CS/MATH 375 – Homework 10

Numerical Integration

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1 Composite Trapezoid and Simpson's Rule

We analyze the function $f(x) = x\sin(x)$ on the interval $[0, \pi]$. The exact integral is

$$\int_0^\pi x \sin x \, dx = \pi.$$

1.1 Trapezoid Error Bound for 10^{-6} Accuracy

The composite trapezoid error is bounded by

$$|E| \le \frac{(b-a)}{12} h^2 \max_{x\in[a,b]} |f''(x)|, \qquad h = \frac{b-a}{n}.$$

Since $f''(x) = 2\cos x - x\sin x$, we numerically find

$$\max |f''(x)| \approx 3.10306.$$

Using this bound with tolerance 10^{-6},

$$n \ge (b-a)\sqrt{\frac{M}{12 \times 10^{-6}}} \approx 1597.55.$$

Thus, $n = 1598$ subintervals are sufficient to guarantee an error below 10^{-6}.

1.2 Composite Trapezoid Results

The following table shows the trapezoid approximations, absolute errors, and observed convergence orders for several values of n:

Table 1: Composite trapezoid results for $f(x) = x \sin x$ on $[0, \pi]$.

n	Approximation	Error	Observed p
4	2.97841660004589	0.163176053543904	—
8	3.10111574857848	0.0404769050113174	2.0113
16	3.13149297320483	0.0100996803849629	2.0028
32	3.13906895090321	0.00252370268658275	2.0007

The observed order $p \approx 2$ confirms the expected second-order accuracy of the composite trapezoid rule.

1.3 Composite Simpson's Rule ($n = 4$ Subintervals)

Using the composite Simpson method with $n = 4$ subintervals (five function evaluations) yields:

$$\text{Approximation} = 3.148755099970, \quad \text{Error} = 7.1624464 \times 10^{-3}.$$

As expected, the Simpson approximation achieves significantly higher accuracy than the trapezoid rule for the same number of subintervals.

2 Degree of Precision for Simpson's Rule

2.1 Exactness for $f(x) = 1, x, x^2$

Simpson's rule integrates exactly any quadratic function, as it is derived from a quadratic interpolant passing through the points a, $(a+b)/2$, and b. Therefore, it is exact for $f(x) = 1, x, x^2$.

2.2 Exactness for $f(x) = x^3$

Substituting $f(x) = x^3$ into Simpson's formula yields the same result as the true integral $\int_a^b x^3 dx = (b^4 - a^4)/4$, confirming exactness for cubic polynomials.

2.3 Linearity

For constants α, β,

$$S(\alpha f + \beta g) = \alpha S(f) + \beta S(g),$$

so Simpson's rule is a linear operator.

2.4 Degree of Precision

Because Simpson's rule is exact for $1, x, x^2, x^3$ but not for x^4, its degree of precision is **3**.

3 MATLAB Code

comp_trap_int.m

```
function I = comp_trap_int(f,a,b,n)
%COMP_TRAP_INT Composite trapezoid rule.
% I = COMP_TRAP_INT(f,a,b,n) approximates the integral of f on [a,b]
% using n subintervals (n >= 1).
%
% Example:
% f = @(x) x.*sin(x); I = comp_trap_int(f,0,pi,8)
%
h = (b - a) / n;
x = a:h:b;
y = arrayfun(f, x);
I = h * (0.5*y(1) + sum(y(2:end-1)) + 0.5*y(end));
end
```

comp_simpson_int.m

```
function I = comp_simpson_int(f,a,b,n)
%COMP_SIMPSON_INT Composite Simpson's rule.
% I = COMP_SIMPSON_INT(f,a,b,n) approximates the integral of f on [a,b]
% using n subintervals (n must be even; n >= 2).
%
% Example:
% f = @(x) x.*sin(x); I = comp_simpson_int(f,0,pi,4)
%
if mod(n,2) ~= 0
    error('n must be even for composite Simpson''s rule');
end
h = (b - a) / n;
x = a:h:b;
y = arrayfun(f, x);
I = (h/3) * ( y(1) + y(end) + ...
    4*sum(y(2:2:end-1)) + 2*sum(y(3:2:end-2)) );
end
```

hw10_run.m

```
function hw10_run()
%HW10_RUN Driver script for CS/MATH 375 HW #10.
% Generates the trapezoid table (n=4,8,16,32), Simpson with n=4,
% and a bound-based estimate for n to reach 1e-6 with trapezoid.

format long g
f = @(x) x.*sin(x);
a = 0; b = pi;
exact = pi;

% --- Part 1(b,c): Trapezoid approximations and observed order ---
nlist = [4 8 16 32];
approx = zeros(size(nlist));
err = zeros(size(nlist));
p = nan(size(nlist));

for k = 1:numel(nlist)
    n = nlist(k);
    approx(k) = comp_trap_int(f,a,b,n);
    err(k) = abs(approx(k) - exact);
    if k > 1
        h_ratio = ((b-a)/nlist(k))/((b-a)/nlist(k-1));
        p(k) = log(err(k)/err(k-1)) / log(h_ratio);
    end
end

T = table(nlist.', approx.', err.', p.', ...
    'VariableNames', {'n','TrapzApprox','Error','Observed_p'});
disp('Composite trapezoid results:');
disp(T);

% --- Part 1(d): Composite Simpson with n=4 ---
n_simp = 4;
simp = comp_simpson_int(f,a,b,n_simp);
simp_err = abs(simp - exact);
```

```
fevals = n_simp + 1;
fprintf('\nComposite Simpson with n=%d subintervals:\n', n_simp);
fprintf(' Approximation = %.12f\n', simp);
fprintf(' Error = %.12e\n', simp_err);
fprintf(' f-evaluations = %d\n', fevals);

% --- Part 1(a): bound on n for 1e-6 accuracy for trapezoid ---
f2 = @(x) 2*cos(x) - x.*sin(x);
xx = linspace(a,b,20001);
M = max(abs(f2(xx)));
tol = 1e-6;
n_bound = ceil( (b-a) * sqrt(M / (12*tol)) );
fprintf('\nTrapezoid error-bound estimate: need n >= %d ', n_bound);
fprintf('to guarantee error <= 1e-6.\n');
end
```