Condensed Matter Physics Meets Python via SageMath

Properties to distinguish quantum phases of matter

Amit Jamadagni

Birla Institute of Technology and Science, Pilani.
Fudan University, Shanghai, China.

f2011620@goa.bits-pilani.ac.in, bitsjamadagni@gmail.com

Abstract

Group theory has been used to classify various phases of matter, but the same cannot be applied to phases of matter at absolute zero which are called as topological phases of matter. In order to classify such topological phases of matter many theories have been proposed like Quantum Double Models, Levin-Wen String Net model, Twisted Quantum Double Model. This presentation aims to introduce Quantum Double Models and some of its related properties like excitations, excitation condensation, identification of various boundaries using the excitation condensation on the boundary. These properties are heavily dependent on the Group Theory constructs, therefore to compute the related properties SageMath has been used. The following properties :

1. Excitation types for Quantum Double $D(G)$
2. Condensation of excitation on various boundaries $K \subset G$
3. Ribbon operators for lattice with boundary

have been computed for Symmetric Group of 2 labels (Toric Code), Symmetric Group of 3 labels (the smallest non-abelian group) though the code can be used for any general finite group. The presentation aims at a particular kind of boundary construct that is a combination of subgroup and trivial cocycle, though in the literature there are more general boundary conditions which involve the non trivial cocycle. The code presented can be easily extended to realize such boundary constructions which use cocycles of some group cohomology.

Quantum Double Models and their properties

Introduction

Given a group G, consider a lattice with each edge being associated with a Hilbert space and indexed by a group element. A site in the lattice is given by a pair of adjacent vertex and face. For a given site (vertex v and face f), define the vertex operator A_v^g and face operator B_s^h as in Figure 1 :

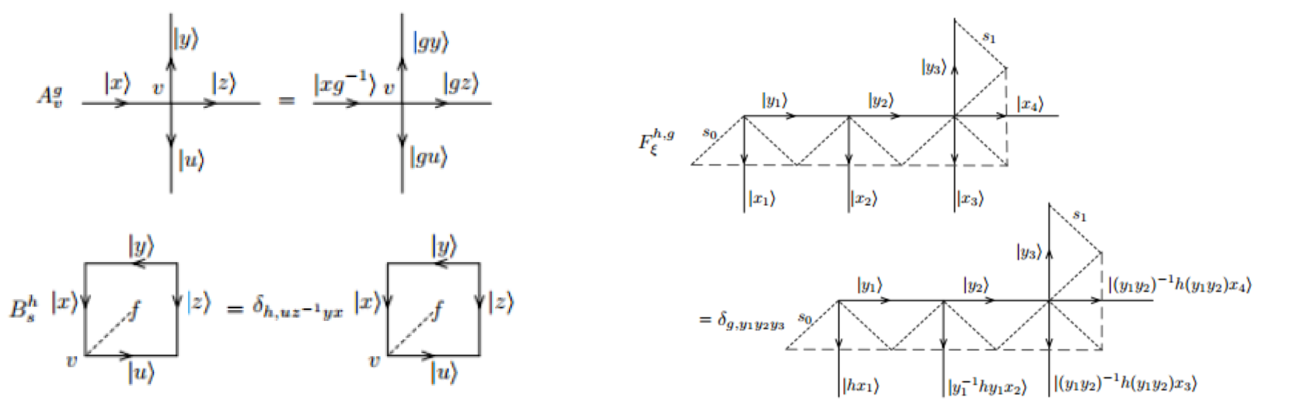

Figure 1: Definition of the A_v^g, B_s^h operators on a arbitrary vertex and face, ribbon operator $F_\xi^{h,g}$ on a arbitrary lattice

. The Hamiltonain for the lattice is given by

$$H = -\Sigma_v A_v - \Sigma_f B_f$$

Ribbon operators, Excitations, Anyon types

A ribbion ξ in the lattice is a sequence of adjacent sites connecting two sites s_0 and s_1. The ribbon operator $F_\xi^{h,g}$ is defined as in figure 1.The application of ribbon operator on the ground state, gives rise to quasi-particle excitations at the end of the ribbon. The excitations are independent of the topology of the ribbon operator. Therefore the excitations can be moved around the lattice by extending/contracting the ribbon. Anyon types for a Kitaev Quantum Double are in one-to-one correspondace with the irreducible representations of the centralizers of the conjugacy classes of the group. SageMath has been used to compute the ribbon operators as well as anyon types for various groups like Z_2, S_3, D_4.

Introduction of Boundaries, Corresponding Ribbon operators, Correspoding Condensates

Consider a lattice as above but with a boundary, that is with a lattice on one half and nothing on the other side, and the edges connecting both the half planes being the boundary. The edges on the boundary are associated with a Hilbert space $C[K]$, where C is the complex field, $K \subset G$ and are indexed by elements of K. The vertex and the face operator for the internal lattice remain as defined in the introduction, but for the boundary the vertex and face operators are defined as follows :

$$A_s^K = \frac{1}{|K|}\Sigma_{k\in K} A_s^k$$
$$B_s^K = \Sigma_{k\in K} B_s^k$$

where s is a site on the boundary.

The hamiltonian of the system with the boundary is given by,

$$H = -\Sigma_v A_v - \Sigma_f B_f - \Sigma_s(A_s^K + B_s^K)$$

A ribbon $T = \Sigma_{h,g} c_{h,g} F_\xi^{h,g}$, connecting the sites in the bulk to the sites on the boundary is given by,

$$T^{(k,g)} = \Sigma_{l\in K} F_\xi^{(lkl^{-1},lg^{-1})}$$

The excitations in a Quantum Double model are given by irreps of the centralizers of the conjugacy class of the group. The character of a particular representation π is given by $\chi_{(\bar{a},\pi)}$:

$$\chi_{(\bar{a},\pi)}(gh^*) = \delta_{h\in\bar{a}}\delta_{gh,hg} tr_\pi(k_h^{-1} g k_h),$$

as $h \in \bar{a}$ is one of the conditions to be satisfied, k_h is some element in G such that $h = k_h a k_h^{-1}$

For $k \in K$ and $g \in G$ define $|\psi_K^{k,g} >= T^{(k,g)}|\psi_K >$ and let $A(K)$ be the span of these vectors. Let $\chi_{A(K)}$ be the character of the representation $A(K)$.

$$\chi_{A(K)}(hg^*) = \frac{1}{|K|}\delta_{gh,hg}\Sigma_{x\in G}\delta_{xgx^{-1}\in K}\delta_{xhx^{-1}\in K}$$

To compute whether a particular excitation condenses at the boundary, the inner product defined as

$$< \chi_1, \chi_2 >= \frac{1}{|G|}\Sigma_{g,h}(\chi_1(gh^*)^*\chi_2(gh^*)$$

of the above characters is observed, if it is positive the excitation associated with the irrep of $D(G)$ condenses at the boundary.

Computations using Group Theory in SageMath

Anyon types for Quantum Double $D(G)$

1. Computing the centralizers of the conjugacy class of the group

```
def centralizer_conjugacy_class_QDM_generic(QDM_group):
  cent_QDM_group = []
  for conj_class in QDM_group.conjugacy_classes():
    centralizer = QDM_group.centralizer(conj_class.an_element())
    cent_QDM_group.append(centralizer)
  return cent_QDM_group
```

2. The character table gives the trace of the irreducible representations

```
def character_table_centralizers(centralizer_generic_group):
  char_table = []
  for subgroup in centralizer_generic_group:
    char_table.append(subgroup.character_table())
  return char_table
```

Developing the machinery to compute the excitations condense on a given boundary

1. Computing the character related to the irreducible representation of the group

```
def character_excitation(G, conjugacy_class, g, h):
  k_h = 0
  for g_1 in G:
    if h*g_1 == g_1*conjugacy_class.an_element():
      k_h = g_1
      break
  if g*h == h*g and k_h != 0:
    return k_h^-1*g*k_h
  else:
    return 0
```

2. Computing the character related to a particular boundary

```
def character_subgroup(G, subgroup, g, h):
  sum = 0
  if h*g == g*h:
    for g_1 in G:
      if g_1*g*g_1^-1 in subgroup and g_1*h*g_1^-1 in subgroup:
            sum = sum + 1
  return sum/len(subgroup)
```

3. Computing the inner product terms of the above characters

```
def inner_product_of_characters(QDM_group, subgroup, conjugacy_class):
  inner_product_terms = []
  for g in QDM_group:
    for h in QDM_group:
      if character_subgroup(QDM_group, subgroup, g, h) != 0 and character_excitation(QDM_group, conjugacy_class, g, h) != 0:
              inner_product_terms.append([character_subgroup(QDM_group, subgroup, g, h), character_excitation(QDM_group, conjugacy_class, g, h)])
  return inner_product_terms
```

Construction of the ribbon operators for lattice with boundary

Ribbon operator connecting the bulk to the boundary

```
def ribbon_operator_constructs(QDM_group, subgroup):
  ribbon_operator_terms = []
  for k in subgroup:
    for g in QDM_group:
      for l in subgroup:
            ribbon_operator_terms.append([k,g,l*k*l^-1, l*g^-1])
  return ribbon_operator_terms
```

Results and Extensions

- Anyon types given any Quantum Double $D(G)$ have been computed.
- Explicit form of the ribbon operators in the presence of boundaries has been presented giving an insight into the condensation of excitations.
- Given $K \subset G$, using the above procedure we can compute the excitations that condense on the boundary, this allows the classification of different boundaries based on the set of anyons condensing on a given boundary K.
- The above results assume that the boundaries are identified by $K \subset G$ and a trivial cocycle of the group cohomology $H^2(K, U(1))$ and can be extended for some (K, ϕ) where $K \subset G$, ϕ non-trivial cocycle in $H^2(K, U(1))$.