CHAPTER 1

(A) $p(w|\vec{x}) = \frac{p(\vec{x},w)}{p(\vec{x})} = \frac{p(\vec{x}|w) \times p(w)}{p(\vec{x})} \propto p(\vec{x}|w) \times p(w)$

This term does not depend on "w"

$p(w|\vec{x}) \propto w^{z}(1-w)^{N-z} \times w^{a-1}(1-w)^{b-1}$ where $z = \sum_{i=1}^{N} x_i$

$\propto w^{z+a-1}(1-w)^{N+b-z-1}$

Notice that posterior is a proper probability density and must add up (integrate) to 1.

This must be a beta dist. with parameters a' and b'

$$\boxed{p(w|\vec{x}) = \frac{w^{a'-1}(1-w)^{b'-1}}{B(a',b')}}$$ where $a' = a+z$ and $b' = N+b-z$

(B) $f_{Y_1,Y_2}(y_1,y_2) = |J|\, f[v_1(y_1,y_2), v_2(y_1,y_2)]$

$= y_2 \frac{1}{\Gamma(a_1)\Gamma(a_2)} x_1^{a_1-1} e^{-x_1} x_2^{a_2-1} e^{-bx_2}$

$= \frac{1}{\Gamma(a_1)\Gamma(a_2)} y_1^{a_1-1} y_2^{a_1+a_2-1}(1-y_1)^{a_2-1} e^{-y_2}$

$y_1 = \frac{x_1}{x_1+x_2}$ $\quad y_2 = x_1+x_2$

$v_1(y_1,y_2) = x_1 = y_1 y_2$

$v_2(y_1,y_2) = x_2 = y_2(1-y_1)$

$J = \begin{vmatrix} y_2 & y_1 \\ -y_2 & 1-y_1 \end{vmatrix} = y_2$

$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1Y_2}(y_1,y_2)\,.\,dy_2$

$f_{Y_2}(y_2) = \int_{-\infty}^{\infty} f_{Y_1Y_2}(y_1,y_2)\,.\,dy_1$

(C) $p(\theta|\vec{x}) = p(\vec{x}|\theta) \times p(\theta)$; $p(\vec{x}|\theta) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma_\theta^2}} \exp\left(-\frac{(x_i-\theta)^2}{2\sigma^2}\right)$

$\propto \exp\left(\frac{-\sum(x_i-\theta)^2}{2\sigma^2}\right) \times \exp\left(-\frac{(\theta-m)^2}{2v}\right)$

$\propto \exp\left(\frac{2\theta N\bar{x} - N\theta^2}{2\sigma^2} - \frac{\theta^2 - 2\theta m}{2v}\right) = \exp\left[-\frac{\theta^2}{2}\underbrace{\left[\frac{1}{v} + \frac{N}{\sigma^2}\right]}_{\frac{1}{\sigma_\theta'^2}} + \theta\underbrace{\left[\frac{m}{\sigma^2} + \frac{N\bar{x}}{\sigma^2}\right]}\right]$

$\theta' = \sigma_\theta'^2\left[\frac{m}{v^2} + \frac{N\bar{x}}{\sigma^2}\right]$

$\propto \exp\left[-\frac{\theta^2}{2\sigma_\theta'^2} + \frac{\theta\theta'}{\sigma_\theta'^2} - \frac{\theta'^2}{2\sigma_\theta'^2}\right] = \exp\left[-\frac{(\theta-\theta')^2}{2\sigma_\theta'^2}\right]$

$\sim \boxed{N(\theta', \sigma_\theta'^2)}$

(D) $f(\vec{x}\mid\theta,w) = \prod_{i=1}^{N} \left(\frac{w}{2\pi}\right)^{1/2} \exp\left\{-\frac{w}{2}(x_i-\theta)^2\right\}$

$$= (2\pi)^{-N/2}(w)^{N/2}\exp\left\{-w\sum_{i=1}^{N}\frac{(x_i-\theta)^2}{2}\right\}$$

$f(w\mid\alpha,\beta) = \beta^{\alpha} w^{\alpha-1}\exp(-\beta w)$

$f(w\mid\vec{x}) \propto f(\vec{x}\mid w)\cdot f(w) = (w)^{\frac{N}{2}+\alpha-1}\exp\left\{-w\left(\beta+\sum_{i=1}^{N}\frac{(x_i-\theta)^2}{2}\right)\right\}$

$= \text{Gamma}(\alpha',\beta')$ where $\boxed{\alpha' = \alpha+\frac{N}{2}}$ and $\boxed{\beta' = \beta+\sum_{i=1}^{N}\frac{(x_i-\theta)^2}{2}}$

(E) $p(\theta\mid\vec{x}) \propto p(\vec{x}\mid\theta)\cdot p(\theta)$

$\propto \exp\left(-\frac{(\theta-m)^2}{2v}\right)\prod_{i=1}^{N}\exp\left(-\frac{(x_i-\theta)^2}{2\sigma_i^2}\right)$ Let $w_v = \frac{1}{v}$ and $w_{\sigma_i} = \frac{1}{\sigma_i^2}$

$\propto \exp\left\{-\frac{(\theta-m)^2}{2}w_v + \sum_{i=1}^{N} -\frac{(x_i-\theta)^2}{2}w_{\sigma_i}\right\}$

$\propto \exp\left\{-\frac{1}{2}\left[(\theta^2-2m\theta+\cancel{m^2})w_v + \sum_{i=1}^{N}(\cancel{x_i^2}-2x_i\theta+\theta^2)w_{\sigma_i}\right]\right\}$

$\propto \exp\left\{-\frac{1}{2}\left[(w_v+\sum w_{\sigma_i})\theta^2 - 2(mw_v+\sum x_i w_{\sigma_i})\theta\right]\right\}$

$\propto \exp\left\{-\frac{1}{2}a\left(\theta-\frac{b}{a}\right)^2\right\}$ where $\boxed{a = w_v+\sum w_{\sigma_i}}$ and $\boxed{b = mw_v+\sum x_i w_{\sigma_i}}$

(F) $p(x) = \int_{-\infty}^{\infty} p(w,x)\,.\,dw = \int_{-\infty}^{\infty} p(x\mid w)\;p(w)\,.\,dw$

$p(w) = \frac{(b/2)^{a/2}}{\Gamma(a/2)}\,w^{a/2-1}\exp\left(-\frac{b}{2}w\right)$

$p(x\mid w) = \left(\frac{w}{2\pi}\right)^{1/2}\exp\left(-\frac{w}{2}(x-m)^2\right)$

$p(x,w) = \frac{(b/2)^{a/2}}{\Gamma(a/2)\sqrt{2\pi}}\,w^{\frac{a}{2}+\frac{1}{2}-1}\exp\left(-w\left(\frac{b}{2}+\frac{(x-m)^2}{2}\right)\right)$

$= \frac{(b/2)^{a/2}}{\Gamma(a/2)\sqrt{2\pi}}\,\frac{\Gamma(a')}{(b')^{a'}}\;\frac{(b')^{a'}}{\Gamma(a')}\,w^{a'-1}\exp(-b'w)$ → This will normalize to 1 in the integral

$p(x) = \int p(x,w)\,.\,dw = \frac{(b/2)^{a/2}}{(b')^{a'}\sqrt{2\pi}}\,\frac{\Gamma(a')}{\Gamma(a/2)}$ where $a' = \frac{a}{2}+\frac{1}{2}$

$b' = \frac{b}{2}+\frac{(x-m)^2}{2}$