taken to be the local value computed in LIGO's second observing run ([5]) and the upper limit on Ω_{ref} is taken to be the one reported in the isotropic search from LIGO's first and second observing runs ([4]). The chirp mass $\mathcal{M}$ is calculated for the case of two $30\ M_{\odot}$ black holes.

Using these parameters, we get an upper limit on the horizon distance

$$\boxed{r_{\mathrm{hor}} \leq 19.7\mathrm{Gpc}}$$

2.1 Considering Distributions of Source Parameters

Up until now, we have only considered chirp masses from a single source type. If instead we would like to consider a full distribution of source types, $N(z)(f_r\frac{dE_{\mathrm{gw}}}{df_r})$ in Equation 1 is replaced by $\sum_i N_i(z)(f_r\frac{dE_{\mathrm{gw},i}}{df_r})$, where the sum is over source types i. For continuous values of source parameters, Equation 6 becomes

$$\rho_c c^2 \Omega_{\mathrm{gw}}(f) = \int_0^{r_{\mathrm{hor}}} \frac{H_0}{c} \int d\vec{\theta} N(D_C, \vec{\theta})(f\frac{dE(f,\vec{\theta})_{\mathrm{gw}}}{df})\mathrm{d}D_C,$$

where $\vec{\theta}$ are the source parameters. Here, we are primarily concerned with the mass distribution of sources, so we only consider $\vec{\theta} = \mathcal{M}$. As before, we

substitute $\frac{dE_{\text{gw}}}{df} = \frac{\mathcal{M}^{5/3}(G\pi)^{2/3}}{3f^{1/3}}$ and assume N is constant in comoving volume. This yields

$$\rho_c c^2 \Omega_{\text{gw}}(f) = r_{\text{hor}} \frac{H_0}{c} \frac{(fG\pi)^{2/3}}{3} \int d\mathcal{M} N(\mathcal{M}) \mathcal{M}^{5/3}. \tag{8}$$

Now all that is left to be done is find an expression for $N(\mathcal{M})$ for a given mass distribution and rate.

$$N(\mathcal{M}) = \int d\mathcal{M} \frac{dN(\mathcal{M})}{d\mathcal{M}}$$
$$= \int d\mathcal{M} \int dt \frac{d\mathcal{R}}{d\mathcal{M}}.$$

For simplicity, we will assume all systems are equal mass and then re-write in terms of an individual component mass, m_1. In this scenario, $\frac{d(\mathcal{R})}{d\mathcal{M}} = 2^{1/5} \frac{d(\mathcal{R})}{dm_1}$ and Equation 8 becomes

$$\rho_c c^2 \Omega_{\text{gw}}(f) = r_{\text{hor}} \frac{H_0}{c} \frac{(fG\pi)^{2/3}}{3(2)^{1/3}} \int dt \int dm_1 \frac{d\mathcal{R}}{dm_1} m_1^{5/3}. \tag{9}$$

We take the mass distribution model $\frac{d\mathcal{R}}{dm_1}$ from Equation 17 of [5] and modify it to get rid of redshift evolution, obtaining

$$\frac{d\mathcal{R}}{dm_1 dm_2} = \mathcal{R}_0 p(m_1, m_2 | \gamma, m_{min}, m_{max})$$

where γ, m_{min}, m_{max} are the power law slope, minimum and maximum possible component masses, respectively, and are determined by the model fit. We combine this with the model from [6] (namely Equation 5) and choose a pairing function that will enforce equal masses to get

$$\int dm_1 \frac{d\mathcal{R}}{dm_1} m_1^{5/3} = \frac{\mathcal{R}_0(2\gamma + 1)}{m_{max}^{2\gamma+1} - m_{min}^{2\gamma+1}} \left[\frac{m_{max}^{2\gamma+8/3} - m_{min}^{2\gamma+8/3}}{(2\gamma + 8/3)} \right].$$

Plugging this into Equation 9, using the values listed in Table 1, and rearranging yields $\boxed{r_{\text{hor}} \leq 39.2 \text{Gpc}}$.

3 Issues

The rate and Ω_{ref} are taken from LIGO analyses which assume an FLRW universe, so using those values might not be consistent with this analysis. However, the rate measurement is done locally, in which the universe is basically Minkowski, so hopefully it is not a large effect. Additionally, Ω_{gw} is measured

based on the strain signal alone - I think it might not have a cosmology folded in.

The number density N is calculated by multiplying the rate by the amount of time it takes GWs to reach us from the horizon distance (namely, $N = \mathcal{R}r_{\text{hor}}/c$). Presumably, the rate at each point should be multiplied by the time it takes GWs to reach us from the radius at that point, or some other time that I just can't think of at this time.

If instead the time used is the time spent observing the background t_{obs}, Equation 7 instead becomes

$$r_{\text{hor}} = \frac{9\Omega_{\text{ref}}H_0 c^3}{8\mathcal{R}t_{\text{obs}}}(f_{\text{ref}})^{-2/3}(G\pi\mathcal{M})^{-5/3}. \tag{10}$$

Using $t_{obs} = 1.5$ yr gives $r_{\text{hor}} = 8.4 \times 10^{11}$ Gpc, which is very large and either indicates an issue with a Euclidian universe or the fact that the stochastic upper limits are very high.

References

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