Journal of Statistical Physics, Vol. 25, No. 3, 1981

The Validity of Nonlinear Langevin Equations

N. G. van Kampen[1]

Received August 22, 1980

In the presence of internal noise the variables describing a system are intrinsically stochastic. If they constitute a Markov process the Ω expansion enables one to extract a deterministic macroscopic equation and to compute the fluctuations in successive approximations. In the lowest or linear noise approximation the fluctuations can be represented by a Langevin equation, provided it is handled appropriately. Higher orders cannot be described by any white noise Langevin equation. The question whether the equation has to be interpreted according to Itô or Stratonovich concerns these higher orders, for which the equation is not valid anyway.

KEY WORDS: Langevin equation; nonlinear noise.

1. FORMULATION OF THE PROBLEM

Consider a physical system whose macroscopic state is determined by a variable x which obeys an equation of motion

$$\dot{x} = f(x) \tag{1}$$

In principle x could stand for a set of variables, but for simplicity we confine ourselves to the case of a single one. In order to take fluctuations into account one often employs the "Langevin approach." This consists in adding a fluctuating term to the macroscopic equation:

$$\dot{x} = f(x) + g(x)w(t) \tag{2}$$

and postulating certain stochastic properties of the random function w. It should be emphasized that this postulate is an essential ingredient of the Langevin approach; without it the equation is merely a definition of w and contains no information from which the fluctuations in x can be calculated.

[1] Institute for Theoretical Physics of the University at Utrecht, Netherlands.

0022-4715/81/0700-0431$03.00/0

One usually postulates w to be Gaussian white noise, normalized by

$$\langle w(t)\rangle = 0, \qquad \langle w(t_1)w(t_2)\rangle = \delta(t_1 - t_2) \tag{3}$$

The choice of $g(x)$ is based on physical considerations regarding the nature of the noise source.

This way of taking fluctuations into account is beset with difficulties. One of them is that if g depends on x the last term has no well-defined meaning unless one supplies an *ad hoc* interpretation rule. Yet it may be stated that this approach leads to correct results in the following three cases.

(a) When the noise is due to an *external* source coupled to the original system (1). This means that g is proportional to a physical coupling parameter and can be switched off. In this case, however, $w(t)$ is never exactly white, so that (2) does have a well-defined meaning as a stochastic differential equation. In principle this equation can be solved and afterwards one may take the limit of vanishing autocorrelation time of w. Practically all engineering problems belong to this class, including the passage of a noisy signal through a nonlinear device. This case presents no conceptual difficulties and is not the subject of this article.

(b) *Internal* noise cannot be switched off but is an inherent feature of the same physical mechanism that causes the evolution (1). Examples are Brownian motion, Ohmic resistance, chemical rate equations. Yet the Langevin approach can be used *in linear approximation* around a stationary solution x^s of (1). This means that $|x - x^s|$ must be small enough to set

$$f(x) = (x - x^s)f'(x^s), \qquad g(x) = g(x^s) \tag{4}$$

No interpretation dilemma arises as g is taken constant. If the stationary state happens to be the thermodynamic equilibrium the value of this constant can be found from the fluctuation–dissipation theorem.

This case also comprises those systems in which the nonlinear terms are of mechanical or kinematic origin, provided that the dissipative processes in them, which cause the fluctuations, may be regarded as linear. An example is hydrodynamics with linear viscosity and heat conduction.

(c) Internal noise around a *nonstationary* solution $x = \varphi(t)$ of (1) can also be described by (2) in the "linear noise approximation." In this approximation the fluctuations are supposed to be so small that for the purpose of computing them one may set

$$f(x) = f(\varphi(t)) + \{x - \varphi(t)\}f'(\varphi(t)), \qquad g(x) = g(\varphi(t)) \tag{5}$$

Thus, for computing the solution $\varphi(t)$ the original nonlinear equation (1) is used, but for the fluctuations f is locally linearized.

Class (b) is a subclass of (c) since (4) is obtained from (5) by substituting the particular solution $\varphi(t) = x^s$. Our task is therefore to show that (5) is correct. More precisely, it constitutes the lowest order in a well-defined systematic approximation scheme. At the same time we shall find expressions for f and g. Moreover we shall show in Section 3 that higher orders cannot be represented by any Langevin equation (2) with (3). Thus *the physical contents of the Langevin equation is exhausted by its approximate solution* (5).

In this limited sense one may use the nonlinear Langevin equation (2), and in fact in many applications it is used in precisely this sense. However, there is a tendency to take the equation (2) more seriously and to think that *all* information contained in it, even beyond the approximation (5), has physical content. Of course, one then has to face the fact that (2) as it stands has no meaning and requires an additional interpretation rule. Our interpretation rule (5), which takes into account that (2) itself is only an approximation, is of course not satisfactory for mathematicians. Hence they developed the Itô and Stratonovich rules.[1,2] There is no point in arguing which one is correct, because both are mathematically consistent and both extend the meaning of (2) beyond its physical validity.

In our case of internal noise the variable x is basically a stochastic process and should be treated as such, rather than as a macroscopic quantity to which fluctuations are added. A fundamental assumption, both of the Langevin approach and of the present one, is that x (or the set of variables x) is so chosen as to be Markovian (with sufficient approximation). That implies that its transition probability density $P(x,t\,|\,x_0,t_0)$ (from x_0 at t_0 to x at $t > t_0$) obeys for fixed x_0, t_0 a master equation

$$\dot{P} = \mathbb{W}P \tag{6}$$

where $\mathbb{W}$ is a linear operator acting on the x dependence of P. The kernel of this operator has the form

$$\mathbb{W}(x\,|\,x') = W(x\,|\,x') - \delta(x - x')\int W(x''\,|\,x)\,dx''$$

where $\mathbb{W}(x\,|\,x') \geqslant 0$ is the transition probability per unit time and represents the information concerning the physical structure of the system. When x only takes discrete values the integrations are replaced with summations. Equation (6) will be our starting point.

The master equation (6) describes the evolution of P as a semigroup. It has to be solved for $t > t_0$ with initial condition

$$P(x, t_0\,|\,x_0, t_0) = \delta(x - x_0) \tag{7}$$

for every possible x_0. Then the Markov process $x(t)$ is completely known, *including its macroscopic behavior* and the fluctuations around it. In particular it will be possible to test the assumptions and results of the Langevin approach. This strategy has been used earlier by Akcasu, and in fact our main results are implicit in his paper.[3] Unfortunately his work has not received the attention it deserves.

In most cases the master equation cannot be solved exactly, specifically in those cases that correspond with a nonlinear macroscopic equation. It is therefore necessary to utilize the expansion in a parameter Ω, which often represents the size of the system. The largest terms yield the macroscopic equation and the next terms the fluctuations in lowest approximation, which is called the "linear noise approximation." *Whenever we shall talk of orders of approximation we are referring to this expansion in* Ω^{-1}. For the higher terms to be actually small it is necessary that Ω be large. For them to remain small in the course of time the macroscopic solution $\varphi(t)$ must be asymptotically stable. This we assume throughout; if it is not so the expansion takes an entirely different form.

The Ω expansion is not just a mathematical device for obtaining approximate solutions of the master equation. *It is also the only consistent way of extracting a macroscopic, nonfluctuating equation* (1) from the stochastic process described by (6). Even if one could solve the master equation exactly one would still not know how to decompose the $P(x, t \mid x_0, t_0)$ obtained into a macroscopic part, obeying a deterministic equation (1), and a fluctuating part. Many authors tacitly identify the macroscopic value of x with its average $\langle x \rangle$, but this average does not, in general, obey a deterministic equation of motion. This is only so if all equations are linear, as in case (b)—which explains why the theory of linear noise is impervious to the conceptual difficulties that have plagued the literature on noise in nonlinear systems.

The Langevin approach, on the other hand, is the embodiment of the idea that the deterministic equation (1) is given first, and that fluctuation can be added afterwards, as in (2). This is true for external noise, but not for internal noise. In the case of internal noise $x(t)$ is basically a stochastic process given by (6), and its decomposition into a macroscopic aspect and fluctuations is man made. The Ω expansion is needed to carry out this decomposition in a consistent manner.

2. JUSTIFICATION OF EQ. (5)

We first summarize the results of the Ω expansion as far as the linear noise approximation. For simplicity we do not write the parameter Ω, but the reader can verify from the literature[4] that our equations represent the first two terms in the expansion. Afterwards we verify that the same result is reproduced by (5).

The information contained in the transition probabilities $W(x \mid x')$ may be expressed alternatively in the jump moments $a_\nu(x)$, defined for $\nu = 0, 1,$

2, . . . by

$$a_\nu(x) = \int (x' - x)^\nu W(x' \mid x)\, dx' \tag{8}$$

The integral extends over the range of possible values of x. The macroscopic equation (1) is

$$\dot{x} = a_1(x) \tag{9}$$

[This statement has to be qualified. In general the jump moments (8) are power series in Ω^{-1}. The equation (9) therefore contains terms of relative order Ω^{-1}, which do not belong to the truly macroscopic description. Hence equation (1) should be identified with the leading term of (9) alone. For our present considerations, however, this complication is irrelevant and we therefore assume that no such higher-order terms are present.]

Let $x = \varphi(t)$ be a solution of (9) with initial value x_0,

$$\dot{\varphi}(t) = a_1(\varphi(t)), \qquad \varphi(t_0) = x_0 \tag{10}$$

The actual $x(t)$ is not equal to this, but is a stochastic variable and may be written

$$x = \varphi(t) + \xi \tag{11}$$

Here ξ is a new stochastic variable, differing from x by the nonstochastic shift $\varphi(t)$. Its average obeys

$$\partial_t \langle \xi \rangle = a_1'(\varphi(t)) \langle \xi \rangle \tag{12}$$

where the prime indicates the derivative. Since the initial value x_0 is taken as initial value of $\varphi(t)$ it is clear that $\langle \xi \rangle$ vanishes at $t = t_0$ and hence at all later times. Thus, *in this approximation* the macroscopic value $\varphi(t)$ is identical with the average $\langle x(t) \rangle$.

The second moment of ξ—which is also the variance of x—obeys

$$\partial_t \langle \xi^2 \rangle = 2a_1'(\varphi(t)) \langle \xi^2 \rangle + a_2(\varphi(t)) \tag{13}$$

According to (7) its initial value at t_0 equals zero. This determines the variance of the fluctuations around the macroscopic solution $\varphi(t)$. Finally, the linear noise approximation has the consequence that the fluctuations are Gaussian, so that they are fully determined by their average and variance.

Equations (10), (12), (13) are the results of the Ω expansion. We now try to construct an equation of Langevin type that reproduces them. First it is clear that the fluctuations are correctly given by

$$\dot{\xi} = a_1'(\varphi(t))\xi + \left[a_2(\varphi(t)) \right]^{1/2} w(t) \tag{14}$$

with Gaussian white $w(t)$ normalized as in (3). This is a *linear* Langevin equation and can be solved without difficulty, even though the coefficients are functions of t. The explicit solution is not needed, however, for showing that (14) is equivalent with the results summarized above. It suffices to note, firstly, that (14) leads to a Gaussian $\xi(t)$ as it should; secondly, that $\langle\xi\rangle$ does indeed obey (12); and, thirdly, that $\langle\xi^2\rangle$ obeys (13). The third point is verified by the familiar computation of Uhlenbeck and Ornstein.[5] One obtains from (14)

$$\langle(\Delta\xi)^2\rangle = \left\langle \left\{ a_1'\xi\,\Delta t + (a_2)^{1/2}\int_0^{\Delta t} w(t')\,dt' \right\}^2 \right\rangle$$

$$= a_2\,\Delta t + \mathcal{O}(\Delta t)^2$$

so that

$$\Delta\langle\xi^2\rangle = 2\langle\xi\,\Delta\xi\rangle + \langle(\Delta\xi)^2\rangle = \left(2a_1'\xi^2 + a_2\right)\Delta t$$

in agreement with (13). Thus, for a given macroscopic solution $\varphi(t)$ the Langevin equation (14) is an entirely adequate description of the fluctuations in linear noise approximation.

The problem, however, is to find an equation (2) for x itself, which is not confined to one particular macroscopic solution φ. It should hold for all x_0, t_0 and imply the macroscopic equation (9) as well. This is actually achieved by setting

$$\dot{x} = a_1(x) + \left[a_2(x)\right]^{1/2} w(t) \tag{15}$$

provided it is interpreted in the way indicated by (5). That is, this equation has to be handled as follows. First neglect the fluctuating term in (15) so as to obtain the macroscopic equation (9). Solve this equation with the required initial value to obtain $\varphi(t)$. Substitute this macroscopic solution for the x in a_2 and linearize a_1 around it:

$$\dot{x} = a_1(\varphi) + (x-\varphi)a_1'(\varphi) + \left[a_2(\varphi)\right]^{1/2} w(t) \tag{16}$$

This is the same equation as (14). The conclusion is that *equation* (15) *does give the same results that follow from the master equation in linear noise approximation, provided it is interpreted in the way specified above*. To this extent (15) is correct.

3. LIMITATION OF THE LANGEVIN EQUATION

It will now be shown that when the approximation is carried to the next order it is no longer possible to reproduce the result by a Langevin equation. To make the consistency of the argument clear it is now necessary to display the powers of Ω explicitly. On the other hand, to avoid lengthy equations we take the special case (b) of fluctuations in equilibrium. Clearly if these do not obey a Langevin equation there is no such equation for the general case either.

Let x be measured on the scale of the elementary jumps, which is independent of Ω. If one is dealing with a homogeneous system of size Ω, such as a well-stirred chemical reaction, the number of jumps is proportional to Ω, but their frequency per unit volume depends on the intensive variable x/Ω. Hence

$$a_\nu(x) = \Omega\alpha_\nu(x/\Omega)$$

where α_ν is a function that no longer contains Ω as implicit parameter. (Actually this is not the most general case, but we shall adopt it for simplicity; it is not hard to generalize it so as to encompass all master equations to which the Ω expansion applies.)

The ansatz (11) now takes the form

$$x = \Omega\varphi(t) + \Omega^{1/2}\xi$$

The macroscopic equation (10) is

$$\dot{\varphi} = \alpha_1(\varphi) \qquad \varphi(t_0) = x_0/\Omega$$

To find the probability distribution $\Pi(\xi, t)$ of the fluctuations around the macroscopic value $\Omega\varphi(t)$ one expands the master equation in powers of $\Omega^{-1/2}$. The result is

$$\begin{aligned}\frac{\partial\Pi}{\partial t} = &-\frac{\partial}{\partial\xi}\left\{\alpha_1' + \Omega^{-1/2}\alpha_1''\xi + \frac{1}{2}\Omega^{-1}\alpha_1'''\xi^2\right\}\Pi \\ &+\frac{1}{2}\frac{\partial^2}{\partial\xi^2}\left\{\alpha_2 + \Omega^{-1/2}\alpha_2'\xi + \frac{1}{2}\Omega^{-1}\alpha_2''\xi^2\right\}\Pi \\ &-\frac{1}{3!}\Omega^{-1/2}\frac{\partial^3}{\partial\xi^3}\left\{\alpha_3 + \Omega^{-1/2}\alpha_3'\xi\right\}\Pi \\ &+\frac{1}{4!}\Omega^{-1}\frac{\partial^4}{\partial\xi^4}\alpha_4\Pi + \mathcal{O}(\Omega^{-3/2})\end{aligned} \tag{17}$$

The coefficients are the $\alpha_\nu(\varphi)$ and their derivatives; they depend on time through $\varphi(t)$. If one omits the terms with $\Omega^{-1/2}$ and higher, (17) reduces to a linear Fokker–Planck equation, from which the equations (12) and (13) in the previous section were obtained.

For our present purpose we include the terms with $\Omega^{-1/2}$ and Ω^{-1}. On the other hand, it suffices to take the special case $\varphi = \varphi^s$, so that the coefficients are constants. Then the first two lines constitute a nonlinear Fokker–Planck equation. The essential point, however, is that *in the same order higher derivatives appear.* This cannot be reproduced by a Langevin equation. For, any equation of type (2), regardless whether it is interpreted by Itô or Stratonovich, is equivalent with a Fokker–Planck equation (without derivatives higher than the second).

To complete the argument we show explicitly that any equation (2) is incompatible with (17). For a Markov process with master equation (6) the change of x during a short time Δt has the moments

$$\begin{aligned}\langle(\Delta x)^{\nu}\rangle_x &= \int (x'-x)^{\nu} P(x',t+\Delta t\,|\,x,t)\,dx' \\ &= \Delta t \int (x'-x)^{\nu} W(x'\,|\,x)\,dx' + o(\Delta t)\end{aligned}$$

Now equation (2) is equivalent with a Fokker–Planck equation

$$\frac{\partial P(x,t\,|\,x_0,t_0)}{\partial t} = \left[-\frac{\partial}{\partial x}F(x) + \frac{1}{2}\frac{\partial^2}{\partial x^2}g(x)^2\right]P$$

where $F=f$ for Itô and $F=f+\frac{1}{2}gg'$ for Stratonovich.[2] In either case, one finds for $\nu=1,2,\ldots$

$$\begin{aligned}\frac{\partial\langle(\Delta x)^{\nu}\rangle}{\partial t} &= \int (x'-x)^{\nu}\left[-\frac{\partial}{\partial x'}F(x') + \frac{1}{2}\frac{\partial^2}{\partial x'^2}g(x')^2\right]P(x',t\,|\,x,t)\,dx' \\ &= \int\left[\nu(x'-x)^{\nu-1}F(x') + \frac{1}{2}\nu(\nu-1)(x'-x)^{\nu-2}g(x')^2\right] \\ &\quad\times\delta(x'-x)\,dx'\end{aligned}$$

This expression vanishes for $\nu \geqslant 3$. Of course this is just the condition that Kolmogorov postulated in order to derive the Fokker–Planck equation,[6] but we needed to know that it is also a necessary condition.

On the other hand, one finds in the same way from (17)

$$\frac{\langle(\Delta x)^3\rangle}{\Delta t} = \Omega^{3/2}\frac{\langle(\Delta\xi)^3\rangle_\xi}{\Delta t} = \Omega\alpha_3 + \Omega^{1/2}\alpha_3'\xi$$

$$\frac{\langle(\Delta x)^4\rangle}{\Delta t} = \Omega^2\frac{\langle(\Delta x)^4\rangle_\xi}{\Delta t} = \Omega\alpha_4$$

The last line does *not* vanish, unless $\alpha_4=0$, that is, unless

$$\int (x'-x)^4 W(x'\,|\,x)\,dx' = 0$$

which is true only if the original $\mathbb{W}$ in (6) is itself a second-order differential operator à la Fokker and Planck. The conclusion is that no Langevin equation (2) is valid beyond the lowest order in the fluctuations, i.e., the linear noise approximation—and to this order it is equivalent to (5), or more explicitly to (13). The Itô–Stratonovich dilemma is moot, because it refers to higher terms, for which (2) is not valid anyway.

4. ALTERNATIVE VERIFICATION OF (16)

In this section we shall employ a reverse approach as an alternative to the method of Section 2. The strategy is to take a master equation, write the corresponding equation (14) with a yet unspecified function $v(t)$ instead of $w(t)$, and derive the stochastic properties of $v(t)$. Our technique is based on the "curtailed generating functional," which has recently been introduced in a similar context.[7] The result will be that in linear noise approximation $v(t)$ is Gaussian white noise, thus confirming (14) and (16).

We carry out this scheme for the special example of the radioactive decay. The master equation for the probability distribution of the number n of active nuclei is

$$\dot{P}(n,t) = (n+1)P(n+1,t) - nP(n,t)$$

As initial distribution at $t=0$ take $P(n,0) = \delta_{n,n_0}$. The first two jump moments are

$$a_1(n) = -n, \qquad a_2(n) = n$$

Hence the corresponding equation (15) is

$$\dot{n} = -n + n^{1/2}v(t) \tag{18}$$

where w is replaced by v to emphasize that we do not anticipate the stochastic properties of this function.

There is still one snag. If $v(t)$ should turn out to be singular, (18) as it stands might have no meaning. Since we do not presume on the nature of $v(t)$, the only possible manner to assign a meaning to (18) is to declare it equivalent to

$$\frac{\dot{n}}{n^{1/2}} = -n^{1/2} + v(t) \tag{19}$$

We therefore have to find the stochastic properties of

$$v(t) = 2e^{-t/2}\frac{d}{dt}e^{t/2}n^{1/2}$$

Thus we have to find the generating functional of v, involving an arbitrary test function $l(t)$,

$$\begin{aligned} G_v[l] &= \left\langle \exp\left[i\int_0^\infty l(t)v(t)\,dt \right] \right\rangle \\ &= \exp\left[-2il(0)(n_0)^{1/2}\right]\left\langle \exp\left\{ -2i\int_0^\infty [n(t)]^{1/2}e^{t/2}\frac{d}{dt}e^{-t/2}l(t)\,dt \right\} \right\rangle \end{aligned}$$

The second factor is the characteristic function of $n^{1/2}$:

$$G_v[l] = \exp\left[-2il(0)(n_0)^{1/2}\right] G_{n^{1/2}}[k] \tag{20}$$

$$k(t) = -2e^{t/2}\frac{d}{dt}e^{-t/2}l(t)$$

To find the characteristic function of $n^{1/2}$ one defines the curtailed characteristic functional $\Gamma(n,t)$—depending not only on the function $k(t)$ but also on two additional variables n,t—by

$$\Gamma(n,t) = \left\langle \exp\left\{ i\int_0^t k(t')\left[n(t')\right]^{1/2} dt' \right\} \delta_{n(t),n} \right\rangle$$

It obeys the "masterly equation"

$$\dot{\Gamma}(n,t) = (n+1)\Gamma(n+1,t) - n\Gamma(n,t) + ik(t)\left[n(t)\right]^{1/2}\Gamma(n,t) \tag{21}$$

with initial value

$$\Gamma(n,0) = \delta_{n,n_0} \tag{22}$$

Once this equation has been solved one obtains the characteristic functional from the identity

$$G_{n^{1/2}}[k] = \lim_{t\to\infty} \sum_{n=0}^{\infty} \Gamma(n,t) \tag{23}$$

As (21) cannot be solved exactly we again employ the Ω expansion, setting

$$n = \Omega\varphi(t) + \Omega^{1/2}\xi, \qquad \Gamma(n,t) = \Omega^{-1/2}\Xi(\xi,t)$$

$$\frac{\partial \Xi}{\partial t} - \Omega^{1/2}\frac{d\varphi}{dt}\frac{\partial \Xi}{\partial \xi} = \left(\Omega^{-1/2}\frac{\partial}{\partial \xi} + \frac{1}{2}\Omega^{-1}\frac{\partial^2}{\partial \xi^2}\right)(\Omega\varphi + \Omega^{1/2}\xi)\Xi \tag{24}$$

$$+ ik(t)\Omega^{1/2}\varphi^{1/2}\left(1 + \frac{1}{2}\Omega^{-1/2}\xi/\varphi\right)\Xi$$

Terms of order $\Omega^{-1/2}$ and higher have been omitted in agreement with the linear noise approximation. To cancel the terms of order $\Omega^{1/2}$ take

$$\varphi(t) = \varphi_0 e^{-t}, \qquad \varphi_0 = n_0/\Omega \tag{25}$$

and substitute

$$\Xi(\xi,t) = \Psi(\xi,t)\exp\left\{ i\Omega^{1/2}\int_0^t k(t')\left[\varphi(t')\right]^{1/2} dt' \right\}$$

$$= \Psi(\xi,t)\exp\left[i(n_0)^{1/2}\int_0^t k(t')e^{-t'/2} dt' \right]$$

One then has to order Ω^0

$$\frac{\partial \Psi}{\partial t} = \frac{1}{2}\varphi\frac{\partial^2 \Psi}{\partial \xi^2} + \frac{\partial}{\partial \xi}\xi\Psi + \frac{1}{2}ik(t)\frac{\xi}{\varphi^{1/2}}\Psi \tag{26}$$

According to (22), (24), and (25) the initial value is

$$\Psi(\xi, 0) = \delta(\xi)$$

Equation (26) can be solved by substituting for Ψ an arbitrary Gaussian, say

$$\Psi(\xi, t) = \frac{\gamma}{(2\pi\alpha)^{1/2}} \exp\left[-\frac{(\xi - \beta)^2}{2\alpha} \right] \tag{27}$$

Substitution in (26) shows that (27) is a solution provided that α, β, γ as functions of t obey three equations, which can be solved to give

$$\alpha(t) = \varphi_0 e^{-t}(1 - e^{-t})$$

$$\beta(t) = -\frac{1}{2} i(\varphi_0)^{1/2} e^{-t} \int_0^t k(t')(e^{t'/2} - e^{-t'/2})\, dt'$$

$$\log \gamma(t) = -\frac{1}{4} \int_0^t e^{-t'/2} k(t')\, dt' \int_0^{t'} k(t'')(e^{t''/2} - e^{-t''/2})\, dt''$$

Inserting all this into (23) one finds

$$G_{n^{1/2}}[k] = \exp\left[i(n_0)^{1/2} \int_0^\infty k(t') e^{-t'/2}\, dt' \right.$$
$$\left. - \frac{1}{4} \int_0^\infty e^{-t'/2} k(t')\, dt' \int_0^{t'} k(t'')(e^{t''/2} - e^{-t''/2})\, dt'' \right]$$

Using (20) one finally obtains after some partial integrations

$$G_v[l] = \exp\left[-\frac{1}{2} \int_0^\infty l(t')^2\, dt' \right]$$

This is precisely the generating functional of Gaussian white noise.

Of course the same calculation can be done for other special examples. In fact, it can be carried out for the general Markov process described by (6) and then constitutes an alternative derivation of the result of Section 2, rather than just a verification for a special case. In order to prove in this way the result of Section 3 one would have to go to order Ω^{-1}. This is possible in principle, but the calculation is formidable.

As we now know that $v(t)$ in (18) is Gaussian white noise, it appears that our interpretation (19) is merely the Stratonovich rule. That may seem surprising since it implies that on averaging (18) one finds that $\langle n \rangle$ does not obey the macroscopic decay law, although that ought to be true for this linear process. In fact, (18) with Stratonovich rule is equivalent to the Itô equation

$$\dot{n} = -n + \tfrac{1}{4} + n^{1/2} w(t) \qquad \text{(Itô)}$$

Hence

$$\partial_t \langle n \rangle = -\langle n \rangle + \tfrac{1}{4}$$

This shows, however, that the error is of relative order Ω^{-1} and therefore beyond the order to which the Langevin equation can claim validity.

REFERENCES

1. K. Itô, *Mem. Amer. Math. Soc.* **4**:1 (1951); R. L. Stratonovich, *Topics in the Theory of Random Noise*, Vol. I (Gordon and Breach, New York, 1963), p. 89 ff.
2. R. E. Mortensen, *J. Stat. Phys.* **1**:271 (1961); N. G. van Kampen, *J. Stat. Phys.* **24**:175 (1981).
3. Z. A. Akcasu, *J. Stat. Phys.* **16**:33 (1977). The reader should beware of the slight difference in the definitions of various quantities.
4. N. G. van Kampen, *Can. J. Phys.* **39**:551 (1961); *Advances in Chemical Physics*, Vol. 34, I. Prigogine and S. A. Rice, eds. (Wiley, New York, 1976), p. 245; R. Kubo, K. Matsuo, and K. Kitahara, *J. Stat. Phys.* **9**:51 (1973).
5. G. E. Uhlenbeck and L. S. Ornstein, *Phys. Rev.* **36**:823 (1930); reprinted in N. Wax, *Noise and Stochastic Processes* (Dover, New York, 1954).
6. A. Kolmogorov, *Math. Ann.* **104**:415 (1931).
7. N. G. van Kampen, *Phys. Lett.* **76A**:104 (1980).