Quantum Field Theory

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To my professor

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Quantum Field Theory

Chapter 1

Classical Scalar Field

1.1 Scalar field theory

In theoretical physics, scalar field theory can refer to a classical or quantum theory of scalar fields. A field which is invariant under any Lorentz transformation is called a "scalar", in contrast to a vector or tensor field. The quanta of the quantized scalar field are spin-zero particles, and as such are bosons.

No fundamental scalar fields have been observed in nature, though the Higgs boson may yet prove the first example. However, scalar fields appear in the effective field theory descriptions of many physical phenomena. An example is the pion, which is actually a "pseudoscalar", which means it is not invariant under parity transformations which invert the spatial directions, distinguishing it from a true scalar, which is parity-invariant. Because of the relative simplicity of the mathematics involved, scalar fields are often the first field introduced to a student of classical or quantum field theory.

In this article, the repeated index notation indicates the Einstein summation convention for summation over repeated indices. The theories described are defined in flat, D-dimensional Minkowski space, with (D-1) spatial dimension and one time dimension and are, by construction, relativistically covariant. The Minkowski space metric, $\eta_{\mu\nu}$, has a particularly simple form: it is diagonal, and here we use the $[-+++]$ sign convention.

1.2 Classical Scalar Field Theory

1.2.1 Linear (free) theory

The most basic scalar field theory is the linear theory. The action for the free relativistic scalar field theory is:

$$\mathscr{S} = \int \mathrm{d}^{D-1}x\mathrm{d}t\mathscr{L} = \int \mathrm{d}^{D-1}x\mathrm{d}t\left[\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right] = \int \mathrm{d}^{D-1}x\mathrm{d}t\left[\frac{1}{2}(-\partial_t\phi)^2 + \right. \tag{1.1}$$

where $\mathscr{L}$ is known as a Lagrangian density. This is an example of a quadratic action, since each of the terms is quadratic in the field, ϕ. The term proportional to m^2 is sometimes known as a mass term, due to its interpretation in the quantized version of this theory in terms of particle mass.

The equation of motion for this theory is obtained by extremizing the action above. It takes the following form, linear in ϕ:

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi - m^2\phi = -\partial_t^2\phi + \nabla^2\phi - m^2\phi = 0 \tag{1.2}$$

Note that this is the same as the Klein-Gordon equation, but that here the interpretation is as a classical field equation, rather than as a quantum mechanical wave equation.

1.2.2 Nonlinear (interacting) theory

The most common generalization of the linear theory above is to add a scalar potential $V(\phi)$ to the equations of motion, where typically, V is a polynomial in ϕ of order 3 or more (often a monomial). Such a theory is sometimes said to be interacting, because the Euler-Lagrange equation is now nonlinear, implying a self-interaction. The action for the most general such theory is

$$\begin{aligned}\mathscr{S} &= \int \mathrm{d}^{D-1}x\mathrm{d}t\mathscr{L} = \int \mathrm{d}^{D-1}x\mathrm{d}t\left[\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \\ &= \int \mathrm{d}^{D-1}x\mathrm{d}t\left[\frac{1}{2}(-\partial_t\phi)^2 + \frac{1}{2}\delta^{ij}\partial_i\phi\partial_j\phi - \frac{1}{2}m^2\phi^2 - \sum_{n=3}^{\infty}\frac{1}{n!}g_n\phi^n\right]\end{aligned} \tag{1.3}$$

The $n!$ factors in the expansion are introduced because they are useful in the Feynman diagram expansion of the quantum theory, as described below. The corresponding Euler-Lagrange equation of motion is:

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi + V'(\phi) = -\partial_t^2\phi + \nabla^2\phi + V'(\phi) = \Box\phi + V'(\phi) = 0 \tag{1.4}$$

We can define the new operator, let say box or $\Box$, which is useful when we want to use general relativity ideas in quantum feild theory.

$$\Box\phi = \eta^{\mu\nu}\partial_\mu\partial_\nu\phi = -\partial_t^2\phi + \nabla^2\phi \tag{1.5}$$

1.3 Dimensional analysis and scaling

Physical quantities in these scalar field theories may have dimensions of length, time or mass, or some combination of the three. However, in a relativistic theory, any quantity t, with dimensions of time, can be 'converted' into a length, $l = ct$, by using the velocity of light, c.

Similarly, any length l is equivalent to an inverse mass, $l = \frac{\hbar}{mc}$, using Planck's constant, $\hbar$. Heuristically, one can think of a time as a length, or either time or length as an inverse mass. In short, one can think of the dimensions of any physical quantity as defined in terms of just one independent dimension, rather than in terms of all three. This is most often termed the mass dimension of the quantity.

One objection is that this theory is classical, and therefore it is not obvious that Planck's constant should be a part of the theory at all. In a sense this is a valid objection, and if desired one can indeed recast the theory without mass dimensions at all. However, this would be at the expense of making the connection with the quantum scalar field slightly more obscure. Given that one has dimensions of mass, Planck's constant is thought of here as an essentially arbitrary fixed quantity with dimensions appropriate to convert between mass and inverse length.

1.3.1 Scaling Dimension

The classical scaling dimension, or mass dimension, Δ, of ϕ describes the transformation of the field under a rescaling of coordinates:

$$x \to \lambda x \tag{1.6}$$

$$\phi \to \lambda^{-\Delta}\phi \tag{1.7}$$

The units of action are the same as the units of $\hbar$, and so the action itself has zero mass dimension. This fixes the scaling dimension of ϕ to be

$$\Delta = \frac{D-2}{2}. \tag{1.8}$$

1.4 Scale Invariance

There is a specific sense in which some scalar field theories are scale-invariant. While the actions above are all constructed to have zero mass dimension, not all actions are invariant under the scaling transformation

$$x \to \lambda x \tag{1.9}$$

$$\phi \to \lambda^{-\Delta}\phi \tag{1.10}$$

The reason that not all actions are invariant is that one usually thinks of the parameters m and g_n as fixed quantities, which are not rescaled under the transformation above. The condition for a scalar field theory to be scale invariant is then quite obvious: all of the parameters appearing in the action should be dimensionless quantities. In other words, a scale invariant theory is one without any fixed length scale (or equivalently, mass scale) in the theory.

For a scalar field theory with D spacetime dimensions, the only dimensionless parameter g_n satisfies $n = \frac{2D}{D-2}$. For example, in $D = 4$ only g_4 is classically dimensionless, and so the only classically scale-invariant scalar field theory in $D = 4$ is the massless ϕ^4 theory. Classical scale invariance normally does not imply quantum scale invariance. See the discussion of the beta function below.

1.4.1 Conformal Invariance

A transformation:

$$x \to \tilde{x}(x) \tag{1.11}$$

is said to be conformal if the transformation satisfies:

$$\frac{\partial \tilde{x^\mu}}{\partial x^\rho}\frac{\partial \tilde{x^\nu}}{\partial x^\sigma}\eta_{\mu\nu} = \lambda^2(x)\eta_{\rho\sigma} \tag{1.12}$$

for some function $\lambda^2(x)$. The conformal group contains as subgroups the isometries of the metric $\eta_{\mu\nu}$ (the Poincar group) and also the scaling transformations considered above. In fact, the scale-invariant theories in the previous section are also conformally-invariant.

1.5 ϕ^4 theory

Massive ϕ^4 theory illustrates a number of interesting phenomena in scalar field theory.

The Lagrangian density is:

$$\mathscr{L} = \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}\delta^{ij}\partial_i\phi\partial_j\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{4!}\phi^4. \tag{1.13}$$

1.5.1 Spontaneous symmetry breaking

This Lagrangian has a Z_2 symmetry under the transformation $\phi \to -\phi$.

This is an example of an internal symmetry, in contrast to a space-time symmetry.

If m^2 is positive, the potential $V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{g}{4!}\phi^4$ has a single minimum, at the origin. The solution $\phi = 0$ is clearly invariant under the Z_2 symmetry. Conversely, if m^2 is negative, then one can readily see that the potential $V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{g}{4!}\phi^4$ has two minima. This is known as a double well potential, and the lowest energy states (known as the vacua, in quantum field theoretical language) in such a theory are not invariant under the Z_2 symmetry of the action (in fact it maps each of the two vacua into the other). In this case, the Z_2 symmetry is said to be spontaneously broken.

1.5.2 Kink solutions

The ϕ^4 theory with a negative m^2 also has a kink solution, which is a canonical example of a soliton. Such a solution is of the form:

$$\phi(\vec{x},t) = \pm \frac{m}{2\sqrt{g}} \tanh\left(\frac{m(x-x_0)}{\sqrt{2}}\right) \tag{1.14}$$

where x is one of the spatial variables (ϕ is taken to be independent of t, and the remaining spatial variables). The solution interpolates between the two different vacua of the double well potential. It is not possible to deform the kink into a constant solution without passing through a solution of infinite energy, and for this reason the kink is said to be stable. For $D > 2$, i.e. theories with more than one spatial dimension, this solution is called a domain wall.

Another well-known example of a scalar field theory with kink solutions is the sine-Gordon theory.

1.6 Complex scalar field theory

In a complex scalar field theory, the scalar field takes values in the complex numbers, rather than the real numbers. The action considered normally takes the form:

$$\mathcal{S} = \int \mathrm{d}^{D-1}x\mathrm{d}t\mathcal{L} = \int \mathrm{d}^{D-1}x\mathrm{d}t \left[\eta^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - V(|\phi|^2)\right] \tag{1.15}$$

This has a $U(1)$ symmetry, whose action on the space of fields rotates $\phi \to e^{i\alpha}\phi$, for some real phase angle α.

As for the real scalar field, spontaneous symmetry breaking is found if m^2 is negative. This gives rise to a Mexican hat potential which is analogous to the double-well potential in real scalar field theory, but now the choice of vacuum breaks a continuous $U(1)$ symmetry instead of a discrete one. This leads to a Goldstone boson.

1.7 $O(N)$ theory

One can express the complex scalar field theory in terms of two real fields, $\phi^1 = Re\phi$ and $\phi^2 = Im\phi$ which transform in the vector representation of the $U(1) = O(2)$ internal symmetry. Although such fields transform as a vector under the internal symmetry, they are still Lorentz scalars. This can be generalised to a theory of N scalar fields transforming in the vector representation of the $O(N)$ symmetry. The Lagrangian for an $O(N)$-invariant scalar field theory is typically of the form:

$$\mathscr{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi \cdot \partial_\nu\phi - V(\phi \cdot \phi) \tag{1.16}$$

using an appropriate $O(N)$-invariant inner product.

1.8 Quantum scalar field theory

In quantum field theory, the fields, and all observables constructed from them, are replaced by quantum operators on a Hilbert space. This Hilbert space is built on a vacuum state, and dynamics are governed by a Hamiltonian, a positive operator which annihilates the vacuum. A construction of a quantum scalar field theory may be found in the canonical quantization article, which uses canonical commutation relations among the fields as a basis for the construction. In brief, the basic variables are the field ϕ and its canonical momentum . Both fields are Hermitian. At spatial points $\vec{x}$, $\vec{y}$ at equal times, the canonical commutation relations are given by

$$[\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0, and [\phi(\vec{x}), \pi(\vec{y})] = i\delta(\vec{x} - \vec{y}) \tag{1.17}$$

and the free Hamiltonian is:

$$H = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2\right] \tag{1.18}$$

A spatial Fourier transform leads to momentum space fields:

$$\tilde{\phi}(\vec{k}) = \int d^3x e^{-i\vec{k}\cdot\vec{x}}\phi(\vec{x}), and \tilde{\pi}(\vec{k}) = \int d^3x e^{-i\vec{k}\cdot\vec{x}}\pi(\vec{x}) \tag{1.19}$$

which are used to define annihilation and creation operators:

$$a(\vec{k}) = \left(E\tilde{\phi}(\vec{k}) + i\tilde{\pi}(\vec{k})\right), a^\dagger(\vec{k}) = \left(E\tilde{\phi}(\vec{k}) - i\tilde{\pi}(\vec{k})\right), \tag{1.20}$$

where $E = \sqrt{k^2 + m^2}$. These operators satisfy the commutation relations:

$$[a(\vec{k}_1), a(\vec{k}_2)] = [a^\dagger(\vec{k}_1), a^\dagger(\vec{k}_2)] = 0, [a(\vec{k}_1), a^\dagger(\vec{k}_2)] = (2\pi)^3 2E\delta(\vec{k}_1 - \vec{k}_2) \tag{1.21}$$

The state $|0>$ annihilated by all of the operators a is identified as the bare vacuum, and a particle with momentum $\vec{k}$ is created by applying $a^\dagger(\vec{k})$ to the vacuum. Applying all possible combinations of creation operators to the vacuum constructs the Hilbert space. This construction is called Fock space. The vacuum is annihilated by the Hamiltonian:

$$H = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} a^\dagger(\vec{k}) a(\vec{k}) \tag{1.22}$$

where the zero-point energy has been removed by Wick ordering.

Interactions can be included by adding an interaction Hamiltonian. For a ϕ^4 theory, this corresponds to adding a Wick ordered term $g : \phi^4 : /4!$ to the Hamiltonian, and integrating over x. Scattering amplitudes may be calculated from this Hamiltonian in the interaction picture. These are constructed in perturbation theory by means of the Dyson series, which gives the time-ordered products, or n-particle Green's functions $\langle 0|\mathscr{T}\{\phi(x_1)\cdots\phi(x_n)\}|0\rangle$ as described in the Dyson series article. The Green's functions may also be obtained from a generating function that is constructed as a solution to the Schwinger-Dyson equation.

1.8.1 Feynman Path Integral

The Feynman diagram expansion may be obtained also from the Feynman path integral formulation. The time ordered vacuum expectation values of polynomials in ϕ, known as the n-particle Green's functions, are constructed by integrating over all possible fields, normalized by the vacuum expectation value with no external fields:

$$\langle 0|\mathscr{T}\{\phi(x_1)\cdots\phi(x_n)\}|0\rangle = \frac{\int \mathscr{D}\phi\phi(x_1)\cdots\phi(x_n)e^{i\int d^4x\left(\frac{1}{2}\partial^\mu\phi\partial_\mu\phi-\frac{m^2}{2}\phi^2-\frac{g}{4!}\phi^4\right)}}{\int \mathscr{D}\phi e^{i\int d^4x\left(\frac{1}{2}\partial^\mu\phi\partial_\mu\phi-\frac{m^2}{2}\phi^2-\frac{g}{4!}\phi^4\right)}} \tag{1.23}$$

All of these Green's functions may be obtained by expanding the exponential in $J(x)\phi(x)$ in the generating function

$$Z[J] = \int \mathscr{D}\phi e^{i\int d^4x\left(\frac{1}{2}\partial^\mu\phi\partial_\mu\phi-\frac{m^2}{2}\phi^2-\frac{g}{4!}\phi^4+J\phi\right)} = Z[0]\sum_{n=0}^{\infty}\frac{i^n J(x_1)\cdots J(x_n)}{n!}\langle 0|\mathscr{T}\{\phi(x_1)\cdots\phi(x_n \tag{1.24}$$

A Wick rotation may be applied to make time imaginary. Changing the signature to $(++++)$ then turns the Feynman integral into a statistical mechanics partition function in Euclidean space:

$$Z[J] = \int \mathscr{D}\phi e^{-\int d^4x\left(\frac{1}{2}(\nabla\phi)^2+\frac{m^2}{2}\phi^2+\frac{g}{4!}\phi^4+J\phi\right)} \tag{1.25}$$

Normally, this is applied to the scattering of particles with fixed momenta, in which case, a Fourier transform is useful, giving instead:

$$\tilde{Z}[\tilde{J}] = \int \mathscr{D}\tilde{\phi} e^{-\int d^4p\left(\frac{1}{2}(p^2+m^2)\tilde{\phi}^2+\frac{\lambda}{4!}\tilde{\phi}^4-\tilde{J}\tilde{\phi}\right)} \tag{1.26}$$

The standard trick to evaluate this functional integral is to write it as a product of exponential factors, schematically:

$$\tilde{Z}[\tilde{J}] \sim \int \mathscr{D}\tilde{\phi}\prod_p\left[e^{-(p^2+m^2)\tilde{\phi}^2/2}e^{-g\tilde{\phi}^4/4!}e^{\tilde{J}\tilde{\phi}}\right] \tag{1.27}$$

The second two exponential factors can be expanded as power series, and the combinatorics of this expansion can be represented graphically. The integral with $\lambda = 0$ can be treated as a product of infinitely many elementary Gaussian integrals, and the result may be expressed as a sum of Feynman diagrams, calculated using the following Feynman rules:

* Each field $\tilde{\phi}(p)$ in the n-point Euclidean Green's function is represented by an external line (half-edge) in the graph, and associated with momentum p.
* Each vertex is represented by a factor $-g$.
* At a given order g^k, all diagrams with n external lines and k vertices are constructed such that the momenta flowing into each vertex is zero. Each internal

line is represented by a propagator $1/(q2+m2)$, where q is the momentum flowing through that line.
* Any unconstrained momenta are integrated over all values.
* The result is divided by a symmetry factor, which is the number of ways the lines and vertices of the graph can be rearranged without changing its connectivity.
* Do not include graphs containing "vacuum bubbles", connected subgraphs with no external lines.

The last rule takes into account the effect of dividing by $\tilde{Z}[0]$. The Minkowski-space Feynman rules are similar, except that each vertex is represented by $-ig$, while each internal line is represented by a propagator $fraci(q^2-m^2+i\varepsilon)$, where the term represents the small Wick rotation needed to make the Minkowski-space Gaussian integral converge.

1.8.2 Renormalization

The integrals over unconstrained momenta, called "loop integrals", in the Feynman graphs typically diverge. This is normally handled by renormalization, which is a procedure of adding divergent counter-terms to the Lagrangian in such a way that the diagrams constructed from the original Lagrangian and counter-terms is finite. A renormalization scale must be introduced in the process, and the coupling constant and mass become dependent upon it.

The dependence of a coupling constant g on the scale λ is encoded by a beta function, $\beta(g)$, defined by the relation:

$$\beta(g) = \lambda \frac{\partial g}{\partial \lambda} \tag{1.28}$$

This dependence on the energy scale is known as the running of the coupling parameter, and theory of this kind of scale-dependence in quantum field theory is described by the renormalization group.

Beta-functions are usually computed in an approximation scheme, most commonly perturbation theory, where one assumes that the coupling constant is small. One can then make an expansion in powers of the coupling parameters and truncate the higher-order terms (also known as higher loop contributions, due to the number of loops in the corresponding Feynman graphs).

The beta-function at one loop (the first perturbative contribution) for the ϕ^4 theory is:

$$\beta(g) = \frac{3}{16\pi^2} g^2 + O(g^3) \tag{1.29}$$

The fact that the sign in front of the lowest-order term is positive suggests that the coupling constant increases with energy. If this behavior persists at large couplings, this would indicate the presence of a Landau pole at finite energy, or quantum triviality. The question can only be answered non-perturbatively, since it involves strong coupling.

A quantum field theory is trivial when the running coupling, computed through its beta function, goes to zero when the cutoff is removed. Consequently, the propagator becomes that of a free particle and the field is no longer interacting. Alternatively, the field theory may be interpreted as an effective theory, in which the cutoff is not removed, giving finite interactions but leading to a Landau pole at some energy scale. For a ϕ^4 interaction, Michael Aizenman proved that the theory is indeed trivial for space-time dimension $D \geq 5$. For $D = 4$ the triviality has yet to be proven rigorously, but lattice computations have confirmed this. (See Landau pole for details and references.) This fact is relevant as the Higgs field, for which triviality bounds are used to set limits on the Higgs mass, based on the new physics must enter at a higher scale (perhaps the Planck scale) to prevent the Landau pole from being reached.

1.9 Scalar Field equation for general metrics

Until here we learnd how to deal with scalar field equation in Minkowski-space. Now we want to focous more about scalar field equation in curved spaces, and solve the equation for some well known curved spaces which has specific metrics, such as Schwarzschild, FRW, and Anti de Sitter metric.

The metric tensor is such an important object in curved space that is given a new symbol, let say $g_{\mu\nu}$, while $\eta_{\mu\nu}$ specificly be used for Minkowski space as a metric symbol. $g_{\mu\nu}$ should obey some rules to be considered as metric. First of all it should be a symmetric $(0,2)$ matrix, or more generaly tensor. Second, it usually taked non-degenerate, it meas that its determinant does not vanish, $g = |g_{\mu\nu}|$. If the determinant is not zero, then obviously the matrix, tensor, has its own inverse, which is uniqe:

$$g^{\mu\nu}g_{\nu\sigma} = \delta^{\nu}_{\sigma} \tag{1.30}$$

In minkowski-space we difined the line elemet:

$$ds = \eta_{\mu\nu}dx^{\mu}dx^{\nu} \tag{1.31}$$

Respactively, it is possible to define the line element in the same way, which is more general and it can be use for any kind of curved space or flat space:

$$ds = g_{\mu\nu}dx^{\mu}dx^{\nu} \tag{1.32}$$

Based on the new metric, we want to deveplope the new scalar field equation of motion. Lagrangian has the form of,

$$\mathscr{L} = \sqrt{-|g_{\mu\nu}|}\left[\frac{1}{2}g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi - \frac{1}{2}m^2\phi^2\right] \tag{1.33}$$

The next step, we have to derive the equation of motion from Lagrangian, equation 1.33,

$$\begin{aligned} 0 =& \frac{\partial\mathscr{L}}{\partial\phi} - \frac{\partial}{\partial\mu}\frac{\mathscr{L}}{\partial\phi_{\mu}} \rightarrow \\ 0 =& \frac{\partial}{\partial\mu}\sqrt{-|g_{\mu\nu}|}g^{\mu\nu}\frac{\partial\phi}{\partial\nu} - \sqrt{-|g_{\mu\nu}|}m^2\phi \end{aligned} \tag{1.34}$$

So the motion equation for scalar field has the form of,

$$0 = \left[\frac{1}{\sqrt{-|g_{\mu\nu}|}}\partial_{\mu}\sqrt{-|g_{\mu\nu}|}g^{\mu\nu}\partial_{\nu} - m^2\right]\phi \tag{1.35}$$

For few sections we just introduce some famous metrics and their tensors and after that we would try to solve the equation of motion for scalar field, equation 2.3.

1.9.1 Schwarzschild Metric

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2 d\Omega_{D-2}^2 \tag{1.36}$$

For example the Schwartzschild black hole of mass M is describedd by the metric:

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2 d\Omega^2 \tag{1.37}$$

The Schwarzschild metric describes the spacetime around a spherically symmetric body, such as a planet, or a black hole. With coordinates $(x^0, x^1, x^2, x^3) = (ct, r, \theta, \phi)$, we can write the metric as:

$$(g_{\mu\nu}) = \begin{pmatrix} -(1 - \frac{2GM}{rc^2}) & 0 & 0 & 0 \\ 0 & (1 - \frac{2GM}{rc^2})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \tag{1.38}$$

The intresting point about the metric of Schwarzschild is that if one puts the mass of black hole equal to zero the metric would become similar to Minkowski's metric. Because when the mass is equal to zero it means that there is no black hole so the metric should be the same as Minkowski's metric.

1.9.2 FRW Metric

Friadmann-Rabertson-Walker, or simpely FRW, metric describe homogeneous, isotropic universe, including, to a good degree of approximation, the portion we have seen of our own universe. The metric, in spherical coordinate, has the form of:

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right) \tag{1.39}$$

and the tensor of FRW metric has the form of:

$$(g_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a(t)}{1-k^2r^2} & 0 & 0 \\ 0 & 0 & a(t)r^2 & 0 \\ 0 & 0 & 0 & a(t)r^2 \sin(\theta)^2 \end{pmatrix} \tag{1.40}$$

In these equations $a(t)$ is the scale factor, cosmic scale factor or sometimes the Robertson-Walker scale factor parameter of the Friedmann equations is a function of time which represents the relative expansion of the universe. Modern observations suggests that the universe is expanding and introducing $a(t)$ is the way the help us to consider the expantion of our univerce in our equations. And the relation of scale factor and Hubble parameter has the form of:

$$H = \frac{\dot{a}(t)}{a(t)} \tag{1.41}$$

1.9.3 Anti de Sitter space

In mathematics and physics, n-dimensional anti de Sitter space, sometimes written AdS_n, is a maximally symmetric Lorentzian manifold with constant negative scalar curvature. It is the Lorentzian analogue of n-dimensional hyperbolic space, just as Minkowski space and de Sitter space are the analogues of Euclidean and elliptical spaces respectively.

It is best known for its role in the AdS/CFT correspondence.

In the language of general relativity, anti de Sitter space is a maximally symmetric, vacuum solution of Einstein's field equation with a negative (attractive) cosmological constant Λ (corresponding to a negative vacuum energy density and positive pressure).

In mathematics, anti de Sitter space is sometimes defined more generally as a space of arbitrary signature (p,q). Generally in physics only the case of one timelike dimension is relevant. Because of differing sign conventions, this may correspond to a signature of either $(n1,1)$ or $(1,n1)$.

A coordinate patch covering part of the space gives the half-space coordinatization of anti de Sitter space. The metric for this patch is:

$$ds^2 = \frac{1}{z^2}\left(-dt^2 + dx^2 + dy^2 + dz^2\right) \tag{1.42}$$

We easily see that this metric is conformally equivalent to a flat half-space Minkowski spacetime.

The constant time slices of this coordinate patch are hyperbolic spaces in the Poincar half-plane metric. In the limit as $z = 0$, this half-space metric reduces to a Minkowski metric $dz^2 = dt^2 - dx^2 - dy^2$; thus, the anti-de Sitter space contains a conformal Minkowski space at infinity (”infinity” having y-coordinate zero in this patch).

In AdS space time is periodic, and the universal cover has non-periodic time. The coordinate patch above covers half of a single period of the spacetime.

Because the conformal infinity of AdS is timelike, specifying the initial data on a spacelike hypersurface would not determine the future evolution uniquely (i.e. deterministically) unless there are boundary conditions associated with the conformal infinity.

The ”half-space” region of anti de Sitter space and its boundary.

Another commonly used coordinate system which covers the entire space is given by the coordinates t, $r \geqslant 0$ and the hyperpolar coordinates α, θ and ϕ.

$$\begin{aligned} ds^2 &= -\left(k^2r^2+1\right)dt^2 + \frac{1}{k^2r^2+1}dr^2 + r^2 d\Omega^2 \\ &= -\left(k^2r^2+1\right)dt^2 + \frac{1}{k^2r^2+1}dr^2 + r^2\left(d\theta^2 + \sin(\theta)^2 d\phi^2\right) \end{aligned} \tag{1.43}$$

The image on the right represents the ”half-space” region of anti deSitter space and its boundary. The interior of the cylinder corresponds to Anti de Sitter spacetime, while its cylindrical boundary corresponds to its conformal boundary. The green shaded region in the interior corresponds to the region of AdS covered by the half-space coordinates and it is bounded by two null, aka lightlike, geodesic hyperplanes; the green shaded area on the surface corresponds to the region of conformal space covered by Minkowski space.

The green shaded region covers half of the AdS space and half of the conformal spacetime; the left ends of the green discs will touch in the same fashion as the right ends.

1.9.4 Scalar Field in Anti de Sitter space

Now we want to find the general solution of the equation of motion, 2.3 , in Anti de Sitter space, from now AdS, space. First we focuse on general solution in Cartesian coordinate system and after that we figure out how to find the general solution of equation of motion for AdS space with Spherical coordinate basis.

Equation 1.42 shows the metric of AdS space in Caretesian coordinate system. Based on equation 1.42 the metric tensor has the form of:

$$(g_{\mu\nu}) = \begin{pmatrix} -\frac{1}{z^2} & 0 & 0 & 0 \\ 0 & \frac{1}{z^2} & 0 & 0 \\ 0 & 0 & \frac{1}{z^2} & 0 \\ 0 & 0 & 0 & \frac{1}{z^2} \end{pmatrix} \tag{1.44}$$

Determinant of tensor $g_{\mu\nu}$, equation 1.44, is $|g_{\mu\nu}| = -\frac{1}{z^8}$. So the for the scalar fields and in AdS space with cartesian coordinate system the equation of motion, equation 2.3, would be,

$$\left[-z^2\frac{\partial^2}{\partial t^2} + z^2\frac{\partial^2}{\partial x^2} + z^2\frac{\partial^2}{\partial y^2} + z^2\frac{\partial^2}{\partial z^2} - 2z\frac{\partial}{\partial z} - m^2\right]\phi = 0 \tag{1.45}$$

and then we can divide both side of equaion by z^2,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{2}{z}\frac{\partial\phi}{\partial z} - \frac{m^2}{z^2}\phi = \frac{\partial^2\phi}{\partial t^2} \tag{1.46}$$

Now, we can use seperation of variables method to find the solution of equation of motion in AdS space with cartesian coordinate system, equation 1.46. Based on this method, $\phi(t,x,y,z) = T(r)X(x)Y(y)Z(z)$, and by substituting the new function in our equation, equation 1.46, and then divide both side of equation by ϕ, $T(r)X(x)Y(y)Z(z)$, we would have:

$$\frac{\frac{\partial^2 X(x)}{\partial x^2}}{X(x)} + \frac{\frac{\partial^2 Y(y)}{\partial y^2}}{Y(y)} + \frac{\frac{\partial^2 Z(z)}{\partial z^2} - \frac{2}{z}\frac{\partial Z(z)}{\partial z} - \frac{m^2}{z^2}Z(z)}{Z(z)} = \frac{\frac{\partial^2 T(t)}{\partial t^2}}{T(t)} \tag{1.47}$$

There are four terms, three spatial terms and one time term, which that each ones depend just on one varibale. So we can easily conclude that, as you may learned on basic courses like partial differential equation, each term should be constant. Lest say C_1, C_2, and C_3 for spatial terms and C for time term. Then we have to solve each term seperately fo find the answers. This equation is very similar to equation of wave, like what we have in electrodynamics. The differenc is that one of our spatial terms is not similar what we had before on wave problems. But we use the same method to find the solution for this partial differencial equation. Let's do the calculations.

For time term, it would be like,

$$C = \frac{\frac{\partial^2 T(t)}{\partial t^2}}{T(t)} \tag{1.48}$$

and then,

$$T(t) = a_1 e^{i\omega t} + a_2 e^{i\omega' t} \tag{1.49}$$

which ω and ω' can be complex number. Here, for simplicity, we assume that $T(t) = a_1 e^{i\omega t}$, then the constant, C, would be equal to $-\omega^2$ and a_1 is just a normalization factor. For the next step we try to find the solution of spatial terms. The first two spatial terms, $X(x)$ and $Y(y)$, are the same as time term and we can solve them like time term, so for this two term we have:

$$\begin{aligned} X(x) &= b_1 e^{ik_1 x} + b_2 e^{k_{12} x} \\ Y(y) &= b_3 e^{ik_2 y} + b_4 e^{k_{22} y} \end{aligned} \tag{1.50}$$

and like the time term for simplicity we can assume that the second terms are zero. So, C_1 and C_2 are k_1^2 and k_2^2 respectively. b_1 and b_3 are just normalization factors. The most important part of the equation is the last spatial time which make difference between Minkowski's space and AdS space. The differencial equation for the second term has the form of,

$$\frac{\partial^2 Z(z)}{\partial z^2} - \frac{2}{z}\frac{\partial Z(z)}{\partial z} - (C_3 + \frac{m^2}{z^2})Z(z) = 0 \tag{1.51}$$

and C_3 is a constant that related with C_2, C_3, and C with the following relationship,

$$C_3 = C - C_1 - C_2 = -[\omega^2 - k_1^2 - k_2^2] = -k^2 \tag{1.52}$$

If we multiply quation by z^2 then it becomes famous Bessel differencial equation, which its answers are Bessel functions.

$$z^2 \frac{\partial^2 Z(z)}{\partial z^2} - 2z\frac{\partial Z(z)}{\partial z} - (z^2 C_3 + m^2)Z(z) = 0 \tag{1.53}$$

and the asnwer of this equation has the form of,

$$Z(z) = d_1 z^{\frac{3}{2}} J_m(\sqrt{k^2} z) + d_2 z^{\frac{3}{2}} Y_m(\sqrt{k^2} z) \tag{1.54}$$

That d_1 and d_2 are just normalization factors. So the complete solution of motion equation for scalr field in AdS space with cartesian coordinate basis has the form of,

$$\phi(t,x,y,z) = A_1 e^{ik_\mu x_\mu} z^{\frac{3}{2}} J_m(\sqrt{k^2} z) + A_2 e^{ik_\mu x_\mu} z^{\frac{3}{2}} Y_m(\sqrt{k^2} z) \tag{1.55}$$

That A_1 and A_2 are just normalization factors and $k_\mu x_\mu$ is $\omega t + k_1 x + k_2 y$ and ω, k_1, and k_2 can be complex numbers.

Now, lets use the same method to find the solution of motion equation for scalar field in AdS space with Spherical coordinate basis. Equation 1.43 shows the metric of AdS space in Spherica coordinate system. Based on equation 1.43 the metric tensor is:

$$(g_{\mu\nu}) = \begin{pmatrix} -\left(k^2 r^2 + 1\right) & 0 & 0 & 0 \\ 0 & \frac{1}{k^2 r^2 + 1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin(\theta)^2 \end{pmatrix} \tag{1.56}$$

Determinant of tensor $g_{\mu\nu}$, equation 1.56, is $|g_{\mu\nu}| = -r^4 \sin(\theta)^2$. So the equation of motion for scalar field in AdS space with spherical coordinate system basis, equation 2.3, has the form of,

$$-\frac{1}{(k^2 r^2 + 1)} \frac{\partial^2 \phi}{\partial t^2} + (k^2 r^2 + 1) \frac{\partial^2 \phi}{\partial r^2} + (4k^2 r + \frac{2}{r}) \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \Omega^2} - m^2 \phi = 0$$

In case of spherical symmetry we can seperate the angular varibales from radial and time variables, r and t. In basic quantum mechanics courses and partial differenctial courses it is proved that for spherical symmetry the eigenvalue of angular part is: $l(l+1)$. So we can divide ϕ into two part, angular part and time-radial part, $\phi(t,r,\theta,\Phi) = X(t,r) Y_l^m(\theta,\Phi)$, also is is possible to show that $\frac{\partial^2 Y_l^m(\theta,\Phi)}{\partial \Omega^2} = l(l+1)$. On the other hand we can assume that function of $X(t,r)$ are sparetable into to function of $R(r)$ and $T(t)$. Then simply we can rewrite equation of motion,

$$-\frac{R(r)}{(k^2r^2+1)}\frac{\partial^2 T(t)}{\partial t^2}+(k^2r^2+1)T(t)\frac{\partial^2 R(r)}{\partial r^2} \tag{1.57}$$
$$+(4k^2r+\frac{2}{r})T(t)\frac{\partial R(r)}{\partial r}+(\frac{l(l+1)}{r^2}-m^2)R(r)T(t)=0$$

or,

$$(k^2r^2+1)^2\frac{\frac{\partial^2 R(r)}{\partial r^2}}{R(r)}+(k^2r^2+1)(4k^2r+\frac{2}{r})\frac{\frac{\partial R(r)}{\partial r}}{R(r)} \tag{1.58}$$
$$+\frac{l(l+1)}{r^2}+(k^2l(l+1)-m^2)=\frac{\frac{\partial^2 T(t)}{\partial t^2}}{T(t)}$$

Becasue one side of the equation is function of time and the other side is function of r then, for having solution, both side should be constant, like C. The solution for time term is:

$$T(t)=A_1e^{i\omega t}+A_2e^{i\omega' t} \tag{1.59}$$

Which for simplicity we assume the the second term is zero then, we have $T(t)=A_1e^{i\omega t}$ that A_1 is a normalization factor, and ω can be a complex number. On the othr hand, our constant, C, would be $-\omega^2$. So differential equation for radial term is,

$$(k^2r^2+1)^2\frac{\partial^2 R(r)}{\partial r^2}+(k^2r^2+1)(4k^2r+\frac{2}{r})\frac{\partial R(r)}{\partial r} \tag{1.60}$$
$$+\left[\frac{l(l+1)}{r^2}+(k^2l(l+1)+\omega^2-m^2)\right]R(r)=0$$

We can solve this differential equation with series method. In series method we assume that, $R(r)=\sum_{-\infty}^{+\infty}a_nr^n$, and all n's should be integer. First we substitute a_0 in our differential equation and we get $a_0=constant$. We do the same thing for $n>0$ and we get $a_n=0$, for $n>0$, because based on boundary condition in large radios the function should approaches to zero, based on this argument all for positive n's we should get zero. Now let's focous on $n<0$. By substituting $\sum a_{-n}r^{-n}$ in equation 1.60 we get,

$$
\begin{aligned}
0 = \sum_{n=0}^{\infty} & a_{-n} n(n+1) k^4 r^{-n+2} + a_{-n} 2k^2 n(n+1) r^{-n} + a_{-n} n(n+1) r^{-n-2} \\
& - a_{-n} 4nk^2 r^{-n+2} - a_{-n} 6nk^2 r^{-n} - a_{-n} n r^{-n-2} \\
& - a_{-n} l(l+1) k^4 r^{-n-2} + a_{-n} \left[k^2 l(l+1) + \omega^2 - m^2\right] r^{-n}
\end{aligned} \tag{1.61}
$$

Then the coefficient of each r^n should be zero, so we have,

$$
\begin{aligned}
0 = & a_{-n-2}(n+2)(n+3) k^4 r^{-n} + a_{-n} 2k^2 n(n+1) r^{-n} + a_{-n}(n-2)(n-1) r^{-n} \\
& - a_{-n-2} 4(n+2) k^2 r^{-n} - a_{-n} 6nk^2 r^{-n} - a_{-n}(n-2) r^{-n} \\
& - a_{-n+2} l(l+1) k^4 r^{-n} + a_{-n} \left[k^2 l(l+1) + \omega^2 - m^2\right] r^{-n}
\end{aligned} \tag{1.62}
$$

Finaly, we can find all a_n's, if we have boundari condition,

$$
a_{-n-2} = \frac{a_{-n} \left[(-2n^2 + 4n - l(l+1)) k^2 + m^2 - \omega^2\right] + a_{-n+2} \left[l(l+1) - n^2 + 3n - 2\right]}{(n^2 + n - 2) k^4} \tag{1.63}
$$

1.10 Questions

1. Derive the partial differential equation for equation of motion in Schwarzschild space with spherical coordinate basis.

2. Derive the partial differential equation for equation of motion in FRW space with spherical coordinate basis.

Chapter 2

Scalar Field in AdS space

2.1 n-dimentional AdS space

In chapter 1 we learned how to solve the scalar field in AdS space in cartesian and spherical coordanate basis. In this chapter, we are looking into this problem further and we discuss about the differenc features of AdS sapce in detail.

The metric of n-dimentional AdS space for m timelike dimention and $n-m$ sapcelike dimention has the form of,

$$ds^2 = \frac{1}{z^2}\left[-\sum_{i=1}^{m} dt_i^2 + \sum_{i=1}^{n-m} dx_i^2\right] \tag{2.1}$$

Which in this equation we assume that x_n would be z coordinate, so the metric tensor has the form of,

$$g_{\mu\nu} = \frac{1}{z^2}\begin{pmatrix} \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix}_{m\times m} & 0 \\ 0 & \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{(n-m)\times(m-n)} \end{pmatrix} \tag{2.2}$$

And the determinant of this tensor is $|g_{\mu\nu}| = \frac{(-1)^m}{z^n}$. Let's use equation of motion for scalar scalar, equation 2.3 which was introduced in chapter 2, to derive the differential equation of evolution of field in n-dimentional AdS space,

$$0 = \left[\frac{1}{\sqrt{-|g_{\mu\nu}|}}\partial_\mu\sqrt{-|g_{\mu\nu}|}g^{\mu\nu}\partial_\nu - m^2\right]\phi \tag{2.3}$$

$$\left[-z^2\sum_{i=1}^{m}\frac{\partial^2}{\partial t_i^2} + z^2\sum_{i=1}^{n-m-1}\frac{\partial^2}{\partial x_i^2} + z^2\frac{\partial^2}{\partial z^2} + (2-n)z\frac{\partial}{\partial z} - m^2\right]\phi = 0 \tag{2.4}$$

and then we can divide both side of equaion by z^2,

$$\sum_{i=1}^{n-m-1}\frac{\partial^2\phi}{\partial x_i^2} + \frac{\partial^2\phi}{\partial z^2} + \frac{2-n}{z}\frac{\partial\phi}{\partial z} - \frac{m^2}{z^2}\phi = \sum_{i=1}^{m}\frac{\partial^2\phi}{\partial t_i^2} \tag{2.5}$$

As we saw in chapter 1 if we use seperation of variables method then there is a simple solution for time and all spetial terms, except z term for Z spatial coordinate. The time and spatial, except z, part of field ϕ has the form of,

$$\phi(x_\mu) = e^{ik.x}Z(z) = e^{ik_\mu x_\mu}Z(z) \tag{2.6}$$

By substituting $\phi(x_\mu)$ in equation 2.5 we have,

$$\frac{\partial^2 Z(z)}{\partial z^2} + \frac{2-n}{z}\frac{\partial Z(z)}{\partial z} + \left[k^2 - \frac{m^2}{z^2}\right]Z(z) = 0 \tag{2.7}$$

Let's say $Z(z) = z^\alpha f(z)$, then $\frac{\partial Z(z)}{\partial z} = \alpha z^{\alpha-1}f(z) + z^\alpha\frac{\partial f(z)}{\partial z}$ and $\frac{\partial^2 Z(z)}{\partial z^2} = \alpha(\alpha-1)z^{\alpha-2}f(z) + 2\alpha z^{\alpha-1}\frac{\partial f(z)}{\partial z} + z^\alpha\frac{\partial^2 f(z)}{\partial z^2}$. Then by substituting these three equation in equation 2.7 we have,

$$\begin{aligned}\alpha(\alpha-1)z^{\alpha-2}f(z) + 2\alpha z^{\alpha-1}\frac{\partial f(z)}{\partial z} + z^\alpha\frac{\partial^2 f(z)}{\partial z^2}\\ + \frac{2-n}{z}\left[\alpha z^{\alpha-1}f(z) + z^\alpha\frac{\partial f(z)}{\partial z}\right] + \left[k^2 - \frac{m^2}{z^2}\right]z^\alpha f(z) = 0\end{aligned} \tag{2.8}$$

And then,

$$z^2\frac{\partial^2 f(z)}{\partial z^2} + (2\alpha+2-n)z\frac{\partial f(z)}{\partial z} + \left[k^2z^2 - (m^2 + \alpha(n-1-\alpha))\right]f(z) = 0 \tag{2.9}$$

By choosing $\alpha = \frac{n-1}{2}$, we get the Bessel differential equation and the general answer has the form of,

$$\begin{aligned}\phi(x_i,z) =& A_1 e^{(ik_\mu x_\mu)} z^{\frac{n-1}{2}} J_\nu(kz) \\ &+ A_2 e^{(ik_\mu x_\mu)} z^{\frac{n-1}{2}} Y_\nu(kz) \quad , \quad \nu = \sqrt{m^2 + \frac{(n-1)^2}{4}}\end{aligned} \tag{2.10}$$

Which k_μ's can be complex numbers and A_1 and A_2 are just normalization facors which depends on boundary condition.

Let's see what happened for our field when it approached to $z = 0$ and $z \to \infty$. Bsed of definition of Bessel function in the Literature,

$$z^{\frac{n-1}{2}} J_\nu(kz) = z^{\frac{n-1}{2}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!\Gamma(i+\nu+1)} (\frac{k}{2}z)^{(2i+\nu)} \tag{2.11}$$

and,

$$z^{\frac{n-1}{2}} Y_\nu(kz) = z^{\frac{n-1}{2}} \frac{J_\nu(kz)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)} \tag{2.12}$$

Then,

$$\lim_{z\to 0} z^{\frac{n-1}{2}} J_\nu(kz) = z^{\frac{n-1}{2}+\nu} \times \frac{k^\nu}{2^\nu \Gamma(\nu+1)} \quad , \quad \nu = \sqrt{m^2 + \frac{n^2-2n+1}{4}} \tag{2.13}$$

and,

$$\lim_{z\to 0} z^{\frac{n-1}{2}} Y_\nu(kz) = -z^{\frac{n-1}{2}-\nu} \times \frac{\Gamma(\nu)2^\nu}{k^\nu \pi} \quad , \quad \nu = \sqrt{m^2 + \frac{n^2-2n+1}{4}} \tag{2.14}$$

or simply,

$$\lim_{z\to 0} z^{\frac{n-1}{2}} J_\nu(kz) = C z^{\frac{n-1}{2}+\sqrt{m^2+\frac{(n+1)^2}{4}}} \tag{2.15}$$

$$\lim_{z\to 0} z^{\frac{n-1}{2}} Y_\nu(kz) = C' z^{\frac{n-1}{2}-\sqrt{m^2+\frac{(n-1)^2}{4}}} \tag{2.16}$$

For case $n = 4$, it would be,

$$\lim_{z\to 0} z^{\frac{n-1}{2}} J_\nu(kz) = C z^{\frac{3}{2}+\sqrt{m^2+\frac{9}{4}}} \tag{2.17}$$

and,

$$\lim_{z\to 0} z^{\frac{n-1}{2}} Y_\nu(kz) = C' z^{\frac{3}{2}-\sqrt{m^2+\frac{9}{4}}} \tag{2.18}$$

2.2 Black hole in Anti de Sitter space

In theoretical physics, an AdS black hole is a black hole solution of general relativity or its extensions which represents an isolated massive object, but with a negative cosmological constant. Such a solution asymptotically approaches Anti de Sitter space at spatial infinity, and is a generalization of the Kerr vacuum solution, which asymptotically approaches Minkowski spacetime at spatial infinity.

In $3+1$ dimensions, the metric is given by,

$$ds^2 = -\left(k^2r^2 + 1 - \frac{C}{r}\right)dt^2 + \frac{1}{k^2r^2+1-\frac{C}{r}}dr^2 + r^2 d\Omega^2 \tag{2.19}$$

where t is the time coordinate, r is the radial coordinate, Ω are the polar coordinates, C is a constant and k is the AdS curvature.

In general, in $d+1$ dimensions, the metric is given by,

$$ds^2 = -\left(k^2r^2 + 1 - \frac{C}{r^{d-2}}\right)dt^2 + \frac{1}{k^2r^2+1-\frac{C}{r^{d-2}}}dr^2 + r^2 d\Omega^2 \tag{2.20}$$

According to the AdS/CFT correspondence, if gravity were quantized, an AdS black hole would be dual to a thermal state on the conformal boundary. In the context of say, AdS/QCD, this would correspond to the deconfinement phase of the quark-gluon plasma.

Also we can write the black hole's metric in $d+1$ dimention AdS space in another format which may be more useful,

$$ds^2 = \frac{u^2}{L^2}\left[-(1-(\frac{u_0}{u})^d)dt^2 + \sum_{i=1}^{d-1} dx_i^2\right] + \frac{L^2du^2}{u^2(1-(\frac{u_0}{u})^d)} \tag{2.21}$$

From now we assume that we have just one time like coordinate, though it would be easy to expand this idea into, let's say m-domantional time like space.

That $u = u_0$ is its horizon and $u \to \infty$ is its boundary. And the metric tensor has the form of,

$$g_{\mu\nu} = \begin{pmatrix} -\frac{u^2}{L^2}\left(1-(\frac{u_0^2}{u^2})^d\right) & & 0 \\ & \begin{pmatrix} \frac{u^2}{L^2} & 0 & \cdots & 0 \\ 0 & \frac{u^2}{L^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{u^2}{L^2} \end{pmatrix}_{d-1\times d-1} & \\ 0 & & \frac{L^2}{u^2\left(1-(\frac{u_0^2}{u^2})^d\right)} \end{pmatrix} \tag{2.22}$$

The determinant of this matrix is $-(\frac{u}{L})^{d-1}$. Now we want to solve the equation of motion for scalar field for this metric.

$$\begin{aligned} 0 = &-\frac{L^2}{u^2(1-(\frac{u_0}{u})^d)}\frac{\partial^2\phi}{\partial t^2} + \frac{L^2}{u^2}\frac{\partial^2\phi}{\partial x_i^2} + \frac{u^2(1-(\frac{u_0}{u})^d)}{L^2}\frac{\partial^2\phi}{\partial u^2} \\ &+ (\frac{L}{u})^{d-1}\left[\frac{u^{d-4}(d-3)}{L^{d-3}(1-(\frac{u_0}{u})^d)} - d\frac{1}{u}(\frac{u_0}{u})^d\frac{u^{d-3}}{L^{d-3}(1-(\frac{u_0}{u})^d)^2}\right]\frac{\partial\phi}{\partial u} \end{aligned} \tag{2.23}$$

We use the same method to solve equation . And we can assume that $\frac{\partial^2\phi}{\partial t^2}$ and $\frac{\partial^2\phi}{\partial x_i^2}$ are all constant. It is the same as we say we want to solve the d-dimentional wave equation for AdS black hole. So we have,

$$\begin{aligned} 0 = &\left[\frac{k_t^2 L^2}{u^2(1-(\frac{u_0}{u})^d)} - k_i^2\frac{L^2}{u^2}\right]f(u) + \frac{u^2(1-(\frac{u_0}{u})^d)}{L^2}\frac{d^2f(u)}{du^2} \\ &+ (\frac{L}{u})^{d-1}\left[\frac{L^2(d-3)}{u^3(1-(\frac{u_0}{u})^d)} - d\frac{1}{u}(\frac{u_0}{u})^d\frac{L^2}{u^2(1-(\frac{u_0}{u})^d)^2}\right]\frac{df(u)}{du} \end{aligned} \tag{2.24}$$

and then,

$$\begin{aligned} 0 = &\left[k_t^2 - k_i^2(1-(\frac{u_0}{u})^d)\right]f(u) + \frac{u^4(1-(\frac{u_0}{u})^d)^2}{L^4}\frac{d^2f(u)}{du^2} \\ &+ L^{d-1}\left[\frac{(d-3)}{u^d} - \frac{du_0^d}{u^{2d}}\frac{1}{(1-(\frac{u_0}{u})^d)}\right]\frac{df(u)}{du} \end{aligned} \tag{2.25}$$

Now let's change the variable u into $\frac{u_0}{w}$. This form would be more useful in this context, because our horizon and boundary changes from $w = 1$ to $w = 0$.

And one can expande the answer near $w = 1$ and/or $w = 0$. The metric has the form of,

$$ds^2 = L^2 \left[\frac{u_0^2}{L^4 w^2} \left[-(1-w^d)dt^2 + \sum_{i=1}^{d-1} dx_i^2 \right] + \frac{d^2 w}{w^2(1-w^d)} \right] \tag{2.26}$$

After rescaling time, and x_i coordinates we get,

$$ds^2 = \frac{L^2}{w^2} \left[-(1-w^d)dt^2 + \sum_{i=1}^{d-1} dx_i^2 + \frac{d^2 w}{w^2(1-w^d)} \right] \tag{2.27}$$

and the metric tensor would be,

$$g_{\mu\nu} = \begin{pmatrix} -\frac{L^2}{w^2}(1-w^d) & & 0 \\ & \begin{pmatrix} \frac{L^2}{w^2} & 0 & \cdots & 0 \\ 0 & \frac{L^2}{w^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{L^2}{w^2} \end{pmatrix}_{d-1\times d-1} & \\ 0 & & \frac{L^2}{w^2(1-w^d)} \end{pmatrix} \tag{2.28}$$

The determinant of this matrix is $-(\frac{L}{w})^{2d+2}$. Now we want to solve the equation of motion of scalar field for this metric.

$$\begin{aligned} 0 = &- \frac{w^2}{L^2(1-w^d)} \frac{\partial^2 \phi}{\partial t^2} + \frac{w^2}{L^2} \frac{\partial^2 \phi}{\partial x_i^2} + \frac{w^2(1-w^d)}{L^2} \frac{\partial^2 \phi}{\partial w^2} \\ &+ \frac{w}{L^2} \left[1 - d - w^d \right] \frac{\partial \phi}{\partial w} - m^2 \phi \end{aligned} \tag{2.29}$$

We use the seperation of variables method to solve equation . And we can assume that $\frac{\partial^2 \phi}{\partial t^2}$ and $\frac{\partial^2 \phi}{\partial x_i^2}$ are all constant. Let's say $\phi(t, x_i, w) = Ae^{ik.x} f(w) + Be^{-ik.x} g(w)$. For finding $f(w)$ and $g(w)$, first we assume that $B = 0$ and the we can find $f(w)$ then we assume that $A = 0$ and we can find $g(w)$. By substituting $\phi(t, x_i, w)$ in the equation and assuming $B = 0$ we get,

$$\begin{aligned} 0 = &\frac{w^2}{L^2(1-w^d)} k_t^2 f(w) - \sum_{i=1}^{d-1} \frac{w^2}{L^2} k_{x_i}^2 f(w) + \frac{w^2(1-w^d)}{L^2} \frac{\partial^2 f(w)}{\partial w^2} \\ &+ \frac{w}{L^2} \left[1 - d - w^d \right] \frac{\partial f(w)}{\partial w} - m^2 f(w) \end{aligned} \tag{2.30}$$

Now let's focous on the behaiviour of our function near boundary and near horizon. When $w \to 0$ it approaches to boundary. And when $w \to 1$ it approaches to horizon. Near boundary our ordinary diffrential equation has the form of,

$$0 = w^2 \frac{\partial^2 f(w)}{\partial w^2} + w(1-d)\frac{\partial f(w)}{\partial w} + \left[(k_t^2 - \sum_{i=1}^{d-1} k_{x_i}^2) w^2 - L^2 m^2 \right] f(w) \quad (2.31)$$

And this is a Bessel function which we have solved it before in section 2.1 we solved this equation and showed that it has two independent answer, $w^{\frac{d}{2}} J_\nu(kw)$ and $w^{\frac{d}{2}} Y_\nu(kw)$ that $\nu = \sqrt{m^2 + \frac{d^2}{4}}$, and also we proved that for $w \to 0$ the function behave,

$$\lim_{w \to 0} w^{\frac{d}{2}} J_\nu(kw) \longrightarrow w^{\frac{d}{2} + \sqrt{m^2 + \frac{d^2}{4}}} \quad (2.32)$$

$$\lim_{w \to 0} w^{\frac{d}{2}} Y_\nu(kw) \longrightarrow w^{\frac{d}{2} - \sqrt{m^2 + \frac{d^2}{4}}} \quad (2.33)$$

Near horizon it is a little more tricky. First we need to expand the coefficient of $f(w)$ and $\frac{\partial f(w)}{\partial w}$ near $w = 1$ and then one should try to find the behaviour of differential equation and function of $f(w)$. Let's divide both side of equation 2.30 by coefficient of $\frac{\partial^2 f(w)}{\partial w^2}$,

$$\begin{aligned} 0 = & \frac{\partial^2 f(w)}{\partial w^2} + \left[\frac{1 - d - w^d}{w(1 - w^d)} \right] \frac{\partial f(w)}{\partial w} \\ & + \left[\frac{1}{(1 - w^d)^2} k_t^2 - \frac{1}{1 - w^d} \sum_{i=1}^{d-1} k_{x_i}^2 - \frac{L^2 m^2}{w^2 (1 - w^d)} \right] f(w) \end{aligned} \quad (2.34)$$

Now one should try to expand the coefficient of $f(w)$ and $\frac{\partial f(w)}{\partial w}$ near $w = 1$. And ignore the terms which approach to zero,

$$\begin{aligned} 0 = & \frac{\partial^2 f(w)}{\partial w^2} + \left[-\frac{d}{1 - w^d} + \frac{1 - d}{2d}(1 - w^d) + \cdots \right] \frac{\partial f(w)}{\partial w} \\ & + \left[\frac{1}{(1 - w^d)^2} k_t^2 - \frac{1}{1 - w^d} \sum_{i=1}^{d-1} k_{x_i}^2 - L^2 m^2 \left(-\frac{1}{1 - w^d} - \frac{2}{d} + \cdots \right) \right] f(w) \end{aligned} \quad (2.35)$$

As one takes the limit of this function when $w \to 1$, $\frac{1}{(1-w^d)^2}$ approaches to ∞ faster than $\frac{1}{1-w^d}$. So close to $w = 1$ the diffrential equation has the form of,

$$0 = \frac{\partial^2 f(w)}{\partial w^2} - \frac{d}{1-w^d}\frac{\partial f(w)}{\partial w} + \frac{k_t^2}{(1-w^d)^2} f(w) \tag{2.36}$$

or,

$$0 = \frac{\partial^2 f(w)}{\partial w^2} - \frac{1}{1-w}\frac{\partial f(w)}{\partial w} + \frac{k_t^2}{(1-w)^2 d^2} f(w) \tag{2.37}$$

So the answer has the form of,

$$f(w) = A(1-w)^{ik_t} + B(1-w)^{-ik_t} = Ae^{ik_t \ln(1-w)} + Be^{-ik_t \ln(1-w)} \tag{2.38}$$

$e^{-ik_t \ln(1-w)}$ is incoming wave which go into the black hole and $e^{ik_t \ln(1-w)}$ is outcoming wave which coming out of black hole. If we choose $A = 0$ then there is no out comeing term so all information, wave, is just going into the black hole and nothing come back.

Chapter 3

Black Hole in Anti de Sitter

3.1 Introduction

In chapter 2 we learned how to deal with AdS space and in last section, section **??**, we solved the scalar field equation for simple black hole in Anti de Sitter space and studied the behavior of waves near horizon and boundary of black hole. In this chapter we want to study black holes in AdS space more generally. For example in the following section we are going to derive the equations for charged black hole and then we are going to study the behavior of wave function, general solution of scalar field near horizon and boundary. Then we apply this idea to some real systems and try to study the behavior of some condens matter systems by applying this idea to our system.

3.2 Charged Black Hole in AdS

In section **??** we saw that the metric for simle black hole has the form of,

$$ds^2 = -\left(k^2r^2 + 1 - \frac{C}{r^{d-2}}\right)dt^2 + \frac{1}{k^2r^2 + 1 - \frac{C}{r^{d-2}}}dr^2 + r^2d\Omega^2 \qquad (3.1)$$

or,

$$ds^2 = \frac{u^2}{L^2}\left[-(1-(\frac{u_0}{u})^d)dt^2 + \sum_{i=1}^{d-1} dx_i^2\right] + \frac{L^2du^2}{u^2(1-(\frac{u_0}{u})^d)} \qquad (3.2)$$

Now we can add charge to our black hole and make it charged black hole. So the metric changes as,

$$ds^2 = \frac{r^2}{R^2}(-f dt^2 + d\vec{x}^2) + \frac{R^2}{r^2}\frac{dr^2}{f} \quad , \quad f = 1 + \frac{Q^2}{r^{2d-2}} - \frac{M}{r^d} \tag{3.3}$$

and we can find the radius of horizon,

$$f(r_0) = 0 \rightarrow M = r_0^d + \frac{Q^2}{r_0^{d-2}} \tag{3.4}$$

Also charge, energy, and entropy density have forms of,

$$\rho = \frac{2(d-2)}{c_d}\frac{Q}{\kappa^2 R^{d-1} g_F} \quad , \quad \varepsilon = \frac{d-1}{2\kappa^2}\frac{M}{R^{d+1}} \quad , \quad s = \frac{2\pi}{\kappa^2}(\frac{r_0}{R})^{d-1} \tag{3.5}$$

And temprature of this system is,

$$T = \frac{dr_0}{4\pi r^2}(1 - \frac{(d-2)Q^2}{dr_0^{2d-2}}) \tag{3.6}$$

For zero temprature, $r_0^{2d-2} = \frac{(d-2)Q^2}{d}$, the redshift factor f gets the form of,

$$f = 1 - (\frac{r_0}{r})^d + \frac{d}{d-2}(\frac{r_0}{r})^d\left((\frac{r_0}{r})^{d-2} - 1\right) \tag{3.7}$$

Near the horizon, $f \rightarrow 0$ or $r \rightarrow r_0$, we can write the taylor series, so we have,

$$f = d(d-1)(\frac{r-r_0}{r_0})^2 + \cdots \tag{3.8}$$

By changing the varibales, $r - r_0 = \frac{\lambda R_2^2}{\zeta}$ and $t = \lambda^{-1}\tau$, which $R_2^2 = \frac{R^2}{d(d-1)}$ we find that the metric becomes,

$$ds^2 = \frac{R_2^2}{\zeta^2}(-d\tau^2 + d\zeta^2) + \frac{r_0^2}{R^2}d\vec{x}^2 \tag{3.9}$$

and for low temprature, by replacing $r_0 - r_* = \lambda\frac{R_2^2}{\zeta_0}$, the metric has the form of,

$$ds^2 = \frac{R_2^2}{\zeta^2}\left(-(1-\frac{\zeta}{\zeta_0})d\tau^2 + \frac{d\zeta^2}{1-\frac{\zeta}{\zeta_0}}\right) + \frac{r_0^2}{R^2}d\vec{x}^2 \tag{3.10}$$

and the temprature is,

$$\begin{aligned} T &= \frac{dr_0}{4\pi R^2}\left(1-\frac{r_*^{2d-2}}{r_0^{2d-2}}\right) \\ &= \frac{dr_0}{4\pi R^2}\frac{(r_0-r_*)(r_0^{2d-1}+r_*r_0^{2d}+\cdots r_*^{2d}r_0+r_*^{2d-1})}{r_0^{2d-2}} \\ &= \frac{dr_0}{4\pi R^2}\frac{2(d-1)(r_0-r_*)r_0^{2d-1}}{r_0^{2d-2}} = \frac{1}{2\pi\zeta_0} \end{aligned} \tag{3.11}$$

For zero temrature and for charged balck hole in AdS space near the horizon the metric tensor has the form of,

$$g_{\mu\nu} = \begin{pmatrix} -\frac{R_2^2}{\zeta^2} & & 0 \\ & \begin{pmatrix} \frac{r_0^2}{R^2} & 0 & \cdots & 0 \\ 0 & \frac{r_0^2}{R^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{r_0^2}{R^2} \end{pmatrix}_{d-1\times d-1} & \\ 0 & & \frac{R_2^2}{\zeta^2} \end{pmatrix} \tag{3.12}$$

and for low temprature has the form of,

$$g_{\mu\nu} = \begin{pmatrix} -\frac{R_2^2}{\zeta^2}\left(1-\frac{\zeta^2}{\zeta_0^2}\right) & & 0 \\ & \begin{pmatrix} \frac{r_0^2}{R^2} & 0 & \cdots & 0 \\ 0 & \frac{r_0^2}{R^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{r_0^2}{R^2} \end{pmatrix}_{d-1\times d-1} & \\ 0 & & \frac{R_2^2}{\zeta^2}\frac{1}{1-\frac{\zeta^2}{\zeta_0^2}} \end{pmatrix} \tag{3.13}$$

and equation of plain wave for zero and low temprature respectively would be,

$$\frac{\zeta^2}{R_2^2}\frac{\partial^2 f(\zeta)}{\partial\zeta^2} + \left[\frac{\zeta^2 k_t^2}{R_2^2} - m^2 - \frac{R^2 k_{\vec{x}}^2}{r_0^2}\right] f(\zeta) = 0 \tag{3.14}$$

and,

$$\frac{\zeta^2(1-\frac{\zeta^2}{\zeta_0^2})}{R_2^2}\frac{\partial^2 f(\zeta)}{\partial\zeta^2}-2\frac{\zeta^3}{R_2^2\zeta_0^2}\frac{\partial f(\zeta)}{\partial\zeta}+\left[\frac{\zeta^2 k_t^2}{R_2^2(1-\frac{\zeta^2}{\zeta_0^2})}-m^2-\frac{R^2k_{\vec{x}}^2}{r_0^2}\right]f(\zeta)=0 \tag{3.15}$$

Now, let's focuse more on our original metric without doing any changing variables or taking any limits. By defining $f=1+\frac{Q^2}{r^{2d-2}}-\frac{M}{r^d}$, the metric tensor of equation 3.3 is,

$$g_{\mu\nu}=\begin{pmatrix} -\frac{r^2}{R^2}f & & 0 \\ & \begin{pmatrix} \frac{r^2}{R^2} & 0 & \cdots & 0 \\ 0 & \frac{r^2}{R^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{r^2}{R^2}\end{pmatrix}_{d-1\times d-1} & \\ 0 & & \frac{R^2}{r^2 f}\end{pmatrix} \tag{3.16}$$

Now we are able to write the euqation of motion and solve it for scalar field, which describes the behaviour of plain wave in Anti de Sitter space with existance of a charged balck hole. Then we can study the behaviour of plain wave near the horizon and boundary and then we are able to find the Green function. So let's take a look at its differential equation,

$$\begin{aligned} 0=&\frac{1}{R^2}\left[(d+1)r-\frac{(d-3)Q^2}{r^{2d-3}}-\frac{M}{r^{d-1}}\right]\frac{\partial\phi(r)}{\partial r} \\ &+\frac{r^2 f}{R^2}\frac{\partial^2\phi(r)}{\partial r^2}+\left[\frac{R^2}{r^2 f}k_t^2-\frac{R^2}{r^2}k_{\vec{x}}^2-m^2\right]\phi(r) \end{aligned} \tag{3.17}$$

Now we are ready to study the behaviour of our wave function near horizon, $r\longrightarrow r_0$, and near boundary, $r\longrightarrow\infty$. For $d>1$, near boundary redshift factor f approaches to 1 and the equation 3.17 would be simplified as,

$$0=r^2\frac{\partial^2\phi(r)}{\partial r^2}+(d+1)r\frac{\partial\phi(r)}{\partial r}-R^2m^2\phi(r) \tag{3.18}$$

By substituting $\phi(r)=r^\alpha$ in our differential equation we get, $0=\alpha^2+\alpha d-R^2m^2$. So the answer near boundary is looks like,

$$\phi(r)=A(k_\mu)r^{\Delta-d}+B(k_\mu)r^{-\Delta} \quad , \quad \Delta=\frac{d}{2}+\sqrt{\frac{d^2}{4}+R^2m^2} \tag{3.19}$$

Horizon dose not have a zero area so its radius is not zero, too. Equation 3.4 gives us the radius of horizon. Also instead of mass of black hole M in redshift factor f we can substitute $r_0^d + \frac{Q^2}{r_0^{d-2}}$, so the redshift factor becomes,

$$f = (1 - \frac{r_0^d}{r^d}) + \frac{Q^2}{r^d}\left[\frac{1}{r^{d-2}} - \frac{1}{r_0^{d-2}}\right] \tag{3.20}$$

$$\begin{aligned} &= (r - r_0)\Bigg[\frac{r^{d-1} + r^{d-2}r_0 + \cdots + r_0^{d-2}r + r_0^{d-1}}{r^d} \\ &\qquad - \frac{Q^2}{r^d}\frac{r^{d-3} + r^{d-4}r_0 + \cdots + r_0^{d-4}r + r_0^{d-3}}{r^{d-2}r_0^{d-2}}\Bigg] \end{aligned} \tag{3.21}$$

By defining $Q^2 \equiv \frac{d}{d-2}r_*^{2d-2}$, near horizon we can simplify our differential equation when $r \to r_0$, and $f \to (r - r_0)\left[\frac{d}{r_0} - \frac{dr_*^{2d-2}}{r_0^{2d-1}}\right]$,

$$0 = \frac{\partial^2\phi(r)}{\partial r^2} + \frac{1}{(r-r_0)}\frac{\partial\phi(r)}{\partial r} + \frac{1}{(r-r_0)^2} \times \frac{R^4k_t^2}{d^2r_0^2\left(1 - \frac{r_*^{2d-2}}{r_0^{2d-2}}\right)^2}\phi(r) \tag{3.22}$$

or simply,

$$0 = \frac{\partial^2\phi(r)}{\partial r^2} + \frac{1}{(r-r_0)}\frac{\partial\phi(r)}{\partial r} + \frac{\beta^2}{(r-r_0)^2}\phi(r) \tag{3.23}$$

when

$$\beta^2 = \frac{R^4k_t^2}{d^2r_0^2\left(1 - \frac{r_*^{2d-2}}{r_0^{2d-2}}\right)^2} \tag{3.24}$$

And answer of this differential equation is:

$$\phi(r) = A(r-r_0)^{\sqrt{-\beta^2}} + B(r-r_0)^{-\sqrt{-\beta^2}} = Ae^{i\beta\ln(r-r_0)} + Be^{-i\beta\ln(r-r_0)} \tag{3.25}$$

3.3 Zero Temperature

The last equations are true when we are far from zero temprature. In this part, we are dealing with zero temperature, and then we will study these equations in low temprature, close to zero temprature. Equation 3.20 is the red shift factor, f, in finit temprature, but in zero temprature its form changes and, as we will see, we get factor of $(r-r_0)^2$. For simplicity, for our purpose we just focus on case $d=4$, 3 conventional space direction and one conventional time direction and an other spacelike direction for having AdS_5/CFT_4 theory. Zero temprature is when $r_0 = r_*$, so by substituting $Q = \frac{d}{d-2} r_0^{2d-2}$ for red shift factor one gets,

$$f = \frac{(d-2)r^{2d-2} - (2d-2)r_0^d r^{d-2} + d r_0^{2d-2}}{(d-2)r^{2d-2}} \quad , \quad d=4$$
$$= \frac{(r-r_0)^2(r+r_0)^2(r^2+2r_0^2)}{r^6} \tag{3.26}$$

And equation of motion for scalar field has the form of,

$$0 = \frac{1}{R^2}\left[\frac{(r-r_0)(r+r_0)(5r^4+5r_0^2r^2+2r_0^4)}{r^5}\right]\frac{\partial\phi(r)}{\partial r} + \frac{r^2 f}{R^2}\frac{\partial^2\phi(r)}{\partial r^2} + \left[\frac{R^2}{r^2 f}k_t^2 - \frac{R^2}{r^2}k_{\vec{x}}^2 - m^2\right]\phi(r) \tag{3.27}$$

Note that in this case, zero temprature, r_0 is exactly equal to r_*. Now we want to find the Green function and behaviour of our function near boundart, $r \to \infty$ and close to horizon, $r \to r_0$. Near boundary nothing would change from before, everything we had in last section is true for zero temprature too and we have the same form of answer for zero tamprature. Because Q, charge of black hole wich is related to temprature, and M, mass of black hole which is related to radius of horizon, do not play any role in the equation of motion, equation 3.17, near boundary, in case of $r \to \infty$. Close to horizon, $r \to r_0$, redshift factor has the form of $f = \frac{12}{r_0^2}(r-r_0)^2$, equation of motion 3.27 gets the form of,

$$0 = (r-r_0)^4\frac{\partial^2\phi(r)}{\partial r^2} + 2(r-r_0)^3\frac{\partial\phi(r)}{\partial r} + \frac{R^4k_t^2}{144}\phi(r) \tag{3.28}$$

and the answer has the form of,

$$\phi(r,\mu) = C(k_\mu)e^{\frac{i\beta}{r-r_0}} + D(k_\mu)e^{\frac{-i\beta}{r-r_0}} \quad , \quad \beta = \frac{R^2k_t}{12} \tag{3.29}$$

$e^{\frac{i\beta}{r-r_0}}$ is incoming wave which is going through black hole and $e^{\frac{-i\beta}{r-r_0}}$ is out going wave which is going out of black hole. In our case, we are mostly interested in case that we have just incoming wave and there is no out going wave which go out of black hole. So one need to choos $B=0$. Then we are able to find the Green's function,

$$G(k_\mu) \equiv \frac{B(k_\mu)}{A(k_\mu)} \quad , \quad \phi(r,k_\mu) = A(k_\mu)r^{\Delta-d} + B(k_\mu)r^{-\Delta} \tag{3.30}$$

3.4 Charged Scalar Field in Charged Black Hole

In this section we try to study behaviour of charged scalar field with charged black hole geometry in AdS space. Then we will learn more about the low friquencies, and then one should suggest low friquency limit to avoid breaking of the theory. Low friquency is breaking because of infinit number of degree of freedom. But let's first look at the action of charged scalar field

$$S = -\int d^{d+1}x\sqrt{-|g|}\left[(D_\mu\phi)^*(D^\mu\phi) + m^2\phi^*\phi\right] \tag{3.31}$$

where,

$$D_\mu = \partial_\mu - iqA_\mu \tag{3.32}$$

Note that action 3.31 just depend on q through:

$$\mu_q \equiv \mu q \tag{3.33}$$

which is the effective chemical potential for the field of charge q. We are interested to work on frequency space so we should take fourier trasformation of $\phi(r,x^\mu)$ and writting,

$$\phi(r,x^\mu) = \int \frac{d^dk}{\sqrt{(2\pi)^d}}\phi(r,k_\mu)e^{ik_\mu x^\mu} \tag{3.34}$$

And from now we work only on frequency space. After taking fourier trasformation we have,

$$\frac{-1}{\sqrt{-|g|}}\partial_r(\sqrt{-|g|}g^{rr}\partial_r\phi) + \left[g^{ii}(k^2-u^2)+m^2\right]\phi \quad , \tag{3.35}$$

where,

$$u(r) = \sqrt{\frac{-g_{ii}}{g_{tt}}}\left(\omega + \mu_q(1-\frac{r_0^{d-2}}{r^{d-2}})\right) \quad , \tag{3.36}$$

If the action is defined by equation 3.31. By considering all above equations then for motion equation we get,

$$\begin{aligned} 0 =& \frac{r^2f}{R^2}\frac{\partial^2\phi(r)}{\partial r^2} + \frac{1}{R^2}\left[(d+1)r - \frac{(d-3)Q^2}{r^{2d-3}} - \frac{M}{r^{d-1}}\right]\frac{\partial\phi(r)}{\partial r} \\ &+ \left[\frac{R^2}{r^2f}\left(k_t + \mu_q(1-\frac{r_0^{d-2}}{r^{d-2}})\right)^2 - \frac{R^2}{r^2}k_{\vec{x}} - m^2\right]\phi(r) \end{aligned} \tag{3.37}$$

Now we are ready to study $\phi(r,k_\mu)$ in finit temprature near boundary and close to horizon. As we have seen the radius of horizon is r_0 and boundary is located at $r\to\infty$. What ever we had on the last sections is true here. And nothing would be changed in our equation near boundary and close to horizon. By analyzing equation of 3.37 we will get the following answers near boundary and close to horizon respectively,

$$\phi(r,k_\mu) = A(k_\mu)r^{\Delta-d} + B(k_\mu)r^{-\Delta} \quad , \quad for \quad r\to\infty \tag{3.38}$$

where $\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2R^2}$ and,

$$\phi(r) = C(k_\mu)e^{i\beta\ln(r-r_0)} + D(k_\mu)e^{-i\beta\ln(r-r_0)} \tag{3.39}$$

where $\beta = \frac{R^2k_t}{d\left(1-\frac{r_*^{2d-2}}{r_0^{2d-2}}\right)}$. In the above equation, as we have seen before, $e^{-i\beta\ln(r-r_0)}$ and $e^{i\beta\ln(r-r_0)}$ are incoming wave, which goes into the black hole, and out going wave, which is going out of black hole. And finaly Green's function is defined,

$$G(k_\mu) \equiv \frac{B(k_\mu)}{A(k_\mu)} \tag{3.40}$$

Let's be more accurate close to the boundary and horizon. We want to expand the equations near boundary and horizon and find more terms to make our theory more accurate. Let's do it first for boundary. But becasue we are intrested in cases which $d=4$, 4-dimentional cases, so here we do not solve the problem for general d-dimentional case. We can rewrite redshift factor,

$$\begin{aligned} f(r) =& \frac{r-r_0}{r^6r_0^2}\Big[(4r_0^7-4r_*^6r_0)+(14r_0^6-2r_*^6)(r-r_0)+20r_0^5(r-r_0)^2 \\ &+15r_0^4(r-r_0)^3+6r_0^3(r-r_0)^4+r_0^2(r-r_0)^5\Big] \end{aligned} \tag{3.41}$$

and the coefficient of $\frac{\partial\phi}{\partial r}$, let's call it $g(r)$, is

$$\begin{aligned} g(r) =& \frac{1}{r^5r_0^2}\Big[(4r_0^8-4r_0^2r_*^6)+(28r_0^7-4r_0r_*^6)(r-r_0)+(74r_0^6-2r_*^6)(r-r_0)^2 \\ &+100r_0^5(r-r_0)^3+75r_0^4(r-r_0)^4+30r_0^3(r-r_0)^5+5r_0^2(r-r_0)^6\Big] \end{aligned} \tag{3.42}$$

In all of our equation ϕ is a function of r, $k_{\vec{x}}$, and $k_t=\omega$ and for simplicity we just write $\phi(r)$. But we are aware that actualy it is function of all of these terms through frourier trasformation. So now we are able to rewrite the equation of motion in $(r-r_0)^n$ terms, when $r \to r_0$,

$$
\begin{aligned}
0 = & \frac{(r-r_0)^2}{R^2 r_0^{14}} \Big[16(r_0^7 - r_*^6 r_0)^2 + 16(r_0^7 - r_*^6 r_0)(14r_0^6 - 2r_*^6)(r-r_0) \\
& \qquad + (14r_0^6 - 2r_*^6)^2 (r-r_0)^2 + \cdots \Big] \frac{\partial^2 \phi(r)}{\partial r^2} \\
& + \frac{(r-r_0)}{R^2 r_0^{14}} \Big[16(r_0^7 - r_*^6 r_0)^2 + 12(r_0^7 - r_*^6 r_0)(14r_0^6 - 2r_*^6)(r-r_0) \\
& \qquad + 2(14r_0^6 - 2r_*^6)^2 (r-r_0)^2 + \cdots \Big] \frac{\partial \phi(r)}{\partial r} \\
& + \Bigg[\frac{R^2}{r_0^2} \left(\omega^2 + 2\omega\mu_q \frac{2}{r_0}(r-r_0) + \mu_q^2 \frac{4}{r_0^2}(r-r_0)^2 + \cdots \right) \\
& \qquad - \frac{r-r_0}{r_0^8} \Big[(4r_0^7 - 4r_*^6 r_0) + (14r_0^6 - 2r_*^6)(r-r_0) + \cdots \Big] \left(\frac{R^2}{r_0^2} k_{\vec{x}}^2 + m^2 \right) \Bigg] \phi(r)
\end{aligned}
\tag{3.43}
$$

The solution has the form of,

$$
\begin{aligned}
\phi(r) = & A(k_\mu, \omega) \left[e^{-i\beta \ln(r-r_0)} + a_1 e^{-i\beta \ln(r-r_0)}(r-r_0) + a_2 e^{-i\beta \ln(r-r_0)}(r-r_0)^2 \right] \\
& + B(k_\mu, \omega) \left[e^{i\beta \ln(r-r_0)} + a_1 e^{i\beta \ln(r-r_0)}(r-r_0) + a_2 e^{i\beta \ln(r-r_0)}(r-r_0)^2 \right]
\end{aligned}
\tag{3.44}
$$

if $\tilde{M} = \frac{R^4}{r_0^2} k_{\vec{x}}^2 + R^2 m^2$, $\tilde{r} = \frac{r_*}{r_0}$, and $n = \mp i\beta$ then,

$$
\begin{aligned}
a_1 = & - \frac{8(7-\tilde{r}^6)(n^2-n) + 6(7-\tilde{r}^6)n + \frac{\omega\mu_q R^4}{r_0^2(1-\tilde{r}^6)} - \tilde{M}}{16(n^2+n) + 16(n+1) + \frac{R^4\omega^2}{r_0^2(1-\tilde{r}^6)}} \times \frac{4}{r_0} \\
a_2 = & - \Bigg[4(7-\tilde{r}^6)^2(n^2-n) + 8(7-\tilde{r}^6)^2 n + 4\frac{R^4\mu_q^2}{r_0^2} - 2(7-\tilde{r}^6)\tilde{M} \\
& \qquad + 4r_0 a_1 \left((1-\tilde{r}^6)(7-\tilde{r}^6)(8n^2+14n+6) + \omega\mu_q \frac{R^4}{r_0^2} - (1-\tilde{r}^6)\tilde{M} \right) \Bigg] \\
& \times \left[16r_0^2(1-\tilde{r}^6)^2(n^2+4n+4) + \omega^2 R^4 \right]^{-1}
\end{aligned}
\tag{3.45}
$$

We can continue whay we have done above to find the higher order temrs of $\phi(r)$ near horizon. For our perpose these terms are more than enough. For the next step we are interested to calculate more terms near boundary. So let's look at equation of motion one more time when we are in n-dimentional space,

$$
\begin{aligned}
0 = & \left[r^4 - \frac{2M}{r^{d-4}} + \frac{(d-3)2Q^2}{r^{2d-6}} + \frac{M^2}{r^{2d-4}} - \frac{Q^2 M}{r^{3d-6}} + \frac{(d-3)Q^4}{r^{4d-8}}\right] \frac{\partial^2 \phi(r)}{\partial r^2} \\
& \left[(d+1)r^3 - \frac{(d+2)M}{r^{d-3}} + \frac{4Q^2}{r^{2d-5}} + \frac{(d-4)MQ^2}{r^{3d-3}} + \frac{M^2}{r^{2d-3}} - \frac{(d-3)Q^4}{r^{3d-5}}\right] \frac{\partial \phi(r)}{\partial r} \\
& \left[\left(R^4\left(k_t + \mu_q(1 - \frac{r_0^{d-2}}{r^{d-2}})\right)^2 - R^4 k_{\vec{x}}^2\right) - R^2 m^2 \left(r^2 + \frac{Q^2}{r^{2d}} - \frac{M}{r^{d-2}}\right)\right] \phi(r)
\end{aligned} \tag{3.46}
$$

For case $d = 4$ answer has the form of,

$$
\begin{aligned}
\phi(r) = & A(k_\mu, \omega) r^{\Delta-4} \left[1 + a_1 \frac{1}{r^2} + a_2 \frac{1}{r^4} + \cdots\right] \\
& B(k_\mu, \omega) r^{-\Delta} \left[1 + a'_1 \frac{1}{r^2} + a'_2 \frac{1}{r^4} + \cdots\right]
\end{aligned} \tag{3.47}
$$

where, $n = \Delta - d$ and $-\Delta$ respectivly for a_1, a_2 and a'_1, a'_2.

$$
\begin{aligned}
a_1, a'_1 = & -\frac{R^4(k_t + \mu_q)^2 - R^4 k_{\vec{x}}}{n^2 - 4 - R^2 m^2} \\
a_2, a'_2 = & \frac{2M(n^2 + 2n) - a_2\left(R^4(k_t + \mu_q)^2 - R^4 k_{\vec{x}}\right) + MR^2 m^2}{n^2 - 4n - R^2 m^2}
\end{aligned} \tag{3.48}
$$

3.4.1 Exeptional Cases

In this part we talk about some exeptional cases, for example when $\Delta - d = -\Delta$ then our both independent answer become dependent and they are not different solutions of our differential equation. And we know that for second order differential equation we have to get two independent solution.

So for case $\Delta - d = -\Delta$, or in other word $\Delta = \frac{d}{2}$ or $-\frac{d^2}{4} = R^2 m^2$, our solution bear boundary has the form of,

$$
\phi(r) = A(k_\mu, \omega) r^{-\Delta} + B(k_\mu, \omega) r^{-\Delta} \ln(r) \tag{3.49}
$$

and for more general cases, when we are looking for more terms near boundary and $\Delta = k$ and k in an integer number and $d = 4$, answer has the form of,

$$\phi(r) = A_1(k_\mu, \omega) r^{\Delta-4} + r^{\Delta-6}\left[A_{21}(k_\mu, \omega) + A_{22}(k_\mu, \omega)\ \ln(r)\right]$$
$$+ r^{\Delta-8}\left[A_{31}(k_\mu, \omega) + A_{32}(k_\mu, \omega)\ \ln(r) + A_{33}(k_\mu, \omega)\ \ln^2(r)\right]$$
$$+\cdots \tag{3.50}$$

which we have just two independent coefficent, lets say A_1 and A_{21}, and the other are dependent to these two coefficient.

3.5 Numerical Approach

Equation of motion of form 3.37 is hard, or in other word is impossible, to solve analyticaly so far, cunsequently for studying the features of Green function one needs to solve the motion equation by numerical methods. In this section, we try to introduce an appropriate numerical method for solving this equation and then we use that method for studying the Green function under different conditions. Our aim is developing a numerical method which use near horizon condition to calculate the Green function.

One of the best numerical method with hight accuracy is Runge-Kutta method. For the first time, this technique were developed and used by the German mathematicians C. Runge and M.W. Kutta. Let's define our initial value problem as following,

$$\phi(k_\mu, \omega, r_i) = \phi_i \quad , \quad \frac{\partial \phi(k_\mu, \omega, r_i)}{\partial r} = \dot{\phi}_i \tag{3.51}$$

For initial condition we want to start from horizon, or it is better to say near horizon, and calculate the Green funtion. In section 3.4 we learned that the solution near horizon has the form of,

$$\phi(k_\mu, \omega, r) = A(k_\mu, \omega) e^{-i\beta \ln(r-r_0)}\left[1 + a_1(r-r_0) + a_2(r-r_0)^2 + \cdots\right]$$
$$+ B(k_\mu, \omega) e^{i\beta \ln(r-r_0)}\left[1 + a_1(r-r_0) + a_2(r-r_0)^2 + \cdots\right] \tag{3.52}$$

Exactly in $r = r_0$ the fuction is not difined so one should start somewhere close to r_0. We will start from point $r = r_0 + h$, lets call it start point, as our initial condition. So for initial condition we have,

$$
\begin{aligned}
\phi_i(k_\mu,\omega,r_0+h) =& A(k_\mu,\omega)e^{-i\beta\ln(h)}\left[1+a_1(h)+a_2(h)^2+\cdots\right] \\
&+B(k_\mu,\omega)e^{i\beta\ln(h)}\left[1+a_1(h)+a_2(h)^2+\cdots\right] \\
\dot{\phi}_i(k_\mu,\omega,r_0+h) =& -i\beta A(k_\mu,\omega)\frac{e^{-i\beta\ln(h)}}{h}\left[1+a_1(h)+a_2(h)^2+\cdots\right] \\
&+i\beta B(k_\mu,\omega)\frac{e^{i\beta\ln(h)}}{h}\left[1+a_1(h)+a_2(h)^2+\cdots\right] \\
&+A(k_\mu,\omega)e^{-i\beta\ln(h)}\left[a_1+2a_2(h)+\cdots\right] \\
&+B(k_\mu,\omega)e^{i\beta\ln(h)}\left[a_1+2a_2(h)+\cdots\right]
\end{aligned}
\tag{3.53}
$$

We will use Runge-Kutta scheme to find the value of $\phi(k_\mu,\omega,r)$ in each point. We will start from start point and continue going forward in r direction untile we get to the aim point, let's say r_f. In each step we use forth order Runge-Kutta scheme to find the next point. If we have the derivative of n^{th} point then we are able to use the following formula to calculate the $(n+1)^{th}$ point,

$$
\begin{aligned}
\phi_{n+1} &= \phi_n + \frac{k_1+2k_2+2k_3+k_4}{6} \\
\dot{\phi}_{n+1} &= \dot{\phi}_n + \frac{l_1+2l_2+2l_3+l_4}{6}
\end{aligned}
\tag{3.54}
$$

where,

$$
\begin{aligned}
RH_{cons}(r,\phi,\dot{\phi}) &= \frac{\partial^2\phi}{\partial r^2} \\
&= -\frac{1}{f}\left[\frac{d+1}{r} - \frac{(d-3)Q^2}{r^{2d-1}} - \frac{M}{r^{d+1}}\right]\dot{\phi} \\
&\quad - \left[\frac{R^4}{r^4 f^2}\left(k_t + \mu_q(1-\frac{r_0^{d-2}}{r^{d-2}})\right)^2 - \frac{R^4}{r^4 f}k_{\vec{x}} - \frac{R^2 m^2}{r^2 f}\right]\phi \\
l_1 &= h\, RH_{cons}(r_n,\phi_n,\dot{\phi}_n) \\
k_1 &= h\, \dot{\phi}_n \\
l_2 &= h\, RH_{cons}(r_n+0.5h,\phi_n+0.5k_1,\dot{\phi}_n+0.5l_1) \\
k_2 &= h\, (\dot{\phi}_n+0.5l_1) \\
l_3 &= h\, RH_{cons}(r_n+0.5h,\phi_n+0.5k_2,\dot{\phi}_n+0.5l_2) \\
k_3 &= h\, (\dot{\phi}_n+0.5l_2) \\
l_4 &= h\, RH_{cons}(r_n+h,\phi_n+k_3,\dot{\phi}+l_3) \\
k_4 &= h\, (\dot{\phi}+0.5l_3)
\end{aligned}
\tag{3.55}
$$

This method is 4^{th} order Runge-Kutta method, say RK4, and its error is $\frac{1}{120}h^5\phi^5(\xi)$ which ξ is between r_n and r_{n+1}, and $\phi^5(\xi)$ is 5^{th} derivative of function ϕ.

3.5.1 Adaptive grid size

In Runge-Kutta scheme it is easy to change the grid size in each step of solution. In each step we can use a independent grid size of h, which means that this grid size do not depent on its last grid size and its following grid size. In prolem like this which in some rigions like near horizon the equations has flactuating features for not loosing the details and get the approprate solutions, one need to have a small grid size and close to boundary which it is a smooth function without any flactuating behaviour small grid size slow down the computational process and make the code time expensive. So one need to find a way to deal with the grid size regard to feature of our function, make the grid size small when we have a fluctuations and/or make the grid size when we have a smooth function. We can change our varibale to new ones to make our function smooth in all regions to use the same grid size. Always this method is not working because it is hard to find an appropriate coordinates to change our function to a simple smooth function. The best way is that introduce a way to deal with grid size directly. In each step it would be good to find an appropriate grid size independently and apply that grid size for that specific step. And we do the same process for each step, then we can controle the error of our answer by changing the grid size into appropriate one.

First of all, we are looking at the error of each step. As we may know the error of 4^{th} order Runge-Kutta is,

$$err = \frac{1}{120} h^5 \phi^5(\xi) \qquad , \qquad \phi^5(\xi) = \frac{\partial^5 \phi(r)}{\partial r^5} \tag{3.56}$$

so we can consider specific error for each step and then we are able to calculate step size. The problem is that we don't know exactly the value of $\phi^5(\xi)$. In maximume $\phi^5(\xi)$ is the maximum of 5^{th} derivative of $\phi(\xi)$ between r_i and r_{i+1}, but even we don't know the maximume value of $\phi^5(\xi)$. So we need to define another constrain for h. We know that for this specific differential equations the solution near horizon is flactuating so rapidly but as it going forward to infinity it gets smoother, by smoother we mean that its higher derivative gets smaller and smaller. So approximatly it is possible to find the maximume of 5^{th} derivative by finding the 5^{th} derivative of approximate solution near horizon. By useing equation 3.19 and considering incomeing wave for 5^{th} derivative we gets,

$$\frac{(10\beta^2 + 5\beta^4) + (24\beta + \beta^5)i}{(r - r_0)^5} \, e^{-i\beta \; ln(r - r_0)} \tag{3.57}$$

If we start our solution from $r_0 + h_1$ then from equation it would be obvious that maximume of 5^{th} derivative is $\sqrt{(10\beta^2 + 5\beta^4)^2 + (24\beta + \beta^5)^2} \; h_1^{-5}$. We can use this fomula and equation 3.56 to calculate the step size h_* near horizon. So for h we get:

$$h_* = h_1 \left[\frac{120 \, err}{\sqrt{(10\beta^2 + 5\beta^4)^2 + (24\beta + \beta^5)^2}} \right]^{-5} \tag{3.58}$$

and for all of our domain we can calculate the grid size h,

$$h = max(h_*, \left[\frac{120 \, err}{\sqrt{(10\beta^2 + 5\beta^4)^2 + (24\beta + \beta^5)^2}} \right]^{-5}, 1.0) = \tag{3.59}$$

Bibliography