Control of Systems with Parameter Uncertainty

KOM 501E

Homework 1

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Our desired system to be controlled was given as

$$\frac{98}{47}+\frac{13 s^5}{9}+s^4\left(\frac{209}{43}+\frac{q_1}{11}\right)+\frac{12 q_1}{11}+\frac{5 q_2}{23}+\frac{q_3}{21}+s\left(\frac{178}{25}+\frac{23 q_1}{40}+q_2+\frac{10 q_3}{23}+\frac{8 q_4}{57}\right)+s^2\left(\frac{683}{57}+\frac{28 q_1}{37}+q_3+\frac{28 q_4}{43}\right)+\frac{q_4}{97}+s^3\left(\frac{323}{29}+\frac{4 q_1}{31}+q_4\right)$$

where the uncertain parameters are bounded as follows: $1 \leq q_1 \leq 80$ and $-1 \leq q_i \leq 1$ (for i=2,3,4).

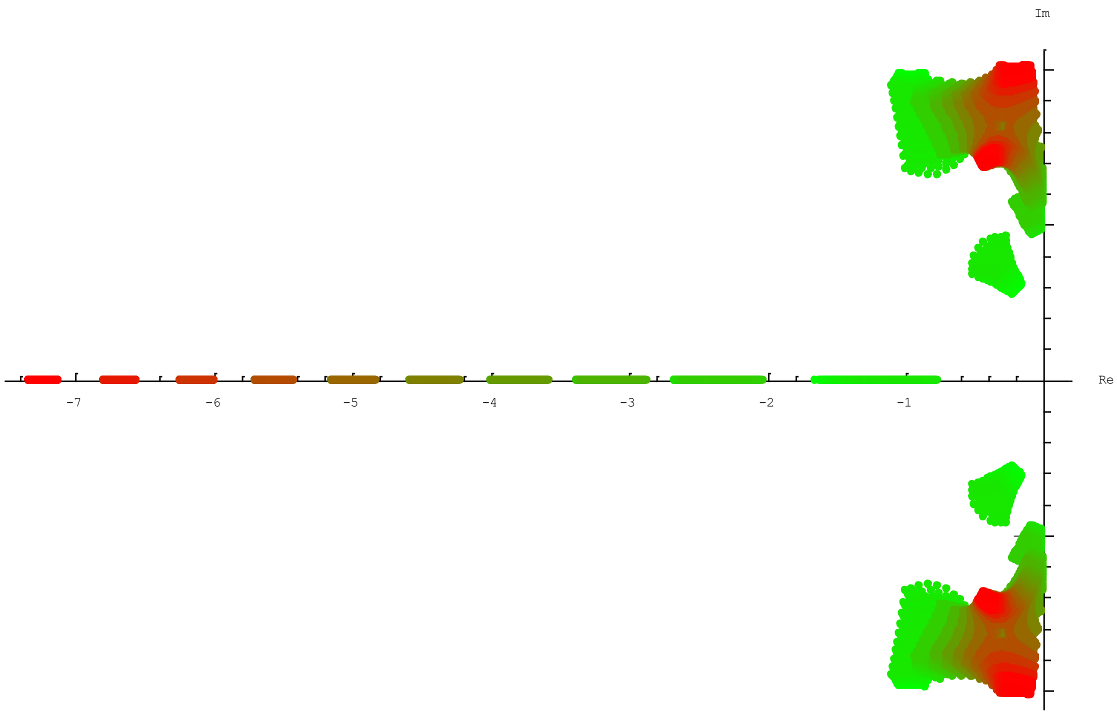
1. Draw the pole-spread of the polynomial (use η_i=10).

The boundries were defined as follows in mathematica:

```
qvec = {{q1, 1, 80}, {q2, -1, 1}, {q3, -1, 1}, {q4, -1, 1}};
qmin=Thread[qvec][[2]]
qmax=Thread[qvec][[3]]
qv=Thread[qvec][[1]]
```

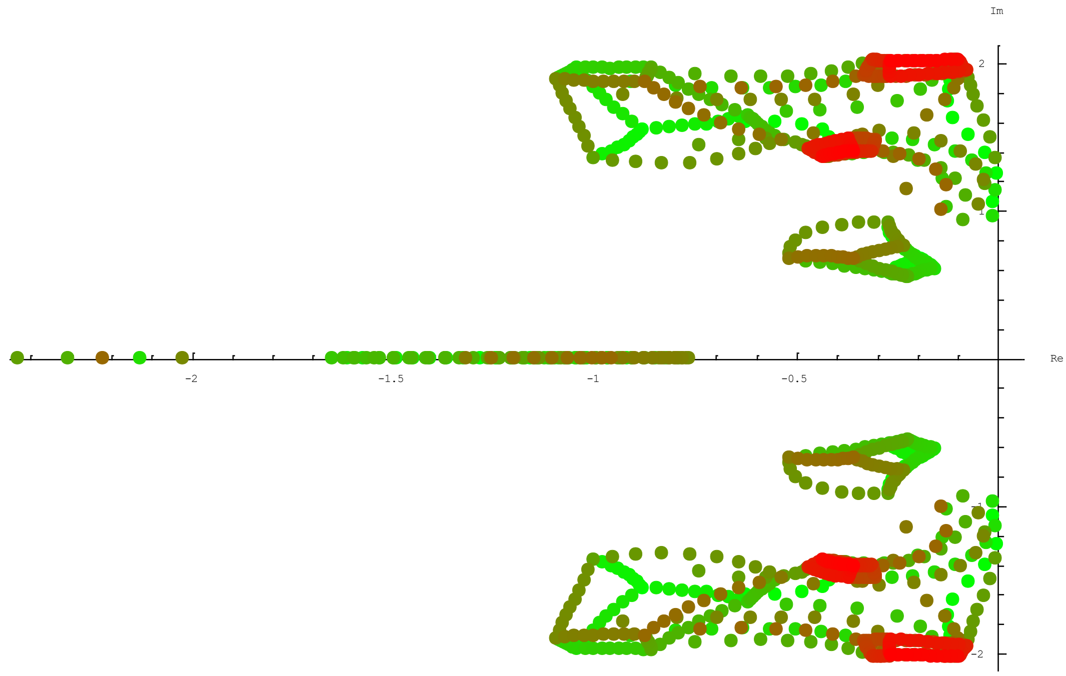
After all, with the help of the macsybox it was possible to draw pole-spread.

```
allgrp=PoleSpread[psq,s,qvec,ScanType->"All",QResolution□10]
```



No. of roots calculated = 10000

```
edgegrp=PoleSpread[psq,s,qvec,ScanType->"Edges",QResolution□10,PointSize□0.015]
```



To make a detailed analysis of the roots, the poles in the right half plane was checked as follows,

```
sp = ScanPoints[qmin, qmax, ScanType -> "Vertices"];
rl = Map[MapThread[Rule, {qv, #}] &, sp];
vpolys = psq /. rl;
RHPPoleList = Map[(Select[s /. Solve[# == 0.0], (Re[##] ≥ 0) &]) &, vpolys];

num = Length[Select[Map[Length, RHPPoleList], (# > 0) &]];
If[num === 0, Print["All vertex polynomials are stable"],
  Print[num, " vertex polynomial(s) is/are unstable"]];

All vertex polynomials are stable
```

The result was "**All vertex polynomials are stable**" and i can state that the mentioned polynomial family is robustly stable for this gridding.

2. Find the Routh table of the polynomial by the help of a symbolic algebra program. (If you are using Mathematica use Reduce[] command together with the restrictions on qi as given above to examine the stability of the polynomial family).

With the help of the following command, it was possible to create the routh tabulation

rth = RouthTabulation[psq]

$$
\begin{aligned}
\Bigg\{ & \frac{13}{9},\ \frac{209}{43}+\frac{q_1}{11},\ \frac{473\left(\frac{23558978}{639711}+\frac{4q_1^2}{341}-\frac{13q_3}{9}+q_1\left(\frac{77399291}{141600591}+\frac{q_4}{11}\right)+\frac{1517q_4}{387}\right)}{2299+43q_1},\ -(-4875406629967989340785\,q_1^3+469312578\,q_1^2\,(20813999598900\,q_2-20492886000600\,q_3-23\,(14286460515199+8242528909200\,q_4))+ \\
& 41080952\,(635789538439416900\,q_2+41558843128195575\,q_3^2-6496575\,q_3\,(42438209068+13194608007\,q_4)-322\,(25708449701258791+6049827683337450\,q_4+228071451532050\,q_4^2))+ \\
& 23199\,q_1\,(43570107649243807800\,q_2-744940600\,q_3\,(-63226924640+6217553223\,q_4)-161\,(7112540229977496839+1357254351100684800\,q_4+18732893138416800\,q_4^2)))\,/ \\
& (146441168088600\,(94677228\,q_1^2-12617\,(-23558978+924027\,q_3-2507601\,q_4)+57\,q_1\,(77399291+12872781\,q_4))), \\
& -(-19221486818751065651463264947784525\,q_1^4+267641923544730\,q_1^3\,(35745010738716455516319+95302737768621940200\,q_2-637846560938691146400\,q_3-1819126825344743250000\,q_4)+ \\
& 16371369\,q_1^2\,(134652370813702826723175600000\,q_2^2-5764129782987949531048800000\,q_3^2-20868765968400\,q_2\,(-23747400698103901+354742949685600\,q_3+5388183094218000\,q_4)- \\
& 4471878421800\,q_3\,(-57304931111863309+30978408504292415\,q_4)+3703\,(-2070575359810266153147174827-324565369206619142772437000\,q_4+24117122889897688858956000\,q_4^2))+ \\
& 10188076096\,(541175432021719909366661100000\,q_2^2+1644239960489943151022756250\,q_3^3-5439818289725\,q_3^2\,(-2508206962499486+558978226888725\,q_4)-2091897150\,q_3 \\
& (269919662626057267169+6435153809711763573 8\,q_4+1697315573699986725\,q_4^2)-77763\,(433795694611156459304 95992+1188101427825177174641935 0\,q_4+629032118333825970073350\,q_4^2+10700848187362278320625\,q_4^3)+ \\
& 45476025\,q_2\,(83159245099519345575\,q_3^2-125600450\,q_3\,(2029142484436+1509575304447\,q_4)-483\,(7727071951415146068+5648234037580067200\,q_4+202674066322159575\,q_4^2)))+ \\
& 550820167128\,q_1\,(40029926268941838807480000\,q_2^2-71574358435875\,q_3^2\,(-826034162453+26742164400\,q_4)-10925\,q_3\,(131167777735910494736239+24716855199564780889877\,q_4+151450405678763749800\,q_4^2)- \\
& 3703\,(-1166514213347323402239934-241669866623205111534025\,q_4-26099588670233433759300\,q_4^2+963945641554543350000\,q_4^3)- \\
& 11400\,q_2\,(2700409675\,q_3\,(-1946004115187+143003724129\,q_4)+161\,(246644308269426165957+98291921109763766725\,q_4+1040428230067158075\,q_4^2))))\,/ \\
& (22567386454200\,(4875406629967989340785\,q_1^3-469312578\,q_1^2\,(20813999598900\,q_2-20492886000600\,q_3-23\,(14286460515199+8242528909200\,q_4))- \\
& 41080952\,(635789538439416900\,q_2+41558843128195575\,q_3^2-6496575\,q_3\,(42438209068+13194608007\,q_4)-322\,(25708449701258791+6049827683337450\,q_4+228071451532050\,q_4^2))- \\
& 23199\,q_1\,(43570107649243807800\,q_2-744940600\,q_3\,(-63226924640+6217553223\,q_4)-161\,(7112540229977496839+1357254351100684800\,q_4+18732893138416800\,q_4^2)))),\ \frac{98}{47}+\frac{12q_1}{11}+\frac{5q_2}{23}+\frac{q_3}{21}+\frac{q_4}{97}\Bigg\}
\end{aligned}
$$

cond=Apply[And,Map[(#>0)&,rth]]

$$
\begin{aligned}
& \frac{209}{43}+\frac{q_1}{11}>0\ \&\&\ \frac{473\left(\frac{23558978}{639711}+\frac{4q_1^2}{341}-\frac{13q_3}{9}+q_1\left(\frac{77399291}{141600591}+\frac{q_4}{11}\right)+\frac{1517q_4}{387}\right)}{2299+43q_1}>0\ \&\& \\
& -(-4875406629967989340785\,q_1^3+469312578\,q_1^2\,(20813999598900\,q_2-20492886000600\,q_3-23\,(14286460515199+8242528909200\,q_4))+41080952\,(635789538439416900\,q_2+ \\
& 41558843128195575\,q_3^2-6496575\,q_3\,(42438209068+13194608007\,q_4)-322\,(25708449701258791+6049827683337450\,q_4+228071451532050\,q_4^2))+ \\
& 23199\,q_1\,(43570107649243807800\,q_2-744940600\,q_3\,(-63226924640+6217553223\,q_4)-161\,(7112540229977496839+1357254351100684800\,q_4+18732893138416800\,q_4^2)))\,/ \\
& (146441168088600\,(94677228\,q_1^2-12617\,(-23558978+924027\,q_3-2507601\,q_4)+57\,q_1\,(77399291+12872781\,q_4)))>0\ \&\& \\
& -(-19221486818751065651463264947784525\,q_1^4+267641923544730\,q_1^3\,(35745010738716455516319+95302737768621940200\,q_2-637846560938691146400\,q_3- \\
& 1819126825344743250000\,q_4)+16371369\,q_1^2\,(134652370813702826723175600000\,q_2^2-5764129782987949531048800000\,q_3^2- \\
& 20868765968400\,q_2\,(-23747400698103901+354742949685600\,q_3+5388183094218000\,q_4)-4471878421800\,q_3\,(-57304931111863309+30978408504292415\,q_4)+ \\
& 3703\,(-2070575359810266153147174827-324565369206619142772437000\,q_4+24117122889897688858956000\,q_4^2))+ \\
& 10188076096\,(541175432021719909366661100000\,q_2^2+1644239960489943151022756250\,q_3^3-5439818289725\,q_3^2\,(-2508206962499486+558978226888725\,q_4)- \\
& 2091897150\,q_3\,(269919662626057267169+64351538097117635738\,q_4+1697315573699986725\,q_4^2)- \\
& 77763\,(43379569461115645930495992+1188101427825177174641935 0\,q_4+629032118333825970073350\,q_4^2+10700848187362278320625\,q_4^3)+45476025\,q_2 \\
& (83159245099519345575\,q_3^2-125600450\,q_3\,(2029142484436+1509575304447\,q_4)-483\,(7727071951415146068+5648234037580067200\,q_4+202674066322159575\,q_4^2)))+ \\
& 550820167128\,q_1\,(40029926268941838807480000\,q_2^2-71574358435875\,q_3^2\,(-826034162453+26742164400\,q_4)- \\
& 10925\,q_3\,(131167777735910494736239+24716855199564780889877\,q_4+151450405678763749800\,q_4^2)- \\
& 3703\,(-1166514213347323402239934-241669866623205111534025\,q_4-26099588670233433759300\,q_4^2+963945641554543350000\,q_4^3)- \\
& 11400\,q_2\,(2700409675\,q_3\,(-1946004115187+143003724129\,q_4)+161\,(246644308269426165957+98291921109763766725\,q_4+1040428230067158075\,q_4^2))))\,/ \\
& (22567386454200\,(4875406629967989340785\,q_1^3-469312578\,q_1^2\,(20813999598900\,q_2-20492886000600\,q_3-23\,(14286460515199+8242528909200\,q_4))- \\
& 41080952\,(635789538439416900\,q_2+41558843128195575\,q_3^2-6496575\,q_3\,(42438209068+13194608007\,q_4)- \\
& 322\,(25708449701258791+6049827683337450\,q_4+228071451532050\,q_4^2))-23199\,q_1\,(43570107649243807800\,q_2-744940600\,q_3\,(-63226924640+6217553223\,q_4)- \\
& 161\,(7112540229977496839+1357254351100684800\,q_4+18732893138416800\,q_4^2))))>0\ \&\&\ \frac{98}{47}+\frac{12q_1}{11}+\frac{5q_2}{23}+\frac{q_3}{21}+\frac{q_4}{97}>0
\end{aligned}
$$

To determine the stability of the polynomial, all of the conditions above has to be satisfied.
For the first condition,

$q_1 > -53$ which is satisfied.

Since i couldn't calculate the rest of the conditions, i couldn't reach a solid solution about the stability of the mentioned polynomial family.

3. Similarly, calculate Hurwitz matrices and the corresponding determinants. Examine the stability of the polynomial family using the Hurwitz determinants.

After defining the p(s,q) polynomial, i assigned the coefficients of the polynomials to different variables and defined the Hurwitz matrix for n=5.

```
a5 = Coefficient[psq, s^5];
a4 = Coefficient[psq, s^4];
a3 = Coefficient[psq, s^3];
a2 = Coefficient[psq, s^2];
a1 = Coefficient[psq, s];
a0 = 12 q1 / 11 + 5 q2 / 23 + q3 / 21 + q4 / 97 + 98 / 47;

H = {{a4, a2, a0, 0, 0}, {a5, a3, a1, 0, 0}, {0, a4, a2, a0, 0}, {0, a5, a3, a1, 0}, {0, 0, a4, a2, a0}};
```

As it is stated in the Frazer and Duncan's theorem, If there exists a stable polynomial within the polynomial family and if determinant of matrix H is not equal to zero, the polynomial family can be called robustly stable. To find a stable polynomial, i just picked up random variable values and checked the stability. As it can be seen below, for these values a polynomial is stable within the polynomial family.

```
psq1 = psq /. q1 → 1 /. q2 → 0 /. q3 → 0 /. q4 → 0
```

$$\frac{1642}{517}+\frac{1539\,s}{200}+\frac{26867\,s^2}{2109}+\frac{10129\,s^3}{899}+\frac{2342\,s^4}{473}+\frac{13\,s^5}{9}$$

```
rth = RouthTabulation[psq1];
cond = Apply[And, Map[(# > 0) &, rth]]
True
```

To examine the second condition, determinant of the Hurwitz matrix was checked.

```
Solve[Det[H] == 0]
```

The numbers that makes the determinant of Hurwitz matrix zero are q_1 -> -50, q_4 -> -202. As these numbers exceeds the boundries of the variables, it can be said that the polynomial family is robustly stable.

4. For q2=q3= q4= -1, examine the stability of the polynomial by the help of Bialas Theorem.

According to Bialas theorem, Lets define H_n^b and H_n^c be Hurwitz matrices of $P_b(s)$ and $P_c(s)$. Our polynomial family is stable if and only if,

- $P_b(s)$ is stable
- The matrix $(H_n^b)^{-1} H_n^c$ has no nonpositive real eigenvalues.

To examine the stability under given conditions and with the help of the Bialas theorem, i created $P_b(s)$ and $P_c(s)$ first.

```
Pbs = psq /. q1 → 1
```

$$\frac{70260559}{24221967}+\frac{1604629\,s}{262200}+\frac{1005542\,s^2}{90687}+\frac{9230\,s^3}{899}+\frac{2342\,s^4}{473}+\frac{13\,s^5}{9}$$

```
Pcs = psq /. q1 → 80
```

$$\frac{2157753715}{24221967}+\frac{1689383\,s}{32775}+\frac{6427154\,s^2}{90687}+\frac{18394\,s^3}{899}+\frac{5739\,s^4}{473}+\frac{13\,s^5}{9}$$

After that, i checked the stability of $P_b(s)$

```
RouthTabulation[Pbs]
```

$$\left\{\frac{13}{9}, \frac{2342}{473}, \frac{140518180283}{19981849149}, \frac{583686036832959548502203 3}{791447940704884666041300}, \frac{330323935375137246347771698281442316177}{131722683611298135829651706512745388600}, \frac{70260559}{24221967}\right\}$$

Since all the elements of routh table are greater then zero, it can be claimed that $P_b(s)$ is stable.

Then for the second condition i created H_n^b and H_n^c matrixes. After that i calculated the eigenvalues of $(H_n^b)^{-1} H_n^c$ matrix.

$$H4c = \left\{\left\{\frac{18394}{899}, \frac{1689383}{32775}, 0, 0\right\}, \left\{\frac{5739}{473}, \frac{6427154}{90687}, \frac{2157753715}{24221967}, 0\right\}, \left\{0, \frac{18394}{899}, \frac{1689383}{32775}, 0\right\}, \left\{0, \frac{5739}{473}, \frac{6427154}{90687}, \frac{2157753715}{24221967}\right\}\right\};$$

$$H4b = \left\{\left\{\frac{9230}{899}, \frac{1604629}{262200}, 0, 0\right\}, \left\{\frac{2342}{473}, \frac{1005542}{90687}, \frac{70260559}{24221967}, 0\right\}, \left\{0, \frac{9230}{899}, \frac{1604629}{262200}, 0\right\}, \left\{0, \frac{2342}{473}, \frac{1005542}{90687}, \frac{70260559}{24221967}\right\}\right\};$$

```
H = Simplify[Eigenvalues[Inverse[H4b].H4c]]
```

$$\left\{\frac{2157753715}{70260559}, \text{Root}[-1618238886221763246730797 72736 - 427253114556986335928993820 00\,\#1 + 76891844239192661030762207343\,\#1^2 + 63766208224700673985588 07218\,\#1^3\,\&, 1],\right.$$
$$\text{Root}[-1618238886221763246730797 72736 - 427253114556986335928993820 00\,\#1 + 76891844239192661030762207343\,\#1^2 + 63766208224700673985588 07218\,\#1^3\,\&, 3],$$
$$\left.\text{Root}[-1618238886221763246730797 72736 - 427253114556986335928993820 00\,\#1 + 76891844239192661030762207343\,\#1^2 + 63766208224700673985588 07218\,\#1^3\,\&, 2]\right\}$$

The system has only one real eigenvalue and it is greater than zero which means the system has no nonpositive eigenvalues. Therefore, the system is robustly stable under these conditions.

5. Set q2=q3=q4= 0. Then, examine if the polynomial p(s − 0.26, q_1) satisfies the interlacing property for q1 = 1, q1 = 46 and q1 = 80. For this purpose, you are required to draw the interlacing plot (plot of even and odd parts of p(jw)).

First of all, i created the characteristic polynomial fort he conditions q2=q3=q4= 0 and p(s − 0.26, q1).

```
psq1 = psq /. q2 → 0 /. q3 → 0 /. q4 → 0 /. s → s - 0.26
```

$$\frac{98}{47} + \frac{13}{9}(-0.26+s)^5 + (-0.26+s)^4\left(\frac{209}{43} + \frac{q_1}{11}\right) + (-0.26+s)^3\left(\frac{323}{29} + \frac{4 q_1}{31}\right) + (-0.26+s)\left(\frac{178}{25} + \frac{23 q_1}{40}\right) + (-0.26+s)^2\left(\frac{683}{57} + \frac{28 q_1}{37}\right) + \frac{12 q_1}{11}$$

After that, before creating the odd and even parts of p(jw), i reached the coefficients.

```
a = CoefficientList[psq1, s]
```

Then i created the odd and even parts of p(jw)

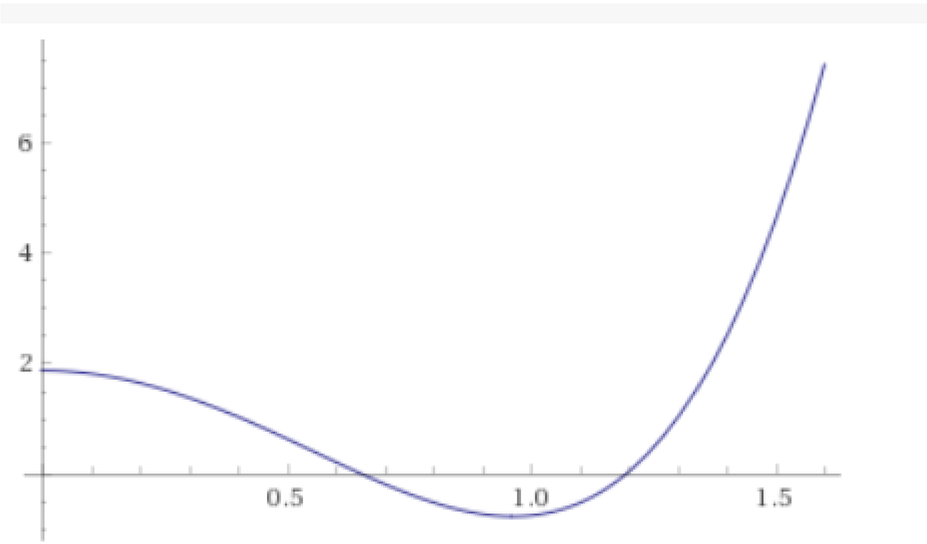
```
Peven = a[[1]] + a[[3]] * s^2 + a[[5]] * s^4;

Podd = a[[2]] * s + a[[4]] * s^3 + a[[6]] * s^5;
```

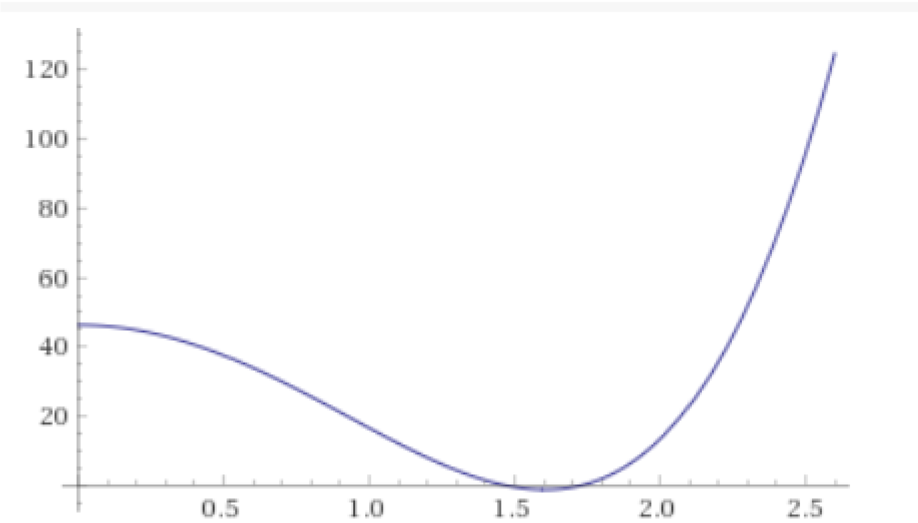
Then for q1 = 1

```
Peven /. s → I * w /. q1 → 1
```

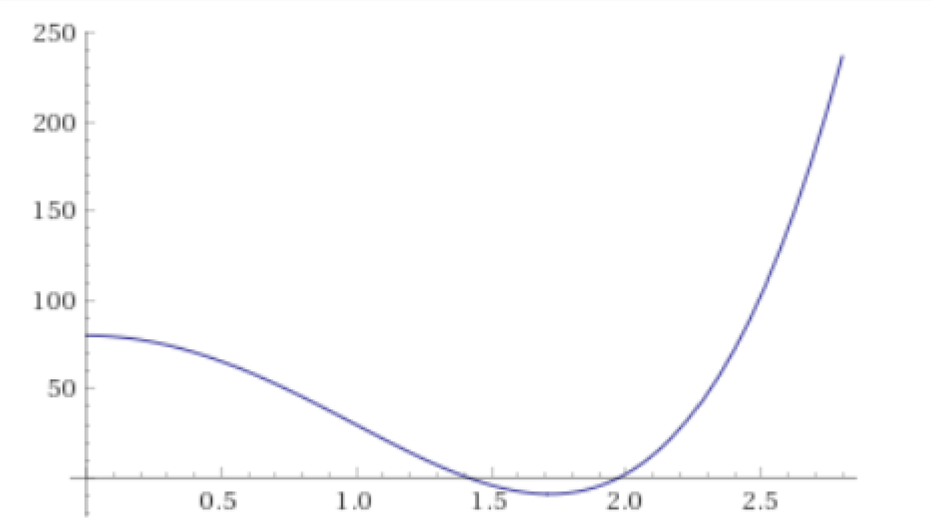
$$1.85937 - 5.70538\, w^2 + 3.0736\, w^4$$



For q1 = 46;



For q1 = 80;



6. **Draw the Mikhailov Plot for the polynomial (p(s,q)) using gridding. Grid the frequency variable (w) between 0 and 5. (You can use line segments to join the points of the frequency plot for each q∈Q). By looking at this graph what can you say about the stability of the polynomial family?**

After defining the characteristic polynomial, i drew the Mikhailov plot within the desired values.

```
qvec = {{q1, 1, 80}, {q2, -1, 1}, {q3, -1, 1}, {q4, -1, 1}};

qmin = Thread[qvec][[2]];
qmax = Thread[qvec][[3]];
qv = Thread[qvec][[1]];

wlist = Table[wfix, {wfix, 0, 1.5, 0.1}];
EdgeValueSet[psq, s, qvec, wlist, ConvexHullOnly → False, FillConvexHull → False, Color → RGBColor[0, 0, 1]]
```

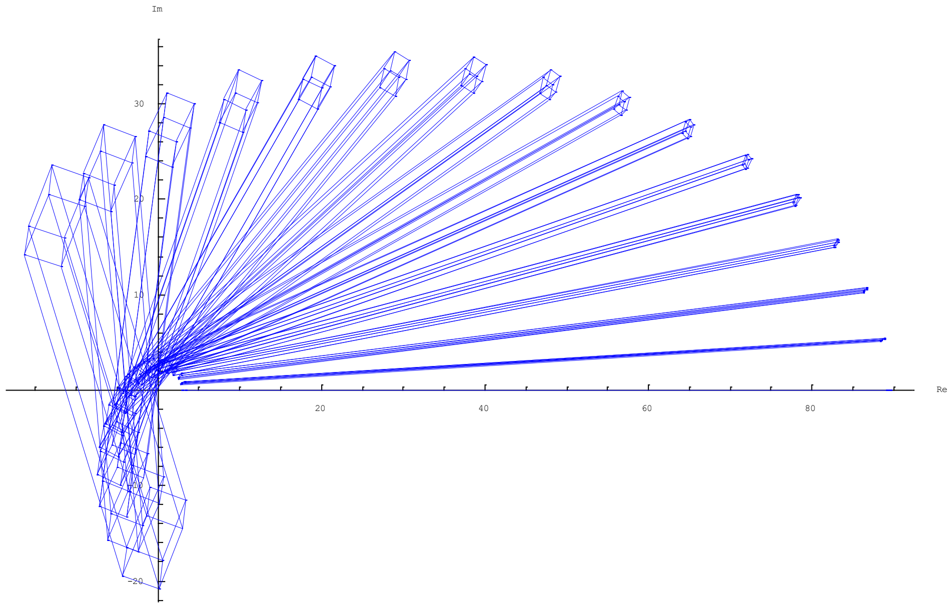


Image 1 for 0<w<1.5

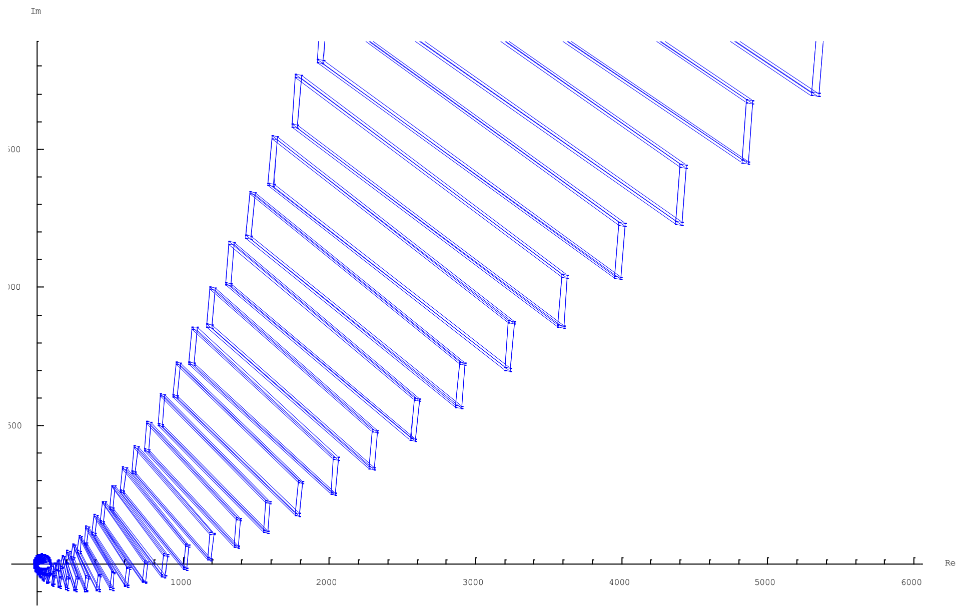


Image 2 for 0<w<5

According to Mikhailov's theorem, plot must be intersect with the axises n times where n is the degree of the characteristic polynomial. As we can see in the graph above, the plot intersects with the axises only 4 times which is less than the degree of the polynomial. This means polynomial family is **not** robustly stable. Even if it seems like this result contradicts with the result of the first question, it should be considered that gridding method was used in the first question and it was not representing the actual system.