Conditionals and Marginals

(B) $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{pmatrix}$; $\Sigma^{-1} = \Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21}^T & \Omega_{22} \end{pmatrix}$; Σ_{11} and Ω_{11} are $p \times p$

Σ_{22} and Ω_{22} are $q \times q$

$\Sigma_{12} = \Sigma_{21}^T$ and $\Omega_{12} = \Omega_{21}^T$ are $p \times q$

$$\Sigma \Sigma^{-1} = \Sigma \Omega = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21}^T & \Sigma_{22} \end{bmatrix} \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21}^T & \Omega_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_{11}\Omega_{11} + \Sigma_{12}\Omega_{12}^T & \Sigma_{11}\Omega_{12} + \Sigma_{12}\Omega_{22} \\ \Sigma_{12}^T\Omega_{11} + \Sigma_{22}\Omega_{12}^T & \Sigma_{12}^T\Omega_{12} + \Sigma_{22}\Omega_{22} \end{bmatrix} = I_n = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}$$

$$\Sigma_{11}\Omega_{11} + \Sigma_{12}\Omega_{12}^T = I_p \longrightarrow \Omega_{11} = \Sigma_{11}^{-1} - \Sigma_{11}^{-1}\Sigma_{12}\Omega_{12}^T$$

$$\Sigma_{11}\Omega_{12} + \Sigma_{12}\Omega_{22} = 0 \longrightarrow \Omega_{12} = -\Sigma_{11}^{-1}\Sigma_{12}\Omega_{22}$$

$$\Sigma_{12}^T\Omega_{11} + \Sigma_{22}\Omega_{12}^T = 0 \longrightarrow \Omega_{12}^T = -\Sigma_{22}^{-1}\Sigma_{12}^T\Omega_{11}$$

$$\Sigma_{12}^T\Omega_{12} + \Sigma_{22}\Omega_{22} = I_q \longrightarrow \Omega_{22} = \Sigma_{22}^{-1} - \Sigma_{22}^{-1}\Sigma_{12}^T\Omega_{12}$$

Plug Ω_{12}^T into Ω_{11}

$$\Omega_{11} = \Sigma_{11}^{-1} + \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T\Omega_{11}$$

$$(I - \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T)\Omega_{11} = \Sigma_{11}^{-1}$$

$$(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T)\Omega_{11} = I_p$$

$$\boxed{\Omega_{11} = (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T)^{-1}}$$

$$\boxed{\Omega_{12}^T = -\Sigma_{22}^{-1}\Sigma_{12}^T(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T)^{-1}}$$

Plug Ω_{12} into Ω_{22}

$$\Omega_{22} = \Sigma_{22}^{-1} + \Sigma_{22}^{-1}\Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12}\Omega_{22}$$

$$(I_q - \Sigma_{22}^{-1}\Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12})\Omega_{22} = \Sigma_{22}^{-1}$$

$$(\Sigma_{22} - \Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12})\Omega_{22} = I_q$$

$$\boxed{\Omega_{22} = (\Sigma_{22} - \Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12})^{-1}}$$

$$\boxed{\Omega_{12} = -\Sigma_{11}^{-1}\Sigma_{12}(\Sigma_{22} - \Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12})^{-1}}$$