The Multivariate Normal Distribution

(A) $\text{cov}(x) = E\{(x-\mu)(x-\mu)^T\} = E\{xx^T - x\mu^T - \mu x^T + \mu\mu^T\}$

$$= E(xx^T) - E(x)\mu^T - \mu E(x^T) + \mu\mu^T$$

$$= E(xx^T) - \mu\mu^T - \mu\mu^T + \mu\mu^T$$

$$= \boxed{E(xx^T) - \mu\mu^T}$$

$$\text{cov}(Ax+b) = E[(Ax+b)(Ax+b)^T] - E[Ax+b][E(Ax+b)]^T$$

$$= E[(Ax+b)(x^TA^T+b^T)] - E[Ax+b][A\mu+b]^T$$

$$= E[Axx^TA^T + Axb^T + bx^TA^T + bb^T] - [A\mu+b][\mu^TA^T+b^T]$$

$$= AE(xx^T)A^T + \cancel{A\mu b^T} + \cancel{b\mu^TA^T} + \cancel{bb^T} - A\mu\mu^TA^T - \cancel{A\mu b^T} - \cancel{b\mu^TA^T} - \cancel{bb^T}$$

$$= A[E(xx^T) - \mu\mu^T]A^T = \boxed{A\,\text{cov}(x)\,A^T}$$

(B) $f(\vec{z}) = f(z_1)\cdot f(z_2) \cdots f(z_p) = \prod_{i=1}^{p} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z_i^2}{2}\right)$

$$= \frac{1}{(2\pi)^{p/2}} \exp\left[-\frac{\sum z_i^2}{2}\right] = \frac{1}{(2\pi)^{p/2}} \exp\left[-\frac{z^Tz}{2}\right]$$

$$M_x(t) = E(e^{tx})$$

$$M_{\vec{z}}(\vec{t}) = E(e^{t^Tz}) = \int_z e^{t^Tz} \frac{1}{(2\pi)^{p/2}} \exp\left[-\frac{z^Tz}{2}\right] d\vec{z}$$

$$= \int_z \frac{1}{(2\pi)^{p/2}} \exp\left[t^Tz - \frac{z^Tz}{2}\right] d\vec{z}$$

$$= \int_z \frac{1}{(2\pi)^{p/2}} \exp\left[\frac{1}{2}t^Tz + \frac{1}{2}z^Tt - \frac{z^Tz}{2} + \frac{1}{2}t^Tt - \frac{1}{2}t^Tt\right] d\vec{z}$$

$$= \int_z \exp\left(\frac{tt^T}{2}\right) \frac{1}{(2\pi)^{p/2}} \exp\left[-\frac{1}{2}(z^Tz - z^Tt - t^Tz + t^Tt)\right]$$

$$= \exp\left(\frac{tt^T}{2}\right) \int_z \underbrace{\frac{1}{(2\pi)^{p/2}} \exp\left(-\frac{1}{2}(z-t)^T(z-t)\right)}_{=1}$$

$$= \exp\left(\frac{tt^T}{2}\right)$$

(C) Affine property: If $X \sim N(\mu, \Sigma)$ then $z = a^T X \sim N(a^T\mu, a\Sigma a^T)$

Find MGF of $z = a^T X$

$$M_z(t) = M_{a^T x}(t) = E\left(e^{t^T a^T x}\right) = M_x(at)$$

$$M_x(at) = e^{(at)^T \mu} \cdot e^{\frac{1}{2}(at)^T \Sigma (at)}$$

$$= \exp\left(t^T \underbrace{a^T\mu}_{\mu_z} + \frac{1}{2} t^2 \underbrace{a^T \Sigma a}_{\sigma_z}\right) = \exp\left(t^T \mu_z + \frac{1}{2} t^T \sigma_z^2 t\right)$$

(D) $M_x(t) = E\left(e^{t^T x}\right) = E\left(e^{t^T[Lz+\mu]}\right) = \exp(t^T\mu)\, E\left[\exp(t^T L z)\right] = \exp(t^T\mu)\, M_z(L^T t)$

$$= \exp(t^T\mu)\, \exp\left(L^T t . 0 + \frac{1}{2} t^T L L^T t\right) = \exp\left(t^T\mu + \frac{1}{2} t^T L L^T t\right)$$

Then X is multivariate normal with $X \sim N(\mu, LL^T)$

(E) Claim $X \sim N(\mu, \Sigma)$ $Az + b \sim X$ where $z \sim N(0_{n \times 1}, I_{n \times n})$

$\Sigma = LL^T$ Propose $A = L$, $b = \mu$

$x = f(z) = Lz + \mu$

$g^{'}(x) = L^{-1}(x - \mu)$

$$f(X = x) = f(z = g(x)) \left| \frac{\partial g(x)}{\partial x} \right|$$

$$= \frac{1}{(2\pi)^{P/2}} \exp\left[-\frac{1}{2} (L^{-1}(x-\mu))^T (L^{-1}(x-\mu))\right] \| L^{-1} \| \longrightarrow \left[(L^{-1})^T (L^{-1})\right]^{1/2}$$

$$= \frac{1}{(2\pi)^{P/2}} \frac{1}{(|\Sigma|)^{1/2}} \exp\left[-\frac{1}{2} (x-\mu)^T \underbrace{(L^{-1})^T (L^{-1})}_{|\Sigma|^{-1}} (x-\mu)\right)$$