L. *On the Calculation of the Equilibrium and Stiffness of Frames.*

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THE theory of the equilibrium and deflections of frameworks subjected to the action of forces is sometimes considered as more complicated than it really is, especially in cases in which the framework is not simply stiff, but is strengthened (or weakened as it may be) by additional connecting pieces.

I have therefore stated a general method of solving all such questions in the least complicated manner. The method is derived from the principle of Conservation of Energy, and is referred to in Lamé's *Leçons sur l'Elasticité*, Leçon 7^{me}, as Clapeyron's Theorem; but I have not yet seen any detailed application of it.

If such questions were attempted, especially in cases of three dimensions, by the regular method of equations of forces, every point would have three equations to determine its equilibrium, so as to give $3s$ equations between e unknown quantities, if s be the number of points and e the number of connexions. There are, however, six equations of equilibrium of the system which must be fulfilled necessarily by the forces, on account of the equality of action and reaction in each piece. Hence if

$$e=3s-6,$$

the effect of any external force will be definite in producing tensions or pressures in the different pieces; but if $e>3s-6$, these forces will be indeterminate. This indeterminateness is got rid of by the introduction of a system of e equations of elasticity connecting the force in each piece with the change in its length. In order, however, to know the changes of length, we require to assume $3s$ displacements of the s points; 6 of these displacements, however, are equivalent to the motion of a rigid body so that we have $3s-6$ displacements of points, e extensions and e forces to determine from $3s-6$ equations of forces, e equations of extensions, and e equations of elasticity; so that the solution is always determinate.

The following method enables us to avoid unnecessary complexity by treating separately all pieces which are additional to those required for making the frame stiff, and by proving the identity in form between the equations of forces and those of extensions by means of the principle of work.

On the Stiffness of Frames.

Geometrical definition of a Frame.—A frame is a system of lines connecting a number of points.

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A stiff frame is one in which the distance between any two points cannot be altered without altering the length of one or more of the connecting lines of the frame.

A frame of s points in space requires *in general* $3s-6$ connecting lines to render it stiff. In those cases in which stiffness can be produced with a smaller number of lines, certain conditions must be fulfilled, rendering the case one of a maximum or minimum value of one or more of its lines. The stiffness of such frames is of an inferior order, as a small disturbing force may produce a displacement infinite in comparison with itself.

A frame of s points in a plane requires in general $2s-3$ connecting lines to render it stiff.

A frame of s points in a line requires $s-1$ connecting lines.

A frame may be either simply stiff, or it may be self-strained by the introduction of additional connecting lines having tensions or pressures along them.

In a frame which is simply stiff, the forces in each connecting line arising from the application of a force of pressure or tension between any two points of the frame may be calculated either by equations of forces, or by drawing diagrams of forces according to known methods.

In general, the lines of connexion in one part of the frame may be affected by the action of this force, while those in other parts of the frame may not be so affected.

Elasticity and Extensibility of a connecting piece.

Let e be the extension produced in a piece by tension-unity acting in it, then e may be called its extensibility. Its elasticity, that is, the force required to produce extension-unity, will be $\frac{1}{e}$. We shall suppose that the effect of pressure in producing compression of the piece is equal to that of tension in producing extension, and we shall use e indifferently for extensibility and compressibility.

Work done against Elasticity.

Since the extension is proportional to the force, the whole work done will be the product of the extension and the mean value of the force; or if x is the extension and F the force,

$$x=e\mathrm{F},$$

$$\text{work}=\frac{1}{2}\,\mathrm{F}x=\frac{1}{2}\,e\mathrm{F}^2=\frac{1}{2}\frac{1}{e}\,x^2.$$

When the piece is inextensible, or $e=0$, then all the work applied at one end is transmitted to the other, and the frame may be regarded as a machine whose efficiency is perfect. Hence the following

Theorem.—If p be the tension of the piece A due to a tension-unity between the points B and C, then an extension-unity taking place in A will bring B and C nearer by a distance p.

For let X be the tension and x the extension of A, Y the tension and y the extension of the line BC; then supposing all the other pieces inextensible, no work will be done except in stretching A, or

$$\frac{1}{2}\mathrm{X}x+\frac{1}{2}\mathrm{Y}y=0.$$

But $\mathrm{X}=p\mathrm{Y}$, therefore $y=-px$, which was to be proved.

Problem I.—A tension F is applied between the points B and C of a frame which is simply stiff; to find the extension of the line joining D and E, all the pieces except A being inextensible, the extensibility of A being e.

Determine the tension in each piece due to unit tension between B and C, and let p be the tension in A due to this cause.

Determine also the tension in each piece due to unit tension between D and E, and let y be the tension in the piece A due to this cause.

Then the actual tension of A is $\mathrm{F}p$, and its extension is $e\mathrm{F}p$, and the extension of the line DE due to this cause is $-\mathrm{F}epq$ by the last theorem.

Cor.—If the other pieces of the frame are extensible, the com plete value of the extension in DE due to a tension F in BC is

$$-\mathrm{F}\Sigma(epq),$$

where $\Sigma(epq)$ means the sum of the products of epq, which are to be found for each piece in the same way as they were found for A.

The extension of the line BC due to a tension F in BC itself will be

$$-\mathrm{F}\Sigma(ep^2),$$

$\Sigma(ep^2)$ may therefore be called the resultant extensibility along BC.

Problem II.—A tension F is applied between B and C; to find the extension between D and E when the frame is not simply stiff, but has additional pieces R, S, T, &c. whose elasticities are known.

Let p and q, as before, be the tensions in the piece A due to unit tensions in BC and DE, and let r, s, t, &c. be the tensions in A due to unit tension in R, S, T, &c.; also let R, S, T be the tensions of R, S, T, and ρ, σ, τ their extensibilities. Then the tension A

$$=\mathrm{F}p+\mathrm{R}r+\mathrm{S}s+\mathrm{T}t+\&c.;$$

the extension of A

$$=e(\mathrm{F}p+\mathrm{R}r+\mathrm{S}s+\mathrm{T}t+\&\mathrm{c.});$$

the extension of R

$$=-\mathrm{F}\Sigma(epr)-\mathrm{R}\Sigma er^2-\mathrm{S}\Sigma ers-\mathrm{T}\Sigma ert+\&\mathrm{c.}=\mathrm{R}\rho;$$

extension of S

$$=-\mathrm{F}\Sigma(eps)-\mathrm{R}\Sigma(ers)-\mathrm{S}\Sigma es^2-\mathrm{T}\Sigma(est)=\mathrm{S}\sigma;$$

extension of T

$$=-\mathrm{F}\Sigma(ept)-\mathrm{R}\Sigma(ert)-\mathrm{S}\Sigma(est)-\mathrm{T}\Sigma(et^2)=\mathrm{T}\tau;$$

also extension of DE

$$=-\mathrm{F}\Sigma(epq)-\mathrm{R}\Sigma(eqr)-\mathrm{S}\Sigma(eqs)-\mathrm{T}\Sigma(eqt)=x,$$

the extension required. Here we have as many equations to determine R, S, T, &c. as there are of these unknown quantities, and by the last equation we determine x the extension of DE from F the tension in BC.

Thus, if there is only one additional connexion R, we find

$$\mathrm{R}=-\mathrm{F}\frac{\Sigma(epr)}{\Sigma(er^2)+\rho},$$

and

$$x=-\mathrm{F}\left\{\Sigma(epq)+\frac{\Sigma(epr)\Sigma(eqr)}{\Sigma(er^2)+\rho}\right\}.$$

If there are two additional connexions R and S, with elasticities ρ and σ,

$$x=\frac{-\mathrm{F}}{\Sigma e(r^2+\rho)\Sigma e(s^2+\sigma)-\big(\Sigma(ers)\big)^2}\left\{\begin{array}{l}\Sigma(epr)\Sigma(ers)\Sigma(eqs)+\Sigma(eps)\Sigma(eqr)\Sigma(ers)+\Sigma(epq)\Sigma e(r^2+\rho)\Sigma e(s^2+\sigma)\\-\Sigma(epr)\Sigma(eqr)\Sigma e(s^2+\sigma)-\Sigma(eps)\Sigma(eqs)\Sigma e(r^2+\rho)-\Sigma(epq)\big(\Sigma(ers)\big)^2\end{array}\right\}$$

The expressions for the extensibility, when there are many additional pieces, are of course very complicated.

It will be observed, however, that p and q always enter into the equations in the same way, so that we may establish the following general

Theorem.—The extension in BC, due to unity of tension along DE, is always equal to the extension in DE due to unity of tension in BC. Hence we have the following method of determining the displacement produced at any joint of a frame due to forces applied at other joints.

1st. Select as many pieces of the frame as are sufficient to render all its points stiff. Call the remaining pieces R, S, T, &c.

2nd. Find the tension on each piece due to unit of tension in the direction of the force proposed to be applied. Call this the value of p for each piece.

3rd. Find the tension on each piece due to unit of tension in the direction of the displacement to be determined. Call this the value of q for each piece.

4th. Find the tension on each piece due to unit of tension along R, S, T, &c., the additional pieces of the frame. Call these the values of r, s, t, &c. for each piece.

5th. Find the extensibility of each piece and call it e, those of the additional pieces being ρ, σ, τ, &c.

6th. R, S, T, &c. are to be determined from the equations

$$R\rho+R\Sigma(er^2)+S(ers)+T\Sigma(ert)+F\Sigma(epr)=0,$$
$$S\sigma+R\Sigma(ers)+S(es^2)+T\Sigma(est)+F\Sigma(eps)=0,$$
$$T\tau+R\Sigma(ert)+S(est)+T\Sigma(et^2)+F\Sigma(ept)=0,$$

as many equations as there are quantities to be found.

7th. x, the extension required, is then found from the equation

$$x=-F\Sigma(epq)-R\Sigma(erq)-S\Sigma(eqs)-T\Sigma(eqt).$$

In structures acted on by weights in which we wish to determine the deflection at any point, we may regard the points of support as the extremities of pieces connecting the structure with the centre of the earth; and if the supports are capable of resisting a horizontal thrust, we must suppose them connected by a piece of equivalent elasticity. The deflection is then the shortening of a piece extending from the given point to the centre of the earth.

Example.—Thus in a triangular or Warren girder of length l, depth d, with a load W placed at distance a from one end, O; to find the deflection at a point distant b from the same end, due to the yielding of a piece of the boom whose extensibility is e, distant x from the same end.

The pressure of the support at $O=W\frac{l-a}{l}$; and if x is less than a, the force at x will be $\frac{W}{dl}x(l-a)$, or

$$p=\frac{x(l-a)}{dl}.$$

If x is greater than a,

$$p=\frac{a(l-x)}{dl}.$$

Similarly, if x is less than b,

$$q=\frac{x(l-b)}{dl};$$

but if x is greater than b,

$$q=\frac{b(l-x)}{dl}.$$

The deflection due to x is therefore Wepq, where the proper values of p and q must be taken according to the relative position of a, b, and x.

If a, b, l, x represent the *number* of the respective pieces, reckoning from the beginning and calling the first joint 0, the second joint and the piece opposite 1, &c., and if L be the length of each piece, and the extensibility of each piece $=e$, then the deflection of b due to W at a will be, by summation of series,

$$=\frac{1}{6}\mathrm{W}e\mathrm{L}^2 . \frac{a(l-b)}{a^2 l}\{2b(l-a)-(b-a)^2+1\}.$$

This is the deflection due to the yielding of all the horizontal pieces. The greater the number of pieces, the less is the importance of the last term.

Let the inclination of the pieces of the web be α, then the force on a piece between 0 and a is $\mathrm{W}\frac{l-a}{l\sin\alpha}$, or

$$p'=\frac{l-a}{l\sin\alpha} \text{ when } x<a,$$

and

$$p'=\frac{a}{l\sin\alpha} \text{ when } x>a.$$

Also

$$q'=\frac{l-b}{l\sin\alpha} \text{ when } x<b,$$

$$q'=\frac{b}{l\sin\alpha} \text{ when } x>b.$$

If e' be the extensibility of a piece of the web, we have to sum $\mathrm{W}\Sigma e'p'q'$ to get the deflection due to the yielding of the web,

$$=\frac{\mathrm{W}e'}{l^2\sin^2\alpha}a(l-b)\{l+2(b-a)\}.$$

LI. *Proceedings of Learned Societies.*

ROYAL INSTITUTION OF GREAT BRITAIN.

Feb. 12, 1864. "ON the Synthesis of Organic Bodies." By J. Alfred Wanklyn, Esq., Professor of Chemistry, London Institution.

On this tray you will see a collection of well-known substances. Compare these substances with one another, and you will be struck with their dissimilarities. Some are solids and crystalline and brittle, others are liquids which are more fluid than water. Some are without colour; others are highly coloured, and are used for dyeing. Some