THE DESIGN OF OPTIMUM MULTIFACTORIAL EXPERIMENTS

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1. Introduction

A problem which often occurs in the design of an experiment in physical or industrial research is that of determining suitable tolerances for the components of a certain assembly; more generally of ascertaining the effect of quantitative or qualitative alterations in the various components upon some measured characteristic of the complete assembly. It is sometimes possible to calculate what this effect should be; but it is to the more general case when this is not so that the methods given below apply. In such a case it might appear to be best to vary the components independently and study separately the effect of each in turn. Such a procedure, however, is wasteful either of labour or accuracy, while to carry out a complete factorial experiment (i.e. to make up assemblies of all possible combinations of the n components) would require L^n assemblies, where L is the number of values (assumed constant) at which each component can appear. For L equal to 2 this number is large for moderate n and quite impracticable for n greater than, say, 10. For larger L the situation is even worse. What is required is a selection of N assemblies from the complete factorial design which will enable the component effects to be estimated with the same accuracy as if attention had been concentrated on varying a single component throughout the N assemblies. Designs are given below for $L = 2$ and all possible $N \leqslant 100$ except $N = 92$ (as yet not known), and for $L = 3, 4, 5, 7$ when $N = L^r$ (for all r).

The following results have been obtained:

(a) When each component appears at L values, all main effects may be determined with the maximum precision possible using N assemblies, if, and only if, L^2 divides N, and certain further conditions are satisfied.

(b) For $L = 2$, the solution of the problem is for practical purposes complete. In designs of the form $N = L^r$, the effects of certain interactions between the components may also be estimated with maximum precision.

The precision naturally increases with the number of assemblies measured, and to this extent depends on the judgement of the experimenter. Before explaining the procedure in detail, some introductory remarks are necessary on the assumptions made and the method of least squares.

2. Experimental effects when $L = 2$

Each component in the assembly appears at two values throughout; it will be convenient to call one of them the nominal and the other the extreme, where the former usually refers to the actual nominal value and the latter to an extreme of the tolerance range for the component in question (the same extreme for each appearance of the component in a given experiment). Denote the measurable characteristics of the components in the assembly (one per component) by $x_1, x_2, \ldots, x_n$ and the measured assembly characteristic by y.

Then

$$y = y(x_1, x_2, \ldots, x_n),$$

where the functional relationship is in general unknown. Suppose that the nominal value of x_i is x_i^0 and of y, y_1. Thus

$$y_1 = y(x_1^0, x_2^0, \ldots, x_n^0).$$

Suppose also that the extreme value of x_i under consideration is x_i'. Then the *main effect* of component 1 is $m_1 = [\Sigma y(x_1', x_2, x_3, \ldots, x_n) - \Sigma y(x_1^0, x_2, x_3, \ldots, x_n)]/2^n$,

where the total number of possible assemblies is 2^n. In each of the two summations above, the indices on the x_i $(i \neq 1)$ range over all possible sets of values. Similarly, $m_2, m_3, \ldots m_n$ are defined. For brevity the above equation will be written:

$$m_1 = [\Sigma y(x_1') - \Sigma y(x_1^0)]/2^n,$$

and in general $\Sigma y\,(x_i' x_j' \ldots x_k^0)$ will represent the function y evaluated with $x_i x_j \ldots x_k$, taking the values shown and summed over all possible sets of values of the variables that have been suppressed. The main effect of a component is thus seen to be the mean effect on the measured assembly characteristic which that component would produce if acting on its own. Proceeding further we define the *interaction* between components 1, 2, 3, ..., p as

$$m_{(123\ldots p)} = [\Sigma y(x_1' x_2' \ldots x_p') - \Sigma\Sigma y(x_1' x_2' \ldots x_{p-1}' x_p^0) + \Sigma\Sigma y(x_1' x_2' \ldots x_{p-2}' x_{p-1}^0 x_p^0) + \ldots + (-1)^p \Sigma y(x_1^0 x_2^0 \ldots x_p^0)]/2^n,$$

where the inner summation is as explained above; the outer extends over the ${}_pC_1$, ${}_pC_2$, etc., selections of 1, 2, 3, etc., indices 0 available. The nature of an interaction has been discussed by Fisher (1942) and others, and our definition accords with the usual one.

If main effects are regarded as being of the first order of small quantities and *if the function y may be differentiated*, the first approximation to $m_{(123\ldots p)}$ is

$$m_{(123\ldots p)} = (\partial^p y/\partial x_1 \partial x_2 \ldots \partial x_p)\,(x_1' - x_1^0)\,(x_2' - x_2^0) \ldots (x_p' - x_p^0),$$

the derivative being *averaged over the values it takes for all sets of values* of the remaining components. This shows that when the variables are measured on a continuous scale we may validly neglect all the interactions above a certain order, for a $(p-1)$th order interaction (one in p components) is of the pth order of smallness. But the justification for this assumption when some of the x_i are qualitative and not quantitative (and it is frequently made) must be found in considerations outside the data which the experiment provides, in commonsense or philosophical grounds.

The grand mean $M = \Sigma y(x_1, x_2, \ldots, x_n)/2^n$ where the summation is over all possible sets of values of the components. In the jth assembly of an actual experiment, some components will be at nominal and some at their extreme values. If the true value of the assembly characteristic is then y_j, it is found on solving the above equations that

$$y_j = M + a_{j1} m_1 + a_{j2} m_2 + \ldots + a_{jn} m_n + a_{j,n+1} m_{(12)} + \ldots + a_{j,2^n} m_{(123\ldots n)}, \qquad (1)$$

where the coefficient of m_i is ± 1 according as the ith component is at extreme or nominal in the jth assembly; the coefficient of $m_{(123\ldots p)}$ is ± 1 according as the number of plus ones among the coefficients of $m_1 m_2 \ldots m_p$ is odd or even. In doing this the signs of the odd-order interactions (involving an even number of factors) have been reversed, but the notation is convenient, for then the coefficients in y_1 are all minus one. It is assumed that y_1 is always one of the selected assemblies, and this is no real restriction upon the design.

3. Least squares and precision

The purpose of the experiment is to estimate those of the quantities m as may not be assumed negligible from a set of measurements $r_1, r_2, \ldots, r_N$. For this we must solve a set of N linear equations represented by (1). The equations always involve M, and therefore to estimate q of the m's it is necessary to make at least $(q+1)$ measurements. If exactly $(q+1)$ assemblies

are measured, there is a unique set of m's satisfying the equations; if more than $(q+1)$ are measured there will be no unique solution and the best estimates are, as is well known, obtained by the method of least squares. This obtains the set of m's which minimizes

$$S = \sum_j (r_j - M - a_{j1}m_1 - a_{j2}m_2 \ldots - a_{jn}m_n \ldots)^2,$$

where r_j is the measurement in the jth assembly whose true value is y_j. Normally $(q+1)$ is much less than 2^n, all high-order interactions being neglected, so that the number of assemblies N may be made much smaller than for the complete factorial design.

As already stated, the greater the number of assemblies measured, the greater the precision with which component effects may be estimated. On account of errors of measurement and the neglect of certain effects the minimum S_0 of S is not zero. In fact $S_0/(N-q-1)$ provides an unbiased estimate s^2 of σ^2, the variance of error of each measurement (assumed the same for all assemblies). The error variance in the estimation of an effect m_i is of the form σ^2/t_i, where t_i is called the precision constant. It depends only on the design of the experiment, and can be increased indefinitely by increasing N. Our object is to find designs which maximize all the t_i simultaneously for given N. They will be called optimum designs. The ratio of m_i to $s/\sqrt{t_i}$ has a t-distribution on the null hypothesis—that the true value of m_i is zero. The effect of increasing the precision is, first, to increase the power of the t-test in detecting any departure of m_i from zero; secondly, to increase the accuracy of its estimation. In the designs given at the end of this paper, for $L = 2$ all main effects may be estimated with maximum precision N (given N assemblies), that is, the standard error of $m_i = \sigma/\sqrt{N}$ provided N is a multiple of 4. In cases where $N = 2^r$ certain interactions may also be estimated with the maximum precision. The choice of N (subject to $N \geqslant q+1$) will depend on the extent to which the experimenter wishes to minimize the effect of his experimental error.

4. Requirements for optimum designs (any L)

I. Consider now the case of n components each of which may take L values. If interactions are neglected, the true values y_i may be assumed linear functions of certain constants representing the main effects, as was proved rigorously for the case $L = 2$. In general let $x_{j(l)}$ represent the effect due to the jth component at its lth value. The true value of the measurement on the ith assembly is

$$y_i = \sum_j x_{j(l)} \begin{bmatrix} i = 1, 2, \ldots, N \\ j = 1, 2, \ldots, n \\ l = 1, 2, \ldots, L \end{bmatrix},$$

where l represents the value at which the jth component appears in the ith assembly. We now introduce certain new variables in terms of which to express the $x_{j(l)}$, as the primary interest is in the *change* of assembly characteristic caused by certain *changes* in the components.

Let Q be a non-singular $L \times L$ matrix whose first column consists entirely of ones, such that $Q = OD$, where O is orthogonal and D diagonal. The condition on the first column of Q implies that $d_{11} = \sqrt{L}$.

Let

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_L \end{bmatrix} = Q^{-1} \begin{bmatrix} x_{1(1)} \\ x_{1(2)} \\ \vdots \\ x_{1(L)} \end{bmatrix} = Q^{-1}X_1,$$

$u_1 = (x_{1(1)} + x_{1(2)} + \ldots + x_{1(L)})/L$ = mean effect of component 1 and $u_2, u_3, \ldots, u_L$ are constants which determine the effect of *changes* in this component upon the assembly characteristic. The orthogonality property will be used later. Therefore

$$X_1 = QU = \begin{bmatrix} u_1 + a_{12}u_2 + a_{13}u_3 + \ldots + a_{1L}u_L \\ u_1 + a_{22}u_2 + a_{23}u_3 + \ldots + a_{2L}u_L \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ u_1 + a_{L2}u_2 + a_{L3}u_3 + \ldots + a_{LL}u_L \end{bmatrix},$$

where the a_{ij} are certain constants. Similarly, introduce variables $v_1, v_2, v_3, \ldots, v_L$ for component 2 and write:

$$X_2 = \begin{bmatrix} x_{2(1)} \\ x_{2(2)} \\ \ldots \\ x_{2(L)} \end{bmatrix} = \begin{bmatrix} v_1 + a_{12}v_2 + a_{13}v_3 + \ldots + a_{1L}v_L \\ v_1 + a_{22}v_2 + a_{23}v_3 + \ldots + a_{2L}v_L \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ v_1 + a_{L2}v_2 + a_{L3}v_3 + \ldots + a_{LL}v_L \end{bmatrix},$$

where v_1 is the mean effect of component 2. And so on. Hence

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \ldots \\ y_N \end{bmatrix} = AX, \quad \text{where} \quad X = \begin{bmatrix} M \\ u_2 \\ \ldots \\ u_L \\ v_2 \\ \ldots \\ v_L \\ \ldots \end{bmatrix},$$

$$M = u_1 + v_1 + \ldots \text{ to } n \text{ terms.}$$

A is a matrix with N rows and $n(L-1)+1$ columns, the first column consisting of ones, and the remainder consisting of the elements a_{ij} belonging to Q. The columns fall into sets (corresponding to the components) of $(L-1)$ after the first, and the rows of the submatrix formed by such a set consist of repetitions of the rows of Q. At this stage renumber the suffices of the a_{ij} so that A may be written (a_{ij}).

II. The vector Y is known. Solving the equations by least squares (assuming $N \geqslant n(L-1)+1$) gives the so-called normal equations $A'Y = A'AX = CX$ say, i.e. $X = C^{-1}A'Y$. If σ^2 is the error variance of a single observation y, it is proved in text-books that

$$\text{var}\,(\theta_k) = |\,C_{kk}\,|\,\sigma^2/|\,C\,|,$$

where θ_k is the kth element of X and C_{kk} the cofactor of c_{kk} in $C = [c_{ij}]$, C being a symmetric $n \times n$ matrix. It is required to minimize $|\,C_{kk}\,|/|\,C\,|$, i.e. to maximize $t = |\,C\,|/|\,C_{kk}\,|$ by suitable choice of design.

Write $c_{ij}/c_{ii}^{\frac{1}{2}}c_{jj}^{\frac{1}{2}} = r_{ij}$ and the matrix $R = [r_{ij}]$, where $r_{ii} = 1$ and $r_{ij} = r_{ji}$.

Now

$$r_{ij} = \sum_r a_{ri}a_{rj} \Big/ \left(\sum_r a_{ri}^2\right)^{\frac{1}{2}} \left(\sum_r a_{rj}^2\right)^{\frac{1}{2}}.$$

If $(a_{1i}, a_{2i}, \ldots, a_{Ni})$ and $(a_{1j}, a_{2j}, \ldots, a_{Nj})$ be interpreted as the co-ordinates of two points P_i and P_j in a Euclidean N-space, then $r_{ij} = \cos P_iOP_j$, where O is the origin and hence $r_{ij}^2 \leqslant 1$.

Now $t = |R|\,S_k/|R_{kk}|$, where $S_k = \sum_r a_{rk}^2$ and so S_k must be fixed otherwise t may be increased indefinitely. This is equivalent to fixing the scale of measurement, the preceding section having dealt with the choice of origin at the mean. Eliminate the pth row and column from $|R|$ and $|R_{kk}|$ by pivotal condensation: multiply the pth column by r_{pj} $(p \neq k)$ and subtract from the jth column for all $j \neq p$. A row of zeros appears in the pth row except in the diagonal place where there is a one. The determinants have been reduced in order by one, and the second is still a principal minor of the first.

Thus
$$\begin{aligned} |r_{ij}| &= |r_{ij} - r_{ip}r_{jp}| \quad \text{(remembering } r_{pj} = r_{jp}\text{)} \\ &= |(1-r_{ip}^2)^{\frac{1}{2}}(1-r_{jp}^2)^{\frac{1}{2}} r_{ij.p}| \end{aligned}$$
defining $r_{ij.p}$ in this manner, where the suffices appearing after the dot represent columns that have been eliminated.

Therefore taking out factors from rows and columns
$$\begin{aligned} t &= S_k \prod_{i \neq p}(1-r_{ip}^2)\,|r_{ij.p}| \Big/ \prod_{i \neq p,k}(1-r_{ip}^2)\,|[r_{ij.p}]_{kk}| \\ &= S_k(1-r_{kp}^2)\,|r_{ij.p}|/|[r_{ij.p}]_{kk}|. \end{aligned}$$

Now $r_{ij.p} = (\cos P_iOP_j - \cos P_iOP_p \cos P_jOP_p)/\sin P_iOP_p \sin P_jOP_p$,
which is the formula for the cosine of the projection of angle P_iOP_j on to the $(N-1)$-space orthogonal to OP_p. Therefore
$$r_{ij.p}^2 \leqslant 1 \quad (i \neq j) \quad \text{and} \quad r_{ii.p} = 1.$$
The method has obtained a ratio of two determinants of the same type as before, and the process is repeated, step by step, until that in the numerator is of the form
$$\begin{vmatrix} 1 & & r_{qk.p_1p_2} & \dots \\ r_{qk.p_1p_2} & \dots & 1 & \end{vmatrix}$$
and the denominator is 1. Row and column p_1, p_2, ..., are eliminated in turn (no p being equal to k), and so
$$t = S_k(1-r_{kp_1}^2)\,(1-r_{kp_2.p_1}^2)\,(1-r_{kp_3.p_1p_2}^2)\dots(1-r_{kq.p_1p_2\dots p_{n-2}}^2).$$

This is a maximum only when $r_{kp} = 0$ for all $p \neq k$ and all k. For equal precision S_k must be constant for all k and $t = S_k$. Therefore $A'A = C = tI$. Hence the designs for which the maximum precision is attained are those which correspond to columns of an orthogonal matrix (apart from an arbitrary multiplier).

At this point it is convenient to prove the formula for the error variance. Let A be the non-square matrix with orthogonal columns of the equations: $Y = AX$. Introduce further columns U so that (A, U) is a square orthogonal matrix, and corresponding dummy variables whose column vector is Z. The least squares solution of the above equations is X_0 given by $A'AX_0 = A'Y$, therefore
$$tIX_0 = A'Y, \quad X_0 = \frac{1}{t}A'Y. \tag{1}$$
The equations $[A, U]\begin{bmatrix} X \\ Z \end{bmatrix} = Y$ have a unique solution, and on multiplying by $\begin{bmatrix} A' \\ U' \end{bmatrix}$,
$$tI\begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} A'Y \\ U'Y \end{bmatrix},$$
so the resulting value of $X = X_0$ as before. The residual vector $E = Y - AX_0 = UZ$.
$$\text{Sum of squares of residuals} = E'E = Z'U'UZ = tZ'Z = t\Sigma z_i^2. \tag{2}$$

III. Consider now any pair of components f and g. Suppose they appear together at their lth and l'th values respectively $w_{ll'}$ times. This defines an $L \times L$ matrix $W = [w_{ll'}]$. The scalar product of a column of A belonging to f by a column of A belonging to g is zero by the orthogonality of A. Let these correspond to the uth and vth columns of Q and revert to the old suffices of a_{ij} corresponding to Q, i.e. $Q = [a_{ij}]$, where $i, j = 1, 2, \ldots, L$.

Then $a_{lu} . w_{ll'} . a_{l'v}$ in dummy suffices equals

$$Q'WQ = \begin{bmatrix} N & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & 0 \end{bmatrix}.$$

The N appears because the first column for f is the same as the first column for g, equal to the first column of A which consists entirely of ones.

Now $Q = OD$ where O is orthogonal and D diagonal, therefore $Q'WQ = DO'WOD$, i.e.

$$O'WO = D^{-1} \begin{bmatrix} N & 0 \\ 0 & 0 \end{bmatrix} D^{-1}$$

$$= \begin{bmatrix} N/d_{11}^2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore

$$W = OO'WOO' = \begin{bmatrix} 1/\sqrt{L} & & \\ 1/\sqrt{L} & \text{other terms} & \\ 1/\sqrt{L} & & \end{bmatrix} \begin{bmatrix} N/L & 0 \\ & \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{L} & 1/\sqrt{L} & \ldots & 1/\sqrt{L} \\ & \text{other terms} & & \\ & & & \end{bmatrix}$$

$$= \begin{bmatrix} N/L^2 & N/L^2 & \ldots & N/L^2 \\ N/L^2 & N/L^2 & \ldots & N/L^2 \\ \ldots & \ldots & \ldots & \ldots \\ N/L^2 & N/L^2 & & N/L^2 \end{bmatrix}$$

Sum of terms in lth row is the number of replications of the lth value of f. Therefore

(i) Each component is replicated at each of its values the same number of times.

(ii) Each pair of components occur together at every combination of values the same number of times.

(iii) The number of assemblies is divisible by the square of the number of values.

The converse—that under these conditions the matrix A is orthogonal—can be proved by reversing these steps. The actual matrix Q chosen is unimportant, and the design can be specified by means of a rectangular array with N rows and n columns containing L different letters $(a, b, c, \ldots, k)$ representing the L values of each component. The problem is then a purely combinatorial one. If $N = KL^2$, the maximum number of columns n is

$$(KL^2 - 1)/(L - 1)$$

or its integral part since $KL^2 \geqslant n(L-1)+1$. We propose to call designs of this type *multifactorial designs*.

Returning to the case of $L = 2$, it is necessary to obtain an orthogonal $4K \times 4K$ matrix A whose first column consists entirely of ones. Choosing $Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ the other columns of A consist of equal numbers of $+1$ and -1. The signs may be changed down the length of certain columns without spoiling the design so that the first row apart from the corner element consists entirely of -1. Then apart from this row (in future called the basic row) and the first column the design consists of a square matrix with $2K$ plus and $(2K-1)$ minus ones in each column and (by orthogonality) row, and such that each pair of columns contains a pair of plus ones in the same row K times. The estimates of component effects are obtained from equation (1) of § 4 (II):

$$X_0 = \frac{1}{4K} A'Y \quad (\text{here } t = 4K).$$

Thus they can be evaluated by addition and subtraction with only one division. This simplicity appears in the illustrative example given in §§ 9 and 10. The dummy variables z_i are similarly evaluated and the estimated error variance is

$$s^2 = \frac{4K}{4K-n-1} \Sigma z_i^2.$$

5. METHODS OF SOLUTION

Certain methods of constructing orthogonal matrices with elements plus or minus one are known (Paley, 1933). They depend upon the theory of finite fields, an outline of which will now be given.

A field F is defined as a set of quantities which is closed with respect to two operations, addition and multiplication (i.e. if a, b in F, so are $a+b, ab$). These quantities satisfy the following laws:

(i) $a+b = b+a$. $\quad ab = ba$. (ii) $a+(b+c) = (a+b)+c$. $\quad a(bc) = (ab)c$. (iii) $a(b+c) = ab+ac$.

(iv) There is an x such that $a+x = b$ for every a, b.

From these it may be proved: (a) There is a unique quantity 0 such that $a+0 = a$ for all a. (b) The quantity x in (iv) is unique. (c) $a.0 = 0$. Finally, we add

(v) There is a y such that $ay = b$ for every b, all $a \neq 0$, to our axioms.

Hence as before: (d) There is a unique quantity 1 such that $a.1 = a$ for all a. (e) There is a unique quantity a^{-1} such that $a.a^{-1} = 1$ $(a \neq 0)$. (f) The quantity y in (v) is unique. $y = a^{-1}b$.

Consider the integers $0, 1, 2, \ldots, (p-1)$ where p is prime, and write $a = b$ if $(a-b)$ is divisible by p. Then this set of integers forms a finite field as may be easily shown. For example, when $p = 5$, the numbers in the field are 0, 1, 2, 3, 4.

$$2+4 = 6 = 1, \quad 2+3 = 5 = 0, \quad 2.3 = 4.4 = 1.$$

Hence 2 and 3 are reciprocals and 4 is its own reciprocal. This field is called the Galois field of order p, $GF(p)$.

Now suppose x is a number algebraic over $GF(p)$, that is, x satisfies an algebraic equation with coefficients in $GF(p)$. Then it defines an algebraic extension of $GF(p)$, namely, all polynomials in x with coefficients in $GF(p)$. If x satisfies an equation irreducible in $GF(p)$ and of degree n, there are p^n distinct polynomials in x. They are of the form

$$f(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} \quad (a_0, a_1, \ldots, a_{n-1} \text{ in } GF(p)).$$

Such an algebraic extension is, in fact, a field. Moreover, all fields of degree n over $GF(p)$ may be shown to be equivalent. There is a member of the extended field α such that $1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{q-2}$ $(q = p^n)$ constitute the non-zero elements of the field and $\alpha^{q-1} = 1$; an extension of *Fermat's theorem*. Any one of these equivalent fields is denoted by $GF(p^n)$. We shall now require various simple theorems.

DEF. If a non-zero element of a finite field is a perfect square $(a = b^2)$ it is called a quadratic residue of the field. All other non-zero elements are non-Q.R.'s.

TH. 1. The numbers of Q.R.'s and non-Q.R.'s are equal $(p > 2)$.

For every $b = \alpha^u$, $\quad a = b^2 = \alpha^{2u} = \alpha^{2u-\lambda(q-1)}$ $(\lambda$ integral$)$.

But $(q-1)$ is even $(p > 2)$. Hence only even powers of α are Q.R.'s.

Therefore there are $\frac{1}{2}(q-1)$ Q.R.'s. and $\frac{1}{2}(q-1)$ non-Q.R.'s.

We now define the Legendre function $\chi(a)$:

$$\chi(0) = 0,$$
$$\chi(a) = +1 \quad \text{when } a \text{ is a Q.R.}$$
$$= -1 \quad \text{when } a \text{ is a non-Q.R.}$$

Th. 1 states that $\sum_a \chi(a) = 0$ (summation over whole field).

TH. 2. $\chi(a)\,\chi(b) = \chi(ab)$.

This is trivial when $a = 0$ or $b = 0$. Otherwise $a = \alpha^u, b = \alpha^v$ and $ab = \alpha^{u+v}$ is a Q.R. if and only if $(u+v)$ is even, i.e. u and v of same parity. This proves the result.

TH. 3. $\left.\begin{aligned}\chi(-1) &= +1 \text{ if } q = 4t+1\\ &= -1 \text{ if } q = 4t-1\end{aligned}\right\}$ for integral t.

For $\alpha^{q-1} = +1$.

Therefore $\alpha^{\frac{1}{2}(q-1)} = \pm 1 = -1$ since powers of α are distinct up to α^{q-1}.

Hence -1 is a Q.R. if and only if $\frac{1}{2}(q-1)$ is even $= 2t$ and $q = 4t+1$.

TH. 4. $\sum_j \chi(j-i_1)\chi(j-i_2) = -1$ (summation over all j in $GF(p^n)$; $p > 2$; $i_1 \neq i_2$),

$$\sum_j \chi(j-i_1)\chi(j-i_2) = \sum_j \chi\{(j-i_1)(j-i_2)\} \quad \text{by Th. 2.}$$

Put $$u = j - \frac{(i_1+i_2)}{2}, \quad u_0 = \frac{i_1-i_2}{2} \neq 0.$$

Expression $= \sum_u \chi(u^2-u_0^2)$ (j is summed over whole field so u will be also).

Put $u = u_0 v \quad (u_0 \neq 0)$.

Expression $= \sum_v \chi\{u_0^2(v^2-1)\}$ (u is summed over whole field so v will be also)

$$= \sum_v \chi(u_0^2)\,\chi(v^2-1) = \sum_v \chi(v^2-1) \quad \text{by Th. 2.}$$

Now if $v^2 - 1 = x^2$, $v^2 - x^2 = 1$,

$$(v-x)(v+x) = 1.$$

If $v+x = y$, $v-x = y^{-1}$.

Therefore $$v = \tfrac{1}{2}(y+y^{-1}), \quad x = \tfrac{1}{2}(y-y^{-1}).$$

Hence the number of values of v for which $\chi(v^2-1) = +1$ or 0 is the number of values of v for which $v = \frac{1}{2}(y+y^{-1})$.

Now if $y+y^{-1}=w+w^{-1}$, $y^2w+w=yw^2+y$, $(y-w)(1-yw)=0$.

Therefore $w=y$ or $y^{-1}.y$ and y^{-1} are distinct unless $y=\pm 1$, $v=\pm 1$ when

$$\chi(v^2-1)=\chi(0)=0.$$

Hence to every one of the $\frac{1}{2}(q-1)$ reciprocal pairs (y, y^{-1}) corresponds a distinct value of v.
Thus there are $\frac{1}{2}(q-3)$ values of v for which $\chi(v^2-1)=+1$ (excluding $v=\pm 1$).
There are two values of v for which $\chi(v^2-1)=0$ $(v=\pm 1)$.
Hence there are $\frac{1}{2}(q-1)$ values of v for which $\chi(v^2-1)=-1$.

Therefore

$$\sum_j \chi(j-i_1)\chi(j-i_2)=\sum_v \chi(v^2-1)=-1.$$

Applications

I. Consider the matrix $A=(a_{ij})$ $(i,j=0,1,2,\ldots,p)$ of order $(p+1)$, where $p=4t-1$.

$$a_{i0}=a_{0j}=+1,$$
$$a_{ij}=\chi(j-i) \quad (i\neq 0,\ j\neq 0,\ i\neq j),$$
$$a_{ii}=-1.$$

The scalar product of 1st and $(i+1)$th rows

$$=a_{00}a_{i0}+a_{0i}a_{ii}+\sum_{j=1}^{p}\chi(j-i)$$
$$=1-1+0=0 \quad \text{(Th. 1)}.$$

Scalar product of (i_1+1)th and (i_2+1)th rows

$$=a_{i_1 0}a_{i_2 0}+a_{i_1 i_1}a_{i_2 i_1}+a_{i_2 i_2}a_{i_1 i_2}+\sum_{j=1}^{p}\chi(j-i_1)\chi(j-i_2)$$
$$=1-\chi(i_1-i_2)-\chi(i_2-i_i)-1 \quad \text{(Th. 4)}$$
$$=0 \text{ since } p=4t-1 \quad \text{(Th. 3)}.$$

Hence the matrix A is orthogonal.

II. To construct A of order $p^n+1=4t$, we associate the rows and columns (except the first) with the elements of $GF(p^n)$ and the proof runs exactly as before.

III. If A is orthogonal $\begin{bmatrix} A & A \\ A & -A \end{bmatrix}$ is also orthogonal and has double the order of A.

Hence an orthogonal matrix A of order $2^h(p^n+1)$ (where $p^n=4t-1$) or 2^h can be constructed by successive doubling.

IV. If $p^n=4t+1$, (p^n+1) is not divisible by 4.
But an A of order $2(p^n+1)$ can be obtained by a slight modification of the method.
Consider the matrix $B=(b_{ij})$ $(i,j=0,1,2,\ldots,p^n)$ of order (p^n+1) $[p^n=4t+1]$.

$$b_{i0}=b_{0j}=+1 \quad (i\neq 0,\ j\neq 0),$$
$$b_{ij}=\chi(u_j-u_i) \quad (i\neq 0,\ j\neq 0)$$

where u_i is the element of $GF(p^n)$ associated with the $(i+1)$th row and column of B, $i=1,2,\ldots,p^n$,

$$b_{00}=0.$$

Scalar product of 1st and $(i+1)$th rows

$$=b_{00}b_{i0}+\sum_{j=1}^{p^n}\chi(u_j-u_i)=0 \quad \text{(Th. 1)}.$$

Scalar product of (i_1+1)th and (i_2+1)th rows

$$= b_{i_1 0}b_{i_2 0} + b_{i_1 i_1}b_{i_2 i_1} + b_{i_2 i_2}b_{i_1 i_2} + \sum_{j=1}^{p^n} \chi(u_j - u_{i_1})\,\chi(u_j - u_{i_2})$$

$$= 1 - 1 = 0 \quad \text{(Th. 4)}.$$

Thus B is orthogonal.

Now replace $+1$ by the submatrix $C = \begin{bmatrix} +1 & +1 \\ +1 & -1 \end{bmatrix}$.

-1 by the submatrix $-C = \begin{bmatrix} -1 & -1 \\ -1 & +1 \end{bmatrix}$.

0 by the submatrix $D = \begin{bmatrix} +1 & -1 \\ -1 & -1 \end{bmatrix}$. The new matrix A thus formed is of order $2(p^n+1)$.

Consider the scalar products of the $(2i_1+1)$th and $(2i_1+2)$th rows with the $(2i_2+1)$th and $(2i_2+2)$th rows. This is a (2×2) matrix $M_{i_1 i_2}$.

Now $$M_{i_1 i_2} = \sum_{j=0}^{p^n} (b_{i_1 j}C)(b_{i_2 j}C') + (D)(b_{i_2 i_1}C') + (b_{i_1 i_2}C)(D') \quad [j \neq i_1,\ j \neq i_2]$$

$$= CC' \sum_{j=0}^{p^n} b_{i_1 j}b_{i_2 j} + X(u_{i_1} - u_{i_2})(DC' + CD') \quad [j \neq i_1,\ j \neq i_2]$$

(by Th. 2 and Th. 3 for $p^n = 4t+1$)

$$= CC'0 + \chi(u_{i_1} - u_{i_2}) \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

since $\sum_{j=0}^{p^n} b_{i_1 j}b_{i_2 j} = 0$ (orthogonality of B) and the omitted terms vanish

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Finally, it is clear that the $(2i_1+1)$th and $(2i_1+2)$th rows are orthogonal to each other. *Hence A is orthogonal and of the type required.*

Thus by successive doubling we may obtain matrices of order $2^h(p^n+1)$ where $p^n = 4t+1$.

V. Summing up we have:

If $N = 2^h(p^n+1) = 4K$, where p is an odd prime or zero, an orthogonal matrix A can be constructed with plus and minus ones.

The matrices constructible by these methods include all values of $N = 4K$ up to 100 excepting 92. Those of order 2^r are structurally the same as the complete factorial design in r factors *if they have been obtained by successive doubling*. These will be called geometrical designs because of their close connexion with finite geometries. It is clear that if two columns of the design represent main effects, and if the interaction column corresponding to them in the complete factorial case is a dummy in the actual experiment, it may be used to estimate the interaction. The condition for a column to be the interaction between p other columns is that it is $\pm D$, where D is a column vector whose ith element is the product of the ith elements of the original p columns. So far interaction columns have only been found in the geometrical designs and in them every interaction between an arbitrary set of columns is a column of the design. It must also be mentioned that the cyclic designs for $N = 2^r$ obtainable by the method of § 8 depending on $GF(2^r)$ are in fact merely permutations of the geometric designs. They are the forms used in the tables for convenience.

6. CASE OF MORE THAN TWO LEVELS

We now provide experimental designs for determining component effects with maximum precision when the number of values L is greater than 2. These solutions cover the cases where the number of assemblies $N = L^r$, L being a prime or a power of a prime and r any positive integer. Two methods are given: in the first, successive columns of the design are formed by simple operations on the preceding columns; in the second, which is of more limited application, the design is specified by one column, all others being cyclic permutations of this.

7. MODIFIED FACTORIAL DESIGNS

The methods given in this section for the construction of multifactorial designs, although discovered independently, are nevertheless identical with those used by Bose & Kishen (1940) to express the generalized interaction for the purpose of confounding certain contrasts with block differences in agricultural experiments. They construct their interactions directly from finite projective geometries without using, as we have done, the intermediate device of orthogonal sets of Latin squares. We shall, however, describe these methods, as they may not be familiar to experimenters in this country, especially not in the way in which we propose using them.

Suppose a complete factorial experiment is carried out for r factors each at L levels (i.e. in this case r components measured at L values) so that L^r assemblies are made. Let the levels be called $0, 1, 2, \ldots, (L-1)$. Then the r main effects define r columns of a design array (with L^r rows) containing these L symbols. Each symbol appears the same number of times in a column as any other. Each combination of symbols for two columns occurs equally frequently. We shall apply the term orthogonal to such a pair of columns. Now let A, B be two orthogonal columns. The interaction AB has $(L-1)^2$ degrees of freedom. Since each column of the array is associated with $(L-1)$ degrees of freedom, a first-order interaction is represented by a set of $(L-1)$ columns which will be called the terms of this interaction. Similarly, an interaction of the mth order (i.e. involving $m+1$ factors) is represented by $(L-1)^m$ columns of the design array.

Now an interaction between two factors is most naturally defined by the following conditions:

(a) Each combination of levels of A and B corresponds to only one level within each term of AB.

(b) The terms of AB are orthogonal to A and B and to one another.

Condition (a) means that if, for instance, in one assembly level 2 of A and level 5 of B occur together, and if a term of AB is defined to appear at level 3 in this case, then whenever A and B occur at these levels together again, this term of AB appears at level 3. Since, owing to the complete factorial basis of the design, every combination of levels of ABC occurs equally often, each combination of levels of A and B occurs equally often with every level of C. But such a combination of levels of A and B fixes the level in a term of AB by condition (a). Hence each level in a term of AB occurs equally often with every level of C. In other words, each interaction term will be orthogonal to the main effects not connected with it.

Now for condition (b). If the rows and columns of a square $L \times L$ array correspond respectively to the L levels of A and B, each cell may be filled up with the level appropriate to a particular interaction term. For any particular term of AB such a square will be Latin, because, regarding a row, the level of A is fixed; all the levels of the interaction term must

occur equally often with this level of A and hence each symbol appears once in every row of the square; similarly it appears once in every column in order that the interaction term may be orthogonal to B. Finally, superimposing the Latin squares for two terms of the same interaction, each symbol belonging to the first term must appear once in the same cell with each symbol of the second term, in order that these two may be orthogonal: thus if conditions (a) and (b) are satisfied the interaction terms are founded upon a completely orthogonal set of Latin squares.

It remains to show that the terms from two different interactions are orthogonal. This follows because every combination of levels of four factors $ABCD$ occurs equally frequently, i.e. each combination of levels of A and B occurs equally often with every combination of levels of C and D. The former correspond to levels in the terms of AB; the latter to levels in terms of CD. Hence a term of AB is orthogonal to a term of CD. Similarly AB and AC may be dealt with. This shows that the first order interaction terms may be joined to the main factors as part of the balanced design. Higher order interactions may be regarded as first order interactions between those of lower order, e.g. $(ABC) = (AB)(C)$, it being understood that $(L-1)$ terms are derived from each term of (AB) taken with the factor C, so that in this case there will be $(L-1)^2$ terms for the second order interaction. This procedure builds up the design by an inductive process, and when the interaction of the $(r-1)$th order has been obtained, it will be complete. The total degrees of freedom in the original factorial design $= L^r - 1$. Hence the number of factors that may be measured if interactions are neglected

$$= \frac{L^r - 1}{L - 1}.$$

It may be remarked that there is nothing new in this treatment of the complete factorial design except the modification of the usual Fisher interactions so that they may be placed on the same footing as main effects.

If L is a prime number, cyclic Latin squares exist forming an orthogonal set: each square is obtained by writing the first row of symbols in standard order, successive rows being obtained by shifting the symbols along ρ places from each row to the next ($\rho = 1, 2, \ldots, (L-1)$). In this case the appropriate column of the design is formed as follows: assuming the first row in the order $0, 1, 2, \ldots, (L-1)$, if x level of A and y level of B occur together, the corresponding level for this interaction term is $(y + \rho x)$, the symbols being reduced with modulus L. The squares for $\rho = 1, 2, \ldots, (L-1)$, give all the interaction terms.

When L is the nth power of a prime p, then we associate the L levels of a factor with the elements of a Galois field, $GF(p^n)$. Suppose these elements to be $u_0, u_1, u_2, \ldots, u_{L-1}$ where u_0 is the zero and u_1 the unity of the field. If then u_x level of A and u_y level of B occur together, the corresponding levels for interaction terms are $u_y + u_\rho u_x$, and the squares for $u_\rho = u_1, u_2, \ldots, u_{L-1}$ give all the terms present. The method of constructing completely orthogonal sets of Latin squares from Galois fields is given in Stevens (1939).

For $L = 6$ it is known that no pair of orthogonal Latin squares exists so it is not amenable to this treatment. The design for $N = 9$, $L = 3$ is given below; the accompanying key refers to the column vectors and the rows are labelled as if belonging to a complete factorial design.

	A	B	$(AB)_1$	$(AB)_2$		A	B	$(AB)_1$	$(AB)_2$		A	B	$(AB)_1$	$(AB)_2$
a_1b_1	0	0	0	0	a_2b_1	1	0	1	1	a_3b_1	2	0	2	2
a_1b_2	0	1	1	2	a_2b_2	1	1	2	0	a_3b_2	2	1	0	1
a_1b_3	0	2	2	1	a_2b_3	1	2	0	2	a_3b_3	2	2	1	0

Key: $(AB)_1 = (A) + (B)$, $(AB)_2 = (A) + 2(B)$.

For $r = 4$, $N = 81$, if the main effects are taken as A, B, C, D, $(ABCD)_1$, then all first order interactions are determinable.

8. CYCLIC SOLUTIONS WHEN L IS A PRIME NUMBER

We shall again be concerned in this section with Galois fields: each element of $GF(p^n)$ will be represented by a set of n ordered numbers, each number being $0, 1, 2, \ldots, (p-1)$ where p is prime. Consider the block B_1 of elements which have the integer r in position s ($r = 0, 1, 2, \ldots, p-1$; $s = 1, 2, 3, \ldots, n$).

For example, in $GF(3^2)$ the block having 2 in position 1 is 20, 21, 22. We require to show that if the elements of this block are multiplied in succession by any element of the field other than $000 \ldots 0t$ ($1 \leqslant t \leqslant p-1$), then the elements of the resulting block B_2 have equal numbers of all possible r in position s. There are in fact p^{n-1} elements in B_1, and we need to prove that B_2 is subdivisible into

p^{n-2} elements having 0 in position s

p^{n-2} elements having 1 in position s

.....

p^{n-2} elements having $p-1$ in position s.

There is no loss in generality if we consider $s = 1$, i.e. we refer now to the first members of all elements of the field. Take now all the elements having 1 in this position. Multiply this block A_1 by any element b of the field and obtain block A_2. Suppose in A_2 that r_0 first members are 0, r_1 first members are 1, ..., and r_{p-1} first members are $p-1$. Clearly $\sum_0^{p-1} r_i = p^{n-1}$.

Case 1. $r_0, r_1, \ldots, r_{p-1}$ all $\neq 0$.

Form the complete block C_1 (first member 0) by subtracting one element of A_1 from all other elements of A_1. If C_1 is multiplied by b we get the block C_2 formed also by subtracting one element of A_2 from all other elements of A_2. We can form the elements of C_2 (first member 0) by subtracting one of the r_0 elements of A_2 (first member 0) from itself and all the other $r_0 - 1$ such elements. Hence there are r_0 elements of C_2 (first member 0).

We can also form the elements of C_2 (first member 0) by subtracting one of the r_1 elements of A_2 (first member 1) from itself and all the other $r_1 - 1$ such elements. Hence there are r_1 elements of C_2 (first member 0).

Hence $r_0 = r_1 = r_2 = \ldots = r_{p-1} = p^{n-2}$.

This result must be true for all other first members, since all elements of the field are obtainable from those in A_1 by addition or subtraction.

Case 2. One or more of $r_0, r_1, r_2, \ldots, r_{p-1} = 0$.

Suppose in fact that $r_i, r_j, \ldots, r_k \neq 0$. Exactly as above, we can show $r_i = r_j = \ldots = r_k$. This leads to a contradiction since p^{n-1} is not divisible by a number less than p, unless we have

Case 3. All except one of $r_0, r_1, \ldots, r_{p-1} = 0$.

Suppose that the first members of all elements in A_2 are w, where $1 \leqslant w \leqslant p-1$. They cannot all be 0 since we can generate the whole field by addition and subtraction among the elements of A_1 and therefore the same among the elements of A_2. This would lead to all first members being 0 which is a contradiction unless b is the zero of the field.

Now we can find m such that $mw \equiv 1 \pmod{p}$. Hence mb (i.e. $b+b+\ldots+b$) times A_1 gives a block all of whose first members are also 1 (block D). Therefore multiplying by $(mb)^{-1}$ gives a block all of whose first members are 1 (block E). Subtract block D from block E (i.e. ith element from ith element) and obtain a block all of whose first members are 0. This must lead as before to first members of all blocks being 0. Hence the multiplier

$$(mb)-(mb)^{-1} = 00\ldots0,$$

therefore $(mb) = (mb)^{-1}$, i.e. $mb = \pm$ the unity of the field.

Hence the only possible b for Case 3 are $00\ldots0t$ where $1 \leqslant t \leqslant p-1$.

Write now the first members of the field elements in the order generated by c, a primitive root of the field, and its powers, i.e.

$$00\ldots00,\ 00\ldots01,\ c,\ c^2,\ \ldots,\ c^{p^n-2} \quad (C^{p^n-1}=1).$$

If these elements are multiplied in turn by $c, c^2, \ldots$, we obtain a cyclic permutation on all elements other than the zero, and by the above theorem any pair of columns satisfies the required symmetrical property. Multiplication by $c^{u(p^n-1)/(p-1)}$ will multiply columns by $00\ldots0t$, where t takes the values $1, 2, \ldots, p-1$, and hence the required property is satisfied only by the powers $c, c^2, \ldots, c^{(p^n-1)/(p-1)}$.

The proof that $c^{u(p^n-1)/(p-1)} = 00\ldots0t$ is as follows:

The element $00\ldots0t$ is expressible in the form C^x, therefore

$$c^{x(p-1)} = (00\ldots0t)^{p-1} \equiv 1 \equiv c^{p^n-1} \equiv c^{u(p^n-1)}.$$

Therefore $$x(p-1) = u(p^n-1).$$

For example, the elements of $GF(3^2)$ written in the order generated by powers of a primitive root are

$$00,\quad 01,\quad 11,\quad 20,\quad 21,\quad 02,\quad 22,\quad 10,\quad 12.$$

Taking the first members, we obtain a cyclic solution for $N = 9$, $L = 3$:

0	0	0	0
0	1	2	2
1	2	2	0
2	2	0	2
2	0	2	1
0	2	1	1
2	1	1	0
1	1	0	1
1	0	1	2

Thus, from the field $GF(p^n)$ we may obtain a cyclic solution for the case $L = p$, $N = p^n$. In the table of designs given below the first column of a cyclic solution is provided corresponding to the Galois fields 2^4, 2^5, 3^2, 3^3, 3^4, 5^2, 5^3 and 7^2. These have been taken from the tables in Stevens (1939), forming the basis of a series of completely orthogonal sets of cyclic Latin squares.

9. Experimental procedure

Suppose that the investigator is presented with an assembly containing 9 components and the problem of determining the effect of each of these in the performance of the whole. He decides upon an experiment in which each component appears at two values throughout and main effects are determined with a precision four times as great as that with which an assembly can be measured; in other words, the appropriate design is that for $L = 2$, $N = 16$. On referring to the table below he finds the design represented symbolically as follows (an explanation appears in a few lines):

$$+\,+\,+\,+\,-\,+\,-\,+\,+\,-\,-\,+\,-\,-\,-$$

The complete design is generated by taking this as the first column (or row), shifting it cyclically one place fourteen times and adding a final row of minus signs, thus:

```
+ − − − + − − + + − + − + + +
+ + − − − + − − + + − + − + +
+ + + − − − + − − + + − + − +
+ + + + − − − + − − + + − + −
− + + + + − − − + − − + + − +
+ − + + + + − − − + − − + + −
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− − + − − + + − + − + + + + −
− − − + − − + + − + − + + + +
− − − − − − − − − − − − − − −
```

The rows of this design may be taken as referring to assemblies and the columns to components. In the case in point there are nine components so that only nine columns are required. Select any nine columns, say the first nine, and obtain:

		Components								
		1	2	3	4	5	6	7	8	9
Assembly	1	+	−	−	−	+	−	−	+	+
	2	+	+	−	−	−	+	−	−	+
	3	+	+	+	−	−	−	+	−	−
	4	+	+	+	+	−	−	−	+	−
	5	−	+	+	+	+	−	−	−	+
	6	+	−	+	+	+	+	−	−	−
	7	−	+	−	+	+	+	+	−	−
	8	+	−	+	−	+	+	+	+	−
	9	+	+	−	+	−	+	+	+	+
	10	−	+	+	−	+	−	+	+	+
	11	−	−	+	+	−	+	−	+	+
	12	+	−	−	+	+	−	+	−	+
	13	−	+	−	−	+	+	−	+	−
	14	−	−	+	−	−	+	+	−	+
	15	−	−	−	+	−	−	+	+	−
	16	−	−	−	−	−	−	−	−	−

The components have been labelled $1, 2, \ldots, 8, 9$: a plus corresponding to component 7 in assembly 3 means that in that assembly component 7 appears at its extreme value; a minus corresponding to component 3 in assembly 12 means that in that assembly component 3 appears at its nominal value; and similarly. It will be seen that each component appears eight times at an extreme value and eight times at nominal, so that the arrangement is perfectly symmetrical. The investigator now proceeds to set up assemblies according to this design, to measure whatever characteristic of them is in mind, and to record the results.

10. ANALYSIS OF THE RESULTS

The results are in the form: measurement on assembly $1 = r_1$, measurement on assembly $2 = r_2$, ..., measurement on assembly $16 = r_{16}$. The effect of component 5, say, is required. Observe now that this component appears as plus in assemblies 1, 5, 6, 7, 8, 10, 12 and 13; and as minus in assemblies 2, 3, 4, 9, 11, 14, 15 and 16. Then the best estimate m_5 of the contribution of component 5 to the assembly characteristic due to its shift in value is

$$m_5 = (r_1 + r_5 + r_6 + r_7 + r_8 + r_{10} + r_{12} + r_{13} - r_2 - r_3 - r_4 - r_9 - r_{11} - r_{14} - r_{15} - r_{16})/16,$$

all observations where component 5 appears as plus being taken positively and where it appears as minus being taken negatively, and the divisor being the number of assemblies

made up. A solution similar to this, and as simple, holds for all designs where $L = 2$, and the general method of which components to put in which assemblies and how to evaluate the effects should now be apparent.

The results provide in addition an estimate of the experimental error, obtained as follows. Suppose that instead of 9 components, 15 had been used, laid out in accordance with the experimental design given above. Then m_{12}, for example, would have been evaluated by the equation

$$m_{12} = (r_2 + r_4 + r_5 + r_8 + r_{12} + r_{13} + r_{14} + r_{15} - r_1 - r_3 - r_6 - r_7 - r_9 - r_{10} - r_{11} - r_{16})/16.$$

In general, with n components, the quantities $m_{n+1}, m_{n+2}, \ldots, m_{4K-1}$ can be evaluated from the equations (number of assemblies $N = KL^2 = 4K$ here). Since there are just n components, these quantities should each be zero. In actual practice this will not be so due to experimental error. The variance due to error is estimated by the formula

$$s^2 = 4K(m_{n+1}^2 + m_{n+2}^2 + \ldots + m_{4K-1}^2)/(4K - n - 1).$$

Here $s^2 = 16(m_{10}^2 + m_{11}^2 + \ldots + m_{15}^2)/6$ and the error variance of $m_i = s_i^2 = s^2/4K$. This formula is, as proved above, equivalent to the usual sum of squares of residuals divided by the degrees of freedom; degrees of freedom for error $= (4K - 1) - n$.

A correction is necessary here. It will not usually be possible to select components whose values are exactly at nominal or extreme. All components will in any case have to be measured and the extent to which they differ from the aimed-at values will affect the values of m_i and s_i^2. Suppose that 'nominal' components are selected from a small range whose centre is the nominal value; and similarly at the extreme. For the ith component the difference in value between nominal and extreme is $2t_i$. If the component differs from the aimed-at value by c_i and if $b_i = c_i/t_i$, then the equations we are solving, instead of being of the form

$$r_j = M + a_{j1}m_1 + a_{j2}m_2 + \ldots + a_{jn}m_n,$$

where the coefficients a_{ij} are $+1$ or -1, are of the form

$$r_j = M + (a_{j1} + b_{j1})\,m_1 + (a_{j2} + b_{j2})\,m_2 + \ldots + (a_{jn} + b_{jn})\,m_n,$$

i.e. $R = (A + B)\,X$ where capital letters refer to the appropriate matrices. An approximate solution for X is obtained from $R = AX$, as above, and closer approximations may be obtained by iteration; a detailed treatment of the method is given in Lindley (1946).

Standard statistical methods now apply in determining the significance of effects and of differences between effects; whether the tolerance on a certain component may be increased and what would happen to the assembly characteristics if this were done; whether it is advisable to reduce the tolerance on another because of the large-scale effect allowed by the existing tolerance; whether the design of the assembly is correct in the sense that if both ends of the tolerance range have been explored the results show that the nominal value of each component is in the optimum position: these questions, and many like them depending on particular circumstances, may now all be answered. Errors may of course in all cases be reduced by replication, but it is suggested that, in order to obtain the best selection from the set of all possible assemblies (the complete factorial experiment) and thus minimize the errors due to interactions between components (here neglected as small), a complete design should be chosen in preference to the repetition of a smaller one. This aspect must not be confused with the fact that certain designs are obtained from smaller ones by the process of doubling, which is an entirely different thing. The designs in the table below (pp. 323, 324) will apply

directly to any experiment requiring less than 100 assemblies; should larger designs be required, they may be constructed by the general methods given.

11. RELATIONSHIP BETWEEN MULTIFACTORIAL AND BALANCED INCOMPLETE BLOCK DESIGNS

We begin for convenience with the definition of a balanced incomplete block design (Fisher & Yates, 1943). In this, v varieties are placed in blocks of k experimental units (k being less than v) such that every two varieties occur together in the same number (λ) of blocks; each variety appears r times in all and the number of blocks is b. Whence $rv = bk$ and

$$\lambda = r(k-1)/(v-1).$$

Consider any of the multifactorial designs for $L = 2$. Let the rows refer to blocks and the columns to varieties; and suppose that a plus sign represents the appearance of a variety in a block, a minus sign the non-appearance. For $N = 4m$ we obtain a balanced incomplete block design with $b = v = 4m-1$, $k = r = 2m$ and $\lambda = m$; the complementary design has $b = v = 4m-1$, $k = r = 2m-1$ and $\lambda = m-1$. The proof follows immediately from the orthogonality of the columns of the multifactorial design.

Now consider a complete multifactorial design F with N rows and L symbols; by complete we mean that the number of columns of F is $(N-1)/(L-1)$. Referring to § 4, suppose that all the elements of the diagonal matrix D are equal to $\sqrt{L}$, so that $Q'Q = L.I$. Let the rows of Q refer to the symbols $0, 1, 2, \ldots, L-1$. In the multifactorial design F using these L symbols replace each by the corresponding row of Q, omitting the 1 contributed by the first column. Add a first column of ones to the resulting matrix and obtain matrix A. Clearly $A'A = N.I$ and hence $AA' = N.I$. In any two rows of F a pair of unequal symbols in the same column contributes -1 to the scalar product of the corresponding rows of A; a pair of equal symbols contributes $+(L-1)$. Supposing that in these two rows of F there are λ pairs of equal symbols in the same column, and remembering the 1 at the beginning of each row of A, we have

$$1 + (L-1)\lambda - 1[(N-1)/(L-1) - \lambda] = 0,$$

whence

$$\lambda = (N-L)/L(L-1).$$

Let the rows of F refer to varieties and let each column represent L blocks, one corresponding to each of the L different symbols. By the result of the previous paragraph every two varieties occur together in the same number of blocks. We therefore obtain a balanced incomplete block design with parameters:

$$r = (N-1)/(L-1); \quad v = N; \quad b = rL; \quad k = N/L; \quad \lambda = (N-L)/L(L-1).$$

When $L = 2$, so that $N = 4m$, we can thus generate a large number of designs. When $L > 2$, we obtain balanced incomplete blocks with parameters

$$\begin{aligned} r &= 1 + L + L^2 + \ldots + L^{h-1}, \\ v &= L^h, \\ b &= L + L^2 + L^3 + \ldots + L^h, \\ k &= L^{h-1}, \\ \lambda &= 1 + L + L^2 + \ldots + L^{h-2}, \end{aligned}$$

where $L = p^m$, p a prime, and $h > 1$.

The balanced incomplete block designs formed from multifactorial designs, for which

$$r = (N-1)/(L-1); \quad v = N; \quad b = rL; \quad k = N/L; \quad \lambda = (N-L)/L(L-1);$$

are in fact of a special kind and have been called by Bose (1942) *affine resolvable*. A balanced incomplete block design is resolvable if we can separate the b blocks into r sets of n blocks each ($b = nr$) such that each variety occurs once among the blocks of a given set; and if in addition either (i) $b+1 = v+r$ or (ii) any two blocks belonging to different sets have the same number of varieties in common, then the other is true and the design is called affine resolvable because of its relation to certain finite Euclidean geometries. Bose has shown

(1) If a resolvable balanced incomplete block design is such that any two blocks belonging to different sets have the same number of varieties in common, then $b+1 = v+r$.

(2) If for a resolvable balanced incomplete block design $b+1 = v+r$ then any two blocks belonging to different sets have the same number of varieties in common. We have shown

(3) If a resolvable incomplete block design (i.e. one with r, v, b, k given but not necessarily balanced in the sense that every two varieties occur together in the same number of blocks) has $b+1 = v+r$ and is such that any two blocks belonging to different sets have the same number of varieties in common, then it is balanced.

To sum up, if a resolvable incomplete block design has any two of the following properties:

(i) any two blocks belonging to different sets have the same number of varieties in common,

(ii) balance,

(iii) $b+1 = v+r$,

then it has the third. The orthogonal matrix method we have used to prove (3) can also be used to provide short proofs of (1) and (2).

Consequently a multifactorial design can be formed from a balanced incomplete block design provided that the latter is resolvable with parameters

$$r = (N-1)/(L-1); \quad v = N; \quad b = rL; \quad k = N/L; \quad \lambda = (N-L)/L(L-1).$$

Bose has pointed out that affine resolvable designs can be constructed from the affine geometry $EG(h, p^m)$ (our notation) by taking varieties as points and blocks as $(h-1)$-flats; this construction gives all the multifactorial designs for $L > 2$ which have so far been obtained.

The most general aspect of the multifactorial design is obtained by considering each assembly as a block and each value of each component as a variety. We obtain a partially balanced incomplete block design (Bose & Nair, 1939) with parameters:

$$r = N/L; \quad v = L(N-1)/(L-1); \quad b = N; \quad k = (N-1)/(L-1);$$

$$\lambda_1 = N/L^2; \quad n_1 = [(N-1)/(L-1)-1]\,L; \quad \lambda_2 = 0; \quad n_2 = (L-1);$$

$$p_{ij}^1 = \begin{bmatrix} [(N-1)/(L-1)-2]\,L & (L-1) \\ (L-1) & 0 \end{bmatrix}; \quad p_{ij}^2 = \begin{bmatrix} [(N-1)/(L-1)-1]\,L & 0 \\ 0 & (L-2) \end{bmatrix}.$$

Although Bose & Nair state that general methods for the construction of partially balanced incomplete block designs are to appear, we have been unable to find them, so that this aspect of the multifactorial design does not yield more solutions of the problem.

12. Summary

Methods are developed to avoid the complete factorial experiment in industrial experimentation when the number of factors is so large that the standard procedure is impractic-

able. By assuming a simplified linear hypothesis, the problem of determining main effects with maximum precision is reduced to a combinatorial one. Practically all useful solutions of this have been found when each factor appears at two levels, but the solutions for more than two levels are fairly limited. The relationship of these solutions to some encountered in balanced incomplete blocks has been discussed.

We are indebted to Mr G. A. Barnard for suggesting the problem and the method of approach by least squares; and to Dr Bronowski for drawing our attention to the useful paper of R. E. A. C. Paley. The work was carried out as part of the programme of the Ministry of Supply, Research and Development (S.R. 17), and appears by kind permission of the Chief Scientific Officer.

TABLE OF DESIGNS

A. *Designs for $L = 2$.* The first row of any cyclic design is given opposite N, the number of assemblies. As stated at the end of § 4 (III) the matrix A here consists of plus and minus ones: these are denoted below by plus and minus signs. There is always a final row of minus signs —the basic assembly—to be added. In the designs for $N = 28, 52, 76, 100$ the square blocks are permuted cyclically amongst themselves; in the three latter cases the extra column has alternate signs throughout apart from the corner element. The larger designs are grouped in fives for convenience.

$N = 8$. + + + − + − −

$N = 12$. + + − + + + − − − + −

$N = 16$. + + + + − + − + + − − + − − −

$N = 20$. + + − − + + + + − + − + − − − − + + −

$N = 24$. + + + + + − + − + + − − + + − − + − + − − − −

$N = 28$. First nine rows

+ − + + + + − − −	− + − − − + − − +	+ + − + − + + − +
+ + − + + + − − −	− − + + − − + − −	− + + + + − + + −
− + + + + + − − −	+ − − − + − − + −	+ − + − + + − + +
− − − + − + + + +	− − + − + − − − +	+ − + + + − + − +
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+ + + − − − + − +	− − + − − + − + −	+ − + + − + + + −
+ + + − − − + + −	+ − − + − − − − +	+ + − + + − − + +
+ + + − − − − + +	− + − − + − + − −	− + + − + + + − +

$N = 32$. − − − − + − + − + + + − + + − − − + + + + + − − + + − + − − +

$N = 36$. (Obtained by trial) − + − + + + − − − + + + + + − + + + − − + − − − − + − + − + + − − + −

$N = 40$. Double design for $N = 20$.

$N = 44$. + + − − + − + − − + + + − + + + + + − − − + − + + + − − − − − + − − − + + − + − + + −

$N = 48$. + + + + + − + + + + − − + − + − + + + − − + − − + + − + + − − − + − + − + + − − − − + − − − −

$N = 52$.

First eleven rows

+	+ − + − + − + − + −	+ − + − + − + − + −	+ − + − + − + − + −	+ − + − + − + − + −	+ − + − + − + − + −
+	− + − − − − − − − −	+ + − − − − + + + +	+ + + + − − + + − −	+ + − − + + − − + +	+ + + + + + − − − −
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+	− − − − − − − + − −	− − + + + + + + − −	− − + + − − + + + +	+ + − − + + + + − −	+ + − − − − + + + +
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+	− − − − − − − − − +	− − − − + + + + + +	+ + − − + + − − + +	− − + + − − + + + +	+ + + + − − − − + +
−	− + − + − + − + + +	− + − + + − + − + −	+ − − + + − − + + −	− + + − − + + − + −	+ − + − − + − + + −

$N = 56$. Double design for $N = 28$.

$N = 60$. + + − + +   + − + − +   − − + − −   + + + − +   + + + − −   + + + + +   − − − − −   + + − − −   − + − − −
+ + − + +   − + − + −   − − + −

$N = 64$. Double design for $N = 32$.

$N = 68$. + + − − +   − + − − +   + − − − +   + + + − +   − + + + +   + + − − +   − − − + −   + + + − +   + − − − −
− − + − +   − − − − +   + + − − +   + − + − +   + −

$N = 72$. + + + + +   + + − + +   + − + − −   + + − + +   + − − − +   + − + − +   + − + − −   − + + + −   + − − + −
+ − − + +   + − − − +   − − + + −   + − − − +   − − − − −   −

$N = 76$. + +−

+	− +	− −	+ +	− −	− −	+ +	+ +	− −	+ +	− −	− −	− −	− −	+ +	+ +	+ +	− −	+ +	+ +	+ +	+ +	− −
−	+ +	− +	+ −	− +	− +	+ −	+ −	− +	+ −	− +	− +	− +	− +	+ −	+ −	+ −	− +	+ −	+ −	+ −	+ −	− +

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+ +	+ +	+ +	− −	− −	− −	− −	+ +	− −	+ +	+ +	− −	− −	+ +	− −
+ −	+ −	+ −	− +	− +	− +	− +	+ −	− +	+ −	+ −	− +	− +	+ −	− +

The first three rows are given; to obtain the complete design the square blocks are permuted cyclically. The first column, apart from the corner element, has alternate signs.

$N = 80$. + + + − +   + − − + +   + + − + −   − + − + +   + + + + −   + + − − −   − + + − −   − + − + −   + − + − +
+ + − − +   + + + − −   + − − − −   − − + − +   + − + − −   − − + + −   − + − −

$N = 84$. + + − + +   − − + − +   + + + − −   − + + − −   − + − + −   + + + + +   + + − + −   − + + + −   + + − − +
− − − + +   − + − − −   − − − − +   − + − + +   + − − + +   + − − − −   + − + + −   − + −

$N = 88$. Double design for $N = 44$.
$N = 92$. This design has not yet been obtained.
$N = 96$. Double design for $N = 48$.
$N = 100$.

+	+ − + − + − + − + − + − + −	+ − + − + − + − + − + − + −	+ − + − + − + − + − + − + −	+ − + − + − + − + − + − + −
+	− + − − − − − − − − − − − −	+ + − − + + + + − − + + − −	+ + − − − − + + + + − − + +	+ + + + + + − − − − − − + +
−	+ + − + − + − + − + − + − +	+ − − + + − + − − + + − − +	+ − − + − + + − + − − + + −	+ − + − + − − + − + − + + −
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B. *Designs for* $L = 3, 5, 7$. The first column is given below and the complete design is formed by permuting it cyclically $(N-1)/(L-1)-1$ times and adding a row of zeros. The corresponding orthogonal matrix A of § 4 (II) is obtained by replacing the component value symbols of the design by the rows of Q (§ 4 (I)) with its first column suppressed.

$N = 9, L = 3.$ 01220211
$N = 27, L = 3.$ 00101 21120 11100 20212 21022 2
$N = 81, L = 3.$ 01111 20121 12120 20221 10201 10012 22021 00200 02222 10212 21210 10112 20102 20021 11012 00100
$N = 25, L = 5.$ 04112 10322 42014 43402 3313
$N = 125, L = 5.$ 02221 04114 13134 12021 10244 31402 00444 20322 32121 32404 22043 31230 40033 34014 41424 21430 34403 11241 03001 11302 33234 34231 01330 12243 2010
$N = 49, L = 7.$ 01262 21605 32335 20413 11430 65155 61024 54425 03646 634

REFERENCES

BOSE, R. C. (1942). A note on the resolvability of balanced incomplete block designs. *Sankhyā*, **6**, 105.

BOSE, R. C. & KISHEN, K. (1940). On the problem of confounding in the general symmetrical factorial design. *Sankhyā*, **5**, 21.

BOSE, R. C. & NAIR, K. R. (1939). Partially balanced incomplete block designs. *Sankhyā*, **4**, 337.

FISHER, R. A. (1942). *The Design of Experiments*, 3rd ed. Oliver and Boyd.

FISHER, R. A. & YATES, F. (1943). *Statistical Tables for Biological, Agricultural and Medical Research*, 2nd ed., p. 14. Oliver and Boyd.

HOTELLING, H. (1944). Some improvements in weighing and other experimental techniques. *Ann. Math. Stat.* **15**, 297.

KISHEN, K. (1945). On the design of experiments for weighing and making other types of measurement. *Ann. Math. Stat.* **16**, 294.

LINDLEY, D. V. (1946). On the solution of some equations in least squares. *Biometrika*, **33**, 326.

PALEY, R. E. A. C. (1933). On orthogonal matrices. *J. Math. Phys.* **12**, 311.

STEVENS, W. L. (1939). The completely orthogonalized Latin square. *Ann. Eugen., Lond.*, **9**, 82.

Added in proof

(1) Since this paper was written, one of the authors (J. P. B.) has obtained a design for $L = 3$, $N = 18$, $n = 7$, by trial and error. It is known that this is the largest value of n possible.

(2) The designs given above for $L = 2$ provide what is effectively a complete solution of the experimental problem considered by Hotelling (1944) and Kishen (1945).