EXERCISES 3

Basic Concepts

$$MSE(\hat{f}, f) = E\{(f(x) - \hat{f}(x))^2\} = E\{f^2(x) + \hat{f}^2(x) - 2f(x).\hat{f}(x)\}$$

$$= f^2(x) + E\{\hat{f}^2(x)\} - E(\hat{f}(x))^2 + E(f(x))^2 - 2f(x)E\{\hat{f}(x)\}$$

$$= [E\{\hat{f}^2(x)\} - E(\hat{f}(x))^2] + [f^2(x) + E(f(x))^2 - 2f(x)E\{\hat{f}(x)\}]$$

$$= V + B^2$$

Curve Fitting by Linear Smoothing

(A) $$MLE \propto \sum_{i=1}^{n} (\hat{\beta} x_i - y_i)^2$$

$$\frac{\partial MLE}{\partial \hat{\beta}} = \sum_{i=1}^{n} 2(\hat{\beta} x_i - y_i) x_i = 0$$

$$\hat{\beta} \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} y_i x_i = 0 \implies \hat{\beta} = \frac{\sum_{i=1}^{n} y_i x_i}{\sum_{i=1}^{n} x_i^2}$$

$$\hat{f}(x^*) = \hat{\beta} x^* = \frac{\sum_{i=1}^{n} x_i . x^* y_i}{\sum_{i=1}^{n} x_i^2} = \sum_{i=1}^{n} w(x_i, x^*) y_i \quad \text{where} \quad w(x_i, x^*) = \frac{x_i x^*}{\sum_{i=1}^{n} x_i^2}$$

Local Polynomial Regression

(C) $$E[\hat{f}(x)] = E[S_x y] = S_x E[y] = S_x . f(x)$$

$$Var[\hat{f}(x)] = S_x^T S_x \sigma^2$$

Local Polynomial Regression

(A) Minimize $\sum_{i=1}^{n} \{ y_i - a_0 - a_1(x_i - x) - a_2(x_i-x)^2 - a_3(x_i-x)^3 - \dots - a_D (x_i-x)^D \} \frac{1}{h} K\left(\frac{x_i - x}{h}\right)$

Define $R_x = \begin{bmatrix} 1 & (x_1-x) & \dots & (x_1-x)^D \\ 1 & (x_2-x) & & \\ \vdots & & & \\ 1 & (x_n-x) & \dots & (x_n-x)^D \end{bmatrix}_{n\times D}$

Define $W_x = \frac{1}{h}\begin{bmatrix} K\left(\frac{x_1-x}{h}\right) & 0 & \dots & 0 \\ 0 & K\left(\frac{x_2-x}{h}\right) & & \\ \vdots & & \ddots & \\ 0 & & & K\left(\frac{x_n-x}{h}\right) \end{bmatrix}_{n\times n}$

Minimize $(y - R_x a)\, W_x\, (y - R_x a)^T$ say $= L_x$

$$\frac{\partial L_x}{\partial a} = 2R_x^T W_x R_x a - 2R_x^T W_x y = 0$$

$$a = (R_x^T W_x R_x)^{-1} R_x^T W_x y$$

(B) If $D=1$ $\quad R_x = \begin{bmatrix} 1 & x_1-x \\ 1 & x_2-x \\ \vdots & \vdots \\ 1 & x_n-x \end{bmatrix}_{n\times 2}$

$$R_x^T W_x y = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1-x & \dots & & x_n-x \end{bmatrix}_{2\times n} [W_x]_{n\times n} \cdot [y]_{n\times 1} = \begin{bmatrix} K\left(\frac{x_1-x}{h}\right), & K\left(\frac{x_2-x}{h}\right) & \dots, & K\left(\frac{x_n-x}{h}\right) \\ (x_1-x)K\left(\frac{x_1-x}{h}\right) & & & (x_n-x)K\left(\frac{x_n-x}{h}\right) \end{bmatrix}_{2\times n} [y]_{n\times 1}$$

$$R_x^T W_x R_x = \begin{bmatrix} \sum K\left(\frac{x_i-x}{h}\right) & \sum K\left(\frac{x_i-x}{h}\right)(x_i-x) \\ \sum K\left(\frac{x_i-x}{h}\right)(x_i-x) & \sum K\left(\frac{x_i-x}{h}\right)(x_i-x)^2 \end{bmatrix}_{2\times 2}$$

$$a = (\text{const}) \underbrace{\begin{bmatrix} \sum K\left(\frac{x_i-x}{h}\right)(x_i-x)^2 & -\sum K\left(\frac{x_i-x}{h}\right)(x_i-x) \\ -\sum K\left(\frac{x_i-x}{h}\right)(x_i-x) & \sum K\left(\frac{x_i-x}{h}\right) \end{bmatrix}_{2\times 2}}_{A} \begin{bmatrix} K\left(\frac{x_1-x}{h}\right) & \dots & K\left(\frac{x_n-x}{h}\right) \\ K\left(\frac{x_1-x}{h}\right)(x_1-x) & \dots & K\left(\frac{x_n-x}{h}\right)(x_n-x) \end{bmatrix}_{2\times n} \cdot \begin{bmatrix} y \end{bmatrix}_{n\times 1}$$

$a = (\text{const})\, [A]_{2\times 2} [B]_{2\times 1}$ where $[B] = \begin{bmatrix} \sum K\left(\frac{x_i-x}{h}\right) y_i \\ \sum K\left(\frac{x_i-x}{h}\right)(x_i-x)(y_i) \end{bmatrix}$

$$= (\text{const}) \begin{bmatrix} s_2 & -s_1 \\ -s_1 & s_0 \end{bmatrix} \begin{bmatrix} \sum K\left(\frac{x_i-x}{h}\right) y_i \\ \sum K\left(\frac{x_i-x}{h}\right)(x_i-x) y_i \end{bmatrix}$$

$$\hat{f}(x) = a_0 \propto s_2 \sum K\left(\frac{x_i-x}{h}\right) y_i - s_1 \sum K\left(\frac{x_i-x}{h}\right)(x_i-x) y_i = \sum \left(K\left(\frac{x_i-x}{h}\right)\left(s_2 - s_1(x_i-x)\right)\right) y_i$$

$$w_i = K\left(\frac{x_i-x}{h}\right) \{ s_2 - (x_i-x) s_1 \}$$