– Conditionals and Marginals

(1) Remember affine property:

If $x \sim N(\mu, C) \rightarrow Ax + b \sim N(A\mu + b, ACA^T)$

Choose such A so that $Ax = x_1 \longrightarrow A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}_{k\times 1} \Rightarrow A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ 0 & & \ddots & 1 \; 0 \end{bmatrix}_{k\times n}$

$$x_1 = Ax \sim N(A\mu, ACA^T)$$

$$\boxed{A\mu = \mu_1}$$

$$AC = \begin{bmatrix} 1 & & \\ & \ddots & 0 \\ 0 & & 1 \end{bmatrix}_{k\times n} \times \begin{bmatrix} C_{11} & C_{12} \cdots & C_{1n} \\ C_{21} & & \\ \vdots & & \\ C_{n1} & \cdots & C_{nn} \end{bmatrix}_{n\times n} = \begin{bmatrix} C_{11} & \cdots & C_{1n} \\ C_{21} & & \\ \vdots & & \\ C_{k1} & \cdots & C_{kn} \end{bmatrix}_{k\times n}$$

$$ACA^T = \begin{bmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & & \\ C_{k1} & \cdots & C_{kn} \end{bmatrix}_{k\times n} \times \begin{bmatrix} 1 & & 0 \\ & \ddots & 1 \\ 0 & & \end{bmatrix}_{n\times k} = \begin{bmatrix} C_{11} & C_{12} \cdots & C_{1k} \\ C_{21} & \cdots & \\ \vdots & & \\ C_{k1} & \cdots & C_{kk} \end{bmatrix}$$

$$= \Sigma_{11}$$

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(B) $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{pmatrix}$; $\Sigma^{-1} = \Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21}^T & \Omega_{22} \end{pmatrix}$; Σ_{11} and Ω_{11} are $p \times p$

Σ_{22} and Ω_{22} are $q \times q$

$\Sigma_{12} = \Sigma_{21}^T$ and $\Omega_{12} = \Omega_{21}^T$ are $p \times q$

$$\Sigma \Sigma^{-1} = \Sigma \Omega = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21}^T & \Sigma_{22} \end{bmatrix} \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21}^T & \Omega_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_{11}\Omega_{11} + \Sigma_{12}\Omega_{12}^T & \Sigma_{11}\Omega_{12} + \Sigma_{12}\Omega_{22} \\ \Sigma_{12}^T\Omega_{11} + \Sigma_{22}\Omega_{12}^T & \Sigma_{12}^T\Omega_{12} + \Sigma_{22}\Omega_{22} \end{bmatrix} = I_n = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}$$

$$\Sigma_{11}\Omega_{11} + \Sigma_{12}\Omega_{12}^T = I_p \longrightarrow \Omega_{11} = \Sigma_{11}^{-1} - \Sigma_{11}^{-1}\Sigma_{12}\Omega_{12}^T$$

$$\Sigma_{11}\Omega_{12} + \Sigma_{12}\Omega_{22} = 0 \longrightarrow \Omega_{12} = -\Sigma_{11}^{-1}\Sigma_{12}\Omega_{22}$$

$$\Sigma_{12}^T\Omega_{11} + \Sigma_{22}\Omega_{12}^T = 0 \longrightarrow \Omega_{12}^T = -\Sigma_{22}^{-1}\Sigma_{12}^T\Omega_{11}$$

$$\Sigma_{12}^T\Omega_{12} + \Sigma_{22}\Omega_{22} = I_q \longrightarrow \Omega_{22} = \Sigma_{22}^{-1} - \Sigma_{22}^{-1}\Sigma_{12}^T\Omega_{12}$$

Plug Ω_{12}^T into Ω_{11}

$$\Omega_{11} = \Sigma_{11}^{-1} + \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T\Omega_{11}$$

$$(I - \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T)\,\Omega_{11} = \Sigma_{11}^{-1}$$

$$(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T)\,\Omega_{11} = I_p$$

$$\boxed{\Omega_{11} = (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T)^{-1}}$$

$$\boxed{\Omega_{12}^T = -\Sigma_{22}^{-1}\Sigma_{12}^T(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T)^{-1}}$$

Plug Ω_{12} into Ω_{22}

$$\Omega_{22} = \Sigma_{22}^{-1} + \Sigma_{22}^{-1}\Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12}\Omega_{22}$$

$$(I_q - \Sigma_{22}^{-1}\Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12})\,\Omega_{22} = \Sigma_{22}^{-1}$$

$$(\Sigma_{22} - \Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12})\,\Omega_{22} = I_q$$

$$\boxed{\Omega_{22} = (\Sigma_{22} - \Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12})^{-1}}$$

$$\boxed{\Omega_{12} = -\Sigma_{11}^{-1}\Sigma_{12}(\Sigma_{22} - \Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12})^{-1}}$$