REAL ANALYSIS

MASTER OF ARTS IN ECONOMIS

AGUSTÍN FARÍAS LOBO

School of Economics and Business, University of Chile
Economics Department

This notes have been written based of the classes made by professor Eugenio Finat in the course "Análisis Real y Optimización No Lineal", an elective course of the Economics M.A. Program at University of Chile. Additional references used are included explicitly in the final pages.

The writer of this notes is a student. Therefore, this material may contain mistakes and should not be used as a main reference when studying any math course.

In case of finding an error, please contact me at *afariaslobo@gmail.com*.

This version was updated on September 28, 2025.

General Index

1 Real and Complex Numbers

We will begin studying the real-number system. Because our focus is the mathematical analysis, we introduce the real numbers highlighting their axioms below they and their properties. To see the notation used in this section, see Section A.

We assume there exists a nonempty set $\mathbb{R}$ of objects, called real numbers, which satisfy ten axioms listed below. These axioms can be naturally divided into three groups: the field axioms, the order axioms, and the completeness axiom.

Along with the set $\mathbb{R}$ we assume the existence of two operations, addition and multiplication, such that for every pair of real numbers x and y the sum $x+y$ and the product xy are real numbers uniquely determined by x and y satisfying the following axioms. Axioms from 1.1 to 1.5 correspond to the field axioms.

<u>Axiom 1.1: Commutative laws.</u> $x+y=y+x$, $xy=yx$.

<u>Axiom 1.2: Associative laws.</u> $x+(y+z)=(x+y)+z$, $(xy)z=x(yz)$.

<u>Axiom 1.3: Distributive law.</u> $x(y+z)=xy+xz$.

<u>Axiom 1.4</u> Given any two real numbers $x,y\in\mathbb{R}$, there exists a $z\in\mathbb{R}$ such that $x+z=y$. This z is denoted by $y-x$, and the number $x-x$ is denoted by 0, independently of the x chosen. We write $-x$ for $0-x$ and call $-x$ the negative of x.

<u>Axiom 1.5</u> There exists at least one real number $x\neq 0$. If x and y are two real numbers with $x\neq 0$, then there exists a real number z such that $xz=y$. This z is denoted y/x, and the number x/x is denoted by 1, independently of the x chosen. We write x^{-1} for $1/x$ if $x\neq 0$, and call x^{-1} the reciprocal of x.

We assume the existence of a relation $<$, which establishes an ordering among the real numbers. Axioms from 1.6 to 1.9 correspond to the order axioms.

<u>Axiom 1.6</u> Exactly one of the relations $x=y$, $x<y$ or $x>y$ holds.[1]

<u>Axiom 1.7</u> If $x<y$, then for every $z\in\mathbb{R}$ we have $x+z<y+z$.

<u>Axiom 1.8</u> If $x,y>0$, then $xy>0$.[2]

[1] The expression $x\leq y$ means that one of the relations $x<y$ and $x=y$ holds.

[2] A real number x is called positive if $x>0$, and negative if $x<0$. The real number x is nonnegative if $x\geq 0$. The set of all positive real numbers is denoted by $\mathbb{R}^+$, and the set of all negative real numbers by $\mathbb{R}^-$.

Axiom 1.9 If $x > y$ and $y > z$, then $x > z$.

Now we present a theorem that it is a direct consequence of the axioms presented above.

THEOREM 1.1. Given $a, b \in \mathbb{R}$ such that $a \leq b + \varepsilon$, $\forall \varepsilon > 0$. Then $a \leq b$.

The set of all points between a and b is called an interval. Nevertheless, it is important to distinguish when intervals include their endpoints. This motivates the following definitions.

Definition 1.1: Open interval. Assume $a < b$. The open interval (a,b) is defined to be the set

$$(a,b) = \{x : \ a < x < b\}.$$

Definition 1.2: Closed interval. Assume $a < b$. The closed interval $[a,b]$ is defined to be the set

$$[a,b] = \{x : \ a \leq x \leq b\}.$$

Definition 1.3: Half-open interval. Assume $a < b$. The half-open interval $[a,b)$ is defined to be the set

$$[a,b) = \{x : \ a \leq x < b\}.$$

Definition 1.4: Infinite interval. Given $a \in \mathbb{R}$, the infinite intervals are defined as

$$(a,\infty) = \{x : \ x > a\}, \quad [a,\infty) = \{x : \ x \geq a\}, \quad (-\infty,a) = \{x : \ x < a\}, \quad (-\infty,a] = \{x : \ x \leq a\}.$$

Integers are a special subset of $\mathbb{R}$. To define this particular set, it is important to define what it is an inductive set.

Definition 1.5: Inductive set. A set of real numbers is called an inductive set if satisfy the two following properties:

a) The number 1 belongs to the set.

b) For every x in the set, the number $x + 1$ also belongs to the set.

Using the notion of inductive set, it is possible to define the positive integers.

Definition 1.6: Positive integer. A real number is called a positive integer if it belongs to every inductive set. The set of positive integers is denoted by $\mathbb{Z}^+$ and it corresponds to the smallest inductive set.

Definition 1.7: Negative integer. A real number is called a negative integer if it is the negative of a positive integer. The set of negative integers is denoted by $\mathbb{Z}^-$.

The set that contains positive integers, negative integers and 0 it is called the set of integers $\mathbb{Z}$.

If $n, d \in \mathbb{Z}$ and $n = cd$ for some $c \in \mathbb{Z}$, we say that d is a divisor of n and that n is a multiple of d. We write this relationship by $d|n$. If $d|a$ and $d|b$, d is a common divisor of a and b.

Definition 1.8: Prime number. An integer n is called a prime if $n > 1$ and if the only positive divisors of n are 1 and n. Every integer $n > 1$ that is not prime is composite. The integer 1 is neither prime nor composite.

PROPOSITION 1.1. Every integer $n > 1$ is either a prime or a product of primes.

PROPOSITION 1.2. Every integer $n > 1$ is either a prime or a product of primes.

PROPOSITION 1.3. Every pair of integers a and b has a common divisor d of the form $d = ax + by$, where x and y are integers. Also, every common divisor of a and b divides this d.

If d is a common divisor of a and b of the form $d = ax + by$, then $-d$ is also a divisor of the same form, $-d = a(-x) + b(-y)$. Of these common divisors, the nonnegative one is called the greatest common divisor of a and b, denoted by $\gcd(a, b)$ or (a, b). If $(a, b) = 1$, a and b are relatively prime.

PROPOSITION 1.4. *Euclid's lemma*. If $a|bc$ and $(a, b) = 1$, then $a|c$.

PROPOSITION 1.5. If a prime p divided ab, then $p|a$ or $p|b$. More generally, if a prime p divides a product $a_1 \cdot \ldots \cdot a_k$, then p divides at least one of the factors.

THEOREM 1.2. *Unique factorization theorem*. Every integer $n > 1$ can be represented as a product of prime factors in only one way, apart from the order of the factors.

Quotients of integers a/b, where $b \neq 0$, are called rational numbers. The set of rational numbers is denoted by $\mathbb{Q}$ and contains $\mathbb{Z}$. All the field and order axioms are satisfied by $\mathbb{Q}$.

Real numbers that are not rational are called irrational numbers. Examples of irrational numbers are $\sqrt{2}$, e, π and e^{π}. Typically is not easy to prove that particular numbers are irrational or not. However, it is not too difficult to establish that certain numbers are indeed irrational.

PROPOSITION 1.6. If n is a positive integer which is not a perfect square, then $\sqrt{n}$ is irrational.

PROPOSITION 1.7. If $e^x = 1 + x + x^2/2! + x^3/3! + ...$, then the number e is irrational.

Definition 1.9: Upper bound. Let S be a set of real numbers. If there is a real number b such that $x \leq b$, $\forall x \in S$, then b is called an upper bound for S. In that case we say that S is bounded above by b.

If an upper bound b is also a member of S, b is called the largest member or the maximum of S. There can be at most one b. If b exists, then we write

$$b = \max S.$$

A set with no upper bound is called unbounded above.

Definition 1.10: Lower bound. Let S be a set of real numbers. If there is a real number b such that $x \geq a$, $\forall x \in S$, then a is called a lower bound for S. In that case we say that S is bounded below by b.

If an upper bound a is also a member of S, a is called the smallest member or the minimum of S. There can be at most one a. If a exists, then we write

$$b = \min S.$$

A set with no lower bound is called unbounded below.

Definition 1.11: Supremum. Let S be a set of real nunber bounded above. A real number b is called a least upper bound for S or a supremum for S if it has the following properties:

a) b is an upper bound for S.

b) No number less than b is an upper bound for S.

If b is the supremum of the set A, we denote

$$b = \sup A.$$

It is not difficult to prove that if a set of real number has a supremum, this is unique.

Definition 1.12: Supremum. Let S be a set of real nunber bounded below. A real number b is called a greatest lower bound for S or a infimum for S if it has the following properties:

a) b is a lower bound for S.

b) No number greater than b is a lower bound for S.

If b is the infimum of the set A, we denote

$$b = \inf A.$$

Again, it is not difficult to prove that if $\inf A$ exists, it is unique.

With the definition 1.12 it is now possible to present the last necessary axiom to construct the real number system.

Axiom 1.10: Completeness axiom. Every nonempty set S of real numbers whicj is bounded above has a supremum. That is, there is a real number b such that $b = \sup S$.

On next we present certain useful properties that are a consequence of the definition we gave to the supremum of a set.

PROPOSITION 1.8. *Approximation property.*. Let S be a nonempty set of real numbers with a supremum, say $b = \sup S$. Then for every $a < b$ there is some x in S such that $a < x \leq b$.

PROPOSITION 1.9. *Additive property.*. Given nonempty subsets $A, B \subset \mathbb{R}$, let C denote the set $C = \{x + y : \ x \in A, \ y \in B\}$. If $\sup A$ and $\sup B$ exist, then $\sup C$ exists and it is given by

$$\sup C = \sup A + \sup B.$$

PROPOSITION 1.10. *Comparison property.*. Given nonempty subsets $A, B \subset \mathbb{R}$ such that $s \leq t$ for every $s \in S$ and $t \in T$. If Thas a supremum then S has a supremum and $\sup S \leq \sup T$.

PROPOSITION 1.11. The set $\mathbb{Z}^+$ of positive integers $1,2,3,\ldots$ is unbounded above.

PROPOSITION 1.12. For every real number x there is a positive integer n such that $x < n$.

PROPOSITION 1.13. *Archimedean property..* If $x > 0$ and if y is an arbitrary real number, there is a positive integer n such that $y < n \cdot x$.

Let r be a real number of the form

$$r = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_n}{10^n}$$

where a_0 is a nonnegative integer and $a_1,\ldots,a_n$ are integers satisfying $0 \leq a_i \leq 9$, $\forall i \in \{1,\ldots,n\}$, can be briefly written as

$$r = a_0.a_1a_2\ldots a_n.$$

The last expression is said to be a finite decimal representation of r. Real numbers such as r are necessarily rational.[3] Nevertheless, not every real number has a finite decimal representation.

PROPOSITION 1.14. *Finite approximation of real numbers..* Let $x \in \mathbb{R}_0^+$. Then for every integer $n \geq 1$ there is a finite decimal $r_n = a_0.a_1a_2\ldots a_n$ such that

$$r_n \leq x < r_n + \frac{1}{10^n}.$$

Definition 1.13: Absolute value. Let x be a real number. The absolute value of x, denoted by $|x|$, is defined as

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$$

PROPOSITION 1.15. If $a \geq 0$, then we have the inequality $|x| \leq a$ if, and only if, $-a \leq x \leq a$.

[3] Note that numbers of the same form of r can be expressed as $r = a/10^n$ where $a \in \mathbb{Z}$ and $n \in \mathbb{N}$.

THEOREM 1.3. *Triangle inequality*. For arbitrary real numbers x and y we have

$$|x+y| \leq |x|+|y|.$$

THEOREM 1.4. *Cauchy-Schwarz inequality*. If $a_1,\ldots,a_n$ and $b_1,\ldots,b_n$ are arbitrary real numbers, we have

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right).$$

In vector notation the Cauchy-Schwarz inequality is given by

$$(a\cdot b)^2 \leq ||a||^2||b||^2,$$

where $a=(a_1,\ldots,a_n)$ and $b=(b_1,\ldots,b_n)$ are two n-dimensional vectors, $a\cdot b=\sum_{k=1}^{n} a_k b_k$ is their dot product and $||a||=(a\cdot a)^{1/2}$ is the length of a.

On next we present an extension of the real number system that adds two "ideal points" denoted $+\infty$ (plus infinity) and $-\infty$ (minus infinity).

Definition 1.14: Extended real number system. The extended real number system $\mathbb{R}^*$ it is the set of real numbers $\mathbb{R}$ together with two symbols $+\infty$ and $-\infty$ which satisfy the following properties:

a) For any $x\in\mathbb{R}$ we have

$$\begin{aligned}
x+(+\infty) &= +\infty,\\
x+(-\infty) &= -\infty,\\
x-(+\infty) &= -\infty,\\
x-(-\infty) &= +\infty,\\
\frac{x}{+\infty} = \frac{x}{-\infty} &= 0.
\end{aligned}$$

b) For any $x\in\mathbb{R}$ such that $x>0$ we have $x(+\infty)=+\infty$ and $x(-\infty)=-\infty$.

c) For any $x\in\mathbb{R}$ such that $x<0$ we have $x(+\infty)=-\infty$ and $x(-\infty)=+\infty$.

d) It is safistied that

$$\begin{aligned}
&(+\infty)+(+\infty)=(+\infty)(+\infty)=(-\infty)(-\infty)=+\infty,\\
&(-\infty)+(-\infty)=(+\infty)(-\infty)=-\infty.
\end{aligned}$$

e) For any $x \in \mathbb{R}$, then $-\infty < x < \infty$.

Following the last definition, it is possible to denote $\mathbb{R}$ as $(-\infty, +\infty)$ and $\mathbb{R}^*$ as $[-\infty, +\infty]$. The points in $\mathbb{R}$ are called finite to distinguish them from the infinite points $+\infty$ and $-\infty$.

Definition 1.15 An open interval $(a, +\infty)$ is called a neighborhood of $+\infty$ or a ball with center $+\infty$. Every open interval $(-\infty, a)$ is called a neighborhood of $-\infty$ or a ball with center $-\infty$.

From axioms 1.6 to 1.9 it is possible to infer that the square of a real number is never negative. Therefore, the equation $x^2 = -1$ will not have a solution among real numbers. To provide solution to equation like the past it is necessary to introduce another type of numbers.

As we will discuss, the "complex numbers" will provide solutions to general algebraic equations of the form

$$a_0 + a_1 x + \ldots + a_n x^n = 0,$$

where $a_0, \ldots, a_n$ are arbitrary real numbers.

Definition 1.16: Complex numbers. A complex number is an ordered pair of real numbers denoted by (x_1, x_2). x_1 is called the real part of the complex number, and x_2 is called the imaginary part.

Two complex numbers, $x = (x_1, x_2)$ and $y = (y_1, y_2)$, are equal if, and only if, $x_1 = y_1$ and $x_2 = y_2$. We define the sum $x + y$ and the product xy by the equations

$$\begin{aligned} x + y &= (x_1 + y_1, x_2 + y_2), \\ xy &= (x_1 y_1 - x_2 y_2, x_1 y_2 + x_2 y_1). \end{aligned}$$

The set of complex numbers is denoted by $\mathbb{C}$.

PROPOSITION 1.16. The operations of addition and multiplication of complex numbers satisfy the commutative, associative and distributive laws.

PROPOSITION 1.17. Let $x = (x_1, x_2)$ be a complex number. Then x satisfy the following equations:

$$\begin{aligned} (x_1, x_2) + (0, 0) &= (x_1, x_2), \\ (x_1, x_2)(0, 0) &= (0, 0), \\ (x_1, x_2)(1, 0) &= (x_1, x_2), \\ (x_1, x_2) + (-x_1, -x_2) &= (0, 0). \end{aligned}$$

PROPOSITION 1.18. Given two complex numbers $x = (x_1, x_2)$ and $y = (y_1, y_2)$, there exists a complex number z such that $x + z = y$. In fact, $z = (y_1 - x_1, y_2 - x_2)$ and z is denoted by $y - z$. Also, the complex number $(-x_1, -x_2)$ is denoted by $-x$.

PROPOSITION 1.19. For any two complex numbers x and y, we have

$$-x(y) = x(-y) = -(xy) = (-1,0)(xy).$$

Definition 1.17 If $x = (x_1, x_2)$ and y are complex numbers with $x \neq 0$, we define

$$x^{-1} = \left(\frac{x_1}{x_1^2 + x_2^2}, -\frac{x_2}{x_1^2 + x_2^2} \right),$$

and $y/x = yx^{-1}$.

PROPOSITION 1.20. If x and y are complex numbers with $x \neq 0$, there exists a complex number z such that $xz = y$, namely, $z = yx^{-1}$.

PROPOSITION 1.21. Let $x = (x_1, 0)$ and $y = (y_1, 0)$ be complex numbers with $y_1 \neq 0$. Then,

$$(x_1,0) + (y_1,0) = (x_1 + y_1, 0) \in \mathbb{R},$$
$$(x_1,0)(y_1,0) = (x_1 y_1, 0) \in \mathbb{R},$$
$$(x_1,0)/(y_1,0) = (x_1/y_1, 0) \in \mathbb{R}.$$

Definition 1.18: Imaginary unit The complex number $(0,1)$ is denoted by i and is called the "imaginary unit".

PROPOSITION 1.22. Every complex number $x = (x_1, x_2)$ can be represented in the form $x = x_1 + ix_2$.

PROPOSITION 1.23. The square of the imaginary unit equals -1.

Definition 1.19: Modulus of a complex number. Let $x = (x_1, x_2)$ be a complex number. The modulus or absolute value of x is the nonnegative real number $|x|$ given by

$$|x| = \sqrt{x_1^2 + x_2^2}.$$

PROPOSITION 1.24. Let $x \in \mathbb{C}$ and $y \in \mathbb{C} \backslash \{0\}$. Then,

a) $|(0,0)| = 0$.

b) $|x| > 0$ if $x \neq 0$.

c) $|xy| = |x|\,|y|$.

d) $|x/y| = |x|/|y|$.

e) $|(x_1, 0)| = |x_1|$.

THEOREM 1.5. *Triangle inequality*. If $x, y \in \mathbb{C}$, then

$$|x + y| \leq |x| + |y|.$$

One characteristic of the complex numbers is that they cannot be ordered in a way that they will satisfy Axioms 1.6 to 1.8. This fact can be proved assuming that i is either less than zero or greater than zero. If Axioms 1.6, 1.7 and 1.8 are satisfied, we will always get to a contradiction in either case.

IF THERE IS TIME TO DO IT, IT IS NECESSARY TO INCLUDE SUBSECTIONS 1.26 TO 1.33 OF APOSTOL BOOK.

References

2 Set theory

Now we study some basic notions and terminology about set theory. Just like it was mentioned in the previous section the focus of these notes is real analysis. Therefore, we will briefly present only the basic concepts necessary for mathematical analysis.

Definition 2.1: Set. A collection of objects seen as a single entity is called a set. The objects contained in a set are called elements or members of that set, a we will say that they belong to or are contained in the set.

Typically a set will be denoted by capital letters and its elements by lower-case letters.

We write $x \in S$ to mean "x is an elementof S", or "x belongs to S". If S is the collection of all x which satisfy a property P, we indicate this by

$$S = \{x : \ x \text{ satisfies } P\}, or \{x | \ x \text{ satisfies } P\}.$$

We say that A is a subset of B, denoted by $A \subseteq B$, when every element of A also belongs to B. We have both $A \subseteq B$ and $B \subseteq A$ if, and only if, A and B have the same elements, in which case they are equal sets: $A = B$.

Definition 2.2: Empty set. The set that contains no elements is called "empty set" and it is denoted by $\emptyset$. We agree to call it a subset of every set.

The standard definition of a set does not take into account the order of its element. For example, $\{a,b\} = \{b,a\}$.

When we wish to consider a set of two elements, a and b, as being ordered we shall enclose the elements in parentheses: (a,b). In this case, a is the first element and b is the second element.

Definition 2.3: Ordered pair. An ordered pair (a,b) is a set given by $(a,b) = \{\{a\},\{a,b\}\}$.

PROPOSITION 2.1. The equality $(a,b) = (c,d)$ satisfies if, and only if, $a = c$ and $b = d$.

Definition 2.4: Cartesian product. Given two sets A and B, set the set of all ordered pairs (a,b) such that $a \in A \ \wedge \ b \in B$ is called the cartesian product of A and B, and is denoted by $A \times B$.

Definition 2.5: Relation. Any set of ordered pairs is called a relation.

Definition 2.6: Domain. Let S be a relation. The set of all elements of x that occurs as the first elements of the ordered pairs (x,y) in S is called the domain of S, denoted as $\mathscr{D}(S)$ or $\text{Dom}(S)$.

Definition 2.7: Range. Let S be a relation. The set of all elements of y that occurs as the second elements of the ordered pairs (x,y) in S is called the range of S, denoted as $\mathscr{R}(S)$ or Range(S).

Definition 2.8: Function. A function f is a set of ordered pairs (x,y), no two of which have the same first member. That is, if $(x,y) \in f$ and $(x,z) \in f$m then $y = z$.

The definition of a function requires that for every x in the domain of f there is exactly one y such that $(x,y) \in f$. It is frequent to call y the "value" of f at x and denote $y = f(x)$.

PROPOSITION 2.2. Two functions f and g are equal if, and only if, $\mathscr{D}(f) = \mathscr{D}(g)$ and $f(x) = g(x), \forall x \in \mathscr{D}(f)$.

When $\mathscr{D}(f)$ is a subset of $\mathbb{R}$, then f is called a function of one real variable. If $\mathscr{D}(f)$ is a subset of $\mathbb{C}$, then f is called a function of a complex variable. If $\mathscr{D}(f)$ is a subset of a cartesian product $A \times B$, then f is called a function of two variables. In the latter case we write $f(a,b)$ instead of $f((a,b))$.

If S is a subset of $\mathscr{D}(f)$, we say that f is defined on S. In this case, the set of $f(x)$ such that $x \in S$ is called the image of S under f, denoted by $f(S)$. If T is any set which contains $f(S)$, then f is also called a mapping from S to T. This is often denoted

$$f : S \to T.$$

If $f(S) = T$, the mapping is said to be onto T.

If functions f and g satisfy $g \subseteq f$, then g is called a restriction of g, and f is called an extension of g. In particular, if S is a subset of $\mathscr{D}(F)$ and if $g(x) = f(x), \forall x \in S$, then g is a restriction of f to S.

Definition 2.9: One-to-one function. Let f be a function defined on S. We say f is one-to-one on S if, and only if, for every x and y in S,

$$f(x) = f(y) \quad \Longrightarrow \quad x = y.$$

These functions are also called injective.

Definition 2.10: Converse. Given a relation S, the new relation $\breve{S}$ defined by

$$\breve{S} = \{(a,b) : \ (b,a) \in S\}$$

is called the converse of S.

Definition 2.11: Inverse function. Let f be a function and consider its converse relation $\breve{f}$, which may or may not be a function. If $\breve{f}$ is also a function, then $\breve{f}$ is the inverse of f and it is denoted by f^{-1}.

PROPOSITION 2.3. If f is one-to-one on its domain, then $\breve{f}$ is also a function.

Definition 2.12: Composite functions. Given two functions f and g such that $\mathscr{R}(f) \subseteq \mathscr{D}(g)$, it is possible to form a new function defined as

$$(g \circ f)(x) = g(f(x)),$$

which is called the composite of g and f.

The composition of functions does not generally satisfies commutativity, but it satisfies the associative law:

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Definition 2.13: Finite sequence. A finite sequence of n terms is a function f whose domain is the set of numbers $\{1, 2, ..., n\}$. The elements of the range of a sequence are called its terms.

Definition 2.14: Ininite sequence. An infinite sequence is a function f whose domain is the set of all positive integers $\{1, 2, 3, ...\}$. The function value f_n is called the nth term of the sequence.

Let $s = \{s_n\}$ be an infinite sequence, and let k be a function whose domain is the set of positive integers and whose range is a subset of positive integers. If k is "order-preserving", then the composite function $s \circ k$ is defined for every $n \in \mathbb{N}$, and every n satisfies

$$(s \circ k)(n) = s_{k(n)}.$$

The composite function $s_{k(n)}$ is called a subsequence of s.

Definition 2.15: Equinumerous sets. Two sets, A and B, are equinumerous or similar if, and only if, there exists a one-to-one function f whose domain is the set A and whose range is the set B. This relation between A and B is denoted by $A \sim B$.

Definition 2.16: Finite set. A set S is called finite if $S \sim \{1, 2, ..., n\}$, where n is the "cardinal number" of S. Logically, infinite sets are sets that are not finite.

An infinite set must be similar to some proper subset of itself, whereas a finite set cannot be similar to any proper subset of itself.

Definition 2.17: Countably infinite set. The set S is countably infinite if it is equinumerous with the set of all positive integers. That is,

$$S \sim \{1, 2, 3, ...\}.$$

If a set is countably infinite, there is a function f that establishes a one-to-one correspondence between positive integers and the elements of S. Therefore, S can be expressed as

$$S = \{f(1), f(2), f(3), ...\}.$$

Countable infinite sets are said to have cardinal number $\aleph_0$.

Definition 2.18: Countable set. The set S is countable if it is either finite or countably infinite. In other case it is called uncountable.

PROPOSITION 2.4. Very subset of a countable set is countable.

PROPOSITION 2.5. The set of all real numbers, $\mathbb{R}$, is uncountable.

PROPOSITION 2.6. The cartesian product $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.

Now it is discussed the topic of set algebra.

Definition 2.19: Union of sets. Given two sets, A_1 and A_2, the union $A_1 \cup A_2$ is the set of those elements which belong either to A_1, or to A_2, or tho both.

It can be shown that the union of sets is a commutative and associative operation:

$$A_1 \cup A_2 = A_2 \cup A_1,$$
$$A_1 \cup (A_2 \cup A_3) = (A_1 \cup A_2) \cup A_3.$$

The definition of union can be extended to finite or infinite collections of sets.

Definition 2.20: Arbitrary union. If F is an arbitrary collection of sets, then the union of all the sets in F is defined to be the set of those elements which belong to at least one of the sets in F. This is denoted by

$$\bigcup_{A \in F} A.$$

If F is a finite collection of sets, $F = \{A_1, ..., A_n\}$, then

$$\bigcup_{A\in F} A = \bigcup_{k=1}^{n} A_k = A_1 \cup ... \cup A_n.$$

If F is an infinite collection of sets, $F = \{A_1, ...\}$, then

$$\bigcup_{A\in F} A = \bigcup_{k=1}^{\infty} A_k = A_1 \cup A_2 \cup ...$$

Definition 2.21: Arbitrary intersection. If F is an arbitrary collection of sets, the intersection of all sets in F is defined to be the set of those elements which belong to every one of the sets in F and is denoted by

$$\bigcap_{A\in F} A.$$

If two sets, A_1 and A_2, do not have elements in common, then $A_1 \cap A_2$ is an empty set, and A_1 and A_2 are called disjoint sets.

If F is a finite collection of sets, $F = \{A_1, ..., A_n\}$, then

$$\bigcap_{A\in F} A = \bigcap_{k=1}^{n} A_k = A_1 \cap ... \cap A_n.$$

If F is an infinite collection of sets, $F = \{A_1, ...\}$, then

$$\bigcap_{A\in F} A = \bigcap_{k=1}^{\infty} A_k = A_1 \cap A_2 \cap ...$$

It can be shown that the intersection operation is commutative and associative.

PROPOSITION 2.7. Let $f : A \to B$ be a function, and let $X_i \subset A$ be a collection of sets indexed by $i \in I$. In that case,

$$f\left(\bigcup_{i\in I} X_i\right) = \bigcup_{i\in I} f(X_i),$$
$$f\left(\bigcap_{i\in I} X_i\right) \subseteq \bigcap_{i\in I} f(X_i).$$

Let $Y_j \subset B$ be a collection of sets indexed by $j \in J$. In that case,

$$f^{-1}\left(\bigcup_{j\in J} Y_j\right) = \bigcup_{j\in J} f^{-1}(Y_j),$$
$$f^{-1}\left(\bigcap_{j\in J} Y_j\right) \subseteq \bigcap_{j\in J} f^{-1}(Y_j).$$

Definition 2.22: Relative complement. Let A and B be sets. The complement of A relative to B, denoted by $B-A$, is defined as

$$B-A = \{x: \ x \in B, \text{ but } x \notin A\}.$$

It is possible to see that if $A \subseteq B$, then $B-(B-A) = A$. Also, if $B \cap A$ is empty, then $B-A = B$.

PROPOSITION 2.8. Let F be a collection of sets. Then, for any set B, we have

$$B - \bigcup_{A\in F} A = \bigcup_{A\in F} (B-A),$$
$$B - \bigcap_{A\in F} A = \bigcap_{A\in F} (B-A).$$

Definition 2.23 F is called a collection of disjoint sets if every two distinct sets in F are disjoint.

PROPOSITION 2.9. Let $F = \{A_1, A_2, ...\}$ be a countable collection of disjoint sets such that each set A_n is countable. Then, the union $\bigcup_{k=1}^{\infty} A_k$ is also countable.

PROPOSITION 2.10. If $F = \{A_1, A_2, ...\}$ is a countable collection of sets, let $G = \{B_1, B_2, ...\}$, where $B_1 = A_1$ and $B_n = A_n - \bigcup_{k=1}^{n-1}$ for $n > 1$. Then G is a collection of disjoint sets, and we have

$$\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} B_k.$$

PROPOSITION 2.11. If F is a countable collection of countable sets, then the union of all sets in F is also a countable set.

---

3 Point set topology

Before get into the topic of metric spaces, it will be useful to understand basic notions about topology, a branch of mathematics that studies abstract sets: sets of arbitrary objects.

Definition 3.1: n-dimensional space. Let $n > 0$ be an integer. An ordered set of n real numbers $x = (x_1, ..., x_n)$ is called an n-dimensional point or a vector with n components. The number x_k is called the kth coordinate of x. The set of all n-dimensional points called a n-dimensional Euclidean space or simply n-space, and is denoted by $\mathbb{R}^n$.

Definition 3.2: Algebraic operations on n-dimensional points. Let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ be in $\mathbb{R}^n$. Let also $\alpha \in \mathbb{R}$ be a scalar. We define the following operations:

- Equality: $x = y$ if, and only if, $x_1 = y_1, ..., x_n = y_n$.
- Sum: $x + y = (x_1 + y_1, ..., x_n + y_n)$.
- Multiplication by scalar: $\alpha x = (\alpha x_1, ..., \alpha x_n)$.
- Zero vector: $0 = (0, ..., 0)$.
- Inner (dot) product: $x \cdot y = \sum_{k=1}^{n} x_k y_k$.
- Norm or length: $||x|| = (x \cdot x)^{1/2} = (\sum_{k=1}^{n} x_k^2)^{1/2}$.

It is important to mention that the norm $||x - y||$ is called the distance between x and y.

PROPOSITION 3.1. *Properties of points in* $\mathbb{R}^n$.. Let x, y and z be points in $\mathbb{R}^n$, and $\alpha \in \mathbb{R}$ a scalar. Then,

a) $||x|| \geq 0$.

b) $||x|| = 0$ if, and only if, $x = 0$.

c) $||\alpha x|| = ||\alpha|| \, ||x||$.

d) $||x - y|| = ||y - x||$.

e) Cauchy-Schwartz inequality: $||x \cdot y|| \leq ||x|| \, ||y||$.

f) Triangle inequality: $||x + y|| \leq ||x|| + ||y||$, and $||x - z|| \leq |x - y|| + ||y - z||$.

Definition 3.3: Unit coordinate vector. The unit coordinate vector u_k in $\mathbb{R}^n$ is the vector whose kth component is 1 and its remaining component are zero. The vectors $u_1, ..., u_n$ are also called basis vectors.

Definition 3.4: Open ball. Let a be a point in $\mathbb{R}^n$ and r be a positive number. The set of all points $x \in \mathbb{R}^n$ such that $||x-a|| < r$ is called an open ball or an open n-ball of radius r and center a.

Formally, an open ball or radius r and center a is given by

$$B(a;r) = \{x : \ ||x-a|| < r\} .$$

Definition 3.5: Interior point. Let S be a subset of $\mathbb{R}^n$, and assume that $a \in S$. Then a is called an interior point of S if there is an open ball with center at a, all of whose points belong to S. The set of all interior points of S is called the interior of S and is denoted by int S.

If $a \in S$ is an interior point, then a can be surrounded by an open ball $B(a;r) \subseteq S$.

Definition 3.6: Open set. A set S in $\mathbb{R}^n$ is called open if all its points are interior points.

In $\mathbb{R}^n$ the empty set and the whole space are open sets. Also, the cartesian product $(a_1,b_1) \times ... \times (a_n,b_n)$ of n one-dimensional open intervals is an open set in $\mathbb{R}^n$ called an n-dimensional open interval, denoted by (a,b) where $a = (a_1,...,a_n)$ and $b = (b_1,...,b_n)$.

PROPOSITION 3.2. The union of any collection of open sets is an open set.

PROPOSITION 3.3. The intersection of a finite collection of open sets is an open set.

Definition 3.7: Closed set. A set S in $\mathbb{R}^n$ is called closed if its complement $\mathbb{R}^n - S$ is open.

PROPOSITION 3.4. The union of a finite collection of closed sets is closed. On the other hand, the intersection of an arbitrary collection of closed sets is closed.

PROPOSITION 3.5. If A is an open set and B is a closed set, then $A-B$ is open and $B-A$ is closed.

Definition 3.8: Component interval. Let S be an open subset of $\mathbb{R}$. An open interval I is called component interval of S if $I \subseteq S$ and if there is no open interval $J \neq I$ such that $I \subseteq J \subseteq S$. In other words, a component interval of S is not a proper subset of any other open interval contained in S.

PROPOSITION 3.6. Every point of a nonempty open set S belongs to one and only one component interval of S.

THEOREM 3.1. *Representation theorem for open sets on the real line*. Every nonempty open set S in $\mathbb{R}$ is the union of a countable collection of disjoint component intervals of S.

Definition 3.9: Adherent point. Let S be a subset of $\mathbb{R}^n$, and x a point in $\mathbb{R}^n$. The point x is not necessarily in S. Then x is said to be adherent to S if every open ball $B(x,r)$ contains at least one point of S.

Definition 3.10: Accumulation point. If $S \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$, then x is called an accumulation point of S if every open ball $B(x,r)$ contains at least one point distinct from x.

In other words, x is an accumulation point of S if, and only if, x adheres to $S - \{x\}$. If $x \in S$ but it is not an accumulation point of S, then x is called an isolated point.

PROPOSITION 3.7. If x is an accumulation point of S, then every open ball $B(x,r)$ contains infinitely many points of S.

PROPOSITION 3.8. A set S in $\mathbb{R}^n$ is closed if, and only if, it contains all its adherent points.

Definition 3.11: Closure. The set of all adherent points for a set S is called the closure of S, denoted by $\overline{S}$.

PROPOSITION 3.9. A set S is closed if, and only if, $S = \overline{S}$.

Definition 3.12: Derived set. The of all accumulation points is called the derived set of S and is denoted by S'.

PROPOSITION 3.10. A set S in $\mathbb{R}^n$ is closed if, and only if, it contains all its accumulation points.

Definition 3.13: Bounded set. A set S in $\mathbb{R}^n$ is bounded if it lies entirely within an open ball $B(a,r)$ for some $r > 0$ and some $a \in \mathbb{R}^n$.

THEOREM 3.2. *Bolzano-Weierstrass theorem.* If a bounded set S in $\mathbb{R}^n$ contains infinitely many points, then there is at least one point in $\mathbb{R}^n$ which is an accumulation point of S.

THEOREM 3.3. *Cantor intersection theorem.* Let $\{Q_1, Q_2, Q_3, \ldots\}$ be a countable collection of nonempty sets in $\mathbb{R}^n$ such that $Q_{k+1} \subseteq Q_k$ for $k \in \{1, 2, \ldots\}$; and each set Q_k is closed and Q_1 is bounded. Then, $\bigcap_{k=1}^{\infty} Q_k$ is closed and nonempty.

Definition 3.14: Covering. A collection of sets F is said to be a covering of a given set S or to cover S if $S \subseteq \bigcup_{A \in F} A$. If F is a collection of open sets, then F is called an open covering of S.

PROPOSITION 3.11. Let $G = \{A_1, A_2, \ldots\}$ denote a countable collection of all open balls having rational radii and centers at points with rational coordinates. Assume $x \in \mathbb{R}^n$ and let S be an open set in $\mathbb{R}^n$ which contains x. Then at least one of the open balls in G contains x and is contained in S. In other words, we will have that $x \in A_k \subseteq S$ for some A_k in G.

THEOREM 3.4. *Lindelöf covering theorem.* Assume $A \subseteq \mathbb{R}^n$ and let F be an open covering of A. Then there is a countable subcollection of F which also covers A.

THEOREM 3.5. *Heine-Borel covering theorem.* Let F be an open covering of a closed and bounded set A in $\mathbb{R}^n$. Then a finite subcollection of F also covers A.

Definition 3.15: Compactness. A set S in $\mathbb{R}^n$ is compact if, and only if, every open covering of S contains a finite subcover, that is, a finite subcollection which also covers S.

PROPOSITION 3.12. Let S be a subset of $\mathbb{R}^n$. Then the following three statements are equivalent:

a) S is compact.

b) S is closed and bounded.

c) Every infinite subset of S has an accumulation point in S.

4 Metric spaces

In this section we aim to describe exhaustively the topic of metric spaces. We start with the definition of a metric space and other related concepts. Then we focus on concepts discussed in the last section in the context of metric spaces.

Definition 4.1: Metric space. A metric space is a nonempty set M of objects together with a function $d : M \times M \to \mathbb{R}$, called the metric of the space, that satisfies the following properties for every $x, y, z \in M$:

a) $d(x,x) = 0$.

b) $d(x,y) > 0$ if $x \neq y$.

c) $d(x,y) = d(y,x)$.

d) $d(x,z) = d(x,y) + d(y,z)$.

Sometimes the metric space is denoted by (M,d) to emphasize that both the set M and the metric d play a role in the definition of a metric space.

Example 4.1 It is possible to demonstrate that $(\mathbb{R}^n, d)$ with $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$d_p(x,y) = \left[\sum_{k=1}^{n} (|x_k - y_k|)^p \right]^{1/p} \quad \text{with } p \in [1, +\infty)$$

is a metric space. Some interesting cases of the distance d_p are

- When $p = 1$, d_1 is the taxicab distance: $d_1(x,y) = \sum_{k=1}^{n} |x_k - y_k|$.
- When $p = 2$, d_2 is the euclidean distance: $d_2(x,y) = (\sum_{k=1}^{n} |x_k - y_k|)^{1/2}$.
- When $p \to +\infty$, d_∞ is the distance of the maximum: $d_\infty(x,y) = \max_{k \in \{1,\dots,n\}} |x_k - y_k|$.

Definition 4.2: Metric subspace. If (M,d) is a metric space and $S \subset M$ is nonempty, then (S,d) is also a metric space with the restriction of d to $S \times S$ as metric. In this case (S,d) is called a metric subspace of (M,d).

Definition 4.3: Ultrametric space. Let M be a metric space with distance $d : M \times M \to \mathbb{R}$. M is an ultrametric space if d satisfies the "ultrametric inequality" or strong triangle inequality:

$$d(x,z) \leq \max\{d(x,y), d(y,z)\}, \ \forall x,y,z \in M.$$

Definition 4.4: Vector space. A vector space over a field F is a nonempty set V which is closed under the addition and the scalar multiplication. Therefore, if $A, B \in V$ and $c \in F$,

$$A + B \in V \qquad \text{and} \qquad cA \in V.$$

Definition 4.5: Normed vector space. V is a normed vector space if V is a $\mathbb{R}$-vector space with a norm $||\cdot|| : V \to \mathbb{R}$. Therefore, $||\cdot||$ satisfies

a) $||v|| \geq 0$, $\forall v \in V$.

b) $||v|| = 0$ if, and only if, $v = 0$.

c) $||av|| = |a| \;\; ||v||$, $\forall a \in \mathbb{R}$ and $v \in V$.

d) $||v+w|| \geq ||v|| + ||w||$, $\forall v, w \in V$.

Example 4.2 If V is a normed vector space, then V is a metric space with distance $d(v,w) = ||v-w||$ induced by the norm $||\cdot||$.

Example 4.3 Let $M = \mathscr{B}[a,b]$ be the space of all bounded functions $f : [a,b] \to \mathbb{R}$. We define the supremum norm (or sup norm) as

$$||f||_\infty = \sup_{x \in [a,b]} |f(x)|.$$

Then, $d(f,g) = ||f-g||_\infty$ defines a distance. Therefore, M is a metric space.

Example 4.4 Let $M = \mathscr{C}[a,b]$ be the space of all continuous functions $f : [a,b] \to \mathbb{R}$. We define the maximum norm[4] as

$$||f||_\infty = \max_{x \in [a,b]} |f(x)|.$$

Then, $d(f,g) = ||f-g||_\infty$ defines a distance. Therefore, M is a metric space.

Example 4.5 Every nonempty set M can be a metric space with the discrete distance $d : M \times M \to \mathbb{R}$ which is defined as

$$d(p,q) = \begin{cases} 1 & \text{if } p \neq q, \\ 0 & \text{if } p = q. \end{cases}$$

In fact, M is an ultrametric space.

4.1 Open and closed sets

Some of the point set topology definitions and results can be extended to arbitrary metric spaces.

Definition 4.6: Open ball in M. The open ball in M with center x_0 and radius $r > 0$ centered in $x_0 \in M$ is the set

$$B(x_0, r) = \{x \in E \mid d(x, x_0) < r\}.$$

[4] Weierstrass theorem allow us to replace the supremum with the maximum

It is possible to see that every open ball in M is nonempty because they always contain their center. Also, every open ball is bounded because it is self contained.

If $S \subseteq M$, a point $a \in S$ is called an interior point if there exists an open ball $B(a,r)$ which lies entirely in S. The interior of S, denoted by int S, is the set of all interior points of S. If all of its point are interior, the set S is open in M. If $M - S$ is open in M, then S is closed in M.

If $S \subseteq M$, a point $x \in M$ is called an adherent point of S if every ball $B(x,r)$ contains at least one point of S. If x adheres to $S - \{x\}$, then x is called an accumulation point of S. The closure of S, denoted $\overline{S}$, is the set of all adherent points of S, and the derived set S' is the set of all accumulation points of S.

PROPOSITION 4.1. Let (S,d) be a metric subspace of (M,d), and let X be a subset of S. Then X is open in S if, and only if, $X = A \cap S$, for some A which is open in M.

PROPOSITION 4.2. Let (S,d) be a metric subspace of (M,d), and let Y be a subset of S. Then Y is closed in S if, and only if, $Y = B \cap S$, for some B which is closed in M.

PROPOSITION 4.3. Let $A_1, A_2, ..., A_n$ be a collection of open sets in M. Then, the set

$$\bigcap_{i=1}^{n} A_i$$

is open.

Let $B_1, B_2, ...$ be an arbitrary collection of open sets in M, indexed by $i \in I$. Then, the set

$$\bigcup_{i \in I} B_i$$

is open.

PROPOSITION 4.4. Let $A_1, A_2, ..., A_n$ be a collection of closed sets in M. Then, the set

$$\bigcup_{i=1}^{n} A_i$$

is closed.

Let $B_1, B_2, \ldots$ be an arbitrary collection of closed sets in M, indexed by $i \in I$. Then, the set

$$\bigcap_{i \in I} B_i$$

is closed.

PROPOSITION 4.5. If A is open and B is closed, then $A - B$ is open and $B - A$ is closed.

PROPOSITION 4.6. For any $S \subseteq M$ the following statements are equivalent:

a) S is closed in M.

b) S contains all its adherent points.

c) S contains all its accumulation points.

d) $S = S'$.

Let (M, d) be a metric subspace and let $S \subseteq M$. A collection F of open subsets of M is said to be an open covering if $S \subseteq \bigcup_{A \in F} A$. The set $S \subseteq M$ is compact if every open covering of S contains a finite subcover. S is called bounded if $S \subseteq B(a, r)$ for some $r > 0$ and some $a \in M$.

PROPOSITION 4.7. Let S be a compact subset of M. Then S is closed and bounded, and every infinite subset of S has an accumulation point in S.

It is worth to mention that in the Euclidean space $\mathbb{R}^n$, if S is closed and bounded, or if every infinite subset of S has an accumulation point in S, then S will be compact. Nevertheless, in a general metric space, a closed and bounded subset of M it is not compact.

PROPOSITION 4.8. Let X be a closed subset of a compact metric space M. Then X is compact.

Definition 4.7: Boundary. Let S be a subset of a metric space M. A point $x \in M$ is called a boundary point if every open ball $B(x, r)$ contains at least one point of S and at least one point of $M - S$. The set of all boundary points of S is called the boundary of S and is denoted by ∂S.

References

A Elementary notation

Let S denote a set or a collection of objects. The notation $x \in S$ means that the object x is in the set S. In other case, we write $x \notin S$ when x is not in the set S.

A set S is a subset of T, written by $S \subset$, if every object in S is also in T. A set it is called nonempty if it contains at least one object.

The notation $\{x : \; x \text{ satisfies } P\}$ is used to designate the set of real numbers x which satisfy property P.