RANDOM FIELDS

AND

GEOMETRY

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ii

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Preface

Since the term 'random field' has a variety of different connotations, ranging from Agriculture to Statistical Mechanics, let's start by clarifying that for us a random field is a stochastic process, usually taking values in a Euclidean space, and defined over a parameter space of dimensionality at least one.

Consequently, random processes defined on countable parameter spaces are of little or no interest to us. Indeed, even processes on $\mathbb{R}^1$ are almost trivial. The parameter spaces we like best are manifolds, although, for much of the time, we shall require no more than they be pseudo-metric spaces.

With this clarification in hand, the next thing that you should know is that this book is one of a pair, and that it is divided into three parts.

It is one of a pair since, as we complete it, we are starting, together with KW (Keith Worsley), on a companion volume tentatively entitled *ARF*, or *Applications of Random Fields*. The current volume – *RFG* – concentrates on the theory and mathematical background of random fields, while *ARF* is intended to do precisely what its title promises. It is to that volume that you should turn if you want to see applications not only of the theory of this book, but of (smooth) random fields in general.

Making a clear split between theory and practice has both advantages and disadvantages. It certainly eased the pressure on us to attempt the almost impossible goal of writing in a style that would be accessible to all. It also, to a large extent, eases the load on you the reader, as you can now choose the volume closest to your interests and so avoid either 'irrelevant' mathematical detail or the 'real world', depending on your outlook and tastes. However these are small gains when compared to the major loss of

creating an apparent dichotomy between two things that should, in principle, go hand in hand: Theory and Application. What is true in principle is particularly true of the topic at hand and to explain why we shall indulge ourselves in a paragraph or two of history.

The precusor to both of the current volumes was the 1981 *The Geometry of Random Fields* (*GRF*) which grew out of RJA's (i.e. Robert Adler's) PhD thesis under Michael Hasofer. The problem that gave birth to the thesis was an applied one, having to do with ground fissures due to water permeating through the earth under a building site. However, both the thesis and *GRF* ended up being directed more to theoreticians than to subject matter researchers. Nevertheless, the topics there found many applications over the past two decades, in disciplines as widespread as Astrophysics and Medical Imaging.

These applications lead to a wide variety of extensions of the material of *GRF* which, while different in extent to what was there, were not really different in kind. However, in the late 1990's KW found himself facing a brain mapping problem on the cerebral cortex (i.e. the brain surface) which involved looking at random fields on manifolds. Jonathan Taylor (JET) looked at this problem and, in somewhat of a repetition of history, took it to an abstract level and wrote a PhD thesis that completely revolutionised[1] the way one should think about problems involving the geometry generated by smooth random fields. This, and subsequent material, makes up Part III of the current book.

In fact, this book is really about its Part III, and it is there that all the new material will be found. Part I is mainly an adaptation of RJA's 1990 IMS lecture notes, *An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes*, considerably corrected and somewhat reworked with the intention of providing all that one needs to know about Gaussian random fields in order to read Part III. In addition, there is a Chapter on stationarity. *En passant*, we also included many things that were not really needed for Part III, so that Part I can be (and often has been) used as the basis of a one quarter course in Gaussian processes. Beyond some issues of exposition, there is nothing new in Part I, and its inclusion is so that the book, as a whole, will be self-contained.

It is also true that there is nothing new in Part II, which is about geometry, both integral and differential. However, the material here is considerably less standard than that of Part I, and we expect that few readers other than professional geometers will be familiar with all of it. In particular, there are two main aims to Part II. One is to set up an analogue of the critical point theory of Marston Morse in the framework of Whitney stratified manifolds. What makes this non-standard (at least in terms of what most students of Mathematics see as part of their graduate education) is

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[1]This verb was chosen by RJA and not JET.

that Morse theory is usually done for smooth manifolds, preferably without boundaries. Whitney stratified manifolds are only piecewise smooth, and are permitted any number of edges, corners, etc. This brings them closer to the objects of Integral Geometry, to which we devote a chapter. While the results of this specific chapter are actually subsumed by what we shall have to say about Whitney stratified manifolds, they have the advantage that they are easy to state and prove without heavy machinery.

The second aim of Part II is to develop Lipschitz-Killing curvatures in the setting of Whitney stratified manifolds. These two topics make for a somewhat unusual combination and there is no easily accessible and succinct[2] alternative to our Part II for learning about them. In the same vein, in order to help novice differential geometers, we have included a one chapter primer on Differential Geometry which runs quickly, and rather unaesthetically, through the basic concepts and notation of this most beautiful part of Mathematics.

Having established that Parts I and II of this book are merely background material, what is it in Part III that justifies 250 pages of preparation? Part III revolves around the *excursion sets* of smooth, $\mathbb{R}^k$ valued random fields f over piecewise smooth manifolds M. Excursion sets are subsets of M given by

$$A_D \;\equiv\; A_D(f, M) \;\overset{\Delta}{=}\; \{t \in M : f(t) \in D\},$$

for $D \subset \mathbb{R}^k$.

A great deal of the sample function behaviour of such fields can be deduced from their excursion sets and a surprising amount from their Euler, or Euler-Poincaré, characteristics, defined in Part II. In particular, if we denote the Euler characteristic of a set A by $\varphi(A)$, then much of Part III is devoted to finding the following expression for their expectation, when f is Gaussian with zero mean and constant variance:

$$(0.0.1) \qquad \mathbb{E}\{\varphi(A_D)\} \;=\; \sum_{j=0}^{\dim M} (2\pi)^{-j/2} \mathcal{L}_j(M) \mathcal{M}_j^k(D).$$

Here the $\mathcal{L}_j(M)$ are the Lipschitz-Killing curvatures of M with respect to a Riemannian metric induced by the random field f, and the $\mathcal{M}_j^k(D)$ are certain Minkowski-like functionals (closely akin to Lipschitz-Killing curvatures) on $\mathbb{R}^k$ under Gauss measure.

If all of this sounds terribly abstract, the truth is that it both is, and is not. It is abstract, because while (0.0.1) has had many precursors over the last 60 years or so, it has never before been established in the generality described above. It is also abstract in that the tools involved in

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[2]The stress here is on 'succinct', as none of the material of Part II is new and there are many excellent texts, some of which we shall list later, that treat it. However, all presume a background knowledge that is beyond what we shall need.

the derivation of (0.0.1) in this setting require some rather heavy machinery from Differential Geometry. However, this level of abstraction has turned out to pay significant dividends, for not only does it yield insight into earlier results that we did not have before, but it also has practical implications. For example, the approach that we shall employ works just as well for non-stationary processes as it does for stationary ones[3]. However, non-stationarity, even on manifolds as simple as $[0,1]^2$, was previously considered essentially intractable. Simply put, this is one of those rare but constantly pursued examples in Mathematics in which abstraction leads not only to a complete and elegant theory, but also to practical consequences.

An extremely simple and very down to earth application of (0.0.1) arises when the manifold is the unit interval $[0,1]$, f is real valued, and $D = [u, \infty)$. In that case, $\mathbb{E}\{\varphi(A_D)\}$ is no more than the mean number of upcrossings of the level u by the process f, along with a boundary correction term. Consequently, modulo the boundary term, (0.0.1) collapses to no more that the famous Rice formula, undoubtedly one of the most important results in the application of smooth stochastic processes.

One of the reasons that Rice's formula is so important is that it has long been used as an approximation, for large u, to the *excursion probability*

$$\mathbb{P}\Big\{ \sup_{t \in [0,1]} f(t) \geq u \Big\},$$

itself an object of major practical importance. The heuristic argument behind this is simple: If f crosses a high level, it is unlikely do so more than once. Thus, in essence, the probability that f crosses the level u is close to the probability that there is an upcrossing of u, along with a boundary correction term. (The correction comes from the fact that one way for $\sup_{t \in [0,1]} f(t)$ to be larger than u is for there to be no upcrossings but $f(0) \geq u$.) Since the number of upcrossings of a high level will always be small, the probability of an upcrossing is well approximated by the mean number of upcrossings. Hence Rice's formula gives an approximation for excursion probabilities.

If (0.0.1) is the main result of Part III, then the second most important result is that, at the same level of generality and for a wide choice of D, we can find a bound for the difference

$$\Big| \mathbb{P}\Big\{ \exists\, t \in M : f(t) \in D \Big\} \; - \; \mathbb{E}\{\varphi(A_D)\} \Big|.$$

A specific case of this occurs when f is takes values in $\mathbb{R}^1$, in which case not only can we show that, for large u,

$$\Big| \mathbb{P}\Big\{ \sup_{t \in M} f(t) \geq u \Big\} \; - \; \mathbb{E}\big\{\varphi\big(A_{[u,\infty)}\big)\big\} \Big|$$

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[3]Still assuming, however, zero mean and constant variance.

is small, but we can provide an upper bound to it which is both sharp and explicit.

Given that the second term here is known from (0.0.1), what this inequility gives is an excellent approximation to Gaussian excursion probabilities in a very wide setting, something that has long been a holy grail of Gaussian process theory.

In the generality in which we shall be working, the bound is determined by geometric properties of the manifold M with the induced Riemmanian metric mentioned above. Furthermore, unlike the handwaving argument described above for the simple one-dimensional case, the new tools provide a fully rigorous proof.

So now you know what awaits you at the end of the path through this book. However, traversing the path has value in itself. Wandering, as it does, through the fields of both Probability and Geometry, it is a path that we imagine not too many of you will have passed over before. We hope that you will enjoy the scenery along the way as much as we have enjoyed describing it. (We also hope, for your sake, that it will be easier and faster in the reading than it was in the writing.)

We are left now with two tasks: Advising how to best read this book, and with acknowledgements.

READING: TO BE ADDED LAST WHEN ALL CHAPTERS ARE WRITTEN.

Both RJA and JET owe debts of gratitude to KW, and we had better acknowledge them now, since we can hardly do it in the Preface of *ARF*.

Beyond our personal debts to KW, not least for getting the two of together, the subject matter of this book also owes him an enormous debt of gratitude. It was during his various extensions and applications of the material of *GRF* that the passage between the old Euclidean theory and its newer manifold version began to take shape. Without his tenacious refusal to leave (applied) problems because the theory (geometry) seemed too hard, the foundations upon which our Part III is based would never have been laid.

Back to the personal level, we also owe debts of gratitude to numerous students at the Technion, UC Santa Barbara, and Stanford who sat through courses as we put this volume together. Their enthusiasm, patience, and refusal to take 'it is easy to see that' for an answer when it was not all that easy to see things, not to mention all the typos and errors that they found, has helped iron a lot of wrinkles out of the final product.

MORE ACKNOWLEDGEMENTS. If you would like to be included here, send us comments and/or corrections!

Finally, don't forget, after you finishing reading this book, to run to library for a copy of *ARF*, to see what all of this theory is *really* good for.

Until such time as *ARF* appears in print, preliminary versions will be available on our home pages, which is where we shall also keep a list of typos and/or corrections for this book after it is published.

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Contents

1
PART I: GAUSSIAN PROCESSES

If you have not yet read the Preface, then please do so now.

Since you have read the Preface, you already know a number of important things about this book, including the fact that Part I is about Gaussian random fields.

The centrality of Gaussian fields to this book is due to two basic factors:

- Gaussian processes have a rich, detailed and very well understood general theory, which makes them beloved by theoreticians.

- In applications of random field theory, as in applications of almost any theory, it is important to have specific, explicit formulae that allow one to predict, to compare theory with experiment, etc. As we shall see in Part III, it will be only for Gaussian (and related, cf. Section 1.4.6 and Chapter 14) fields that it is possible to derive such formulae in the setting of excursion sets.

The main reason behind both these facts is the convenient analytic form of the multivariate Gaussian density, and the related properties of Gaussian fields. This is what Part I is about.

There are five main collections of basic results which will be of interest to us. Rather interestingly, although later in the book we shall be interested in Gaussian fields defined over various types of manifolds, the basic theory of Gaussian fields is actually independent of the specific geometric structure of the parameter space. Indeed, after decades of polishing, even proofs gain little in the way of simplification by restricting to special cases even as simple as $\mathbb{R}$. Thus, at least for a while, we can and shall work in as wide as

possible generality, working with fields defined only on topological spaces to which we shall assign a natural (pseudo) metric induced by the covariance structure of the field.

The first set of results that we require, along with related information, form Chapter 1 and are encapsulated in the various versions in Theorems 1.3.3 and 1.3.5 and their Corollaries. These give a sufficient condition, in terms of metric entropy, ensuring the sample path boundedness and continuity of a Gaussian field along with providing information about moduli of continuity. While this entropy condition is also necessary for *stationary* fields, this is not the case in general, and so for completeness we look briefly at the majorising measure version of this theory in Section 1.5. However, it will be a rare reader of *this* book who will ever need the more general theory.

To put the seemingly abstract entropy conditions into focus, these results will be followed by a Section with a goodly number of extremely varied examples. Despite the fact that these cover only the tip of a very large iceberg, their diversity shows the power of the abstract approach, in that all can be treated via the general theory without further probabilistic arguments. The reader who is not interested in the general Gaussian theory, and cares mainly about the geometry of fields on $\mathbb{R}^N$ or on smooth manifolds, need only read Sections 1.4.1 and 1.4.2 on continuity and differentiability in this scenario, along with the early parts of Section 1.4.3, needed for understanding the spectral representation of Chapter 5.

Chapter 2 contains the Borell-TIS (Borell-Tsirelson-Ibragimov-Sudakov) inequality and Slepian inequalities (along with some of their relatives). The Borell-TIS inequality gives a universal bound for the *excursion probability*

$$(1.0.1) \qquad \qquad \mathbb{P}\left\{\sup_{t \in T} f(t) \geq u\right\},$$

$u > 0$, for *any* centered, continuous Gaussian field. As such, it is a truly basic tool of Gaussian processes, somewhat akin to Chebychev's inequality in Statistics or maximal inequalities in Martingale Theory. Slepian's inequality and its relatives are just as important and basic, and allow one to use relationships between covariance functions of Gaussian fields to compare excursion probabilities and expectations of suprema.

The main result of Chapter 3 is Theorem 3.1.1, which gives an expansion for a Gaussian field in terms of deterministic eigenfunctions with independent $N(0,1)$ coefficients. A special case of this expansion is the Karhunen-Loève expansion of Section 3.2, with which we expect many readers will already be familiar. Together with the spectral representations of Chapter 5, they make up what are probably the most important tools in the Gaussian modeller's box of tricks. However, these expansions are also an extremely important theoretical tool, whose development has far reaching consequences.

Chapter 4 serves as a basic introduction to what is also one of the central topics of Part III, the computation of the excursion probabilities of (1.0.1) for a zero mean Gaussian field and a general parameter space T. In Part III we shall develop highly detailed expansions of the form

$$\mathbb{P}\left\{\sup_{t \in T} f(t) \geq u\right\} = u^{\alpha} e^{-u^2/2\sigma_T^2} \sum_{j=0}^{n} C_j u^{-j} + \text{error},$$

for large u, appropriate parameters α, σ_T^2, n and C_j that depend on both f and T. However, to do this, we shall have to place specific assumptions on the parameter space T, in particular assuming that it be a piecewise smooth manifold[1]. Without these assumptions, the best that one can do is to identify α, σ_T^2 and, ocassionally C_0, and this is what Chapter 4 will do. Furthermore, so as to keep the treatment down to a reasonable length, we shall generally concentrate only on upper bounds, rather than expansions, for Gaussian excursion probabilities[2].

Chapter 5, the last of Part I, is somewhat different to the others in that it is not really about Gaussian processes, but about stationarity and isotropy in general. The main reason for the generality is that limiting oneself to the Gaussian scenario gains us so little that it is not worthwhile doing so. The results here, however, will be crucial for many of the detailed calculations of Part III.

Finally, we repeat what was already said in the Preface, that there is nothing new in Part I beyond perhaps the way some things are presented, and that as a treatment of the basic theory of Gaussian fields it is *not* meant to be exhaustive. There are now many books covering various aspects of this theory, including those by Bogachev [29], Dudley [53], Fernique [64], Hida and Hitsuda [73], Janson [81], Ledoux and Talagrand [92], Lifshits [98] and Piterbarg [117]. In terms of what will be important to us, [53] and [92] stand out from the pack, perhaps augmented with Talagrand's review [145]. Finally, while not as exhaustive[3] as the others, you might find RJA's lecture notes [4], augmented with the corrections in Section 2.1 below, a user-friendly introduction to the subject.

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[1]For those of you who are already comfortable with the theory of stratified manifolds, our 'piecewise smooth manifolds' are Whitney stratified manifold with convex support cones.

[2]There is also a well developed theory of a Poisson limit nature for probabilities of the form $\mathbb{P}\left\{\sup_{t \in T_u} f(t) \geq \eta(x,u)\right\}$, for which $\eta(x,u)$ and the size of T_u grow (to infinity) with u. In this case, one searches for growth rates that give a limit dependent on x, but not u. You can find more about this in Aldous [11], which places the Gaussian theory within a much wider framework of limit results, or in Leadbetter, Lindgren and Rootzén [90] and Piterbarg [117] which give more detailed and more rigoruous accounts for the Gaussian and Gaussian related situation.

[3]Nor, perhaps, as exhausting.

1
Gaussian fields

1.1 Random fields

As you read in the Preface, for us a *random field* is simply a stochastic process, usually taking values in a Euclidean space, and defined over a parameter space of dimensionality at least one. Although we shall be rather loose about exchanging the terms 'field' and 'process', in general we shall use 'field' when the geometric structure of the parameter space is important to us, and shall use 'process' otherwise. 'Random' and 'stochastic' are, of course, completely synonymous.

Here is a formal defintion:

Definition 1.1.1 *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a complete probability space and* T^1 *a topological space. Then a measurable mapping* $f : \Omega \to \mathbb{R}^T$ *is called a real valued random field[2]. Measurable mappings from* Ω *to* $(\mathbb{R}^T)^d$, $d > 1$, *are*

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[1]The use of T comes from the prehistory of Gaussian processes, and probably stands for 'time'. While the whole point of this book is to get away from the totally ordered structure of $\mathbb{R}$, the notation is too deeply entombed in the collective psyche of probabilists to change it now. Later on, however, when we move to manifolds as parameter spaces, we shall emphasise this by replacing T by M. Nevertheless, points in M will still be denoted by t. We hereby make the appropriate apologies to geometers.

[2]More on notation: While we shall follow the standard convention of denoting random variables by upper case Latin characters, we shall use lower case to denote random functions. The reason for this will be become clear in Parts II and III, where we shall need the former for tangent vectors.

called vector valued random fields. If $T \subset \mathbb{R}^N$, we call f an (N, d) random field, and if $d = 1$ simply an N-dimensional random field.

We shall generally not distinguish between

$$f_t \equiv f(t) \equiv f(t, \omega) \equiv (f(\omega))(t),$$

etc., unless there is some special need to do so. Throughout, we shall demand that all random fields are separable [47], which implies conditions on both T and f.

Definition 1.1.2 *An $\mathbb{R}^d$-valued random field f, on a topological space T, is called separable if there exists a countable dense subset $D \subset T$ and a fixed event N with $\mathbb{P}\{N\} = 0$ such that, for any closed $B \subset \mathbb{R}^d$ and open $I \subset T$,*

$$\{\omega : f(t, \omega) \in B, \ \forall t \in I\} \ \Delta \ \{\omega : f(t, \omega) \in B, \ \forall t \in I \cap D\} \ \subset \ N.$$

Here Δ denotes the usual symmetric difference operator

$$(1.1.1) \qquad\qquad A \Delta B = (A \cap B^c) \cup (A^c \cap B),$$

where A^c is the complement of A.

This assumption holds throughout the book, and we shall rarely mention it again. It solves many measurability problems of a kind that are simply not important for us. For example, without separability, it is not necessarily the case that the supremum of a random field is a well-defined random variable.

1.2   Gaussian variables and fields

A real-valued random variable X is said to be *Gaussian* (or *normally distributed*) if it has the density function

$$\varphi(x) \ \overset{\Delta}{=} \ \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2}, \qquad x \in \mathbb{R},$$

for some $m \in \mathbb{R}$ and $\sigma > 0$. It is elementary calculus that the mean of X is m and the variance σ^2, and that the characteristic function is given by

$$\phi(\theta) \ = \ \mathbb{E}\left\{e^{i\theta X}\right\} \ = \ e^{i\theta m - \theta^2/2\sigma^2}.$$

We abbreviate this by writing $X \sim N(m, \sigma^2)$. The case $m = 0$, $\sigma^2 = 1$ is rather special and in this situation we say that X has a *standard* normal distribution. In general, if a random variable or process has zero mean we call it *centered*.

Since the indefinite integral of φ is not a simple function, we also need notation (Φ) for the distribution function and (Ψ) for the tail probability function of a standard normal variable:

$$(1.2.1) \qquad \Phi(x) \triangleq 1 - \Psi(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} \, du.$$

While Φ and Ψ may not be explicit, there are simple, and rather important, bounds which hold for every $x > 0$ and become sharp very quickly as x grows. In particular, in terms of Ψ we have[3]

$$(1.2.2) \qquad \left(\frac{1}{x} - \frac{1}{x^3}\right) \varphi(x) \;<\; \Psi(x) \;<\; \frac{1}{x}\varphi(x),$$

An $\mathbb{R}^d$-valued random variable X is said to be *multivariate Gaussian* if, for every $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$, the real valued variable $\langle \alpha, X' \rangle = \sum_{i=1}^{d} \alpha_i X_i$ is Gaussian[4]. In this case there exists a mean vector $m \in \mathbb{R}^d$ with $m_j = \mathbb{E}\{X_j\}$ and a non-negative definite[5] $d \times d$ covariance matrix C, with elements $c_{ij} = \mathbb{E}\{(X_i - m_i)(X_j - m_j)\}$, such that the probability density of X is given by

$$(1.2.3) \qquad \varphi(x) \;=\; \frac{1}{(2\pi)^{d/2}|C|^{1/2}} e^{-\frac{1}{2}(x-m)C^{-1}(x-m)'},$$

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[3] The upper bound in (1.2.2) follows from the observation that

$$\int_{x}^{-\infty} e^{-u^2/2} \, du \;\leq\; \int_{x}^{-\infty} \frac{u}{x} e^{-u^2/2} \, du \;=\; x^{-1} e^{-x^2/2}$$

after integration by parts. For the lower bound, make the substitution $u \to x + v/x$ to note

$$
\begin{aligned}
\int_{x}^{-\infty} e^{-u^2/2} \, du \;&=\; \int_{0}^{\infty} e^{-(x^2 + 2v + v^2/x^2)/2} x^{-1} \, dv \\
&=\; x^{-1} e^{-x^2/2} \int_{0}^{\infty} e^{-(2v + v^2/x^2)/2} \, dv \\
&=\; x^{-1} e^{-x^2/2} \int_{0}^{\infty} e^{-v}(1 - v^2/(2x^2)) \, dv,
\end{aligned}
$$

on using the fact that $e^{-z} > 1 - z$ for all $z \geq 0$. It i snoe trivial to check that the remaining integral is at least $1 - x^{-2}$.

[4] Note: Throughout the book, vectors are taken to be row vectors and a prime indicates transposition. The inner product between x and y in $\mathbb{R}^d$ is denoted by $\langle x, y \rangle$ or, occasionally, by $x \cdot y$.

[5] A $d \times d$ matrix C is called non-negative definite (or positive semi-definite) if $xCx' \geq 0$ for all $x \in \mathbb{R}^d$. A function $C : T \times T \to \mathbb{R}$ is called non-negative definite if the matrices $(C(t_i, t_j))_{i,j=1}^{n}$ are non-negative definite for all $1 \leq n < \infty$ and all $(t_1, \ldots, t_n) \in T^n$.

where $|C| = \det C$ is the determinant[6] of C. Consistently with the one-dimensional case, we write this as $X \sim N(m, C)$, or $X \sim N_d(m, C)$ if we need to emphasise the dimension.

In view of (1.2.3) we have that Gaussian distributions are completely determined by their first and second order moments and that uncorrelated Gaussian variables are independent. Both of these facts will be of crucial importance later on.

While the definitions are fresh, note for later use that it is relatively straightforward to check from (1.2.3) that the characteristic function of a multivariate Gaussian X is given by

$$(1.2.4) \qquad \phi(\theta) \;=\; \mathbb{E}\{e^{i\langle\theta,X\rangle}\} \;=\; e^{i\langle\theta,m\rangle - \frac{1}{2}\theta C \theta'}.$$

where $\theta \in \mathbb{R}^d$.

One consequence of the simple structure of ϕ is the fact that if $\{X_n\}_{n\geq 1}$ is an L^2 convergent[7] sequence of Gaussian vectors, then the limit X must also be Gaussian. Furthermore, if $X_n \sim N(m_n, C_n)$, then

$$(1.2.5) \qquad |m_n - m|^2 \to 0, \quad \text{and} \quad \|C_n - C\|^2 \to 0,$$

as $n \to \infty$, where m and C are the mean and covariance matrix of the limiting Gaussian. The norm on vectors is Euclidean and that on matrices any of the usual. The proofs involve only (1.2.4) and the continuity theorem for convergence of random variables.

One immediate consequence of either (1.2.3) or (1.2.4) is that if A is any $d \times d$ matrix and $X \sim N_d(m, C)$, then

$$(1.2.6) \qquad\qquad AX \;\sim\; N(mA,\, A'CA).$$

A judicious choice of A then allows us to compute conditional distributions as well. If $n < d$ make the partitions

$$
\begin{aligned}
X &= \left(X^1, X^2\right) = \left((X_1, \ldots, X_n),\, (X_{n+1}, \ldots X_d)\right), \\
m &= \left(m^1, m^2\right) = \left((m_1, \ldots, m_n),\, (m_{n+1}, \ldots m_d)\right), \\
C &= \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},
\end{aligned}
$$

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[6]At various palces we shall use the notation $|\ |$ to denote any of 'absolute value', 'Euclidean norm', 'determinant' or 'Lebesgue measure', depending on the argument, in a natural fashion. The notation $\|\ \|$ is used only for either the norm of complex numbers or for special norms, when it usually appears with a subscript.

[7]That is, there exists a random vector X such that $\mathbb{E}\{|X_n - X|^2\} \to 0$ as $n \to \infty$.

where C_{11} is an $n \times n$ matrix. Then each X^i is $N(m^i, C_{ii})$ and the conditional distribution[8] of X^i given X^j is also Gaussian, with mean vector

$$(1.2.7) \qquad m_{i|j} = m^i + C_{ij}C_{jj}^{-1}(X^j - m^j)'$$

and covariance matrix

$$(1.2.8) \qquad C_{i|j} = C_{ii} - C_{ij}C_{jj}^{-1}C_{ji}.$$

We can now define a real valued *Gaussian random field* to be a random field f on a parameter set T for which the (finite dimensional) distributions of $(f_{t_1}, \ldots, f_{t_n})$ are multivariate Gaussian for each $1 \le n < \infty$ and each $(t_1, \ldots, t_n) \in T^n$. The functions $m(t) = \mathbb{E}\{f(t)\}$ and

$$C(s, t) = \mathbb{E}\left\{(f_s - m_s)(f_t - m_t)\right\}$$

are called the *mean* and *covariance functions* of f. Multivariate[9] Gaussian fields taking values in $\mathbb{R}^d$ are fields for which $\langle \alpha, f_t' \rangle$ is a real valued Gaussian field for every $\alpha \in \mathbb{R}^d$.

In fact, one can also go in the other direction as well. Given *any* set T, a function $m : T \to \mathbb{R}$, and a non-negative definite function $C : T \times T \to \mathbb{R}$ there exists[10] a Gaussian process on T with mean function m and covariance function C.

Putting all this together, we have the important principle that, *for a Gaussian process, everything about it is determined by the mean and covariance functions.* The fact that no real structure is required of the parameter space T is what makes Gaussian fields such a useful model for random processes on general spaces. To build an appreciation for this, you may want to jump ahead to Section 1.4 to look at some examples. However, you will get more out of that section if you first bear with us to answer one of the most fundamental questions in the theory of Gaussian processes: When are Gaussian processes (almost surely) bounded and/or continuous?

---

[8]To prove this, take

$$A = \begin{pmatrix} I_n & -C_{12}C_{22} \\ 0 & I_{d-n} \end{pmatrix}$$

and define $Y = (Y^1, Y^2) = AX$. Check using (1.2.6) that Y^1 and $Y^2 \equiv X^2$ are independent and use this to obtain (1.2.7) and (1.2.8) for $i = 1$, $j = 2$.

[9]Similarly, Gaussian fields taking values in a Banach space B are fields for which $\alpha(f_t)$ is a real valued Gaussian field for every α in the topological dual B^* of B. The covariance function is then replaced by a family of operators $C_{st} : B^* \to B$, for which $Cov(\alpha(f_t), \beta(f_s)) = \beta(C_{st}\alpha)$, for $\alpha, \beta \in B^*$.

[10]This is a consequence of the Kolmogorov existence theorem, which, at this level of generality, can be found in Dudley [52]. Such a process is a random variable in $\mathbb{R}^T$ and may have terrible properties, including lack of measurability in t. However, it will *always* exist.

1.3   Boundedness and continuity

The aim of this section is to develop a useful *sufficient* condition for a centered Gaussian field on a parameter space T to be almost surely bounded and/or continuous; i.e. to determine conditions for which

$$\mathbb{P}\{\sup_{t\in T}|f(t)| < \infty\} = 1 \quad \text{or} \quad \mathbb{P}\{\lim_{s\to t}|f(t) - f(s)| = 0, \ \forall t \in T\} = 1.$$

Of course, in order to talk about continuity – i.e. for the notation $s \to t$ above to have some meaning – it is necessary that T have some topology, so we assume that (T, τ) is a metric space, and that continuity is in terms of the τ-topology. Our first step is to show that τ is irrelevant to the question of continuity[11]. This is rather useful, since we shall also soon show that boundedness and continuity are essentially the same problem for Gaussian fields, and formulating the boundedness question requires no topological demands on T.

   To start, define a new metric d on T by

$$(1.3.1) \qquad\qquad d(s,t) \triangleq \left\{ \mathbb{E}\left[(f(s) - f(t))^2 \right] \right\}^{\frac{1}{2}},$$

in a notation that will henceforth remain fixed. Actually, d is only a pseudo-metric, since although it satisfies all the other demands of a metric, $d(s,t) = 0$ does not necessarily[12] imply that $s = t$. Nevertheless, we shall abuse terminology by calling d the *canonical metric* for T and/or f.

   It will be convenient for us to always assume that d is continuous in the τ-topology and we shall indeed do so, since in the environment of f continuity that will interest us, d continuity costs us nothing. To see this, suppose that $\sup_T \mathbb{E}\{f_t^2\} < \infty$, and that f is a.s. continuous. Then

$$
\begin{aligned}
\lim_{s\to t} d^2(s,t) &= \lim_{s\to t} \mathbb{E}\left\{ (f(s) - f(t))^2 \right\} \\
&= \mathbb{E}\left\{ \lim_{s\to t} (f(s) - f(t))^2 \right\} \\
&= 0,
\end{aligned}
$$

the exchange of limit and expectation justified by uniform integrability. In other words, a.s. continuity of f implies the continuity of d.

   Here is the Lemma establishing the irrelevance of τ to the continuity question.

---

[11] However, τ will come back into the picture when we talk about moduli of continuity later in this section.

[12] For a counter-example, think of a periodic process on $\mathbb{R}$, with period p. Then $d(s,t) = 0$ implies no more than $s - t = kp$ for some $k \in \mathbb{Z}$.

Lemma 1.3.1 *Let f be a centered Gaussian process on a compact metric space (T, τ). Then f is a.s. continuous with respect to the τ-topology if, and only if, it is a.s. continuous with respect to the d (pseudo) topology. More precisely, with probability one, for all $t \in T$,*

$$\lim_{s:\,\tau(s,t)\to 0} |f(s) - f(t)| = 0 \iff \lim_{s:\,d(s,t)\to 0} |f(s) - f(t)| = 0$$

Proof. Since d is (always) assumed continuous in the τ topology, it is immediate that if f is d-continuous then it is τ-continuous.

Suppose, therefore, that f is τ-continuous. For $\eta \geq 0$, let

$$A_\eta = \{(s,t) \in T \times T : d(s,t) \leq \eta\}.$$

Since d is continuous, this is a τ-closed subset of $T \times T$. Furthermore, $\bigcap_{\eta > 0} A_\eta = A_0$. Fix $\varepsilon > 0$. Then, by the τ-compactness of T, there is a finite set $B \subset A_0$ (the number of whose elements will in general depend on ε) such that

$$\bigcup_{(s',t')\in B} \left\{ (s,t) \in T \times T : \max\left(\tau(s,s'),\, \tau(t,t') \right) \leq \varepsilon \right\}$$

covers A_η for some $\eta = \eta(\varepsilon) > 0$, with $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$. . That is, whenever $(s,t) \in A_\eta$ there is a $(s',t') \in B$ with $\tau(s,s'),\, \tau(t,t') \leq \varepsilon$. Note that

$$|f_t - f_s| \leq |f_s - f_{s'}| + |f_{s'} - f_{t'}| + |f_{t'} - f_t|.$$

Since $(s',t') \in B \subset A_0$, we have $f(s') = f(t')$ a.s. Thus

$$\sup_{d(s,t)\leq\eta(\varepsilon)} |f_t - f_s| \leq 2 \sup_{\tau(s,t)\leq\varepsilon} |f_t - f_s|,$$

and the τ-continuity of f implies its d-continuity. $\qquad\square$

The astute reader will have noted that in the statement of Lemma 1.3.1 the parameter space T was quietly assumed to be compact, and that this additional assumption was needed in the proof. Indeed, from now on we shall assume that this is always the case, and shall rely on it heavily. Fortunately, however, it is not a serious problem. As far as continuity is concerned, if T is σ-compact[13] then a.s. continuity on its compact subsets immediately implies a.s. continuity over T itself. We shall not go beyond σ-compact spaces in this book. The same is not true for boundedness, nor should it be[14]. However, we shall see that, at least on compact T, boundedness and continuity are equivalent problems. Furthermore, both depend on how 'large'

---

[13]T is σ-compact if it can be represented as the countable union of compact sets.

[14]Think of simple Brownian motion on $\mathbb{R}_+$. While bounded on every finite interval, it is a consequence of the law of the iterated logarithm that it is unbounded on $\mathbb{R}_+$.

is the parameter set T, when size is measured in a metric that comes from the process itself[15]. One way to measure the size of T is via the notion of metric entropy.

Definition 1.3.2 *Let f be a centered Gaussian field on T, and d the canonical metric (1.3.1). Assume that T is d-compact, and write*

$$(1.3.2) \qquad B_d(t, \varepsilon) \triangleq \{s \in T : d(s, t) \leq \varepsilon\}$$

for the d ball centered on $t \in T$ and of radius ε. Let $N(T, d : \varepsilon) \equiv N(\varepsilon)$ denote the smallest number of such balls that cover T, and set

$$(1.3.3) \qquad H(T, d : \varepsilon) \equiv H(\varepsilon) = \ln(N(\varepsilon)).$$

Then N and H are called the (metric) entropy and log-entropy functions for T (or f). We shall refer to any condition or result based on N or H as an entropy condition/result.

Note that since we are assuming that T is d-compact, it follows that $H(\varepsilon) < \infty$ for all $\varepsilon > 0$. The same need not be (nor generally is) true for $\lim_{\varepsilon \to 0} H(\varepsilon)$. Furthermore, note for later use that if we define

$$(1.3.4) \qquad \operatorname{diam}(T) = \sup_{s,t \in T} d(s, t),$$

then $N(\varepsilon) = 1$ and so $H(\varepsilon) = 0$ for $\varepsilon \geq \operatorname{diam}(T)$.

Here then are the main results of this section, all of which will be proven soon.

Theorem 1.3.3 *Let f be a centered Gaussian field on a d-compact T, d the canonical metric, and H the corresponding entropy. Then there exists a universal constant K such that*

$$(1.3.5) \qquad \mathbb{E}\left\{\sup_{t \in T} f_t\right\} \leq K \int_0^{\operatorname{diam}(T)/2} H^{1/2}(\varepsilon)\, d\varepsilon.$$

This result has immediate consequences for continuity. Define the *modulus of continuity* ω_F of a real-valued function F on a metric space (T, τ) as

$$(1.3.6) \qquad \omega_F(\delta) \equiv \omega_{F,\tau}(\delta) \triangleq \sup_{\tau(s,t) \leq \delta} |F(t) - F(s)|, \qquad \delta > 0.$$

The modulus of continuity of f can be thought of as the supremum of the random field $f_{s,t} = f_t - f_s$ over a certain neighbourhood of $T \times T$, in

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[15]A particularly illuminating example of this comes from Brownian noise processes of Section 1.4.3. There we shall see that the *same* process can be continuous, or discontinuous, depending on how we specify its parameter set.

that

(1.3.7)
$$\omega_{f,\tau}(\delta) \;=\; \sup_{\substack{(s,t)\,\in\,T\times T \\ \tau(s,t)\leq\delta}} f(s,t).$$

(Note we can drop the absolute value sign since the supremum here is always non-negative.)

More precisely, write $d^{(2)}$ for the canonical metric of $f_{s,t}$ on $T\times T$. Then

$$
\begin{aligned}
d^{(2)}\bigl((s,t),(s',t')\bigr) \;&=\; \left[\mathbb{E}\left\{\bigl((f_t - f_s) - (f_{t'} - f_{s'})\bigr)^2\right\}\right]^{1/2} \\
&\leq\; 2\max\bigl(d(s,t),\,d(s',t')\bigr),
\end{aligned}
$$

and so
$$N(T\times T,\,d^{(2)}:\delta) \;\leq\; N(T,d:\delta/2).$$

From these observations, it is immediate that Theorem 1.3.3 implies

Corollary 1.3.4 *Under the conditions of Theorem 1.3.3 there exists a universal constant K such that*

(1.3.8)
$$\mathbb{E}\left\{\omega_{f,d}(\delta)\right\} \;\leq\; K\int_0^\delta H^{1/2}(\varepsilon)\,d\varepsilon.$$

Note that this is not quite enough to establish the a.s. continuity of f. Continuity is, however, not far away, since the same construction used to prove Theorem 1.3.3 will also give us the following, which, with the elementary tools we have at hand at the moment[16], neither follows from, nor directly implies, (1.3.8).

Theorem 1.3.5 *Under the conditions of Theorem 1.3.3 there exists a random $\eta \in (0,\infty)$ and a universal constant K such that*

(1.3.9)
$$\omega_{f,d}(\delta) \;\leq\; K\int_0^\delta H^{1/2}(\varepsilon)\,d\varepsilon,$$

for all $\delta < \eta$.

Note that (1.3.9) is expressed in terms of the d modulus of continuity. Translating this to a result for the τ modulus is trivial.

We shall see later that if f is stationary[17] then the convergence of the entropy integral is also necessary for continuity and that continuity and boundedness always occur together (Theorem 1.5.4). Now, however, we

---

[16]See, however Theorem 2.1.3 below, to see what one can do with better tools.

[17]We shall treat stationarity in detail in Chapter 5. For the moment all you need to know is that under stationarity the expectation $\mathbb{E}\{f(t)\}$ is constant while the covariance $\mathbb{E}\{f(t)f(s)\}$ is a function of $s-t$ only.

shall prove Theorems 1.3.3 and 1.3.5 following the approach of Talagrand [145]. The original proof of Theorem 1.3.5 is due to Dudley [48], and, in fact, things have not really changed very much since then. Immediately following the proofs, in Section 1.4, we shall look at examples, to see how entropy arguments work in practice. You may want to skip to the examples before going through the proofs first time around.

We start with the following almost trivial, but important, observations.

Observation 1.3.6 *If f is a separable process on T then $\sup_{t \in T} f_t$ is a well defined (i.e. measurable) random variable.*

Measurability follows directly from Definition 1.1.2 of separability which gave us a countable dense subset $D \subset T$ for which

$$\sup_{t \in T} f_t \;=\; \sup_{t \in D} f_t.$$

The supremum of a countable set of measurable random variables is always measurable.

One can actually manage without separability for the rest of this section, in which case

$$\sup \left\{ \mathbb{E} \left[\sup_{t \in F} f_t \right] \;:\; F \subset T, \; F \text{ finite} \right\}$$

should be taken as the definition of $\mathbb{E}\{\sup_T f_t\}$.

Observation 1.3.7 *If f is a separable process on T and X a centered random variable (not necessarily independent of f), then*

$$\mathbb{E}\left\{ \sup_{t \in T} (f_t + X) \right\} \;=\; \mathbb{E}\left\{ \sup_{t \in T} f_t \right\}.$$

As trite as this observation is, it ceases to be valid if, for example, we investigate $\sup_t |f_t + X|$ rather than $\sup_t (f_t + X)$.

Proof of Theorem 1.3.3 Fix a point $t_o \in T$ and consider $f_t - f_{t_o}$. In view of Observation 1.3.7, we can work with $\mathbb{E}\{\sup_T (f_t - f_{t_o})\}$ rather than $\mathbb{E}\{\sup_T f_t\}$. Furthermore, in view of separability, it will suffice to take these suprema over the countable separating set $D \subset T$. To save on notation, we might therefore just as well assume that T is countable, which we now do.

We shall represent the difference $f_t - f_{t_o}$ via a telescoping sum, in what is called a *chaining* argument, and is in essence an approximation technique. We shall keep track of the accuracy of the approximations via entropy and simple union bounds.

To build the approximations, first fix some $r \geq 2$ and choose the largest $i \in \mathbb{Z}$ such that $\operatorname{diam}(T) \leq r^{-i}$, where the $\operatorname{diam}(T)$ is measured in terms of the canonical metric of f. For $j > i$, take a finite subset Π_j of T such that

$$\sup_{t \in T} \; \inf_{s \in \Pi_j} d(s,t) \;\leq\; r^{-j},$$

(possible by d-compactness) and define a mapping $\pi_j : T \to \Pi_j$ satisfying

(1.3.10) $$\sup_{t \in T} d(t, \pi_j(t)) \leq r^{-j}.$$

For consistency, set $\Pi_i = \{t_o\}$ and $\pi_i(t) = t_o$ for all t. Consistent with the notations of Definition 1.3.2, we can choose Π_j to have no more than $N_j \triangleq N(r^{-j})$ points, and so entropy has now entered into the argument.

The idea of this construction is that the points $\pi_j(t)$ are successive approximations to t, and that as we move along the 'chain' $\pi_j(t)$ we have the decomposition[18]

(1.3.11) $$f_t - f_{t_o} = \sum_{j > i} f_{\pi_j(t)} - f_{\pi_{j-1}(t)}.$$

To start the proof we look for a bound on the tail of the distribution of $\sup_T(f_t - f_{t_o})$, for which we need a little notation (carefully chosen to make the last lines of the proof come out neatly). Define

$$M_j = N_j N_{j-1}, \qquad a_j = 2^{3/2} r^{-j+1} \sqrt{\ln(2^{j-i} M_j)}, \qquad S = \sum_{j > i} a_j.$$

Then M_j is the maximum number of possible pairs $(\pi_j(t), \pi_{j-1}(t))$ as t varies through T and a_j was chosen so as to make later formulae simplify. Recall (1.2.2), which implies

(1.3.12) $$\mathbb{P}\{X \geq u\} \leq e^{-u^2/2\sigma^2},$$

---

[18] We actually need to check two things: that this potentially infinite sum in (1.3.11) is well defined, with probability one, and that it indeed converges to $f_t - f_{t_o}$. The first is straightforward, the second requires a little measure theoretic legerdemain.

By (1.3.12) and (1.3.10), it follows that

$$\mathbb{P}\left\{\left|f_{\pi_j(t)} - f_{\pi_{j-1}(t)}\right| \geq \left(r/\sqrt{2}\right)^{-j}\right\} \leq \exp\left(\frac{-(r/\sqrt{2})^{-2j}}{2(2r^{-j+1})^2}\right)$$
$$= \exp\left(-2^j/8r^2\right),$$

which is emminently summable. By Borel-Cantelli, and recalling that $r \geq 2$, we have that the sum in (1.3.11) converges absolutely, with probability one.

For the second step, fix a $t \in T$ and consider the sum

$$f_t^{(n)} \triangleq f_{t_o} + \sum_{j=i+1}^{n} f_{\pi_j(t)} - f_{\pi_{j-1}(t)}$$
$$\equiv f_{\pi_n(t)}.$$

From the construction of the π_j it is immediate that $\mathbb{E}\{|f(t) - f_t^{(n)}|^2\} \to 0$ as $n \to \infty$, and so the fact that $f_t^{(n)}$ converges almost surely implies that it must also converge to f_t. Since this must be true for any countable subset of points in T, separability gives that it is true throughout T. (cf. Definition 1.1.2.)

for $X \sim N(0, \sigma^2)$ and $u > 0$. Applying (1.3.12) we have, for all $u > 0$,

(1.3.13)

$$\mathbb{P}\left\{\exists\, t \in T : f_{\pi_j(t)} - f_{\pi_{j-1}(t)} > u a_j\right\} \leq M_j \exp\left(\frac{-u^2 a_j^2}{2(2r^{-j+1})^2}\right)$$

and so

$$\mathbb{P}\left\{\sup_{t \in T} f_t - f_{t_o} \geq uS\right\} \leq \sum_{j>i} M_j \exp\left(\frac{-u^2 a_j^2}{2(2r^{-j+1})^2}\right)$$

$$= \sum_{j>i} M_j \left(2^{j-i} M_j\right)^{-u^2}.$$

For $u > 1$ this is at most

$$\sum_{j>i} \left(2^{j-i}\right)^{-u^2} \leq 2^{-u^2} \sum_{j>i} 2^{-(j-i)+1}$$

$$= 2 \cdot 2^{-u^2}.$$

The basic relationship that, for non-negative random variables X,

$$\mathbb{E}\{X\} = \int_0^\infty \mathbb{P}\{X \geq u\}\, du,$$

together with the observation that $\sup_{t \in T}(f_t - f_{t_o}) \geq 0$ since $t_o \in T$, immediately yields

(1.3.14) $$\mathbb{E}\left\{\sup_{t \in T} f_t\right\} \leq KS,$$

with $K = 1 + 2\int_0^\infty 2^{-u^2}\, du$. Thus, all that remains to complete the proof is to compute S.

Using the definition of S, along with the elementary inequality that $\sqrt{ab} \leq \sqrt{a} + \sqrt{b}$, gives

$$S \leq 2^{3/2} \sum_{j>i} r^{-j+1} \left(\sqrt{j-i}\,\sqrt{\ln 2} + \sqrt{\ln N_j} + \sqrt{\ln N_{j-1}}\right)$$

$$\leq K \left(r^{-i} + \sum_{j \geq i} r^{-j} \sqrt{\ln N_j}\right)$$

$$\leq K \sum_{j \geq i} r^{-j} \sqrt{\ln N_j},$$

where K is a constant that may change from line to line, but depends only on r. The last inequality follows from absorbing the lone term of r^{-i} into

the second term of the sum, possible since the very definition of i implies that $N_{i+1} \geq 2$, and changing the multiplicative constant K accordingly.

Recalling now the definition of N_j as $N(r^{-j})$, we have that

$$\varepsilon \leq r^{-j} \;\Rightarrow\; N(\varepsilon) \geq N_j.$$

Thus

$$\int_0^{r^{-i}} \sqrt{\ln N(\varepsilon)}\, d\varepsilon \;\geq\; \sum_{j \geq i} \left(r^{-j} - r^{-j-1}\right)\sqrt{\ln N_j}$$
$$= \; K \sum_{j \geq i} r^{-j}\sqrt{\ln N_j}.$$

Putting this together with the bound on S and substituting into (1.3.14) gives

$$\mathbb{E}\left\{\sup_{t \in T} f_t\right\} \;\leq\; K \int_0^{r^{-i}} H^{1/2}(\varepsilon)\, d\varepsilon.$$

Finally, note that $N(\varepsilon) = 1$ (and so $H(\varepsilon) = 0$) for $\varepsilon \geq 2r^{-i} \geq \mathrm{diam}(T)$, to establish (1.3.5) and so complete the proof. $\qquad\square$

Proof of Theorem 1.3.5 The proof starts with the same construction as in the proof of Theorem 1.3.3. Note that from the same principles behind the telescoping sum (1.3.11) defining $f_t - f_{t_o}$ we have that for all $s, t \in T$ and $J > i$,

(1.3.15)
$$f_t - f_s \;=\; f_{\pi_J(t)} - f_{\pi_J(s)}$$
$$+ \sum_{j > J} \left[f_{\pi_j(t)} - f_{\pi_{j-1}(t)}\right] - \sum_{j > J} \left[f_{\pi_j(s)} - f_{\pi_{j-1}(s)}\right].$$

From (1.3.12) we have that for all $u > 0$,

$$\mathbb{P}\left\{\left|f_{\pi_j(t)} - f_{\pi_j(s)}\right| \geq u\, d(\pi_j(t), \pi_j(s))\right\} \;\leq\; e^{-u^2/2}.$$

Arguing as we did to obtain (1.3.13), and the line or two following, we now see that

$$\mathbb{P}\left\{\exists\, s, t \in T : \left|f_{\pi_j(t)} - f_{\pi_j(s)}\right| \geq \sqrt{2}\, d(\pi_j(t), \pi_j(s)) \sqrt{\ln(2^{j-i} N_j^2)}\right\}$$
$$\leq\; 2^{i-j}.$$

Since this is a summable series, Borel-Cantelli gives the existence of a random $j_o > i$ for which, with probability one,

$$j > j_o \;\Rightarrow\; \left|f_{\pi_j(t)} - f_{\pi_j(s)}\right| \leq \sqrt{2}\, d(\pi_j(t), \pi_j(s))\sqrt{\ln(2^{j-i} N_j^2)}$$

for all $s, t \in T$.

Essentially the same argument also gives that

$$j > j_o \;\Rightarrow\; \left| f_{\pi_j(t)} - f_{\pi_{j-1}(t)} \right| \;\leq\; \sqrt{2}\, d(\pi_j(t), \pi_{j-1}(t)) \sqrt{\ln(2^{j-i} M_j)}$$

for all $t \in T$.

Putting these into (1.3.15) gives that

$$\begin{aligned}
|f_t - f_s| \;\leq\;\; & K d(\pi_{j_o}(t), \pi_{j_o}(s)) \sqrt{\ln(2^{j_o - i} N_{j_o}^2)} \\
& + K \sum_{j > j_o} d(\pi_j(t), \pi_{j-1}(t)) \sqrt{\ln(2^{j-i} M_j)}.
\end{aligned}$$

Note that $d(\pi_j(t), \pi_{j-1}(t)) \leq 2r^{-j+1}$ and

$$\begin{aligned}
d(\pi_{j_o}(t), \pi_{j_o}(s)) \;\;&\leq\;\; d(s, t) + 2r^{-j_o} \\
&\leq\;\; 3r^{-j_o}
\end{aligned}$$

if we take $d(s,t) \leq \eta \overset{\Delta}{=} r^{-j_o}$. The above sums can be turned into integrals just as we did at the end of the previous proof, which leads to (1.3.9) and so completes the argument. □

Before leaving to look at some examples, you should note one rather crucial fact: The *only* Gaussian ingredient in the preceeding two proofs was the basic inequality (1.3.12) giving $\exp(-u^2/2)$ as a tail bound for a single $N(0,1)$ random variable. The remainder of the proof used little more than the union bound on probabilities and some clever juggling. Furthermore, it does not take a lot to of effort to see that the square root in the entropy integrals such as (1.3.5) is related to 'inverting' the square in $\exp(-u^2/2)$ while the logarithm comes from 'inverting' the exponential. If this makes you feel that there is a far more general, non-Gaussian theory behind all this, and that it is not going to be very different to the Gaussian one, then you are right. A brief explanation of how it works is in Section 1.4.6.

1.4   Examples

Our choice of examples was determined primarily by the type of random fields that we need later in the book. Thus we shall start by looking at random fields on $\mathbb{R}^N$, followed by looking at how to turn questions of differentiability into questions of continuity. Following these, we shall look at the Brownian family of processes, needed for setting up the spectral representation theory of Chapter 5. As mentioned above, this family is also very instructive in understanding *why* the finiteness of entropy integrals is a natural condition for coninuity. The section on generalised fields is useful

as a recipe for providing examples, and the section on set indexed processes is there for fun. Neither of these will appear elsewhere in the book. The non-Gaussian examples of Section 1.4.6 are worth knowing about already at this stage.

1.4.1   Fields on $\mathbb{R}^N$

Returning to Euclidean space after the abstraction of entropy on general metric spaces, it is natural to expect that conditions for continuity and boundedness will become so simple to both state and prove that there was really no need to introduce such abstruse general concepts.

This expectation is both true and false. It turns out that avoiding the notion of entropy does not make it any easier to establish continuity theorems, and, indeed, reliance on the specific geometry of the parameter space often confounds the basic issues. On the other hand, the following important result is easy to state without specifically referring to any abstract notions. To formulate it, let f_t be a centered Gaussian process on a compact $T \subset \mathbb{R}^N$ and define

$$(1.4.1) \qquad p^2(u) \; = \; \sup_{|s-t| \leq u} \mathbb{E} \left\{ |f_s - f_t|^2 \right\},$$

where $|\cdot|$ is the usual Euclidean metric. If f is stationary, then

$$(1.4.2) \qquad p^2(u) \; = \; 2 \sup_{|t| \leq u} [C(0) - C(t)].$$

Theorem 1.4.1 *If, for some $\delta > 0$, either*

$$(1.4.3) \quad \int_0^\delta (-\ln u)^{\frac{1}{2}} \, dp(u) \; < \; \infty \quad \text{or} \quad \int_\delta^\infty p \left(e^{-u^2} \right) \, du \; < \; \infty,$$

then f is continuous and bounded on T with probability one. A sufficient condition for either integral in (1.4.3) to be finite is that, for some $0 < C < \infty$ and $\alpha, \eta > 0$,

$$(1.4.4) \qquad \mathbb{E} \left\{ |f_s - f_t|^2 \right\} \; \leq \; \frac{C}{|\log |s-t||^{1+\alpha}},$$

for all s, t with $|s - t| < \eta$. Furthermore, there exists a constant K, dependent only on the dimension N, and a random $\delta_o > 0$ such that, for all $\delta < \delta_o$,

$$(1.4.5) \qquad \omega_f(\delta) \; \leq \; K \int_0^{p(\delta)} (-\ln u)^{\frac{1}{2}} \, dp(u),$$

where the modulus of continuity ω_f is taken with respect to the Euclidean metric. A similar bound, in the spirit of (1.3.8), holds for $\mathbb{E}\{\omega_f(\delta)\}$.

Proof. Note first that since $p(u)$ is obviously non-decreasing in u, the Riemann-Stieljes integral (1.4.3) is well defined. The proof that both integrals in (1.4.3) converge and diverge together and that the convergence of both is assured by (1.4.4) is simple calculus and left to the reader. Of more significance is relating these integrals to the entropy integrals of Theorems 1.3.3 and 1.3.5 and Corollary 1.3.4. Indeed, all the claims of the theorem will follow from these results if we show that

$$(1.4.6) \qquad \int_0^\delta H^{1/2}(\varepsilon)\,d\varepsilon \;\leq\; K \int_0^{p(\delta)} (-\ln u)^{\frac{1}{2}}\,dp(u)$$

for small enough δ.

Since T is compact, we can enclose it in a N-cube C_L of side length $L = \max_{i=1,\dots,N} \sup_{s,t \in T} |t_i - s_i|$. Since p is non-decreasing, there is no problem in defining

$$p^{-1}(\varepsilon) \;\stackrel{\Delta}{=}\; \sup\{u : p(u) \leq \varepsilon\}.$$

Now note that, for each $\varepsilon > 0$, the cube C_L, and so T, can be covered by $[1 + L\sqrt{N}/(2p^{-1}(\varepsilon))]^N$ (Euclidean) N-balls, each of which has radius no more than ε in the canonical metric d. Thus,

$$
\begin{aligned}
\int_0^\delta H^{1/2}(\varepsilon)\,d\varepsilon \;&\leq\; \sqrt{N} \int_0^\delta \left(\ln(1 + L\sqrt{N}/(2p^{-1}(\varepsilon))) \right)^{\frac{1}{2}} d\varepsilon \\
&=\; \sqrt{N} \int_0^{p(\delta)} \left(\ln(1 + L\sqrt{N}/2u) \right)^{\frac{1}{2}} dp(u) \\
&\leq\; 2\sqrt{N} \int_0^{p(\delta)} (-\ln u)^{\frac{1}{2}}\,dp(u)
\end{aligned}
$$

for small enough δ. This completes the proof. $\qquad\square$

The various sufficient conditions for continuity of Theorem 1.4.1 are quite sharp, but not necessary. There are two stages at which necessity is lost. One is simply that entropy conditions, in general, need not be necessary in the non-stationary case. The second is that something is lost in the passage from entropy to the conditions on p.

As an example of the latter, take the continuous, centered Gaussian process f on $\mathbb{R}$ with covariance function $C(t) = \exp(-t^2/2)$ and, for $t > 0$ define $g(t) = f(\ln(1/t))$. It is easy to check that f is a.s. continuous, as is g, since it is obtained from f via a continuous transformation. It is also easy to check that f and g have identical entropy functions. However, while the p function for f satisfies (1.4.3), this is not true for that of g.

Despite these drawbacks, the results of Theorem 1.4.1 are, from a practical point of view, reasonably definitive. For example, we shall see below (Corollary 1.5.5) that if f is stationary and

$$(1.4.7) \qquad \frac{K_1}{(-\ln|t|)^{1+\alpha_1}} \;\leq\; C(0) - C(t) \;\leq\; \frac{K_2}{(-\ln|t|)^{1+\alpha_2}},$$

for $|t|$ small enough, then f will be sample path continuous if $\alpha_2 > 0$ and discontinuous if $\alpha_1 < 0$.

Before leaving the Euclidean case, it is also instructive to see how the above conditions on covariance functions translate to conditions on the spectral measure ν of (5.4.1) when f is stationary. Although this involves using concepts yet to be defined, this is a natural place to describe the result. If you are familiar with spectral theory, the following will be easy to follow. If not, then you can return here after reading Chapter 5.

The translation is via standard Tauberian theory, which translates the behaviour of C at the origin to that of ν at infinity. (cf. for example, [26]). A typical result is the following, again in the centered, Gaussian case: If the integral

$$\int_0^\infty \big(\log(1 + \lambda) \big)^{1+\alpha} \, \nu(d\lambda)$$

converges for some $\alpha > 0$ then f is a.s. continuous, while if it diverges for some $\alpha < 0$ then f is a.s. discontinuous.

In other words, it is the 'high frequency oscillations' in the spectral representation that are controlling the continuity/discontinuity dichotomy. This is hardly surprising. What is perhaps somewhat more surprising, since we have seen that for Gaussian processes continuity and boundedness come together, is that it is these same oscillations that are controlling boundedness as well.

1.4.2 Differentiability on $\mathbb{R}^N$

We shall stay with Euclidean $T \subset \mathbb{R}^N$ for the moment, and look at the question of a.s. differentiability of centered, Gaussian f. We have already considered the issue of L^2 differentiability in Section 5.4. There, for $(t, t') \in T \times \times^k \mathbb{R}^N$, we defined the k-th order derivative in the direction t' as the L^2 limit

$$D_{L^2}^k f(t, t') = \lim_{h \to 0} F(t, ht'),$$

where $F(t, t')$ is the symmetrized difference

$$F(t, t') \;=\; \frac{1}{\prod_{i=1}^k |t_i'|} \sum_{s \in \{0,1\}^k} (-1)^{k - \sum_{i=1}^k s_i} f\left(t + \sum_{i=1}^k s_i t_i' \right).$$

We also noted there that when $D_{L^2}^k f$ exists, it is a Gaussian field on $T \times \otimes^k \mathbb{R}^n$, since L^2 limits of Gaussian variables are always Gaussian.

While L^2 existence was fine for what we needed in Chapter 1, later on we shall need to know when a.s. derivatives exist, and whether or not they are a.s. continuous. The general structure that we have so far built actually makes this a very simple question to answer.

To see why, endow the space $\mathbb{R}^N \times \otimes^k \mathbb{R}^N$ with the norm

$$\|(s, s')\|_{n,k} \triangleq |s| + \|s'\|_{\otimes^k \mathbb{R}^N} = |s| + \left(\sum_{i=1}^{k} |s_i'|^2 \right)^{1/2},$$

and write $B_{n,k}(y, h)$ for the h-ball centered at $y = (t, t')$ in the metric induced by $\|\cdot\|_{n,k}$. Furthermore, write

$$T_{k,\rho} \triangleq T \times \{t' : \|t'\|_{\otimes^k \mathbb{R}^N} \in (1-\rho, 1+\rho)\}$$

for the product of T with the ρ-tube around the unit sphere in $\otimes^k \mathbb{R}^N$. This is enough to allow us to formulate

Theorem 1.4.2 *Suppose f is a centered Gaussian random field on an open $T \in \mathbb{R}^N$, possessing k-th order derivatives in the L^2 sense. Suppose, furthermore, that there exists $0 < K < \infty$, and $\rho, \delta, h_0 > 0$ such that for $0 < \eta_1, \eta_2, h < h_0$,*

$$(1.4.8) \quad \mathbb{E} \left\{ [F(t, \eta_1 t') - F(s, \eta_2 s')]^2 \right\}$$
$$< K \left(-\ln \left(\|(t, t') - (s, s')\|_{n,k} + |\eta_1 - \eta_2| \right) \right)^{-(1+\delta)},$$

for all
$$((t, t'), (s, s')) \in T_{k,\rho} \times T_{k,\rho} : (s, s') \in B_{n,k}((t, t'), h).$$

Then, with probability one, f is k times continuously differentiable; viz. $f \in C^k(T)$.

Proof. Recalling that we have assumed the existence of L^2 derivatives, we can define the Gaussian field

$$\widehat{F}(t, t', \eta) = \begin{cases} F(t, \eta t') & \eta \neq 0, \\ D_{L^2}^k f(t, t') & \eta = 0, \end{cases}$$

on $\widehat{T} \triangleq T_{k,\rho} \times (-h, h)$, an open subset of the finite dimensional vector space $\mathbb{R}^N \times \otimes^k \mathbb{R}^N \times \mathbb{R}$ with norm

$$\|(t, t', \eta)\|_{n,k,1} = \|(t, t')\|_{n,k} + |\eta|.$$

Whether or not $f \in C^k(T)$ is clearly the same issue as whether or not $\widehat{F} \in C(\widehat{T})$, with the issue of the continuity of f really only being on the hyperplane where $\eta = 0$. But this puts us back into the setting of Theorem 1.4.1, and it is easy to check that condition (1.4.4) there translates to (1.4.8) in the current scenario. □

1.4.3   The Brownian family of processes

Now recall the Brownian noise processes from Section 1.4.3. There we saw that the *same* process could be continuous, or discontinuous, depending on how we specified its parameter set. We shall now see that the difference between the parameter sets had nothing to do with their geometrical properties, but rather their 'size' as measured in terms of d, via the tool of metric entropy.

Perhaps the most basic of all random fields is a collection of independent Gaussian random variables. While it is simple to construct such random fields for finite and even countable parameter sets, deep technical difficulties obstruct the construction for uncountable parameter sets. The path that we shall take around these difficulties involves the introduction of random measures which, at least in the Gaussian case, are straightforward to formulate.

Let $(T, \mathcal{T}, \nu)$ be a σ-finite measure space and denote by $\mathcal{T}_\nu$ the collection of sets of $\mathcal{T}$ of finite ν measure. A *Gaussian noise* based on ν, or 'Gaussian ν-noise' is a random field $W : \mathcal{T}_\nu \to \mathbb{R}$ such that, for all $A, B \in \mathcal{T}_\nu$,

(1.4.9) $W(A) \sim N(0, \nu(A))$.

(1.4.10) $A \cap B = \emptyset \Rightarrow W(A \cup B) = W(A) + W(B)$ a.s.

(1.4.11) $A \cap B = \emptyset \Rightarrow W(A)$ and $W(B)$ are independent.

Property (1.4.10) encourages one to think of W[19] as a random (signed) measure, although it is not generally σ-finite. We describe (1.4.11) by saying that W has *independent increments*

Theorem 1.4.3 *If $(T, \mathcal{T}, \nu)$ is a measure space, then there exists a real valued Gaussian noise, defined for all $A \in \mathcal{T}_\nu$, satisfying* (1.4.9)–(1.4.11).

Proof. In view of the closing remarks of the preceeding Section all we need do is provide an appropriate covariance function on $\mathcal{T}_\nu \times \mathcal{T}_\nu$. Try

(1.4.12) $C_\nu(A, B) \triangleq \nu(A \cap B)$.

This is non-negative definite, since for any $A_i \in \mathcal{T}_\nu$ and $\alpha_i \in \mathbb{R}$

$$
\begin{aligned}
\sum_{i,j} \alpha_i C_\nu(A_i, A_j) \alpha_j &= \sum_{i,j} \alpha_i \alpha_j \int_T I_{A_i}(x) I_{A_j}(x) \, \nu(dx) \\
&= \int_T \Big(\sum_i \alpha_i I_{A_i}(x) \Big)^2 \nu(dx) \\
&\geq 0.
\end{aligned}
$$

---

[19]While the notation 'W' is inconsistent our determination to use lower case Latin characters for random functions, we retain it as a tribute to Norbert Wiener, who is the mathematical father of these processes.

Consequently there exists a centered Gaussian random field on $\mathcal{T}_\nu$ with covariance C_ν. It is immediate that this field satisfies (1.4.9)–(1.4.11) and so we are done.   □

A particularly simple example of a Gaussian noise arises when $T = \mathbb{Z}$. Take ν a discrete measure of the form $\nu(A) = \sum_k a_k \delta_k(A)$, where the a_k are non-negative constants and δ_k is the Dirac measure on $\{k\}$. For $\mathcal{T}$ take all subsets of $\mathbb{Z}$. In this case, the Gaussian noise can actually be defined on points $t \in T$ and extended to a signed measure on sets in $\mathcal{T}_\nu$ by additivity. What we get is a collection $\{W_k\}_{k \in \mathbb{Z}}$ of independent, centered, Gaussian variables, with $\mathbb{E}\{W_k^2\} = a_k$. If the a_k are all equal, this is classical Gaussian 'white' noise on the integers.

A more interesting case is $T = \mathbb{R}^N$, $\mathcal{T} = \mathbb{B}^N$, the Borel σ-algebra on $\mathbb{R}^N$ and $\nu(\cdot) = |\cdot|$, Lebesgue measure. This gives us a Gaussian white noise defined on the Borel subsets of $\mathbb{R}^N$ of finite Lebesgue measure, which is also a field with orthogonal increments in the sense of (1.4.11). It is generally called the *set indexed Brownian sheet*. It is not possible, in this case, to assign non-trivial values to given points $t \in \mathbb{R}^N$, as was the case in the previous example.

It also turns out that working with the Brownian sheet on all of $\mathbb{B}^N$ is not really the right thing to do, since, as will follow from Theorem 1.4.5, this process is rather badly behaved. Restricting the parameter space to various classes of subsets of $\mathbb{B}^N$ is the right approach. Doing so gives us a number of interesting examples, with which the remainder of this section is concerned.

As a first step, restrict W to rectangles of the form $[0, t] \subset \mathbb{R}^N_+$, where $t \in \mathbb{R}^N_+ = \{(t_1, ..., t_N) : t_i \geq 0\}$. It then makes sense to define a random field on $\mathbb{R}^N_+$ itself via the equivalence

$$(1.4.13) \qquad\qquad W(t) \;=\; W([0, t]).$$

W_t is called the Brownian sheet on $\mathbb{R}^N_+$, or *multiparameter Brownian motion*. It is easy to check that this field is the centered Gaussian process with covariance

$$(1.4.14) \qquad \mathbb{E}\{W_s W_t\} \;=\; (s_1 \wedge t_1) \times \cdots \times (s_N \wedge t_N),$$

where $s \wedge t \overset{\Delta}{=} \min(s, t)$.

When $N = 1$, W is the standard Brownian motion on $[0, \infty)$. When $N > 1$, if we fix $N - k$ of the indices, it is a scaled k-parameter Brownian sheet in the remaining variables. (This is easily checked via the covariance function.) Also, when $N > 1$, it follows immediately from (1.4.14) that $W_t = 0$ when $\min_k t_k = 0$; i.e. when t is on one of the axes. It is this image, with $N = 2$, of a sheet tucked in at two sides and given a good shake, that led Ron Pyke [119] to introduce the name.
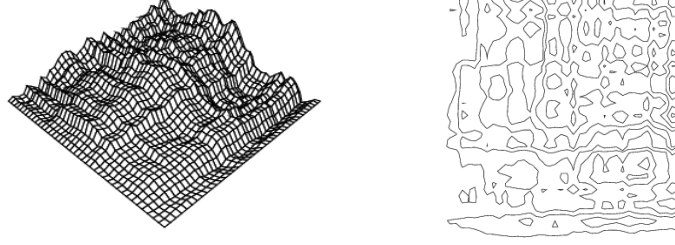
FIGURE 1.4.1. A simulated Brownian sheet on $[0,1]^2$, along with its contour lines at the zero level.

A simple simulation of a Brownian sheet, along with its contour lines, is in Figure 1.4.1.

One of the rather interesting aspects of the contour lines of Figure 1.4.1 is that they are predominantly parallel to the axes. There is a rather deep reason for this, and it has generated a rather massive literature. Many fascinating geometrical properties of the Brownian sheet have been discovered over the years (e.g. [35, 41, 41, 41, 42] and references therein) and a description of the potential theoretical aspects of the Brownian sheet is well covered in [85] where you will also find more references. Nevertheless, the geometrical properties of fields of this kind fall beyond the scope of our interests, since we shall be concerned with the geometrical properties of smooth (i.e. at least differentiable) processes only. Since the Brownian motion on $\mathbb{R}_+^1$ is well known to be non-differentiable at all points, it follows from the above comments relating the sheet to the one dimensional case that Brownian sheets too are far from smooth.

Nevertheless, we shall still have need of these processes, primarily since they hold roughly the same place in the theory of multi-parameter stochastic processes that the standard Brownian motion does in one dimension. The Brownian sheet is a multi-parameter martingale (e.g. [35, 160, 161]) and forms the basis of the multiparameter stochastic calculus. There is a nice review of its basic properties in [155], which also develops its central rôle in the theory of stochastic partial differential equations, and describes in what sense it is valid to describe the derivative

$$\frac{\partial^N W(t_1,\ldots,t_N)}{\partial t_1 \ldots \partial t_N}$$

as Gaussian white noise.

The most basic of the sample path properties of the Brownian sheets are the continuity results of the following theorem. Introduce a partial order on $\mathbb{R}^k$ by writing $s \prec (\preceq) t$ if $s_i < (\leq) t_i$ for all $i = 1,..,k$, and for $s \preceq t$ let $\Delta(s,t) = \prod_1^N [s_i, t_i]$. Although $W(\Delta(s,t))$ is already well defined via the original set indexed process, it also helpful to think of it as the 'increment'

of the point indexed W_t over $\Delta(s,t)$; *viz.*

(1.4.15)

$$W(\Delta(s,t)) \;=\; \sum_{\alpha \in \{0,1\}^N} (-1)^{N - \sum_{i=1}^N \alpha_i}\, W\left(s + \sum_{i=1}^N \alpha_i(t_i - s_i)\right).$$

We call $W(\Delta(s,t))$ the *rectangle indexed* version of W.

Theorem 1.4.4 *The point and rectangle indexed Brownian sheets are continuous[20] over compact $T \subset \mathbb{R}^N$.*

Proof. We need only consider the point indexed sheet, since by (1.4.15) its continuity immediately implies that of the rectangle indexed version. Furthermore, we lose nothing by taking $T = [0,1]^N$. Thus, consider

$$w(s,t) \;\overset{\Delta}{=}\; \mathbb{E}\left\{|W(s) - W(t)|^2\right\}$$

for $s,t \in [0,1]^N$. We shall show that

(1.4.16) $$w(s,t) \;\le\; 2N\,|t - s|$$

from which it follows, in the notation of (1.4.1), that $p^2(u) \le 2Nu$. Since $\int_1^\infty e^{-u^2/2}\, du < \infty$, Theorem 1.4.1 (cf. (1.4.3)) immediately yields the continuity and boundedness of W.

To establish (1.4.16) write $u \vee v$ for $\max(u,v)$ and note that

$$w(s,t) \;\le\; 2\prod_{i=1}^N (s_i \vee t_i) \;-\; 2\prod_{i=1}^N (s_i \wedge t_i).$$

Set $a = 2\prod_{i=1}^{N-1}(s_i \vee t_i)$ and $b = 2\prod_{i=1}^{N-1}(s_i \wedge t_i)$. Then $2 \ge a > b$ and

$$w(s,t) \le a(s_N \vee t_N) - b(s_N \wedge t_N).$$

If $s_N > t_N$ the right-hand side is equal to

$$as_N - bt_N \;=\; a(s_N - t_N) \;+\; t_N(a - b)$$
$$\le\; 2|s_N - t_N| \;+\; |a - b|.$$

Similarly, if $s_N < t_N$ the right-hand side equals

$$at_N - bs_N \;=\; a(t_N - s_N) \;+\; s_N(a - b)$$
$$\le\; 2|t_N - s_N| \;+\; |a - b|,$$

---

[20] For the rectangle indexed case, we obviously need a metric on the sets forming the parameter space. The symmetric difference metric of (1.1.1) is natural, and so we use it.

so that

$$w(s,t) \;\leq\; 2|t_N - s_N| \;+\; \prod_{i=1}^{N-1}(s_i \vee t_i) \;-\; \prod_{i=1}^{N-1}(s_i \wedge t_i).$$

Continuing this process yields

$$w(s,t) \;\leq\; 2\sum_{i=1}^{N}|t_i - s_i| \;\leq\; 2N|t - s|,$$

which establishes (1.4.16) and so the theorem. $\square$

In the framework of the general set indexed sheet, Theorem 1.4.4 states that the Brownian sheet is continuous over $\mathcal{A} = \{\text{all rectangles in } T\}$ for compact T, and so bounded. This is far from a trivial result, for enlarging the parameter set, for the *same* process, can lead to unboundedness. The easiest way to see this is with an example.

An interesting, but quite simple example is given by the class of lower layers in $[0,1]^2$. A set A in $\mathbb{R}^N$ is called a *lower layer* if $s \prec t$ and $t \in A$ implies $s \in A$. In essence, restricted to $[0,1]^2$ these are sets bounded by the two axes and a non-increasing line. A specific example in given in Figure 1.4.2, which is part of the proof of the next theorem.

Theorem 1.4.5 *The Brownian sheet on lower layers in $[0,1]^2$ is discontinuous and unbounded with probability one.*

Proof. We start by constructing some examples of lower layers. Write a generic point in $[0,1]^2$ as (s,t) and let T_{01} be the upper right triangle of $[0,1]^2$; i.e. those points for which which $s \leq 1$ and $t \leq 1 \leq s + t$. Let C_{01} be the largest square in T_{01}; i.e. those points for which which $\frac{1}{2} < s \leq 1$ and $\frac{1}{2} \leq t \leq 1$.

Continuing this process, for $n = 1, 2, \ldots$, and $j = 1, ..., 2^n$, let T_{nj} be the right triangle defined by $s + t \geq 1$, $(j-1)2^{-n} \leq s < j2^{-n}$, and $1 - j2^{-n} < t \leq 1 - (j-1)2^{-n}$. Let C_{nj} be the square filling the upper right corner of T_{nj}, in which $(2j-1)2^{-(n+1)} \leq s < j2^{-n}$ and $1 - (2j-1)2^{-(n+1)} \leq t < 1 - (j-1)2^{-n}$.

The class of lower layers in $[0,1]^2$ certainly includes all sets made up by taking those points that lie between the axes and one of the step-like structures of Figure 1.4.2, where each step comes from the horizontal and vertical sides of some T_{nj} with, perhaps, different n.

Note that since the squares C_{nj} are disjoint for all n and j, the random variables $W(C_{nj})$ are independent. Also $|C_{nj}| = 4^{-(n+1)}$ for all n, j.

Let D be the negative diagonal $\{(s,t) \in [0,1]^2 : s + t = 1\}$ and $L_{nj} = D \cap T_{nj}$. For each $n \geq 1$, each point $p = (s,t) \in D$ belongs to exactly one such interval $L_{n,j(n,p)}$ for some unique $j(n,p)$.
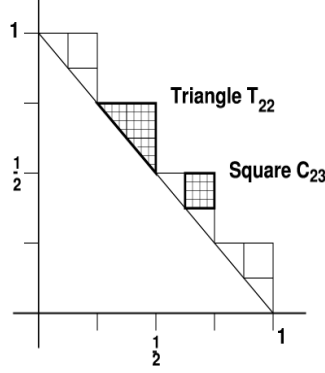
FIGURE 1.4.2. Construction of some lower layers.

For each $p \in D$ and $M < \infty$ the events

$$E_{np} \stackrel{\triangle}{=} \{W(C_{n,j(n,p)}) > M2^{-(n+1)}\}$$

are independent for $n = 0, 1, 2, ...$, and since $W(C_{nj})/2^{-(n+1)}$ is standard normal for all n and j they also have the same positive probability. Thus, for each p we have that, for almost all ω, the events E_{np} occur infinitely often. Let $n(p) = n(p, \omega)$ be the least such n.

Since the events $E_{np}(\omega)$ are measurable jointly in p and ω, Fubini's theorem implies that, with probability one, for almost all $p \in D$ (with respect to Lebesgue measure on D) some E_{np} occurs, and $n(p) < \infty$. Let

$$V_\omega = \bigcup_{p \in D} T_{n(p),j(n(p),p)},$$

$$A_\omega \stackrel{\triangle}{=} \{(s, t) : s + t \le 1\} \cup V_\omega,$$

$$B_\omega \stackrel{\triangle}{=} A_\omega \setminus \bigcup_{p \in D} C_{n(p),j(n(p),p)}.$$

Then A_ω and B_ω are lower layers. Furthermore, almost all $p \in D$ belong to an interval of length $2^{\frac{1}{2}-n(p)}$ which is the hypothenuse of a triangle with the square $C_p = C_{n(p),j(n(p),p)}$ in its upper right corner, for which $2W(C_p) > M2^{-n(p)}$. Consequently,

$$W(A_\omega) - W(B_\omega) \ge \sum_p M2^{-n(p)}/2,$$

where the sum is over those $p \in D$ corresponding to distinct intervals $L_{n(p),j(n(p),p)}$. Since the union of the countably many such intervals is almost all of the diagonal, the sum of $\sum 2^{-n(p)}$ is precisely 1.

Hence $W(A_\omega) - W(B_\omega) \geq M/2$, implying that $\max\{|W(A_\omega)|, |W(B_\omega)|\} \geq M/4$. Sending $M \to \infty$ we see that W is unbounded and so, *a fortiori*, discontinuous with probability one over lower layers in $[0,1]^2$.   □

The above argument is due to Dudley [51] and a similar argument shows that W is unbounded over the convex subsets of $[0,1]^3$. *A fortiori*, W is also unbounded over the convex subsets of $[0,1]^k$ for all $k \geq 4$ and (just to make sure that you don't confuse sample path properties with topological properties of the parameter space) W *is* continuous over convex subsets of the unit square. For details see [53].

These examples should be enough to convince you that the relationship between a Gaussian process and its parameter space is, as far as continuity and boundedness are concerned, an important and delicate subject.

1.4.4 Generalised fields

We start with an example: Take a centered, Gaussian random field f on $\mathbb{R}^N$, with covariance function C. Let $\mathcal{F}$ be a family of functions on $\mathbb{R}^N$, and for $\varphi \in \mathcal{F}$ define

(1.4.17) $$f(\varphi) \;=\; \int_{\mathbb{R}^N} \varphi(t)\, f(t)\, dt.$$

We thus obtain a centered Gaussian process indexed by functions in $\mathcal{F}$, whose covariance functional is given by

(1.4.18) $$C(\varphi, \psi) \;=\; \mathbb{E}\{f(\varphi)\, f(\psi)\} \;=\; \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \varphi(s) C(s,t) \psi(t)\, ds dt.$$

The above construction only makes sense when $C(t,t) < \infty$, for otherwise f has infinite variance and the integrand in (1.4.17) is not defined. Nevertheless, there are occasions when (1.4.18) makes sense, even though $C(t,t) = \infty$. In this case we shall refer to C as a *covariance kernel*, rather than covariance function.

Indeed, we have already been in this scenario twice. In Section 5.3 we looked at moving averages of Gaussian ν-noise, in which case $\mathcal{F}$ was made up of translations of the form $\varphi(s) = F(t-s)$, for $F \in L^2(\nu)$. When treating the spectral representation of stationary processes, we took $\mathcal{F}$ as the complex exponentials $\exp(it \cdot \lambda)$ for fields on $\mathbb{R}^N$ and as the family of characters in Section 5.4 for fields on a general group. Consider the basic spectral distribution theorem, Theorem 5.4.1, which was written in the setting of $\mathcal{C}$-valued fields. For simplicity, assume that ν has a spectral density g. Then (5.4.1) gave us that stationary covariances over $\mathbb{R}^N$ can be

formally written as

$$
\begin{aligned}
C(s,t) &= \int_{\mathbb{R}^N} e^{i(t-s)\cdot\lambda}\, g(\lambda)\, d\lambda \\
&= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} e^{it\cdot\lambda_1} \left[g^{1/2}(\lambda_1)\, \delta(\lambda_1,\lambda_2)\, g^{1/2}(\lambda_2) \right] e^{-is\cdot\lambda_2}\, d\lambda_1 d\lambda_2,
\end{aligned}
$$

which is in the form of (1.4.18), with $\varphi = e^{it\cdot}$, $\psi = e^{-is\cdot}$. The covariance kernel in the integrand now involves the Dirac delta function, which is certainly not finite on the diagonal. Although it was never formally acknowledged as such, it was this issue that led to our having to be careful in defining the stochastic integral in the spectral representation of Theorem 5.4.2.

While moving averages and stationary processes afford two classes of examples, there are many more, some of which we shall describe at the end of this section. In particular, given any positive definite function C on $\mathbb{R}^N \times \mathbb{R}^N$, not necessarily finite, one can define a *function indexed* process on

$$
\mathcal{F}_C \triangleq \left\{ \varphi : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \varphi(s) C(s,t) \varphi(t)\, ds dt \ < \ \infty \right\}.
$$

The proof requires no more than checking that given such a C on $\mathbb{R}^N \times \mathbb{R}^N$, the corresponding C defined by (1.4.18) determines a *finite* positive definite, and so covariance, function on $\mathcal{F}_C \times \mathcal{F}_C$.

In general, function indexed processes of this kind, for which the covariance kernel of (1.4.18) is infinite on the diagonal, are known as *generalised random fields*[21].

The question that we shall now look at is when such processes are continuous and bounded. The answer involves a considerable amount of work, but it is worthwhile at some stage to go through the argument carefully. It is really the only non-Euclidean case for which we shall give an involved entropy calculation in reasonably complete detail.

To make our life a little simpler, while still covering most of the important examples, we shall assume that the divergence of the covariance kernel near the diagonal is bounded as follows:

$$
(1.4.19) \qquad\qquad C(s,t) \ \le \ \frac{C}{|s-t|^\alpha},
$$

for all $|s-t| \le \delta$, for some $\delta > 0$, $C < \infty$ and $\alpha > 0$.

---

[21] The terminology comes from deterministic analogues. There are many partial differential equations for which no pointwise solution exists, but 'smoothed' or 'weak' versions of the equations do have solutions. This is analogous to the non-existence of a pointwise-defined $f(t)$ in (1.4.17), while a function indexed version of f does make sense. The function φ plays the rôle of a local mollifier, or smoothing function

We now start describing potential function spaces to serve as parameter spaces for continuous generalised Gaussian fields with such covariance kernels.

Take $T \subset \mathbb{R}^N$ compact, $q > 0$, and $p = \lfloor q \rfloor$. Let $C_0, \ldots, C_p$ and C_q be finite, positive constants, and let $\mathcal{F}^{(q)} = \mathcal{F}^{(q)}(T, C_0, \ldots, C_p, C_q)$ be the class of functions on T whose partial derivatives of orders $1, \ldots, p$ are bounded by $C_0, \ldots, C_p$, and for which the partial derivatives of order p satisfy Hölder conditions of order $q - p$ with constant C_q. Thus for each $\varphi \in \mathcal{F}^{(q)}$ and $t, t + \tau \in T$,

$$(1.4.20) \qquad \varphi(t + \tau) \; = \; \sum_{n=0}^{p} \frac{\Phi_n(t, \tau)}{n!} \; + \; \Delta(t, \tau),$$

where each $\Phi_n(t, \tau)$ is a homogeneous polynomial of degree n in $\tau = (\tau_1, \ldots, \tau_N)$ of the form

$$(1.4.21) \qquad \Phi_n(t, \tau) \; = \; \sum_{j_1=1}^{N} \cdots \sum_{j_n=1}^{N} \frac{\partial^n \varphi(t)}{\partial t_{j_1} \ldots \partial t_{j_n}} \, \tau_{j_1} \cdots \tau_{j_n},$$

and where

$$(1.4.22) \qquad \sup_{t \in T} \left| \frac{\partial^n \varphi(t)}{\partial t_{j_1} \ldots \partial t_{j_n}} \right| \; \leq \; C_n \quad \text{and} \quad |\Delta(t, \tau)| \; \leq \; C_q |\tau|^q.$$

Two things are obvious in the above setup, in which you should think of the dimension N as fixed. Firstly, the larger the α in (1.4.19) the rougher the process with this covariance will be. Secondly, the larger q is the smaller the family $\mathcal{F}^{(q)}$ will be, and thus the more likely that a Gaussian process defined on $\mathcal{F}^{(q)}$ will be continuous. Thus it seems reasonable to expect that a result of the following kind should be true.

Theorem 1.4.6 *A centered Gaussian process with covariance function satisfying (1.4.18) and covariance kernel satisfying (1.4.19) will be continuous on $\mathcal{F}^{(q)}(T, C_0, \ldots, C_p, C_q)$ if $\alpha < N$ and*

$$q \; > \; \frac{1 + \alpha - N}{2}.$$

A few comments are in order before we start the proof. Firstly, note that since we have not specified any other metric on $\mathcal{F}^{(q)}$, the continuity claim of the theorem is in relation to the topology induced by the canonical metric d. There are, of course, more natural metrics on $\mathcal{F}^{(q)}$, but recall from Lemma 1.3.1 that mere continuity is independent of the metric, as long as $C(\varphi, \psi)$ is continuous in φ and ψ. More detailed information on moduli of continuity will follow immediately from Theorem 1.3.5, the relationship of the chosen metric to d, and the entropy bound (1.4.4) below.

Although we suggested above thinking of the dimension N as fixed, if you prefer to let it vary while keeping α and q fixed, you will find that the larger N is, the less derivatives we require of our test functions to ensure continuity on $\mathcal{F}^{(q)}$. While at first this seems counter-intuitive, you should remember that as N increases the degree of the singularity in (1.4.19) decreases (for fixed α) and so the result is, in fact, reasonable.

Finally, while the reason for the assumption $N > \alpha$ should be obvious from the proof, it is worthwhile noting already that it is precisely this condition that gives us a process with finite variance, since, for $\varphi \in \mathcal{F}^{(q)}$ with $\|\varphi\|_\infty < M$

$$
\begin{aligned}
\mathbb{E}\{f^2(\varphi)\} &= \int_T \int_T \varphi(s) C(s,t) \varphi(t)\, ds dt \\
&\leq M^2 \int_T \int_T C(s,t)\, ds dt \\
&\leq CM^2 \int_T \int_T |t-s|^{-\alpha}\, ds dt.
\end{aligned}
$$

Since $T \subset \mathbb{R}^N$ is compact, a transformation to polar coordinates easily shows that the last integral is finite only if $N > \alpha$.

Proof of Theorem 1.4.6 The proof relies on showing that the usual entropy integral converges, where the entropy of $\mathcal{F}^{(q)}$ is measured in the canonical metric d, where

(1.4.23)

$$
d^2(\varphi, \psi) = \int_T \int_T (\varphi(s) - \psi(s))\, C(s,t)\, (\varphi(t) - \psi(t))\, ds dt.
$$

We shall obtain a bound on the entropy by explicitly constructing, for each $\varepsilon > 0$, a finite family $\mathcal{F}_\varepsilon^{(q)}$ of functions that serve as an ε-net for $\mathcal{F}^{(q)}$ in the d metric. To make life notationally easier, we shall assume throughout that $T = [0,1]^N$.

Fix $\varepsilon > 0$ and set

(1.4.24) $\delta = \delta(\varepsilon) = \varepsilon^{1/(q+(N-\alpha)/2)}.$

Let $\mathbb{Z}_\delta$ denote the grid of the $(1 + \lfloor \delta^{-1} \rfloor)^N$ points in $[0,1]^N$ of the form

(1.4.25) $t_\eta = (\eta_1 \delta, \ldots, \eta_N \delta), \quad \eta_i = 0, 1, \ldots, \lfloor \delta^{-1} \rfloor, \quad i = 1, \ldots, N.$

Set

$$
\delta_n = \delta_n(\varepsilon) = \delta^{q-n}, \qquad n = 0, \ldots, p = \lfloor q \rfloor,
$$

and for each $\varphi \in \mathcal{F}^{(q)}$, $n = 0, \ldots, p$, and t_η of the form (1.4.25) let $A_\eta^{(n)}(\varphi)$ denote the vector formed by taking the integer part of δ_n^{-1} times the partial

derivatives of φ of order n evaluated at the point t_η. (The index η here is, of course, N-dimensional.) Thus, a typical element of $A_\eta^{(n)}$ is of the form

$$(1.4.26) \quad \left\{ A_\eta^{(n)}(\varphi) \right\}_i \; = \; \left\lfloor \frac{\varphi^{(n_1,\ldots,n_N)}(t_\eta)}{\delta_n} \right\rfloor, \quad n_1 + \cdots + n_N \; = \; n,$$

where we have written $\varphi^{(n_1,\ldots,n_N)}$ for the derivative $\partial^n \varphi / \partial^{n_1} x_1 \ldots \partial^{n_N} x_N$, and the index i runs from 1 to $\binom{n+N-1}{N-1}$, the number of partitions of n into N parts.

Finally, for $\varphi \in \mathcal{F}^{(q)}$, let $A^{(n)} = A^{(n)}(\varphi)$ denote the vector valued function on $\mathbb{Z}_\delta$ defined by $A^{(n)}(t_\eta) = A_\eta^{(n)}(\varphi)$. For each $\varphi \in \mathcal{F}^{(q)}$, let $F_{A^{(n)}(\varphi)}$ denote the set of $\psi \in \mathcal{F}^{(q)}$ with fixed matrix $A^{(n)}(\varphi)$. Our first task will be to show that the d-radius of $F_{A^{(n)}(\varphi)}$ is not greater than $C\varepsilon$, where C is a constant dependent only on q and N. All that will then remain will be to calculate how many different collections $F_{A^{(n)}(\varphi)}$ are required to cover $\mathcal{F}^{(q)}$. In other words, we need to find how many φ's are needed to approximate, in terms of the metric (1.4.23), all functions in $\mathcal{F}^{(q)}$.

Thus, take $\varphi_1, \varphi_2 \in F_{A^{(n)}(\varphi)}$, and set

$$(1.4.27) \qquad\qquad \varphi \; = \; \varphi_1 - \varphi_2.$$

Let $\| \; \|_d$ be the norm induced on $\mathcal{F}^{(q)}$ by the metric d, and $\| \; \|_\infty$ the usual sup norm. Then

$$\|\varphi\|_d^2 \; = \; \int_{[0,1]^N} \int_{[0,1]^N} \varphi(s) C(s,t) \varphi(t) \, ds dt, \qquad \|\varphi\|_\infty \; = \; \sup_{[0,1]^k} |\varphi(t)|.$$

We have to show that the φ of (1.4.27) has d-norm less than $C\varepsilon$.

Note first, however, that in view of the definition (1.4.26) of the matrix A_δ we have that for each $t_\eta \in \mathbb{Z}_\delta$ and each partial derivative $\varphi^{(n_1,\ldots,n_N)}$ of such a φ of order $n_1 + \cdots + n_N = n \leq p$ that

$$\varphi^{(n_1,\ldots,n_N)}(t_\eta) \; \leq \; \delta_n.$$

Putting this inequality together with the Taylor expansion (1.4.20)–(1.4.21) we find that for all $t \in [0,1]^N$

$$\begin{aligned} |\varphi(t)| \; &\leq \; \sum_{n=0}^{p} \frac{N^n}{n!} \, \delta_n \, \delta^n \; + \; C\delta^q \\ &= \; C(N,p) \, \delta^q, \end{aligned}$$

the last line following from the definition of the δ_n and the fact that each polynomial Φ_n of (1.4.21) has less than N^n distinct terms.

Thus, for φ of the form (1.4.27),

$$(1.4.28) \qquad\qquad \|\varphi\|_\infty \; \leq \; C\delta^q.$$

We now turn to $\|\varphi\|_d$. With δ as above, set

$$D_\delta = \left\{ (s,t) \subset [0,1]^N \times [0,1]^N : \max_{i=1,\dots,N} |s_i - t_i| \le \delta \right\}.$$

Then

$$(1.4.29) \quad \|\varphi\|_d^2 = \int_{[0,1]^N \times [0,1]^N} \varphi(s) C(s,t) \varphi(t) \, ds dt$$

$$= \int_{D_\delta} \varphi(s) C(s,t) \varphi(t) \, ds dt$$

$$+ \int_{([0,1]^N \times [0,1]^N) \setminus D_\delta} \varphi(s) C(s,t) \varphi(t) \, ds dt$$

$$= I_1(\delta) + I_2(\delta).$$

Consider the first integral. Letting C change from line to line where necessary, we have from (1.4.19) and (1.4.28) that

$$(1.4.30) \quad I_1(\delta) \le C \delta^{2q} \int_{[0,1]^N} ds \int \cdots \int_{s_i - \delta \le t_i \le s_i + \delta} dt \, |s - t|^{-\alpha}$$

$$\le C \delta^{2q} \int_{[-\delta,\delta]^N} |t|^{-\alpha} \, dt$$

$$\le C \delta^{(2q+N-\alpha)},$$

the last inequality coming from an evaluation via polar coordinates, and requiring the condition $\alpha < N$.

Similarly, again relying on the fact that $\alpha < N$, it is easy to check that $I_2(\delta)$ is also bounded above by $C \delta^{(2q+N-\alpha)}$. Substituting this fact and (1.4.30) into (1.4.29), and applying (1.4.24), we finally obtain that for φ satisfying (1.4.27)

$$(1.4.31) \quad \|\varphi\|_d < C \delta^{\frac{1}{2}(2q+N-\alpha)} = C\varepsilon.$$

That is, the d-radius of each set $F_{A^{(n)}(\varphi)}$ is no greater than a uniform constant times ε.

It remains to determine how many collections $F_{A^{(n)}(\varphi)}$ are required to cover $\mathcal{F}^{(q)}$. Since this is a calculation that is now independent of both Gaussian processes in general, and the above covariance function in particular, we shall only outline how it is done. The details, which require somewhat cumbersome notation, can be found in Kolmogorov and Tihomirov [87], which is a basic reference for general entropy computations.

Consider, for fixed δ, the matrix A_δ, parameterised, as in (1.4.25), by $\eta_i = 0, 1, \dots, [\delta^{-1}]$, $i = 1, \dots, N$, and $n = 0, 1, \dots, p$. Fix, for the moment, $\eta_2 = \cdots = \eta_N = 0$. It is clear from the restrictions (1.4.22), (1.4.26),

the definition of δ_n, and the fact that each vector $A_\eta^{(n)}$ has no more than $\binom{n+N-1}{N-1}$ distinct elements, that there are no more than

$$O\left(\frac{1}{\delta_0}\left(\frac{1}{\delta_1}\right)^{\binom{N}{N-1}}\cdots\left(\frac{1}{\delta_p}\right)^{\binom{p+N}{N-1}}\right) \;=\; O\left(\delta^{-\xi}\right)$$

(for an appropriate and eventually unimportant ξ) ways to fill in the row of A_δ corresponding to $(n_1,\ldots,n_N)=(0,\ldots,0)$.

What remains to show is that because of the rigid continuity conditions on the functions in $\mathcal{F}^{(q)}$, there exists an absolute constant $M = M(q,C_0,\ldots,C_p,C_q)$, such that once this first row is determined, there are no more than M ways to complete the row corresponding to $(n_1,\ldots,n_N) = (1,\ldots,0)$, and similarly no more than M^2 ways to complete the row corresponding to $(n_1,\ldots,n_N) = (2,\ldots,0)$, etc. Thus, all told, there are no more than

(1.4.32) $$O\left(\delta^{-\xi}\,M^{N(1+\delta^{-1})}\right)$$

ways to fill the matrix A_δ, and thus we have a bound for the number of different collections $F_{A^{(n)}(\varphi)}$.

Modulo a constant, it now follows from (1.4.24), (1.4.31) and (1.4.32) that the log entropy function for our process is bounded above by

$$C_1 \;+\; \frac{C_2\,\xi}{(q+(N-\alpha)/2)}\ln\left(\frac{1}{\varepsilon}\right) \;+\; C_3\left(\frac{1}{\varepsilon}\right)^{1/(q+(N-\alpha)/2)}.$$

Since this is integrable if $q > (1+\alpha-N)/2$, we are done. $\square$

Before leaving function indexed processes, there are a number of comments that are worth making, that relate them to other problems both within and outside of the theory of Gaussian processes.

Firstly, in most of the literature pertaining to generalised Gaussian fields the parameter space used is the Schwartz space $\mathcal{S}$ of infinitely differentiable functions decaying faster than any polynomial at infinity. Since this is a very small class of functions (at least in comparison to the classes $\mathcal{F}^{(q)}$ that Theorem 1.4.6 deals with) continuity over $\mathcal{S}$ is automatically assured and therefore not often explicitly treated. However, considerations of continuity and smaller parameter spaces are of relevence in the treatment of infinite dimensional diffusions arising as the solutions of stochastic partial differential equations, in which solutions over very specific parameter spaces are often sought. For more on this see, for example, [78, 156, 155].

Secondly, some words on our choice of (1.4.19) as a condition on the covariance kernel $C(s,t)$. When $\alpha = N - 2$, $N > 2$, then the class of generalised fields that we are considering here includes the so called 'free field' of Euclidean quantum field theory. (When $k = 2$ the free field has a covariance kernel with a logarithmic singularity at 0, and when $k = 1$

the free field is no longer generalised, but is the real valued, stationary Markov Gaussian process with covariance function $C(t) = e^{-\beta|t|}$, for some $\beta > 0$.) This process, along with a large number of related generalised fields whose covariance kernels satisfy similar conditions, possess a type of multi-dimensional Markov property. For details on this see, for example, Dynkin [54, 55, 56, 57] and Adler and Epstein [7] and references therein. For structural and renormalisation properties of generalised fields of this kind, presented among a much wider class of examples, see Dobrushin [46], who also treats a large variety of non-Gaussian fields.

Our divergence assumption (1.4.19) leaves out a class of examples important to the theory of empirical processes, in which the covariance kernel is the product of a Dirac delta δ and a bounded 'density', g, in the sense that

$$\mathbb{E}\{f(\varphi)f(\psi)\} \quad = \quad \int \varphi(t)\psi(t)\,g(t)\,dt$$
$$\text{``} = \text{''} \quad \iint \varphi(s)\left[g^{1/2}(s)\delta(s,t)g^{1/2}(t)\right]\psi(t)\,dt.$$

As we have already noted, such processes arose as the stochastic integrals $W(\varphi)$ of Section 5.2, for which W was a Gaussian ν-noise where ν is (now) a probability measure with density g. For more on this setting, in which the computations are similar in spirit to those made above, see Dudley [53].

Finally, it is worth noting that much of what has been said above regarding generalised fields – i.e. function indexed processes – can be easily extended to Gaussian processes indexed by families of measures. For example, if we consider the function φ in (1.4.17) to be the (positive) density of a measure μ on $\mathbb{R}^N$, then by analogy it makes sense to write

$$f(\mu) \quad = \quad \int_{\mathbb{R}^N} f(t)\,\mu(dt),$$

with the corresponding covariance functional

$$C(\mu,\nu) \quad = \quad \mathbb{E}\{f(\mu)f(\nu)\} \quad = \quad \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\mu(ds)\,C(s,t)\,\nu(dt).$$

Again, as was the case for generalised Gaussian fields, the process $X(\mu)$ may be well defined even if the covariance kernel C diverges on the diagonal. In fact, $f(\mu)$ will be well defined for all $\mu \in M_C$, where

$$M_C \quad \triangleq \quad \left\{\mu : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\mu(ds)\,C(s,t)\,\mu(dt) \quad < \quad \infty\right\}.$$

Similar arguments to those used above to characterise the continuity of a family of Gaussian fields on $\mathcal{F}^{(q)}$ can be used to ascertain continuity of measure indexed processes on suitably smooth classes of measures. We

leave both the details and an attempt to formulate the appropriate results to the interested reader[22].

1.4.5   Set indexed processes

We have already met some set indexed processes in dealing with the Brownian family of processes in Section 1.4.3, in which we concentrated on Gaussian ν-noise (cf. (1.4.9)–(1.4.11)) indexed by sets. We saw, for example, that the while the Brownian sheet indexed by rectangles was continuous (Theorem 1.4.4) when indexed by lower layers in $[0,1]^2$ it was discontinuous and unbounded (Theorem 1.4.5).

In this section we shall remain in the setting of ν-noise, and look at two classes of set indexed processes. The first will be Euclidean, and the parameter spaces will be classes of sets in compact $T \subset \mathbb{R}^N$ with smooth boundaries. For this example we also add the assumption that ν has a density bounded away from zero and infinity on T. For the second eaxmple, we look at Vapnik-Červonenkis classes of sets, of singular importance in statistical learning theory and image processing, and characterised by certain combinatorial properties. Here the ambient space (in which the sets lie) can be any measure space. We shall skimp on proofs when they have nothing qualitatively new to offer. In any case, all that we have to say is done in full detail in Dudley [50, 53], where you can also find a far more comprehensive treatment and many more examples.

Our first family is actually closely related to the family $\mathcal{F}^{(q)}$ of functions we have just studied in detail. While we shall need some of the language of manifolds and homotopy to describe this example, which will only be developed later in Chapter 6, it will be basic enough that the average reader should have no trouble following the argument.

With S^{N-1}, as usual, denoting the unit sphere in $\mathbb{R}^N$, recall the basic fact that we can cover it by two patches V_1 and V_2, each of which maps via a C^∞ diffeomorphism $F_j : V_j \to B^{N-1}$ to the open ball $B^{N-1} = \{t \in \mathbb{R}^{N-1} : |t|^2 < 1\}$.

Adapting slightly the notation of the previous example, let $\mathcal{F}^{(q)}(V_j, M)$ be the set of all real valued functions φ on V_j such that

$$\varphi \circ F_j^{-1} \in \mathcal{F}^{(q)}(B^{N-1}, M, \ldots, M)$$

(cf. (1.4.20)–(1.4.22)). Furthermore, let $\mathcal{F}^{(q)}(S^{N-1}, M)$ denote the set of all real valued functions φ on S^{N-1} such that the restriction of φ to V_j

---

[22]Actually, to the best of our knowledge, this specific example has never been treated in the literature, and so it would be a rather interesting problem to work out how to optimally, and naturally, formulate the requisite smoothness conditions. The general idea of how to proceed can be gleaned from the treatment of set-indexed processes in the following section.

is in $\mathcal{F}^{(q)}(V_j, M)$. Now taking the N-fold Cartesian product of copies of $\mathcal{F}^{(q)}(S^{N-1}, M)$, we obtain a family of functions from S^{N-1} to $\mathbb{R}^N$, which we denote by $D(N, q, M)$, where the D stands for Dudley, who introduced this family in [49].

Each $\varphi \in D(N, q, M)$ defines an $(N-1)$-dimensional surface in $\mathbb{R}^N$, and a simple algebraic geometric construction[23] enables one to 'fill in' the interior of this surface to obtain a set I_φ. We shall denote the family of sets obtained in this fashion by $I(N, q, M)$, and call them the *Dudley sets with q-times differentiable boundaries.*

Theorem 1.4.7 *The Brownian sheet is continuous on a bounded collection of Dudley sets in $\mathbb{R}^N$ with q times differentiable boundaries if $q > N-1 \geq 1$. If $N-1 > 1 \geq q > 0$ or if $N-1 > q \geq 1$ then the Brownian sheet is unbounded with probability one.*

Outline of Proof. The proof of the unboundedness part of the result is beyond us, and so you are referred to [53]. As far as the proof of continuity is concerned, what we need are the following inequalities for the log-entropy, on the basis of which continuity follows from Theorem 1.3.5.

$$H\big(I(N, q, M), \varepsilon\big) \;\leq\; \begin{cases} C\varepsilon^{-2(N-1)/(Nq-N+1)} & (N-1)/N < q \leq 1, \\ C\varepsilon^{-2(N-1)/q} & 1 \leq q. \end{cases}$$

These inequalities rely on the "simple algebraic geometric construction" noted above, and so we shall not bring them in detail. The details are in [53]. The basic idea, however, requires little more than noting that there are basically as many sets in $I(N, q, M)$ as there are functions in $D(N, q, M)$, and we have already seen, in the previous example, how to count the number of functions in $D(N, q, M)$.

There are also equivalent lower bounds for the log-entropy for certain values of N and q, but these are not important to us. $\qquad\square$

We now turn to the so called *Vapnick-Červonenkis*, or VC, sets, due, not surprisingly, to Vapnick and Červonenkis [153, 154]. These sets arise in a very natural way in many areas including statistical learning theory and image analysis. The recent book [152] by Vapnick is a good place to see why.

The arguments involved in entropy calculations for VC classes of sets are of an essentially combinatoric nature, and so somewhat different to those we have met so far. We shall therefore look at them somewhat more

---

[23]The construction works as follows: For given $\varphi : S^{N-1} \to \mathbb{R}^N$ in $D(N, q, M)$, let R_φ be its range and A_φ be the set of all $t \in \mathbb{R}^N$, $t \notin R_\varphi$, such that among mappings of S^{N-1} into $\mathbb{R}^N \setminus \{t\}$, φ is not homotopic (cf. Definition 6.1.3) to any constant map $\psi(s) \equiv r \neq t$. Then define $I_\varphi = R_\varphi \cup A_\varphi$. For an example, try untangling this description for φ the identity map from S^{N-1} to itself to see that what results is $I_\varphi = B^N$.

closely than we did for Dudley sets. For more details, however, including a discussion of the importance of VC classes to the problem of finding universal Donsker classes in the theory of empirical processes, see [53].

Let $(E, \mathcal{E})$ be a measure space. Given a class $\mathcal{C}$ of sets in $\mathcal{E}$ and a finite set $F \subset E$, let $\Delta^{\mathcal{C}}(F)$ be the number of different sets $A \cap F$ for $A \in \mathcal{C}$. If the number of such sets is $2^{|F|}$, then $\mathcal{C}$ is said to *shatter* F. For $n = 1, 2, \ldots,$ set

$$m^{\mathcal{C}}(n) \triangleq \max\{\Delta^{\mathcal{C}}(F) : F \text{ has } n \text{ elements}\}.$$

Clearly, $m^{\mathcal{C}}(n) \leq 2^n$ for all n. Also, set

(1.4.33)

$$V(\mathcal{C}) \triangleq \begin{cases} \inf\{n : m^{\mathcal{C}}(n) < 2^n\} & \text{if } m^{\mathcal{C}}(n) < 2^n \text{ for some } n, \\ \infty & \text{if } m^{\mathcal{C}}(n) = 2^n \text{ for all } n. \end{cases}$$

The class $\mathcal{C}$ is called a Vapnik-Červonenkis class if $m^{\mathcal{C}}(n) < 2^n$ for *some* n; i.e. if $V(\mathcal{C}) < \infty$. The number $V(\mathcal{C})$ is called the VC index of $\mathcal{C}$, and $V(\mathcal{C}) - 1$ is the largest cardinality of a set shattered by $\mathcal{C}$.

Two extreme but easy examples which you can check for yourself are $E = \mathbb{R}$ and $\mathcal{C}$ all half lines of the form $(-\infty, t]$, for which $m^{\mathcal{C}}(n) = n + 1$ and $V(\mathcal{C}) = 2$, and $E = [0, 1]$ with $\mathcal{C}$ all the open sets in $[0, 1]$. Here $m^{\mathcal{C}}(n) = 2^n$ for all n and so $V(\mathcal{C}) = \infty$ and $\mathcal{C}$ is not a VC class.

A more instructive example, that also leads into the general theory we are after, is $E = \mathbb{R}^N$ and $\mathcal{C}$ is the collection of half-spaces of $\mathbb{R}^N$. Let $\Phi(N, n)$ be the maximal number of components into which it is possible to partition $\mathbb{R}^N$ via n hyperplanes. Then, by definition, $m^{\mathcal{C}}(n) = \Phi(N, n)$. It is not hard to see that Φ must satisfy the recurrence relation

(1.4.34) $$\Phi(N, n) = \Phi(N, n-1) + \Phi(N-1, n-1),$$

with the boundary conditions $\Phi(0, n) = \Phi(N, 0) = 1$.

To see this, note that if $\mathbb{R}^N$ has already been partitioned into $\Phi(N, n-1)$ subsets via $n - 1$ $((N-1)$-dimensional) hyperplanes, $H_1, \ldots, H_{n-1}$, then adding one more hyperplane H_n will cut in half as many of these subsets as intersect H_n. There can be no more such subsets, however, than the maximal number of subsets formed on H_n by partitioning with the $n - 1$ $(N-2)$-dimensional hyperplanes $H_1 \cap H_n, \ldots, H_{n-1} \cap H_n$; i.e. $\Phi(N-1, n-1)$. Hence (1.4.34).

Induction then shows that

$$\Phi(N, n) = \begin{cases} \sum_{j=0}^{N} \binom{n}{j} & \text{if } n > N, \\ 2^n & \text{if } n \leq N, \end{cases}$$

where we adopt the usual convention that $\binom{n}{j} = 0$ if $n < j$.

From either the above or (1.4.34) you can now check that

(1.4.35) $$\Phi(N, n) \leq n^N + 1, \qquad \text{for all } N, n > 0.$$

It thus follows, from (1.4.33), that the half-spaces of $\mathbb{R}^N$ form a VC class for all N.

What is somewhat more surprising, however, is that an inequality akin to (1.4.35), which we developed only for this special example, holds in wide (even non-Euclidean) generality.

Lemma 1.4.8 *Let $\mathcal{C}$ be a collection of sets in $\mathcal{E}$ such that $V(\mathcal{C}) \leq v$. Then*

$$(1.4.36) \quad m^{\mathcal{C}}(n) \; < \; \Phi(v, n) \; \leq \; n^v + 1, \qquad \text{for all } n \geq v.$$

Since the proof of this result is combinatoric rather than probabilistic, and will be of no further interest to us, you are referred to either of [153, 53] for a proof.

The importance of Lemma 1.4.8 is that it enables us to obtain bounds on the entropy function for Gaussian ν-noise over VC classes that are independent of ν.

Theorem 1.4.9 *Let W be the Gaussian ν-noise on a probability space $(E, \mathcal{E}, \nu)$. Let $\mathcal{C}$ be a Vapnik-Červonenkis class of sets in $\mathcal{E}$ with $V(\mathcal{C}) = v$. Then there exists a constant $K = K(v)$ (not depending on ν) such that for $0 < \varepsilon \leq \frac{1}{2}$, the entropy function for W satisfies*

$$N(\mathcal{C}, \varepsilon) \; \leq \; K\varepsilon^{-2v}|\ln \varepsilon|^v.$$

Proof. We start with a little counting, and then turn to the entropy calculation proper. The counting argument is designed to tell us something about the maximum number of $\mathcal{C}$ sets that are a certain minimum distance from one another and can be packed into E.

Fix $\varepsilon > 0$ and suppose $A_1, \ldots, A_m \in \mathcal{C}$, $m \geq 2$, and $\nu(A_i \Delta A_j) \geq \varepsilon$ for $i \neq j$. We need an upper bound on m. Sampling with replacement, select n points at random from E. The ν-probability that at least one of the sets $A_i \Delta A_j$ contains none of these n points is at most

$$(1.4.37) \qquad\qquad\qquad \binom{m}{2}(1 - \varepsilon)^n.$$

Choose $n = n(m, \varepsilon)$ large enough so that this bound is less than 1. Then

$$P\{\text{all symmetric differences } A_i \Delta A_j \text{ are non-empty}\} \; > \; 0,$$

and so for at least one configuration of the n sample points the class $\mathcal{C}$ picks out at least m distinct subsets. (Since, with positive probability, given any two of the A_i there is at least one point not in both of them.) Thus, by (1.4.36),

$$(1.4.38) \qquad m \; \leq \; m^{\mathcal{C}}(n) \; \leq \; n^v \; = \; \big(n(m, \varepsilon)\big)^v.$$

Take now the smallest n for which (1.4.37) is less than 1. For this n we have $m^2(1-\varepsilon)^{n-1} \geq 2$, so that

$$n - 1 \ \leq \ \frac{2\ln m - \ln 2}{|\ln(1-\varepsilon)|},$$

and $n \leq (2\ln m)/\varepsilon$. Furthermore, by (1.4.38), $m \leq (2\ln m)^v \varepsilon^{-v}$.

For some $m_0 = m_0(v) < \infty$, $(2\ln m)^v \leq m^{1/(v+1)}$ for $m \geq m_0$, and then $m \leq \varepsilon^{-v-1}$, so $\ln m \leq (v+1)|\ln \varepsilon|$. Hence

$$m \ \leq \ K(v)\,\varepsilon^{-v}|\ln\varepsilon|^v, \qquad \text{for } 0 < \varepsilon \leq \tfrac{1}{2},$$

if $K(v) = \max(m_0, 2^{v+1}(v+1)^v)$.

This concludes the counting part of the proof. We can now do the entropy calculation. Recall that the canonical distance between on sets of $\mathcal{E}$ is given by $d_\nu(A, B) = [\nu(A\Delta B)]^{\frac{1}{2}}$.

Fix $\varepsilon > 0$. In view of the above, there can be no more than $m = K(v)\varepsilon^{2v}|2\ln\varepsilon|^v$ sets $A_1, \ldots, A_m$ in $\mathcal{C}$ for which $d_\nu(A_i, A_j) \geq \varepsilon$ for all i, j. Take an ε-neighbourhood of each of the A_i in the d_ν metric. (Each such neighbourhood is a collection of sets in $\mathcal{E}$.) The union of these neighbourhoods covers $\mathcal{C}$, and so we have constructed an ε-net of the required size, and are done. □

An immediate consequence of the entropy bound of Theorem 1.4.9 is

Corollary 1.4.10 *Let W be Gaussian ν-noise based over a probability space $(E, \mathcal{E}, \nu)$. Then W is continuous over any Vapnik-Červonenkis class of sets in $\mathcal{E}$.*

1.4.6   Non-Gaussian processes

A natural question to ask is whether or not the results and methods that we have seen in this section extend naturally to non-Gaussian fields. We already noted, immediately after the proof of the central Theorem 1.3.5, that the proof there only used normality once, and so the general techniques of entropy should be extendable to a far wider setting.

For most of the processes that will concern us, this will not be terribly relevant. In Chapter 14 we shall concentrate on random fields which can be written in the form

$$f(t) \ = \ F\left(g^1(t), \ldots, g^d(t)\right),$$

where the g^i are i.i.d. Gaussian and $F : \mathbb{R}^d \to \mathbb{R}$ is smooth. In this setting, continuity and boundedness of the non-Gaussian f follow deterministically from similar properties on F and the g^i, and so no additional theory is needed.

Nevertheless, there are many processes that are not attainable in this way, for which one might *a priori* expect that the random geometry of Part III of this book might apply. In particular, we are thinking of the smooth stable fields of Samorodnitsky and Taqqu [129]. With this in mind, and for completeness, we state Theorem 1.4.11 below. However, other than the 'function of Gaussian' non-Gaussian scenario, we know of no cases for which this random geometry has even the beginnings of a parallel theory.

To set up the basic result, let f_t be a random process defined on a metric space (T, τ) and taking values in a Banach space $(B, \| \ \|_B)$. Since we are no longer in the Gaussian case, there is no reason to assume that there is any more a canonical metric of T to replace τ. Also, recall that a function $\varphi : \mathbb{R} \to \mathbb{R}$ is called a *Young function* if it is even, continuous, convex, and satisfies

$$\lim_{x \to 0} \frac{\varphi(x)}{x} \ = \ 0, \qquad \lim_{x \to \infty} \frac{\varphi(x)}{x} \ = \ \infty.$$

Theorem 1.4.11 *Take f as above, and assume that the real valued process $\|f_t - f_s\|_B$ is separable. Let N_τ be the metric entropy function for T with respect to the metric τ. If there exists an $\alpha \in (0, 1]$ and a Young function φ such that the following two conditions are satisfied, then f is continuous with probability one.*

$$\mathbb{E} \left\{ \varphi \left(\frac{\|f(t) - f(s)\|_B^\alpha}{\tau(s, t)} \right) \right\} \ \leq \ 1,$$
$$\int_{N_\tau(u) > 1} \varphi^{-1} \left(N_\tau(u) \right) \, du \ < \ \infty.$$

The best place to read about this is in Ledoux and Talagrand [92].

1.5   Majorising measures

Back in Section 1.3 we used the notion of entropy integrals to determine sufficient conditions for the boundedness and continuity of Gaussian fields. We claimed there that these arguments were sharp for stationary processes, but need not be sharp in general. That is, there are processes which *are* a.s. continuous, but whose entropy integrals diverge. We also noted that the path to solving these issues lay via *majorising measures*, and so we shall now explain what these are, and how they work.

Our plan here is to give only the briefest of introductions to majorising measures. You can find the full theory in the book by Ledoux and Talagrand [92] and the more recent papers [143, 145, 146] by Talagrand. In particular, [145] gives a very user friendly introduction to the subject, and the proof of Theorem 1.5.1 below is taken from there.

We include this section for mathematical completeness[24], and will not use its results anywhere else in the book. It is here mainly to give you an idea of how to improve on the weaknesses of entropy arguments, and to whet your appetite to turn to the sources for more. Here is the main result.

Theorem 1.5.1 *If f is a centered Gaussian process on (d-compact) T, then there exists a universal constant K such that*

$$(1.5.1) \quad \mathbb{E}\left\{\sup_{t \in T} f_t\right\} \leq K \inf_{\mu} \sup_{t \in T} \int_0^\infty \sqrt{\ln \frac{1}{\mu(B_d(t, \varepsilon))}} \, d\varepsilon,$$

where B_d is the d-ball of (1.3.2) and the infimum on μ is taken over all probability measures μ on T.
Furthermore, if f is a.s. bounded, then there exists a probability measure μ and a universal constant K' such that

$$(1.5.2) \quad K' \sup_{t \in T} \int_0^\infty \sqrt{\ln \frac{1}{\mu(B_d(t, \varepsilon))}} \, d\varepsilon \leq \mathbb{E}\left\{\sup_{t \in T} f_t\right\}.$$

A measure μ for which the integrals above are finite for all t is called a majorising measure.

Note that the upper limit to the integrals in the theorem is really $\mathrm{diam}(T)$, since

$$\varepsilon > \mathrm{diam}(T) \quad \Rightarrow \quad T \subset B_d(t, \varepsilon) \quad \Rightarrow \quad \mu(B_d(t, \varepsilon)) = 1,$$

and so the integrand is zero beyond this limit.

Theorem 1.5.1 is the majorising measure version of Theorem 1.3.3[25], which gave an upper bound for $\mathbb{E}\{\sup_t f_t\}$ based on an entropy integral, but which did not have a corresponding lower bound. Theorem 1.5.1, however, takes much more work to prove than its entropic relative. Nevertheless, by building on what we have already done, it is not all that hard to see where the upper bound (1.5.1) comes from.

Outline of proof. Start by re-reading the proof of Theorem 1.3.3 as far as (1.3.13). The argument leading up to (1.3.13) was that, eventually, increments $f_{\pi_j(t)} - f_{\pi_{j-1}}(t)$ would be smaller than $u a_j$. However, on the way to this we could have been less wasteful in a number of our arguments.

For example, we could have easily arranged things so that

$$(1.5.3) \quad \forall \, s, t \in T, \qquad \pi_j(t) = \pi_j(s) \quad \Rightarrow \quad \pi_{j-1}(t) = \pi_{j-1}(s),$$

---

[24]'Mathematical completeness' should be understood in a relative sense, since our proofs here will most definitely be incomplete!

[25]There is a similiar extension of Theorem 1.3.5, but we shall not bother with it.

which would have given us fewer increments to control. We could have also assumed that

(1.5.4) $\forall\, t \in \Pi_j, \qquad \pi_{j(t)} \;=\; t,$

so that, by (1.5.3),

(1.5.5) $\pi_{j-1}(t) \;=\; \pi_{j-1}\left(\pi_j(t)\right).$

So let's assume both (1.5.3) and (1.5.4). Then controlling the increments $f_{\pi_j(t)} - f_{\pi_{j-1}}(t)$ actually means controlling the the increments $f_t - f_{\pi_{j-1}}(t)$ for $t \in \Pi_j$. There are only N_j such increments, which improves on our previous estimate of $N_j N_{j-1}$. This does not make much of a difference, but what does, and this is the core of the majorising measure argument, is replacing the original a_j by families of non-negative numbers $\{a_j(t)\}_{t \in \Pi_j}$, and then to then ask when

(1.5.6) $\forall\, t \in \Pi_j, \qquad f_t - f_{\pi_{j-1}(t)} \;\leq\; u a_j(t)$

for large enough j. Note that under (1.5.6)

$$\forall\, t \in T, \qquad f_t - f_{t_o} \;\leq\; uS,$$

where

$$S \;\triangleq\; \sup_{t \in T} \sum_{j > i} a_j(\pi_j(t)).$$

Thus

$$\mathbb{P}\left\{\sup_{t \in T}(f_t - f_{t_o}) \geq uS\right\} \;\leq\; \sum_{j > i} \sum_{t \in \Pi_j'} \mathbb{P}\left\{f_t - f_{\pi_{j-1}(t)} \geq u a_j(t)\right\}$$

(1.5.7) $\leq\; \sum_{j > i} \sum_{t \in \Pi_j'} 2\exp\left(\dfrac{-u^2 a_j^2(t)}{(2r^{-j+1})^2}\right),$

where $\Pi_j' \triangleq \Pi_j \setminus \bigcup_{i < k \leq j-1} \Pi_k$. (The move from the Π_j to the disjoint Π_j' is crucial, and made possible by (1.5.5).) This bound is informative only if the right hand side is less than or equal to one. Let's see how to ensure this when $u = 1$. Setting

(1.5.8) $w_j(t) \;=\; 2\exp\left(\dfrac{-a_j^2(t)}{(2r^{-j+1})^2}\right)$

we want to have $\sum_j \sum_{t \in \Pi_j'} w_j(t) \leq 1$. We are now getting close to our majorising measure.

Recall that T has long ago been assumed countable. Suppose we have a probability measure μ supported on T and for all $j > i$ and all $t \in \Pi_j$ set $w_j(t) = \mu(\{t\})$. Undo (1.5.8) to see that this means that we need to take

$$a_j(t) \;=\; 2r^{-j+1}\sqrt{\ln \frac{2}{\mu(\{t\})}}.$$

With this choice, the last sum in (1.5.7) is no more than 2^{1-u^2}, and S is given by

$$S \;=\; 2\sup_{t \in T}\sum_{j>i} r^{-j+1}\sqrt{\ln \frac{2}{\mu(\{\pi_j(t)\})}},$$

all of which ensures that for the *arbitrary* measure μ we now have

(1.5.9) $$\mathbb{E}\left\{\sup_{t \in T} f_t\right\} \;\leq\; K\sup_{t \in T}\sum_{j>i} r^{-j+1}\sqrt{\ln \frac{2}{\mu(\{\pi_j(t)\})}}.$$

This is, in essence, the majorising measure upper bound. To make it look more like (1.5.1), note that each map π_j defines a partition $\mathcal{A}_j$ of T comprising of the sets

$$A_t \;\overset{\Delta}{=}\; \{s \in T : \pi_j(s) = t\}, \qquad t \in \Pi_j.$$

With $A_j(t)$ denoting the unique element of $\mathcal{A}_j$ that contains $t \in T$, it is not too hard (but also not trivial) to reformulate (1.5.9) to obtain

(1.5.10) $$\mathbb{E}\left\{\sup_{t \in T} f_t\right\} \;\leq\; K\sup_{t \in T}\sum_{j>i} r^{-j+1}\sqrt{\ln \frac{2}{\mu(\{A_j(t)\})}}.$$

which is now starting to look a lot more like (1.5.1). To see why this reformulation works, you should go [145], which you should now be able to read without even bothering about notational changes. You will also find there how to turn (1.5.10) into (1.5.1), and also how to get the lower bound (1.5.2). All of this takes quite a bit of work, but at least now you should have some idea of how majorising measures arose.   □

Despite the elegance of Theorem 1.5.1, it is not always easy, given a specific Gaussian process, to find the 'right' majorising measure for it. To circumvent this problem, Talagrand recently [146] gave a recipe on how to wield the technique without the need to explicitly compute a majorising measure. However, we are already familiar with one situation in which there is a simple recipe for building majorising measures. This is when entropy integrals are finite.

Thus, let be $H(\varepsilon)$ be our old friend (1.3.3), and set

$$(1.5.11) \qquad g(t) \triangleq \sqrt{\ln \frac{1}{t}}, \qquad 0 < t \leq 1.$$

Then here is a useful result linking entropy and majorising measures.

Lemma 1.5.2 *If $\int_0^\infty H^{1/2}(\varepsilon) \, d\varepsilon < \infty$, then there exists a majorising measure μ and a universal constant K such that*

(1.5.12)

$$\sup_{t \in T} \int_0^\eta g\big(\mu\big(B(t,\varepsilon)\big)\big) \, d\varepsilon \; < \; K\Big(\eta|\log \eta| \; + \; \int_0^\eta H^{1/2}(\varepsilon) \, d\epsilon\Big),$$

for all $\eta > 0$.

This Lemma, together with Theorem 1.5.1, provides an alternative proof of Theorems 1.3.3 and 1.3.5, since the additional term of $\eta|\log \eta|$ in (1.5.12) can be absorbed into the integral (for small enough η) by changing K. In other words, the Lemma shows how entropy results follow from those for majorising measures. What is particularly interesting, however, is the proof of the Lemma, which actually shows how to construct a majorising measure when the entropy integral is *a priori* known to be finite.

Proof. For convenience we assume that $\operatorname{diam}(T) = 1$. For $n \geq 0$, let $\{A_{n,1}, ..., A_{n,N(2^{-n})}\}$ be a minimal family of d-balls of radius 2^{-n} which cover T. Set

$$(1.5.13) \qquad B_{n,k} \; = \; A_{n,k} \setminus \bigcup_{j<k} A_{n,j},$$

so that $\mathcal{B}_n = \{B_{n,1}, ..., B_{n,N(2^{-n})}\}$ is a partition of T and each B_i is contained in a d-ball of radius 2^{-n}. For each pair n, k choose a point $t_{n,k} \in B_{n,k}$ and then define a probability measure μ on T by

$$\mu(A) \; = \; \sum_{n=0}^\infty 2^{-(n+1)} \left(N(2^{-n})\right)^{-1} \sum_{k=1}^{N(2^{-n})} \delta_{t_{n,k}}(A),$$

where δ_{t_k} is the measure giving unit mass to t_k. This will be our majorising measure. To check that it satisfies (1.5.12), note first that if $\varepsilon \in (2^{-(n+1)}, 2^{-n}]$ then

$$\mu\left(B(t,\varepsilon)\right) \; \geq \; \left(2^{n+1} N(2^{-(n+1)})\right)^{-1},$$

for all $t \in T$. Consequently,

$$
\begin{aligned}
\int_0^{2^{-n}} \sqrt{\ln \frac{1}{\mu(B(t,\varepsilon))}}\, d\varepsilon \;\; &\leq \;\; \sum_{k=n+1}^{\infty} 2^{-k} \left(\ln \left(2^k N(2^{-k}) \right) \right)^{\frac{1}{2}} \\
&\leq \;\; \sum_{k=n+1}^{\infty} 2^{-k} \left(k \ln 2 \right)^{\frac{1}{2}} + 2 \int_0^{2^{-n}} \left(\ln(N(\varepsilon)) \right)^{\frac{1}{2}} d\varepsilon \\
&\leq \;\; (n+2) 2^{-n} \sqrt{\ln 2} + 2 \int_0^{2^{-n}} H^{1/2}(\varepsilon)\, d\varepsilon,
\end{aligned}
$$

the last line following from a little elementary algebra.

   This establishes (1.5.12) for dyadic η. The passage to general η follows via a monotonicity argument. $\qquad\square$

   Another class of examples for which it is easy to find a majorising measure is given by that of stationary fields over compact Abelian groups. Once again, we shall assume that you are either familiar with the notion of stationarity or will return after having read Chapter 5. Under stationarity it is perhaps not suprisingly that Haar measure is a majorising measure.

Theorem 1.5.3 *If f is stationary over a compact Abelian group T, then (1.5.1) and (1.5.2) hold with μ taken to be normalised Haar measure on T.*

   A very similar result holds if T is only locally compact. You can find the details in [92].

Proof. Since (1.5.1) is true for any probability measure on T, it also holds for Haar measure. Thus we need only prove the lower bound (1.5.2).

   Thus, assume that f is bounded, so that by Theorem 1.5.1 there exists a majorising measure μ satisfying (1.5.2). We need to show that μ can be replaced by Haar measure on T, which we denote by ν.

   Set

$$
D_\mu \;\overset{\Delta}{=}\; \sup \left\{ \eta : \mu\left(B(t,\eta) \right) < 1/2, \text{ for all } t \in T \right\},
$$

with D_ν defined analogously. With g as at (1.5.11), (1.5.2) can be rewritten as

$$
\int_0^{D_m} g\left(\mu(B(t,\varepsilon)) \right) d\varepsilon \;\leq\; K \mathbb{E} \left\{ \sup_{t \in T} f_t \right\},
$$

for all $t \in T$. Let τ be a random variable with distribution ν; i.e. τ is uniform on T. For each $\varepsilon > 0$, set $Z(\varepsilon) = \mu(B(\tau, \varepsilon))$. Then, for any $t_o \in T$,

$$
\begin{aligned}
\mathbb{E}\{Z(\varepsilon)\} &= \int_T \mu\big(B(t, \varepsilon)\big)\, \nu(dt) \\
&= \int_T \mu\big(t + B(t_o, \varepsilon)\big)\, \nu(dt) \\
&= \int_T \nu\big(t + B(t_o, \varepsilon)\big)\, \mu(dt) \\
&= \nu\big(B(t_o, \varepsilon)\big),
\end{aligned}
$$

where the second equality comes from the stationarity of f and the third and fourth from the properties of Haar measures.

Now note that $g(x)$ is convex over $x \in (0, \frac{1}{2})$, so that it is possible to define a function $\widehat{g}$ that agrees with it on $(0, \frac{1}{2})$, is bounded on $(\frac{1}{2}, \infty)$, and convex on all of $\mathbb{R}_+$. By Jensen's inequality,

$$
\widehat{g}\left(\mathbb{E}\{Z(\varepsilon)\}\right) \leq \mathbb{E}\left\{\widehat{g}\big(Z(\varepsilon)\big)\right\}.
$$

That is,

$$
\widehat{g}\big(\nu\big(B(t_o, \varepsilon)\big)\big) \leq \int_T \widehat{g}\big(\mu(B(t, \varepsilon)\big)\, \nu(dt).
$$

Finally, set $\Lambda = \min(D_\mu, D_\nu)$. Then

$$
\begin{aligned}
\int_0^\Lambda \widehat{g}\big(\nu(B(t_o, \varepsilon)\big)\, d\varepsilon &\leq \int_0^\Lambda d\varepsilon \int_T \widehat{g}\big(\mu(B(t, \varepsilon)\big)\, \nu(dt) \\
&= \int_T \int_0^\Lambda \nu(dt)\widehat{g}\big(\mu(B(t, \varepsilon)\big)\, d\varepsilon \\
&\leq K\mathbb{E}\left\{\sup_{t \in T} f_t\right\}.
\end{aligned}
$$

This is the crux of (1.5.2). The rest is left to you.   □

This more or completes our discussion of majorising measures. However, we still have two promises to fulfill before completing this chapter. The first of these is to show why entropy conditions are necessary, as well as sufficient, for boundedness and continuity if f is stationary. This is Theorem 1.5.4 below. Its proof is really a little dishonest, since it relies on the main majorising measure result Theorem 1.5.1, which we have not proven. It could in fact be proven without resort to majorising measures, and was done so originally. However, for a chapter that was supposed to be only an 'introduction' to the general theory of Gaussian processes, we have already used up more than enough space, and so shall make do with what we have at hand already.

Theorem 1.5.4 *Let f be a centered stationary Gaussian process on a compact group T. Then the following three conditions are equivalent:*

(1.5.14) $\qquad\qquad\qquad$ *f is a.s. continuous on T,*

(1.5.15) $\qquad\qquad\qquad$ *f is a.s. bounded on T,*

(1.5.16) $\qquad\qquad\qquad \displaystyle\int_0^\infty H^{1/2}(\varepsilon)\,d\varepsilon \ < \ \infty.$

Proof. That (1.5.14) implies (1.5.15) is obvious. That (1.5.16) implies (1.5.14) is Lemma 1.5.2 together with Theorem 1.5.1. Thus it suffices to show that (1.5.15) implies (1.5.16), which we shall now do.

Note firstly that by Theorem 1.5.3 we know that

(1.5.17) $\qquad\qquad \displaystyle\sup_{t\in T}\int_0^\infty g\big(\mu\big(B(t,\varepsilon)\big)\big)\,d\varepsilon \ < \ \infty$

for μ normalised Haar measure on T. Furthermore, by stationarity, the value of the integral must be independent of t.

For $\varepsilon \in (0,1)$ let $M(\varepsilon)$ be the maximal number of points $\{t_k\}_{k=1}^{M(\varepsilon)}$ in T for which

$$\min_{1\le j,k\le M(\varepsilon)} d(t_j,t_k) \ > \ \varepsilon.$$

It is easy to check that

$$N(\varepsilon) \ \le \ M(\varepsilon) \ \le \ N(\varepsilon/2).$$

Thus, since μ is a probability measure, we must have

$$\mu\big(B(t,\varepsilon)\big) \ \le \ \big(N(2\varepsilon)\big)^{-1}.$$

Consequently, by (1.5.17) and, in particular, its independence on t

$$\infty \ > \ \int_0^\infty g\big(\mu\big(B(t,\varepsilon)\big)\big)\,d\varepsilon \ \ge \ \int_0^\infty \big(\ln N(2\varepsilon)\big)^{\frac12}\,d\varepsilon$$

$$= \ 2\int_0^\infty H^{1/2}(\varepsilon)\,d\varepsilon,$$

which proves the theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Our second and final responsibility is to prove that claims we made back at (1.4.7) are correct. That they are is the statement of the following corollary to Theorem 1.5.4.

Corollary 1.5.5 *If f is centered and stationary on open $T \subset \mathbb{R}^N$ with covariance function C, and*

(1.5.18) $\qquad \dfrac{K_1}{(-\ln|t|)^{1+\alpha_1}} \ \le \ C(0) - C(t) \ \le \ \dfrac{K_2}{(-\ln|t|)^{1+\alpha_2}},$

for $|t|$ small enough, then f will be sample path continuous if $\alpha_2 > 0$ and discontinuous if $\alpha_1 < 0$.

Proof. Recall from Theorem 1.4.1 that the basic energy integral in (1.5.16) converges or diverges together with

$$(1.5.19) \qquad\qquad \int_\delta^\infty p\left(e^{-u^2}\right) du,$$

where

$$p^2(u) \;=\; 2 \sup_{|t| \leq u} \left[C(0) - C(t)\right].$$

(cf. (1.4.2).) Applying the bounds in (1.5.18) to evaluate (1.5.19) and applying the extension of Theorem 1.5.3 to non-compact groups easily proves the corollary.   □

2
Gaussian inequalities

Basic Statistics has its Chebychev inequality, Martingale Theory has its maximal inequalities, Markov processes have large deviations, but all pale in comparison to the power and simplicity of the coresponding basic inequality of Gaussian processes. This inequality discovered independently, and established with very different proofs, by Borell [31] and Tsirelson, Ibragimov and Sudakov (TIS) [150]. For brevity, we shall call it the Borell-TIS inequality. In the following section we shall treat it in some detail.

A much older inequality, that allows comparison between the suprema of different Gaussian processes, is due to Slepian, and we shall describe it and some of its newer relatives in Section 2.2.

2.1 Borell-TIS inequality

One of the first facts we learnt about Gaussian processes was that if $X \sim N(0, \sigma^2)$ then, for all $u > 0$,

(2.1.1)
$$\left(\frac{\sigma}{\sqrt{2\pi}u} - \frac{\sigma^3}{\sqrt{2\pi}u^3} \right) e^{-\frac{1}{2}u^2/\sigma^2} \leq \mathbb{P}\{X > u\} \leq \frac{\sigma}{\sqrt{2\pi}u} e^{-\frac{1}{2}u^2/\sigma^2}.$$

(cf. (1.2.2).) One immediate consequence of this is that

(2.1.2)
$$\lim_{u \to \infty} u^{-2} \ln \mathbb{P}\{X > u\} = -(2\sigma^2)^{-1}.$$

There is a classical result of Landau and Shepp [89] and Marcus and Shepp [103] that gives a result closely related to (2.1.2), but for the supremum of a general centered Gaussian process. If we assume that f_t is a.s. bounded, then they showed that

$$(2.1.3) \qquad \lim_{u \to \infty} u^{-2} \, \ln \mathbb{P} \left\{ \sup_{t \in T} f_t > u \right\} \;=\; -(2\sigma_T^2)^{-1},$$

where

$$\sigma_T^2 \;\triangleq\; \sup_{t \in T} \mathbb{E}\{f_t^2\}$$

is a notation that will remain with us throughout this section. An immediate consequence of (2.1.3) is that, for all $\varepsilon > 0$ and large enough u,

$$(2.1.4) \qquad \mathbb{P} \left\{ \sup_{t \in T} f_t > u \right\} \;\leq\; e^{\varepsilon u^2 - u^2/2\sigma_T^2}.$$

Since $\varepsilon > 0$ is arbitrary, comparing (2.1.4) and (2.1.1) we reach the rather surprising conclusion that the supremum of a centered, bounded Gaussian process behaves much like a single Gaussian variable with a suitably chosen variance.

In Chapter 10 we shall work very hard to close the gap between (2.1.1) and (2.1.4) (i.e. between u^{-1} and $e^{\varepsilon u^2}$), however now we want to see from where (2.1.4) comes.

In fact, all of the above inequalities are special cases of a *non-asymptotic* result due, as mentioned above, independently, and with very different proofs, to Borell [31] and Tsirelson, Ibragimov and Sudakov (TIS) [150].

Theorem 2.1.1 (Borell-TIS inequality) *Let f_t be a centered Gaussian process, a.s. bounded on T. Write $\|f\| = \|f\|_T = \sup_{t \in T} f_t$. Then $\mathbb{E}\{\|f\|\} < \infty$, and, for all $u > 0$,*

$$(2.1.5) \qquad \mathbb{P}\left\{ \|f\| - \mathbb{E}\{\|f\|\} > u \right\} \;\leq\; e^{-u^2/2\sigma_T^2}.$$

Before we look at the proof of (2.1.5), which we refer to as the Borell-TIS inequality[1], we take a moment to look at some of its conseqences, which are many and major. It is no exaggeration to say that this inequality is today the single most important tool in the general theory of Gaussian processes.

An immediate and trivial consequence of (2.1.5) is that, for all $u > \mathbb{E}\{\|f\|\}$,

$$\mathbb{P}\{ \|f\| > u \} \;\leq\; e^{-(u - \mathbb{E}\|f\|)^2/2\sigma_T^2},$$

---

[1] Actually, Theorem 2.1.1 is not in the same form as Borell's original inequality, in which $\mathbb{E}\{\|f\|\}$ was replaced by the median of $\|f\|$. However, the two forms are equivalent. For this and other variations of (2.1.5), including extensions to Banach space valued processes for which $\| \ \|$ is the norm, see the more detailed treatments of [29, 53, 64, 92, 98]. To see how the Borell-TIS inequality fits into the wider theory of concentration inequalities, see the recent book [91] by Ledoux.

so that both (2.1.3) and (2.1.4) follow from the Borell-TIS inequality.

Indeed, a far stronger result is true, for (2.1.4) can now be replaced by

$$\mathbb{P}\left\{\,\|f\| > u\,\right\} \leq e^{Cu - u^2/2\sigma_T^2},$$

where C is a constant depending only on $\mathbb{E}\{\|X\|\}$, and we know how to at least bound this quantity from Theorem 1.3.5.

Note that, despite the misleading notation, $\|\ \| \equiv \sup$ is not a norm, and that very often one needs bounds on the tail of $\sup_t |f_t|$, which does give a norm. However, symmetry immediately gives

$$(2.1.6) \qquad \mathbb{P}\left\{\sup_t |f_t| > u\right\} \leq 2\,\mathbb{P}\left\{\sup_t f_t > u\right\},$$

so that the Borell-TIS inequality helps out here as well.

Here is a somewhat more significant consequence of the Borell-TIS inequality.

Theorem 2.1.2 *For f centered, Gaussian,*

$$(2.1.7) \qquad \mathbb{P}\{\|f\| < \infty\} = 1 \qquad \Longleftrightarrow \qquad \mathbb{E}\{\|f\|\} < \infty$$
$$\Longleftrightarrow \qquad \mathbb{E}\left\{e^{\alpha\|f\|^2}\right\} < \infty$$

for sufficiently small α.

Proof. The existence of the exponential moments of $\|f\|$ implies the existence of $\mathbb{E}\{\|f\|\}$, and this in turn implies the a.s. finiteness of $\|f\|$. Furthermore, since by Theorem 2.1.1 we already know that the a.s. finiteness of $\|f\|$ entails that of $\mathbb{E}\{\|f\|\}$, all that remains is to prove is that the a.s. finiteness of $\|f\|$ also implies the existence of exponential moments.

But this is an easy consequence of the Borell-TIS inequality, since, with both $\|f\|$ and $\mathbb{E}\{\|f\|\}$ now finite,

$$
\begin{aligned}
\mathbb{E}\left\{e^{\alpha\|f\|^2}\right\} &= \int_0^\infty \mathbb{P}\left\{e^{\alpha\|f\|^2} > u\right\} du \\
&\leq e^\alpha + \mathbb{E}\{\|f\|\} + \int_{e^\alpha \vee \mathbb{E}\{\|f\|\}}^\infty \mathbb{P}\left\{\|f\| > \sqrt{\ln u^{1/\alpha}}\right\} du \\
&\leq e^\alpha + \mathbb{E}\{\|f\|\} \\
&\quad + 2\int_{e^\alpha \vee \mathbb{E}\{\|f\|\}}^\infty \exp\left(\frac{-\left(\sqrt{\ln u^{1/\alpha}} - \mathbb{E}\{\|f\|\}\right)^2}{2\sigma_T^2}\right) du \\
&\leq \mathbb{E}\{\|f\|\} \\
&\quad + 4\alpha \int_0^\infty u \exp\left(\frac{-(u - \mathbb{E}\{\|f\|\})^2}{2\sigma_T^2}\right) \exp\left\{\alpha u^2\right\} du,
\end{aligned}
$$

which is clearly finite for small enough α. $\square$

Recall Theorems 1.3.3 and 1.3.5 which established, respectively, the a.s. boundedness of $\|f\|$ and a bound on the modulus of continuity $\omega_{f,d}(\delta)$ under essentially identical entropy conditions. It was rather irritating back there that we had to establish each result independently, since it is "obvious" that one should imply the other. A simple application of the Borell-TIS inequality almost does this.

Theorem 2.1.3 *Suppose that f is a.s. bounded on T. Then f is also a.s uniformly continuous (with respect to the canonical metric d) if, and only if,*

$$(2.1.8) \qquad \lim_{\eta \to 0} \phi(\eta) \;=\; 0,$$

where

$$(2.1.9) \qquad \phi(\eta) \;\triangleq\; \mathbb{E}\left\{ \sup_{d(s,t)<\eta} (f_s - f_t) \right\}.$$

Furthermore, under (2.1.8), for all $\varepsilon > 0$ there exists an a.s. finite random variable $\eta > 0$ such that

$$(2.1.10) \qquad \omega_{f,d}(\delta) \;\leq\; \phi(\delta)\,|\ln\phi(\delta)|^{\varepsilon},$$

for all $\delta \leq \eta$.

Proof. We start with necessity. For almost every ω we have

$$\lim_{\eta \to 0} \sup_{d(s,t)<\eta} |f_s(\omega) - f_t(\omega)| \;=\; 0.$$

But (2.1.8) now follows from dominated convergence and Theorem 2.1.2.

For sufficiency, note that from (2.1.8) we can find a sequence $\{\delta_n\}$ with $\delta_n \to 0$ such that $\phi(\delta_n) \leq 2^{-n}$. Set $\delta'_n = \min(\delta_n, 2^{-n})$, and consider the event

$$A_n \;=\; \left\{ \sup_{d(s,t)<\delta'_n} |f_s - f_t| \;>\; 2^{-n/2} \right\}.$$

The Borell-TIS inequality (cf. (2.1.6)) gives that, for $n \geq 3$,

$$\begin{aligned}
\mathbb{P}\{A_n\} &\leq& 2\exp\left(-\tfrac{1}{2}(2^{-n/2} - 2^{-n})^2/2^{-2n}\right) \\
&\leq& K\exp\left(-2^{n-1}\right).
\end{aligned}$$

Since $P\{A_n\}$ is an admirably summable series, Borel-Cantelli gives us that f is a.s. uniformly d-continuous, as required.

To complete the proof we need to establish the bound (2.1.10) on $\omega_{f,d}$. Note first that

$$
\begin{aligned}
\mathrm{diam}(S) &= \sup_{s,t \in S} \left(\mathbb{E}\{|f_t - f_s|^2\} \right)^{1/2} \\
&= \sqrt{2\pi} \sup_{s,t \in S} \mathbb{E}\{|f_t - f_s|\} \\
&\leq \sqrt{2\pi}\, \mathbb{E}\left\{ \sup_{s,t \in S} f_t - f_s \right\} \\
&= \sqrt{2\pi}\, \phi(\mathrm{diam}(S)),
\end{aligned}
$$

where the second line is an elementary Gaussian computation and the third uses the fact that $\sup_{s,t \in S}(f_t - f_s)$ is non-negative. Consequently, we have that

$$(2.1.11) \qquad\qquad \delta \leq \sqrt{2\pi}\, \phi(\delta),$$

for all $\delta > 0$.

Now define the numbers

$$
\delta_n = \sup \left\{ \delta : \phi(\delta) \leq e^{-n} \right\}
$$

and, for $\varepsilon > 0$, the events

$$
B_n = \left\{ \sup_{d(s,t) < \delta_n} |f_s - f_t| > \phi(\delta_n)\, |\ln \phi(\delta_n)|^{\varepsilon/2} \right\}.
$$

Then the Borell-TIS inequality gives

$$
\begin{aligned}
\mathbb{P}\{B_n\} &\leq 4 \exp\left\{ -\tfrac{1}{2} \left(|\ln \phi(\delta_n)|^{\varepsilon/2} - 1 \right)^2 \frac{\phi^2(\delta_n)}{\delta_n^2} \right\} \\
&\leq K_1 \exp\left\{ -K_2 n^\epsilon \right\},
\end{aligned}
$$

the second inequality following from (2.1.11) and the definition of δ_n.

Since $\sum_n \mathbb{P}\{B_n\} < \infty$, we have that for $n \geq N(\omega)$

$$
\omega_{f,d}(\delta_n) \leq \phi(\delta_n) \, |\ln \phi(\delta_n)|^{\varepsilon/2}.
$$

Monotonicity of $\omega_{f,d}$, along with separability, complete the proof. $\square$

We now turn to the proof of the Borell-TIS inequality. There are essentially three quite different ways to tackle this proof. Borell's original proof relied on isoperimetric inequalities. While isoperimetric inequalities may be natural for a book with the word "geometry" in its title, we shall avoid them, since they involve setting up a number of concepts for which we shall have no other need. The proof of Tsirelson, Ibragimov and Sudakov used

Itô's formula from stochastic calculus. This is one of our[2] favourite proofs, since as one of the too few links between the Markovian and Gaussian worlds of stochastic processes, it is to be prized.

   We shall, however, take a more direct route, which we learnt about from the excellent collection of exercises [37], although its roots are much older. The first step in this route involves the two following lemmas, which are of independent interest.

Lemma 2.1.4 *Let X and Y be independent k-dimensional vectors of centered, unit variance, independent, Gaussian variables. If $f, g : \mathbb{R}^k \to \mathbb{R}$ are bounded C^2 functions then*

(2.1.12)

$$\mathrm{Cov}\,(f(X), g(X)) \;=\; \int_0^1 \mathbb{E}\left\{\left\langle \nabla f(X), \nabla g(\alpha X + \sqrt{1 - \alpha^2} Y)\right\rangle\right\} d\alpha,$$

where $\nabla f(x) = \left(\frac{\partial f}{\partial x_i} f(x)\right)_{i=1,\ldots,k}$.

Proof. It suffices to prove the lemma with $f(x) = e^{i\langle t, x\rangle}$ and $g(x) = e^{i\langle s, x\rangle}$, with $s, t, x \in \mathbb{R}^k$. Standard approximation arguments (which is where the requirement that f is C^2 appears) will do the rest. Write

$$\varphi(t) \;\triangleq\; \mathbb{E}\left\{e^{i\langle t, X\rangle}\right\} \;=\; e^{|t|^2/2}.$$

(cf. (1.2.4).) It is then trivial that

$$\mathrm{Cov}\,(f(X), g(X)) \;=\; \varphi(s + t) - \varphi(s)\varphi(t).$$

---

[2]Or at least one of RA's favourite proofs. Indeed this approach was used in RA's lecture notes [4]. There, however, there is a problem, for in the third line from the bottom of page 46 appear the words "To complete the proof note simply that...". The word "simply" is simply out of place, and, in fact, Lemma 2.2 there is false as stated. A correction (with the help Amir Dembo) appears, *en passant*, in the 'Proof of Theorem 2.1.1' below.

On the other hand, computing the integral in (2.1.12) gives

$$\int_0^1 \mathbb{E}\left\{ \left\langle \nabla f(X), \nabla g(\alpha X + \sqrt{1-\alpha^2}Y) \right\rangle \right\} d\alpha$$

$$= \int_0^1 d\alpha \, \mathbb{E}\left\{ \left\langle \left(it_j e^{i\langle t, X\rangle} \right)_j , \left(is_j e^{i\langle s, \alpha X + \sqrt{1-\alpha^2}Y\rangle} \right)_j \right\rangle \right\}$$

$$= -\int_0^1 d\alpha \sum_j \mathbb{E}\left\{ e^{i\langle t+\alpha s, X\rangle} \right\} \mathbb{E}\left\{ e^{i\langle s, \sqrt{1-\alpha^2}Y\rangle} \right\}$$

$$= -\int_0^1 d\alpha \, \langle s, t\rangle \, e^{(|t|^2 + 2\alpha\langle s,t\rangle + |s|^2)/2}$$

$$= -\varphi(s)\varphi(t)\left(1 - e^{\langle s,t\rangle} \right)$$

$$= \varphi(s+t) - \varphi(s)\varphi(t),$$

which is all that we need.   □

Lemma 2.1.5 *Let X be as in Lemma 2.1.4. If $h : \mathbb{R}^k \to \mathbb{R}$ is Lipschitz, with Lipschitz constant 1 – i.e. $|h(x) - h(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}^k$ – and if $\mathbb{E}\{h(X)\} = 0$ then, for all $t > 0$,*

$$(2.1.13) \qquad\qquad \mathbb{E}\left\{ e^{th(X)} \right\} \leq e^{t^2/2}.$$

Proof. Let Y be an independent copy of X and α a uniform random variable on $[0,1]$. Define the pair (X, Z_α) via

$$(X, Z_\alpha) \triangleq \left(X, \alpha X + \sqrt{1-\alpha^2}Y \right).$$

Take h as in the statement of the lemma, $t \geq 0$ fixed, and define $g = e^{th}$. Applying (2.1.12) gives

$$\mathbb{E}\{h(X)g(X)\} = \int_0^1 \mathbb{E}\{\langle \nabla g(X), \nabla h(Z_\alpha)\rangle\} \, d\alpha$$

$$= t \int_0^1 \mathbb{E}\left\{ \langle \nabla h(X), \nabla h(Z_\alpha)\rangle \, e^{th(X)} \right\} d\alpha$$

$$\leq t\mathbb{E}\left\{ e^{th(X)} \right\},$$

using the Lipschitz property of h. Let u be the function defined by

$$e^{u(t)} = \mathbb{E}\left\{ e^{th(X)} \right\}.$$

Then

$$\mathbb{E}\left\{ h(X)e^{th(X)} \right\} = u'(t) \, e^{u(t)},$$

so that from the preceeding inequality $u'(t) \leq t$. Since $u(0) = 0$ it follows that $u(t) \leq t^2/2$ and we are done.   □

The following lemma gives the crucial step towards proving the Borell-TIS inequality.

Lemma 2.1.6 *Let X be a k-dimensional vector of centered, unit variance, independent, Gaussian variables. If $h : \mathbb{R}^k \to \mathbb{R}$ is Lipschitz, with Lipschitz constant σ then, for all $u > 0$,*

$$(2.1.14) \qquad \mathbb{P}\left\{h(X) - \mathbb{E}\{h(X)\} > u\right\} \leq e^{-\frac{1}{2}u^2/\sigma^2}.$$

Proof. By scaling it suffices to prove the result for $\sigma = 1$. A Chebyschev argument and (2.1.13) therefore immediately yield that for every $t, u > 0$

$$\mathbb{P}\left\{h(X) - \mathbb{E}\{h(X)\} > u\right\} \leq e^{\frac{1}{2}t^2 - tu}.$$

Taking the optimal choice of $t = u$ gives (2.1.14) and we are done.   □

We now have all we need for the

Proof of Theorem 2.1.1 We have two things to prove. Firstly, Theorem 2.1.1 will follow immediately from Lemma 2.1.6 in the case of finite T and f having i.i.d. components once we show that sup(.), or max(.) in this case, is Lipschitz. We shall show this, and lift the i.i.d. restriction, in one step. The second part of the proof involves lifting the result from finite to general T.

Thus, suppose T is finite, in which case we can write it as $\{1, \ldots, k\}$. Let C be the $k \times k$ covariance matrix of f on T, with components $c_{ij} = \mathbb{E}\{f_i f_j\}$, so that

$$\sigma_T^2 = \sup_{1 \leq i \leq k} c_{ii} = \sup_{1 \leq i \leq k} \mathbb{E}\{f_i^2\}.$$

Let W be a vector of independent, standard Gaussian variables, and A such that $A' \cdot A = C$. Thus $f \overset{\mathcal{L}}{=} AW$, and $\max_i f_i \overset{\mathcal{L}}{=} \max_i (AW)_i$, where $\overset{\mathcal{L}}{=}$ indicates equivalence in distribution (law).

Consider the function $h(x) = \max_i (Ax)_i$. Then

$$
\begin{aligned}
\left| \max_i (Ax)_i - \max_i (Ay)_i \right| &= \left| \max_i (e_i Ax) - \max_i (e_i Ay) \right| \\
&\leq \max_i \left| e_i A(x - y) \right| \\
&\leq \max_i \left| e_i A \right| \cdot |x - y|,
\end{aligned}
$$

where, as usual, e_i is the vector with 1 in position i and zeroes elsewhere. The first inequality above is elementary and the second is Cauchy-Schwartz. But

$$|e_i A|^2 = e_i' A' A e_i = e_i' C e_i = c_{ii},$$

so that

$$\left| \max_i (Ax)_i - \max_i (Ay)_i \right| \leq \sigma_T \left| x - y \right|.$$

In view of the equivalence in law of $\max_i f_i$ and $\max_i (AW)_i$ and Lemma 2.1.6, this establishes the theorem for finite T.

We now turn to lifting the result from finite to general T. This is, almost, an easy exercise in approximation. For each $n > 0$ let T_n be a finite subset of T such that $T_n \subset T_{n+1}$ and T_n increases to a dense subset of T. By separability,

$$\sup_{t \in T_n} f_t \;\overset{a.s.}{\to}\; \sup_{t \in T} f_t,$$

and, since the convergence is monotone, we also have that

$$\mathbb{P}\left\{ \sup_{t \in T_n} f_t \geq u \right\} \to \mathbb{P}\left\{ \sup_{t \in T} f_t \geq u \right\} \quad \text{and} \quad \mathbb{E}\left\{ \sup_{t \in T_n} f_t \right\} \to \mathbb{E}\left\{ \sup_{t \in T} f_t \right\}.$$

Since $\sigma_{T_n}^2 \to \sigma_T^2 < \infty$, (again monotonely) this would be enough to prove the general version of the Borell-TIS inequality from the finite T version if only we knew that the one worrisome term, $\mathbb{E}\{\sup_T f_t\}$, were definitely finite, as claimed in the statement of the theorem. Thus if we show that the assumed a.s. finiteness of $\|f\|$ implies also the finiteness of its mean, we shall have a complete proof to both parts of the theorem.

We proceed by contradiction. Thus, assume $E\{\|f\|\} = \infty$, and choose $u_o > 0$ such that

$$e^{-u_o^2/\sigma_T^2} \leq \tfrac{1}{4} \quad \text{and} \quad \mathbb{P}\left\{ \sup_{t \in T} f_t < u_o \right\} \geq \tfrac{3}{4}.$$

Now choose $n \geq 1$ such that $\mathbb{E}\{\|f\|_{T_n}\} > 2u_o$, possible since $\mathbb{E}\{\|f\|_{T_n}\} \to \mathbb{E}\{\|f\|_T\} = \infty$. The Borell-TIS inequality on the finite space T_n then gives

$$
\begin{aligned}
\tfrac{1}{2} \;\geq\; 2\,e^{-u_o^2/\sigma_T^2} \quad &\geq\quad 2\,e^{-u_o^2/\sigma_{T_n}^2} \\
&\geq\quad \mathbb{P}\left\{ \left| \|f\|_{T_n} - \mathbb{E}\{\|f\|_{T_n}\} \right| > u_o \right\} \\
&\geq\quad \mathbb{P}\left\{ \mathbb{E}\{\|f\|_{T_n}\} - \|f\|_T > u_o \right\} \\
&\geq\quad \mathbb{P}\left\{ \|f\|_T < u_o \right\} \\
&\geq\quad \tfrac{3}{4}.
\end{aligned}
$$

This provides the required contradiction, and so we are done.   □

2.2   Comparison inequalities

The theory of Gaussian processes is rich in comparison inequalities, where by this term we mean results of the form "if f is a 'rougher' process than

g, and both are defined over the same parameter space, then $\|f\|$ will be 'larger' than $\|g\|$". The most basic of these is Slepian's inequality.

Theorem 2.2.1 (Slepian's inequality) *If f and g are a.s. bounded, centered Gaussian processes on T such that $\mathbb{E}\{f_t^2\} = \mathbb{E}\{g_t^2\}$ for all $t \in T$ and*

$$(2.2.1) \qquad \mathbb{E}\left\{(f_t - f_s)^2\right\} \;\leq\; \mathbb{E}\left\{(g_t - g_s)^2\right\}$$

for all $s, t \in T$, then for all real u

$$(2.2.2) \qquad \mathbb{P}\{\|f\| > u\} \;\leq\; \mathbb{P}\{\|g\| > u\}.$$

Furthermore,

$$(2.2.3) \qquad \mathbb{E}\{\|f\|\} \leq \; \mathbb{E}\{\|g\|\}.$$

Slepian's inequality is so natural, that it hardly seems to require a proof, and hardly the rather analytic, non-probabilistic one that will follow. To see that there is more to the story than meets the eye, one need only note that (2.2.2) does *not* follow from (2.2.1) if we replace[3] $\sup_T f_t$ by $\sup_T |f_t|$ and $\sup_T g_t$ by $\sup_T |g_t|$.

Slepian's inequality is based on the following technical lemma. the proof of which, in all its important details, goes back to Slepian's original paper [135].

Lemma 2.2.2 *Let $f_1, ..., f_k$ be centered Gaussian variables with covariance matrix $C = (c_{ij})_{i,j=1}^k$, $c_{ij} = \mathbb{E}\{f_i f_j\}$. Let $h : \mathbb{R}^k \to \mathbb{R}$ be C^2, and assume that, together with its derivatives, it satisfies a $o(|x|^d)$ growth condition at infinity for some finite d[4]. Let*

$$(2.2.4) \qquad \mathcal{H}(C) \;=\; \mathbb{E}\left\{h(f_1, \ldots, f_k)\right\},$$

and assume that for a pair (i, j), $1 \leq i < j \leq k$

$$(2.2.5) \qquad \frac{\partial^2 h(x)}{\partial x_i \partial x_j} \;\geq\; 0$$

for all $x \in \mathbb{R}^k$. Then $\mathcal{H}(C)$ is an increasing function of c_{ij}.

---

[3] For a counterexample to "Slepian's inequality for absolute values" take $T = \{1, 2\}$, with f_1 and f_2 standard normal with correlation ρ. Writing $P_\rho(u)$ for the probability under correlation ρ that $\max(|f_1|, |f_2|) > u$, it is easy to check that, for all $u > 0$, $P_{-1}(u) < P_0(u)$, while $P_0(u) > P_1(u)$, which negates the monotonicity required by Slepian's inequality.

[4] We could actually manage with h twice differentiable only in the sense of distributions. This would save the approximation argument following (2.2.7) below, and would give a neater, albeit slightly more demanding, proof of Slepian's inequality, as in [92].

Proof. We have to show that

$$\frac{\partial \mathcal{H}(C)}{\partial c_{ij}} \geq 0$$

whenever $\partial^2 h / \partial x_i \partial x_j \geq 0$.

   To make our lives a little easier we assume that C is non-singular, so that it makes sense to write $\varphi(x) = \varphi_C(x)$ for the centered Gaussian density on $\mathbb{R}^k$ with covariance matrix C. Straightforward algebra shows that[5]

$$(2.2.6) \quad \frac{\partial \varphi}{\partial c_{ii}} = \frac{1}{2}\frac{\partial^2 \varphi}{\partial x_i^2}, \qquad \frac{\partial \varphi}{\partial c_{ij}} = \frac{\partial^2 \varphi}{\partial x_i \partial x_j}, \qquad i \neq j.$$

   Applying this and our assumptions on h to justify two integrations by parts, we obtain

$$\frac{\partial \mathcal{H}(C)}{\partial c_{ij}} = \int_{\mathbb{R}^k} h(x)\,\frac{\partial \varphi(x)}{\partial c_{ij}}\,dx = \int_{\mathbb{R}^k} \frac{\partial^2 h(x)}{\partial x_i \partial x_j}\,\varphi(x)\,dx \geq 0.$$

This completes the proof for the case of non-singular C. The general case can be handled by approximating a singular C via a sequence of non-singular covariance matrices. □

Proof of Theorem 2.2.1 By separability, and the final argument in the proof of the Borell-TIS inequality, it suffices to prove (2.2.2) for T finite. Note that since $\mathbb{E}\{f_t^2\} = \mathbb{E}\{g_t^2\}$ for all $t \in T$, (2.2.1) implies that $\mathbb{E}\{f_s f_t\} \geq \mathbb{E}\{g_s g_t\}$ for all $s, t \in T$. Let $h(x) = \prod_{i=1}^{k} h_i(x_i)$, where each h_i is a positive non-increasing, C^2 function satisfying the growth conditions placed on h in the statement of Lemma 2.2.2, and k is the number of points in T. Note that, for $i \neq j$

$$\frac{\partial^2 h(x)}{\partial x_i \partial x_j} = h_i'(x_i)h_j'(x_j)\prod_{\substack{n \neq i \\ n \neq j}} h_n(x_n) \geq 0,$$

since both h_i' and h_j' are non-positive. It therefore follows from Lemma 2.2.2 that

$$(2.2.7) \qquad \mathbb{E}\left\{\prod_{i=1}^{k} h_i(f_i)\right\} \geq \mathbb{E}\left\{\prod_{i=1}^{k} h_i(g_i)\right\}.$$

Now take $\{h_i^{(n)}\}_{n=1}^{\infty}$ to be a sequence of positive, non-increasing, C^2 approximations to the indicator function of the interval $(-\infty, \lambda]$, to derive that

$$\mathbb{P}\{\|f\| < u\} \geq \mathbb{P}\{\|g\| < u\},$$

---

[5]This is, of course, the heat equation of PDE theory.

which implies (2.2.2).

To complete the proof, all that remains is to show that (2.2.2) implies (2.2.3). But this is a simple consequence of integration by parts, since

$$
\begin{aligned}
\mathbb{E}\{\|f\|\} &= \int_0^\infty \mathbb{P}\{\|f\| > u\}\,du \;-\; \int_{-\infty}^0 \mathbb{P}\{\|f\| < u\}\,du \\
&\leq \int_0^\infty \mathbb{P}\{\|g\| > u\}\,du \;-\; \int_{-\infty}^0 \mathbb{P}\{\|g\| < u\}\,du \\
&= \mathbb{E}\{\|g\|\}.
\end{aligned}
$$

This completes the proof. $\square$

As mentioned above, there are many extensions of Slepian's inequality, the most important of which is probably the following.

Theorem 2.2.3 (Sudakov-Fernique inequality) *Let f and g be a.s. bounded Gaussian processes on T. If*

$$\mathbb{E}\{f_t\} \;=\; \mathbb{E}\{g_t\}$$

and

$$\mathbb{E}\left\{(f_t - f_s)^2\right\} \;\leq\; \mathbb{E}\left\{(g_t - g_s)^2\right\}$$

for all $s, t \in T$, then

(2.2.8) $$\mathbb{E}\{\|f\|\} \leq \mathbb{E}\{\|g\|\}.$$

In other words, a Slepian-like inequality holds without a need to assume either zero mean or identical variance for the compared processes. However, in this case we have only the weaker ordering of expectations of (2.2.3) and not the stochastic domination of (2.2.2).

The original version of this inequality assumed zero means, and its proof involved considerable calculus, in spirit not unlike that in the proof of Slepian's inequality. (cf. [63] or [80]). The proof that we give is due to Sourav Chatterjee [36] who, at the time of writing, is a gifted young graduate student at Stanford. It starts with a simple and well known lemma.

Lemma 2.2.4 *Let $X = (X_1, ..., X_k)$ be a vector of centered Gaussian variables with arbitrary covariance matrix, and let $h : \mathbb{R}^k \to \mathbb{R}$ be C^1, with h and its first order derivatives satisfying an $o(|x|^d)$ growth condition at infinity for some finite d. Then, for $1 \leq i \leq k$,*

(2.2.9) $$\mathbb{E}\{X_i h(X)\} \;=\; \sum_{j=1}^k \mathbb{E}\{X_i X_j\}\,\mathbb{E}\{h_j(X)\}.$$

where $h_j \triangleq \partial h / \partial x_j$.

Proof. Assume firstly that the X_j are independent and have unit variance. Then,

$$(2.2.10) \qquad \mathbb{E}\left\{X_i h(X)\right\} = \int_{\mathbb{R}^k} x_i h(x) e^{-|x|^2/2}\, dx$$

$$= \int_{\mathbb{R}^k} h_i(x) e^{-|x|^2/2}\, dx$$

$$= \mathbb{E}\left\{h_i(X)\right\},$$

where the second equality is via a simple integration by parts. This is (2.2.9) for this case.

For the general case, write $C = A^T A$ for the covariance matrix of X, and define $h'(x) = h(Ax)$. Let $X' = (X'_1, \ldots, X'_k)$ be a vector of independent, standard normal variables. Then, $X \sim AX'$, and writing a_{mn} and a^T_{mn} for the elements of A and A^T and applying (2.2.10),

$$\mathbb{E}\left\{X_i h(X)\right\} = \mathbb{E}\left\{\sum_{m=1}^{k} a_{im} X'_m h'(X')\right\}$$

$$= \sum_{m=1}^{k} a_{im} \mathbb{E}\left\{h'_m(X')\right\}$$

$$= \sum_{m=1}^{k} a_{im} \mathbb{E}\left\{\sum_{j=1}^{k} a_{jm} h_j(X)\right\}$$

$$= \sum_{j=1}^{k} \mathbb{E}\left\{h_j(X)\right\} \sum_{m=1}^{k} a_{im} a^T_{mj}$$

$$= \sum_{j=1}^{k} \mathbb{E}\left\{X_i X_j\right\} \mathbb{E}\left\{h_j(X)\right\},$$

as required. $\qquad\square$

The Sudakov-Fernique inequality on finite spaces is a special case of the following result, whose proof, *en passant,* actually provides more. For general (non-finite) spaces the arguments that we have previously used to go from finite to general parameter work here as well, giving the classic Sudakov-Fernique inequality with no extras.

Theorem 2.2.5 *Let $X = (X_1, \ldots, X_k)$ and $Y = (Y_1, \ldots, Y_k)$ be Gaussian vectors with $\mathbb{E}\{X_i\} = \mathbb{E}\{Y_i\}$ for all i. Set*

$$\gamma^X_{ij} = \mathbb{E}\left\{(X_i - X_j)^2\right\}, \qquad \gamma^Y_{ij} = \mathbb{E}\left\{(Y_i - Y_j)^2\right\}$$

and

$$\gamma \triangleq \sup_{1 \le i,j \le k} \left|\gamma^X_{ij} - \gamma^Y_{ij}\right|.$$

Then

(2.2.11) $\qquad \left| \mathbb{E}\left\{ \max_i X_i \right\} - \mathbb{E}\left\{ \max_i Y_i \right\} \right| \;\leq\; \sqrt{2\gamma \log k}.$

If, in addition, $\gamma_{ij}^X \leq \gamma_{ij}^Y$ for $1 \leq i, j \leq k$, then

(2.2.12) $\qquad \mathbb{E}\left\{ \max_i X_i \right\} \;\leq \mathbb{E}\left\{ \max_i Y_i \right\}.$

Proof. Without loss of generality, we may assume that X and Y are defined on the same probability space and are independent. Fix $\beta > 0$ and define $h_\beta : \mathbb{R}^k \to \mathbb{R}$ by

$$h_\beta(x) \;\triangleq\; \beta^{-1} \log \left(\sum_{i=1}^{k} e^{\beta x_i} \right).$$

Note that

$$
\begin{aligned}
\max_i x_i \;&=\; \beta^{-1} \log \left(e^{\beta \max_i x_i} \right) \\
&\leq\; \beta^{-1} \log \left(\sum_{i=1}^{k} e^{\beta x_i} \right) \\
&\leq\; \beta^{-1} \log \left(k e^{\beta \max_i x_i} \right) \\
&=\; \beta^{-1} \log k \;+\; \max_i x_i.
\end{aligned}
$$

Thus

(2.2.13) $\qquad \sup_{x \in \mathbb{R}^k} \left| h_\beta(x) - \max_i x_i \right| \;\leq\; \beta^{-1} \log k.$

Now let $\mu_i = \mathbb{E}\{X_i\} = \mathbb{E}\{Y_i\}$ and set

$$\widetilde{X}_i \;=\; X_i - \mu_i, \quad \widetilde{Y}_i \;=\; Y_i - \mu_i.$$

Furthermore, write $\sigma_{ij}^X = \mathbb{E}\{\widetilde{X}_i \widetilde{X}_j\}$, $\sigma_{ij}^Y = \mathbb{E}\{\widetilde{X}_i \widetilde{X}_j\}$ and, for $t \in [0,1]$ define the random vector Z_t by

$$Z_{t,i} \;\triangleq\; \sqrt{1-t}\,\widetilde{X}_i \;+\; \sqrt{t}\,\widetilde{Y}_i \;+\; \mu_i.$$

Now fix $\beta > 0$ and, for all $t \in [0,1]$, set

$$\varphi(t) \;\triangleq\; \mathbb{E}\left\{ h_\beta(Z_t) \right\}.$$

Then φ is differentiable, and

$$\varphi'(t) \;=\; \mathbb{E}\left\{ \sum_{i=1}^{k} \frac{\partial h_\beta}{\partial x_i}(Z_t) \left(\frac{\widetilde{Y}_i}{2\sqrt{t}} \;-\; \frac{\widetilde{X}_i}{2\sqrt{1-t}} \right) \right\}.$$

Again, for any i, Lemma 2.2.4 gives us that

$$\mathbb{E}\left\{\frac{\partial h_\beta}{\partial x_i}(Z_t)\widetilde{X}_i\right\} \;=\; \sqrt{1-t}\sum_{j=1}^{k}\sigma_{ij}^X\,\mathbb{E}\left\{\frac{\partial^2 h_\beta}{\partial x_i\partial x_j}(Z_t)\right\}$$

and

$$\mathbb{E}\left\{\frac{\partial h_\beta}{\partial x_i}(Z_t)\widetilde{Y}_i\right\} \;=\; \sqrt{t}\sum_{j=1}^{k}\sigma_{ij}^Y\,\mathbb{E}\left\{\frac{\partial^2 h_\beta}{\partial x_i\partial x_j}(Z_t)\right\}$$

so that

$$(2.2.14)\qquad \varphi'(t) \;=\; \tfrac{1}{2}\sum_{i,j=1}^{k}\mathbb{E}\left\{\frac{\partial^2 h_\beta}{\partial x_i\partial x_j}(Z_t)\right\}\left(\sigma_{ij}^Y-\sigma_{ij}^X\right).$$

In a moment we shall show that if $\gamma_{ij}^X \le \gamma_{ij}^Y$ then $\varphi' \ge 0$ throughout its range. This will prove (2.2.12), since it implies that, for all $\beta > 0$,

$$\mathbb{E}\left\{h_\beta(Y)\right\} \;=\; \varphi(1) \;\ge\; \varphi(0) \;=\; \mathbb{E}\left\{h_\beta(X)\right\}.$$

Taking $\beta \to \infty$ and applying (2.2.13) establishes (2.2.12). The more delicate inequality (2.2.11) requires more information on φ, which comes from a little elementary calculus.

Note that

$$\frac{\partial h_\beta}{\partial x_i}(x) \;=\; \frac{e^{\beta x_i}}{\sum_{j=1}^{k}e^{\beta x_j}} \;\overset{\Delta}{=}\; p_i(x),$$

where, for each $x \in \mathbb{R}^k$, the $p_i(x)$ define a probability measure on $\{1,\ldots,k\}$. It is then straightforward to check that

$$\frac{\partial^2 h_\beta}{\partial x_i\partial x_j}(x) \;=\; \begin{cases}\beta\left(p_i(x)-p_i^2(x)\right) & i=j,\\ -\beta p_i(x)p_j(x) & i\ne j.\end{cases}$$

Thus,

$$(2.2.15)\quad \sum_{i,j=1}^{k}\frac{\partial^2 h_\beta}{\partial x_i\partial x_j}(x)\left(\sigma_{ij}^Y-\sigma_{ij}^X\right)$$

$$=\; \beta\sum_{i=1}^{k}p_i(x)\left(\sigma_{ij}^Y-\sigma_{ij}^X\right) \;-\; \beta\sum_{i,j=1}^{k}p_i(x)p_j(x)\left(\sigma_{ij}^Y-\sigma_{ij}^X\right)$$

$$=\; \frac{\beta}{2}\sum_{i,j=1}^{k}p_i(x)p_j(x)\left[\left(\sigma_{ii}^Y+\sigma_{jj}^Y-2\sigma_{ij}^Y\right)-\left(\sigma_{ii}^X+\sigma_{jj}^X-2\sigma_{ij}^X\right)\right]$$

$$=\; \frac{\beta}{2}\sum_{i,j=1}^{k}p_i(x)p_j(x)\left(\gamma_{ij}^Y-\gamma_{ij}^X\right).$$

The second equality here uses the fact that $\sum_i p_i(x) = 1$, while the third one relies on the fact that

$$
\begin{aligned}
\sigma_{ii}^X + \sigma_{jj}^X - 2\sigma_{ij}^X &= \mathbb{E}\left\{(\widetilde{X}_i - \widetilde{X}_j)^2\right\} \\
&= \mathbb{E}\left\{(X_i - X_j)^2\right\} - (\mu_i - \mu_j)^2 \\
&= \gamma_{ij}^X - (\mu_i - \mu_j)^2,
\end{aligned}
$$

along with a similar equality for the σ^Y.

Substituting (2.2.15) into (2.2.14) gives us that $\varphi'(t) \geq 0$ whenever $\gamma_{ij}^X \leq \gamma_{ij}^Y$, as we claimed above.

All that remains, therefore, is to establish (2.2.11), which we do by getting a bound on $\varphi'(t)$.

Combining (2.2.15) with the definition of γ, it is immediate that

$$
\left| \sum_{i,j=1}^k \frac{\partial^2 h_\beta}{\partial x_i \partial x_j}(x) \left(\sigma_{ij}^Y - \sigma_{ij}^X\right) \right| \leq \frac{\beta\gamma}{2} \sum_{i,j=1}^k p_i(x)p_j(x) = \frac{\beta\gamma}{2}.
$$

Substituting this into (2.2.14) gives

$$
\left| \mathbb{E}\left\{h_\beta(Y)\right\} - \mathbb{E}\left\{h_\beta(X)\right\} \right| = |\varphi(1) - \varphi(0)| \leq \frac{\beta\gamma}{4}.
$$

Combining this with (2.2.13) gives

$$
\left| \mathbb{E}\left\{\max_i X_i\right\} - \mathbb{E}\left\{\max_i Y_i\right\} \right| \leq \frac{\beta\gamma}{4} + \frac{2\log k}{\beta},
$$

and choosing $\beta = \sqrt{8\log k/\gamma}$ gives (2.2.11), as required.   □

There are many extensions to Sudakov-Fernique inequality that we shall not need in this book, but you can find in the references in the description of Part I. From those sources you can also find out how to extend the above arguments to find conditions on covariance functions that allow statements of the form

$$
\mathbb{P}\{\min_{1\leq i\leq n} \max_{1\leq j\leq m} X_{ij} \geq u\} \geq \mathbb{P}\{\min_{1\leq i\leq n} \max_{1\leq j\leq m} Y_{ij} > u\},
$$

along with even more extensive variations due, originally, to Gordon [66]. Gordon [67] also shows how to extend the essentially Gaussian computations above to elliptically contoured distributions.

3
Orthogonal expansions

While most of what we shall have to say in this brief chapter is rather theoretical, it actually covers one of the most important practical aspects of Gaussian modelling. The basic result is Theorem 3.1.1, which states that *every* centered Gaussian process with a continuous covariance function has an expansion of the form

$$(3.0.1) \qquad\qquad f(t) \; = \; \sum_{n=1}^{\infty} \xi_n \, \varphi_n(t),$$

where the ξ_n are i.i.d. $N(0,1)$, and the φ_n are certain functions on T determined by the covariance function C of f. In general, the convergence in (3.0.1) is in $L^2(\mathbb{P})$ for each $t \in T$, but (Theorem 3.1.2) if f is a.s. continuous then the convergence is uniform over T, with probability one.

There are many theoretical conclusions that follow from this representation. For one example, note that since continuity of C will imply that of the φ_n (cf. Lemma (3.1.4)) it follows from (3.0.1) that sample path continuity of f is a 'tail event' on the σ-algebra determined by the ξ_n, from which one can show that centered Gaussian processes are either continuous with probability one, or discontinous with probability one. There is no middle ground. A wide variety of additional zero-one laws also follow from this representation. Perhaps the most notable is the following,

$$\mathbb{P}\left\{\lim_{s \to t} f_s = f_t \text{ for all } t \in T\right\} \; = \; 1$$

$$\iff \quad \mathbb{P}\left\{\lim_{s \to t} f_s = f_t\right\} \; = \; 1 \text{ for each } t \in T.$$

That is, under our ubiquitous assumption that the covariance function C is continuous, pointwise and global a.s. continuity are equivalent for Gaussian processes. A proof, in the spirit of what follows, can be found in [4].

The practical implications of (3.0.1) are mainly in the area of simulation. By truncating the sum (3.0.1) at a point appropriate to the problem at hand, one needs 'only' to determine the φ_n. However, these arise as the orthonormal basis of a particular Hilbert space, and can generally be found by solving an eigenfunction problem involving C. In particular, if T is a nice subset of $\mathbb{R}^N$, this leads to the famous Karhunen-Loève expansion that we shall treat briefly in Section 3.2. While even in the Euclidean situation there are only a handful of situations for which this eigenfunction problem can be solved analytically, from the point of view of computing it is a standard problem, and the approach is practical.

3.1 The general theory

As usual, we shall take T to be compact in the canonical metric d generated by f.

The first step towards establishing the expansion (3.0.1) lies in setting up the so-called *reproducing kernel Hilbert space* (RKHS) of a centered Gaussian process with covariance function C.

In essence, the RKHS is made up of functions that have about the same smoothness properties that $C(s,t)$ has, as a function in t for fixed s, or vice versa. Start with

$$ S \;=\; \left\{ u : T \to \mathbb{R} : u(\cdot) = \sum_{i=1}^{n} a_i C(s_i, \cdot), \ a_i \text{ real}, \ s_i \in T, \ n \geq 1 \right\}. $$

Define an inner product on S by

$$
\begin{aligned}
(u,v)_H &= \left(\sum_{i=1}^{n} a_i C(s_i, \cdot), \sum_{j=1}^{m} b_j C(t_j, \cdot) \right)_H \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j C(s_i, t_j).
\end{aligned}
$$

The fact that C is non-negative definite implies $(u,u)_H \geq 0$ for all $u \in S$. Furthermore, note that the inner product (3.1.1) has the following unusual

property:

$$(3.1.1) \qquad (u, C(t, \cdot))_H \;\; = \;\; \left(\sum_{i=1}^{n} a_i C(s_i, \cdot), \, C(t, \cdot) \right)_H$$

$$= \;\; \sum_{i=1}^{n} a_i C(s_i, t)$$

$$= \;\; u(t).$$

This is the *reproducing kernel* property.

For the sake of exposition, assume that the covariance function, C, is positive definite (rather than merely non-negative definite) so that $(u, u)_H = 0$ if, and only if, $u(t) \equiv 0$. In this case (3.1.1) defines a norm $\|u\|_H = (u, u)_H^{1/2}$. For $\{u_n\}_{n \geq 1}$ a sequence in S we have

$$|u_n(t) - u_m(t)| \;\; = \;\; |(u_n - u_m, C(t, \cdot))_H|$$

$$\leq \;\; \|u_n - u_m\|_H \, \|C(t, \cdot)\|_H$$

$$\leq \;\; \|u_n - u_m\|_H \, C(t, t),$$

the last line following directly from (3.1.1). Thus it follows that if $\{u_n\}$ is Cauchy in $\|\cdot\|_H$ then it is pointwise Cauchy. The closure of S under this norm is a space of real-valued functions, denoted by $H(C)$, and called the RKHS of f or of C, since every $u \in H(C)$ satisfies (3.1.1) by the separability of $H(C)$. (The separability of $H(C)$ follows from the separability of T and the assumption that C is continuous.)

Since all this seems at first rather abstract, consider two concrete examples. Take $T = \{1, \ldots, n\}$, finite, and f centered Gaussian with covariance matrix $C = (c_{ij})$, $c_{ij} = \mathbb{E}\{f_i f_j\}$. Let $C^{-1} = (c^{ij})$ denote the inverse of C, which exists by positive definiteness. Then the RKHS of f is made up of all n-dimensional vectors $u = (u_1, \ldots, u_n)$ with inner product

$$(u, v)_H \;\; = \;\; \sum_{i=1}^{n} \sum_{j=1}^{n} u_i \, c^{ij} \, v_j.$$

To prove this, we need only check that the reproducing kernel property (3.1.1) holds[1]. But, with $\delta(i, j)$ the Kronecker delta function, and C_k de-

---

[1] A simple proof by contradiction shows that there can never be two different inner products on S with the reproducing kernel property.

noting the k-th row of C,

$$
\begin{aligned}
(u, C_k)_H &= \sum_{i=1}^{n}\sum_{j=1}^{n} u_i\, c^{ij}\, c_{kj} \\
&= \sum_{i=1}^{n} u_i\, \delta(i,k) \\
&= u_k,
\end{aligned}
$$

as required.

For a slightly more interesting example, take $f = W$ to be standard Brownian motion on $T = [0,1]$, so that $C(s,t) = \min(s,t)$. Note that $C(s,\cdot)$ is differentiable everywhere except at s, so that following the heuristics developed above we expect that $H(C)$ should be made up of a subset of functions that are differentiable almost everywhere.

To both make this statement more precise, and prove it, we start by looking at the space S. Thus, let

$$
u(t) = \sum_{i=1}^{n} a_i C(s_i, t), \qquad v(t) = \sum_{i=1}^{n} b_i C(t_i, t),
$$

be two elements of S, with inner product

$$
(u, v)_H = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j \min(s_i, t_j).
$$

Since the derivative of $C(s,t)$ with respect to t is $\mathbb{1}_{[0,s]}(t)$, the derivative of u is $\sum_{i=1}^{n} a_i \mathbb{1}_{[0,s_i]}(t)$. Therefore,

$$
\begin{aligned}
(u, v)_H &= \sum\sum a_i b_j \int_0^1 \mathbb{1}_{[0,s_i]}(t)\mathbb{1}_{[0,t_j]}(t)\, dt \\
&= \int_0^1 \left(\sum a_i\, \mathbb{1}_{[0,s_i]}(t) \sum b_j\, \mathbb{1}_{[0,t_j]}(t) \right) dt \\
&= \int_0^1 \dot{u}(t)\dot{v}(t)\, dt.
\end{aligned}
$$

With S under control, we can now look for an appropriate candidate for the RKHS. Define

$$
(3.1.2) \qquad H = \left\{ u : u(t) = \int_0^t \dot{u}(s)\, ds, \ \int_0^1 (\dot{u}(s))^2\, ds < \infty \right\},
$$

equipped with the inner product

$$
(3.1.3) \qquad (u, v)_H = \int_0^1 \dot{u}(s)\, \dot{v}(s)\, ds.
$$

Since it is immediate that $C(s, \cdot) \in H$ for $t \in [0,1]$, and

$$(u, C(t, \cdot))_H \;=\; \int_0^1 \dot{u}(s)\, \mathbb{1}_{[0,t]}(s)\, ds,$$

it follows that the H defined by (3.1.2) is indeed our RKHS. This H is also known, in the setting of diffusion processes, as a *Cameron-Martin* space.

With a couple of examples under our belt, we can now return to our main task: setting up the expansion (3.0.1). The first step is finding a countable orthonormal basis for the separable Hilbert space $H(C)$. We start with $\mathcal{H} = \text{span}\{f_t,\ t \in \mathbb{R}^N\}$, with the (covariance) inner product that $\mathcal{H}$ inherits as a subspace of $L^2(\mathbb{P})$. Next we define a linear mapping $\Theta : S \to \mathcal{H}$ by

$$\Theta(u) \;=\; \Theta\left(\sum_{i=1}^n a_i\, C(t_i, \cdot)\right) \;=\; \sum_{i=1}^n a_i f(t_i).$$

Clearly $\Theta(u)$ is Gaussian for each $u \in S$.

Since Θ is norm preserving, it extends to all of $H(C)$ with range equal to all of $\mathcal{H}$, with all limits remaining Gaussian. This extension is called the *canonical isomorphism* between these spaces.

Since $H(C)$ is separable, we now also know that $\mathcal{H}$ is, and proceed to to build an orthonormal basis for it. If $\{\varphi_n\}_{n \geq 1}$ is an orthonormal basis for $H(C)$, then setting $\xi_n = \Theta(\varphi_n)$ gives $\{\xi_n\}_{n \geq 1}$ as an orthonormal basis for $\mathcal{H}$. In particular, we must have that the ξ_n are $N(0,1)$ and

$$(3.1.4) \qquad\qquad f_t \;=\; \sum_{n=1}^\infty \xi_n \mathbb{E}\{f_t \xi_n\},$$

where the series converges in $L^2(\mathbb{P})$. Since Θ was an isometry, it follows from (3.1.4) that

$$(3.1.5) \qquad\qquad \mathbb{E}\{f_t \xi_n\} \;=\; (C(t, \cdot), \varphi_n)_H$$
$$= \;\; \varphi_n(t),$$

the last equality coming from the reproducing kernel property of $H(C)$. Putting (3.1.5) together with (3.1.4) now establishes the following central result.

Theorem 3.1.1 *If $\{\varphi_n\}_{n \geq 1}$ is an orthonormal basis for $H(C)$, then f has the L^2-representation*

$$(3.1.6) \qquad\qquad f_t \;=\; \sum_{n=1}^\infty \xi_n \varphi_n(t),$$

where $\{\xi_n\}_{n \geq 1}$ is the orthonormal sequence of centered Gaussian variables given by $\xi_n = \Theta(\varphi_n)$.

The equivalence in (3.1.6) is only in L^2; i.e. the sum is, in general, convergent, for each t, only in mean square. The following result shows that much more is true if we know that f is a.s. continuous.

Theorem 3.1.2 *If f is a.s. continuous, then the sum in (3.1.6) converges uniformly on T with probability one[2] .*

We need two preliminary results before we can prove Theorem 3.1.2. The first is a convergence result due to Itô and Nisio [79] which, since it is not really part of a basic probability course we state in full, and the second an easy Lemma.

Lemma 3.1.3 *Let $\{Z_n\}_{n\geq 1}$ be a sequence of symmetric independent random variables, taking values in a separable, real Banach space B, equipped with the norm topology. Let $X_n = \sum_{i=1}^n Z_i$. Then X_n converges with probability one if, and only if, there exists a B-valued random variable X such that $\langle X_n, x^* \rangle \to \langle X, x^* \rangle$ in probability for every $x^* \in B^*$, the topological dual of B.*

Lemma 3.1.4 *Let $\{\varphi_n\}_{n\geq 1}$ be an orthonormal basis for $H(C)$. Then each φ_n is continuous and*

$$(3.1.7) \qquad\qquad \sum_{n=1}^{\infty} \varphi_n^2(t)$$

converges uniformly in $t \in T$ to $C(t,t)$.

Proof. Note that

$$
\begin{aligned}
|\varphi_n(s) - \varphi_n(t)| &= |(C(s,\cdot), \varphi_n(\cdot))_H - (C(t,\cdot), \varphi_n(\cdot))_H| \\
&= |([C(s,\cdot) - C(t,\cdot)], \varphi_n(\cdot))_H| \\
&\leq \|\varphi_n\|_H \, \|C(s,\cdot) - C(t,\cdot)\|_H \\
&= \|C(s,\cdot) - C(t,\cdot)\|_H \\
&= C(s,s) - 2C(s,t) + C(t,t),
\end{aligned}
$$

where the first and last equalities use the reproducing kernel property and the one inequality is Cauchy-Schwartz. Since C is pointwise continuous it now follows from the definition of $\| \ \|_H$ on S that each φ_n is continuous over T.

---

[2]There is also a converse to Theorem 3.1.2, that the a.s. uniform convergence of a sum like (3.1.6) implies the continuity of f, and some of the earliest derivations (e.g. [65]) of sufficient conditions for continuity actually used this approach. Entropy based arguments, however, turn out to give much easier proofs.

To establish the uniform convergence of (3.1.7), note that the orthonormal expansion and the reproducing kernel property imply

$$(3.1.8) \qquad C(t, \cdot) \;=\; \sum_{n=1}^{\infty} \varphi_n(\cdot)\big(C(t,\cdot), \varphi_n\big)_H$$

$$\;=\; \sum_{n=1}^{\infty} \varphi_n(\cdot)\varphi_n(t),$$

convergence of the sum being in the $\|\cdot\|_H$ norm. Hence, $\sum_{n=1}^{\infty} \varphi_n^2(t)$ converges to $C(t,t)$ for every $t \in T$. Furthermore, the convergence is monotone, and so it follows that it is also uniform ($\equiv$ Dini's theorem). $\qquad\square$

Proof of Theorem 3.1.2 We know that, for each $t \in T$, $\sum_{n=1}^{\infty} \xi_n \varphi_n(t)$ is a sum of independent variables converging in $L^2(\mathbb{P})$. Thus, by Lemma 3.1.3, applied to real-valued random variables, it converges with probability one to a limit, which we denote by f_t. The limit process is, by assumption, almost surely continuous.

Now, consider both f and each function $\xi_n \varphi_n$ as random variables in the Banach space $C(T)$, with sup-norm topology. Elements of the dual $C^*(T)$ are therefore finite, signed, Borel measures μ on T, and $\langle f, \mu \rangle = \int f \, d\mu$. Define

$$f_n(\cdot) \;=\; \sum_{i=1}^{n} \xi_i \varphi_i(\cdot) \;=\; \sum_{i=1}^{n} \Theta(\varphi_i)\varphi_i(\cdot).$$

By Lemma 3.1.3, it suffices to show that for every $\mu \in C^*(T)$ the random variables $\langle f_n, \mu \rangle$ converge in probability to $\langle f, \mu \rangle$. However,

$$\mathbb{E}\left\{ |\langle f_n, \mu \rangle - \langle f, \mu \rangle| \right\} \;=\; \mathbb{E}\left\{ \left| \int_T (f_n(t) - f(t)) \, \mu(dt) \right| \right\}$$

$$\leq\; \int_T \mathbb{E}\left\{ |f_n(t) - f(t)| \right\} \, |\mu|(dt)$$

$$\leq\; \int_T \left[\mathbb{E}\left(f_n(t) - f(t) \right)^2 \right]^{\frac{1}{2}} \, |\mu|(dt)$$

$$=\; \int_T \left(\sum_{j=n+1}^{\infty} \varphi_j^2(t) \right)^{\frac{1}{2}} \, |\mu|(dt),$$

where $|\mu|(A)$ is the total variation measure for μ.

Since $\sum_{j=n+1}^{\infty} \varphi_j^2(t) \to 0$ uniformly in $t \in T$ by Lemma 3.1.4, the last expression above tends to zero as $n \to \infty$. Since this implies the convergence in probability of $\langle f_n, \mu \rangle$ to $\langle f, \mu \rangle$, we are done. $\qquad\square$

3.2   The Karhunen-Loève expansion

As we have already noted, applying the orthogonal expansion (3.1.6) in practice relies on being able to find the orthonormal functions φ_n. When T is a compact subset of $\mathbb{R}^N$ there is a special way in which to do this, leading to what is known as the Karhunen-Loève expansion.

For simplicity, take $T = [0,1]^N$. Let $\lambda_1 \geq \lambda_2 \geq \ldots$, and $\psi_1, \psi_2, \ldots$, be, respectively, the eigenvalues and normalised eigenfunctions of the operator $\mathcal{C} : L^2(T) \to L^2(T)$ defined by $(\mathcal{C}\psi)(t) = \int_T C(s,t)\psi(s)\,ds$. That is, the λ_n and ψ_n solve the integral equation

$$(3.2.9) \qquad \int_T C(s,t)\psi(s)\,ds \;=\; \lambda\psi(t),$$

with the normalisation

$$\int_T \psi_n(t)\psi_m(t)\,dt \;=\; \begin{cases} 1 & n = m, \\ 0 & n \neq m. \end{cases}$$

These eigenfunctions lead to a natural expansion of C, known as Mercer's Theorem. (cf. [123, 172] for a proof.)

Theorem 3.2.1 (Mercer) *Let C, $\{\lambda_n\}_{n\geq 1}$ and $\{\psi_n\}_{n\geq 1}$ be as above. Then*

$$(3.2.10) \qquad C(s,t) \;=\; \sum_{n=1}^{\infty} \lambda_n \psi_n(s)\psi_n(t)\,,$$

where the series converges absolutely and uniformly on $[0,1]^N \times [0,1]^N$.

The key to the Karhunen-Loève expansion is the following result.

Lemma 3.2.2 *For f on $[0,1]^N$ as above, $\{\sqrt{\lambda_n}\psi_n\}$ is a complete orthonormal system in $H(C)$.*

Proof. Set $\varphi_n = \sqrt{\lambda_n}\psi_n$ and define

$$H \;=\; \left\{ h : h(t) = \sum_{n=1}^{\infty} a_n \varphi_n(t),\ t \in [0,1]^N,\ \sum_{n=1}^{\infty} a_n^2 < \infty \right\}.$$

Give H the inner product

$$(h,g)_H \;=\; \sum_{n=1}^{\infty} a_n b_n,$$

where $h = \sum a_n \varphi_n$ and $g = \sum b_n \varphi_n$.

To check that H has the reproducing kernel property, note that

$$
\begin{aligned}
\big(h(\cdot), C(t, \cdot)\big)_H &= \Big(\sum_{n=1}^{\infty} a_n \varphi_n(\cdot), \sum_{n=1}^{\infty} \sqrt{\lambda_n} \psi_n(t) \varphi_n(\cdot) \Big) \\
&= \sum_{n=1}^{\infty} \sqrt{\lambda_n} a_n \psi_n(t) \\
&= h(t).
\end{aligned}
$$

It remains to be checked that H is in fact a Hilbert space, and that $\{\sqrt{\lambda_n} \psi_n\}$ is both complete and orthonormal. But all this is standard, given Mercer's Theorem.   □

We can now start rewriting things to get the expansion we want. Remaining with the basic notation of Mercer's theorem, we have that the RKHS, $H(C)$, consists of all square integrable functions h on $[0,1]^N$ for which

$$
\sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left| \int_T h(t) \psi_n(t) \, dt \right|^2 < \infty,
$$

with inner product

$$
(h, g)_H = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \int_T h(t) \psi_n(t) \, dt \int_T g(t) \psi_n(t) \, dt.
$$

The Karhunen-Loève expansion of f is obtained by setting $\varphi_n = \lambda_n^{\frac{1}{2}} \psi_n$ in the orthonormal expansion (3.1.6), so that

$$
(3.2.11) \qquad\qquad f_t = \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}} \xi_n \, \psi_n(t),
$$

where the ξ_n are i.i.d. $N(0,1)$.

As simple as this approach might seem, it is generally limited by the fact that it is usually not easy to analytically solve the integral equation (3.2.9). There is, however, at least one classic, example for which this can be done, the Brownian sheet of Section 1.4.3, which we recall has covariance function

$$
(3.2.12) \qquad \mathbb{E}\{W_s W_t\} = (s_1 \wedge t_1) \times \cdots \times (s_N \wedge t_N),
$$

on $I^N = [0,1]^N$.

For the moment, we set $N = 1$ and so have the standard Brownian motion on $[0,1]$, for which we have already characterised the corresponding RKHS as the Cameron-Martin space.

For Brownian motion (3.2.9) becomes

$$
\begin{aligned}
\lambda \psi(t) &= \int_0^1 \min(s, t) \, \psi(s) \, ds \\
&= \int_0^t s \, \psi(s) \, ds + t \int_t^1 \psi(s) \, ds.
\end{aligned}
$$

Differentiating both sides with respect to t gives

$$
\begin{aligned}
\lambda \psi'(t) &= \int_t^1 \psi(s)\, ds, \\
\lambda \psi''(t) &= -\psi(t),
\end{aligned}
$$

together with the boundary conditions $\psi(0) = 0$ and $\psi'(1) = 0$

The solutions of this pair of differential equations are given by

$$
(3.2.13) \quad \psi_n(t) = \sqrt{2} \sin\left(\tfrac{1}{2}(2n+1)\pi t \right), \quad \lambda_n = \left(\frac{2}{(2n+1)\pi} \right)^2,
$$

as is easily verified by substitution. Thus, the Karhunen-Loève expansion of Brownian motion on $[0,1]$ is given by

$$
W_t = \frac{\sqrt{2}}{\pi} \sum_{n=0}^{\infty} \xi_n \left(\frac{2}{2n+1} \right) \sin\left(\tfrac{1}{2}(2n+1)\pi t \right).
$$

Returning now to the general Brownian sheet, it is now clear from the product form (3.2.12) of the covariance function that rather taking a single sum in the Karhunen-Loève expansion, it is natural to work with an N-dimensional sum and a multi-index $n = (n_1, \ldots, n_N)$. With this in mind, it follows then from (3.2.13) that the eigenfunctions and eigenvalues are given by

$$
\begin{aligned}
\psi_n(t) &= 2^{N/2} \prod_{i=1}^{N} \sin\left(\tfrac{1}{2}(2n_i + 1)\pi t_i \right), \\
\lambda_n &= \prod_{i=1}^{N} \left(\frac{2}{(2n_i + 1)\pi} \right)^2.
\end{aligned}
$$

As an aside, there is also an elegant expansion of the Brownian sheet using the Haar functions, which also works in the set-indexed setting, due to Ron Pyke [120].

Another class of examples that can almost be handled via the Karhunen-Loève approach is that of stationary fields defined on all of (non-compact) $\mathbb{R}^N$. Since what is about to come is rather imprecise, we shall allow ourselves the standard notational luxury of the stationary theory of taking our field to be complex valued, despite the fact that all that we have done in this chapter was for real valued processes. In this case the covariance function is $C(s,t) = \mathbb{E}\{f_s \overline{f_t}\}$ and it is a function of $t - s$ only. It is then easy to find eigenfunctions for (3.2.9) via complex exponentials. Note that, for any

$\lambda \in \mathbb{R}^N$, the function $e^{i\langle t,\lambda \rangle}$ (a function of $t \in \mathbb{R}^N$) satisfies

$$\int_{\mathbb{R}^N} C(s,t)\, e^{i\langle s,\lambda \rangle}\, ds \; = \; \int_{\mathbb{R}^N} C(t-s)\, e^{i\langle s,\lambda \rangle}\, ds$$
$$= \; e^{i\langle t,\lambda \rangle} \int_{\mathbb{R}^N} C(u)\, e^{-i\langle u,\lambda \rangle}\, du$$
$$= \; K_\lambda\, e^{i\langle t,\lambda \rangle},$$

for some, possibly zero, K_λ.

Suppose that $K_\lambda \neq 0$ for only a countable number of $\lambda \in \mathbb{R}^N$. Then the Mercer expansion (3.2.10) can be written as

$$(3.2.14) \qquad\qquad C(t) \; = \; \sum_\lambda K_\lambda\, e^{i\langle t,\lambda \rangle}$$

while the Karhunen-Loève expansion becomes

$$(3.2.15) \qquad\qquad f(t) \; = \; \sum_\lambda K_\lambda^{1/2} \xi_\lambda\, e^{i\langle t,\lambda \rangle}.$$

These are, respectively, special cases of the Spectral Distribution Theorem (cf. (5.4.1)) and the Spectral Representation Theorem (cf. (5.4.6)) of Chapter 5 when the spectral measure is discrete.

Despite the minor irregularity of assuming that f is complex valued, the above argument is completely rigorous. On the other hand, what follows is not.

If $K_\lambda \neq 0$ on an uncountable set then the situation becomes more delicate, but is nevertheless worth looking at. In this case, one could imagine replacing the summations in (3.2.14) and (3.2.15) by integrals, to obtain

$$C(t) \; = \; \int_{\mathbb{R}^N} K_\lambda\, e^{i\langle t,\lambda \rangle}\, d\lambda$$

and

$$(3.2.16) \qquad\qquad f(t) \; = \; \int_{\mathbb{R}^N} K_\lambda^{1/2} \xi_\lambda\, e^{i\langle t,\lambda \rangle}\, d\lambda.$$

Everything is well defined in the first of these integrals, but in the second we have the problem that the ξ_λ should be independent for each λ and it is well known that there is no measurable way to construct such a process.

Nevertheless, we shall see in Chapter 5 that there are ways to get around these problems and that, when properly formulated, (3.2.16) actually makes sense.

4
Excursion probabilities

As we have already mentioned more than once before, one of the oldest and most important problems in the study of stochastic proceses of any kind is to evaluate the *excursion probabilities*

$$\mathbb{P}\left\{\sup_{t \in T} f(t) \geq u\right\},$$

where f is a random process over some parameter set T. As usual, we shall restrict ourselves to the case in which f is centered and Gaussian and T is compact for the canonical metric of (1.3.1).

Computing excursion probabilities for general Gaussian processes is a surprisingly difficult problem. Even the simple case of $T = [0, 1]$ is hard. In this one-dimensional case, there is hope for obtaining an explicit expression for the excursion probability if f is Markovian. For example, if f is Brownian motion then, as any elementary textbook on stochastic processes will tell you, the so-called "reflection principle" easily yields that $\mathbb{P}\left\{\sup_{0 \leq t \leq 1} f(t) \geq u\right\} = 2\Psi(u)$, where Ψ is the Gaussian tail probability (1.2.1).

However, if we turn to stationary processes on $\mathbb{R}$, then there are only five[1] nontrivial[2] Gaussian processes for which the excursion probability has a known analytic form, and the resulting formulae, while amenable to numerical computation, are not generally very pretty.

In each of these five examples the process is either Markovian, or close to Markovian, and this is what makes the computation feasible. Similarly, none of the processes has even a mean square derivative, and so none are particularly smooth. When looking at smooth processes, or processes over parameter spaces richer than the unit interval, the very specific tools that work for these very special cases fail completely. What one can do in the general case is the subject of this chapter and of Chapter 13.

Our ultimate aim will be to develop, when possible and appropriate, expansions of the form

$$(4.0.1) \quad \mathbb{P}\left\{ \sup_{t \in T} f(t) \geq u \right\} = u^\alpha e^{-u^2/2\sigma_T^2} \left[\sum_{j=0}^{n} C_j u^{-j} + \text{error} \right],$$

for large u, appropriate parameters α, σ_T^2, n and C_j that depend on both f and T. Furthermore, we would like to be able to identify the constants C_j and also to be able to say something about the error term.

In Chapter 13 we shall indeed establish such a result, with a full identification of all constants. However, we shall have to make two major assumptions. One is that the parameter space T is a piecewise smooth manifold, and the second is that f is almost surely C^2. The current chapter will make fewer assumptions, and will concentrate on general approaches relying heavily on the concept of entropy from Chapter 1. As a consequence, we shall generally only be able to identify the first term in (4.0.1). Even

---

[1] For details, see [136, 40] and the review [27]. The five covariance functions are

$$C(t) = e^{-|t|},$$

$$C(t) = \begin{cases} 1 - |t| & |t| \leq 1 \\ 0 & |t| \geq 1, \end{cases}$$

$$C(t) = \tfrac{3}{2} \exp[-\tfrac{1}{3}|t|]\{1 - \tfrac{1}{3}\exp[-2|t|/\sqrt{3}]\},$$

$$C(t) = \begin{cases} 1 - \alpha|t| & |t| \leq 1 \\ C(t-1) & -\infty < t < \infty, \end{cases}$$

$$C(t) = \begin{cases} 1 - |t|/(1-\beta), & |t| \leq 1 \\ -\beta/(1-\beta), & 1 < |t| < (1-\beta)/\beta \end{cases},$$

where, in the last case, $\beta \in (0,1)$, and the computation of the excursion probability is known only for $f(0)$ conditioned to be 0.

[2] There is also one close-to-trivial but extremely enlightening case, with covariance function $\cos(\omega t)$, for which elementary calculations give the excursion probability. We we shall discuss this in some detail in Section 13.4.4.

then, while we shall be able to determine α and σ_T^2, only under rare cases shall we also be able to identify C_0.

Furthermore, so as to keep the treatment down to a reasonable length, we shall generally concentrate on upper bounds, rather than expansions, for Gaussian excursion probabilities.

This chapter contains six sections. The first four have just been described. The remaining two are both brief, and are meant as quick introductions to techniques which can lead to sharper results under additional assumptions. They lie between the general results of the first four sections and the very sharp results, for smooth Gaussian fields on structured domains, of Chapter 13.

4.1   Entropy bounds

As we saw in Section 2.1, an immediate consequence of the Borell-TIS inequality was the fact that, for all $\varepsilon > 0$ and large enough u,

$$(4.1.1) \qquad \mathbb{P}\left\{\sup_{t \in T} f_t \geq u\right\} \leq e^{\varepsilon u^2 - u^2/2\sigma_T^2}.$$

where $\sigma_T^2 = \sup_T \mathbb{E}\{f_t^2\}$. (cf. (2.1.4).) This is in the classic form of a large deviation result, and holds with no assumptions beyond the almost sure boundedness of f. Our aim is to improve on (4.1.1) by placing side conditions on the entropy (cf. Definition 1.3.2) of f over T. Here is a very easy, and definitely sub-optimal, result.

Theorem 4.1.1 *Let be f a centered, a.s. continuous Gaussian field over T with entropy function N. If $N(\varepsilon) \leq K\varepsilon^{-\alpha}$, then, for all sufficiently large u,*

$$(4.1.2) \qquad \mathbb{P}\left\{\sup_{t \in T} f(t) \geq u\right\} \leq C_\alpha\, u^{\alpha+\eta}\, e^{-u^2/2\sigma_T^2},$$

for every $\eta > 0$, where $C_\alpha = C(K, \alpha, \sigma_T^2)$ is a finite constant.

Proof. Take $\varepsilon > 0$ and define

$$\mu(t, \varepsilon) = \mathbb{E}\left\{\sup_{s \in B_d(t, \varepsilon)} f_s\right\},$$

and

$$(4.1.3) \qquad \mu(\varepsilon) = \sup_{t \in T} \mu(t, \varepsilon),$$

where $B_d(t, \varepsilon)$ is a ball of radius ε around t in the canonical metric d of (1.3.1). Since $N(\varepsilon)$ balls of radius ε cover T, it is an immediate consequence of the Borell-TIS inequality that, for $u > \mu(\varepsilon)$,

$$(4.1.4) \qquad \mathbb{P}\left\{\sup_{t \in T} f(t) \geq u\right\} \leq N(\varepsilon)\, e^{-\frac{1}{2}(u - \mu(\varepsilon))^2 / \sigma_T^2}.$$

Allowing $C = C(\alpha)$ to denote a constant, dependent only on α, that may change from line to line, we have from Theorem 1.3.3 that

$$(4.1.5) \qquad \begin{aligned} \mu(t, \varepsilon) &\leq C \int_0^\varepsilon \big(\log N(\varepsilon)\big)^{\frac{1}{2}}\, d\varepsilon \\ &\leq C \int_0^\varepsilon \big(\log(1/\varepsilon)\big)^{\frac{1}{2}}\, d\varepsilon \\ &\leq C \varepsilon \sqrt{\log(1/\varepsilon)}. \end{aligned}$$

Set $\varepsilon = \varepsilon(u) = u^{-1}$, choose u large enough so that $u > Cu^{-1}\sqrt{\log u}$ and substitute into (4.1.4) to obtain

$$(4.1.6) \qquad \begin{aligned} \mathbb{P}\left\{\sup_{t \in T} f(t) \geq u\right\} &\leq C_1\, u^\alpha\, e^{-\frac{1}{2}\left(u - C_2 u^{-1}\sqrt{\log u}\,\right)^2 / \sigma_T^2} \\ &\leq C_3\, u^\alpha\, e^{C_4 \sqrt{\log u}}\, e^{-u^2/2\sigma_T^2}. \end{aligned}$$

Since for $\eta > 0$ and u large enough $e^{C\sqrt{\log u}} < u^\eta$, this gives us (4.1.2) and so completes the proof. $\qquad\square$

The bound in (4.1.2) can be improved slightly, to $C(u \log u)^\alpha e^{-u^2/2\sigma_T^2}$, by choosing $\varepsilon = \varepsilon(u)$ in the above proof to satisfy $u^{-1} = \varepsilon(\log(1/\varepsilon))^{1/2}$. The gain, however, is rather small, in view of the fact that far stronger results hold. In particular, in a series of papers [126, 127, 141, 128, 144] by Samorodnitsky and Talagrand, with a leap-frogging of ideas and techniques[3], the results of the following three theorems, among others, were obtained. In each, f is *a priori* assumed to have continuous sample paths with probability one, $\sigma_T^2 \overset{\Delta}{=} \sup_T \mathbb{E}\{f_t^2\}$ and, at each appearance, K represents a universal constant which may differ between appearances.

Theorem 4.1.2 *If for some $A > \sigma_T$, some $\alpha > 0$ and some $\varepsilon_0 \in [0, \sigma_T]$ we have*

$$(4.1.7) \qquad N(T, d, \varepsilon) \leq (A/\varepsilon)^\alpha$$

---

[3]While the Samorodnitsky and Talagrand papers are what lead to the final form presented here, the key arguments are much older, and two of the key players were Simeon Berman (e.g. [20, 21, 22, 23]) and Michel Weber (e.g. [157]).

for all $\varepsilon < \varepsilon_0$, then for $u \geq \sigma_T^2[(1 + \sqrt{\alpha})/\varepsilon_0]$ we also have

$$(4.1.8) \qquad \mathbb{P}\left\{\sup_{t \in T} f_t \geq u\right\} \leq \left(\frac{KAu}{\sqrt{\alpha}\sigma_T^2}\right)^\alpha \Psi\left(\frac{u}{\sigma_T}\right),$$

where K is a universal constant[4].

Theorem 4.1.3 *Set*

$$(4.1.9) \qquad T_\delta = \left\{t \in T : \mathbb{E}\{f_t^2\} \geq \sigma_T^2 - \delta^2\right\}.$$

Suppose there exist $\alpha > \beta > 1$ such that, for all $\delta > 0$, $\varepsilon \in (0, \delta(1 + \sqrt{\alpha})/\sqrt{\beta})$, we have

$$(4.1.10) \qquad N(T_\delta, d, \varepsilon) \leq A\delta^\beta \varepsilon^{-\alpha}.$$

Then, for $u \geq 2\sigma_T\sqrt{\beta}$,

$$(4.1.11) \quad \mathbb{P}\left\{\sup_{t \in T} f_t \geq u\right\} \leq \frac{A\beta^{\beta/2}}{\alpha^{\alpha/2}} K^{\alpha+\beta} \left(\frac{u}{\sigma_T^2}\right)^{\alpha-\beta} \Psi\left(\frac{u}{\sigma_T}\right).$$

Theorem 4.1.4 *Suppose there exist $A, B > 0$ and $\alpha \in (0, 2)$ so that*

$$N(T, \varepsilon) < A\exp(B\varepsilon^{-\alpha})$$

Then, for all $u > 0$,

$$\mathbb{P}\left\{\sup_{t \in T} f_t > u\right\} \leq K_1 \exp\left(K_2 u^{2\alpha/(2+\alpha)}\right) \Psi\left(\frac{u}{\sigma_T}\right),$$

where K_1 and K_2 are universal.

In a moment we shall turn to proofs of the first two of these results[5], adopting (and occassionally correcting typo's and oversights) the proofs in [144]. Note that (4.1.10) raises an interesting question if we (formally) take $\alpha = \beta$. In that case the upper bound is of the form $C(\alpha, \beta)\Psi(u/\sigma_T)$, and this also serves as a trivial lower bound when $C = 1$. Thus it is natural to ask if there are scenarios in which one can in fact also take $C = 1$ in

---

[4]Note that at first sight the right hand side of (4.1.8) seems to scale incorrectly, since in the power term we would expect to see u/σ_T rather than u/σ_T^2. This is an artifact of the condition $A > \sigma_T$, which implies that increasing σ_T requires increasing A as well, thus ensuring correct scaling. The same phenomenon appears in (4.1.11), albeit a little more subtly.

[5]A proof of a lower bound for (4.1.11) can be found in [127]. You can also find quite general arguments, for both upper and lower bounds and for non-Gaussian as well as Gaussian *stationary* processes in a number of papers by Albin (e.g. [10]). We shall see more in the way of lower bounds for specific cases soon.

the upper bound. It turns out that there are, and we shall look at these in Section 4.2.

Theorems 4.1.2 and 4.1.3 treat entropy functions for which the growth of N in ε is of a power form, which, at least for processes indexed by points in Euclidean spaces or smooth manifolds, are the most common. Theorem 4.1.4, which we shall not prove, handles the scenario of of exponential entropy and is a qualitatively different result. Note that you cannot set $\alpha = 0$ in this result to recover either Theorem 4.1.2 or 4.1.3. The upper bound given here is, under mild side conditions, also a lower bound.

Before reading further, you might want to turn now to Section 4.3 to see some concrete examples for which the above results apply. Here, however, we shall give the proofs of Theorems 4.1.2 and 4.1.3. The central idea is old, and goes back at least to the works of Berman [18, 19] and Pickands [114, 115, 116] in the mid to late 1960's, which themselves have roots in Cramér's classic paper [39].

The basic approach lies in looking for a subset $T_{\max} \subset T$ (very often a unique point in T) where the maximal variance is achieved, and then studying two things: the 'size' of $T_{\max}$ (e.g. as measured in terms of metric entropy) and how rapidly $E\{f_t^2\}$ decays as one moves out of $T_{\max}$. The underlying idea is that the supremum of the process is most likely to occur in or near the region of maximal variance, and the rate of decay of $E\{f_t^2\}$ outside that region determines how to best account for the impact of nearby regions.

A key Lemma, on which all the proofs we shall give rely, is the following result of [144]. Recall, throughout, the assumption of Chapter 1 that the parameter space T is compact for the pseudometric d, which we shall assume is still in place.

Lemma 4.1.5 *Let f be a centered, a.s. continuous Gaussian process on T and X a centered Gaussian variable such that the family $\{X, f_t, t \in T\}$ is jointly Gaussian. Assume that*

$$(4.1.12) \qquad\qquad \mathbb{E}\left\{(f_t - X)\, X\right\} \;\leq\; 0$$

for all $t \in T$. Set

$$\mu \;=\; \mathbb{E}\left\{\sup_{t \in T} f_t\right\}, \quad \sigma^2 \;=\; \mathbb{E}\left\{X^2\right\}, \quad a^2 \;=\; \sup_{t \in T} \mathbb{E}\left\{(f_t - X)^2\right\}.$$

Then, if $a < \sigma$ and $u > \mu$,

$$\mathbb{P}\left\{\sup_{t \in T} f_t \geq u\right\} \;\leq\; \frac{1}{2}\exp\left(-\frac{(u-\mu)^2}{2a^2}\right)$$

$$(4.1.13) \qquad\qquad + \frac{a}{\sqrt{a^2+\sigma^2}}\exp\left(-\frac{(u-\mu)^2}{2(a^2+\sigma^2)}\right) + \Psi\left(\frac{u-\mu}{\sigma}\right).$$

If, in addition, $u > \max(2\mu, \mu + \sigma)$ then

(4.1.14)
$$\mathbb{P}\left\{\sup_{t \in T} f_t \geq u\right\} \;\leq\; \Psi\left(\frac{u}{\sigma}\right)\left[1 + K\frac{au}{\sigma^2}\exp\left(\frac{a^2u^2}{2\sigma^4}\right)\right]\exp\left(\frac{2u\mu}{\sigma^2}\right),$$

where K is a universal constant.

In a typical application of Lemma 4.1.5 X is usually the value of f at a point $t_o \in T$ of maximal variance. In that case (4.1.12) trivially holds, and $a < \sigma$ will also hold if T is sufficiently small.

Proof. While the proof is rather technical, the idea behind it is actually quite simple. It starts by defining the function a_t by

$$a_t \;=\; \frac{\mathbb{E}\{(f_t - X)X\}}{\sigma^2} \;\leq\; 0,$$

and the random process g_t by

$$f_t \;=\; g_t \;+\; (1 + a_t)\, X.$$

Note that $\mathbb{E}\{g_t X\} = 0$. That is, X and the full process g are independent.

The argument then works as follows. There are essentially three ways for $\sup_T f_t$ to be larger than u. One is for $\sup_T g_t$ to be large. One is for each of X and $\sup_T g_t$ to be large enough for the sum to be large, and the last is for X to be large. The three terms in (4.1.13) correspond, in order, to these three cases, and each is obtained by conditioning on the distribution of X and then using the Borell-TIS inequality to handle $\sup_T g_t$. That there is only one term in (4.1.14) comes from the fact that here u is even larger, and then the last term actually dominates all others. The reason for this is that from the assumption $a_t \leq 0$ and $a < \sigma$ it follows that $E\{X^2\} > \sup_T E\{g_t^2\}$, and so it is 'easier' for X to reach larger values than it is for g. Now read on.

It is trivial to check that, for all $s, t \in T$,

$$\mathbb{E}\left\{(g_t - g_s)^2\right\} \;\leq\; \mathbb{E}\left\{(f_t - f_s)^2\right\},$$

so that by the Sudakov-Fernique inequality (Theorem 2.2.3)

(4.1.15) $$\mathbb{E}\left\{\sup_{t \in T} g_t\right\} \;\leq\; \mathbb{E}\left\{\sup_{t \in T} f_t\right\} \;=\; \mu.$$

Preparing for the conditioning argument, write

$$\mathbb{P}\left\{\sup_{t \in T} f_t \geq u \,\Big|\, X = x\right\} \;=\; \mathbb{P}\left\{\sup_{t \in T}(f_t - X) \geq u - x \,\Big|\, X = x\right\}$$

$$=\; \mathbb{P}\left\{\sup_{t \in T}(g_t + a_t X) \geq u - x \,\Big|\, X = x\right\}.$$

For $x > 0$, using the fact that $a_t \le 0$, we have

$$\sup_{t \in T} (g_t + a_t x) \le \sup_{t \in T} g_t,$$

from which it trivially follows that, for $x \ge 0$,

$$\mathbb{P}\left\{\sup_{t \in T} f_t \ge u \,\middle|\, X = x\right\} \le \mathbb{P}\left\{\sup_{t \in T} g_t \ge u - x \,\middle|\, X = x\right\}.$$

However, since X and the process g are independent we can drop the last conditioning, and since

$$\mathbb{E}\left\{g_t^2\right\} \le \mathbb{E}\left\{(g_t + a_t X)^2\right\} = \mathbb{E}\left\{(f_t - X)^2\right\} \le a^2$$

it follows from (4.1.15) and the Borell-TIS inequality (Theorem 2.1.1) that, for $0 \le x \le u - \mu$,

$$(4.1.16) \quad \mathbb{P}\left\{\sup_{t \in T} f_t \ge u \,\middle|\, X = x\right\} \le \exp\left(-\frac{(u - x - \mu)^2}{2a^2}\right).$$

We get a slightly different bound for $x \le 0$. Note that since

$$|a_t| = \frac{|\mathbb{E}\left\{(f_t - X)X\right\}|}{\sigma^2} \le \frac{a}{\sigma} \le 1,$$

we have

$$\sup_{t \in T} (g_t + a_t x) \le |x| + \sup_{t \in T} g_t = -x + \sup_{t \in T} g_t.$$

Thus, again applying the Borell-TIS inequality, we have

$$\mathbb{P}\left\{\sup_{t \in T} (g_t + a_t X) \ge u - x \,\middle|\, X = x\right\} \le \mathbb{P}\left\{\sup_{t \in T} g_t \ge u\right\}$$

$$(4.1.17) \qquad\qquad\qquad\qquad \le \exp\left(-\frac{(u - \mu)^2}{2a^2}\right),$$

which is independent of x. We now have all we need to establish (4.1.13). Write, for $u \ge \mu$

$$\mathbb{P}\left\{\sup_{t \in T} f_t \ge u\right\} = \int_{-\infty}^{\infty} \mathbb{P}\left\{\sup_{t \in T} f_t \ge u \,\middle|\, X = x\right\} \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi}\sigma}\, dx$$

$$= \int_{-\infty}^{0} + \int_{0}^{u-\mu} + \int_{u-\mu}^{\infty} \ldots$$

$$\overset{\Delta}{=} I_1 + I_2 + I_3.$$

The term I_3 is trivially bounded by the last term in (4.1.13). For I_1, bound the integrand by (4.1.17) and integrate out x to obtain the first term on

the right hand side of (4.1.13). For the intermediate term, apply (4.1.16) and note, with $s = u - \mu$, that

$$
\begin{aligned}
I_2 &\leq \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left(-\frac{(s-x)^2}{2a^2} - \frac{x^2}{2\sigma^2}\right) dx \\
&= \frac{a}{\sqrt{a^2 + \sigma^2}} \exp\left(-\frac{s^2}{2(\sigma^2 + a^2)}\right),
\end{aligned}
$$

evaluating the integral by completing the square. Thus the first part of the lemma, (4.1.13), is established.

It remains to show (4.1.14). We start by observing that

$$
\exp\left(-\frac{(u-\mu)^2}{2a^2}\right) = \exp\left(-\frac{(u-\mu)^2}{2(a^2+\sigma^2)}\right) \exp\left(-\frac{(u-\mu)^2\sigma^2}{2a^2(a^2+\sigma^2)}\right).
$$

Since $\sigma > a$ and $u > \mu + \sigma$,

$$
\exp\left(-\frac{(u-\mu)^2\sigma^2}{2a^2(a^2+\sigma^2)}\right) \leq \exp\left(-\frac{\sigma^2}{4a^2}\right) \leq \frac{a}{\sigma}.
$$

Thus the sum of the first two terms on the right hand side of (4.1.13) is bounded above by

$$
\frac{2a}{\sigma} \exp\left(-\frac{(u-\mu)^2}{2(a^2+\sigma^2)}\right).
$$

Since

$$
\frac{1}{a^2 + \sigma^2} \geq \frac{1}{\sigma^2} - \frac{a^2}{\sigma^4},
$$

the previous expression is at most

$$
\begin{aligned}
\frac{2a}{\sigma} &\exp\left(-\frac{(u-\mu)^2}{2\sigma^2} + \frac{u^2 a^2}{2\sigma^4}\right) \\
&\leq \frac{2au(u-\mu)}{\sigma^3} \exp\left(\frac{u^2 a^2}{2\sigma^4}\right) \exp\left(-\frac{(u-\mu)^2}{2\sigma^2}\right),
\end{aligned}
$$

on applying the various inequalities between a, σ, u and μ. Apply the basic Gaussian tail bound (1.2.2) to bound the above by

$$
\frac{Kua}{\sigma^2} \Psi\left(\frac{u-\mu}{\sigma}\right) \exp\left(\frac{u^2 a^2}{2\sigma^4}\right),
$$

for a universal K. To complete the argument note that for $x \geq 1$, $y \geq 0$,

$$
\Psi(x - y) \leq e^{2xy}\Psi(x).
$$

(This is easily proven from the the fact that $f(y) \stackrel{\Delta}{=} e^{2xy}\Psi(x) - \Psi(x-y)$ satisfies $f(0) = 0$ and has positive derivative, since

$$f'(y) \;=\; 2xe^{2xy}\Psi(x) - \frac{1}{\sqrt{2\pi}}e^{-(x-y)^2/2} \;\geq\; 0,$$

again applying the Gaussian tail bound (1.2.2).)   □

While Lemma 4.1.5 will be the main tool for the proofs of Theorems 4.1.2 and 4.1.3 we first need a counting result about partitioning our (pseudo) metric space (T, d).

Lemma 4.1.6 *Let (T, d) be a metric space and take p, q integers, $p < q$. Let $\mathcal{P}_q$ be a partition of T of sets of diameter no more than 4^{-q+1}. Take a set of integers k_l, $p \leq l \leq q$. Then there exists an increasing sequence $\{\mathcal{P}_l\}_{p \leq l \leq q}$ of partitions of T with the following properties:*

(4.1.18) $\mathcal{P}_{l+1}$ *is a refinement of $\mathcal{P}_l$. i.e. For each $A \in \mathcal{P}_{l+1}$ there is a $A' \in \mathcal{P}_l$ such that $A \subseteq A'$.*

(4.1.19) *For each set $A \in \mathcal{P}_l$, $\mathrm{diam}(A) \leq 4^{-l+1}$.*

(4.1.20) *Each set of $\mathcal{P}_l$ contains at most k_l sets of $\mathcal{P}_{l+1}$.*

(4.1.21) $|\mathcal{P}_l| \;\leq\; N\left(4^{-l}\right) + \dfrac{|\mathcal{P}_{l+1}|}{k_l}, \quad$ *for all $l < q$,*

where $N(\varepsilon)$ is the metric entropy function for (T, d) and $|\mathcal{P}_l|$ is the number of sets in $\mathcal{P}_l$.

Proof. We shall construct the partitions $\mathcal{P}_l$ by decreasing induction over l; i.e. We show how to construct $\mathcal{P}_l$ once $\mathcal{P}_{l+1}$ is given.

Set $N = N(4^{-l})$ and consider points $\{t_i\}_{1 \leq i \leq N}$ of T so that

$$\sup_{t \in T} \inf_i d(t, t_i) \leq 4^{-l}.$$

For $i \leq N$ let A_i be the union of all the sets in $\mathcal{P}_{l+1}$ that intersect $B_d(t_i, 4^{-l})$. Thus, since the sets in $\mathcal{P}_{l+1}$ have diameter no larger than 4^{-l}, A_i has diameter at most

$$2 \cdot 4^{-l} + 2 \cdot 4^{-l} \;=\; 4^{-l+1}.$$

Now define $C_i = A_i \setminus \bigcup_{j < i} A_j$. These form a partition of T that is coarser than $\mathcal{P}_{l+1}$. Some of these sets might contain more that k_l sets of $\mathcal{P}_{l+1}$. In such a case, these sets can be repartitioned into smaller sets, all but at most one of which is the union of exactly k_l of the sets in $\mathcal{P}_{l+1}$. The exception will contain fewer than k_l of the sets in $\mathcal{P}_{l+1}$. This constructs $\mathcal{P}_l$ and (4.1.21) is immediate.   □

Corollary 4.1.7 *Suppose that in Lemma 4.1.6 we have, for some $A, \alpha > 0$, $N(4^{-l}) \leq (A4^l)^\alpha$ if $l \geq p$ and $|\mathcal{P}_q| \leq (A4^q)^\alpha$. Then, if $k_l \geq 2 \cdot 4^\alpha$ for all l, we have $|\mathcal{P}_l| \leq 2(A4^l)^\alpha$.*

Proof. Using the construction given in the proof of the Lemma, and by decreasing induction over l,

$$|\mathcal{P}_l| \; \leq \; (A4^l)^\alpha \; + \; \frac{2\left(A4^{l+1}\right)^\alpha}{k_l} \; \leq \; 2\left(A4^l\right)^\alpha.$$

$\square$

Proof of Theorem 4.1.2. The argument is by partitioning, much as for the proof of Theorem 4.1.1. The improvement in the result relies on being more careful as to how the partitioning is carried out, and from a judicious application of Lemma 4.1.5, which we did not previously have.

Start by choosing $a < \sigma_T$ and $\mu > 0$. Then choose a partition $\{T_i\}_{i \leq N}$ of T into N compact sets, each of diameter no more than a and for each of which $\mathbb{E}\left\{\sup_{T_i} f_t\right\} \leq \mu$. Then, for $u > \max\left(2\mu, \mu + \sigma_T\right)$, it follows from (4.1.14) (by taking X to be f taken at a point of maximal variance in each T_i) that

$$(4.1.22) \quad \mathbb{P}\left\{\sup_{t \in T} f_t \geq u\right\}$$
$$\leq \; N\left(1 + K\frac{au}{\sigma_T^2}\exp\left(\frac{a^2 u^2}{2\sigma_T^4}\right)\right)\exp\left(\frac{2u\mu}{\sigma_T^2}\right)\Psi\left(\frac{u}{\sigma_T}\right),$$

where K is a universal constant. The free variables here are μ and a, and the problem is that $N \to \infty$ as $a, \mu \to 0$. The trick therefore is to find a sequence of partitions for which N does not grow too fast.

As usual, our first step is to assume that T is finite and once again apply the argument at the end of the proof of the Borell-TIS inequality for moving from finite to infinite T^6. Thus we can, and do, assume that T is finite, and can now start the serious part of the proof.

Since T is finite, we can consider it as a partition of itself, and find a q large enough so that $d(s, t) \geq 4^{-q+1}$ for all $s, t \in T$. Applying Corollary 4.1.7 with $k_l = \lfloor 2 \cdot 4^\alpha \rfloor + 1 \leq 3 \cdot 4^\alpha$ and, for each $l \leq q$, with $4^{-l} \leq \varepsilon_0$, we find a further partition of T into $N \leq 2(A4^l)^\alpha$ sets $\{T_i\}_{i \leq N}$ such that, for $m \geq l$,

$$N\left(T_i, d, 4^{-m+1}\right) \; \leq (3 \cdot 4^\alpha)^{m-l}$$

---

[6]The one issue that is slightly more involved this time is whether or not a bound on the entropy function for the original parameter space T translates to an equivalent bound for the process defined on a finite subset of T. This, however, is easy to check.

for all $i \leq N$. The last inequality follows from repeated applications of (4.1.20). Using this bound in Theorem 1.3.3, it is a simple calculation to see that

$$(4.1.23) \qquad \mathbb{E}\left\{\sup_{t \in T_i} f_t\right\} \leq K\sqrt{\alpha}4^{-l},$$

for some universal K.

Now take for l the smallest integer such that $4^{-l} \leq \sqrt{\alpha}\sigma_T^2/4Ku$, with the K of (4.1.23). Assuming, for no loss of generality, that $K \geq 1$, we have $4^{-l} \leq \varepsilon_0$, provided that $u \geq \sqrt{\alpha}\sigma_T^2/\varepsilon_0$. Since $4^{-l} \geq \sqrt{\alpha}\sigma_T^2/16Ku$ we have $N \leq (16KAu/\sqrt{\alpha}\sigma_T^2)^\alpha$. We also have, in the notation of Theorem 4.1.2, that

$$\mu \leq \frac{\alpha\sigma_T^2}{4u}, \qquad a \leq 4^{-l+1} \leq \frac{\sqrt{\alpha}\sigma_T^2}{u}.$$

Now observe that, since $u > \sqrt{\alpha}\sigma_T$, we have $\mu \leq \sqrt{\alpha}\sigma_T \leq u/4$ and $a < \sigma_T$, so that the result now follows from (4.1.22), as promised. $\square$

Proof of Theorem 4.1.3.   As before, the proof uses a partitioning argument. This time, however, we start by partioning the space into regions depending on the size of $\mathbb{E}\{f_t^2\}$. In each of these regions we apply Theorem 4.1.2 to bound the supremum, so that the hard part of the bounding has already been done. The rest of the proof is really just accounting, to make sure that the various pieces add up appropriately.

To start, fix $u > 2\sigma_T\sqrt{\beta}$. Set $\delta_0 = 0$, $\delta_1 = \sqrt{\beta}\sigma_T^2/u$ and, for $k \geq 1$, set $\delta_k = 2^{k-1}\delta_1$. For $k \geq 1$ set $U_k = T_{\delta_k} \setminus T_{\delta_{k-1}}$. Set $\varepsilon_0 = \delta_1(1+\sqrt{\alpha})/\sqrt{\beta}$, and note

$$\frac{\sigma_T^2(1+\sqrt{\alpha})}{\varepsilon_0} \leq \frac{\sigma_T^2\sqrt{\beta}}{\delta_1} = u.$$

Thus, setting $\sigma_k^2 = \sigma_T^2 - \delta_{k-1}^2$, and applying Theorem 4.1.2 to U_k, we obtain

$$\mathbb{P}\left\{\sup_{t \in U_k} \geq u\right\} \leq A\delta_k^\beta \left(\frac{Ku}{\sqrt{\alpha}\sigma_k^2}\right)^\alpha \Psi\left(\frac{u}{\sigma_k}\right).$$

Let k_0 be the largest integer such that $\delta_{k_0-1} \leq \sigma_T/2$. Since $u \geq 2\sqrt{\beta}\sigma_T$, we have that $k_0 \geq 2$ and $\delta_{k_0-1} \geq \sigma_T/4$. For $k \leq k_0$, we have $\sigma_k^2 \geq 3\sigma_T^2/4$ so that

$$\mathbb{P}\left\{\sup_{t \in U_k} f_t \geq u\right\} \leq A\delta_k^\beta \left(\frac{Ku}{\sqrt{\alpha}\sigma_T^2}\right)^\alpha \left(\frac{\sigma_T}{u}\right) \exp\left(-\frac{u^2}{2\left(\sigma_T^2 - \delta_{k-1}^2\right)}\right).$$

Since we also have that

$$\frac{1}{\sigma_T^2 - \delta_{k-1}^2} \geq \frac{1}{\sigma_T^2} + \frac{\delta_{k-1}^2}{\sigma_T^4},$$

we see that

$$\mathbb{P}\left\{\sup_{t\in U_k} f_t \geq u\right\} \leq A\delta_k^\beta \left(\frac{Ku}{\sqrt{\alpha}\sigma_T^2}\right)^\alpha \exp\left(-\frac{u^2\delta_{k-1}^2}{2\sigma_T^4}\right) \Psi\left(\frac{u}{\sigma_T}\right).$$

Note that

$$\sum_{k\geq 1} \delta_k^\beta \exp\left(-\frac{u^2\delta_{k-1}^2}{2\sigma_T^4}\right) = \delta_1^\beta + \sum_{k\geq 2} \delta_k^\beta \exp\left(-\frac{u^2\delta_{k-1}^2}{2\sigma_T^4}\right)$$

$$\leq \delta_1^\beta \left(1 + \sum_{k\geq 2} 2^{\beta k} \exp\left(-\beta 2^{2k-3}\right)\right)$$

$$\leq (K\delta_1)^\beta,$$

the last line requiring $\beta > 1$. Consequently, if we set $T' = \bigcup_{k\leq k_0} U_k$, we have, recalling the value of δ_1, that

$$(4.1.24) \quad \mathbb{P}\left\{\sup_{t\in T'} f_t \geq u\right\} \leq A\frac{\beta^{\beta/2}}{\alpha^{\alpha/2}} K^{\beta+\alpha} \left(\frac{u}{\sigma_T^2}\right)^{\alpha-\beta} \Psi\left(\frac{u}{\sigma_T}\right).$$

For $t \notin T'$ we have

$$\mathbb{E}\left\{f_t^2\right\} \leq \sigma_T^2 - \delta_{k_0-1}^2 \leq \sigma_T^2 - \sigma_T^2/16 = 15\sigma_T^2/16.$$

Thus, if we now apply the entropy bound (4.1.10), we see that for $u > \sigma_T\sqrt{\beta}$ we have

$$(4.1.25) \quad \mathbb{P}\left\{\sup_{t\in T\setminus T'} f_t \geq u\right\} \leq A\sigma_T^\beta \left(\frac{Ku}{\sqrt{\alpha}\sigma_T^2}\right)^\alpha \Psi\left(\frac{4u}{\sigma_T\sqrt{15}}\right).$$

To conclude, it suffices to check that for $x \geq \sqrt{\beta}$ we have $\Psi(4x/\sqrt{15}) \leq (K\beta)^{\beta/2}x^{-\beta}\Psi(x)$, so that the left hand side of (4.1.25) is dominated by the left hand side of (4.1.11). □

4.2 Processes with a unique point of maximal variance

We shall now look at a rather interesting special case. We know already from the discussion following Theorem 4.1.3 that there would seem to be cases in which the trivial lower bound

$$\mathbb{P}\left\{\sup_{t\in T} f(t) \geq u\right\} \geq \sup_{t\in T} \mathbb{P}\left\{f(t) \geq u\right\} = \Psi\left(\frac{u}{\sigma_T}\right)$$

comes close (up to a constant) as serving as an upper bound as well.

A class of examples in which this holds with the constant 1 arises when there is a *unique* point $t_o \in T$ at which the maximal variance is achieved and some additional side conditions are assumed. In that case, we have the following elegant result of Talagrand [142].

Theorem 4.2.1 *Suppose f is a centered, bounded, Gaussian process on T, and that there is a unique point $t_o \in T$ such that*

$$(4.2.26) \qquad \mathbb{E}\left\{f_{t_o}^2\right\} \;=\; \sigma_T^2 \;=\; \sup_T \mathbb{E}\left\{f_t^2\right\}.$$

For $\delta > 0$, set

$$T_\delta \;=\; \left\{t \in T : \mathbb{E}\left\{f_t f_{t_o}\right\} \geq \sigma_T^2 - \delta^2\right\}.$$

Suppose that

$$(4.2.27) \qquad \lim_{\delta \to 0} \frac{\mathbb{E}\left\{\sup_{t \in T_\delta} f_t\right\}}{\delta} \;=\; 0.$$

Then

$$(4.2.28) \qquad \lim_{u \to \infty} \frac{\mathbb{P}\left\{\sup_{t \in T} f(t) \geq u\right\}}{\Psi(u/\sigma_T)} \;=\; 1.$$

There is actually a converse to this result, which states that (4.2.28) implies the existence of a $t_o \in T$ for which (4.2.26) and (4.2.27) hold. You can find a proof of the converse in [142].

Proof. Since it is a triviality that

$$\mathbb{P}\left\{\sup_{t \in T} f_t \geq u\right\} \;\geq\; \mathbb{P}\left\{f_{t_o} \geq u\right\} \;=\; \Psi\left(\frac{u}{\sigma_T}\right)$$

the theorem will be proven once we establish (4.2.28) with a limsup rather than lim.

To start, take $\eta \in (0, 1)$ and, by (4.2.27), δ_0 small enough so that

$$(4.2.29) \qquad \delta \;\leq\; 2\delta_0 \;\Rightarrow\; \mathbb{E}\left\{\sup_{t \in T_\delta} f_t\right\} \;\leq\; \eta^2 \delta.$$

We can and do assume that $\delta_0 \leq \sigma_T \eta^2$. Note also that

$$\sup_{t \notin T_{\delta_0}} \mathbb{E}\left\{f_t^2\right\} \;<\; \sigma_T^2,$$

since otherwise there would be a $t \notin T_{\delta_0}$ (so that $t \neq t_o$) for which $\mathbb{E}\left\{f_t^2\right\} = \sigma_T^2$, which contradicts the assumption of there being a unique point of

maximal variance. It therefore follows immediately from the Borell-TIS inequality that

$$(4.2.30) \qquad \lim_{u \to \infty} \frac{\mathbb{P}\left\{\sup_{t \in T \setminus T_{\delta_0}} f_t \geq u\right\}}{\Psi(u/\sigma_T)} = 0,$$

and so to complete the proof it will suffice to show that, for some universal K, and for u large enough,

$$(4.2.31) \qquad \mathbb{P}\left\{\sup_{t \in T_{\delta_0}} f_t \geq u\right\} \leq \Psi\left(\frac{u}{\sigma_T}\right)(1 + K\eta).$$

Fix u and set $\alpha = \sigma_T^2/\eta u$. Adopting and adapting the main idea in the proof of Theorem 4.1.3, define a non-decreasing sequence of subsets of T, and a sequence of 'annuli', by setting $V_{-1} = \emptyset$ and, for $k \geq 0$,

$$V_k = T_{2^k \alpha}, \qquad U_k = V_k \setminus V_{k-1}.$$

If p is the smallest integer for which $2^p \alpha \geq \delta_0$, then

$$T_{\delta_0} \subset \bigcup_{0 \leq k \leq p} U_k,$$

and we shall now try to obtain the bound in (4.2.31) by obtaining a similar bound for each U_k and then combining them via a union bound.

Setting $\mu_k = \mathbb{E}\left\{\sup_{t \in V_k} f_t\right\}$, (4.2.29) implies that $\mu_k \leq \alpha\eta^2 2^k$ for $k \leq p$. Setting

$$\omega_k = \sup_{t \in V_k}\left(\mathbb{E}\left\{(f_t - f_{t_o})^2\right\}\right)^{1/2},$$

we have

$$(4.2.32) \qquad \omega_k = \sup_{t \in V_k} K\,\mathbb{E}\left\{|f_t - f_{t_o}|\right\}$$

$$\leq K\,\mathbb{E}\left\{\sup_{t \in V_k} |f_t - f_{t_o}|\right\}$$

$$\leq K\mu_k$$

$$\leq K\alpha\eta^2 2^k,$$

the first line being a standard Gaussian identity and the second inequality coming from the fact that $t_o \in V_k$.

By (4.1.14) with $X = f_{t_o}$ we therefore have

$$\mathbb{P}\left\{\sup_{t \in U_0} f_t \geq u\right\} \leq \Psi\left(\frac{u}{\sigma_T}\right)\left(1 + K\eta e^{K\eta^2/2}\right)e^{2\eta}$$

$$\leq \Psi\left(\frac{u}{\sigma_T}\right)(1 + K\eta).$$

We use a similar argument for the remaining U_k. Take the random variable X of Lemma 4.1.5 to be $X_k = (1 - (\alpha 2^{k-1})^2/\sigma_T^2)f_{t_o}$. It is then easy to check that the definition of T_δ implies that (4.1.12) holds. Furthermore, since $k \leq p$ and $\delta_0 \leq \sigma_T \eta^2$,

$$\left(\mathbb{E}\left\{(X_k - f_{t_o})^2\right\}\right)^{1/2} \;=\; \frac{(\alpha 2^{k-1})^2}{\sigma_T} \;\leq\; \frac{\delta_0}{\sigma_T}\alpha 2^{k-1} \;\leq\; \eta^2 \alpha 2^k.$$

It follows from (4.2.32) and the triangle inequality that if we set

$$a_k^2 \;=\; \sup_{t \in V_k} \mathbb{E}\left\{(f_t - X_k)^2\right\},$$

then $a_k \leq K\eta^2 \alpha 2^k$. Thus, by (4.1.14) and the fact that $\eta < 1$ we have

$$\mathbb{P}\left\{\sup_{t \in U_k} f_t \geq u\right\}$$

$$\leq \;\; \Psi\left(\frac{u}{\sigma_T\,(1 - (\alpha 2^{k-1})^2/\sigma_T^2)}\right)\left(1 + K\eta 2^k e^{K\eta^2 2^{2k}}\right)e^{K\eta 2^k}$$

$$\leq \;\; \Psi\left(\frac{u}{\sigma_T\,(1 - (\alpha 2^{k-1})^2/\sigma_T^2)}\right)e^{K\eta 2^{2k}}.$$

Using the fact that for $x \leq 1$ we know $(1 - x)^{-1} \geq 1 + x$ we have

$$\frac{u}{\sigma_T\,(1 - (\alpha 2^{k-1})^2/\sigma_T^2)} \;\geq\; \frac{u}{\sigma_T} + \frac{(\alpha 2^{k-1})^2\,u}{2\sigma_T^3}.$$

Also, since a simple change of variables shows that

$$\Psi(x + y) \;\leq\; e^{-xy}\Psi(x),$$

for $y \geq 0$, it follows, on recalling the definition of α, that

$$\Psi\left(\frac{u}{\sigma_T\,(1 - (\alpha 2^{k-1})^2/\sigma_T^2)}\right) \;\leq\; \Psi\left(\frac{u}{\sigma_T}\right)\exp\left(-\frac{(\alpha 2^{k-1})^2\,u^2}{\sigma_T^4}\right)$$

$$\leq \;\; \Psi\left(\frac{u}{\sigma_T}\right)\exp\left(-\frac{2^{2k}}{4\eta}\right).$$

Putting all this together, we thus have

$$\sum_{1 \leq k \leq p}\mathbb{P}\left\{\sup_{t \in U_k} f_t \geq u\right\} \;\leq\; \Psi\left(\frac{u}{\sigma_T}\right)\sum_{1 \leq k \leq p}\exp\left(-2^{2k}\left(\frac{1}{4\eta} - K\eta\right)\right).$$

Since the last sum can be made arbitrarily small by taking η small enough we have established (4.2.31) and so the theorem. $\qquad\square$

4.3   Examples

Theorem 4.2.1 provided us with conditions under which Gaussian processes have supremum distribution, at least in the tail, which behaves the same as that of a single Gaussian variable. Here is a class of examples for which this happens.

Example 4.3.1 *Let f_t be a centered Gaussian random field on $\mathbb{R}^N$, with $f(0) = 0$ and with stationary and isotropic increments, in the sense that the distribution of $f(t + s) - f(s)$ depends only on $|t|$. Suppose that*

$$p^2(t) \triangleq \mathbb{E}\left\{(f_{t+s} - f_s)^2\right\} = \mathbb{E}\left\{f_t^2\right\}$$

is convex, and that

$$(4.3.33) \qquad\qquad \lim_{t \to 0} \frac{p^2(t)}{|t|} = 0.$$

With some abuse of notation, let $T = [0, T]$ be a N-dimensional rectangle. Then

$$(4.3.34) \qquad\qquad \lim_{u \to \infty} \frac{\mathbb{P}\left\{\sup_{t \in T} f_t \geq u\right\}}{\Psi(u/\sigma_T)} = 1,$$

where $\sigma_T^2 = \sup_{t \in T} \mathbb{E}\left\{f^2(t)\right\} = \mathbb{E}\left\{f^2(T)\right\}$.

Without doubt the best known class of random fields statisfying the conditions of the example are the so-called *fractional Brownian motions* or *fields*, which have covariance function

$$C(s, t) = \tfrac{1}{2}\left(|t|^{2\alpha} + |s|^{2\alpha} - |t - s|^{2\alpha}.\right).$$

While these processes have isotropic increments for all $\alpha \in (0, 1)$, it is only when $\alpha > \tfrac{1}{2}$ that p^2 is convex. More information on these processes can be found, for example, in [129].

Proof. The proof requires no more than checking that conditions (4.2.26) and (4.2.27) of Theorem 4.2.1 hold. Since f has stationary increments, (4.2.26) is clearly satisfied by taking $t_o = T$. Thus we need only show that (4.2.27) holds; *viz.*

$$\lim_{\delta \to 0} \frac{\mathbb{E}\left\{\sup_{t \in T_\delta} f_t\right\}}{\delta} = 0.$$

However, since $p^2(t)$ is convex, it has a left derivative at each point, and so it is easy to check (draw a picture) that for $t \in T_\delta$ we have $|T - t| \leq K\delta^2$

for some finite K. Thus, since f has isotropic increments, it suffices to show that

$$\lim_{\delta \to 0} \frac{\mathbb{E}\left\{\sup_{0 \leq t \leq \delta^2} |f_t|\right\}}{\delta} = 0.$$

But this follows immediately from an entropy bound such as Theorem 1.3.3 along with the inequality (1.4.6) and condition (4.3.33). $\square$

We now look at two applications of Theorem 4.1.3. The first, which treats a non-stationary process on $I^N = [0,1]^N$, is designed to show how sample roughness and non-stationarity interact to determine excursion probabilities. Setting $\beta = \infty$ gives a result for stationary processes, and setting $\alpha = 2$ gives a result for (relatively) smooth processes of a form that we shall study in far more detail in Chapter 13.

The second example shows how to use the general theory to handle a Brownian sheet problem.

Example 4.3.2 *Let f be a centered, unit variance, stationary, Gaussian process on I^N with covariance function C satisfying*

$$(4.3.35) \qquad\qquad 1 - C(t) \leq a|t|^\alpha,$$

when $|t| \leq \gamma$, for some $a, \gamma > 0$ and $\alpha \in [0, 2]$. Let $\sigma(t)$ be positive, continuous, and nondecreasing (under the usual partial order) on I^N such that

$$(4.3.36) \qquad\qquad |\sigma(t) - \sigma(s)| \leq b|t - s|^\beta$$

for all $s, t \in I^N$ and some $b, \beta \geq 0$, and define a new, nonstationary, process h by

$$(4.3.37) \qquad\qquad h(t) = \sigma(t)f(t), \qquad t \in I^N.$$

Finally, let σ, without a parameter, denote $\sigma(1, \ldots, 1)$. Then there exists a (computable) finite $C = C(N, a, \alpha, b, \beta, \sigma) > 0$ such that, for sufficiently large u,

$$(4.3.38) \qquad \mathbb{P}\left\{\sup_{t \in I^N} h(t) > u\right\} \leq \begin{cases} Cu^{-\frac{2}{\beta} + \frac{2N}{\alpha}} \Psi\left(\frac{u}{\sigma}\right) & 0 < \alpha < 2\beta, \\ C\Psi\left(\frac{u}{\sigma}\right) & 0 < 2\beta \leq \alpha. \end{cases}$$

Proof. Note firstly that if σ is strictly increasing, then h has a unique point of maximum variance, at $t_o = (1, \ldots, 1)$. Nevertheless, even in this case, the assumptions we have made are not strong enough to imply that (4.2.27) holds, so that Theorem 4.2.1 (which is designed for examples where there is a unique point of maximal variance) need not apply. Consequently, we apply Theorem 4.1.3, so that the proof relies on finding a good bound for the entropy $N(T_\delta, d, \varepsilon)$ of the set

$$T_\delta = \left\{t \in I^N : \mathbb{E}\{h_t^2\} = \sigma^2(t) \geq \sigma^2 - \delta^2\right\}.$$

It is an easy calculation from (4.3.36) that T_δ can be covered by a cube of side length no more than $C\delta^{2/\beta}b^{-1/\beta}$, where $C = C(N)$ is a dimension dependent constant.

Furthermore, in view of the definition (4.3.37) of h,

$$(4.3.39) \quad \mathbb{E}\left\{(h_t - h_s)^2\right\} \;\; = \;\; (\sigma_t - \sigma_s)^2 \;+\; 2\sigma_t\sigma_s\left(1 - C(t-s)\right)$$
$$\leq \;\; b^2|t-s|^{2\beta} \;+\; 2\sigma^2 a|t-s|^\alpha,$$

the inequality holding for all $|t-s| < \gamma$, by (4.3.37).

Moving to the canonical metric on I^N in order to calculate entropies, it follows that, if $0 < \alpha < 2\beta$, then (4.3.39) implies

$$d(s,t) \;\; \leq \;\; C\,|t-s|^{\alpha/2}.$$

Combining this with the comments above on the size of T_δ, we have that

$$N(T_\delta, d, \varepsilon) \;\; \leq \;\; C\varepsilon^{-2N/\alpha}\delta^{2/\beta}$$

The first case in (4.3.38) now follows from Theorem 4.1.3. The second is similar and left to you. □

Our last example deals with the pinned Brownian sheet over N-dimensional rectangles. Recall from Section 1.4.3 that the Brownian sheet is a Gaussian noise defined on the Borel subsets of $\mathbb{R}^N$. The *set-indexed, pinned Brownian sheet*, based on a probability measure ν on the Borel sets of I^N, is the set-indexed, zero mean Gaussian process B with covariance function

$$(4.3.40) \qquad \mathbb{E}\left\{B(A)B(A')\right\} \;=\; \nu(A \cap A') - \nu(A)\nu(A').$$

The *point-indexed, pinned Brownian sheet* is obtained by looking only at rectangles of the form $[0,t]$ for some $t \in I^N$, in which case we can index it simply by t. Both the set and point indexed pinned sheets play a crucial rôle in the theory of empirical processes.

Example 4.3.3 *Let B be the set-indexed, pinned Brownian sheet based on a probability measure ν and defined over the collection $\mathcal{A}_N$ of N-dimensional rectangles in I^N. Assume that ν has a bounded density that is everywhere positive[7]. Then there exists a finite $C = C(\nu) > 0$ such that, for large enough u,*

$$(4.3.41) \qquad \mathbb{P}\left\{\sup_{A \in \mathcal{A}_N} B(A) > u\right\} \;\; \leq \;\; Cu^{2(2N-1)}e^{-2u^2}.$$

---

[7]The condition on ν can be relaxed to demanding that ν has a strictly positive density on some interval $[s-\varepsilon, t+\varepsilon]$ $(s, t, \varepsilon \in I^N)$ for which $\nu([s,t]) \geq \frac{1}{2}$, or, indeed, even further to those used by [6] in obtaining corresponding bounds for the point-indexed sheet. See Samorodnitsky [127, 128] for details.

For the point-indexed pinned sheet[8]

$$(4.3.42) \qquad \mathbb{P}\left\{ \sup_{t \in I^N} B(t) > u \right\} \leq Cu^{2(N-1)}e^{-2u^2}.$$

Proof. The first point to note is that, since ν is a probability measure, it follows from (4.3.40) that

$$\sup_{A \in \mathcal{A}_N} \mathbb{E}\left\{ B(A)^2 \right\} = \sup_{t \in I^N} \mathbb{E}\left\{ B(t)^2 \right\} = 1/4.$$

Assuming all its conditions hold, Theorem 4.1.3 therefore immediately gives the exponent of the exponentials in (4.3.41) and (4.3.42). The remainder of the proof involves an unentertaining amount of straightforward algebra, so we shall suffice with only an outline. In particular, we shall discuss only (4.3.41). The point-indexed result (4.3.42) is easier.

As in the proof of Example 4.3.2 we need to find a bound for the entropy of $N(T_\delta, d, \varepsilon)$, where T_δ is now given by

$$\begin{aligned} T_\delta &= \left\{ A \in \mathcal{A}_N : \mathbb{E}\left\{ B^2(A) \right\} \geq \tfrac{1}{4} - \delta^2 \right\} \\ &= \left\{ A \in \mathcal{A}_N : \nu(A) - \nu^2(A) \geq \tfrac{1}{4} - \delta^2 \right\}. \end{aligned}$$

If ν is Lebesgue measure, which we now assume to keep the notation manageable, this simplifies to

$$(4.3.43) \quad T_\delta = \left\{ A \in \mathcal{A}_N : \tfrac{1}{2}(1 - \delta) \leq \nu(A) \leq \tfrac{1}{2}(1 + \delta) \right\}.$$

---

[8] Using the 'double sum' techniques of Section 4.5 below, one can do considerably better than (4.3.42). In a series of works [74, 59, 60, 151] various versions of this technique were used to show that

$$\mathbb{P}\left\{ \sup_{t \in I^N} B(t) > u \right\} \asymp \frac{(4 \ln 2)^{N-1}}{(N-1)!} u^{2(N-1)} e^{-2u^2}$$

where $a(u) \asymp b(u) \iff \lim_{u \to \infty} a(u)/b(u)$ is well defined. (Note: the factor $(N-1)!$ is missing in [59, 60].)

In the one dimensional case there is no loss of generality in taking ν to be uniform measure on $[0,1]$, in which case B is the Brownian bridge. We can then do better than (4.3.42). In particular, any standard graduate probability textbook will prove – via an iterative use of the classic reflection principle – that

$$\mathbb{P}\left\{ \sup_{t \in [0,1]} |B(t)| > u \right\} = 2 \sum_{k=1}^{\infty} (-1)^{k-1} e^{-2k^2 u^2},$$

the first term of which, of course, we have in (4.3.42), albeit without a precise constant. Note that the factor of 2 here comes from the fact that we have looked at the supremem of the absolute value of B. The fact that it is precisely 2 comes from the 'well known' fact that high maxima and low minima of Gaussian processes are asymptotically indpendent (e.g. [90]).

To approximate the sets in T_δ in terms of the canonical metric, take N-dimensional rectangles whose endpoints sit on the points of the lattice

$$L_\varepsilon^N \;=\; \left\{t \in [0,1]^N : t_i = n_i \varepsilon^2, \; n_i \in \left(0, 1, \ldots, [\varepsilon^{-2}]\right), \; i = 1, \ldots, N\right\}.$$

We can choose each of the first $N-1$ coordinates n_i in at most ε^{-2} ways, and the last (in view of (4.3.43)) in at most $C_N \delta \varepsilon^{-2}$ ways, so that

$$N\left(T_\delta, d, \varepsilon\right) \;\leq\; C_N' \varepsilon^{-4N} \delta.$$

Substituting this into (4.1.10) and (4.1.11) is all we need in order to establish (4.3.41) and so complete the argument.

If ν is not Lebesgue measure, but has an everywhere positive, bounded density, the same argument works with a few more constants.   □

4.4   Extensions

There is almost no limit to the number of extensions and variations that exist of the results of the preceeding two sections. Obviously, the more one is prepared to assume, the more one can obtain.

For example, in the notation of Theorem 4.2.1, set

$$L(h) \;=\; \mathbb{E}\left\{\sup_{t \in T_h} (f_t - \mathbb{E}\{f_t f_{t_o}\}f_{t_o})\right\}.$$

It is not hard to see that that condition (4.2.27) of Theorem 4.2.1 is equivalent to

$$\lim_{h \to 0} \frac{L(h)}{h} \;=\; 0.$$

Dobric, Marcus and Weber [45] have shown that if we assume a little more about $L(h)$ we can improve Theorem 4.2.1 as follows.

Theorem 4.4.1 *Suppose there exists a unique $t_o \in T$ such that $\mathbb{E}\left\{f_{t_o}^2\right\} = 1 = \sup_T \mathbb{E}\left\{f_t^2\right\}$, and two functions ω_1, ω_2, concave for $h \in [0, \bar{h}]$ for some $\bar{h} > 0$ with $\omega_i(0) = 0$. Define*

$$h_i(u) \;=\; \sup\left\{h : \frac{\omega_i(h)}{h^2} = u\right\}, \qquad i = 1, 2.$$

If

$$\omega_1(h) \;\leq\; L(h) \;\leq\; \omega_2(h)$$

and

$$\sup_{t \in T_h} \mathbb{E}\left\{[f_t - f_{t_o}\mathbb{E}(f_t f_{t_o})]^2\right\} \;<\; (2 - \varepsilon)h^2$$

for $h \in [0, \bar{h}]$ for some $\varepsilon > 0$, then there exist constants C_1 and C_2 such that for all u large enough

$$e^{C_1 u \omega_1(h_1(u))} \;\leq\; \frac{\mathbb{P}\{\sup_T f_t \geq u\}}{\Psi(u)} \;\leq\; e^{C_2 u \omega_2(h_2(u))}.$$

If, for example,

$$\limsup_{h \to 0} \frac{L(h)}{\omega_1(h)} \;\geq\; 1,$$

then, for all $\varepsilon > 0$,

$$\limsup_{u \to \infty} \frac{\mathbb{P}\{\sup_{t \in T} f(t) \geq u\}}{\Psi(u) \exp(k_1(1 - \varepsilon) u \omega_1(h_1(u)))} \;\geq\; 1.$$

You can find a proof of this result in [45]. It differs in detail, but not in kind, from what we have seen here.

There are also asymptotic bounds of a somewhat different nature due to Lifshits [97], who, in a quite general setting, has shown that

$$\mathbb{P}\left\{\sup_{t \in T} f_t > u\right\} \;\asymp\; (2 - p)^{1/2} \, \Psi_p\left(\frac{u^{2-p} - du^{1-p}}{p\sigma_T^2}\right) \Psi\left(\frac{u - d}{\sigma_T}\right)$$
$$\times \exp\left\{-\frac{(2 - p)u^2}{2p\sigma_T^2} - \frac{d(p - 1)u}{p\sigma_T^2} + \frac{d^2}{2\sigma_T^2}\right\},$$

for all $1 \leq p < 2$, where $\Psi_p(x) \equiv \mathbb{E}\{\exp(x\sup_{t \in T} f_t^p)\}$ and d a constant (not easily) determined by f. Of course, this result is somewhat circular, since one needs the Laplace transform of $\sup_{t \in T} f_t^p$ before being able to compute anything. Nevertheless, the above inequality leads to some nice theoretical results about ℓ^p-valued Gaussian processes.

We close this section with a much less general case than we have hitherto considered. Suppose that f is a stationary, zero mean Gaussian field on $\mathbb{R}^N$ and has a covariance function which, near the origin can be written as

$$(4.4.44) \qquad\qquad C(t) \;=\; 1 - t\Lambda t' \,+\, o(|t|^2),$$

where $\Lambda = \{\lambda_{ij}\}_{1 \leq i,j \leq N}$ is the matrix of second order spectral moments (cf. (5.5.2)). In this case, one can show that

$$(4.4.45) \qquad \lim_{u \to \infty} \frac{\mathbb{P}\left\{\sup_{t \in [0,T]^N} f_t \geq u\right\}}{u^N \Psi(u)} \;=\; \frac{(\det\Lambda)^{1/2}}{(2\pi)^{N/2}}$$

Why this result should be true is something we shall soon investigate in detail and over parameter spaces far more general than cubes in Part III of the book. Indeed, we shall obtain far more precise results, of the ilk of (4.0.1).

4.5   The double sum method

Since we have mentioned it already a number of times, it is only reasonable that we at least also briefly describe the so-called *double sum method* for computing asymptotic exceedence probabilities.

The description will be very brief, since the topic is treated in detail in both [90] and [117]. In particular, Piterbarg's monograph [117] has an excellent and remarkably readable treatment of this rather technical material in the setting of random fields.

In essence, the technique is not qualitatively different from what we have been doing throughout this chapter. The differences lie only in the details and in the concentration on lower, as well as upper, bounds for exceedence probabilities. The basic idea is to break up the parameter space T into a finite union of small sets T_k, where the size of the T_k generally depends on the exceedence level u. The T_k need not be disjoint, although any overlap should be small in relation to their sizes. While initially there is no need to assume any particular structure for T, this usually is needed for the most precise results.

It is then elementary that

$$
\sum_k \mathbb{P}\left\{ \sup_{t \in T_k} f(t) \geq u \right\} \;\geq\; \mathbb{P}\left\{ \sup_{t \in T} f(t) \geq u \right\}
$$

$$
(4.5.1) \qquad\qquad \geq \;\; \sum_k \mathbb{P}\left\{ \sup_{t \in T_k} f(t) \geq u \right\}
$$

$$
-\sum\sum_{j \neq k} \mathbb{P}\left\{ \sup_{t \in T_j} f(t) \geq u, \sup_{t \in T_k} f(t) \geq u \right\}
$$

The single summations are treated much as we did before, by choosing a point $t_k \in T_k$, writing

$$
\mathbb{P}\left\{ \sup_{t \in T_k} f(t) \geq u \right\} \;=\; \int_{-\infty}^{\infty} \mathbb{P}\left\{ \sup_{t \in T_k} f(t) \geq u \big| f(t_k) = u \right\} p_{t_k}(x)\, dx,
$$

and arguing much as we did in the proof of Lemma 4.1.5.

The crux of the double sum method lies in showing that the remaining double sum is of lower order than the single sum and, where possible, estimating its size.

If one could write the joint probabilities in (4.5.1) as the product

$$
\mathbb{P}\left\{ \sup_{t \in T_j} f(t) \geq u, \sup_{t \in T_k} f(t) \geq u \right\} \;=\; \mathbb{P}\left\{ \sup_{t \in T_j} f(t) \geq u \right\} \mathbb{P}\left\{ \sup_{t \in T_k} f(t) \geq u \right\},
$$

then we would be basically done, since then the double sum term would then easily be seen to be of lower order than the single sum. Such independence

obviously does not hold, but if we choose the sizes of the T_k in such a fashion that a 'typical component' of an excursion set is considerably smaller than this size, and manage to show that high extrema are independent of one another, then we are well on the way to a proof.

The details, which are heavy, are all in Piterbarg [117]. Piterbarg's monograph is also an excellent source of worked examples, and includes a number of rigorous computations of excursion probabilities for many interesting examples of processes and fields.

4.6   Local maxima and excursion probabilities

We close this chapter with somewhat of a red herring, in that we shall describe an approach that makes a lot of sense, that has been used and developed with considerable success by Piterbarg [117] and others, but which we shall nevertheless not explore further. Despite this, however, it does give a good heuristic feeling for results we shall develop in Part III, particularly those of Chapter 13.

We leave the general setting of the previous sections, and concentrate on smooth random fields over structured parameter spaces. In particular, we shall adopt the regularity conditions of Chapter 12, in which f will be a smooth Gaussian process defined over a Whitney stratified manifold M. Since we shall only get around to defining these manifolds in Section 8.1, you can either return to the current discussion after reading that section, or simply assume that 'a N-dimensional Whitney stratified manifold' is no more than another way of saying 'a unit cube $I^N \subset \mathbb{R}^N$'. You will not lose much in the way of intuition in doing so.

Furthermore, since the current section deals with a general idea rather than specific results, we shall not be more precise about exactly what conditions we require and shall simply assume that everything we write is well defined. The appropriate rigour will come in Part III of the book.

Our aim is to connect the excursion probability of f with the mean number of its local maxima of a certain type.

We start by noting that a N-dimensional Whitney stratified manifold M can be written as the disjoint union

$$(4.6.1) \qquad\qquad M \;=\; \bigcup_{k=0}^{N} \partial_k M,$$

of open, k-dimensional submanifolds (cf. (9.3.5)). (If you are working with I^N, then this is just a decomposition of I^N into k-dimensional facets, so that $\partial_N M = M^\circ$ is the interior of the cube, $\partial_{N-1} M$ the union of the interiors of the $N-1$ dimensional faces, and so forth, down to $\partial_0 M$, which is made up of the 2^N vertices of the cube.)

Let

$$M_u^E(M) \;\; \stackrel{\Delta}{=} \;\; \#\,(\text{extended outward maxima of } f \text{ in } M \text{ with } f_t \ge u).$$

In the terminology of Section 9.1, M_u^E is more precisely defined as the number of extended outward critical points $t \in \partial_k M \subset M$ of f for which the Hessian of $f|_{\partial_k M}$ has maximal index. For the example of I^N, these points are the local maxima $t \in \partial_k M$, $k = 0, \ldots, N$, of $f|_{\partial_k M}$ for which $f(t_n) \uparrow f(t)$ whenever the $t_n \in \partial_{k+1} M$ converge to t monotonely (for the usual partial order on $\mathbb{R}^N$) along a direction normal to $\partial_k M$ at t.

Since the notion of extended local maxima[9] includes the 0-dimensional sets $\partial_0 M$ (i.e. to the 'corners' of M) it is not hard to see that that, if $\sup_{t \in M} f_t \ge u$, then f must have at least one extended outward maximum on M, and *vice versa*. (cf. the argument in Section 13.1.1 if you want details.) Consequently, we can argue as follows:

$$(4.6.2) \qquad \mathbb{P}\left\{ \sup_{t \in M} f_t \ge u \right\} \;=\; \mathbb{P}\left\{ M_u^E(M) \ge 1 \right\}$$

$$\le\; \sum_{k=0}^{N} \mathbb{P}\left\{ M_u^E(\partial_k M) \ge 1 \right\}$$

$$\le\; \sum_{k=0}^{N} \mathbb{E}\left\{ M_u^E(\partial_k M) \right\}.$$

This gives a simple upper bound for excursion probabilities in terms of the mean number of outward extended maxima. For a lower bound, note

$$(4.6.3) \quad \left\{ \sup_{t \in M} f_t \ge u \right\}$$
$$\Longleftrightarrow \;\; \left\{ M_u^E(M^\circ) \ge 1, \; M_u^E(\partial M) = 0 \right\} \cup \left\{ M_u^E(\partial M) \ge 1 \right\},$$

---

[9] Note that if we write

$$M_u(M) \;\; \stackrel{\Delta}{=} \;\; \#\,(\text{local maxima of } f \text{ in } M \text{ with } f_t \ge u)$$

then one can actually carry through all of the following argument with M_u^E replaced by M_u. In fact, this was Piterbarg's original argument (cf. [117]). Since it is always true that $M_u \ge M_u^E$, this means that the upper bound given in (4.6.6) below is poorer with this change. Since the lower bound involves a difference of two terms, each of which is larger with this change, it is unclear which variation of the argument is better, although one expects the version with M_u^E rather than M_u to give the tighter bound. In some sense, the difference is of academic interest only, since neither bound can be explicitly evaluated. When we turn to the explicit computations for the Gaussian case in Chapter 13 we shall, in any case, adopt a somewhat different approach.

where $\partial M = \bigcup_{k=0}^{N-1} \partial_k M$. Now assume that f, as a function on the N-dimensional manifold M, does not have any critical points on ∂M. Write

$$\left\{ M_u^E(M^\circ) \geq 1, \ M_u^E(\partial M) = 0 \right\}$$
$$= \left\{ M_u^E(M^\circ) = 1, M_u^E(\partial M) = 0 \right\} \cup \left\{ M_u^E(M^\circ) \geq 2, M_u^E(\partial M) = 0 \right\}.$$

To compute the probabilities of the above two events, set

$$p_k = \mathbb{P}\left\{ M_u^E(M^\circ) = k \right\},$$

and note that

$$\mathbb{E}\left\{ M_u^E(M^\circ) \right\} = \mathbb{P}\left\{ M_u^E(M^\circ) = 1 \right\} + \sum_{k=2}^{\infty} k p_k,$$

so that

$$\mathbb{P}\left\{ M_u^E(M) = 1, M_u^E(\partial M) = 0 \right\}$$
$$= \mathbb{P}\left\{ M_u^E(M) = 1 \right\} - \mathbb{P}\left\{ M_u^E(M^\circ) = 1, M_u^E(\partial M) \geq 1 \right\}$$
$$= \mathbb{E}\left\{ M_u^E(M^\circ) \right\} - \sum_{k=2}^{\infty} k p_k - \mathbb{P}\left\{ M_u^E(M^\circ) = 1, M_u^E(\partial M) \geq 1 \right\}.$$

In a similar vein,

$$\mathbb{P}\left\{ M_u^E(M^\circ) \geq 2, M_u^E(\partial M) = 0 \right\}$$
$$= \sum_{k=2}^{\infty} p_k - \mathbb{P}\left\{ M_u^E(M^\circ) \geq 2, M_u^E(\partial M) \geq 1 \right\}.$$

Putting the last two equalities together with (4.6.3) immediately gives us that

(4.6.4)

$$\mathbb{P}\left\{ \sup_{t \in M} f_t \geq u \right\} = \mathbb{E}\left\{ M_u^E(M^\circ) \right\} - \sum_{k=2}^{\infty} (k-1) p_k + \mathbb{P}\left\{ M_u^E(\partial M) \geq 1 \right\}$$
$$- \mathbb{P}\left\{ M_u^E(M^\circ) \geq 1, M_u^E(\partial M) \geq 1 \right\}$$
$$\geq \mathbb{E}\left\{ M_u^E(M^\circ) \right\} - \tfrac{1}{2} \mathbb{E}\left\{ M_u^E(M^\circ)[M_u^E(M^\circ) - 1] \right\}$$

(4.6.5)

on noting that $k - 1 < k(k-1)/2$ for $k \geq 2$. Iterate this argument through $\partial_k M$, $0 \leq k \leq N-1$ and combine it with (4.6.2) to obtain

Theorem 4.6.1 *Let f be almost surely C^2 on a C^2 Whitney stratified manifold M. With the preceeding notation, assume $f|_{\partial_k M}$ has, almost surely,*

no critical points on $\bigcup_{j=0}^{k-1} \partial_j M$. Furthermore, assume that that the random variables $M_u^E(\partial_k M)$ have finite second moments for all k and given u. Then

$$(4.6.6) \quad \sum_{k=0}^{N} \mathbb{E}\left\{M_u^E(\partial_k M)\right\}$$

$$\geq \; \mathbb{P}\left\{\sup_{t \in M} f_t \geq u\right\}$$

$$\geq \; \sum_{k=0}^{N} \left[\mathbb{E}\left\{M_u^E(\partial_k M)\right\} - \tfrac{1}{2}\mathbb{E}\left\{M_u^E(\partial_k M)[M_u^E(\partial_k M) - 1]\right\}\right].$$

Of course, we have not really proven the Theorem 4.6.1 with the rigor that it deserves, so you should feel free to call it a conjecture rather than a theorem.

What you should note about this theorem/conjecture is that it makes no distributional assumptions on f beyond ensuring that all the terms it involves are well defined and finite. It certainly does not require that f be Gaussian. Consequently, it covers a level of generality far beyond the Gaussian theory we have treated so far.

It also makes sense that it would be quite a general phenomenon that the negative terms in (4.6.6) will be of smaller order that the others. After all, for $M_u^E(\partial_k M)[M_u^E(\partial_k M) - 1]$ to be non-zero, there must be at least two extended outward maxima of $f|_{\partial_k M}$ above the level u on $\partial_k M$, and this is unlikely to occur if u is large.

Thus Theorem 4.6.1 seems to hold a lot of promise for approximating extremal probabilities in general, assuming that we could actually compute explicit expressions for the simple expectations in (4.6.6) along with some useful bounds for the product expectations.

In Chapter 11 we shall work hard to obtain generic expressions that, in principle, allow us to compute all these expectations. In particular, we shall have Theorem 11.1.1 for the simple means and Theorem 11.4.1 for the factorial moments in (4.6.6). Unfortunately, however, carrying out the computations in practice will turn out to be impossible, even in the case when f is Gaussian. Consequently, we shall be forced to take a less direct path to approximating excursion probabilities, by evaluating mean Euler characteristics which actually approximate the first order expectations in (4.6.6). The details of this argument will appear in Chapter 13.

First of all, however, we shall need to spend quite some time in Part II of this book on studying geometry.

5
Stationary fields

Stationarity has always been the backbone of almost all examples in the theory of Gaussian processes for which specific computations were possible. As described in the Preface, one of the main reasons we shall be studying Gaussian processes on manifolds is to get around this assumption. Nevertheless, despite the fact that we shall ultimately try to avoid it, we invest a chapter on the topic for two reasons:

- Many of the results of Part III are *significantly* easier to interpret when specialised down to cases under which stationarity holds.

- Even in the non-stationary case, many of the detailed computations of Part III can be considered as deriving from a 'local conversion to pseudo-stationarity', or, even moreso, to pseudo-isotropy. This will be taken care of there via the 'induced Riemannian metric' defined in Section 12.2. Knowledge of what happens under stationarity is therefore important for knowing what to do in the general case.

We anticipate that many of our readers will be familiar with most of the material of this chapter and so will skip it and return only when specific details are required later. For the newcomers, you should be warned that our treatment is only full enough to meet our specific needs and that both style and content are occasionally a little eclectic. In other words, you should go elsewhere for fuller, more standard treatments. References will be given along the way.

The most important classic results of this chapter are the Spectral Distribution and Spectral Representation Theorems for $\mathbb{R}^N$ of Section 5.4. How-

ever, the most important results for us will be some of the consequences of these theorems for relationships between spectral moments which are concentrated in Section 5.5. This is the one section of this chapter that you will almost definitely need to come back to, even if you have decided that you are familiar enough with the stationarity to skip this chapter for the moment.

5.1   Basic stationarity

Although our primary interest lies in the study of real valued random fields it is mathematically more convenient to discuss stationarity in the framework of complex valued processes. Hence, unless otherwise stated, we shall assume throughout this chapter that $f(t) = (f_R(t) + if_I(t))$ takes values in the complex plane $\mathcal{C}$ and that $\mathbb{E}\{\|f(t)\|^2\} = \mathbb{E}\{f_R^2(t) + f_I^2(t)\} < \infty$. (Both f_R and f_I are, obviously, to be real valued.) As for a definition of normality in the complex scenario, we first define a complex random variable to be Gaussian if the vector of its two components is bivariate Gaussian[1]. A complex process f is Gaussian if $\sum_i \alpha_{t_i} f_{t_i}$ is a complex Gaussian variable for all sequences $\{t_i\}$ and complex $\{\alpha_{t_i}\}$.

We also need some additional assumptions on the parameter space T. In particular, we need that it have a group structure[2] and an operation with respect to which the field is stationary. Consequently, we now assume that T has such a structure, '$+$' represents the binary operation on T and '$-$' repesents inversion. As usual, $t - s = t + (-s)$. For the moment, we need no further assumptions on the group.

Since $f_t \in \mathcal{C}$, it follows that the mean function $m(t) = \mathbb{E}\{f(t)\}$ is also complex valued, as is the covariance function, which we redefine for the complex case as

$$(5.1.1) \qquad C(s,t) \triangleq \mathbb{E}\left\{[f(s) - m(s)]\,\overline{[\overline{f(t)} - \overline{m(t)}]}\right\},$$

with the bar denoting complex conjugation.

Some basic properties of covariance functions follow immediately from (5.1.1):

- $C(s,t) = \overline{C(t,s)}$, which becomes the simple symmetry $C(s,t) = C(t,s)$ if f (and so C) is real valued.

---

[1]It therefore folows that a complex Gaussian variable $X = X_R + iX_I$ is defined by five parameters: $\mathbb{E}\{X_I\}$, $\mathbb{E}\{X_R\}$, $\mathbb{E}\{X_I^2\}$, $\mathbb{E}\{X_R^2\}$ and $\mathbb{E}\{X_I X_R\}$.

[2]If stationarity is a new concept for you, you will do well by reading this section the first time taking $T = \mathbb{R}^N$ with the usual notions of addition and subtraction.

- For any $k \geq 1$, $t_1, \ldots, t_k \in T$ and $z_1, \ldots, z_k \in \mathcal{C}$, the Hermitian form $\sum_{i=1}^{k} \sum_{j=1}^{k} C(t_i, t_j) z_i \overline{z}_j$ is always real and non-negative. We summarise this, as before, by saying that C is non-negative definite.

(The second property follows from the equivalence of the double sum to $\mathbb{E}\{\| \sum_{i=1}^{k} [f(t_i) - m(t_i)] z_i \|^2\}$.)

Suppose for the moment that T is Abelian. A random field f is called *strictly homogeneous* or *strictly stationary* over T, with respect to the group operation $+$, if its finite dimensional distributions are invariant under this operation. That is, for any $k \geq 1$ and any set of points $\tau, t_1, \ldots, t_k \in T$

$$(5.1.2) \qquad (f(t_1), \ldots, f(t_k)) \stackrel{\mathcal{L}}{=} (f(t_1 + \tau), \ldots, f(t_k + \tau)).$$

An immediate consequence of strict stationarity is that $m(t)$ is constant and $C(s, t)$ is a function of the difference $s - t$ only.

Going the other way, if for a random field with $\mathbb{E}\{\|f(t)\|^2\} < \infty$ for all $t \in T$, we have that $m(t)$ is constant and $C(s, t)$ is a function of the difference $s - t$ only, then we call f simply *stationary* or *homogeneous*, occassionally adding either of the two adjectives 'weakly' or 'second-order'.

Note that none of the above required the Gaussianity we have assumed up until now on f. If, however, we do add the assumption of Gaussianity, it immediately follows from the structure (1.2.3) of the multivariate Gaussian density[3] that a weakly stationary Gaussian field will also be strictly stationary if $C'(s, t) = \mathbb{E}\{[f(s) - m(s)] [f(t) - m(t)]\}$ is also a function only of $s - t$. If f is real valued, then since $C \equiv C'$ it follows that all weakly stationary real valued Gaussian fields are also strictly stationary and the issue of qualifying adjectives is moot[4].

If T is not Abelian we must distinguish between left and right stationary. We say that a random field f on T is *right-stationary* if (5.1.2) holds and that f is *left-stationary* if $f'(t) \stackrel{\Delta}{=} f(-t)$ is right-stationary. The corresponding conditions on the covariance function change accordingly.

In order to build examples of stationary processes, we need to make a brief excursion into (Gaussian) stochastic integration.

---

[3] Formally, (1.2.3) is not quite enough. Since we are currently treating complex valued processes, the k-dimensional marginal distributions of f now involve $2k$-dimensional Gaussian vectors. (The real and imaginary parts each require k dimensions.)

[4] The reason for needing the additional condition in the complex case should be intuitively clear: C alone will not even determine the variance of each of f_I and f_R but only their sum. However, together with C' both of these parameters, as well as the covariance between f_I and f_R, can be computed. See [106] for further details.

5.2   Stochastic integration

We return to the setting of Section 1.4.3, so that we have a σ-finite[5] measure space $(T, \mathcal{T}, \nu)$, along with the Gaussian ν-noise W defined over $\mathcal{T}$. Our aim will be to establish the existence of integrals of the form

$$(5.2.1) \qquad\qquad \int_T f(t)\, W(dt),$$

for *deterministic* $f \in L^2(\nu)$ and, eventually, complex W. Appropriate choices of f will give us examples of stationary Gaussian fields, many of which we shall meet in the following sections.

Before starting, it is only fair to note that we shall be working with two and a bit assumptions. The 'bit' is that for the moment we shall only treat real f and W. We shall, painlessly, lift that assumption soon. Of the other two, one is rather restrictive and one not, but neither of importance to us. The non-restrictive assumption is the Gaussian nature of the process W. Indeed, since all of what follows is based only on L^2 theory, we can, and so shall, temporarily drop the assumption that W is a Gaussian noise and replace conditions (1.4.9)–(1.4.11) with the following three requirements for all $A, B \in \mathcal{T}$.

$$(5.2.2) \qquad \mathbb{E}\{W(A)\} = 0, \quad \mathbb{E}\{[W(A)]^2\} = \nu(A).$$

$$(5.2.3) \qquad A \cap B = \emptyset \;\Rightarrow\; W(A \cup B) = W(A) + W(B) \text{ a.s.}$$

$$(5.2.4) \qquad A \cap B = \emptyset \;\Rightarrow\; \mathbb{E}\{W(A)W(B)\} = 0.$$

Note that in the Gaussian case (5.2.4) is really equivalent to the seemingly stronger (1.4.11), since zero covariance and independence are then equivalent.

The second restriction is that the integrand f in (5.2.1) is deterministic. Removing this assumption would lead us to having to define the Itô integral which is a construction for which we shall have no need.

Since, by (5.2.3), W is a finitely additive (signed) measure, (5.2.1) is evocative of Lebesgue integration. Consequently, we start by defining the the stochastic version for simple functions

$$(5.2.5) \qquad\qquad f(t) \;=\; \sum_1^n a_i \mathbb{1}_{A_i}(t),$$

---

[5] Note that 'σ-finite' includes 'countable' and 'finite', in which case ν will be a discrete measure and all of the theory we are about to develop is really much simpler. As we shall see later, these are more than just important special cases and so should not be forgotten as you read this section. Sometimes it is all too easy to lose sight of the important simple cases in the generality of the treatment.

where $A_1, \ldots, A_n$ are disjoint, $\mathcal{T}$ measurable sets in T, by writing

$$(5.2.6) \qquad W(f) \equiv \int_T f(t)\, W(dt) \triangleq \sum_1^n a_i\, W(A_i).$$

It follows immediately from (5.2.2) and (5.2.4) that in this case $W(f)$ has zero mean and variance given by $\sum a_i^2\, \nu(A_i)$. Think of $W(f)$ as a mapping from simple functions in $L^2(T, \mathcal{T}, \nu)$ to random variables[6] in $L^2(\mathbb{P}) \equiv L^2(\Omega, \mathcal{F}, \mathbb{P})$. The remainder of the construction involves extending this mapping to a full isomorphism from $L^2(\nu) \equiv L^2(T, \mathcal{T}, \nu)$ onto a subspace of $L^2(\mathbb{P})$. We shall use this isomorphism to define the integral.

Let $\mathcal{S} = \mathcal{S}(T)$ denote the class of simple functions of the form (5.2.5) for some finite n. Note first there is no problem with the consistency of the definition (5.2.6) over different representations of f. Furthermore, W clearly defines a linear mapping on $\mathcal{S}$ which preserves inner products. To see this, write $f, g \in \mathcal{S}$ as

$$f(t) = \sum_1^n a_i \mathbb{1}_{A_i}(t), \quad g(t) = \sum_1^n b_i \mathbb{1}_{A_i}(t);$$

in terms of the same partition, to see that the $L^2(\mathbb{P})$ inner product between $W(f)$ and $W(g)$ is given by

$$
\begin{aligned}
(5.2.7) \quad \langle W(f), W(g) \rangle_{L^2(\mathbb{P})} &= \mathbb{E}\left\{ \sum_1^n a_i W(A_i) \cdot \sum_1^n b_i W(A_i) \right\} \\
&= \sum_1^n a_i b_i \mathbb{E}\left\{ [W(A_i)]^2 \right\} \\
&= \int_T f(t) g(t)\, \nu(dt) \\
&= \langle f, g \rangle_{L^2(\nu)}
\end{aligned}
$$

the second line following from (5.2.4) and the second last from (5.2.2) and the definition of the Lebesgue integral.

Since $\mathcal{S}$ is dense in $L^2(\nu)$ (e.g. [125]) for each $f \in L^2(\nu)$ there is a sequence $\{f_n\}$ of simple functions such that $\|f_n - f\|_{L^2(\nu)} \to 0$ as $n \to \infty$. Using this, for such f we define $W(f)$ as the mean square limit

$$(5.2.8) \qquad\qquad W(f) \triangleq \lim_{n \to \infty} W(f_n).$$

Furthermore, by (5.2.7), the limit is independent of the approximating sequence $\{f_n\}$ and the mapping W so defined is linear and preserves inner

---

[6]Note that if W is Gaussian, then so is $W(f)$.

products; i.e. the mapping is an isomorphism. We take this mapping as our definition of the integral and so we now not only have existence, but from (5.2.7) we have that, for all $f, g \in L^2(\nu)$,

$$(5.2.9) \qquad \mathbb{E}\{W(f)\, W(g)\} \;=\; \int_T f(t)g(t)\, \nu(dt).$$

Note also that since L^2 limits of Gaussian random variables remain Gaussian (cf. (1.2.5) and the discussion above it) under the additional assumption that W is a Gaussian noise it follows that $W(f)$ is also Gaussian.

   With our integral defined, we can now start looking at some examples of what can be done with it.

5.3   Moving averages

We now return to the main setting of this chapter, in which T is an Abelian group under the binary operation $+$ and $-$ represents inversion. Let ν be a Haar measure on $(T, \mathcal{T})$ (assumed to be σ-finite) and take $F : T \to \mathbb{R}$ in $L^2(\nu)$. If W is a ν-noise on $\mathcal{T}$, then the random process

$$(5.3.1) \qquad\qquad f(t) \;\triangleq\; \int_T F(t - s)\, W(ds),$$

is called a *moving average* of W, and we have the following simple result:

Lemma 5.3.1 *Under the preceeding conditions, f is a stationary random field on T. Furthermore, if W is a Gaussian noise, then f is also Gaussian.*

Proof. To establish stationarity we must prove that

$$\mathbb{E}\left\{f(t)f(s)\right\} \;=\; C(t - s)$$

for some C. However, from (5.2.9) and the invariance of ν under the group operation,

$$
\begin{aligned}
\mathbb{E}\left\{f(t)f(s)\right\} \;&=\; \int_T F(t-u)F(s-u)\, \nu(du) \\
&=\; \int_T F(t-s+v)F(v)\, \nu(dv) \\
&\triangleq\; C(t-s),
\end{aligned}
$$

and we are done.

   If W is Gaussian, then we have already noted when defining stochastic integrals that $f(t) = W(F(t-\cdot))$ is a real-valued Gaussian random variable for each t. The same arguments also show that f is Gaussian as a *process*. $\square$

A similar but slightly more sophisticated construction also yields a more general class of examples, in which we think of the elements g of a group G acting on the elements t of an underlying space T. This will force us to change notation a little and, for the argument to be appreciated in full, to assume that you also know a little about manifolds. If you do not, then you can return to this example later, after having read Chapter 6, or simply take the manifold to be $\mathbb{R}^N$. In either case, you may still want to read the very concrete and quite simple examples at the end of this section now.

Thus, taking the elements g of a group G acting on the elements t of an underlying space T, we denote the identity element of G by e and the left and right multiplication maps by L_g and R_g. We also write $I_g = L_g \circ R_g^{-1}$ for the inner automorphism of G induced by g.

Since we are now working in more generality, we shall also drop the commutativity assumption that has been in force so far. This necessitates some additional definitions, since we we must distinguish between left and right stationarity. We say that a random field f on G is strictly *left-stationary* if for all n, all $(g_1, \ldots, g_n)$, and any g_0,

$$(f(g_1), \ldots, f(g_n)) \overset{\mathcal{L}}{=} (f \circ L_{g_0}(g_1), \ldots, f \circ L_{g_0}(g_n)).$$

It is called strictly *right-stationary* if $f'(g) \overset{\Delta}{=} f(g^{-1})$ is strictly left-stationary and strictly *bi-stationary*, or simply strictly *stationary*, if it is both left and right strictly stationary. As before, if f is Gaussian and has constant mean and covariance function C satisfying

$$(5.3.2) \qquad\qquad C(g_1, g_2) = C'(g_1^{-1} g_2),$$

for some $C' : G \to \mathbb{R}$, then f is strictly left-stationary. Similarly, if C satisfies

$$(5.3.3) \qquad\qquad C(g_1, g_2) = C''(g_1 g_2^{-1})$$

for some C'', then f is right-stationary. If f is not Gaussian, but has constant mean and (5.3.2) holds, then f is *weakly* left-stationary. Weak right-stationarity and stationarity are defined analogously.

We can now start collecting the building blocks of the construction, which will be of a left-stationary Gaussian random field on a group G. An almost identical argument will construct a right-stationary field. It is then easy to see that this construction will give a bi-stationary field on G only if it is unimodular, i.e. if any left Haar measure on G is also right invariant.

We first add the condition that G be a *Lie group*; i.e. a group that is also a C^∞ manifold such that the maps taking g to g^{-1} and (g_1, g_2) to $g_1 g_2$ are both C^∞. We say G has a smooth (C^∞) (left) action on a smooth (C^∞) manifold T if there exists a map $\theta : G \times T \to T$ satisfying, for all $t \in T$ and $g_1, g_2 \in G$,

$$\theta(e, t) = t$$
$$\theta(g_2, \theta(g_1, t)) = \theta(g_2 g_1, t).$$

We write $\theta_g : T \to T$ for the partial mapping $\theta_g(t) = \theta(g, t)$. Suppose ν is a measure on T, let $\theta_{g*}(\nu)$ be the *push-forward* of ν under the map θ_g; i.e $\theta_{g*}(\nu)$ is given by

$$\int_A \theta_{g*}(\nu) \;=\; \int_{\theta_{g^{-1}}(A)} \nu.$$

Furthermore, we assume that $\theta_{g*}(\nu)$ is absolutely continuous with respect to ν, with Radon-Nikodym derivative

(5.3.4) $$D(g) \;\triangleq\; \frac{d\theta_{g*}(\nu)}{d\nu}(t),$$

independent of t. We call such a measure ν *left relatively invariant* under G. It is easy to see that $D(g)$ is a C^∞ homomorphism from G into the multiplicative group of positive real numbers, i.e. $D(g_1 g_2) = D(g_1) D(g_2)$. We say that ν is *left invariant* with respect to G if, and only if, it is left relatively-invariant and $D \equiv 1$.

Here, finally, is the result.

Lemma 5.3.2 *Suppose G acts smoothly on a smooth manifold T and ν is left relatively invariant under G. Let D be as in (5.3.4) and let W be Gaussian ν-noise on T. Then, for any $F \in L^2(T, \nu)$,*

$$f(g) \;=\; \frac{1}{\sqrt{D(g)}} W(F \circ \theta_{g^{-1}}),$$

is a left stationary Gaussian random field on G.

Proof. We must prove that

$$\mathbb{E}\{f(g_1) f(g_2)\} \;=\; C(g_1^{-1} g_2)$$

for some $C : G \to \mathbb{R}$. From the definition of W, we have,

$$
\begin{aligned}
\mathbb{E}\{f(g_1) f(g_2)\} &= \frac{1}{\sqrt{D(g_1) D(g_2)}} \int_T F\left(\theta_{g_1^{-1}}(t)\right) F\left(\theta_{g_2^{-1}}(t)\right) \nu(dt) \\
&= \frac{1}{\sqrt{D(g_1) D(g_2)}} \int_T F\left(\theta_{g_1^{-1}}(\theta_{g_2}(t))\right) F(t)\, \theta_{g_2*}(\nu)(dt) \\
&= \frac{D(g_2)}{\sqrt{D(g_1) D(g_2)}} \int_T F\left(\theta_{g_1^{-1} g_2}(t)\right) F(t)\, \nu(dt) \\
&= \sqrt{D(g_1^{-1} g_2)} \int_T F\left(\theta_{g_1^{-1} g_2}(t)\right) F(t)\, \nu(dt) \\
&\triangleq C(g_1^{-1} g_2)
\end{aligned}
$$

This completes the proof. $\qquad\square$

It is easy to find simple examples to which Lemma 5.3.2 applies. The most natural generic example of a Lie group acting on a manifold is its action on itself. In particular, any right Haar measure is left relatively-invariant, and this is a way to generate stationary processes. To apply Lemma 5.3.2 in this setting one needs only to start with a Gaussian noise based on a Haar measure on G. In fact, this is the example (5.3.1) with which we started this section.

A richer but still concrete example of a group G acting on a manifold T is given by the group of rigid motions $G_N = GL(N, \mathbb{R}) \times \mathbb{R}^N$ acting[7] on $T = \mathbb{R}^N$. For $g = (A, t)$ and $s \in \mathbb{R}^N$, set

$$\theta(A, t)(s) \;=\; As + t.$$

In this example it is easy to see that Lebesgue measure $\lambda_N(dt) = dt$ is relatively invariant with respect to G with $D(g) = \det A$. For an even more concrete example, take compact, Borel $B \subset \mathbb{R}^N$ and $F(s) = \mathbb{1}_B(s)$. It then follows from Lemma 5.3.2 that

$$f(A, t) \;=\; \frac{W\left(A^{-1}(B - t)\right)}{\sqrt{|\det(A)|}}$$

is a stationary process with variance $\lambda_N(B)$ and covariance function

$$C((A_1, t_1), (A_2, t_2)) \;=\; \frac{\lambda_N\left(A_1 A_2^{-1}(B - (t_1 - t_2))\right)}{\sqrt{|\det(A_1 A_2^{-1})|}},$$

where we have adopted the usual notations that $B + t = \{s + t : s \in B\}$ and $AB = \{At : t \in B\}$.

Examples of this kind have been used widely in practice. For examples involving the statistics of brain mapping see, for example, [133, 134].

5.4   Spectral representations on $\mathbb{R}^N$

The moving averages of the previous section gave us examples of stationary fields that were rather easy to generate in quite general situations from Gaussian noise. Now, however, we want to look at a general way of generating *all* stationary fields, via the so-called *spectral representation*. This is quite a simple task when the parameter set is $\mathbb{R}^N$, but rather more involved when a general group is taken as the parameter space and issues of group representations arise. Thus we shall start with the Euclidean case which we treat in detail and then discuss some aspects of the general case in the

---

[7] Recall that $GL(N, \mathbb{R})$ is the (general linear) group of invertible transformations of $\mathbb{R}^N$.

following section. In both cases, while an understanding of the spectral representation is a powerful tool for understanding stationarity and a variety of sample path properties of stationary fields, it is not necessary for what comes later in the book.

We return to the setting of complex valued fields, take $T = \mathbb{R}^N$, and assume, as usual, that $\mathbb{E}\{f_t\} = 0$. Furthermore, since we are now working only with stationary processes, it makes sense to misuse notation somewhat and write

$$C(t - s) \;=\; C(s, t) \;=\; \mathbb{E}\{f(s)\overline{f(t)}\}.$$

We call C, which is now a function of one parameter only, non-negative definite if $\sum_{i,j=1}^n z_i C(t_i - t_j)\overline{z_j} \geq 0$ for all $n \geq 1$, $t_1, \ldots, t_n \in \mathbb{R}^N$ and $z_1, \ldots, z_n \in \mathcal{C}$. Then we have the following result, which dates back to Bochner [28], in the setting of (non-stochastic) Fourier analysis, a proof of which can be found in almost any text on Fourier analysis.

Theorem 5.4.1 (Spectral distribution theorem) *A continuous function $C : \mathbb{R}^N \to \mathcal{C}$ is non-negative definite (i.e. a covariance function) if and only if there exists a finite measure ν on the Borel σ-field $\mathbb{B}^N$ such that*

$$(5.4.1) \qquad\qquad C(t) \;=\; \int_{\mathbb{R}^N} e^{i\langle t, \lambda \rangle}\, \nu(d\lambda),$$

for all $t \in \mathbb{R}^N$.

With randomness in mind, we write $\sigma^2 = C(0) = \nu(\mathbb{R}^N)$. The measure ν is called the *spectral measure* (for C) and the function $F : \mathbb{R}^N \to [0, \sigma^2]$ given by

$$F(\lambda) \triangleq \nu\left(\prod_{i=1}^N (-\infty, \lambda_i]\right), \qquad \lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N,$$

is called the *spectral distribution function*[8]. When F is absolutely continuous the corresponding density is called the *spectral density*.

The spectral distribution theorem is a purely analytic result and would have nothing to do with random fields were it not for the fact that covariance functions are non-negative definite. Understanding of the result comes from the spectral representation theorem (Theorem 5.4.2) for which we need to extend somewhat the stochastic integration of Section 5.2.

---

[8]Of course, unless ν is a probability measure, so that $\sigma^2 = 1$, F is not a distribution function in the usual usage of the term.

As usual, we start with a measure ν on $\mathbb{R}^N$, and define a *complex ν-noise* to be a $\mathbb{C}$-valued, set indexed process satisfying

$$(5.4.2) \qquad \mathbb{E}\{W(A)\} = 0, \quad \mathbb{E}\{W(A)\overline{W(A)}\} = \nu(A).$$

$$(5.4.3) \qquad A \cap B = \emptyset \;\Rightarrow\; W(A \cup B) = W(A) + W(B) \text{ a.s.}$$

$$(5.4.4) \qquad A \cap B = \emptyset \;\Rightarrow\; \mathbb{E}\{W(A)\overline{W(B)}\} = 0,$$

where the bar denotes complex conjugation[9].

It is then a straightforward exercise to extend the construction of Section 5.2 to obtain an L^2 stochastic integral

$$W(f) \;=\; \int_{\mathbb{R}^N} f(\lambda)\,W(d\lambda)$$

for $f : \mathbb{R}^N \to \mathbb{C}$ with $\|f\| \in L^2(\nu)$, and satisfying

$$(5.4.5) \qquad \mathbb{E}\{W(f)\overline{W(g)}\} \;=\; \int_{\mathbb{R}^N} f(\lambda)\overline{g(\lambda)}\,\nu(d\lambda).$$

This construction allows us to state the following important result.

Theorem 5.4.2 (Spectral representation theorem) *Let ν be a finite measure on $\mathbb{R}^N$ and W a complex ν-noise. Then the complex valued random field*

$$(5.4.6) \qquad f(t) \;=\; \int_{\mathbb{R}^N} e^{i\langle t,\lambda\rangle}\,W(d\lambda)$$

has covariance

$$(5.4.7) \qquad C(s,t) \;=\; \int_{\mathbb{R}^N} e^{i\langle (s-t),\lambda\rangle}\,\nu(d\lambda)$$

and so is (weakly) stationary. If W is Gaussian, then so is f.

Furthermore, to every mean square continuous, centered, (Gaussian) stationary random field f on $\mathbb{R}^N$ with covariance function C and spectral measure ν (cf. Theorem 5.4.1) there corresponds a complex (Gaussian) ν-noise W on $\mathbb{R}^N$ such that (5.4.6) holds in mean square for each $t \in \mathbb{R}^N$.

In both cases, W is called the spectral process corresponding to f.

---

[9]You should note for later reference that if we split a complex ν-noise into its real and imaginary parts, W_R and W_I say, it does not follow from (5.4.4) that $A \cap B = \emptyset$ implies any of $\mathbb{E}\{W(A)W(B)\} = 0$, $\mathbb{E}\{W_R(A)W_R(B)\} = 0$, $\mathbb{E}\{W_I(A)W_I(B)\} = 0$, or $\mathbb{E}\{W_I(A)W_R(B)\} = 0$. Indeed, this is most definitely not the case for the complex W of Theorem 5.4.2 when, for example, the stationary process f there is real valued. See the discussion below on 'real' spectral representations, in which the above problem is addressed by taking W to be defined on a half space rather than all of $\mathbb{R}^N$.

Proof. The fact that (5.4.6) generates a stationary field with covariance (5.4.7) is an immediate consequence of (5.4.5). What needs to be proven is the statement that *all* stationary fields can be represented as in (5.4.6). We shall only sketch the basic idea of the proof, leaving the details to the reader. (They can be found in almost any book on time series – our favourite is [33] – for processes on either $\mathbb{Z}$ or $\mathbb{R}$ and the extension to $\mathbb{R}^N$ is trivial.)

For the first step, set up a mapping Θ from[10] $\mathcal{H} \stackrel{\Delta}{=} \text{span}\{f_t, \ t \in \mathbb{R}^N\} \subset L^2(\mathbb{P})$ to $\mathcal{K} \stackrel{\Delta}{=} \text{span}\{e^{it\cdot}, \ t \in \mathbb{R}^N\} \subset L^2(\nu)$ via

$$(5.4.8) \qquad \Theta\left(\sum_{j=1}^{n} a_j f(t_j)\right) = \sum_{j=1}^{n} a_j e^{it_j \cdot}.$$

for all $n \geq 1$, $a_j \in \mathcal{C}$ and $t_j \in \mathbb{R}^N$. A simple computation shows that this gives an isomorphism between $\mathcal{H}$ and $\mathcal{K}$, which can then be extended to an isomorphism between their closures $\overline{\mathcal{H}}$ and $\overline{\mathcal{K}}$.

Since indicator functions are in $\overline{\mathcal{K}}$, we can define a process W on $\mathbb{R}^N$ by setting

$$(5.4.9) \qquad W(\lambda) = \Theta^{-1}\left(\mathbb{1}_{(-\infty,\lambda]}\right),$$

where $(-\infty, \lambda] = \prod_{j=1}^{N}(-\infty, \lambda_j]$. Working through the isomorphisms shows that W is a complex ν-noise and that $\int_{\mathbb{R}^N} \exp(i\langle t, \lambda \rangle)\, W(d\lambda)$ is ($L^2(\mathbb{P})$ indistinguishable from) f_t. $\qquad\square$

There is also an inverse[11] to (5.4.6), expressing W as an integral involving f, but we shall have no need of it and so now turn to some consequences of Theorems 5.4.1 and 5.4.2.

When the basic field f is real, it is natural to expect a 'real' spectral representation, and this is in fact the case, although notationally it is still generally more convenient to use the complex formulation. Note firstly that if f is real, then the covariance function is a symmetric function ($C(t) =$

---

[10]Actually, we really define $\overline{\mathcal{H}}$ to be the closure in $L^2(\mathbb{P})$ of all sums of the form $\sum_{i=1}^{n} a_i f(t_i)$ for $a_i \in \mathbb{C}$ and $t_i \in T$, thinned out by identifying all elements indistinguishable in $L^2(\mathbb{P})$: i.e. elements U, V for which $\mathbb{E}\{\|(U - V)\|^2\} = 0$. Recall that we already met this space, as well as the mapping Θ, when treating orthonormal expansions in Section 3.1.

[11]Formally, the inverse relationship is based on (5.4.9), but it behaves like regular Fourier inversion. For example, if $\Delta(\lambda, \eta)$ is a rectangle in $\mathbb{R}^N$ which has a boundary of zero ν measure, then

$$W(\Delta(\lambda, \eta)) = \lim_{K \to \infty} (2\pi)^{-N} \int_{-K}^{K} \cdots \int_{-K}^{K} \prod_{k=1}^{N} \frac{e^{-i\lambda_k t_k} - e^{-i\eta_k t_k}}{-it_k} f(t)\, dt.$$

As is usual, if $\nu(\Delta(\lambda, \eta)) \neq 0$, then additional boundary terms need to be added.

$C(-t)$) and so it follows from the spectral distribution theorem (cf. (5.4.1)) that the spectral measure ν must also be symmetric. Introduce three[12] new measures, on $\mathbb{R}_+ \times \mathbb{R}^{N-1}$, by

$$
\begin{aligned}
\nu_1(A) &= \nu(A \cap \{\lambda \in \mathbb{R}^N : \lambda_1 > 0\}) \\
\nu_2(A) &= \nu(A \cap \{\lambda \in \mathbb{R}^N : \lambda_1 = 0\}) \\
\mu(A) &= 2\nu_1(A) + \nu_2(A)
\end{aligned}
$$

We can now rewrite[13] (5.4.1) in real form, as

$$
(5.4.10) \qquad C(t) = \int_{\mathbb{R}_+ \times \mathbb{R}^{N-1}} \cos(\langle \lambda, t \rangle) \, \mu(d\lambda).
$$

There is also a corresponding real form of the spectral representation (5.4.6). The fact that the spectral representation yields a real valued process also implies certain symmetries[14] on the spectral process W. In particular, it turns out that there are two independent *real* valued μ-noises, W_1 and W_2, such that[15]

$$
(5.4.11) \qquad f_t = \int_{\mathbb{R}_+ \times \mathbb{R}^{N-1}} \cos(\langle \lambda, t \rangle) \, W_1(d\lambda)
$$
$$
+ \int_{\mathbb{R}_+ \times \mathbb{R}^{N-1}} \sin(\langle \lambda, t \rangle) \, W_2(d\lambda).
$$

It is easy to check that f so defined has the right covariance function.

5.5 Spectral moments

Since they will be very important later on, we now take a closer look at spectral measures and, in particular, their moments. It turns out that these contain a lot of simple, but very useful, information. Given the the spectral representation (5.4.7); *viz.*

$$
(5.5.1) \qquad C(t) = \int_{\mathbb{R}^N} e^{i\langle t, \lambda \rangle} \, \nu(d\lambda),
$$

---

[12]Note that if ν is absolutely continuous with respect to Lebesgue measure (so that there is a spectral density) one of these, ν_2, will be identically zero.

[13]There is nothing special about the half-space $\lambda_1 \geq 0$ taken in this representation. Any half space will do.

[14]To rigorously establish this we really need the inverse to (5.4.6), expressing W as an integral involving f, which we do not have.

[15]In one dimension, it is customary to take W_1 as a μ-noise and W_2 as a $(2\nu_1)$-noise, which at first glance is different to what we have. However, noting that, when $N = 1$, $\sin(\lambda t)W_2(d\lambda) = 0$ when $\lambda = 0$, it is clear that the two definitions in fact coincide in this case.

we define the *spectral moments*

$$(5.5.2) \qquad \lambda_{i_1 \ldots i_N} \triangleq \int_{\mathbb{R}^N} \lambda_1^{i_1} \cdots \lambda_N^{i_N} \; \nu(d\lambda),$$

for all $(i_1, \ldots, i_N)$ with $i_j \geq 0$. Recalling that stationarity implies that $C(t) = \overline{C(-t)}$ and $\nu(A) = \nu(-A)$, it follows that the odd ordered spectral moments, when they exist, are zero; i.e.

$$(5.5.3) \qquad \lambda_{i_1 \ldots i_N} = 0, \qquad \text{if } \sum_{j=1}^{N} i_j \text{ is odd.}$$

Furthermore, it is immediate from (5.5.1) that successive differentiation of both sides with respect to the t_i connects the various partial derivatives of C at zero with the the spectral moments. To see why this is important, we need first to define the L^2, or *mean square (partial) derivatives* of a random field.

Choose a point $t \in \mathbb{R}^N$ and a sequence of k 'directions' $t'_1, \ldots, t'_k$ of $\mathbb{R}^N$, so that $t' = (t'_1, \ldots, t'_k) \in \otimes^k \mathbb{R}^N$, the tensor product $\mathbb{R}^N \times \cdots \times \mathbb{R}^N$. We say that f has a k-th order derivative at t, in the direction t', which we denote by $D_{L^2}^k f(t, t')$, if the limit

$$(5.5.4) \qquad D_{L^2}^k f(t, t') \triangleq \lim_{h \to 0} F(t, ht')$$

exists in L^2, where $F(t, t')$ is the symmetrized difference

$$F(t, t') = \frac{1}{\prod_{i=1}^{k} |t'_i|} \sum_{s \in \{0,1\}^k} (-1)^{k - \sum_{i=1}^{k} s_i} f\left(t + \sum_{i=1}^{k} s_i t'_i\right).$$

A simple sufficient condition for L^2 differentiability of order k in all directions and throughout a region $T \in \mathbb{R}^N$ is that

$$(5.5.5) \qquad \lim_{|t'|, |s'| \to 0} \mathbb{E}\left\{F(t, t')F(s, s')\right\}$$

exists[16] for all $s, t \in T$ and sequences $s', t' \in \otimes^k \mathbb{R}^N$. Note that if f is Gaussian then so are its L^2 derivatives, when they exist.

By choosing $t' = (e_{i_1}, \ldots, e_{i_k})$, where e_i is the vector with i-th element 1 and all others zero, we can talk of the mean square derivatives

$$\frac{\partial^k}{\partial t_{i_1} \ldots \partial t_{i_k}} f(t) \triangleq D_{L^2}^k f\left(t, (e_{i_1}, \ldots, e_{i_k})\right)$$

---

[16] This is an immediate consequence of the fact that a sequence X_n of random variables converges in L^2 if, and only if, $\mathbb{E}\{X_n X_m\}$ converges to a constant as $n, m \to \infty$.

of f of various orders.

It is then straightforward to see that the the covariance function of such partial derivatives must be given by

(5.5.6)
$$\mathbb{E}\left\{\frac{\partial^k f(s)}{\partial s_{i_1}\partial s_{i_1}\ldots\partial s_{i_k}}\,\frac{\partial^k f(t)}{\partial t_{i_1}\partial t_{i_1}\ldots\partial t_{i_k}}\right\} = \frac{\partial^{2k}C(s,t)}{\partial s_{i_1}\partial t_{i_1}\ldots\partial s_{i_k}\partial t_{i_k}}.$$

The corresponding variances have a nice interpretation in terms of spectral moments when f is stationary. For example, if f has mean square partial derivatives of orders $\alpha+\beta$ and $\gamma+\delta$ for $\alpha,\beta,\gamma,\delta \in \{0,1,2,\ldots\}$, then[17]

(5.5.7)
$$\mathbb{E}\left\{\frac{\partial^{\alpha+\beta}f(t)}{\partial^\alpha t_i\partial^\beta t_j}\cdot\frac{\partial^{\gamma+\delta}f(t)}{\partial^\gamma t_k\partial^\delta t_l}\right\} = (-1)^{\alpha+\beta}\frac{\partial^{\alpha+\beta+\gamma+\delta}}{\partial^\alpha t_i\partial^\beta t_j\partial^\gamma t_k\partial^\delta t_l}C(t)\Big|_{t=0}$$
$$= (-1)^{\alpha+\beta}\,i^{\alpha+\beta+\gamma+\delta}\int_{\mathbb{R}^N}\lambda_i^\alpha\lambda_j^\beta\lambda_k^\gamma\lambda_l^\delta\,\nu(d\lambda).$$

Here are some important special cases of the above, for which we adopt the shorthand $f_j = \partial f/\partial t_j$ and $f_{ij} = \partial^2 f/\partial t_i\partial t_j$ along with a corresponding shorthand for the partial derivatives of C.

(i) f_j has covariance function $-C_{jj}$ and thus variance $\lambda_{2e_j} = -C_{jj}(0)$, where e_j, as usual, is the vector with a 1 in the j-th position and zero elsewhere.

(ii) In view of (5.5.3), and taking $\beta = \gamma = \delta = 0,\ \alpha = 1$ in (5.5.7)

(5.5.8) $f(t)$ and $f_j(t)$ are uncorrelated,

for all j and all t. If f is Gaussian, this is equivalent to independence. Note that (5.5.8) does *not* imply that f and f_j are uncorrelated *as processes*. In general, for $s \neq t$, we will have that $\mathbb{E}\{f(s)f_j(t)\} = -C_j(s-t) \neq 0$.

(iii) Taking $\alpha = \gamma = \delta = 1,\ \beta = 0$ in (5.5.7) gives that

(5.5.9) $f_i(t)$ and $f_{jk}(t)$ are uncorrelated

---

[17] If you decide to check this for yourself using (5.4.6) and (5.4.7) – which is a worthwhile exercise – make certain that you recall the fact that the covariance function is defined as $\mathbb{E}\{f(s)\overline{f(t)}\}$, or you will make the same mistake RJA did in [3] and forget the factor of $(-1)^{\alpha+\beta}$ in the first line. Also, note that although (5.5.7) seems to have some asymmetries in the powers, these disappear due to the fact that all odd ordered spectral moments, like all odd ordered derivatives of C, are identically zero.

for all i, j, k and all t.

This concludes our initial discussion of stationarity for random fields on $\mathbb{R}^N$. In the following section we investigate what happens under slightly weaker conditions and after that what happens when the additional assumption of isotropy is added. In Section 5.8 we shall see how all of this is really part of a far more general theory for fields defined on groups. In Section 3.1 we shall see that, at least for Gaussian fields restricted to bounded regions, that spectral theory is closely related to a far wider theory of orthogonal expansions that works for general – i.e. not necessarily Euclidean – parameter spaces

5.6   Constant variance

It will be important for us in later chapters that some of the relationships of the previous section continue to hold under a condition weaker than stationarity. Of particular interest is knowing when (5.5.8) holds; i.e. when $f(t)$ and $f_j(t)$ are uncorrelated.

Suppose that f has constant variance, $\sigma^2 = C(t,t)$, throughout its domain of definition, and that its L^2 first-order derivatives all exist. In this case, analagously to (5.5.7), we have that

$$(5.6.1) \qquad \mathbb{E}\left\{f(t)\, f_j(t)\right\} \;=\; \left.\frac{\partial}{\partial t_j}C(t,s)\right|_{s=t}$$
$$=\; \left.\frac{\partial}{\partial s_j}C(t,s)\right|_{s=t}.$$

Since constant variance implies that $\partial/\partial t_j C(t,t) \equiv 0$, the above two equalities imply that both partial derivatives there must be identically zero. That is, f and its first order derivatives are uncorrelated.

One can, of course, continue in this fashion. If first derivatives have constant variance, then they, in turn, will be uncorrelated with second derivatives, in the sense that f_i will be uncorrelated with f_{ij} for all i, j. It will not necessarily be true, however, that f_i and f_{jk} will be uncorrelated if $i \neq j$ and $i \neq k$. This will, however, be true if the covariance matrix of all first order derivatives is constant.

5.7   Isotropy

An interesting special class of homogeneous random fields on $\mathbb{R}^N$ that often arises in applications in which there is no special meaning attached to the coordinate system being used is the class of *isotropic* fields. These are

characterized[18] by the property that the covariance function depends only on the Euclidean length $|t|$ of the vector t so that

(5.7.1) $$C(t) \;=\; C(|t|)\,.$$

Isotropy has a number of surprisingly varied implications for both the covariance and spectral distribution functions, and is actually somewhat more limiting than it might at first seem. For example, we have the following result, due to Matérn [104].

Theorem 5.7.1 *If $C(t)$ is the covariance function of a centered, isotropic random field f on $\mathbb{R}^N$, then $C(t) \geq -C(0)/N$ for all t.*

Proof. Isotropy implies that C can be written as a function on $\mathbb{R}_+$ only. Let τ be any positive real. We shall show that $C(\tau) \geq -C(0)/N$.

Choose any $t_1, \ldots, t_{N+1}$ in $\mathbb{R}^N$ for which $|t_i - t_j| = \tau$ for all $i \neq j$. Then, by (5.7.1),

$$\mathbb{E}\left\{ \left| \sum_{k=1}^{N+1} f(t_k) \right|^2 \right\} \;=\; (N+1)[C(0) + NC(\tau)].$$

Since this must be positive, the result follows.   □

The restriction of isotropy also has significant simplifying consequences for the spectral measure ν of (5.4.1). Let $\theta : \mathbb{R}^N \to \mathbb{R}^N$ be a rotation, so that $|\theta(t)| = |t|$ for all t. Isotropy then implies $C(t) = C(\theta(t))$ and so the Spectral Distribution Theorem implies

(5.7.2)
$$
\begin{aligned}
\int_{\mathbb{R}^N} e^{i\langle t, \lambda \rangle}\, \nu(d\lambda) &= \int_{\mathbb{R}^N} e^{i\langle \theta(t), \lambda \rangle}\, \nu(d\lambda) \\
&= \int_{\mathbb{R}^N} e^{i\langle t, \theta(\lambda) \rangle}\, \nu(d\lambda) \\
&= \int_{\mathbb{R}^N} e^{i\langle t, \lambda \rangle}\, \nu_\theta(d\lambda),
\end{aligned}
$$

where ν_θ is the push-forward of ν by θ defined by $\nu_\theta(A) = \nu(\theta^{-1}A)$. Since the above holds for all t it follows that $\nu \equiv \nu_\theta$; i.e. ν, like C, is invariant under rotation. Furthermore, if ν is absolutely continuous, then its density is also dependent only on the modulus of its argument.

An interesting consequence of this symmetry is that an isotropic field cannot have all the probability of its spectral measure concentrated in one

---

[18]Isotropy can also be defined in the non-stationary case, the defining property then being that the distribution of f is invariant under rotation. Under stationarity, this is equivalent to (5.7.1). Without stationarity, however, this definition does not imply (5.7.1). We shall treat isotropy *only* in the scenario of stationarity.

small region in $\mathbb{R}^N$ away from the origin. In particular, it is not possible to have a spectral measure degenerate at one point, unless that point is the origin. The closest the spectral measure of an isotropic field can come to this sort of behaviour is to have all its probability concentrated in an annulus of the form

$$\{\lambda \in \mathbb{R}^N : a \leq |\lambda| \leq b\}.$$

In such a case it is not hard to see that that the field itself is then composed of a 'sum' of waves travelling in all directions but with wavelengths between $2\pi/b$ and $2\pi/a$ only.

Another consequence of isotropy is the spherical symmetry of the spectral measure significantly simplifies the structure of the spectral moments and so the correlations between various derivatives of f. In particular, it follows immediately from (5.5.7) that

$$(5.7.3) \qquad \mathbb{E}\left\{f_i(t)f_j(t)\right\} \; = \; -\mathbb{E}\left\{f(t)f_{ij}(t)\right\} \; = \; \lambda_2 \delta_{ij}$$

where δ_{ij} is the Kronecker delta and $\lambda_2 \stackrel{\Delta}{=} \int_{\mathbb{R}^N} \lambda_i^2 \, \nu(d\lambda)$, which is independent of the value of i. Consequently, if f is Gaussian, then the first order derivatives of f are independent of one another, as they are of f itself.

Since isotropy has such a limiting effect in the spectrum, it is natural to ask how the spectral distribution and representation theorems are affected under isotropy. The following result, due originally to Schoenberg [132] (in a somewhat different setting) and Yaglom [168], describes what happens.

Theorem 5.7.2 *For C to be the covariance function of a mean square continuous, isotropic, random field on $\mathbb{R}^N$ it is necessary and sufficient that*

$$(5.7.4) \qquad C(t) \; = \; \int_0^\infty \frac{J_{(N-2)/2}(\lambda|t|)}{(\lambda|t|)^{(N-2)/2}} \, \mu(d\lambda),$$

where μ is a finite measure on $\mathbb{R}_+$ and J_m is the Bessel function of the first kind of order m; viz.

$$J_m(x) \; = \; \sum_{k=0}^{\infty} (-1)^k \frac{(x/2)^{2k+m}}{k!\,\Gamma(k+m+1)}.$$

Proof. The proof consists in simplifying the basic spectral representation by using the symmetry properties of ν.

We commence by converting to polar coordinates, $(\lambda, \theta_1, \ldots, \theta_{N-1})$, $\lambda \geq 0$, $(\theta_1, \ldots, \theta_{N-1}) \in S^{N-1}$, where S^{N-1} is the unit sphere in $\mathbb{R}^N$. Define a measure μ on $\mathbb{R}_+$ by setting $\mu([0, \lambda]) = \nu(B^N(\lambda))$, and extending as usual, where $B^N(\lambda)$ is the N-ball of radius λ and ν is the spectral measure of (5.4.1).

Then, on substituting into (5.4.1) with $t = (|t|, 0, \dots, 0)$ and performing the coordinate transformation, we obtain

$$C(|t|) \;=\; \int_0^\infty \int_{S^{N-1}} \exp(i|t|\lambda \cos\theta_{N-1}) \, \sigma(d\theta)\mu(d\lambda)$$

where σ is surface area measure on S^{N-1}. Integrating out $\theta_1, \dots, \theta_{N-2}$ it follows that

$$C(|t|) \;=\; s_{N-2} \int_0^\infty \int_0^\pi e^{i\lambda|t|\cos\theta_{N-1}} \left(\sin\theta_{N-1}\right)^{N-2} d\theta_{N-1}\mu(d\lambda)$$

where

(5.7.5)
$$s_N \;\triangleq\; \frac{2\pi^{N/2}}{\Gamma(N/2)}, \qquad N \geq 1,$$

is the surface area[19] of S^{N-1}.

The inside integral can be evaluated in terms of Bessel functions to yield

$$\int_0^\pi e^{i\lambda|t|\cos\theta} \sin^{N-2}\theta \; d\theta \;=\; \frac{J_{(N-2)/2}(\lambda|t|)}{(\lambda|t|)^{(N-2)/2}}$$

which, on absorbing s_{N-2} into μ, completes the proof.   □

For small values of the dimension N, (5.7.4) can be simplified even further. For example, substituting $N = 2$ into (5.7.4) yields that in this case

$$C(t) \;=\; \int_0^\infty J_0(\lambda|t|) \, \mu(d\lambda),$$

while substituting $N = 3$ and evaluating the inner integral easily yields that in this case

$$C(t) \;=\; \int_0^\infty \frac{\sin(\lambda|t|)}{\lambda|t|} \, \mu(d\lambda).$$

Given the fact that the covariance function of an isotropic field takes such a special form, it is natural to seek a corresponding form for the spectral representation of the field itself. Such a representation does in fact exist and we shall now describe it, albeit without giving any proofs. These can be found, for example, in the book by Wong [159], or as special cases in the review by Yaglom [169], which is described in Section 5.8 below. Another way to verify it would be to check that the representation given in

---

[19]When $N = 1$, we have the 'boundary' of the 'unit sphere' $[-1, 1] \in \mathbb{R}$. This is made up of the two points ± 1, which, in counting measure, has measure 2. Hence $s_1 = 2$ makes sense.

Theorem 5.7.3 below yields the covariance structure of (5.7.4). Since this is essentially an exercise in the manipulation of special functions and not of intrinsic probabilistic interest, we shall avoid the temptation to carry it out.

The spectral representation of isotropic fields on $\mathbb{R}^N$ is based on the so-called *spherical harmonics*[20] on the $(N-1)$-sphere, which form an orthonormal basis for the space of square integrable functions on S^{N-1} equipped with the usual surface measure. We shall denote them by $\{h_{ml}^{(N-1)}, l = 1, \ldots, d_m, m = 0, 1, \ldots\}$ where the d_m are known combinatorial coefficients[21].

Now use the spectral decomposition

$$f(t) \;=\; \int_{\mathbb{R}^N} e^{i\langle t,\lambda\rangle}\, W(d\lambda)$$

to define a family of noises on $\mathbb{R}_+$ by

$$W_{ml}(A) \;=\; \int_A \int_{S^{N-1}} h_{ml}^{(N-1)}(\theta)\, W(d\lambda, d\theta)$$

where, once again, we work in polar coordinates. Note that since W is a ν-noise, where ν is the spectral measure, information about the covariance of f has been coded into the W_{ml}. From this family, define a family of mutually uncorrelated, stationary, one-dimensional processes $\{f_{ml}\}$ by

$$f_{ml}(r) \;=\; \int_0^\infty \frac{J_{m+(N-2)/2}(\lambda r)}{(\lambda r)^{(N-2)/2}}\, W_{ml}(d\lambda),$$

where, as in the spectral representation (5.4.6), one has to justify the existence of this L^2 stochastic integral. These are all the components we need in order to state the following.

Theorem 5.7.3 *A centered, mean square continuous, isotropic random field on $\mathbb{R}^N$ can be represented by*

$$(5.7.6) \qquad f(t) \;=\; f(r,\theta) \;=\; \sum_{m=0}^{\infty} \sum_{l=1}^{d_m} f_{ml}(r) h_{ml}^{(N-1)}(\theta).$$

---

[20] We shall also avoid giving details about spherical harmonics. A brief treatment would add little to understanding them. The kind of treatment required to, for example, get the code correct in programming a simulation of an isotropic field using the representations that follow will, in any case, send you back to the basic reference of Erdélyi [58] followed by some patience in sorting out a software help reference. A quick web search will yield you many interactive, coloured examples of these functions within seconds.

[21] The spherical harmonics on S^{N-1} are often written as $\{h_{m,l_1,\ldots,l_{N-2},\pm l_{N-1}}^{(N-1)}\}$, where $0 \le l_{N-1} \le \cdots \le l_1 \le m$. The constants d_m in our representation can be computed from this.

In other words, isotropic random fields can be decomposed into a countable number of mutually uncorrelated stationary processes with a one-dimensional parameter, a result which one would not intuitively expect. As noted above, there is still a hidden spectral process in (5.7.6), entering via the W_{ml} and f_{ml}. This makes for an important difference between (5.7.6) and the similar looking Karhunen-Loève expansion we saw Section 3.2. Another difference lies in the fact that while it is possible to truncate the expansion (5.7.6) to a finite number of terms and retain isotropy, this is not true of the standard Karhunen-Loève expansion. In particular, isotropic fields can never have finite Karhunen-Loève expansions. For a heuristic argument as to why this is the case, recall from Section 5.7 that under isotropy the spectral measure must be invariant under rotations, and so cannot be supported on a finite, or even countable, number of points. Consequently, one also needs a uncountable number of independent variables in the spectral noise process to generate the process via (5.4.6). However a process with a finite Karhunen-Loève expansion provides only a finite number of such variables, which can never be enough.

An interesting corollary of Theorem 5.7.3 is obtained by fixing r in (5.7.6). We then have a simple representation in terms of uncorrelated random coefficients $f_{ml}(r)$ and spherical harmonics of an isotropic random field on the N-sphere of radius r. If the random field is Gaussian, then the coefficients are actually independent, and we will, essentially, have generated a Karhunen-Loève expansion. You can find many more results of this kind, with much more detail, in [95].

One can keep playing these games for more and more special cases. For example, it is not uncommon in applications to find random fields that are functions of 'space' x and 'time' t, so that the parameter set is most conveniently written as $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^N$. Such processes are often stationary in t and isotropic in x, in the sense that

$$\mathbb{E}\{f(s, u)\overline{f(s + t, u + x)}\} \ = \ C(t, |x|),$$

where C is now a function from $\mathbb{R}^2$ to $\mathbb{C}$. In such a situation the methods of the previous proof suffice to show that C can be written in the form

$$C(t, x) \ = \ \int_{-\infty}^{\infty} \int_0^{\infty} e^{it\nu} G_N(\lambda x) \, \mu(d\nu, d\lambda),$$

where

$$G_N(x) = \left(\frac{2}{x}\right)^{(N-2)/2} \Gamma\left(\frac{N}{2}\right) J_{(N-2)/2}(x)$$

and μ is a measure on the half-plane $\mathbb{R}_+ \times \mathbb{R}^N$.

By now it should be starting to become evident that all of these representations must be special cases of some general theory, that might also be able to cover non-Euclidean parameter spaces. This is indeed the case, although for reasons that will soon be explained the general theory is such that, ultimately, each special case requires almost individual treatment.

5.8   Stationarity over groups

We have already seen in Section 5.3 that the appropriate setting for stationarity is when the parameter set has a group structure. In this case it made sense, in general, to talk about left and right stationarity (cf. (5.3.2) and (5.3.3)). Simple 'stationarity' requires both of these and so makes most sense if the group is Abelian (commutative).

   In essence, the spectral representation of a random field over a group is intimately related to the representation theory of the group. This, of course, is far from being a simple subject. Furthermore, its level of difficulty depends very much on the group in question and so it is correspondingly not easy to give a general spectral theory for random fields over groups. The most general results in this area are in the paper by Yaglom [169] already mentioned above and the remainder of this section is taken from there[22].

   We shall make life simpler by assuming for the rest of this section that T is a *locally compact, Abelian* (LCA) group. As before, we shall denote the binary group operation by $+$ while $-$ denotes inversion. The Fourier analysis of LCA groups is well developed (e.g. [124]) and based on *characters*. A homomorphism γ from T to the multiplicative group of complex numbers is called a character if $\|\gamma(t)\| = 1$ for all $t \in T$ and if

$$\gamma(t + s) \;=\; \gamma(t)\,\gamma(s), \qquad s, t \in T.$$

If $T = \mathbb{R}^N$ under the usual addition, then the characters are given by the family

$$\left\{ \gamma_\lambda(t) \;=\; e^{i\langle t, \lambda \rangle} \right\}_{\lambda \in \mathbb{R}^N}$$

of complex exponential functions, which were at the core of the spectral theory of fields over $\mathbb{R}^N$. If $T = \mathbb{Z}^N$, again under addition, the characters are as for $T = \mathbb{R}^N$, but λ is restricted to $[-\pi, \pi]^N$. If $T = \mathbb{R}^N$ under rotation rather than addition, then the characters are the spherical harmonics on S^{N-1}.

   The set of all continuous characters also forms a group, Γ say, called the *dual group* with composition defined by

$$(\gamma_1 + \gamma_2)(t) \;=\; \gamma_1(t)\,\gamma_2(t).$$

There is also a natural topology on Γ (cf. [124]) which gives Γ a LCA structure and under which the map $(\gamma, t) \overset{\Delta}{=} \gamma(t) : \Gamma \times T \to \mathcal{C}$ is continuous.

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[22]There is also a very readable, albeit less exhaustive, treatment in Hannan [72]. In addition, Letac [96] has an elegant exposition based on Gelfand pairs and Banach algebras for processes indexed by unimodular groups, which, in a certain sense, give a generalisation of isotropic fields over $\mathbb{R}^N$. Ylinen [170] has a theory for noncommutative locally compact groups that extends the results in [169].

The spectral distribution theorem in this setting can now be written as

$$(5.8.1) \qquad\qquad C(t) \ = \ \int_\Gamma (\gamma, t)\, \nu(d\gamma),$$

where the finite spectral measure ν is on the σ-algebra generated by the topology on Γ. The spectral representation theorem can be correspondingly written as

$$(5.8.2) \qquad\qquad f(t) \ = \ \int_\Gamma (\gamma, t)\, W(d\gamma),$$

where W is a ν-noise on Γ.

Special cases now follow from basic group theoretic results. For example, if T is discrete, then Γ is compact, as we noted already for the special case of $T = \mathbb{Z}^N$. Consequently, the integral in the spectral representation (5.8.2) is actually a sum and as such will be familiar to every student of Time Series. Alternatively, if T is compact, Γ is discrete. This implies that ν must be a sum of point masses and W is actually a discrete collection of independent random variables.

An interesting and important special case that we have already met is $T = S^{N-1}$. Treating T as both an N-sphere and the rotation group $O(N)$ we write the sum $\theta_1 + \theta_2$, for $\theta_1, \theta_2 \in S^{N-1}$, for the rotation of θ_1 by θ_2, the latter considered as an element of $O(N)$. In this case the characters are again the spherical harmonics and we have that

$$(5.8.3) \qquad\qquad f(\theta) \ = \ \sum_{m=0}^{\infty} \sum_{l=1}^{d_m} W_{ml}\, h_{ml}^{(N-1)}(\theta)$$

where the W_{ml} are uncorrelated with variance depending only on m. This, of course, is simply (5.7.6) once again, derived from a more general setting. Similarly, the covariance function can be written as

$$(5.8.4) \qquad C(\theta_1, \theta_2) \ = \ C(\theta_{12}) \ = \ \sum_{m=0}^{\infty} \sigma_m^2 C_m^{(N-1)/2}(\cos(\theta_{12})),$$

where θ_{12} is the angular distance between θ_1 and θ_2, and the C_m^N are the Gegenbauer polynomials.

Other examples for the LCA situation follow in a similar fashion from (5.8.1) and (5.8.2) by knowing the structure of the dual group Γ.

The general situation is much harder and, as has already been noted, relies heavily on knowing the representation of T. In essence, given a representation of G on a $GL(H)$ for some Hilbert space H, one constructs a (left or right) stationary random field on G via the canonical white noise on H. The construction of Lemma 5.3.2 with $T = G$ and $H = L^2(G, \mu)$ can be thought of as a special example of this approach. For further details, you should go to the references given earlier in this section.

6
PART II: GEOMETRY

If you have not yet read the Preface, then please do so now.

Since you have read the Preface, you already know that central to much of what we shall be looking at in Part III is the geometry of *excursion sets*.

$$A_u \equiv A_u(f, T) \stackrel{\Delta}{=} \{t \in T : f(t) \geq u\} \equiv f^{-1}([u, \infty))$$

for random fields f over general parameter spaces T.

In order to do this, we are going to need quite a bit of geometry. In fact, we are going to need a lot more than one might expect, since the answers to the relatively simple questions that we shall be asking end up involving concepts that do not, at first sight, seem to have much to do with the original questions.

In Part III we shall see that an extremely convenient description of the geometric properties of A is its *Euler*, or *Euler-Poincaré, characteristic*. In many cases, the Euler characteristic is determined by the fact that it is the unique integer valued functional φ, defined on collections of nice A, satisfying

$$\varphi(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ 1 & \text{if } A \text{ is ball-like}, \end{cases}$$

where by "ball-like" we mean homotopically equivalent[1] to the unit N-ball, $B^N = \{t \in \mathbb{R}^N : |t| \leq 1\}$, and with the additivity property that

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[1] The notion of homotopic equivalence is formalised below by Definition 6.1.4. However, for the moment, "ball-like" will be just as useful a concept.

$$\varphi(A \cup B) \;=\; \varphi(A) + \varphi(B) - \varphi(A \cap B).$$

More global descriptions follow from this. For example, if A is a nice set in $\mathbb{R}^2$, then $\varphi(A)$ is simply the number of connected components of A minus the number of holes in it. In $\mathbb{R}^3$, $\varphi(A)$ is given by the number of connected components, minus the number of "handles", plus the number of holes. Thus, for example, the Euler characteristics of a baseball, a tennis ball, and a coffee cup are, respectively, 1, 2, and 0.

One of the basic properties of the Euler characteristic is that it is determined by the homology class of a set. That is, smooth transformations that do not change the basic 'shape' of a set do not change its Euler characteristic. Finer geometric information, which will change under such transformations, lies in the so-called *Minkowski functionals*. However, even if we were to decide to do without this finer information, we would still arrive to a scenario involving it. We shall see, again in Part III, that the expected value of $\varphi\left(A_u(f, T)\right)$ will, in general, involve the Minkowski functionals of T. This is what was meant above about simple questions leading to complicated answers.

There are basically two different approaches to developing the geometry that we shall require. The first works well for sets in $\mathbb{R}^N$ that are made up of the finite union of simple building blocks, such as convex sets. For many of our readers, we imagine that this will suffice. The basic framework here is that of Integral Geometry.

The second, more fundamental approach is via the Differential Geometry of abstract manifolds. As described in the Preface, this more general setting has very concrete applications and, moreover, often provides more powerful and elegant proofs for a number of problems related to random fields even on the "flat" manifold $\mathbb{R}^N$. This approach is crucial if you want to understand the full theory. Furthermore, since some of the proofs of later results, even in the Integral Geometric setting, are more natural in the manifold scenario, it is essential if you need to see full proofs. However, the jump in level of mathematical sophistication between the two approaches is significant, so that unless you feel very much at home in the world of manifolds you are best advised to read the Integral Geometric story first.

Chapter 6 handles all the Integral Geometry that we shall need. The treatment there is detailed, complete, and fully self-contained. This is not the case when we turn to Differential Geometry, where the theory is too rich to treat in full. We start with a quick and nasty revision of basic Differential Geometry in Chapter 7, most of which is standard graduate course material. Chapter 8 treats piecewise smooth manifolds, which provide the link between the geometry of Chapter 6 with its sharp corners and edges and the smooth manifolds of Chapter 7. In Chapter 9 we look at Morse Theory in the piecewise smooth setting. While Morse Theory in the smooth

setting is, once again, standard material, the piecewise smooth case is less widely known.

In the passage from Integral to Differential Geometry a number of things will happen. Among them, the space T (time, or multi-dimensional time) will become M (manifold[2]) and the Minkowski functionals will become the *Lipschitz-Killing curvatures.*

While Lipschitz-Killing curvatures are well known objects in global Differential Geometry, we do not imagine that they will be too well known to the probabilist reader. Hence we also devote Chapter 10 to a study of the so-called *tube formulae* originally due to Hermann Weyl [158]. Not only should this chapter help you understand what Lipschitz-Killing curvatures are, but it serves as important preliminary material to a different approach to that which we have adopted for computing the excursion probability

$$\mathbb{P}\Big\{ \sup_{t \in M} f(t) \geq u \Big\}$$

which is treated in detail in Chapter 13 of Part III.

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[2]A true transition from T to M would also have the elements t of T becoming elements p (points) of M. However, as we have already mentioned, this seems to be too heavy a psychological price for a probabilist to pay, so we shall remain with points $t \in M$. For this mortal sin of misnotation we again beg forgiveness from our geometer colleagues.

6
Integral Geometry

Our aim in this chapter is to develop a framework for handling *excursion sets*, which we now redefine in a non-stochastic framework.

Definition 6.0.1 *Let f be a measurable, real valued, function on some measurable space, and T be a measurable subset of that space. Then, for each $u \in \mathbb{R}$,*

$$(6.0.1) \qquad A_u \equiv A_u(f, T) \stackrel{\Delta}{=} \{t \in T : f(t) \geq u\},$$

is called the excursion set of f in T over the level u.

Throughout this chapter, T will be a 'nice' (in a sense to be specified soon) subset of $\mathbb{R}^N$, and our tools will be those of Integral Geometry.

6.1 Basic integral geometry

Quite comprehensive studies of Integral Geometry are available in the monographs of Hadwiger [70] and Santaló [130], although virtually all the results we shall derive can also be found in the paper by Hadwiger [71]. Other very useful and somewhat more modern and readable references are Schneider [?] and Klain and Rota [86].

Essentially, we shall be interested in the study of a class of geometric objects known as *basic complexes*. Later on, we shall show that, with probability one, the excursion sets of a wide variety of random fields belong to

this class, and the concepts that we are about to discuss are relevant to random fields. We commence with some definitions and simple results, all of which are due to Hadwiger [71].

Assume that we have equipped $\mathbb{R}^N$ with a Cartesian coordinate system, so that the N vectors e_j (with 1 in the jth position and zeros elsewhere) serve as an orthonormal basis. Throughout this and the following two sections, everything that we shall have to say will be dependent on this choice of basis. The restriction to a particular coordinate system will disappear in the coordinate free approach based on manifolds.

We call a k-dimensional affine subspace of the form

$$ E \; = \; \{t \in \mathbb{R}^N : t_j = a_j, \; j \in J, \; -\infty < t_j < \infty, \; j \notin J\} $$

a *(coordinate) k-plane* of $\mathbb{R}^N$ if J is a subset of size $N - k$ of $\{1, \ldots, N\}$ and $a_j, \; j \in J$, are fixed.

We shall call a compact set B in $\mathbb{R}^N$ *basic* if the intersections $E \cap B$ are simply connected for every k-plane E of $\mathbb{R}^N, k = 1, \ldots, N$. Note that this includes the case $E = \mathbb{R}^N$. These sets, as their name implies, will form the basic building blocks from which we shall construct more complicated and interesting structures. It is obvious that the empty set $\emptyset$ is basic, as is any convex set. Indeed, convex sets remain basic under rotation, a property which characterizes them. Note that it follows from the definition that, if B is basic, then so is $E \cap B$ for any k-plane E.

A set $A \subset \mathbb{R}^N$ is called a *basic complex* if it can be represented as the union of a finite number of basic sets, $B_1, \ldots B_m$, for which the intersections $B_{\nu_1} \cap \cdots \cap B_{\nu_k}$ are basic for any combination of indices $\nu_1, \ldots, \nu_k, \; k = 1, \ldots, m$. The collection of basic sets

$$ p \; = \; p(A) \; = \; \{B_1, \ldots, B_m\} $$

is called a *partition* of A, and their number, m, is called the *order* of the partition. Clearly, partitions are in no way unique.

The class of all basic complexes, which we shall denote by $\mathcal{C}_B$, or $\mathcal{C}_B^N$ if we need to emphasis its dimension, possesses a variety of useful properties. For example, if $A \in \mathcal{C}_B$ then $E \cap A \in \mathcal{C}_B$ for every k-plane E. In fact, if E is a k-plane with $k \le N$ and $A \in \mathcal{C}_B^N$, we have $E \cap A \in \mathcal{C}_B^k$. To prove this it suffices to note that if

$$ p \; = \; \{B_1, \ldots, B_m\} $$

is a partition of A then

$$ p' \; = \; \{E \cap B_1, \ldots, E \cap B_m\} $$

is a partition of $E \cap A$, and, since $E \cap B_j$ is a k-dimensional basic for all j, $E \cap A \in \mathcal{C}_B^k$.

Another useful property of $\mathcal{C}_B$ is that it is invariant under those rotations of $\mathbb{R}^N$ that map the vectors $e_1, \ldots, e_N$ onto one another. It is not, however, invariant under all rotations, a point to which we shall return later[1]. Furthermore, $\mathcal{C}_B$ is additive, in the sense that $A, B \in \mathcal{C}_B$ and $A \cap B = \emptyset$ imply[2] $A \cup B \in \mathcal{C}_B$.

These and similar properties make the class of basic complexes quite large, and, in view of the fact that convex sets are basic, implies that they include the convex ring. (i.e. the collection of all sets formed via finite union and intersection of convex sets.)

With a class of sets in mind, we can now begin our search for a way to describe the shape of sets by looking for an integer valued functional wtih the following two basic properties:

$$(6.1.1) \qquad \varphi(A) \;=\; \begin{cases} 0, & \text{if } A = \emptyset, \\ 1, & \text{if } A \neq \emptyset \text{ is basic,} \end{cases}$$

and

$$(6.1.2) \qquad \varphi(A \cup B) \;=\; \varphi(A) + \varphi(B) - \varphi(A \cap B),$$

whenever $A, \; B, \; A \cup B, \; A \cap B \in \mathcal{C}_B$.

An important result of Integral Geometry states that not only does a functional possessing these two properties exist, but it is *uniquely* determined by them. We shall prove this by obtaining an explicit formulation for φ, which will also be useful in its own right.

Let $p = p(A)$ be a partition of order m of some $A \in \mathcal{C}_B$ into basics. Define the *characteristic of the partition* to be

---

[1] A note for the purist: As noted earlier, our definition of $\mathcal{C}_B$ is dependent on the choice of basis, which is what loses us rotation invariance. An easy counterexample in $\mathcal{C}_B^2$ is the descending staircase $\bigcup_{j=1}^{\infty} B_j$, where $B_j = \{(x,y) : 0 \leq x \leq 2^{-j}, \; 1 - 2^{1-j} \leq y \leq 1 - 2^{-j}\}$. This is actually a basic with relation to the natural axes, but not even a basic complex if the axes are rotated 45 degrees, since then it cannot be represented as the union of a *finite* number of basics. Hadwiger's [71] original definition of basic complexes was basis independent but covered a smaller class of sets. In essence, basic sets were defined as above (but relative to a coordinate system) and basic complexes were required to have a representation as a finite union of basics for *every* choice of coordinate system. Thus our descending staircase is not a basic complex in his scenario. While more restrictive, this gives a rotation invariant theory. The reasons for our choice will become clearer later on when we return to a stochastic setting. See, in particular, Theorem 11.2.3 and the comments following it. See also the axis free approach of Section 9.2.

[2] Note that if $A \cap B \neq \emptyset$ then $A \cup B$ is not necessarily a basic complex. For a counterexample, take A to be the descending staircase of the previous footnote, and let B be the line segment $\{(x,y) : y = 1 - x, \; x \in [0,1]\}$. There is no way to represent $A \cup B$ as the union of a *finite* number of basics, essentially because of the infinite number of single point intersections between A and B, or, equivalently, the infinite number of holes in $A \cup B$.

$$(6.1.3) \quad \kappa(A,p) \quad = \quad \sum^{(1)} \epsilon(B_j) - \sum^{(2)} \epsilon(B_{j_1} \cap B_{j_2}) + \ldots$$

$$+(-1)^{n+1} \sum^{(n)} \epsilon(B_{j_1} \cap \cdots \cap B_{j_n}) + \ldots$$

$$+(-1)^{m+1} \epsilon(B_1 \cap \ldots \cap B_m)$$

where $\sum^{(n)}$ denotes summation over all subsets $\{j_1, \ldots, j_n\}$ of $\{1, \ldots, m\}$, $1 \le n \le m$, and ϵ is an indicator function for basic sets, defined by

$$\epsilon(A) \quad = \quad \begin{cases} 0, & \text{if } A = \emptyset, \\ 1, & \text{if } A \ne \emptyset \text{ is basic.} \end{cases}$$

Then, if a functional φ satisfying (6.1.1) and (6.1.2) does in fact exist, it follows iteratively from these conditions and the definition of basic complexes that, for any $A \in \mathcal{C}_B$ and any partition p,

$$(6.1.4) \qquad\qquad \varphi(A) \quad = \quad \kappa(A,p).$$

Thus, given existence, uniqueness of φ will follow if we can show that $\kappa(A,p)$ is independent of the partition p.

Theorem 6.1.1 *Let $A \subset \mathbb{R}^N$ be a basic complex and $p = p(A)$ a partition of A. Then the quantity $\kappa(A,p)$ is independent of p. If we denote this by $\varphi(A)$, then φ satisfies (6.1.1) and (6.1.2), and $\varphi(A)$ is called the Euler characteristic of A.*

Proof. The main issue is that of existence, which we establish by induction. When $N = 1$, basics are simply closed intervals or points, or the empty set. Then if we write $N(A)$ to denote the number of disjoint intervals and isolated points in a set $A \subset \mathcal{C}_B^1$, setting

$$\varphi(A) \quad = \quad N(A)$$

yields a function satisfying $\varphi(A) = \kappa(A,p)$ for every p and for which (6.1.1) and (6.1.2) are clearly satisfied.

Now let $N > 1$ and assume that for all spaces of dimension k less than N we have a functional φ^k on $\mathcal{C}_B^k$ for which $\varphi^k(A) = \kappa(A,p)$ for all $A \in \mathcal{C}_B^k$ and every partition p of A. Choose one of the vectors e_j, and for $x \in (-\infty, \infty)$ let E_x (which depends on j) denote the $(N-1)$-plane

$$(6.1.5) \qquad\qquad E_x \overset{\Delta}{=} \{t \in \mathbb{R}^N : t_j = x\}.$$

Let $A \in \mathcal{C}_B^N$ and let $p = p(A) = \{B_1, \ldots, B_m\}$ be one of its partitions. Then clearly the projections onto E_0 of the cross-sections $A \cap E_x$ are all

in $\mathcal{C}_B^{N-1}$, so that there exists a partition-independent functional φ_x defined on $\{A \cap E_x, A \in \mathcal{C}_B^N\}$ determined by

$$\varphi_x(A \cap E_x) \; = \; \varphi^{N-1}(\text{projection of } A \cap E_x \text{ onto } E_0).$$

Via φ_x we can define a new partition-independent function f by

$$(6.1.6) \qquad\qquad f(A, x) \; = \; \varphi_x(A \cap E_x).$$

However, by the induction hypothesis and (6.1.6), we have from (6.1.3) that

$$f(A, x) \; = \; \sum\nolimits^{(1)} \epsilon(B_{j_1} \cap E_x) - \sum\nolimits^{(2)} \epsilon(B_{j_1} \cap B_{j_2} \cap E_x) + \dots.$$

Consider just one of the right-hand terms, writing

$$\epsilon(x) \; = \; \epsilon(B_{j_1} \cap \dots \cap B_{j_k} \cap E_x).$$

Assume that the intersection $B_{j_1} \cap \dots \cap B_{j_k}$ is non-empty. Otherwise there is nothing to prove. Since $\epsilon(x)$ is zero when the intersection of the B_{j_i} with E_x is empty and one otherwise, we have for some finite a and b that $\epsilon(x) = 1$ if $a \le x \le b$ and $\epsilon(x) = 0$ otherwise. Thus $\epsilon(x)$ is a step function, taking at most two values. Hence $f(A, x)$, being the sum of a finite number of such functions, is again a step function, with a finite number of discontinuities. Thus the left-hand limits

$$(6.1.7) \qquad\qquad f(A, x^-) \; = \; \lim_{y \downarrow 0} f(A, x - y)$$

always exist. Now define a function h, which is non-zero at only a finite number of points x, by

$$(6.1.8) \qquad\qquad h(A, x) \; = \; f(A, x) - f(A, x^-)$$

and define

$$(6.1.9) \qquad\qquad \varphi(A) \; = \; \sum_x h(A, x),$$

where the summation is over the finite number of x for which the summand is non-zero. Note that since f is independent of p so are h and φ.

Thus we have defined a functional on $\mathcal{C}_B$, and we need only check that (6.1.1) and (6.1.2) are satisfied to complete this section of the proof. Firstly, note that if B is a basic and $B \ne \emptyset$, and if a and b are the extreme points of the linear set formed by projecting B onto e_j, then $f(B, x) = 1$ if $a \le x \le b$ and equals zero otherwise. Thus $h(B, a) = 1$, while $h(B, x) = 0$, $x \ne a$, so that $\varphi(B) = 1$. This is (6.1.1) since $\varphi(\emptyset) = 0$ is obvious. Now let

$A, B, A \cup B, A \cap B$ all belong to $\mathcal{C}_B$. Then the projections onto E_0 of the intersections

$$A \cap E_x, \qquad B \cap E_x, \qquad (A \cup B) \cap E_x, \qquad A \cap B \cap E_x$$

all belong to $\mathcal{C}_B^{N-1}$ and so by (6.1.6) and the induction hypothesis

$$f(A \cup B, x) \;=\; f(A, x) + f(B, x) - f(A \cap B, x).$$

Replacing x by x^- and performing a subtraction akin to that in (6.1.8) we obtain

$$h(A \cup B, x) \;=\; h(A, x) + h(B, x) - h(A \cap B, x).$$

Summing over x gives

$$\varphi(A \cup B) \;=\; \varphi(A) + \varphi(B) - \varphi(A \cap B)$$

so that (6.1.2) is established and we have the existence of our functional φ. It may, however, depend on the partition p used in its definition.

For uniqueness, note that since, by (6.1.4), $\kappa(A, p') = \varphi(A)$ for any partition p', we have that $\kappa(A, p')$ is independent of p' and so that $\varphi(A)$ is independent of p. That is, we have the claimed uniqueness of φ.

Finally, since $\kappa(A, p)$ is independent of the particular choice of the vector e_j appearing in the proof, so is φ.   □

The proof of Theorem 6.1.1 actually contains more than we have claimed in the statement of the result, since, in developing the proof, we actually obtained an alternative way of computing $\varphi(A)$ for any $A \in \mathcal{C}_B$. This is given explicitly in the following theorem, for which E_x is as defined at (6.1.5).

Theorem 6.1.2 *For basic complexes $A \in \mathcal{C}_B^N$, the Euler characteristic φ, as defined by (6.1.4), has the following equivalent iterative definition:*

$$(6.1.10) \quad \varphi(A) \;=\; \begin{cases} \text{number of disjoint intervals in } A, & \text{if } N = 1, \\ \sum_x \{\varphi(A \cap E_x) - \varphi(A \cap E_{x^-})\}, & \text{if } N > 1, \end{cases}$$

where

$$\varphi(A \cap E_{x^-}) \;=\; \lim_{y \downarrow 0} \varphi(A \cap E_{x-y})$$

and the summation is over all real x for which the summand is non-zero.

This theorem is a simple consequence of (6.1.9) and requires no further proof. Note that it also follows from the proof of Theorem 6.1.1 (cf. the final sentence there) that the choice of vector e_j used to define E_x is irrelevant[3].

---

[3]It is also not hard to show that if A is a basic complex with respect to each of two coordinate systems (which are not simple relabellings of one another) then $\varphi(A)$ will be the same for both. However, this is taking us back to original formulism of [71], which we have already decided is beyond what we need here.

The importance of Theorem 6.1.2 lies in the iterative formulation it gives for φ, for, using this, we shall show in a moment how to obtain yet another formulation that makes the Euler characteristic of a random excursion set amenable to probabilistic investigation.

Figure 6.1.1 shows an example of this iterative procedure in $\mathbb{R}^2$. Here the vertical axis is taken to define the horizontal 1-planes E_x. The values of $\varphi(A \cap E_x)$ appear closest to the vertical axis, with the values of h to their left. Note in particular the set with the hole 'in the middle'. It is on sets like this, and their counterparts in higher dimensions, that the characteristic φ and the number of connected components of the set differ. In this example they are, respectively, zero and one. For the moment, ignore the arrows and what they are pointing at.
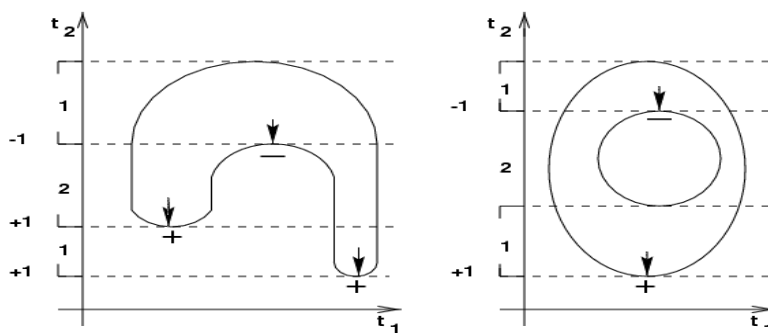
FIGURE 6.1.1. Computing the Euler characteristic.

To understand how this works in higher dimensions, you should try to visualise some N-dimensional examples to convince yourself that for the N-dimensional unit ball, B^N, and the N-dimensional unit sphere, S^{N-1}:

$$\varphi(B^N) \;=\; 1, \qquad \varphi(S^{N-1}) \;=\; 1 + (-1)^{N-1}.$$

It is somewhat less easy (and, indeed, quite deep in higher dimensions) to see that, if $K_{N,k}$ denotes B^N with k non-intersecting cylindrical holes drilled through it, then, since both $K_{N,k}$ and its boundary belong to $\mathcal{C}_B^N$,

$$\varphi(K_{N,k}) \;=\; 1 + (-1)^N k,$$

while

$$\varphi(\partial K_{N,k}) \;=\; [1 + (-1)^{N-1}](1 - k).$$

Finally, if we write $\bar{K}_{N,k}$ to denote B^N with k 'handles' attached, then,

$$\varphi(\bar{K}_{N,k}) \;=\; 1 - k.$$

In view of (**??**), knowing the Euler characteristic of balls means that we know it for all basics. However, this invariance goes well beyond balls,

since it can also be shown that the Euler characteristic is the same for all homotopically equivalent sets; i.e. sets which are geometry "alike" in that they can be deformed into one another in a continuous fashion. Here is a precise definition of this equivalence.

Definition 6.1.3 *Let f and g be two mappings from A to B, both subsets of a Hausdorff space[4] X. If there exists a continuous mapping, $F : A \times [0,1] \to X$ with the following three properties, then we say f is homotopic or deformable to g:*

$$
\begin{aligned}
F(t,0) &= f(t) & \forall t \in A, \\
F(t,\tau) &\in X & \forall t \in A,\ \tau \in [0,1], \\
F(t,1) &= g(t) & \forall t \in A.
\end{aligned}
$$

Definition 6.1.4 *Let A and B be subsets of (possibly different) Hausdorff spaces. If there exist continuous $f : A \to B$ and $g : B \to A$ for which the composite maps $g \circ f : A \to A$ and $f \circ g : B \to B$ are deformable, respectively, to the identity maps on A and B, then A and B are called homotopically equivalent.*

6.2   Excursion sets again

We now return briefly to the setting of excursion sets. Our aim in this section will be to find a way to compute their Euler characteristics directly from the function f, without having to look at the sets themselves. In particular, we will need to find a method that depends only on *local* properties of f, since later, in the random setting, this (via finite dimensional distributions) will be all that will be available to us for probabilistic computation.

Since the main purpose of this section is to develop understanding and since we shall ultimately use the techniques of the critical point theory developed in Chapter 9 to redo everything that will be done here in far greater generality, we shall not give all the details of all the proofs. The interested reader can find most of them either in GRF [3] or the review [5]. Furthermore, we shall restrict the parameter set T to being a bounded rectangle of the form

$$
(6.2.1) \quad T = [s,t] \triangleq \prod_{i=1}^{N} [s_i, t_i], \qquad -\infty < s_i < t_i < \infty.
$$

For our first definition, we need to decompose T into a disjoint union of open sets, starting with its interior, its faces, edges etc. More precisely,

---

[4]A Hausdorff space is a topological space T for which, given any two distinct points $s, t \in T$ there are open sets $U, V \subset T$ with $s \in U$, $t \in V$ and $U \cap V = \emptyset$.

a *face* J, of dimension k, is defined by fixing a subset $\sigma(J)$ of $\{1, \ldots, N\}$ of size k and a sequence of $N - k$ zeroes and ones, which we write as $\varepsilon(J) = \{\varepsilon_1, \ldots, \varepsilon_{N-k}\}$, so that

$$(6.2.2) \quad J \;=\; \{v \in T : v_j = (1 - \varepsilon_j)s_j + \varepsilon_j t_j, \text{ if } j \in \sigma^c(J),$$
$$s_j < v_j < t_j, \text{ if } j \in \sigma(J)\},$$

where $\sigma^c(J)$ is the complement of $\sigma(J)$ in $\{1, \ldots, N\}$.

In anticipation of later notation, we write $\partial_k T$ for the collection of faces of dimension k in T. This is known as the *k-skeleton* of T. Then $\partial_N T$ contains only T°, while $\partial_0 T$ contains the 2^N vertices of the rectangle. In general, $\partial_k T$ has

$$(6.2.3) \qquad\qquad J_k \;\triangleq\; 2^{N-k} \binom{N}{k}$$

elements. Not also that the usual boundary of T, ∂T, is given by the disjoint union

$$\partial T \;=\; \bigcup_{k=0}^{N-1} \bigcup_{J \in \partial_k T} J.$$

We start with some simple assumptions on f and, as usual, write the first and second order partial derivatives of f as $f_j = \partial f / \partial t_j$, $f_{ij} = \partial^2 f / \partial t_i \partial t_j$.

Definition 6.2.1 *Let T be a bounded rectangle in $\mathbb{R}^N$ and let f be a real valued function defined on an open neighbourhood of T.*

Then, if for a fixed $u \in \mathbb{R}$ the following three conditions are satisfied for each face J of T for which $N \in \sigma(J)$, we say that f is suitably regular with respect to T at the level u.

(6.2.4) *f has continuous partial derivatives of up to second order in an open neighbourhood of T.*

(6.2.5) *$f_{|J}$ has no critical points at the level u.*

(6.2.6) *If $J \in \partial_k T$ and $D_J(t)$ denotes the symmetric $(k-1) \times (k-1)$ matrix with elements f_{ij}, $i, j \in \sigma(J) \setminus \{N\}$, then there are no $t \in J$ for which $0 = f(t) - u = \det D_J(t) = f_j(t)$ for all $j \in \sigma(J) \setminus \{N\}$.*

The first two conditions of suitable regularity are meant to ensure that the boundary $\partial A_u = \{t \in T : f(t) = u\}$ of the excursion set be smooth in the interior T° of T and that its intersections with ∂T also be smooth. The last condition is a little more subtle, since it relates to the curvature of ∂A_u both in the interior of T and on its boundary.

The main importance of suitable regularity is that it gives us the following:

Theorem 6.2.2 *Let* $f : \mathbb{R}^N \to \mathbb{R}^1$ *be suitably regular with respect to a bounded rectangle* T *at level* u*. Then the excursion set* $A_u(f, T)$ *is a basic complex.*

The proof of Theorem 6.2.2 is not terribly long, but since it is not crucial to what will follow can be skipped at first reading. The reasoning behind it is all in Figure 6.2.1, and after understanding this you can skip to the examples immediately following the proof without losing much. In any case, this lemma will reappear in a more general form as Lemma 11.1.13.

For those of you remaining with us, we start with a lemma:

Lemma 6.2.3 *Let* $f : \mathbb{R}^N \to \mathbb{R}^1$ *be suitably regular with respect to a bounded rectangle* T *at the level* u*. Then there are only finitely many* $t \in T$ *for which*

$$(6.2.7) \qquad f(t) - u \;=\; f_1(t) \;=\; \ldots \;=\; f_{N-1}(t) \;=\; 0.$$

Proof. To start, let $g = (g^1, \ldots, g^N) : T \to \mathbb{R}^N$ be the function defined by

$$(6.2.8) \;\; g^1(t) \;=\; f(t) - u, \quad g^j(t) \;=\; f_{j-1}(\, t), \quad j = 2, \ldots, N.$$

Let $t \in T$ satisfy (6.2.7). Then, by (6.2.5), $t \notin \partial T$; i.e. t is in the interior of T. We shall show that there is an open neighbourhood of t throughout which no other point satisfies (6.2.7), which we rewrite as $g(t) = 0$. This would imply that the points in T satisfying (6.2.7) are isolated and thus, from the compactness of T, we would have that there are only a finite number of them. This, of course, would prove the lemma.

The inverse mapping theorem[5] implies that such a neighbourhood will exist if the $N \times N$ matrix $(\partial g^i / \partial t_j)$ has a non-zero determinant at t. However, this matrix has the following elements:

$$\frac{\partial g^1}{\partial t_j} \;=\; f_j(t), \qquad \text{for } j = 1, \ldots, N,$$

$$\frac{\partial g^i}{\partial t_j} \;=\; f_{i-1,j}(t), \qquad \text{for } i = 2, \ldots, N, \; j = 1, \ldots, N.$$

Since t satisfies (6.2.7) all elements in the first row of this matrix, other than the N-th, are zero. Expanding the determinant along this row gives

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[5] The inverse mapping theorem goes as follows: Let $U \subset \mathbb{R}^N$ be open and $g = (g^1, \ldots, g^N) : U \to \mathbb{R}^N$ be a function possessing continuous first-order partial derivatives $\partial g^i / \partial t_j, i, j = 1, \ldots, N$. Then if the matrix $(\partial g^i / \partial t_j)$ has a non-zero determinant at some point $t \in U$, there exist open neighbourhoods U_1 and V_1 of t and $g(t)$, respectively, and a function $g^{-1} : V_1 \to U_1$, for which

$$g^{-1}(g(t)) \;=\; t \qquad \text{and} \qquad g(g^{-1}(s)) = s,$$

for all $t \in U_1$ and $s \in V_1$.

us that it is equal to

(6.2.9) $$(-1)^{N-1} f_N(t) \det D(t),$$

where $D(t)$ is as defined at (6.2.6). Since (6.2.7) is satisfied, (6.2.5) and (6.2.6) imply, respectively, that neither $f_N(t)$ nor the determinant of $D(t)$ is zero, which, in view of (6.2.9), is all that is required. □

Proof of Theorem 6.2.2 When $N = 1$ we are dealing throughout with finite collections of intervals, and so the result is trivial.

Now take the case $N = 2$. We need to show how to write A_u as a finite union of basic sets.

Consider the set of points $t \in T$ satisfying either

(6.2.10) $$f(t) - u = f_1(t) = 0$$

or

(6.2.11) $$f(t) - u = f_2(t) = 0.$$

For each such point draw a line containing the point and parallel to either the horizontal or vertical axis, depending, respectively, on whether (6.2.10) or (6.2.11) holds. These lines form a grid over T, and it is easy to check that the connected regions of A_u within each cell of this grid are basic. Furthermore, these sets have intersections which are either straight lines, points, or empty, and Lemma 6.2.3 (applied for the original axes and a 90° rotation of them) guarantees that there are only a finite number of them, so that they form a partition for A_u. (An example of this partitioning procedure is shown in Figure 6.2.1. The dots mark the points where either (6.2.10) or (6.2.11) holds.) This provides the required partition, and we are done.
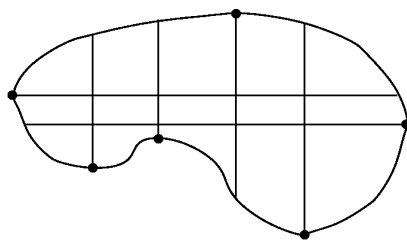

FIGURE 6.2.1. Partitioning excursion sets into basic components

For $N > 2$ essentially the same argument works, using partitioning $(N-1)$ planes passing through points at which

$$f(t) - u = f_1(t) = \cdots = f_{N-1}(t).$$

Lemma 6.2.3 again guarantees the finiteness of the partition. The details are left to you[6].   □

We now attack the problem of obtaining a simple way of computing the Euler characteristic of A_u. As you are about to see, this involves looking at each $A_u \cap J$, $J \in \partial_k T$, $0 \le k \le N$, separately. We start with the simple example $T = I^2$, in which $\partial_2 T = T^o$, $\partial_1 T$ is composed of four open intervals parallel to the axes, and $\partial_0 T$ contains the four vertices of the square. Since this is a particularly simple case, we shall pool $\partial_1 T$ and $\partial_0 T$, and handle them together as ∂T.

Thus, let $f : \mathbb{R}^2 \to \mathbb{R}^1$ be suitably regular with respect to I^2 at the level u. Consider the summation (6.1.10) defining $\varphi(A_u(f, I^2))$; *viz.*

$$(6.2.12) \quad \varphi(A_u) = \sum_{x \in (0,1]} \{\varphi(A_u \cap E_x) - \varphi(A_u \cap E_{x^-})\},$$

where now E_x is simply the straight line $t_2 = x$ and so $n_x \stackrel{\Delta}{=} \varphi(A_u \cap E_x)$ counts the number of distinct intervals in the cross-section $A_u \cap E_x$. The values of x contributing to the sum correspond to values of x where n_x changes.

It is immediate from the continuity of f that contributions to $\varphi(A_u)$ can only occur when E_x is tangential to ∂A_u (Type I contributions) or when $f(0, x) = u$ or $f(1, x) = u$ (Type II contributions). Consider the former case first.

If E_x is tangential to ∂A_u at a point t, then $f_1(t) = 0$. Furthermore, since $f(t) = u$ on ∂A_u, we must have that $f_2(t) \ne 0$, as a consequence of suitable regularity. Thus, in the neighbourhood of such a point and on the curve ∂A_u, t_2 can be expressed as an implicit function of t_1 by

$$f(t_1, g(t_1)) = u.$$

---

[6]You should at least try the three dimensional case, to get a feel for the source of the conditions on the various faces of T in the definition of suitable regularity.

The implicit function theorem[7] gives us

$$\frac{dt_2}{dt_1} = -\frac{f_1(t)}{f_2(t)},$$

so that applying what we have just noted about the tangency of E_x to ∂A_u we have for each contribution of Type I to (6.2.12) that there must be a point $t \in I^2$ satisfying

(6.2.13) $f(t) = u,$

and

(6.2.14) $f_1(t) = 0.$

Furthermore, since the limit in (6.2.12) is one sided, continuity considerations imply that contributing points must also satisfy

(6.2.15) $f_2(t) > 0.$

Conversely, for each point satisfying (6.2.13) to (6.2.15) there is a unit contribution of Type I to $\varphi(A_u)$. Note that there is no contribution of Type I to $\varphi(A_u)$ from points on the boundary of I^2 because of the regularity condition (6.2.5). Thus we have a one–one correspondence between unit contributions of Type I to $\varphi(A_u)$ and points in the interior of I^2 satisfying (6.2.13) to (6.2.15). It is also easy to see that contributions of $+1$ will correspond to points for which $f_{11}(t) < 0$ and contributions of -1 to points for which $f_{11}(t) > 0$. Furthermore, because of (6.2.6) there are no contributing points for which $f_{11}(t) = 0$.

Consider now Type II contributions to $\varphi(A)$, which is best done by looking first at Figure 6.2.2.

The eight partial and complete disks there lead to a total Euler characteristic of 8. The three sets A, B and C are accounted for by Type I contributions, since in each case the above analysis will count $+1$ at their lowest points. We need to account for the remaining five sets, which means

---

[7] The implicit function theorem goes as follows: Let $U \subset \mathbb{R}^N$ be open and $F : U \to \mathbb{R}$ possess continuous first-order partial derivatives. Suppose at $t^* \in U, F(t^*) = u$ and $F_N(\ t^*) \neq 0$. Then the equation

$$F(t_1, \ldots, t_{N-1}, g(t_1, \ldots, t_{N-1})) = u$$

defines an implicit function g which possesses continuous, first-order partial derivatives in some interval containing $(t_1^*, \ldots, t_{N-1}^*)$, and such that $g(t_1^*, \ldots, t_{N-1}^*) = t_N^*$. The partial derivatives of g are given by

$$\frac{\partial g}{\partial t_j} = -\frac{F_j}{F_N}, \qquad \text{for } j = 1, \ldots, N-1.$$

Furthermore, if F is k times differentiable, so is g.
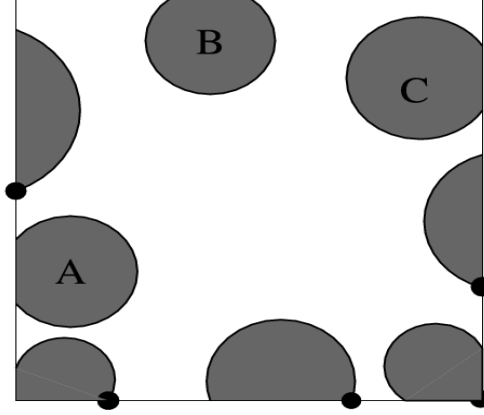
FIGURE 6.2.2. Contributing points to the Euler characteristic

running along ∂I^2 and counting points there. In fact, what we need to do is count $+1$ at the points marked with a $\bullet$. There is never a need to count -1 on the boundary. Note that on the two vertical sections of ∂I^2 we count $+1$ whenever we enter the set (at its intersection with ∂I^2) from below. There is never a contribution from the top side of I^2. For the bottom, we need to count the number of connected components of its intersection with the excursion set, which can be done either on "entering" or "leaving" the set in the positive t_1 direction. We choose the latter, and so must also count a $+1$ if the point $(1,0)$ is covered.

Putting all this together, we have a Type II contribution to $\varphi(A)$ whenever one of the following four sets conditions is satisfied:

(6.2.16)
$$
\begin{cases}
t = (t_1, 0), & f(t) = u, & f_1(t) < 0, \\
t = (0, t_2), & f(t) = u, & f_2(t) > 0, \\
t = (1, t_2), & f(t) = u, & f_2(t) > 0, \\
t = (1, 0), & f(t) > u.
\end{cases}
$$

The above argument has established the following, which in Chapter 9, with completely different techniques and a much more sophisticated language, will be extended to parameter sets in $\mathbb{R}^N$ and on C^2 manifolds with piecewise smooth boundaries:

Theorem 6.2.4 *Let f be suitably regular with respect to I^2 at the level u. Then the Euler characteristic of its excursion set $A_u(f, I^2)$ is given by the number of points t in the interior of I^2 satisfying*

(6.2.17) $f(t) - u \;=\; f_1(t) \;=\; 0, \ \ f_2(t) > 0, \ and \ f_{11}(t) \;<\; 0,$

minus the number of points satisfying

(6.2.18) $f(t) - u \;=\; f_1(t) \;=\; 0, \ \ f_2(t) > 0, \ and \ f_{11}(t) \;>\; 0,$

plus the number of points on the boundary of I^2 satisfying one of the four sets of conditions in (6.2.16).

This is what we have been looking for in a doubly simple case: The ambient dimension was only 2, and the set T a simple square. There is another proof of Theorem 6.2.4 in Section 9.4, built on Morse Theory. There you will also find a generalisation of this result to I^N for all finite N, although the final point set representation is a little different to that of (6.2.16). Morse Theory is also the key to treating far more general parameter spaces. Nevertheless, what we have developed so far, along with some ingenuity, does let us treat some more general cases, for which we give an example. You should be able to fill in the details of the computation by yourself, although some hand waving may be necessary. Here is the example:
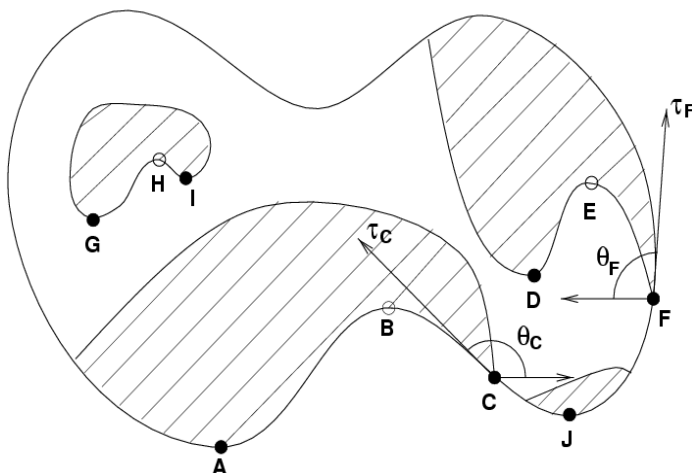

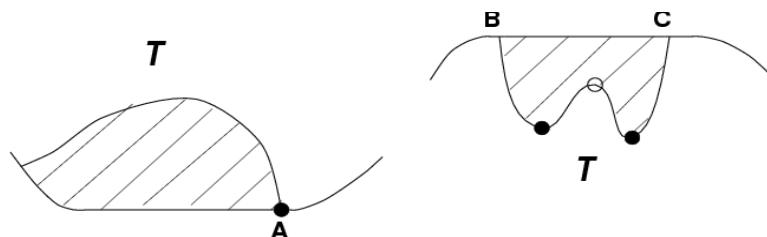FIGURE 6.2.3. Computing the Euler characteristic for a general shape.

FIGURE 6.2.4. Points on a horizontal part of the boundary.

Consider the parameter space T to be the surrounding dumbell shape of Figure 6.2.3. Of the four components of the excursion set (the hatched objects) three intersect ∂T. We already know how to characterise the small

component in the top left: We count a $+1$ for each $\bullet$, -1 for for each $\circ$, and sum. The problem is what to do with the remaining two components. Worsley [165] showed that what needs to be done is to again subtract from the number of points in the interior of T that satisfy (6.2.17) the number satisfying (6.2.18) and then to go along ∂T and a $+1$ for each point marked with a $\bullet$.

  More precisely, the rules for marking are as follows,

(a) If t is in the interior of T, then apply the criteria (6.2.17) and (6.2.18) exactly as before, marking $\bullet$ $(+1)$ and $\circ$ (-1) respectively.

(b) If $t \in \partial T \cap \partial A_u$, and the tangent to ∂T is *not* parallel to the horizontal axis, then let $f_{\mathrm{up}}(t)$ be the derivative of f in the direction of the tangent to ∂T pointing in the positive t_2 direction. (Two such tangent vectors appear as τ_C and τ_F in Figure 6.2.3.) Furthermore, take the derivative of f with respect to t_1 in the direction pointing *into* T. Call this $f_\perp$. (It will equal either f_1 or $-f_1$, depending on whether the angles θ in Figure 6.2.3 from the horizontal to the τ vector develop in a counter-clockwise or clockwise direction, respectively.) Now mark t as a $\bullet$ (and so count as $+1$) if $f_\perp(t) < 0$ and $f_{\mathrm{up}}(t) > 0$. There are no $\circ$ points in this class.

(c) If $t \in \partial T \cap \partial A_u$, and the tangent to ∂T *is* parallel to the horizontal axis, but t is not included in an open interval ∂T which is parallel to this axis, then proceed as in (b), simply defining $f_\perp$ to be f_1 if the tangent is above ∂T and $-f_1$ otherwise.

(d) If $t \in \partial T \cap \partial A_u$ belongs to an open interval of ∂T which is parallel to the horizontal axis (as in Figure 6.2.4) then mark it as a $\bullet$ if T is above ∂T and $f_1(t) < 0$. (Thus, as in Figure 6.2.4, points such as B and C by which A_u "hangs" from ∂T will never be counted.)

(e) Finally, if $t \in \partial T \cap A_u$, has not already been marked, and coincides with one of the points that contribute to the Euler characteristic of T itself (e.g. A, B and J in Figure 6.2.3) then mark it exactly as it was marked in computing $\varphi(T)$.

All told, this can be summarised as

Theorem 6.2.5 (Worsley [165]) *Let $T \subset \mathbb{R}^2$ be compact with boundary ∂T that is twice differentiable everywhere except, at most, at a finite number of points. Let f be suitably regular for T at the level u. Let $\chi(A_u(f,T))$ be the number of points in the interior of T satisfying (6.2.17) minus the number satisfying (6.2.18).*

Denote the number of points satisfying (b)–(d) above as $\chi_{\partial T}$, and denote the sum of the contributions to $\varphi(T)$ of those points described in (e) by $\varphi_{\partial T}$. Then

(6.2.19) $$\varphi(A) = \chi(A) + \chi_{\partial T} + \varphi_{\partial T}.$$

Theorem 6.2.5 can be extended to three dimensions (also in [165]). In principle, it is not too hard to guess what has to be done in higher dimensions as well. Determining whether a point in the interior of T and on ∂A_u contributes a $+1$ or -1 will depend on the curvature of ∂A_u, while if $t \in \partial T$ both this curvature and that of ∂T will have rôles to play.

It is clear that these kinds of arguments are going to get rather messy very quickly, and a different approach is advisable. This is provided via the critical point theory of Differential Topology, which we shall develop in Chapter 9. However, before doing this we want to develop some more geometry in the still relatively simple scenario of Euclidean space and describe how all of this relates to random fields.

6.3   Intrinsic volumes

The Euler characteristic of Section 6.1 arose as the unique additive functional (cf. (6.1.2)) that assigned the value 1 to sets homotopic to a ball, and 0 to the empty set. It turned out to be integer valued, although we did not demand this in the beginning and has an interpretation in terms of 'counting' the various topological components of a set. But there is more to life than mere counting and one would also like to be able to say things about the volume of sets, the surface area of their boundaries, their curvatures, etc. In this vien, it is natural to look for a class of N additional, position and rotation invariant, non-negative, functionals $\{\mathcal{L}_j\}_{j=1}^N$, which are also additive in the sense of (6.1.2) and scale with dimensionality in the sense that

$$(6.3.1) \qquad \mathcal{L}_j(\lambda A) \; = \; \lambda^j \mathcal{L}_j(A), \qquad \lambda > 0,$$

where $\lambda A \stackrel{\Delta}{=} \{t : t = \lambda s, \ s \in A\}$.

Such functionals exist, and together with $\mathcal{L}_0 \stackrel{\Delta}{=} \varphi$, the Euler characteristic itself, make up what are known as the *intrinsic volumes* defined on a suitable class of sets A. The reason why the intrinsic volumes are of crucial importance to us will become clear when we get to the discussion following Theorem 6.3.1, which shows that many probabilistic computations for random fields are intimately related to them. They can be defined in a number of ways, one of which is a consequence of *Steiner's formula* [86, 130], which, for *convex*[8] subsets of $\mathbb{R}^N$, goes as follows:

For $A \subset \mathbb{R}^N$ and $\rho > 0$, let

$$(6.3.2) \qquad \mathrm{Tube}(A, \rho) \; = \; \{x \in \mathbb{R}^N : d(x, A) \leq \rho\}$$

---

[8]Moving from basic complexes down to the very special case of convex sets is a severe restriction in terms of what we need, and we shall lift this restriction soon. Nevertheless, convex sets are a comfortable scenario in which to first meet intrinsic volumes.

be the 'tube' of radius ρ, or 'ρ-tube', around A, where

$$d(x, A) \;\triangleq\; \inf_{y \in A} |x - y|$$

is the usual Euclidean distance from the point x to the set A. An example is given in Figure 6.3.1, in which A is the inner triangle and $\mathrm{Tube}(A, \rho)$ the larger triangular object with rounded-off corners.

With λ_N denoting, as usual, Lebesgue measure in $\mathbb{R}^N$, Steiner's formula states[9] that $\lambda_N(\mathrm{Tube}(A, \rho))$ has a finite expansion in powers of ρ. In particular,

$$(6.3.3) \qquad \lambda_N\left(\mathrm{Tube}(A, \rho)\right) \;=\; \sum_{j=0}^{N} \omega_{N-j} \rho^{N-j} \mathcal{L}_j(A),$$

where

$$(6.3.4) \qquad \omega_j = \lambda_j(B(0,1)) = \frac{\pi^{j/2}}{\Gamma\!\left(\frac{j}{2} + 1\right)} = \frac{s_j}{j}$$

is the volume of the unit ball in $\mathbb{R}^j$. (Recall that s_j was the corresponding surface area. cf. (5.7.5).)

We shall see a proof of (6.3.3) later on in Chapter 10 in a far more general scenario, but it is easy to see from Figure 6.3.1 from where the result comes.
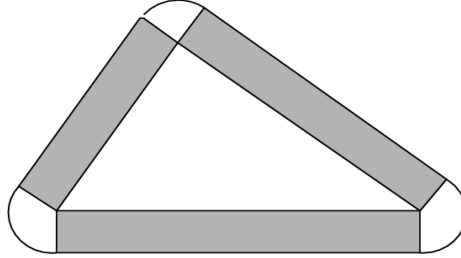
FIGURE 6.3.1. The tube around a triangle.

To find the area (i.e. 2-dimensional volume) of the enlarged triangle, one needs only to sum three terms:

- The area of the original, inner triangle.

- The area of the three rectangles. Note that this is the perimeter (i.e. 'surface area') of the triangle multiplied by ρ.

---

[9]There is a more general version of Steiner's formula for the case in which T and its tube are embedded in $\mathbb{R}^{N'}$, where $N' > N = \dim(A)$. In that case (6.3.2) still holds, but with N replaced by N'. See Theorem 10.5.6 for more details in the manifold setting.

- The area of the three corner sectors. Note that the union of these sectors will always give a disk of Euler characteristic 1 and radius ρ.

In other words,

$$\text{area}\left(\text{Tube}(A, \rho)\right) \;=\; \pi \rho^2 \varphi(A) \;+\; \rho\,\text{perimeter}(A) \;+\; \text{area}(A).$$

Comparing this to (6.3.3) it now takes only a little thought to guess what the intrinsic volumes must measure. If the ambient space is $\mathbb{R}^2$, then $\mathcal{L}_2$ measures area, $\mathcal{L}_1$ measures boundary length, while $\mathcal{L}_0$ gives the Euler characteristic. In $\mathbb{R}^3$, $\mathcal{L}_3$ that measures volume, $\mathcal{L}_2$ measures surface area, $\mathcal{L}_1$ is a measure of cross-sectional diameter, and $\mathcal{L}_0$ is again the Euler characteristic. In higher dimensions, it takes some imagination, but $\mathcal{L}_N$ and $\mathcal{L}_{N-1}$ are readily seen to measure volume and surface area, while $\mathcal{L}_0$ is always the Euler characteristic. Precisely why this happens, how it involves the curvature of the set and its boundary, and what happens in less familiar spaces forms much of the content of Section 7.6 and is treated again, in fuller detail, in Chapter 10.

In the meantime, you can try checking for yourself, directly from (6.3.3) and a little first-principles geometry that for a N-dimensional cube of side length T the intrinsic volumes are given by

$$(6.3.5) \qquad \mathcal{L}_j\left([0,T]^N\right) \;=\; \binom{N}{j} T^j.$$

It is also not hard to see that for N-dimensional rectangles

$$(6.3.6) \qquad \mathcal{L}_j\left(\prod_{i=1}^{N}[0,T_i]\right) \;=\; \sum T_{i_1}\ldots T_{i_j},$$

where the sum is taken over the $\binom{N}{j}$ different choices of subscripts $i_1,\ldots,i_j$.

For handling $B^N(T)$, the ball of radius T, it useful to go beyond first principles. Noting that $\text{Tube}(B^N(T),\rho) = B^N(T+\rho)$, we have

$$\begin{aligned}
\lambda_N\left(\text{Tube}\left(B^N(T),\rho\right)\right) \;&=\; (T+\rho)^N \omega_N \\
&=\; \sum_{j=0}^{N} \binom{N}{j} T^j \rho^{N-j} \omega_N \\
&=\; \sum_{j=0}^{N} \omega_{N-j}\rho^{N-j}\binom{N}{j}T^j \frac{\omega_N}{\omega_{N-j}}.
\end{aligned}$$

Comparing this to Steiner's formula (6.3.3) it is immediate that

$$(6.3.7) \qquad \mathcal{L}_j\left(B^N(T)\right) \;=\; \binom{N}{j} T^j \frac{\omega_N}{\omega_{N-j}}.$$

For $S^{N-1}(T)$ the sphere of radius T in $\mathbb{R}^N$, a similar argument, using the fact that

$$\text{Tube}\left(S^{N-1}(T), \rho\right) \;=\; B^N(T+\rho) - B^N(T-\rho)$$

yields

$$(6.3.8) \quad \mathcal{L}_j\left(S^{N-1}(T)\right) \;=\; 2\binom{N}{j}\frac{\omega_N}{\omega_{N-j}}T^j \;=\; 2\binom{N-1}{j}\frac{s_N}{s_{N-j}}T^j$$

if $N - 1 - j$ is even, and 0 otherwise.

Further examples can be found, for example, in [130].

A useful normalization of the intrinsic volumes are the so-called *Minkowski functionals*, defined as

$$(6.3.9) \qquad\qquad \mathcal{M}_j(A) = (j!\,\omega_j)\,\mathcal{L}_{N-j}(A)$$

so that, when expressed in terms of the $\mathcal{M}_j$, Steiner's formula now reads like a Taylor series expansion

$$(6.3.10) \qquad\qquad \lambda_N\left(\text{Tube}(A, \rho)\right) \;=\; \sum_{j=0}^{N}\frac{\rho^j}{j!}\mathcal{M}_j(A).$$

It is an important and rather deep fact, due to Weyl in the manifold setting [158] (see also [69, 86]) that the $\mathcal{L}_j$ are independent of the ambient space in which sets sit. Because of the reversed numbering system and the choice of constants, this is not true of the Minkowski functionals[10].

There is another way to define intrinsic volumes that works just as well for our main scenario of basic complexes as it does for convex sets, based on the idea of *kinematic density*. Let $G_N = \mathbb{R}^N \times O(N)$ be the group of rigid motions on $\mathbb{R}^N$, and equip it with Haar measure, normalized to be Lebesgue measure on $\mathbb{R}^N$ and the invariant probability measure on $O(N)$. A formula of Hadwiger states that

$$(6.3.11) \qquad \mathcal{L}_j(A) \;=\; \begin{bmatrix} N \\ j \end{bmatrix} \int_{G_N} \varphi\left(A \cap gE_j\right) \mu_N(dg),$$

where E_j is any j-dimensional affine subspace of $\mathbb{R}^N$ and

$$(6.3.12) \qquad \begin{bmatrix} N \\ j \end{bmatrix} = \frac{[N]!}{[N-j]![j]!} = \binom{N}{j}\frac{\omega_N}{\omega_{N-j}\,\omega_j},$$

---

[10] To see why the $\mathcal{M}_j$ are dependent on the ambient space, let i_{NM} be the standard inclusion of $\mathbb{R}^N$ into $\mathbb{R}^M$, $M \geq N$, defined by $i_{NM}(x) = (x_1, \ldots, x_N, 0, \ldots, 0) \in \mathbb{R}^M$ for $x \in \mathbb{R}^N$. Consider $M > N$, and note that the polynomials $\lambda_M(\text{Tube}(A, \rho))$ and $\lambda_M(\text{Tube}(i_{NM}(A), \rho))$, lead to different geometric interpretations. For example, for a curve C in $\mathbb{R}^2$, $\mathcal{M}_1(C)$ will be proportional to the arc length of C and $\mathcal{M}_2(C) = 0$, while $\mathcal{M}_2(i_{2,3}(C))$, rather than $\mathcal{M}_1(i_{2,3}(C))$, is proportional to arc length.

where $[N]! = N! \, \omega_N$, and φ is our old friend, the Euler characteristic. This representation is important, and we shall return to it later.

Note, however, that although over the class of convex sets the two definitions of intrinsic volumes are equivalent, Steiner's formula no longer holds for these intrinsic volumes over the class of basic complexes, or even over the convex ring. For a counterexample, take the union of a vertical and horizontal line as in Figure 6.3.2 and consider the tube around it. Blindly applying Steiner's formula, one finds that there are two problems: The area of the quarter disk on the lower left corner has been forgotten, and the area marked with a black square has been double counted. Since these two area are different, they do not cancel to give the right volume for the tube.

FIGURE 6.3.2. A tube around the the letter L.

The way around this will be to develop a somewhat different notion, that of Lipschitz-Killing curvatures, which we shall do in Chapters 7 and 8 in the setting on manifolds, and for which we shall develop in Chapter 10 an extension of Steiner's formula. One of the key differences between Lipschitz-Killing curvatures and the intrinsic volumes that we have looked at so far will be that the former will also be allowed to take negative values. This will allow, for example, for the subtraction of the area of the double counted square in Figure 6.3.2.

The time has now come to explain what all of the above has to do with random fields. For this we need one more result from Integral Geometry, due to Hadwiger [70], which also seems to be the only place to find a proof. However, unless you have a weakness for classical German, you should turn to Schneider [?] to read more about it. In Klain and Rota [86], a proof is given for continuous, invariant functionals on the convex ring, which, as noted previously, is a subset of the basic complexes. Zahle [173] has the most general result in terms of the classes of sets covered, although with continuity replacing monotonicity among the conditions of the theorem.

Theorem 6.3.1 *Let ψ be a real valued function on basic complexes in $\mathbb{R}^N$, invariant under rigid motions, additive (in the sense of (6.1.2)) and monotone, in the sense that $A \subseteq B \Rightarrow \psi(A) \leq \psi(B)$. Then*

$$(6.3.13) \qquad \psi(A) \; = \; \sum_{j=0}^{N} c_j \mathcal{L}_j(A),$$

where $c_0, \ldots, c_N$ are non-negative (ψ-dependent) constants.

Now take an isotropic field f on $\mathbb{R}^N$ and consider the set-indexed functional

$$(6.3.14) \qquad \psi(A) \; \triangleq \; \mathbb{P}\left\{\sup_{t \in A} f(t) > u\right\}.$$

Then ψ is clearly monotone and rigid-motion invariant. Unfortunately, it is not quite additive, since even if A and B are disjoint,

$$(6.3.15) \;\; \psi(A \cup B) \;\; = \;\; \psi(A) \; + \psi(B)$$
$$- \mathbb{P}\left\{\left(\sup_{t \in A} f(t) > u\right) \cap \left(\sup_{t \in B} f(t) > u\right)\right\}.$$

However, if u is large then we expect each of the terms in (6.3.15) to be small, with the last term on the right hand side to be of smaller order than each of the others[11]. Arguing heuristically, it is therefore not unreasonable to expect that there might be an invariant, additive *and* monotone functional $\widetilde{\psi}_u$ for which

$$(6.3.16) \qquad \mathbb{P}\left\{\sup_{t \in A} f(t) > u\right\} \;\; = \;\; \widetilde{\psi}_u(A) \;\; + \;\; o\left(\widetilde{\psi}_u(A)\right),$$

in which case, by Theorem 6.3.1, we would be able to conclude that

$$(6.3.17) \;\; \mathbb{P}\left\{\sup_{t \in A} f(t) > u\right\} \;\; = \;\; \sum_{j=0}^{N} c_j(u)\, \mathcal{L}_j(A) \;\; + \;\; \text{lower order terms},$$

for some functions $c_j(u)$.

The fact that such a function does indeed exist, and is given by

$$\widetilde{\psi}_u \;\; = \;\; \mathbb{E}\left\{\varphi\left(A_u(f, A)\right)\right\},$$

where φ is the Euler characteristic and A_u an excursion set, is one of the main punch lines of this book and of the last few years of research in Gaussian random fields. Proving this, in wide generality, and computing the coefficients c_j in (6.3.17) as functions of u is what much of Part II of this book is about.

If you are interested mainly in nice Euclidean parameter spaces, and do not need to see the details of all the proofs, you can now comfortably skip the rest of Part II, with the exception of Section 9.4 which gives a version of

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[11] This would happen, for example, if A and B were sufficiently distant for the values of f in A and B to be close to independent. It turns out that, at least for Gaussian f, these heuristics are true even if A and B are close, as long as u is large enough. We shall not treat these results, but you can find examples in [17] and [90].

Theorem 6.2.4 in general dimensions. The same is true if you have a solid background in Differential Geometry, even if you plan to follow all the proofs later on. You can return later when the need to confirm notation arises.

7
Differential Geometry

As we have said more than once, this chapter is intended to serve as a rapid and non-comprehensive introduction to Differential Geometry, basically in the format of a 'glossary of terms'. They will be familiar to those who have studied Differential Geometry, and hopefully informative enough to allow the uninitiated[1] to follow the calculations in later chapters. However, to go beyond merely following and develop a real understanding of what is going on, it will be necessary to learn the material from its classical sources[2].

Essentially all that we have to say is 'well known', in the sense that it appears in textbooks of Differential Geometry. Thus, the reader familiar with these books will be able to skim the remainder of this chapter, needing only to pick up our notation and emphases.

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[1] For first timers we have added numerous footnotes and simple examples along the way that are meant to help them through the general theory. In general, we shall be satisfied with the exposition as long as we have not achieved the double failure of both boring the expert and bamboozling the novice. We also apologize to the experts for attempting the impossible: to reduce their beautiful subject matter to a 'glossary'.

[2] For the record, the books that we found most useful were Boothby [30], Jost [83], Millman and Parker [107], Morita [109] and O'Neill [112]. The two recent books by Lee [93, 94] stand out from the pack as being particularly comprehensive and easy to read, and we highly recommend them as the right place to start.

7.1   Manifolds

A differentiable manifold is a mathematical generalisation, or abstraction, of objects such as curves and surfaces in $\mathbb{R}^N$. Intuitively, it is a smooth set with a locally defined, Euclidean, coordinate system.

Manifolds without boundary

We call M a *topological N-manifold* if M is a locally compact Hausdorff space such that, for every $t \in M$, there exists an open $U \subset M$ containing t, an open $\tilde{U} \subset \mathbb{R}^N$, and a homeomorphism $\varphi : U \to \tilde{U}$.

A *(coordinate) chart* on M is a pair (φ, U), where, as above, $U \subset M$ is open and $\varphi : U \to \varphi(U) \subset \mathbb{R}^N$ is a homeomorphism. A collection $\{\varphi_i : U_i \to \mathbb{R}^N\}_{i \in I}$ of charts is called C^k *compatible* if

$$(7.1.1) \qquad \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

is a C^k diffeomorphism[3] for every $i, j \in I$ for which $U_i \cap U_j \neq \emptyset$.

If a collection of charts gives a covering of M – i.e. $\bigcup_{i \in I} U_i = M$ – then it is called an (C^k) *atlas* for M. An atlas is called *maximal* if it is not contained in any strictly larger atlas with homeomorphisms satisfying (7.1.1). A topological N manifold, together with a C^k maximal atlas, is called a C^k *differential manifold*. The maximal atlas is usually referred to as the *differential structure* of M.

Manifolds with boundary

If some of the U_i are subsets of the 'half-space' $\mathbb{H}^N \triangleq \mathbb{R}^{N-1} \times \mathbb{R}_+$ in $\mathbb{R}^N$, then we talk about a *manifold with boundary*, rather than a simple manifold. A manifold with boundary can be thought of as a disjoint union of two manifolds: ∂M, its boundary, an $N - 1$ dimensional manifold, and M°, its interior, an N-dimensional manifold.

Continuous and differentiable functions on manifolds

The notions of continuous and differentiable functions on a C^k manifold M are straightforward. A function $f : M \to \mathbb{R}$ is said to be of class C^k if $f \circ \varphi^{-1}$ is of class C^k, in the usual Euclidean sense, for every chart in the atlas.

Tangent spaces

For a manifold M embedded in $\mathbb{R}^N$, such as a curve or a surface, it is straightforward to envision what is meant by a *tangent vector* at a point $t \in M$. It is no more than a vector with origin at t, sitting in the tangent plane to M at t. Given such a vector, v, one can differentiate functions

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[3]i.e. Both $\varphi_i \circ \varphi_j^{-1}$ and its inverse $\varphi_j \circ \varphi_i^{-1}$ are k times differentiable as functions from subsets of $\mathbb{R}^N$ to subsets of $\mathbb{R}^N$.

$f : M \to \mathbb{R}$ along the direction v. Thus, to each v there corresponds a local derivative. For abstract manifolds, the basic notion is not that of a vector sitting in a tangent plane, but rather that of a differential operator.

To see how this works, we start with the simplest case, in which $M = \mathbb{R}^N$ and everything reduces to little more than renaming familiar objects. Here we can manage with the atlas containing the single chart (M, i_{NN}), where i_{NN} is the inclusion map. We change notation after the inclusion, writing $x = i_{NN}(t)$ and $M' = i_{NN}(M) \ (= \mathbb{R}^N)$. To every vector X_x with origin $x \in M'$, we can assign a linear map from $C^1(M')$ to $\mathbb{R}$ as follows:

If $f \in C^1(M)$ and X_x is a vector of the form $X_x = x + \sum_{i=1}^{N} a_i e_i$, where $\{e_i\}_{1 \le i \le N}$ is the standard basis for $\mathbb{R}^N$, we define the first order differential operator[4] X_x by its action on f:

$$(7.1.2) \qquad\qquad X_x f \;=\; \sum_{i=1}^{N} a_i \left.\frac{\partial f}{\partial x_i}\right|_x .$$

It is elementary calculus that X_x satisfies the product rule $X_x(fg) = f X_x g + g X_x f$, and that for any two functions f, g that agree on a neighbourhood of x we have $X_x f = X_x g$.

For each $x \in U$, identifying $\left.\frac{\partial}{\partial x_i}\right|_x$ with e_i, we see that

$$\left\{ \left.\frac{\partial}{\partial x_i}\right|_x \right\}_{1 \le i \le N}$$

forms a basis for a N-dimensional space of first-order differential operators, which we call the *tangent space* at x and denote by $T_x \mathbb{R}^N$. Returning to M, and identifying $T_t \mathbb{R}^N$ with $T_x \mathbb{R}^N$, we have now defined the tangent space here as well.

Before turning to the case of an abstract manifold M, we need to be a little more precise about the term 'first order differential operator' used above. While it may be obvious what is meant in $\mathbb{R}^N$, we need to be a little more careful in general. Loosely speaking, given an $x \in \mathbb{R}^N$ and a neighbourhood U of x a *first-order differential operator* is a linear operator on $C^1(U)$ that satisfies the product rule and depends only on the derivatives of a function when evaluated at x. The looseness in the term "depends only on the derivatives of a function when evaluated at x" can be made precise by requiring that the operator is actually defined on the *germ*[5] of C^1 functions at x. As this definition is local, it extends easily to manifolds.

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[4] Hopefully, the standard usage of X_x to denote both a vector and a differential operator will not cause too much confusion. In this simple case, they clearly amount to essentially the same thing. In general, they are the same by definition.

[5] The germs of f at x are the (equivalence) class of functions g for which $f \equiv g$ over some open neighbourhood U of x. All local properties of f at x depend only on the germs to which f belongs.

Thus, choosing $t \in M$ and a chart (U, φ) around t, we can define similarly a first order differential operator at $t \in M$ as a a linear operator (on the germs of C^1 functions) that satisfies the product rule.

Formally, if $x \in \varphi(U) \subset \mathbb{R}^N$ for some chart (U, φ), then we can lift the basis of $T_x \mathbb{R}^N$ (built as in the Euclidean case) to the point $t = \varphi^{-1}(x)$ via $\varphi_{*,t}^{-1}$, the so-called differential or *push-forward* of φ^{-1} at t. The map $\varphi_{*,t}^{-1}$ is a linear map which maps the first order differential operators $(\partial/\partial x_i)|_{\varphi(t)}$, $1 \leq i \leq N$, acting on $C^1(\varphi(U))$ to first order differential operators on $C^1(U)$ and we define it by

$$(7.1.3) \qquad \left(\varphi_{*,t}^{-1}(X_{\varphi(t)}) \right) f \;=\; X_{\varphi(t)} \left(f \circ \varphi^{-1} \right)$$

for any $f \in C^1(M)$. Since $\varphi_{*,t}^{-1}$ is linear it follows that the set

$$(7.1.4) \qquad \left\{ \varphi_{*,t}^{-1} \left(\frac{\partial}{\partial x_i} \Big|_{\varphi(t)} \right) \right\}_{1 \leq i \leq N}$$

is linearly independent. We define the space it spans to be $T_t M$, the *tangent space of M at t*. Its elements X_t, while being differential operators, are called the *tangent vectors* at t.

With some abuse of notation, in a chart (U, φ) with a coordinate system $(x_1, \ldots, x_N)$ for $\varphi(U)$ the basis (7.1.4) is usually written as

$$(7.1.5) \qquad \left\{ \frac{\partial}{\partial x_i} \Big|_t \right\}_{1 \leq i \leq N},$$

and is referred to as the *natural basis* for $T_t M$ in the chart (U, φ).

Chain rule: the push-forward

The chain rule of vector calculus is a formula for differentiating the composition $h \circ g$ for $g \in C^1(\mathbb{R}^k; \mathbb{R}^j)$, $h \in C^1(\mathbb{R}^j; \mathbb{R}^l)$. In differential geometry, the chain rule is expressed through the push-forward of differentiable functions. Specifically, given two manifolds M and N and any $g \in C^1(M; N)^6$ we define its *push-forward* g_* by defining it in any charts (U, φ) and (V, ψ) of M and N, such that $g(U) \cap V \neq \emptyset$. We define g_* by

$$(g_* X_t) h \;=\; X_t(h \circ g)$$

for any $h \in C^1(N)$.

Vector fields

As each $X_t \in T_t M$ is a linear map on $C^1(M)$ satisfying the product rule,

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[6]$C^k(M; N)$, the space of 'k times differentiable functions from M to N', is defined analogously to $C^1(M)$. Thus $f \in C^k(M; N)$ if, for all $t \in M$, there is a chart (U_M, φ_M) for M, a neighbourhood V of t with $V \subset U_M$, such that $f(V) \subset U_N$ for some chart (U_N, φ_N) for N, for which the composite map $\varphi_N \circ f \circ (\varphi_M)^{-1} : \varphi_M(V) \to \varphi_N(f(V))$ is C^k in the usual, Euclidean, sense.

we can construct a first-order differential operator X, called a *vector field*, that takes C^k functions $(k \geq 1)$ to real-valued functions, as follows:

$$(Xf)_t \; = \; X_t f \qquad \text{for some } X_t \in T_t M.$$

In other words, a vector field is a map that assigns, to each $t \in M$, a tangent vector $X_t \in T_t M$. Thus, in a chart (U, φ) with coordinates $(x_1, \ldots, x_N)$, a vector field can be represented (cf. (7.1.5)) as

(7.1.6)
$$X_t \; = \; \sum_{i=1}^{N} a_i(t) \left. \frac{\partial}{\partial x_i} \right|_t .$$

If the a_i are C^k functions, we can talk about C^k vector fields. Note that, for $j \geq 1$, a C^k vector field maps $C^j(M)$ to $C^{\min(j-1,k)}(M)$.

Vector bundles
A C^l *vector bundle* is a triple (E, M, F), along with a map $\pi : E \to M$, where E and M are, respectively, $N + q$ and N dimensional manifolds of class at least C^l and F is a q dimensional vector space. The manifold E is locally a product. That is, every $t \in M$ has a neighbourhood U such that there is a homeomorphism $\varphi_U : U \times F \to \pi^{-1}(U)$ satisfying $(\pi \circ \varphi_U)(t, f_U) = t$, for all $f_U \in F$. Furthermore, for any two such overlapping neighbourhoods U, V with $t \in U \cap V$, we require that

$$\varphi_U(t, f_U) \; = \; \varphi_V(t, f_V) \iff f_U \; = \; g_{UV}(t)\,(f_V),$$

where $g_{UV}(t) : F \to F$ is a non-degenerate linear transformation. The functions $g_{UV} : U \cap V \to GL(F)$ are called the *transition functions* of the bundle (E, M, F). The manifold M is called the *base manifold* and the vector space $\pi^{-1}(t)$, which is isomorphic to F (as a vector space), is called the *fiber* over t. Usually, it is clear from the context what M and F are, so we refer to the bundle as E. A C^j, $j \leq l$, *section* of a vector bundle is a C^j map $s : M \to E$ such that $\pi \circ s = \text{id}_M$, where id_M is the identity map on M. In other words, if one thinks of a vector bundle as assigning, to each point of $t \in M$, the entire vector space $\pi^{-1}(t)$, then a C^j section is a rule for choosing from this space in a smooth (C^j) fashion.

In a similar fashion, one can define *fiber bundles* when the fibers F are not vector spaces, although we shall not go through such a construction yet. Two such fiber bundles which will be of interest to us are the *sphere bundle* $S(M)$ and the *orthonormal frame bundle* $\mathcal{O}(M)$ of a Riemannian manifold (M, g). These will be described below once we have the definition of a Riemannian manifold.

Tangent bundles
Perhaps the most natural example of a vector bundle is obtained from the

collection of all tangent spaces $T_t M$, $t \in M$. This can be parametrized in a natural way into a manifold, $T(M)$, called the *tangent bundle*,

In a chart (U, φ), any tangent vector X_t, $t \in U$, can be represented as in (7.1.6), so the set of all tangent vectors at points $t \in U$ is a $2N$-dimensional space, locally parametrized by $\tilde{\varphi}(X_t) = (x_1(t), \ldots, x_N(t); a_1(t), \ldots, a_N(t))$. Call this E_t, and call the projection of E_t onto its last N coordinates F_t. Denote the union (over $t \in M$) of the E_t by E, and of the F_t by F, and define the natural projection $\pi : E \to M$ given by $\pi(X_t) = t$, for $X_t \in E_t$. The triple (E, M, F), along with π, defines a vector bundle, which we call the *tangent bundle* of M and denote by $T(M)$.

The tangent bundle of a manifold is itself a manifold, and as such can be given a differential structure in the same way that we did for M. To see this, suppose M is a C^k manifold and note that an atlas $\{U_i, \varphi_i\}_{i \in I}$ on M determines a covering on $T(M)$, the charts $\{\pi^{-1}(U_i), \tilde{\varphi}_i\}_{i \in I}$ of which determine a topology on $T(M)$, the smallest topology such that $\tilde{\varphi}_{|\pi^{-1}(U)}$ are homeomorphisms. In any two charts (U, φ) with coordinates $(x_1, \ldots, x_n)$ and (V, ψ) with coordinates $(y_1, \ldots, y_n)$ with $U \cap V \neq \emptyset$, a point $X_t \in T(M)$ is represented by

$$
\begin{aligned}
X_t &= (x_1(t), \ldots, x_N(t); a_1(t), \ldots, a_N(t)) \\
&= (y_1(t), \ldots, y_N(t); b_1(t), \ldots, b_N(t)),
\end{aligned}
$$

where we have the relation

$$
(7.1.7) \qquad b_i(t) = \sum_{j=1}^{N} a_j(t) \frac{\partial y_i}{\partial x_j},
$$

and $\partial y_i / \partial x_j$ is the first order partial derivative of the real valued function y_i with respect to xJj.

Since $\phi_U \circ \phi_V^{-1}$ is a C^k diffeomorphism, we have that $\tilde{\phi}_U \circ \tilde{\phi}_V^{-1}$ is a C^{k-1} diffeomorphism, having lost one order of differentiation because of the partial derivatives in (7.1.7). In summary, the atlas $\{U_i, \phi_i\}_{i \in I}$ determines a C^{k-1} differential structure on $T(M)$ as claimed.

7.2   Tensor calculus

Tensors are basically linear operators on vector spaces. They have a multitude of uses, but there are essentially only two main approaches to setting up the (somewhat heavy) definitions and notations that go along with them. When tensors appear in Applied Mathematics or Physics, the definitions involve high dimensional arrays that transform (as the ambient space is transformed) according to certain definite rules. The approach we shall adopt, however, will be the more modern Differential Geometric one, in

which the transformation rules result from an underlying algebraic structure via which tensors are defined. This approach is neater and serves two of our purposes: We can fit everything we need into only six pages, and the approach is essentially coordinate free[7]. The latter is of obvious importance for the manifold setting. The downside of this approach is that if this is the first time you see this material, it is very easy to get lost among the trees without seeing the forest. Thus, since one of our main uses of tensors will be for volume (and, later, curvature) computations on manifolds, we shall accompany the definitions with a series of footnotes showing how all of this relates to simple volume computations in Euclidean space. Since manifolds are locally Euclidean, this might help a first-timer get some feeling for what is going on.

Tensors and exterior algebras

Given an N-dimensional vector space V and its dual V^*, a map ω is called a *tensor* of order (n, m) if

$$\omega \in L(\underbrace{V \oplus \cdots \oplus V}_{n \text{ times}} \oplus \underbrace{V^* \oplus \cdots \oplus V^*}_{m \text{ times}}; \mathbb{R})$$

where $L(E; F)$ denotes the set of (multi) linear[8] mappings between two vector spaces E and F. We denote the space of tensors of order (n, m) by $\mathcal{T}_m^n$, where n is said to be the *covariant* order and m the *contravariant* order[9].

Let $\mathcal{T}(V) = \oplus_{i,j=1}^\infty \mathcal{T}_i^j(V)$ be the direct sum of all the tensor spaces $\mathcal{T}_j^i(V)$. The bilinear associative *tensor product* $\otimes : \mathcal{T}(V) \times \mathcal{T}(V) \to \mathcal{T}(V)$, is defined on $\mathcal{T}_j^i \times \mathcal{T}_l^k$ as

$$(\alpha \otimes \beta) \left(v_1, \ldots, v_{i+k}, v_1^*, \ldots, v_{j+l}^*\right)$$
$$\overset{\Delta}{=} \alpha \left(v_1, \ldots, v_i, v_1^*, \ldots, v_j^*\right) \times \beta \left(v_{i+1}, \ldots, v_{i+k}, v_{j+1}^*, \ldots, v_{j+l}^*\right).$$

The algebra $(\mathcal{T}(V), \otimes)$ can be split into two subalgebras, the covariant and contravariant tensors $\mathcal{T}^*(V) = \oplus_{i=1}^\infty \mathcal{T}_0^i(V)$ and $\mathcal{T}_*(V) = \oplus_{i=1}^\infty \mathcal{T}_i^0(V)$.

---

[7]Ultimately, however, we shall need to use the array notation when we come to handling specific examples. It appears, for example, in the connection forms of (7.3.9) and the specific computations of Section 7.7.

[8]A multi-linear linear mapping is linear, separately, in each variable. In general, we shall not bother with the prefix.

[9]For a basic example, let $V = \mathbb{R}^3$. There are three, very elementary, covariant tensors of order 1 operating on vectors $v = (v_1, v_2, v_3)$. These are $\theta_1(v) = v_1$, $\theta_2(v) = v_2$ and $\theta_3(v) = v_3$. Thus θ_j measures the 'length' of v in direction j, or, equivalently, the 'length' of the projection of v onto the j-th axis, where 'length' is signed. The fact that the length may be signed is crucial to all that follows. Not also that, since $\mathbb{R}^3$ is its own dual, we could also treat the same θ_i as contravariant tensors of order 1.

Alternating and symmetric covariant tensors

A covariant tensor γ of order k (a $(k,0)$-tensor) is said to be *alternating* if

$$\gamma(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \;=\; \varepsilon_\sigma \gamma(v_1, \ldots, v_k) \qquad \forall\, \sigma \in S(k),$$

where $S(k)$ is the symmetric group of permutations of k letters and ε_σ is the sign of the permutation σ. It is called *symmetric* if

$$\gamma(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \;=\; \gamma(v_1, \ldots, v_k) \qquad \forall\, \sigma \in S(k).$$

For example, computing the determinant of the matrix formed from N-vectors gives an alternating covariant tensor of order N on $\mathbb{R}^N$.

For $k \geq 0$, we denote by $\Lambda^k(V)$ (respectively, $\mathrm{Sym}(\mathcal{T}_0^k(V))$) the space of alternating (symmetric) covariant k-tensors on V, and by

$$\overset{*}{\bigwedge}(V) \;=\; \oplus_{k=0}^\infty \Lambda^k(V), \qquad \mathrm{Sym}^*(V) = \oplus_{k=0}^\infty \mathrm{Sym}(\mathcal{T}_0^k(V))$$

their direct sums. If $k = 0$, then both $\Lambda^0(V)$ and $\mathrm{Sym}(\mathcal{T}_0^0(V))$ are isomorphic to $\mathbb{R}$. Note that $\bigwedge^*(V)$ is actually a finite dimensional vector space as $\Lambda^k(V) \equiv \{0\}$ if $k > N$.

There exist natural projections $\mathcal{A} : \mathcal{T}^*(V) \to \bigwedge^*(V)$ and $\mathcal{S} : \mathcal{T}^*(V) \to \mathrm{Sym}^*(V)$ defined on each $\mathcal{T}^k(V)$ by

$$\mathcal{A}\gamma(v_1, \ldots, v_k) \;=\; \frac{1}{k!} \sum_{\sigma \in S(k)} \varepsilon_\sigma \gamma(v_{\sigma(1)}, \ldots, v_{\sigma(k)}),$$

$$\mathcal{S}\gamma(v_1, \ldots, v_k) \;=\; \frac{1}{k!} \sum_{\sigma \in S(k)} \gamma(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

Grassman algebra

The bilinear, associative *wedge product*[10], $\wedge : \bigwedge^*(V) \times \bigwedge^*(V) \to \bigwedge^*(V)$ is

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[10]Continuing the example of Footnote 9, take $u, v \in \mathbb{R}^3$ and check that this definition gives $(\theta_1 \wedge \theta_2)(u, v) = (u_1 v_2 - v_1 u_2)$, which is, up to a sign, the area of the parallelogram with corners $0, \pi(u), \pi(v)$ and $\pi(u) + \pi(v)$, where $\pi(u)$ is the projection of u onto the plane spanned by the first two coordinate axes. That is, this particular alternating covariant tensor of order 2 performs an area computation, where 'areas' may be negative. Now take $u, v, w \in \mathbb{R}^3$, and take the wedge product of $\theta_1 \wedge \theta_2$ with θ_3. A little algebra gives that that $(\theta_1 \wedge \theta_2 \wedge \theta_3)(u, v, w) = \det\langle u, v, w \rangle$ where $\langle u, v, w \rangle$ is the matrix with u in the first column, etc. In other words, it is the (signed) volume of the parallelopiped three of whose edges are u, v and w. Extending this example to N dimensions, you should already be able to guess a number of important facts, including:

(i) Alternating covariant tensors of order n have to a lot to do with computing n-dimensional (signed) volumes.

(ii) The wedge product is a way of combining lower dimensional tensors to generate higher dimensional ones.

defined on $\Lambda^r(V) \times \Lambda^s(V)$ by

$$(7.2.1) \qquad \alpha \wedge \beta \; = \; \frac{(r+s)!}{r! \, s!} \mathcal{A}(\alpha \otimes \beta).$$

The algebra $(\bigwedge^*(V), \wedge)$ is called the *Grassman* or *exterior algebra* of V.

Induced inner products on $\mathcal{T}^*(V)$

Any inner product on V induces, in a natural way, an inner product on $\mathcal{T}^*(V)$, and, correspondingly on any subspace of $\mathcal{T}^*(V)$. We shall be particulary interested in $\bigwedge^*(V)$ and so give the details for this case.

Specifically, given an inner product on V and an orthonormal basis $B_{V^*} = \{\theta_1, \ldots, \theta_N\}$ for V^*, then

$$(7.2.2) \;\; B_{\bigwedge^*(V)} \; = \; \{\theta_0\} \cup \bigcup_{j=1}^{N} \left\{ \theta_{i_1} \wedge \cdots \wedge \theta_{i_j} : i_1 < i_2 < \cdots < i_j \right\},$$

where θ_0 is a 0-form, is a basis[11] for $\bigwedge^*(V)$, a vector space of dimension $\sum_{k=0}^{N} \binom{N}{k} = 2^N$.

The unique inner product on $\bigwedge^*(V)$ which makes this basis orthogonal is the 'corresponding' inner product to which we referred above.

Algebra of double forms

Define $\Lambda^{n,m}(V)$ to be the linear span of the image of $\Lambda^n(V) \times \Lambda^m(V)$ under the map $\otimes$, and define, with some abuse of notation, $\bigwedge^*(V) \otimes \bigwedge^*(V) = \oplus_{n,m=0}^{\infty} \Lambda^{n,m}(V)$. An element of $\Lambda^{n,m}(V)$ is called a *double form of type* (n,m). Note that a (n,m) double form is alternating in its first n and last m variables.

The *double wedge product* $\cdot$ on $\bigwedge^*(V) \otimes \bigwedge^*(V)$ is defined on tensors of the form $\gamma = \alpha \otimes \beta$ by

$$(\alpha \otimes \beta) \cdot (\alpha' \otimes \beta') \; = \; (\alpha \wedge \alpha') \otimes (\beta \wedge \beta'),$$

(iii) Our earlier observation that $\Lambda^k(V) \equiv \{0\}$ if $k > N$ translates, in terms of volumes, to the trivial observation that the k-volume of an N-dimensional object is zero if $k > N$.

(iv) Since area computations and determinants are intimately related, so will be tensors and determinants.

[11] The notation of the example of Footnote 9 should now be clear. The 1-tensor θ_j defined by $\theta_j(v) = v_j$ is in fact the j-th basis element of V^* if V is given the usual basis $\{e_j\}_{j=1}^{N}$. Thus, in view of the fact that $B_{\bigwedge^*(V)}$ is a basis for $\bigwedge^*(V)$, the examples of tensors of order 2 and 3 that we built are actually archetypical.

and then extended by linearity. For $\gamma \in \Lambda^{n,m}(V)$ and $\theta \in \Lambda^{p,q}(V)$ this gives $\gamma \cdot \theta \in \Lambda^{n+p,m+q}(V)$ for which

(7.2.3)

$$
\begin{aligned}
&(\gamma \cdot \theta)\left((u_1, \ldots, u_{n+p}), (v_1, \ldots, v_{m+q})\right) \\
&= \frac{1}{n!m!p!q!} \sum_{\substack{\sigma \in S(n+p) \\ \rho \in S(m+q)}} \varepsilon_\sigma \varepsilon_\rho \Big[\gamma\left((u_{\sigma_1}, \ldots, u_{\sigma_n}), (v_{\rho_1}, \ldots, v_{\rho_m})\right) \\
&\qquad\qquad\qquad\qquad \times \theta\left(\left(u_{\sigma_{n+1}}, \ldots, u_{\sigma_{n+p}}\right), \left(v_{\rho_{m+1}}, \ldots, v_{\rho_{m+q}}\right)\right) \Big].
\end{aligned}
$$

Note that a double form of type $(n, 0)$ is simply an alternating covariant tensor so that, comparing (7.2.3) with (7.2.1), it is clear that the restriction of $\cdot$ to $\oplus_{j=0}^N \Lambda^{j,0}(V)$ is just the usual wedge product.

We shall be most interested in the restriction of this product to

$$
\textstyle\bigwedge^{*,*}(V) = \bigoplus_{j=0}^{\infty} \Lambda^{j,j}(V).
$$

The pair $(\bigwedge^{*,*}(V), \cdot)$ is then a commutative (sub) algebra[12].

For a double form $\gamma \in \bigwedge^{*,*}(V)$, we define the polynomial γ^j as the product of γ with itself j times for $j \geq 1$, and set $\gamma^0 = 1$. If γ is of type (k, k) then γ^j is of type (jk, jk). Furthermore, for powers (7.2.3) simplifies somewhat. In particular, if γ is a $(1, 1)$ double form, then it is easy to check from (7.2.3) that

$$
(7.2.4) \quad \gamma^j\left((u_1, \ldots, u_j), (v_1, \ldots, v_j)\right) = j! \det\left(\gamma\left(u_k, v_\ell\right)_{k,\ell=1,\ldots,j}\right).
$$

Trace of a double form

As described above, any inner product on V induces a corresponding inner product on $\bigwedge^*(V)$. Therefore, any inner product induces a real-valued map on $\bigwedge^{*,*}(V)$, called the *trace*, defined on tensors of the form $\gamma = \alpha \otimes \beta$ by

$$
\mathrm{Tr}(\gamma) = \langle \alpha, \beta \rangle
$$

and then extended by linearity to the remainder of $\bigwedge^*(V)$. Given an orthonormal basis $(v_1, \ldots, v_N)$ of V, a simple calculation shows that, for $\gamma \in \Lambda^{k,k}(V)$, $k \geq 1$, we have the following rather useful expression for Tr:

$$
(7.2.5) \quad \mathrm{Tr}(\gamma) = \frac{1}{k!} \sum_{a_1, \ldots, a_k = 1}^{N} \gamma\left((v_{a_1}, \ldots, v_{a_k}), (v_{a_1}, \ldots, v_{a_k})\right).
$$

If $\gamma \in \Lambda^{0,0}(V)$, then $\gamma \in \mathbb{R}$ and we define $\mathrm{Tr}(\gamma) = \gamma$. One can also check that while the above seemingly depends on the choice of basis, the trace

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[12]The product is not commutative on all of $\bigwedge^*(V) \otimes \bigwedge^*(V)$ since, in general, for $\alpha \in \Lambda^{n,m}(V)$ and $\beta \in \Lambda^{p,q}(V)$ we have $\alpha \cdot \beta = (-1)^{np+mq}\beta \cdot \alpha$.

operator is, in fact, basis independent, a property generally referred to as *invariance*.

There is also a useful extension to (7.2.5) for powers of symmetric forms in $\gamma \in \Lambda^{1,1}$. Using (7.2.4) to compute γ^j, and (7.2.5) to compute its trace, it is easy to check that

$$(7.2.6) \qquad \mathrm{Tr}\left(\gamma^j\right) \;=\; j! \det{}_j \left(\gamma\left(v_i, v_j\right)\right)_{i,j=1,\ldots N},$$

where, for a matrix A,

$$(7.2.7) \qquad \det{}_j(A) \;\overset{\Delta}{=}\; \text{Sum over all } j \times j \text{ principle minors of } A.$$

When there is more than one inner product space in consideration, say V_1 and V_2, we shall denote their traces by Tr^{V_1} and Tr^{V_2}.

For a more complete description of the properties of traces, none of which we shall need, you could try Section 2 of [61]. This is not easy reading, but is worth the effort.

A useful trace formula
To each $\gamma \in \Lambda^{k,k}(V)$ there corresponds a linear map $T_\gamma : \Lambda^k(V) \to \Lambda^k(V)$. The correspondence is unique, and so one can think of γ and T_γ as equivalent objects. To define T_γ, take a basis element of $\Lambda^k(V)$ of the form $\theta_{i_1} \wedge \cdots \wedge \theta_{i_k}$ (cf. (7.2.2)) and define its image under T_γ by setting

$$(7.2.8) \quad T_\gamma(\theta_{i_1} \wedge \cdots \wedge \theta_{i_k})(w_1,\ldots,w_k) \;=\; \gamma(v_{i_1},\ldots,v_{i_k},w_1,\ldots,w_k)$$

and extending linearly to all of $\Lambda^k(V)$.

It is a simple computation to check that if $k = 1$ and we write

$$(7.2.9) \qquad\qquad I \;=\; \sum_{i=1}^{N} \theta_i \otimes \theta_i$$

then T_I is the identity from $\Lambda^1(V)$ to $\Lambda^1(V)$. In general, T_{I^k} is the identity from $\Lambda^k(V)$ to $\Lambda^k(V)$.

We shall have need of the following useful formula (cf. [61], p. 425) in Part III when we calculate the expected Euler characteristic of a Gaussian random field on M. For $\gamma \in \Lambda^{k,k}(V)$, and $0 \le j \le N - k$,

$$(7.2.10) \qquad\qquad \mathrm{Tr}(\gamma I^j) \;=\; \frac{(N-k)!}{(N-k-j)!}\,\mathrm{Tr}(\gamma).$$

Tensor bundles, differential forms and Grassman bundles
Having defined the tangent bundle $T(M)$ and, for a fixed vector space V, the tensor spaces $\mathcal{T}_m^n(V), \bigwedge{}^*(V)$, it is a simple matter to define tensor bundles over M.

The basic observation is that, given a C^k manifold M, its tangent space $T_t M$ with dual $T_t^* M$, we can carry out the constructions of tensors on $V = T_t M$, at every $t \in M$, exactly as we did in the previous section. Then, just as we defined vector fields as maps $t \mapsto X_t \in T_t M$, $t \in M$, we can define tensor fields, covariant tensor fields, alternating covariant tensor fields, etc.

It is also quite simple to associate a differential structure with these objects, following the argument at the end of Section 7.1. In particular, since tensor fields of order (n, m) are determined, locally, in the same way that vector fields are defined, any atlas $\{U_i, \phi_i\}_{i \in I}$ for M determines[13] a C^{k-1} differential structure[14] on the collection

$$\mathcal{T}_m^n(M) \triangleq \bigcup_{t \in M} \mathcal{T}_m^n(T_t M),$$

which itself is called the *tensor bundle of order* (n, m) over M.

Constructing the Grassman algebra on each $T_t M$ and taking the union

$$\bigwedge{}^*(M) \triangleq \bigcup_{t \in M} \bigwedge{}^*(T_t M)$$

gives what is known as the *Grassman bundle* of M. As was the case for vector fields, one can define $C^j, j \leq k-1$ sections of $\bigwedge^*(M)$, and these are called the C^j *differential forms*[15] of mixed degree.

Similarly, carrying out the same construction over the $\Lambda^k(T_t M)$ gives the *bundle of (differential) k-forms* on M, while carrying it out for $\bigwedge^{*,*}(T_t M)$ yields the *bundle of double (differential) forms* on M.

7.3   Riemannian manifolds

If you followed the footnotes while we were developing the notion of tensors, you will have noted that we related them to the elementary notions of area

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[13]The finite dimensionality of spaces $\mathcal{T}_m^n(T_t M)$, noted earlier, is crucial to make this construction work.

[14] Included in this 'differential structure' is a set of rules describing how tensor fields transform under transformations of the local coordinate systems, akin to we had for simple vector fields in (7.1.7).

In fact, there is a lot hidden in this seemingly simple sequence of constructions. In particular, recall that in our discussion of tensors we mentioned that the tensors of Applied Mathematics and Physics are defined via arrays which transform according to very specific rules. However, nowhere in the path we have chosen have these demands explicitly appeared. They are, however, implicit in the constructions that we have just carried out.

[15]The reason for the additional of the adjective 'differential' will be explained later. cf. (7.3.12) and the discussion following it.

and volume in the simple Euclidean setting of $M = \mathbb{R}^N$. In general, of course, they are somewhat more complicated. In fact, if you think back, we do not as yet even have a notion of simple distance on M, let alone notions of volume. For this, we need to add a Riemannian structure to the manifold.

Riemannian manifolds

Formally, a C^{k-1} *Riemannian metric* g on a C^k manifold M is a C^{k-1} section of $\mathrm{Sym}(\mathcal{T}_0^2(M))$, such that for each t in M, g_t is positive definite; *viz.* $g_t(X_t, X_t) \geq 0$ for all $t \in M$ and $X_t \in T_tM$, with equality if, and only if, $X_t = 0$. Thus a C^{k-1} Riemannian metric determines a family of smoothly (C^{k-1}) varying inner products on the tangent spaces T_tM.

A C^k manifold M together with a C^{k-1} Riemannian metric g is called a C^k *Riemannian manifold* (M, g).

Sphere and orthonormal frame bundles

The first thing that a Riemannian metric gives us is the length $g_t(X_t, X_t)$ of tangent vectors $X_t \in T_tM$, and this allows us to define two rather important tangent bundles.

The *sphere bundle* $S(M)$ of (M, g) is the set of all unit tangent vectors of M; i.e. elements $X_t \in T(M)$, for some $t \in M$, with $g_t(X_t, X_t) = 1$. This is an example of a bundle with fibers that are not vector spaces, since the fibers are the spheres $S(T_tM)$.

Another example is the *orthonormal frame bundle*, $\mathcal{O}(M)$, the set of all sets of N unit tangent vectors $(X_t^1, \ldots, X_t^N)$ of M such that $(X_t^1, \ldots, X_t^N)$ form an orthonormal basis for T_tM.

Geodesic metric induced by a Riemannian metric

Despite its name, a Riemannian metric is not a true metric on M. However it does induce a metric τ_g on M. Since g determines the length of a tangent vector, we can define the length of a C^1 curve $c : [0, 1] \to M$ by

$$L(c) = \int_{[0,1]} \sqrt{g_t(c', c')(t)} \, dt$$

and define the metric τ_g by

(7.3.1) $$\tau_g(s, t) = \inf_{c \in D^1([0,1]; M)_{(s,t)}} L(c)$$

where $D^1([0, 1]; M)_{(s,t)}$ is the set of all piecewise C^1 maps $c : [0, 1] \to M$ with $c(0) = s$, $c(1) = t$. When the metric g is unambiguous, we write τ_g as τ.

A curve in M connecting two points $s, t \in M$, along which the infimum in (7.3.1) is achieved, is called a *geodesic* connecting them. Geodesics need not be unique.

Gradient of $f \in C^1((M, g))$

The *gradient* of f is the unique continuous vector field on M such that

(7.3.2)
$$g_t(\nabla f_t, X_t) = X_t f$$

for every vector field X.

Differentiating vector fields: connections on vector bundles

Differentiating vector fields on a manifold M is considerably more subtle on manifolds than in simple Euclidean space and since it is crucial to all that follows we shall spend some time discussing it. In fact, it is already a delicate issue if M is a simple surface in $\mathbb{R}^3$, and so we look at this case first.

Suppose X and Y are two vector fields, and we want to 'differentiate' Y, at the point $t \in M$, in the direction X_t. Following the usual procedure, we need to 'move' a distance δ in the direction X_t and compute the limit of the ratios $(Y_t - Y_{t+\delta X_t})/\delta$. There are three problems here. The first is that there is no guarantee (and it will not generally be the case) that $t + \delta X_t$ lies on M, so that $Y_{t+\delta X_t}$ is not well defined. Thus we have to find a way of 'moving in the direction X_t' while staying on M. This in itself is not a problem, and can be achieved by moving along any C^1 curve c with $c(0) = t$, $\dot{c}(0) = X_t$. Then, instead of $Y_{t+\delta X_t}$, which may not be well defined, we consider $Y_{c(\delta)}$ and so need to evaluate the limit

(7.3.3)
$$\lim_{\delta \to 0} \frac{Y_{c(\delta)} - Y_t}{\delta}.$$

The second problem is that $Y_{c(\delta)} \in T_{c(\delta)}\mathbb{R}^3$ and $Y_t \in T_t\mathbb{R}^3$, so that the above difference is also not well defined. However, fixing a basis for $\mathbb{R}^3$, we can use the natural identifications of $T_t\mathbb{R}^3$ and $T_{c(\delta)}\mathbb{R}^3$ with $\mathbb{R}^3$ to define the difference $Y_{c(\delta)} - Y_t$ and so, under appropriate conditions on Y, to also define the limit in (7.3.3).

The third problem is clearest in the case in which Y is also a tangent vector field, and for reasons that will become clearer later we decide we would like a notion of derivative that also yields a tangent vector in T_tM. The above construction will not necessarily do this, even when (7.3.3) exists and yields a vector in $T_t\mathbb{R}^3$. This last problem can, however, be easily solved by projecting (using the natural basis for $T_t\mathbb{R}^3$) the limit (7.3.3) onto $T_tM \subset T_t\mathbb{R}^3$.

This solves the problem of differentiating vector fields when M is embedded in Euclidean space. That is, the above recipe yields a way to differentiate vector fields Y on M in a direction $X_t \in T_tM$ and come up with a new (tangent) vector field on M. In constructing a solution, we had to overcome three problems:

1. How to make sense of $Y_{t+\delta X_t}$, which we resolved by taking a C^1 curve through t with $c(0) = t, \dot{c}(0) = X_t$.

2. How to make sense of the difference $Y_{c(\delta)} - Y_t$, which we resolved by 'moving' the vector $Y_{c(\delta)} \in T_{c(\delta)}\mathbb{R}^3$ to $T_t\mathbb{R}^3$.

3. Ensuring that the resulting derivative was somehow in $T_t M$, which we resolved by projecting (7.3.3) onto $T_t M$.

Of these, the first is easily generalizable to manifolds. The second and third are somewhat more complicated. Instead of trying to generalize these steps, we begin with a local definition of the derivative of a vector field.

In any given chart (U, φ) on an N-dimensional manifold M, it is possible to define a way of differentiating vector fields on U. Namely, given N^3 functions $\{\Gamma_{ij}^k : U \to \mathbb{R}\}_{1 \leq i,j,k \leq N}$, we define

$$ D_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \;=\; \sum_{k=1}^{N} \Gamma_{ij}^k \frac{\partial}{\partial x_k}, $$

where this should be thought of as differentiating the vector field $\partial/\partial x_j$ in the direction $\partial/\partial x_i$.

This actually determines a way to differentiate vector fields on U, once we demand that D be linear and obey the product rule. That is, for differentiable[16] functions a_i, b_j, $1 \leq i, j \leq N$, we require that

$$ D_{\sum_{i=1}^{N} a_i \frac{\partial}{\partial x_i}} \left(\sum_{j=1}^{N} b_j \frac{\partial}{\partial x_j} \right) \;=\; \sum_{i=1}^{N} a_i \left(\sum_{j=1}^{N} \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{k=1}^{n} b_j \Gamma_{ij}^k \frac{\partial}{\partial x_k} \right). $$

The construction of the 'derivative' D is an example on $T(M)$ of the more general concept of a *connection* on a vector bundle E. It is clear from the construction that D depends on the choice of functions Γ_{ij}^k, known as the *Christoffel symbols* (of the second kind) of the connection D, and so different choices will lead to different connections. In general, a connection on a vector bundle E with base manifold M is a bilinear map

$$ D : \; C^k(E) \times C^q(T(M)) \;\to\; C^{\min(k-1,q)}(E) $$

satisfying

$$ \begin{aligned} (7.3.4) \qquad D(f\omega, X) &= (Xf)\omega + fD(\omega, X), &\forall f \in C^1(M), \\ (7.3.5) \qquad D(\omega, fX) &= fD(\omega, X), &\forall f : M \to \mathbb{R}. \end{aligned} $$

Unlike our definition of a derivative for vector fields on $M \subset \mathbb{R}^3$ (which you can check determines a unique connection on the tangent bundle $T(M)$) we have no natural, geometrically motivated choice for Γ_{ij}^k on a general manifold. However, looking a little more closely at the Euclidean case, it

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[16] Actually, only the b_j need be differentiable.

is clear that we used the natural Riemannian structure on $\mathbb{R}^3$ to project the vector field (7.3.3) to T_tM. In this case we were implicitly endowing M with the Riemannian metric it inherits as a submanifold of $\mathbb{R}^3$. It turns out that the Riemannian metric is the missing ingredient needed to specify the 'natural' connection on $T(M)$ for a C^3 Riemannian manifold (M,g).

It is easy to check that our procedure for differentiating vector fields on $M \subset \mathbb{R}^l$ does indeed determine[17] a connection D_M, say, on $T(M)$ by setting $D_M(X,Y) = D_XY$. Furthermore, it is straightforward to check, by working in a chart (U,φ) with natural basis $\{\partial/\partial x_i\}_{1 \leq i \leq N}$, that D_M satisfies the following two conditions

$$D_M\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = D_M\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}\right)$$

$$\frac{\partial}{\partial x_i}\left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right\rangle = \left\langle D_M\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}\right), \frac{\partial}{\partial x_k}\right\rangle$$
$$+ \left\langle \frac{\partial}{\partial x_j}, D_M\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_i}\right)\right\rangle,$$

where $\langle \, , \, \rangle$ is the usual Euclidean inner product.

It is not hard to show that the above two equations actually determine the connection that we constructed, which is now called the *Levi-Civita connection* and denoted by ∇[18]. The advantage of using these equations over our previous construction is that they in no way use the embedding of M in $\mathbb{R}^l$ and so present a purely 'intrinsic' (to M) approach.

This approach also works for general C^3 Riemannian manifolds (M,g), and yields the Levi-Civita connection determined by the two conditions

(7.3.6) ∇ is *torsion free*, i.e.
$$\nabla_X Y - \nabla_Y X - [X,Y] = 0$$

(7.3.7) ∇ is *compatible* with the metric g, i.e.
$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

where $[X,Y]f = XYf - YXf$ is the so called *Lie bracket*[19] of the vector fields X and Y. The Levi-Civita connection uses slightly different notation, and we write $\nabla_X Y$ instead of $D(Y,X)$, where both $X,Y \in T(M)$.

As the Levi-Civita connection is indeed a connection, it has the rather useful property that $X,Y \in T(M)$ implies that $\nabla_X Y \in T(M)$, so that this

---

[17]Although we only discussed $l = 3$, it is immediate that the same construction works for general l.

[18]Hopefully, the double usage of ∇ for both gradient and Riemannian connection will not cause too many difficulties. Note that, like the gradient, the connnection 'knows' about the Riemannian metric g.

[19]The Lie bracket measures the failure of partial derivatives to commute, and is always zero in the Euclidean case.

notion of differentiation of tangent vector fields in the direction of tangent vectors never takes us out of the tangent bundle[20]. If $M = \mathbb{R}^N$ and g is the usual Euclidean inner product, then it is easy to check that $\nabla_X Y$ is no more that the directional derivative of the vector field Y in the directions given by X. In this case $\nabla_X Y$ is known as the *flat connection*.

It is a matter of simple algebra to derive from (7.3.6) and (7.3.7) that, for C^1 vector fields X, Y, Z

$$
(7.3.8) \quad 2g(\nabla_X Y, Z) \;=\; Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\
+ \, g(Z, [X,Y]) + g(Y, [Z,X]) + g(X, [Z,Y]).
$$

This equation is known as *Koszul's formula*. Its importance lies in the fact that the right hand side depends only on the metric g and differential-topological notions. Consequently, it gives a coordinate free formula that actually determines ∇.

Since $\nabla_X Y \in T(M)$ it must have a representation in terms of local coordinates. To write this out, we we need the notion of a C^k *orthonormal frame field* $\{E_i\}_{1 \le i \le N}$, which is a C^k section of the orthonormal (with respect to g) frame bundle $\mathcal{O}(M)$. Then there is a collection of N^2 1-forms $\{\theta_i^j\}_{i,j=1}^N$, known as *connection forms*, which determine the Riemannian connection via the following two relationships and linearity:

$$
(7.3.9) \qquad \nabla_X E_i \;=\; \sum_{j=1}^N \theta_i^j(X)\, E_j,
$$

$$
(7.3.10) \quad \nabla_X(fY) \;=\; (Xf)Y \,+\, f\nabla_X Y, \qquad f \in C^1(M).
$$

We shall see in detail how to compute the θ_i^j for some examples in Section 7.7. In general, they are determined by (7.3.15) below, in which $\{\theta_i\}_{1 \le i \le N}$ denotes the orthonormal *dual frame* corresponding to $\{E_i\}_{1 \le i \le N}$. To understand (7.3.15) we need one more concept, that of the *(exterior) differential* of a k-form.

If $f : M \to \mathbb{R}$ is C^1, then its *exterior derivative* or *differential, df,* is the 1-form defined by

$$
(7.3.11) \qquad\qquad df(E_{it}) \;=\; f_i(t) \;\overset{\Delta}{=}\; E_{it}f.
$$

---

[20]In fact, if this were not true, (7.3.7) would make little sense, since g is defined only on $T(M)$. Note also that, since $g(Y,Z) \in C^1(M)$, the expression $Xg(Y,Z)$ in (7.3.7) is also well defined. Actually, (7.3.7) is a little misleading, and would be more consistent with what has gone before if it were written as

$$
X_t g(Y_t, Z_t) \;=\; g_t((\nabla_X Y)_t, Z_t) + g_t(Y_t, (\nabla_X Z)_t).
$$

The preponderence of t's here is what leads to the shorthand of (7.3.7).

Alternatively, df is just the push-forward of $f : M \to \mathbb{R}$. Consequently, we can write df, using the dual frame, as

$$df \;=\; \sum_{i=1}^{N} f_i \, \theta_i.$$

If $\theta = \sum_{i=1}^{N} h_i \theta_i$ is a 1-form, then its exterior derivative is the 2-form

$$(7.3.12) \qquad d\theta \;\triangleq\; \sum_{i=1}^{N} dh_i \wedge \theta_i.$$

Note that the exterior derivative of a 0-form (i.e. a function) gave a 1-form, and that of a 1-form gave a 2-form. There is a general notion of exterior differentiation, which in general takes k-forms to $(k+1)$-forms[21], but which we shall not need.

We do now, however, have enough to define the 1-forms θ_i^j of (7.3.9)–(7.3.10). They are the unique set of N^2 1-forms satisfying the following two requirements:

$$(7.3.13) \qquad d\theta^i \;-\; \sum_{j=1}^{N} \theta^j \wedge \theta_j^i \;=\; 0,$$

$$(7.3.14) \qquad \theta_i^j \;+\; \theta_j^i \;=\; 0.$$

The Riemannian metric g is implicit in these equations in that, as usual, it determines the notion of orthonormality and so the choice of the θ_i. In fact, (7.3.13) is a consequence of the compatibility requirement (7.3.7), while (7.3.14) is a consequence of the requirement (7.3.6) that the connection be torsion free.

Since it will ocassionally be useful, we note that (7.3.14) takes the following form if the tangent frame and its dual are not taken to be orthonormal:

$$(7.3.15) \qquad dg_{ij} \;=\; \sum_{k=1}^{N} \left(\theta_k^i g_{kj} \;+\; \theta_k^j g_{ki} \right),$$

where $g_{ij} = (E_i, E_j) = g(E_i, E_j)$.

Exponential map and geodesics

As we noted earlier, a geodesic from t to s on (M, g) is a curve that locally minimizes arc length among all curves between t and s. It is straightforward to show that a geodesic $c : (-\delta, \delta) \to M$ satisfies

$$\nabla_{\dot{c}} \dot{c} \;=\; 0.$$

---

[21] This is why we used the terminology of 'differential forms' when discussing Grassman bundles at the end of Section 7.1.

Hence, for every $\dot{c}(0) \in T_t M$, there is a unique geodesic leaving t in the direction $\dot{c}(0)/|\dot{c}(0)|$, obtained by solving the above second order differential equation.

This procedure is formalized by the exponential map $\exp_{(M,g)} : T_t M \to M$ of a Riemannian manifold (M, g), which sends a tangent vector $X_t \in T_t M$ to the geodesic of length $|X_t|$ in the direction $X_t/|X_t|$.

The exponential map also allows us to choose a special coordinate system for charts. An orthonormal basis $\{E_i\}$ for $T_t M$ gives an isomorphism $E : \mathbb{R}^N \to T_t M$ by $E(x_1, \ldots, x_N) = \sum_i x_i E_i$. If U is a small enough neighbourhood of t in M we can use E to define the coordinate chart

$$(7.3.16) \qquad \varphi_t \triangleq E^{-1} \circ \exp^{-1} : U \to \mathbb{R}^N.$$

Any such coordinates are called *(Riemannian) normal coordinates* centered at t.

Hessian of $f \in C^2((M, g))$

The *(covariant) Hessian* $\nabla^2 f$ of a function $f \in C^2(\widetilde{M})$ on a Riemannian manifold $(\widetilde{M}, g)$ is the bilinear symmetric map from $C^1(T(\widetilde{M})) \times C^1(T(\widetilde{M}))$ to $C^0(\widetilde{M})$ (i.e. it is a double differential form) defined by[22]

$$(7.3.17) \quad \nabla^2 f(X, Y) \triangleq XYf - \nabla_X Yf = g(\nabla_X \nabla f, Y),$$

where ∇_X is the Levi-Civita connection of $(\widetilde{M}, g)$. The equality here is a consequence of (7.3.7) and (7.3.2).

Note the obvious but important point that if t is such that $\nabla f(t) = 0$[23] then $Xf(t) = 0$ for all $X \in T_t M$ (cf. (7.3.2)) and so by (7.3.17) it follows that $\nabla^2 f(X, Y)(t) = XYf(t)$. Consequently, at these points the Hessian is independent of the metric g.

7.4   Integration on manifolds

We now have the tools and vocabulary to start making mathematical sense out of our earlier comments linking tensors and differential forms to volume computations. However, rather than computing only volumes, we shall need general measures over manifolds. Since a manifold M is a locally compact

---

[22]$\nabla^2 f$ could also have been defined to be $\nabla(\nabla f)$, in which case the first relationship in (7.3.17) becomes a consequence rather than a definition. Recall that in the simple Euclidean case the Hessian is defined to the $N \times N$ matrix $H_f = (\partial^2 f/\partial x_i \partial x_j)_{i,j=1}^N$. Using H_f to define the two-form $\nabla^2 f(X, Y) = XH_f Y'$, (7.3.17) follows from simple calculus.

[23]These are the *critical points* of f, and will play a central rôle in the Morse theory of Chapter 9 and throughout Part III.

topological space, there is no problem defining measures over it and, by Riesz's theorem, these are positive, bounded, linear functionals on $C_c^0(M)$, the c denoting compact support. This description, however, does not have much of a geometric flavour to it and so we shall take a different approach. The branch of mathematics of "adding a geometric flavour" to this description is known, appropriately, as *Geometric Measure Theory* [62, 108]. We shall not, however, have need for this heavy machinery.

Integration of differential forms

Recall that, for any two manifolds M and N, a C^1 map $g : M \to N$ induces a mapping $g_* : T(M) \to T(N)$, referred to as the push-forward, (cf. (7.1.3)). Just as g_* replaces the chain rule of vector calculus, another related mapping $g^* : \bigwedge^*(T(N)) \to \bigwedge^*(T(M))$, called the *pull-back*, replaces the change of variables formula. Actually, the pull-back is more than just a generalization of the change of variables formula, although for our purposes it will suffice to think of it that way. We define g^* on $(k,0)$-tensors by

$$(7.4.1) \qquad (g^*\alpha)(X_1, \ldots, X_k) \triangleq \alpha(g_* X_1, \ldots, g_* X_k).$$

The pull-back has many desirable properties, the main ones being that it commutes with both the wedge product of forms and the exterior derivative d. (cf. (7.3.12).)

With the notion of a pull-back defined, we can add one more small piece of notation. Take a chart (U, φ) of M, and recall the notation of (7.1.5) in which we used $\{\partial/\partial x_i\}_{1 \le i \le N}$ to denote both the natural, Euclidean basis of $\varphi(U)$ and its push-forward to $T(U)$. We now do the same with the notation

$$(7.4.2) \qquad\qquad \{dx_1, \ldots, dx_N\},$$

which we use to denote both the natural, dual coordinate basis in $\mathbb{R}^N$ (so that $dx_i(v) = v_i$) and its pull-back under φ.

Now we can start defining integration, all of which hinges on a single definition: If $A \subset \mathbb{R}^N$, and $f : A \to \mathbb{R}$ is Lebesgue integrable over A, then we define

$$(7.4.3) \qquad \int_A f \, dx_1 \wedge \cdots \wedge dx_N \triangleq \int_A f(x) \, dx,$$

where $\{dx_i\}_{1 \le i \le N}$, as above, is the natural, dual coordinate basis in $\mathbb{R}^N$, and the right hand side is simply Lebesgue integration.

Since the wedge products in the left hand side of (7.4.3) generate all N-forms on $\mathbb{R}^N$ (cf. (7.2.2)) this and additivity solves the problem of integrating N-forms on $\mathbb{R}^N$ in full generality.

Now we turn to manifolds. For a given chart, (U, φ), and an N-form α on U, we define the integral of α over $V \subset U$ as

$$(7.4.4) \qquad\qquad \int_V \alpha \triangleq \int_{\varphi(V)} \left(\varphi^{-1}\right)^* \alpha$$

where the right hand side is defined by virtue of (7.4.3).

To extend the integral beyond single charts, we require a new condition, that of *orientability*. A C^k manifold M is said to be *orientable* if there is there is an atlas $\{U_i, \varphi_i\}_{i \in I}$ for M such that, for any pair of charts $(U_i, \varphi_i), (U_j, \varphi_j)$ with $U_i \cap U_j \neq \phi$, the Jacobian of the map $\varphi_i \circ \varphi_j^{-1}$ has a positive determinant. For orientable manifolds it is straightforward to extend the integral (7.4.4) to general domains by additivity. This leads to the notion of an *integrable N-form*, essentially any N-form for which (7.4.4) can be extended to all of M as a σ-finite signed measure. In particular, every integrable form α determines such a measure, μ_α say, by setting

$$(7.4.5) \qquad \mu_\alpha(A) \;=\; \int_A \alpha.$$

Given an oriented manifold M with atlas as above, one can also define the notion of an *oriented (orthonormal) frame field* over M. This is a (orthonormal) frame field $\{E_{1t}, \ldots, E_{Nt}\}$ over M for which, for each chart (U, φ) in the atlas, and at each $t \in U$, the push-forward $\{\varphi_* E_{1t}, \ldots, \varphi_* E_{Nt}\}$ can be transformed to the standard basis of $\mathbb{R}^N$ via a transformation with positive determinant.

Given an oriented orthonormal frame field, there is a unique volume form, which we denote by either Ω or by Vol_g if we want to emphasise the dependence on the metric g, which plays the rôle of Lebesgue measure on M and which is defined by the requirement that

$$(7.4.6) \qquad \left(\mathrm{Vol}_g\right)_t (E_{1t}, \ldots, E_{Nt}) \;\equiv\; \Omega_t (E_{1t}, \ldots, E_{Nt}) \;=\; +1.$$

From this, and the fact that Ω is a differential form, it follows that, for $X \in T_t M$,

$$(7.4.7) \qquad \left(\mathrm{Vol}_g\right)_t (X) \;\equiv\; \det\left(\{g_t(X_i, E_{jt})\}_{1 \le i,j \le N}\right).$$

The integral of Ω is comparatively easy to compute. For a fixed (oriented) chart (U, φ), with natural basis $\{\partial/\partial x_i\}_{1 \le i \le N}$, write

$$(7.4.8) \qquad g_{ij}(t) \;=\; g_t \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right),$$

where g is the Riemannian metric. This determines, for each t, a positive definite matrix $(g_{ij}(t))_{1 \le i,j \le N}$. Then, for $A \subset U$,

$$
\begin{aligned}
(7.4.9) \qquad \int_A \Omega \;&\equiv\; \int_{\varphi(A)} (\varphi^{-1})^* \Omega \\
&=\; \int_{\varphi(A)} \sqrt{\det(g_{ij} \circ \varphi^{-1})} \, dx_1 \wedge \cdots \wedge dx_N \\
&=\; \int_{\varphi(A)} \sqrt{\det(g_{ij} \circ \varphi^{-1})} \, dx,
\end{aligned}
$$

where the crucial intermediate term comes from (7.4.3)–(7.4.8) and some algebra.

Retaining the orientation of M determined by Ω, both this integral and that in (7.4.4) can be extended to larger subsets of M by additivity[24]. This yields a σ-finite measure μ_Ω associated to Ω, called *Riemannian measure*.

For obvious reasons, we shall often write the volume form Ω as $dx_1 \wedge \cdots \wedge dx_N$, where the 1-forms dx_i are the (dual) basis of $T^*(M)$.

An important point to note is that Vol_g agrees with the N-dimensional Hausdorff measure[25] associated with the metric τ induced by g, and so we shall also often write it as $\mathcal{H}_N$. In this case we shall usually write integrals as $\int_M h(t)d\mathcal{H}_N(t)$ rather than $\int_M h\mathcal{H}_N$, which would be more consistent with our notation so far.

We now return to the issue of orientability. In setting up the volume form Ω, we first fixed an orthonormal frame field and then demanded that $\Omega_t(E_{1t}, \ldots, E_{Nt}) = +1$, for all $t \in M$. (cf. (7.4.6).) We shall denote M, along with this orientation, by M^+. However, there is another orientation of M for which $\Omega = -1$ when evaluated on the orthonormal basis. We write this manifold as M^-. On an orientable manifold, there are only two such possibilities.

In fact, it is not just Ω which can be used to determine an orientation. Any non-vanishing continuous N-form α on an orientable manifold can be used to determine one of two orientations, with the orientation being determined by the sign of α on the orthonormal basis at any point on M. We can thus talk about the 'orientation induced by α'.

---

[24] The last line of (7.4.9), when written out in longhand, should be familiar to most readers as an extension of the formula giving the 'surface area' of a regular N-dimensional surface. The extension lies in the fact that an arbitrary Riemannian metric now appears, whereas in the familiar case there is only one natural candidate for g.

[25] If M is a N-dimensional manifold, treat (M, τ) as a metric space, where τ is the geodesic metric given by (7.3.1). The diameter of a set $S \subset M$ is then $\mathrm{diam}(S) = \sup\{\tau(s,t) : s, t \in S\}$ and, for integral n, the Hausdorff n-measure of $A \subset M$ is defined by

$$\mathcal{H}_n(A) \triangleq \omega_n 2^{-n} \liminf_{\varepsilon \downarrow 0} \sum_i (\mathrm{diam}\ B_i)^n,$$

where, for each $\varepsilon > 0$, the infimum is taken over all collections $\{B_i\}$ of open τ-balls in M whose union covers A and for which $\mathrm{diam}(B_i) \leq \varepsilon$. As usual, $\omega_n = \pi^{n/2}/\Gamma(n/2+1)$ is the volume of the unit ball in $\mathbb{R}^n$. For the moment, we need only the case $n = N$. When both are defined, and the underlying metric is Euclidean, Hausdorff and Lebesgue measures agree.

Later we shall need a related concept, that of *Hausdorff dimension*. If $A \subset M$, the Hausdorff dimension of A is defined as

$$\dim(A) \triangleq \inf\left\{\alpha : \liminf_{\varepsilon \downarrow 0} \sum_i (\mathrm{diam}\ B_i)^\alpha = 0\right\},$$

where the conditions on the infimum and the B_i are as above.

With an analogue for Lebesgue measure in hand, we can set up the analogues of Borel measurability and (Lebesgue) integrability in the usual ways. Furthermore, it follows from (7.4.3) that there is an inherent smoothness[26] in the above construction. In particular, if M is compact, then for any continuous, non-vanishing N-form α which induces the same orientation on M as does Ω, there is an Ω-integrable function $d\alpha/d\Omega$ for which

$$\int_M \alpha \;=\; \int_M \frac{d\alpha}{d\Omega}\, \Omega.$$

For obvious reasons, we call $d\alpha/d\Omega$ the *Radon-Nikodym derivative* of α.

Coarea formula

The following important result, due to Federer [61] and known as his *coarea formula*, allows us to break up integrals over manifolds into iterated integrals over submanifolds of lower dimension. Consider a differentiable[27] map $f : M \to N$ between two Riemannian manifolds, with $m = \dim(M) \geq n = \dim(N)$. At each $t \in M$ its push-forward, $f_*(t)$, is a linear map between two Hilbert spaces and can be written as

$$(7.4.10) \qquad \sum_{j=1}^{n} \tilde{\theta}_j(f(t)) \sum_{i=1}^{m} A_{ij}^f(t) \theta_i(t)$$

for orthonormal bases $\{\theta_j(t)\}_{i=1}^m$ of $(T_t M)^*$ and $\{\tilde{\theta}_j(f(t))\}_{j=1}^n$ of $T_{f(t)} N$.

Writing $\{\partial/\partial t_1, \ldots, \partial/\partial t_m\}$ for the corresponding basis of $T_t M$, the A_{ij}^f are given by

$$(7.4.11) \qquad A_{ij}^f(t) \;=\; (\partial/\partial t_i) f_j(t).$$

Similarly, the pull-back f^*, evaluated at $t \in M$, is a linear map from $\bigwedge^*(T_{f(t)} N)$ to $\bigwedge^*(T_t M)$. If we restrict $f^*(t)$ to $\Lambda^n(T_{f(t)} N)$, it can be identified with an element in the linear span of $\Lambda^n(T_{f(t)} N) \times \Lambda^n(T_t M)$.

Both the push-forward and the pull-back share the same norm, which we write as $Jf(t)$ and which can be calculated by taking the square root of the sum of squares of the determinants of the $n \times n$ minors of the matrix A^f of (7.4.10). Alternatively, it is given by

$$(7.4.12) \qquad Jf(t) \;\triangleq\; \sqrt{\det\left(\langle \nabla f_i(t), \nabla f_j(t)\rangle\right)}.$$

---

[26] This smoothness implies, for example, that N-forms cannot be supported, as measures, on lower dimensional sub-manifolds of M.

[27] Federer's setting is actually a little more general than this, since he works with Lipschitz mappings. In doing so he needs to replace derivatives with "approximate derivatives" throughout and ensure that some of the claims that we make are made only for almost all points.

Federer's coarea formula [61] states that for Lipschitz maps f and $g \in L^1(M, \mathcal{B}(M), \mathcal{H}_m)$,

$$(7.4.13) \qquad \int_M g(t)\, Jf(t)\, d\mathcal{H}_m(t) \ = \ \int_N d\mathcal{H}_n(u) \int_{f^{-1}(u)} g(s)\, d\mathcal{H}_{m-n}(s).$$

Simplifying matters a little bit, consider two special cases. If $M = \mathbb{R}^N$ and $N = \mathbb{R}$, it is easy to see that $Jf = |\nabla f|$, so that

$$(7.4.14) \qquad \int_{\mathbb{R}^N} g(t)\, |\nabla f(t)|\, dt \ = \ \int_{\mathbb{R}} du \int_{f^{-1}\{u\}} g(s)\, d\mathcal{H}_{N-1}(s).$$

There is not a great deal of simplification in this case beyond the fact that it is easy to see what the functional J is. On the other hand, if $M = N = \mathbb{R}^N$, then $Jf = |\det \nabla f|$, and

$$(7.4.15) \qquad \int_{\mathbb{R}^N} g(t)\, |\det \nabla f(t)|\, dt \ = \ \int_{\mathbb{R}^N} du \int_{f^{-1}\{u\}} g(s)\, d\mathcal{H}_0(s)$$

$$= \ \int_{\mathbb{R}^N} \left(\sum_{t:\, f(t)=u} g(t) \right)\, du.$$

We shall return to (7.4.15) in Section 11.3 and to the coarea formula in general in Chapter 14 where it will play a very important rôle in our calculations.

7.5   Curvature

We now come to what is probably the most central of all concepts in Differential Geometry, that of curvature. In essence, much of what we have done so far in developing the calculus of manifolds can be seen as no more than setting up the basic tools for handling the ideas to follow.

Curvature is the essence that makes the local properties of manifolds inherently different from simple Euclidean space, where curvature is always zero. Since there are many very different manifolds, and many different Riemannian metrics, there are a number of ways to measure curvature. In particular, curvature can be measured in a somewhat richer fashion for manifolds embedded in ambient spaces of higher dimension than it can for manifolds for which no such embedding is given. A simple example of this is given in Figure 7.5.1, where you should think of the left hand circle S^1 as being embedded in the plane while the right hand circle exists without any embedding. In the embedded case there are notions of 'up' and 'down', with the two arrows at the top and bottom of the circle pointing 'up'. In one case the circle curves 'away' from the arrow, in the other, 'towards'

it, so that any reasonable definition of curvature has to be different at the two points. However, for the non-embedded case, in which there is nothing external to the circle, the curvature must be the same everywhere. In what follows, we shall capture the first, richer, notion of curvature via the *second fundamental form* of the manifold, and the second via its *curvature tensor* which is related to the second fundamental form through the Gauss equation (7.5.8) below.
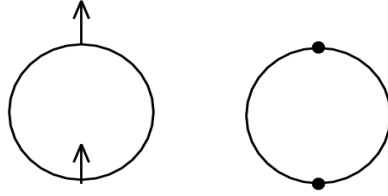
FIGURE 7.5.1. Embedded and intrinsic circles

Riemannian curvature tensor
We start with the (Riemannian) curvature tensor. While less informative than the second fundamental form, the fact that it is *intrinsic* (i.e. it is uniquely determiend by the Riemannian metric) actually makes it a more central concept to Riemannian geometry.

Much in the same way that the Lie bracket $[X, Y] = XY - YX$ measures the failure of partial derivatives to commute, the Riemannian curvature tensor, R, measures the failure of covariant derivatives to commute. A relatively simple computation shows that for vector fields X, Y it is not generally true that $\nabla_X \nabla_Y - \nabla_Y \nabla_X = 0$. However, rather than taking this difference as a measure of curvature, to ensure linearity, it is convenient to define the *(Riemannian) curvature operator*[28]

$$(7.5.1) \qquad R(X, Y) \triangleq \nabla_X \nabla_Y \ - \ \nabla_Y \nabla_X \ - \ \nabla_{[X,Y]}.$$

Note that if $[X, Y] = 0$, as is the case when X_t and Y_t are coordinate vectors in the natural basis of some chart, then $R(X, Y) = \nabla_X \nabla_Y \ - \ \nabla_Y \nabla_X$, and so is the first measure of lack of commutativity of ∇_X and ∇_Y mentioned above.

The *(Riemannian) curvature tensor*, also denoted by R, is defined by

$$(7.5.2) \ \ R(X, Y, Z, W) \ \triangleq \ g\left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \ W\right)$$
$$= \ g(R(X, Y)Z, \ W),$$

---

[28]Note that the curvature operator depends on the underlying Riemannian metric g via the dependence of the connection on g.

where the R in the second line is, obviously, the curvature operator. It is easy to check that for $\mathbb{R}^N$, equipped with the standard Euclidean metric, $R \equiv 0$[29].

The definition (7.5.2) of R is not terribly illuminating, although one can read it as 'the amount, in terms of g and in the direction W, by which ∇_X and ∇_Y fail to commute when applied to Z'. To get a better idea of what is going on, it is useful to think of planar sections of $T_t M$.

For any $t \in M$, we call the span of two linearly independent vectors $X_t, Y_t \in T_t M$ the *planar section* spanned by X_t and Y_t, and denote it by $\pi(X_t, Y_t)$. Such a planar section is determined by any pair of orthonormal vectors E_{1t}, E_{2t} in $\pi(X_t, Y_t)$, and we call

$$(7.5.3) \quad \kappa(\pi) \overset{\Delta}{=} -R\left(E_{1t}, E_{2t}, E_{1t}, E_{2t}\right) = R\left(E_{1t}, E_{2t}, E_{2t}, E_{1t}\right),$$

the *sectional curvature* of the planar section. It is independent of the choice of basis. Sectional curvatures are somewhat easier to understand than the curvature tensor, but essentially equivalent, since it is easy to check from the symmetry properties of the curvature tensor that it is uniquely determined by the sectional curvatures.

We shall later need a further representation of R, somewhat reminiscent of the representation (7.3.9) for the Riemannian connection. The way that R was been defined in (7.5.2) it is clearly a covariant tensor of order 4. However, it is not a difficult computation, based on (7.3.9), to see that it can also be expressed as a symmetric double form of order (2,2). In particular, if g is C^2 and $\{\theta_i\}_{1 \leq i \leq N}$ is the dual of a C^2 orthonormal frame field, then $R \in C^0(\Lambda^{2,2}(M))$ and can be written as

$$(7.5.4) \qquad\qquad R = \frac{1}{2} \sum_{i,j=1}^{N} \Omega_{ij} \otimes \left(\theta_i \wedge \theta_j\right),$$

where the Ω_{ij} are skew symmetric C^0 differential 2-forms ($\Omega_{ij} = -\Omega_{ji}$) known as the *curvature forms* for the section $\{E_i\}_{1 \leq i \leq N}$ and are defined by

$$(7.5.5) \qquad\qquad \Omega_{ij}(X,Y) = R\left(E_i, E_j, X, Y\right)$$

for vector fields X, Y.

Second Fundamental Form

Let (M, g) be a Riemannian manifold embedded[30] in an ambient Rieman-

---

[29]Manifolds for which $R \equiv 0$ are called *flat spaces*. However, not only Euclidean space is flat. It is easy to check, for example, that the cylinder $S^1 \times \mathbb{R}$ is also flat when considered as a manifold embedded in $\mathbb{R}^3$ with the usual Euclidean metric.

[30] We have used this term often already, albeit in a descriptive sense. The time has come to define it properly: Suppose $f : M \to \widehat{M}$ is C^1. Take $t \in M$ and charts (U, ϕ)

nian manifold $(\widehat{M}, \widehat{g})$ of co-dimension[31] at least one. Write ∇ for the Levi-Civita connection on M and $\widehat{\nabla}$ for the Levi-Civita connection on $\widehat{M}$. If $t \in M$, then the *normal space* to M in $\widehat{M}$ at t is

$$(7.5.6) \quad T_t^{\perp}M \; \triangleq \; \left\{ X_t \in T_t\widehat{M} : \widehat{g}_t\,(X_t, Y_t) = 0 \text{ for all } Y_t \in T_tM \right\},$$

and we also write $T^{\perp}(M) = \bigcup_t T_t^{\perp}M$. Note that since $T_t\widehat{M} = T_tM \oplus T_t^{\perp}M$ for each $t \in M$, it makes sense to talk about tangential and normal components of an element of $T_t\widehat{M}$ and so of the orthogonal projections

$$P_{TM} : \pi_{\widehat{M}}^{-1}M \to T(M), \quad P_{TM}^{\perp} : \pi_{\widehat{M}}^{-1}M \to T^{\perp}(M),$$

where $\pi_{\widehat{M}}$ is the canonical projection on the tangent bundle $T(\widehat{M})$ and $T(M)$ and $T^{\perp}(M)$ are embedded sub-bundles of $T(\widehat{M})$.

The *second fundamental form* of M in $\widehat{M}$ can now be defined to be the operator S from $T(M) \times T(M)$ to $T^{\perp}(M)$ satisfying

$$(7.5.7) \quad S(X, Y) \; \triangleq \; P_{TM}^{\perp}\left(\widehat{\nabla}_X Y\right) \; = \; \widehat{\nabla}_X Y - \nabla_X Y,$$

where the equality here is known as *Gauss's formula*.

When (M, g) isometrically embedded in $(\widehat{M}, \widehat{g})$ there is a useful relation between the second fundamental form and the curvature tensors R of (M, g) and $\widehat{R}$ of $(\widehat{M}, \widehat{g})$, known as the *Gauss equation*, and given by

$$(7.5.8) \quad \widehat{g}(S(X, Z), S(Y, W)) - \widehat{g}(S(X, W), S(Y, W))$$
$$= \; 2\left(\widehat{R}(X, Y, Z, W) - R(X, Y, Z, W)\right).$$

In the special case that $\widehat{M}$ is a manifold with smooth boundary $\partial\widehat{M} = M$, the above simplifies to

$$(7.5.9)$$
$$S^2((X, Y), (Z, W)) \; = \; -2\left(R^{\partial\widehat{M}}(X, Y, Z, W) - R^{\widehat{M}}(X, Y, Z, W)\right).$$

---

and (V, ψ) containing t and $f(t)$, respectively. The *rank* of f at t is defined to be the rank of the mapping $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ between Euclidean spaces. If f is everywhere of rank $\dim M$ then it is called an *immersion*. If $\dim M = \dim \widehat{M}$ then it is called a *submersion*. Note that this is a purely local property of M.

If, furthermore, f is a one-one homeomorphism of M onto its image $f(M)$ (with its topology as a subset of $\widehat{M}$) then we call f an *embedding* of M in $\widehat{M}$ and refer to M as an *embedded (sub-)manifold* and to $\widehat{M}$ as the *ambient* manifold. This is a global property, and amounts to the fact that M cannot 'intersect' itself on $\widehat{M}$.

Finally, let M and $\widehat{M}$ be Riemannian manifolds with metrics g and $\widehat{g}$, respectively. Then we say that (M, g) is an *isometrically embedded Riemannian manifold* of $(\widehat{M}, \widehat{g})$ if, in addition to the above, $g = f^*\widehat{g}$, where $f^*\widehat{g}$ is the pull-back of $\widehat{g}$. (cf. (7.4.1).)

[31] The *codimension* of M in $\widehat{M}$ is $\dim(\widehat{M}) - \dim(M)$.

Now let ν denote a unit normal vector field on M, so that $\nu_t \in T_t^{\perp}M$ for all $t \in M$. Then the *scalar second fundamental form* of M in $\widehat{M}$ for ν is defined, for $X, Y \in T(M)$, by

$$(7.5.10) \qquad S_{\nu}(X,Y) \; \stackrel{\Delta}{=} \; \widehat{g}\left(S(X,Y),\nu\right),$$

where the internal S on the right hand side refers to the second fundamental form (7.5.7). Note that, despite its name, the scalar fundamental form is not a differential form, since it is symmetric (rather than alternating) in its arguments. When there is no possibility of confusion we shall drop the qualifier 'scalar' and refer also to S_{ν} as the second fundamental form.

In view of the fact that $\widehat{\nabla}$ is compatible with the metric g (cf. (7.3.7)) we also have the so-called *Weingarten equation* that, for $X, Y \in T(M)$,

$$(7.5.11) \qquad S_{\nu}(X,Y) \; = \; \widehat{g}(\widehat{\nabla}_X Y, \nu) \; = \; -\widehat{g}(Y, \widehat{\nabla}_X \nu).$$

As we already noted, S_{ν} is a symmetric 2 tensor, so that, as for the curvature tensor, we can view it as a symmetric section of $\bigwedge^{*,*}(M)$. As such, it contains a lot of information about the embedding of M in $\widehat{M}$ and thus curvature information about M itself. For example, fix ν and use the second fundamental form to define an operator $\mathcal{S}_{\nu} : T(M) \to T(M)$ by $\widehat{g}(\mathcal{S}_{\nu}(X), Y) = S_{\nu}(X,Y)$ for all $Y \in T(M)$. Then $\mathcal{S}_{\nu}$ is known as the *shape operator*. It has N real eigenvalues, known as the *principal curvatures* of M in the direction ν, and the corresponding eigenvalues are known as the *principal curvature vectors*.

All of the above becomes quite familiar and particularly useful if M is a simple surface determined by the graph of $f : \mathbb{R}^N \to \mathbb{R}$, with the usual Euclidean metric. In this case, the principle curvatures are simply the eigenvalues of the Hessian $(\partial^2 f / \partial x_i \partial x_j)_{i,j=1}^N$. (cf. (7.3.17).) In particular, if (M, g) is a surface in $\mathbb{R}^3$ with the induced Euclidean metric, then the product of these eigenvalues is known as *Gaussian curvature* and it is Gauss's *Theorema Egregium* that it is an isometry invariant of (M, g): i.e. it is independent of the embedding of M in $\mathbb{R}^3$. In the next section we shall see that the Gaussian curvature is not unique as far as isometric invariance goes and other invariants can be obtained as integrals of mixed powers of curvature and second fundamental forms over manifolds.

7.6   Intrinsic volumes for Riemannian manifolds

In Section 6.3, in the context of Integral Geometry, we described the notions of *intrinsic volumes* and *Minkowski functionals* and how they could be used, via Steiner's formula (6.3.3), to find an expression for the tube around a convex Euclidean set. In Chapter 10 we shall extend these results

considerably, looking at tubes around submanifolds of some larger manifold. In doing so we shall encounter extensions of intrinsic volumes known as Lipschitz-Killing curvatures, and study them in some depth.

However, while the detailed calculations of Chapter 10 are not needed for Part III, the same is not true of Lipschitz-Killing curvatures, which will have a crucial rôle to play there. Hence we define them now, with no further motivation.

Let (M, g) be a C^2 Riemannian manifold. For each $t \in M$, the Riemannian curvature tensor R_t given by (7.5.2) is in $\Lambda^{2,2}(T_t M)$, so that for $j \leq \dim(M)/2$ it makes sense to talk about the j-th power R_t^j of R_t. We can also take the trace

$$\mathrm{Tr}^M(R^j)(t) \;\triangleq\; \mathrm{Tr}^{T_t M}(R_t^j).$$

Integrating these powers over M gives the *Lipschitz-Killing curvature measures* of (M, g) defined, for measurable subsets U of M, by

(7.6.1) $\mathcal{L}_j(M, U)$

$$= \begin{cases} \frac{(-2\pi)^{-(N-j)/2}}{\left(\frac{N-j}{2}\right)!} \int_U \mathrm{Tr}^M\left(R^{(N-j)/2}\right) \mathrm{Vol}_g & \text{if } N - j \text{ is even,} \\ 0 & \text{if } N - j \text{ is odd.} \end{cases}$$

The *Lipschitz-Killing curvature* of (M, g) defined as

(7.6.2) $$\mathcal{L}_j(M) \;=\; \mathcal{L}_j(M, M).$$

While (7.6.1) is a tidy formula, the integral is not always easy to compute. Perhaps the easiest example is given by the Lipschitz-Killing curvatures of S^N, which you should be able to compute directly from (7.6.1). We shall carry out the computation at the end of the following section, after having looked a little more carefully at some of the components making up the trace in the integrand.

7.7   A Euclidean example

We close this chapter with what might be considered a 'concrete' example, taking as our manifold M a compact C^2 domain T in $\mathbb{R}^N$ endowed with a Riemannian metric g. We shall show how to explicitly compute both the curvature tensor R and the second fundamental form S, as well as traces of their powers. This section can be skipped on first reading, although you will probably want to come back to it once we start looking at specific examples in later chapters.

Christoffel Symbols
We first met Christoffel symbols back in Section 7.3, where we used them

to motivate the construction of Riemannian connections. Now, however, we want to work in the opposite direction. That is, given M and g, which uniquely define a Levi-Civita connection, we want to represent this connection in a form conducive to performing computations. In doing this we shall recover the Christoffel symbols of Section 7.3 as well as developing a related class of symbols.

We start with $\{E_i\}_{1 \leq i \leq N}$, the standard[32] coordinate vector fields on $\mathbb{R}^N$. This also gives the natural basis in the global chart $(\mathbb{R}^N, i)$, where i is the inclusion map. We now define[33] the so-called *Christoffel symbols of the first kind* of ∇,

$$(7.7.1) \qquad \Gamma_{ijk} \stackrel{\Delta}{=} g(\nabla_{E_i} E_j, E_k), \qquad 1 \leq i, j, k \leq N.$$

We further define

$$(7.7.2) \qquad\qquad\qquad g_{ij} = g(E_i, E_j).$$

Despite the possibility of some confusion, we also denote the corresponding matrix function by g, doubling up on the notation for the metric.

With this notation it now follows via a number of successive applications of (7.3.6) and (7.3.7) that

$$(7.7.3) \qquad \Gamma_{ijk} = \left(E_j g_{ik} - E_k g_{ij} + E_i g_{jk}\right)/2.$$

We need two more pieces of notation, the elements g^{ij} of the inverse matrix g^{-1} and the *Christoffel symbols of the second kind* of ∇, defined by

$$\Gamma_{ij}^k = \sum_{s=1}^N g^{ks} \Gamma_{ijs}.$$

Riemannian Curvature
With the definitions above, it is now an easy and standard exercise to show

---

[32] Note that while the E_i might be 'standard' there is no reason why they should be the 'right' coordinate system to use for a given g. In particular, they are not orthonormal any more, since g_{ij} of (7.7.2) need not be a Kronecker delta. Thus, although we start here, we shall soon leave this choice of basis for an orthonormal one.

[33] An alternative, and somewhat better motivated, definition of the Γ_{ijk} comes by taking the vector fields $\{E_i\}$ to be orthonormal with respect to the metric g. In that case, they can be defined via their rôle in determining the Riemannian connection through the set of N^2 equations $\nabla_{E_i} E_j = \sum_{k=1}^N \Gamma_{ijk} E_k$. Taking this as a definition, it is easy to see that (7.7.1) must also hold. In general, the Christoffel symbols are dependent on the choice of basis.

that, if the metric g is C^2, then

$$
(7.7.4) \quad R^E_{ijkl} \triangleq R((E_i, E_j), (E_k, E_l))
$$

$$
= \sum_{s=1}^{N} \left[g_{sl} \left(E_i \left(\Gamma^s_{jk} \right) - E_j \left(\Gamma^s_{ik} \right) \right) + \Gamma_{isl} \Gamma^s_{jk} - \Gamma_{jsl} \Gamma^s_{ik} \right]
$$

$$
= E_i \Gamma_{jkl} - E_j \Gamma_{ikl} + \sum_{s,t=1}^{N} \left(\Gamma_{iks} \, g^{st} \, \Gamma_{jlt} - \Gamma_{jks} \, g^{st} \, \Gamma_{ilt} \right).
$$

Returning to the definition of the curvature tensor, and writing $\{de_i\}_{1 \le i \le N}$ for the dual basis of $\{E_i\}_{1 \le i \le N}$, it now follows (after some algebra) that the curvature tensor itself can be written as

$$
(7.7.5) \qquad R = \frac{1}{4} \sum_{i,j,k,l=1}^{N} R^E_{ijkl} \, (de_i \wedge de_j) \otimes (de_k \wedge de_l).
$$

Next, we express the curvature tensor in an orthonormal frame field. To this end, let $X = \{X_i\}_{1 \le i \le N}$ be be a section of $\mathcal{O}(T)$, having dual frames $\{\theta_i\}_{1 \le i \le N}$, so that

$$
(7.7.6) \qquad\qquad \theta_i = \sum_{i'=1}^{N} g^{\frac{1}{2}}_{ii'} de_{i'},
$$

where $g^{\frac{1}{2}}$ is given by

$$
(g^{\frac{1}{2}})_{ij} = g(E_i, X_j)
$$

and the notation comes from the easily verified fact that $g^{\frac{1}{2}} (g^{\frac{1}{2}})' = g$, so that $g^{\frac{1}{2}}$ is a square root of g.

It follows that

$$
(7.7.7) \qquad R = \frac{1}{4} \sum_{i,j,k,l=1}^{N} R^X_{ijkl} \, (\theta_i \wedge \theta_j) \otimes (\theta_k \wedge \theta_l),
$$

where

$$
R^X_{ijkl} = \sum_{i',j',k',l'=1}^{N} R^E_{i'j'k'l'} \, g^{-\frac{1}{2}}_{ii'} g^{-\frac{1}{2}}_{jj'} g^{-\frac{1}{2}}_{kk'} g^{-\frac{1}{2}}_{ll'}
$$

$$
= R \left((X_i, X_j), (X_k, X_l) \right)
$$

and you are free to interpret the $g^{-\frac{1}{2}}_{ij}$ as either the elements of $(g^{\frac{1}{2}})^{-1}$ or of a square root of g^{-1}.

In (7.7.7) we now have a quite computable representation of the curvature tensor for any orthonormal basis. Given this, we also have the curvature

forms $\Omega_{ij}(X, Y) = R(E_i, E_j, X, Y)$ of (7.5.4) and so, via (7.5.5), we can rewrite the curvature tensor as

$$R \;=\; \frac{1}{2}\, \Omega_{ij} \otimes (\theta_i \wedge \theta_j).$$

With the product and general notation of (7.2.3) we can thus write R^k as

$$R^k \;=\; \frac{1}{2^k} \sum_{i_1,\ldots,i_{2k}=1}^{N} \left(\wedge_{l=1}^{k}\Omega_{i_{2l-1}i_{2l}}\right) \otimes \left(\wedge_{l=1}^{k}(\theta_{i_{2l-1}} \wedge \theta_{i_{2l}})\right),$$

where

$$\wedge_{l=1}^{k}\Omega_{i_{2l-1}i_{2l}}(X_{a_1},\ldots,X_{a_{2k}})$$

$$= \frac{1}{2^k} \sum_{\sigma \in S(2k)} \varepsilon_\sigma \prod_{l=1}^{k} \Omega_{i_{2l}i_{2l-1}}(X_{a_{\sigma(2l-1)}}, X_{a_{\sigma(2l)}})$$

$$= \frac{1}{2^k} \sum_{\sigma \in S(2k)} \varepsilon_\sigma \prod_{l=1}^{k} R^X_{i_{2l-1}i_{2l}a_{\sigma(2l-1)}a_{\sigma(2l)}}.$$

It follows that

$$R^k\left((X_{a_1},\ldots,X_{a_{2k}}),(X_{a_1},\ldots,X_{a_{2k}})\right)$$

$$= \frac{1}{2^{2k}} \sum_{i_1,\ldots,i_{2k}=1}^{N} \delta^{(a_1,\ldots,a_{2k})}_{(i_1,\ldots,i_{2k})} \left(\sum_{\sigma \in S(2k)} \varepsilon_\sigma \prod_{l=1}^{k} R^X_{i_{2l-1}i_{2l}a_{\sigma(2l-1)}a_{\sigma(2l)}} \right),$$

where, for all m,

$$\delta^{(c_1,\ldots,c_m)}_{(b_1,\ldots,b_m)} = \begin{cases} \varepsilon_\sigma & \text{if } c = \sigma(b), \text{ for some } \sigma \in S(m) \\ 0 & \text{otherwise.} \end{cases}$$

We are finally in a position to write down an expression for the trace of R^k, as defined by (7.2.5):

(7.7.8)

$$\mathrm{Tr}(R^k) \;=\; \frac{1}{(2k)!} \sum_{a_1,\ldots,a_{2k}=1}^{N} R^k\left((X_{a_1},\ldots,X_{a_{2k}}),(X_{a_1},\ldots,X_{a_{2k}})\right)$$

$$= \frac{1}{2^{2k}} \sum_{a_1,\ldots,a_{2k}=1}^{n} \left(\sum_{\sigma \in S(2k)} \varepsilon_\sigma \prod_{l=1}^{k} R^X_{a_{2l-1}a_{2l}a_{\sigma(2l-1)}a_{\sigma(2l)}} \right).$$

This is the equation we have been searching for to give a 'concrete' example of the general theory.

A little thought will show that while the above was presented as an example of a computation on $\mathbb{R}^N$ it is, in fact, far more general. Indeed, you can reread the above, replacing $\mathbb{R}^N$ by a general manifold and the E_i by a family of local coordinate systems that are in some sense 'natural' for computations. Then (7.7.8) still holds, as do all the equations leading up to it. Thus the title of this section is somewhat of a misnomer, since the computations actually have nothing to do with Euclidean spaces!

The Lipschitz-Killing curvatures of S^{N-1}

We already know from (6.3.8) that the intrinsic volumes of $S^{N-1}(T)$, the sphere of radius T in $\mathbb{R}^N$, are given by

$$(7.7.9) \qquad \mathcal{L}_j\left(S^{N-1}(T)\right) \;=\; 2\binom{N-1}{j}\frac{s_N}{s_{N-j}}T^j$$

if $N-1-j$ is even, and 0 otherwise. The argument that lead to this was integral geometric, and involved no more than knowing how to compute the volume of N balls and some trivial algebra.

We shall now rederive this from the integral (7.6.1) but shall have to work a little harder. The advantage, of course, is that while the harder calculation generalises to more complex cases, the simple one does not.

Since it well known that $S^{N-1}(T)$ has constant curvature, the trace in (7.6.1) is constant. Furthermore, since the volume element there is simply spherical measure it is immediate that (7.6.1) reduces to

$$\mathcal{L}_j\left(S^{N-1}(T)\right) \;=\; \frac{(-2\pi)^{-(N-1-j)/2}}{\left(\frac{N-1-j}{2}\right)!}T^{N-1}s_N\mathrm{Tr}^M\left(R_t^{(N-1-j)/2}\right)$$

when $N-1-j=0$, and zero otherwise. Here t is any point on $S^{N-1}(T)$. Thus, all we need to do is compute the trace. We shall indicate how to do this, and leave the remaining algebra needed to obtain (7.7.9) to you.

The key point to note is that if X, Y, U, V are unit vectors in $T_t S^{N-1}(T)$, then

$$(7.7.10) \qquad R(X,Y,U,V) \;=\; \begin{cases} -T^{-2} & \text{if } (X,Y)=(U,V), \\ T^{-2} & \text{if } (X,Y)=(V,U), \\ 0 & \text{otherwise.} \end{cases}$$

This can be checked, for example, by taking a parametric representation of $S^{N-1}(T)$ to compute the Christoffel symbols of the first kind of (7.7.1) as the first step of the computations described above. Alternatively, at least for the non-zero cases, you can use the 'well known' (and easily checkable) result that the curvature of $S^2(T)$ is T^{-2} and then use (7.5.3) to go from two to general dimensions.

Once we have (7.7.10) then (7.7.8) and a little counting gives

$$\mathrm{Tr}\left(R^k\right) \;=\; (-1)^k T^{-2k} 2^{-k}\frac{(N-1)!}{(N-1-2k)!}.$$

This, along with some more algebra, will yield (7.7.9), as required.

Second Fundamental Form
We now need repeat the above for the scalar second fundamental form S_ν and also go a little further, deriving an expression for the trace of mixed powers of the form $R^k S_\nu^j$, $(j \leq N - 2k)$ on ∂T. We shall need these soon for computing the Lipschitz-Killing curvatures of manifolds with boundaries. (cf. (10.7.1).)

We start much as we did for the curvature tensor, by choosing a convenient set of bases. However, this time the 'natural' Euclidean basis is no longer natural, since our primary task is to parameterise the surface ∂T in a convenient fashion.

Thus, this time we start with $\{E_i^*\}_{1 \leq i \leq N-1}$, the natural basis determined by some atlas on ∂T. This generates $T(\partial T)$. It is then straightforward to enlarge this to a section $E^* = \{E_i^*\}_{1 \leq i \leq N}$ of $S(T)$, the sphere bundle of T, in such a way that, on ∂T, $E_N^* = \nu$, the inward pointing unit normal vector field on ∂T.

It then follows from the definition of the scalar second fundamental form (cf. (7.5.10)) that

$$(7.7.11) \qquad S_\nu = \sum_{i=1}^{N-1} \gamma_{iN} \otimes de_i^*,$$

where, by (7.5.11), the connection forms γ_{ij} (on $\mathbb{R}^N$) satisfy

$$(7.7.12) \qquad \gamma_{ij}(Y) = g(\nabla_Y E_i^*, E_j^*).$$

If we now define $\Gamma_{ijk}^* = g(\nabla_{E_i^*} E_j^*, E_k^*)$, then the connection forms γ_{iN} can be expressed[34] as

$$(7.7.13) \qquad \gamma_{iN} = \sum_{j=1}^{N-1} \Gamma_{jiN}^* \, de_j^*.$$

---

[34] It is often possible to write things in a format that is computationally more convenient. In particular, if the metric is Euclidean and if it is possible to explicitly determine functions a_{ij} so that

$$E_{it}^* = \sum_{k=1}^{N} a_{ik}(t) E_{kt},$$

then it follows trivially from the definition of the Γ_{jiN}^* that

$$\Gamma_{jiN}^*(t) = \sum_{k,l,m=1}^{N} a_{jk}(t) \frac{\partial}{\partial t_k} \big(a_{Nl}(t) \, a_{im}(t) \, g_{ml}(t) \big).$$

If, as for the curvature tensor, we now choose a smooth section X of $\mathcal{O}(T)$ with dual frames θ, such that, on ∂T, $X_N = \nu$, similar calculations yield that

$$S_\nu \;=\; \sum_{i,j=1}^{N-1} S_{ij}^X\, \theta_i \otimes \theta_j \;=\; \sum_{i=1}^{N-1} \theta_{iN} \otimes \theta_i,$$

where

$$S_{ij}^X \;\triangleq\; \sum_{i',j'=1}^{N-1} g_{ii'}^{-\frac{1}{2}} g_{jj'}^{-\frac{1}{2}} \Gamma_{j'i'N}, \qquad \theta_{iN} \;\triangleq\; \sum_{i=1}^{N-1} S_{ij}^X \theta_j.$$

Finally, on setting $p = 2k + j$, it follows that

(7.7.14)

$$R^k S_\nu^j \;=\; \frac{1}{2^k} \sum_{a_1,\ldots,a_p=1}^{N-1} \left(\textstyle\bigwedge_{l=1}^{k} \Omega_{a_{2l-1},a_{2l}}\right) \wedge \left(\textstyle\bigwedge_{m=1}^{j} \theta_{a_{2k+m}N}\right) \otimes \left(\textstyle\bigwedge_{l=1}^{p} \theta_{a_l}\right),$$

a formula we shall need in Chapter 12.

8
Piecewise smooth manifolds

So far, all that we have had to say about manifolds and calculus on manifolds has been of a local nature; i.e. it depended only on what was happening in individual charts. However, looking back at what we did in Sections 6.1–6.3 in the setting of Integral Geometry, this is not going to solve our main problem, which is understanding the global structure of excursion sets of random fields now defined over manifolds.

In order to handle this, we are going to need a reasonably heavy investment in notation leading to the definition of what we shall call *piecewise smooth spaces*. The investment will be justified by ultimately producing results, in Part III, that are both elegant and applicable. To understand the need for piecewise smooth spaces two simple examples should suffice, the sphere S^2, which is a C^∞ manifold without boundary, and the unit cube I^3, a flat manifold with a boundary made up of six faces which intersect at twelve edges, themselves intersecting at eight vertices. The cube, faces, edges and vertices are themselves flat C^∞ manifolds, of dimensions 3,2,1 and 0, respectively.

In the first case, if $f \in C^k(S^2)$, the excursion set $A_u(S^2, f)$ is made of smooth subsets of S^2, each one bounded by a C^k curve. In the second case, for $f \in C^k(I^3)$, while the individual components of $A_u(I^3, f)$ will have a C^k boundary away from ∂I^3, their boundaries will also have faces, edges and vertices where they intersect with ∂I^3. We already know from Section 6.2 that when we attempt to find point set representations for the Euler characteristics of excursion sets these boundary intersections are important. (e.g. (6.2.16) for the case of I^2.) This is even the case if the boundary of the parameter set is itself smooth (e.g. Theorem 6.2.5). Consequently, as

soon as we permit as parameter spaces manifolds with boundaries, we are going to require techniques to understand how these boundaries intersect with excursion sets. The current chapter will develop what we need in the setting of piecewise smooth spaces[1]. If you are interested only in parameter spaces which are manifolds without boundary then you can go directly to Chapter 9 and skip the current one. However, you will then have to forgo fully understanding how to handle excursion sets over parameter spaces as simple as cubes, which, from the point of view of applications, is a rather significant loss.

Finally, we note that as much of the material of this chapter is not completely standard material in differential geometry, even those readers comfortable with Chapter 7 may find it useful, at least for establishing notation.

8.1   Whitney stratified spaces

What we shall ultimately refer to as piecewise smooth spaces will be Whitney stratified spaces satisfying a local convexity condition. The basic reference for these is the monograph by Goresky and Macpherson [68], although we shall occassionaly also have need of material that can be found in Pflaum [113].

We start with a topological subspace M of a C^k ambient manifold $\widetilde{M}$. The term 'stratified' refers to a decomposition of M into strata which we shall take to be smooth manifolds. A Whitney-stratified space is a stratified subspace[2] $M \subset \widetilde{M}$ with some additional regularity constraints imposed. These regularity conditions are essential in setting up Morse theory for stratified spaces, which we describe in Chapter 9.

More formally, a C^l, $l \le k$, *stratified space*, or *decomposed space*, $M \subset \widetilde{M}$ is a subspace of $\widetilde{M}$ along with a partition $\mathcal{Z}$ of M such that

1. Each piece, or *stratum*, $S \in \mathcal{Z}$ is an embedded C^l submanifold of $\widetilde{M}$, without boundary.

2. For $R, S \in \mathcal{Z}$, if $R \cap \overline{S} \neq \emptyset$ then $R \subset \overline{S}$, and R is said to be *incident* to S.

We shall use the terminologies 'stratified space' and *stratified manifold* interchangeably.

---

[1]An alternative approach is via 'sets of finite reach' as developed in [61, 62] and its generalisations as in [173]. While this gives an approach both powerful and mathematically elegant, it involves a heavier background investment than is justified for our purposes.

[2]We shall henceforth drop the term 'subspace' in favour of space, with the understanding that, for our purposes, there is *always* some ambient $\widetilde{M}$ in which M is embedded

The partition $\mathcal{Z}$ has a natural partial order, namely $R \preceq S$ if $R \subset \overline{S}$. Thus we can treat $\mathcal{Z}$ either as a partition of M, or as an abstract partially ordered set.

To fix a concrete example of a stratified space take $M = I^3$. The natural decomposition is to decompose I^3 into the zero-dimensional vertices, the one-dimensional edges, the two-dimensional faces and the three dimensional interior. Similarly, if $M = \partial I^3$, the natural stratification is to decompose ∂I^3 into the vertices, the edges and the faces.

In general, a stratified space $M \subset \widetilde{M}$ with decomposition $\mathcal{Z}$ can be written as

$$M = \bigcup_{l=0}^{\dim M} \partial_l M$$

where $\partial_l M$ is the *l-dimensional boundary* of M made up from the disjoint union of a finite number of l-dimensional manifolds. Given $0 \le j \le \dim M$, the space

$$\bigcup_{l=0}^{j} \partial_l M$$

is again a decomposed space, with partition $\{S \in \mathcal{Z} : \dim(S) \le j\}$.

Although there is a natural decomposition of I^3 there are also many others, some of which are similar and some of which are not. For example, in the 'natural' decomposition we made above, we could further decompose the three dimensional interior of I^3 into the interior minus the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and this point. For an even simpler example note that $\mathbb{R}$ can be decomposed in many ways, three of which are

$$\begin{aligned}
\mathcal{Z}_1 &= \{(-\infty, 0), \{0\}, (0, \infty)\}, \\
\mathcal{Z}_2 &= \{\mathbb{R} \setminus \{0\}, \{0\}\}, \\
\mathcal{Z}_3 &= \mathbb{R}.
\end{aligned}$$

For the first two, the decompositions look locally the same at every point in $\mathbb{R}$. However, $\mathcal{Z}_3$ is fundamentally different at 0. *A priori*, this means that some of the expressions for the intrinsic volumes described in Section 10.7 seem as if they may depend on the decomposition of M. That this is not the case is a somewhat subtle issue which we shall return to when the need arises[3].

So far, we have not imposed any regularity on how the respective l-dimensional boundaries $\partial_l M$ are 'glued' together. In the case of I^3, and

---

[3] This point is treated in detail in [113], where a distinction is also made between a decomposition and a stratification, where the latter are equivalence classes of decompositions. However, since any decomposition uniquely determines a stratification, for our purposes we can assume that we are given a decomposition $\mathcal{Z}$ of M and talk about the stratification induced by $\mathcal{Z}$. [113] also raises the possibility of parameterizing strata by something known as 'depth' rather than by dimension.

the natural decomposition described above, the pieces all fit together in a nice fashion. For more general parameter spaces, we shall have to impose some further regularity on M.

Returning to I^3 for motivation, note that every point $t \in I^3$ has a neighbourhood that is isomorphic[4] to the product of an open neighbourhood of the stratum that contains t and a cone. For example, each vertex of I^3 has a neighbourhood that is isomorphic to a closed octant in $\mathbb{R}^3$, and every point t in an edge of I^3 has a neighbourhood isomorphic to the product of an open interval around 0 and a closed quadrant in $\mathbb{R}^2$.

This 'locally conic' property of I^3 will be essential to the Morse theory developed in Chapter 9 and for general stratified spaces follows from Whitney's conditions (A) and (B) below. These conditions are regularity conditions imposed on a stratified space that, in particular, imply that each point $t \in M$ has a neighbourhood isomorphic to the product of an open subset of the stratum S containing t and a cone $\mathrm{Cone}(L_S)$ over a stratified space L_S[5], the link of M at t.

A stratified space $(M, \mathcal{Z})$, is said to satisfy Whitney condition (A) at $t \in S$ if the following holds for every $S \preceq \widetilde{S}$.

> (A) If $t_n \to t \in S$ such that $t_n \in \widetilde{S}$ for all n and the sequence of tangent spaces $T_{t_n}\widetilde{S}$ converges in the Grassmanian bundle of $\dim(\widetilde{S})$-dimensional tangent (sub)spaces of $\widetilde{M}$[6] to some $\tau \subset T_t\widetilde{M}$, then $\tau \supset T_tS$. A limit of such tangent spaces is called a *generalised tangent space* of M at t.

In a chart (U, φ) on $\widetilde{M}$ containing t, a stratified space $(M, \mathcal{Z})$ is said to satisfy Whitney condition (B) at $t \in S$ if the following is satisfied for every $S \preceq \widetilde{S}$:

> (B) Let $t_n \to t$ and $s_n \to t$ be such that $t_n \in S, s_n \in \widetilde{S}$, for all n. Further, suppose that the sequence of line segments $\overline{\varphi(t_n)\varphi(s_n)}$ converges in projective space to a line ℓ and the sequence of tangent spaces $T_{s_n}\widetilde{S}$ converges in the Grassmannian to a subspace $\tau \subset T_t\widetilde{M}$. Then, $\varphi_*^{-1}(\ell) \subset \tau$.

---

[4]Two stratified spaces $(M_1, \mathcal{Z}_1)$ and $(M_2, \mathcal{Z}_2)$ of class C^l are said to be *isomorphic* if there exists a map $H : M_1 \to M_2$ that is an isomorphism of the partially ordered sets $\mathcal{Z}_1$ and $\mathcal{Z}_2$. That is, for each $S \in \mathcal{Z}_1$, $H(S) \in \mathcal{Z}_2$ and this map is an isomorphism when S and $H(S)$ are considered elements of the partially ordered sets $\mathcal{Z}_1$ and $\mathcal{Z}_2$, respectively. Furthermore, H is such that, for each $S \in \mathcal{Z}_1$, $H_{|S} : S \to H(S)$ is a diffeomorphism of class C^l.

[5]The cone $\mathrm{Cone}(L)$ over a topological space L is defined as the quotient space $Y = L \times [0, 1)$ where $y_1 \sim y_2 \iff y_1 = (x_1, 0), y_2 = (x_2, 0)$ for $x_1, x_2 \in L$

[6]The Grassmanian bundle referred to here is the natural one, namely the union of the Grassmanian manifolds of $\dim(\widetilde{S})$-dimensional subspaces of $T_t\widetilde{M}$ where the union is taken over all $t \in \widetilde{M}$, and the collection is parameterized similarly to the tangent bundle.

It is easy to show that condition (B) implies condition (A), and it is not too much more difficult to show that whether or not condition (B) is satisfied at t is independent of the chart. (cf. [113].) A stratified space $(M, \mathcal{Z})$ is called a *Whitney stratified space* if it satisfies Whitney condition (B) (and hence, condition (A)) at every $t \in M$.

Examples of Whitney stratified spaces abound, and include

- Finite simplicial complexes.

- Closed semialgebraic subsets of Euclidean spaces.

- Closed subanalytic subsets of Euclidean spaces.

- Basic complexes with C^2 boundary.

The last of these, of course, provides the link between this Chapter and the Integral Geometry of Chapter 6.

Furthermore, Whitney stratified spaces have many desirable properties, among them

- Whitney stratified spaces can be triangulated.

- Whitney stratified spaces have a well defined Euler characteristic.

- The intersection of two Whitney stratified spaces is generically a Whitney stratified space, whose strata are the intersections of the strata of the two spaces.

These three properties will all be of key importance to us, so we shall discuss them a little, including giving rough definitions of the terms we have used in stating them.

By a *triangulation* of a Whitney stratified space M we mean a covering of M by diffeomorphic images of simplices of dimension no more than $\dim(M)$, such that, if two such images have a non-empty intersection then the pre-images of the intersection must be full facets (sub-simplices) of each of the original simplices.

The simplices of the triangulation can be joined to form a single simplicial complex by joining them along the facets where their images intersected on M. We call this a *simplicial complex generated by M*, and denote it by $\mathcal{S}_M$. There is, of course, no uniqueness in such a triangulation. We can then define the *Euler*, or *Euler-Poincaré characteristic* of M as the alternating sum

$$(8.1.1) \qquad \varphi(M) \triangleq \sum_{j=0}^{\dim M} (-1)^{(N-j)} \alpha_j \left(\mathcal{S}_M \right),$$

where $\alpha_j \left(\mathcal{S}_M \right)$ is the number of j-dimensional facets in $\mathcal{S}_M$. Despite the fact that there is no uniqueness for triangulations and so the right hand

side here would seem to depend on $\mathcal{S}_M$, it is a basic theory of Algebraic Topology that the Euler characteristic is well defined and independent of the triangulation.

Finally, we turn to the issue of tranverse intersections. We say that two Whitney stratified submanifolds, M_1 and M_2, subsets of the same ambient N-dimensional manifold $\widetilde{M}$ *intersect transversally* if, for each pair (j, k) with $0 \leq j \leq \dim(M_1)$ and $0 \leq k \leq \dim(M_2)$, and each $t \in \partial_j M_1 \cap \partial_k M_2$, the dimension of

$$(8.1.2) \qquad \operatorname{span} \left\{ X_t + Y_t : X_t \in T_t^\perp \partial_j M_1, \; Y_t \in T_t^\perp \partial_k M_2 \right\}$$

is equal to $2N - j - k \geq 0$.

Now suppose that $f \in C^k(\widetilde{M})$ is such that its excursion set over $\widetilde{M}$, $A_u(f, \widetilde{M}) = f^{-1}[u, \infty)$ is a Whitney stratified manifold. If M is a Whitney stratified submanifold of $\widetilde{M}$, and if $A_u(f, \widetilde{M})$ and M intersect transversally, then $A_u(f, M)$ will also be a Whitney stratified manifold with a stratification inherited from those of $A_u(f, \widetilde{M})$ and M.

The punchline of the above paragraph is that if we start with Whitney stratified manifolds as the parameter spaces of random fields, and if our random fields behave well, then excursion sets will also be Whitney stratified manifolds. This closure, together with the fact that Whitney stratified manifolds provide a natural setting for both smooth and angular parameter spaces, is one of the main reasons that this class of sets will be, for us, the right choice for Part III.

8.2   Piecewise smooth spaces

For much of what follows, we are going to have to slightly limit Whitney stratified manifolds to those which are, in an appropriate sense, locally convex.

To make this precise, consider the lines ℓ appearing in condition (B) defining a Whitney stratified manifold. While these have no direction associated with them, by fixing $t_n = t, \forall n$, we can think of the vector $\varphi(t) - \varphi(s_n)$ as a line segment emanating from t in the direction of s_n. The set of all limiting directions

$$(8.2.1) \quad \mathcal{S}_t M \quad \triangleq \quad \left\{ X_t \in T_t \widetilde{M} : \exists\, \delta > 0, \; c \in C^1 \left((-\delta, \delta), \widetilde{M} \right), \right.$$
$$\left. c(s) \in M \;\; \text{for all } s \in [0, \delta), \; c(0) = t, \; \dot{c}(0) = X_t \right\}$$

is a cone in $T_t \widetilde{M}$ known as the *support cone* of M at t.

Figure 8.2.1 shows a part of the (shaded) support cones for two domains in $\mathbb{R}^2$, where the bases of the cones are the points at the base of the concavity in each domain. Note that while neither of the domains is itself convex,

the smooth domain always has convex support cones, while the domain with the concave cusp does not.
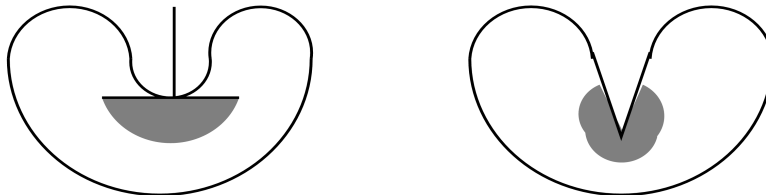

FIGURE 8.2.1. Convex and non-convex support cones

It turns out that C^l Whitney stratified spaces $(M, \mathcal{Z})$ have simple local representations, in that every point $t \in S \subset M$ has a neighbourhood $U \subset \widetilde{M}$ such that $U \cap M$ is homeomorphic to $(U \cap S) \times \text{Cone}(L_S)$, where L_S itself is a compact Whitney stratified space (cf. [113]). Furthermore, the homeomorphism can be chosen to be C^l on each stratum.

For an example, consider the important special case of convex polytopes in Euclidean spaces. Such spaces have canonical stratifications given by the facets. In this case, for a given $t \in M$, the 'link' L_S is naturally identified with the unit normal vectors of the supporting hyperplanes at t, chosen to point towards the interior of M. Furthermore, if we denote the span of a facet S by $[S]$ it is clear that $[S] \times \text{Cone}(L_S)$ is $\mathcal{S}_t M$. (cf. [131].)

In general, the set $(U \cap S) \times \text{Cone}(L_S)$ can be thought of as a subset of $\mathcal{S}_t M$, if we choose an appropriate chart (φ, U) on $\widetilde{M}$. In this sense, the set M is locally approximated by (homeomorphic to) the disjoint union

$$\bigcup_{t \in S} \mathcal{S}_t M$$

of support cones[7].

We now move from support cones to their duals, for which we need to assume that $\widetilde{M}$ has a Riemannian structure. This we do, writing $\widetilde{g}$ for the Riemannian metric on $\widetilde{M}$ and g or $\widetilde{g}_{|\partial_j M}$ for the metric it induces on M or $\partial_j M$.

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[7]In general, the homemorphism can not necessarily be chosen to be smooth in the sense of smooth mappings between stratified spaces [113]. This is illustrated in the following simple example from [113]. Consider the graph

$$\Gamma = \{(t, |t|^{1/3}), t \in (-1, 1)\},$$

which is a Whitney stratified space. Γ is homeomorphic to $\text{Cone}(\{-1, 1\})$, which we can identify with the graph

$$\{(t, |t|), t \in (-1, 1)\}.$$

However, there is no diffeomorphism of the plane that maps Γ to $\text{Cone}(\{-1, 1\})$ – this implies that any homeomorphism $\psi : \Gamma \to \text{Cone}(\{-1, 1\})$ is not smooth as defined in [113]. Nevertheless, it still preserves topological properties.

This allows us to define a dual to each support cone $\mathcal{S}_t M$, known as the *normal cone* of t at M and defined by

$$(8.2.2) \qquad N_t M \;\triangleq\; \left\{ X_t \in T_t \widetilde{M} : \widetilde{g}(X_t, Y_t) \le 0 \;\; \text{for all } Y_t \in \mathcal{S}_t M \right\}$$

In the two domains in Figure 8.2.1, assuming the usual Euclidean metric, the normal cones at the base of the concavity are the outward normal in the first case, and empty in the other.

A more interesting example is given in Figure 8.2.2 for the tetrahedron. In this case we have shown the (truncated) normal cones at (a) a vertex (b) a point on an edge, and (c) a point on a face. What remains is a point in the interior of the tetrahedron, for which the normal cone is empty. Note that in each case the dimension of the normal cone is the codimension of the stratum in which the base point sits, with respect to that of the ambient manifold (in this case, $\mathbb{R}^3$).

(a)   (b)   (c)
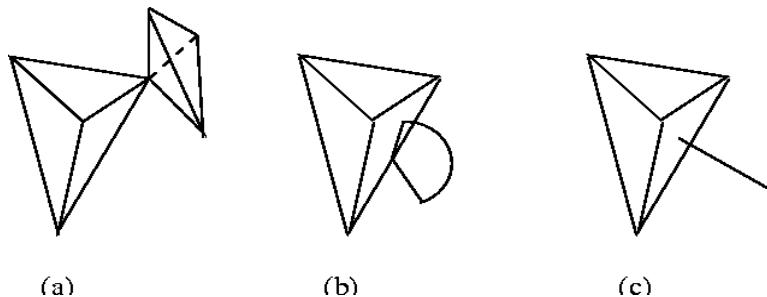
FIGURE 8.2.2. Normal cones for a tetrahedron.

We now have all we need to define those Whitney stratified spaces that we shall work with in the future.

Definition 8.2.1 *A Whitney stratified space* $(M, \mathcal{Z})$ *is called a piecewise smooth space if the support cone* $\mathcal{S}_t M$ *is convex for every* $t \in M$.

Under this definition, the smooth domain of Figure 8.2.1 is a piecewise smooth space, while the domain with concave cusp is not. These examples are actually quite generic, since the main import of the convex support cone assumption is to exclude sharp, concave cusps. Similarly, I^N is a piecewise smooth space, but ∂I^N is not.

9
Critical point theory

In the preceding chapter we set up the two main geometric tools that we shall need in Part III of the book. Piecewise smooth manifolds will serve there as parameter spaces for our random fields, as well as appearing in the proofs, and Lipschitz-Killing curvatures will appear in the answers to the questions we shall ask. Between the questions and the answers will lie considerable computation, and the main geometric tool that we shall need there is the topic of this short chapter.

Critical point theory, also known as Morse theory, is a technique for describing various global topological characteristics of manifolds via the local behaviour, at critical points, of functions defined over the sets. We have already seen a version of this back in Section 6.2, where we obtained point set representations for the Euler characteristic of excursion sets. (cf. Theorems 6.2.4 and 6.2.5, which gave point set representations for excursion sets in $\mathbb{R}^2$, over squares and over bounded sets with C^2 boundaries.) Our aim now is to set up an analagous set of results for the excursion sets of C^2 functions defined over C^3 piecewise smooth spaces. We shall show in Section 9.4 how to specialise these back down to the known, Euclidean examples of Section 6.2.

A full development of this theory for manifolds, which goes well beyond what we shall need, is in the classic treatise of Morse and Cairns [110], but you can also find a very readable introduction to this theory in the recent monograph of Matsumoto [105]. The standard theory, however, concentrates on smooth manifolds, as opposed to the piecewise smooth case that we need. The standard reference in this case is the excellent monograph of Goresky and MacPherson [68].

9.1   Critical points

We begin with a general definition of critical points. Let $(M, \mathcal{Z})$ be a C^l Whitney stratified space embedded in a C^k ambient N-manifold $\widetilde{M}$. For $\widetilde{f} \in C^2(\widetilde{M})$, a *critical point* of $\widetilde{f}$ is a point $t \in \widetilde{M}$ such that $\nabla \widetilde{f}_t = 0$. Points which are not critical are called *regular*.

Now take M to be compact, N-dimensional and C^2 piecewise smooth, embedded in a C^3 ambient manifold $(\widetilde{M}, \widetilde{g})$, writing, as usual, g for the indiced metric on M. Extending the notion of critical points to $f = \widetilde{f}_{|M}$ requires taking note of the fact that the various boundaries in M are of different dimension and so, in essence, involves repeating the above definition for each $f_{|\partial_j M}$. However, our heavy investment in notation now starts to pay dividends, since it is easy to see from the general definition that a point $t \in \partial_j M$, for some $0 \leq j \leq N$, is a critical point if, and only if,

$$(9.1.1) \qquad\qquad \nabla \widetilde{f}_t \in T_t^{\perp} \partial_j M.$$

(cf. (7.5.6) for notation.) Thus we need work only with the single function f and not, explicitly at least, with its various restrictions[1].

This definition implies that all points in $\partial_0 M$ are to be considered as critical points, and, when $\dim(M) = \dim(\widetilde{M})$, that critical points of $f_{|\partial_N M} \equiv \widetilde{f}_{|M^\circ}$ are just critical points of $\widetilde{f}$ in the sense of the initial definition.

We call the set

$$\bigcup_{j=0}^{N} \left\{ t \in \partial_j M : \nabla \widetilde{f}_t \in T_t^{\perp} \partial_j M \right\}$$

the *set of critical points of* $f_{|M}$. All other points are known as *regular* points.

A critical point $t \in \partial_j M$ of $f_{|M}$ is called *non-degenerate* if the covariant Hessian $\nabla^2 f_{|T_t \partial_j M}$ is non-degenerate, when considered as a bilinear mapping. A function $f \in C^2(\widetilde{M})$ is said to be non-degenerate on M if all the critical points of $f_{|M}$ are non-degenerate. The *tangential Morse index* $\iota_{f, \partial_j M}(t)$ of a non-degenerate critical point $t \in \partial_j M$ of $f_{|M}$ is the dimension of the largest subspace L of $T_t \partial_j M$ such that $\nabla^2 f(t)\big|_L$ is negative definite. Thus, a point of tangential Morse index zero is a local minimum of f on $\partial_j M$, while a point of index j is a local maximum. Other indices correspond to saddle points of various kinds. Before defining Morse functions on stratified Whitney spaces, we need one further concept: the normal Morse index, which, as its name suggests, complements the tangential Morse index.

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[1]This assumes, however, that one remembers where all these spaces are sitting, or (9.1.1) makes little sense. Assuming that the ambient space $\widetilde{M}$ is N-dimensional ($N = \dim(M)$) we have, on the one hand, that $\nabla \widetilde{f}$ is also N-dimensional, whereas $T_t^{\perp} \partial_j M$ is $(N-j)$-dimensional. It is important, therefore, to think of $T_t^{\perp} \partial_j M$ as a subspace of $T_t^{\perp} M$ for the inclusion to make sense. Overall, all of these spaces are dependent on $\widetilde{M}$ and its Riemannian metric.

9.2   The normal Morse index

The *normal Morse index*, α, is a measure of local change in the topology of the manifold. In that sense it is very much like the Euler-Poincaré functional φ of (6.1.10). However, unlike φ, it records this change for any normal direction, and later we shall take averages over all directions. In this way we overcome the drawback of the theory of Chapter 6, which was axis dependent. (cf. Footnote 1 there.)

Beyond its use in Morse theory, α will also play a key rôle in the definition of Lipschitz-Killing curvature measures for Whitney stratified spaces in Chapter 10.

Let $(M, \mathcal{Z})$ be Whitney stratified submanifold of an ambient Riemannian manifold $(\widetilde{M}, \widetilde{g})$ and write

$$T^\perp M \;=\; \bigcup_{S \in \mathcal{Z}} T^\perp S$$

for the the *stratified normal bundle* of $(M, \mathcal{Z})$. (cf. (7.5.6).)

We next define the *normal slice* at $t \in S \subset M$ as

$$\mathcal{N}_t \;=\; \exp_{(\widetilde{M}, \widetilde{g})} \left(\left\{ X_t \in T_t^\perp S : |X_t| \leq \rho \right\} \right),$$

where ρ is chosen small enough so that the exponential map is a diffemorphism when restricted to $|X_t| \leq \rho$.

Taking normal coordinates (U_t, φ_t) (cf. (7.3.16)) on $\widetilde{M}$ at t, it follows that

$$\varphi_t(\mathcal{N}_t) \;=\; L_t \cap \varphi_t(U_t),$$

for some linear subspace $L_t \subset \mathbb{R}^{\dim(\widetilde{M})}$ which can be thought of as $T_t^\perp S$.

Fix a unit vector $\nu \in T_t^\perp S$, and choose $(\delta_0, \varepsilon_0)$ small enough so that the topology of

$$\varphi_t(U_t \cap M) \cap B_{\mathbb{R}^{\dim(\widetilde{M})}}(0, \delta) \cap \left\{ t \in \mathbb{R}^{\dim(\widetilde{M})} : \langle t, \nu \rangle = -\varepsilon \right\} \;\subset\; \mathbb{R}^{\dim(\widetilde{M})}$$

is unchanged for $0 < \delta < \delta_0$ and $0 < \varepsilon < \varepsilon_0$. This set is easily seen to be a Whitney stratified manifold, and so has a well defined Euler characteristic, as in (8.1.1). For $0 < \delta < \delta_0$ and $0 < \varepsilon < \varepsilon_0$, denote it by $\chi(\nu)$.

It is not immediately clear that $\alpha(\nu)$ is actually well defined, i.e. that the above Euler characteristic depends only on the vector $\nu \in T^\perp S$ (more precisely on its identification with a co-vector canonically determined by $\tilde{g}^2$) and not on the choice of chart φ_t (which depends on the Riemannian

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[2] Note that the Morse index can actually be defined independently of the Riemannian metric, although then we would need to define it on the stratified co-normal bundle of M. As all our examples are Riemannian, we prefer to think of α as an integer valued map on the stratified normal bundle instead.

metric $\tilde{g}$). However, this is indeed the case, as is established in Theorem 7.5.1 of [68].

Another, more delicate, problem may arise when ν 'annihilates' a generalised tangent space of M at t, in the sense that $\tilde{g}(\nu, X) = 0$ for all X in the space. Such situations are, however, rather atypical, and so we shall ultimately assume that our sets are such that this problem never arises, almost surely.

Vectors ν for which there exists such generalised tangent spaces are referred as *degenerate tangent vectors*[3]. More precisely, for fixed t, the set of degenerate tangent vectors is a closed cone in $T_t \widetilde{M}$ and the collection of all degenerate tangent vectors,

$$(9.2.1) \quad \bigcup_{t \in M} \left\{ \nu_t \in T_t \widetilde{M} : \nu_t \text{ is a degenerate tangent vector of } M \text{ at } t \right\},$$

glued together across each $t \in M$ is a locally conic space. We shall ultimately assume that this space has Hausdorff measure less then N, the dimension of $\widetilde{M}$. This is necessary for the Morse theory of excursion sets of random fields on manifolds, discussed in Chapter 12 as well as to define the Lipschitz-Killing curvatures of Whitney stratified spaces in Chapter 10.

Finally, we define the *Morse index* of M at t in the direction ν to be

$$\alpha(\nu) \triangleq 1 - \chi(\nu).$$

This definition is not as complex as it first seems, and you should try some examples. For example, if $M = [-1, 1] \times \{0\} \cup \{0\} \times [-1, 1]$ is a $+$ in the plane, then $\alpha(\nu) = -1$ for all $\nu \in T_0 \mathbb{R}^2$ unless ν is parallel to one of the axes. In this case $\alpha(\nu) = 0$. If, on the other hand $M = [0, 1] \times \{0\} \cup \{0\} \times [0, 1]$, then $\alpha(\nu)$ can only take one of the values -1, 0 or 1, depending on the angle $\theta = \theta(\nu)$ that ν makes with the x-axis. Specifically,

$$\alpha(\nu) = \begin{cases} 1 & \theta \in (0, \pi/2), \\ 0 & \theta \in [\pi/2, \pi] \cup [3\pi/2, 2\pi], \\ -1 & \theta \in (\pi, 3\pi/2). \end{cases}$$

The Morse index takes a particularly simple form for convex polytopes. In this case it is easy to check that

$$(9.2.2) \qquad \alpha(\nu) = \begin{cases} 1 & -\nu \in (N_t M)^\circ \\ 0 & \text{otherwise,} \end{cases}$$

where $N_t M$ is the normal cone of $(8.2.2)$ under the Euclidean metric.

---

[3]For example, if we decompose $\mathbb{R} \subset \mathbb{R}^2$ as $(-\infty, 0) \times \{0\}, \{(0,0)\}, (0, \infty) \times \{0\}$, then the set of degenerate tangent vectors at 0 correspond to the y-axis, as there is only one generalised tangent space, the x-axis.

In fact, (9.2.2) also holds if M is a Whitney stratified manifold with convex support cones; *viz.* if M is a piecewise smooth space in the sense of the previous section. We shall use this fact heavily in Part III.

9.3   Morse's theorem for stratified spaces

In this section, we conclude the generalisation of the point set representation of the Euler characteristic we discussed in Sections 6.1 and 6.2 to Whitney stratified spaces. The point set representation is given by Morse's theorem below.

The last definition we need before stating Morse's theorem is that of a Morse function. A function $f \in C^2(\widetilde{M})$ is called a *Morse function* on M if, for each k, $f_{|\partial_k M}$ it is non-degenerate on $\partial_k M$ and the restriction of f to $\overline{\partial_k M} = \bigcup_{j=0}^{k} \partial_j M$ has no critical points on $\bigcup_{j=0}^{k-1} \partial_j M$. This is equivalent to requiring that, for each critical point t of $f_{|\partial_k M}$, $\nabla f_{|\partial_k M, t}$ is a non-degenerate tangent vector. (cf. (9.2.1) and the discussion preceeding it.)

With all the definitions cleared up, we have the necessary ingredients to state the following version of Morse's Theorem, due to Goresky and MacPherson [68].

Theorem 9.3.1 (Morse's Theorem) *Let $(M, \mathcal{Z})$ be a C^2 Whitney stratified space embedded in a C^3 Riemannian manifold $(\widetilde{M}, \widetilde{g})$ and $\widetilde{f} \in C^2(\widetilde{M})$ be a Morse function on M. Then, setting $f = \widetilde{f}_{|M}$*

(9.3.1)

$$\varphi(M) \;=\; \sum_{j=0}^{N} \sum_{\{t \in \partial_j M : \nabla \widetilde{f}_t \in T_t^{\perp} \partial_j M\}} (-1)^{\iota_{f, \partial_j M}(t)} \alpha(P_{T_t \partial_j M}^{\perp} \nabla \widetilde{f}_t),$$

where $P_{T_t \partial_j M}^{\perp}$ is orthogonal projection onto $(T_t \partial_j M)^{\perp}$ and $\varphi(M)$ is the Euler characteristic of M.

If the support cones $\mathcal{S}_t M$ are convex for each $t \in M$, i.e. M is a piecewise smooth space, then, following the discussion on convex polytopes, the above theorem reads as follows:

Corollary 9.3.2 (Morse's Theorem for Piecewise Smooth Spaces) *Let $(M, \mathcal{Z})$ be a C^2 piecewise smooth space embedded in a C^3 Riemannian manifold $(\widetilde{M}, \widetilde{g})$ and $f \in C^2(\widetilde{M})$ be a Morse function on M. Then,*

$$(9.3.2) \quad \varphi(M) \;=\; \sum_{j=0}^{N} \sum_{\{t \in \partial_j M : \nabla \widetilde{f}_t \in T_t^{\perp} \partial_j M\}} (-1)^{\iota_{f, \partial_j M}(t)} \mathbb{1}_{\{-\nabla \widetilde{f}_t \in N_t M\}},$$

where $\varphi(M)$ is the Euler characteristic of M.

The points counted in the above corollary are given a special name

Definition 9.3.3 *In the above setup, we call a point $t \in M$ an extended inward critical point of $f_{|M}$ if, and only if,*

$$-\nabla f_t \in N_t(M),$$

where $N_t(M)$ is the normal cone of (8.2.2). Similarly, t is an extended outward critical point if, and only if, $\nabla f_t \in N_t(M)$.

Morse's Theorem is a deep and important result in Differential Topology and is actually somewhat more general than as stated here, since in its full form it also gives a series of inequalities linking the Betti numbers[4] of M to the critical points of Morse functions on M. We shall make no attempt to prove Morse's Theorem, which, given our definition of the Euler characteristic, relies on arguments of Homology Theory and Algebraic Geometry. As mentioned above, [105] and [110] have all the details. Theorem 9.3.1, as presented here, is essentially proven in [139], although the notation and terminology there is a little different to ours.

Soon, in Section 9.4, we shall see how all of this impacts on simple Euclidean examples, in which case we shall recover the Integral Geometric results of Section 6.2 for which we *did* have full proofs.

What we shall prove now is the following corollary, which is actually what we shall be using in the future. The proof is included since, unlike that for Morse's Theorem itself, it does not seem to appear in the literature.

Corollary 9.3.4 *Let M and $\widetilde{M}$ be as in Morse's Theorem. Let $f \in C^2(\widetilde{M})$ be a Morse function on M, and let $u \in \mathbb{R}$ be a regular value of $f_{|\partial_j M}$ for all $j = 0, \ldots, N$. Then,*

$$(9.3.3) \quad \varphi\left(M \cap \widetilde{f}^{-1}[u, +\infty)\right)$$

$$= \sum_{j=0}^{N} \sum_{\{t \in \partial_j M : \widetilde{f}_t \geq u, \nabla \widetilde{f}_t \in T_t^{\perp} \partial_j M\}} (-1)^{\iota - f, \partial_j M(t)} \alpha(-P^{\perp}_{T_t \partial_j M} \nabla \widetilde{f}_t).$$

If M is a piecewise smooth space, i.e. the support cones $\mathcal{S}_t M$ are convex for each $t \in M$, then

$$(9.3.4) \quad \varphi\left(M \cap \widetilde{f}^{-1}[u, +\infty)\right)$$

$$= \sum_{j=0}^{N} \sum_{\{t \in \partial_j M : f_t > u, \nabla \widetilde{f}_t \in T_t^{\perp} \partial_j M\}} (-1)^{\iota - f, \partial_j M(t)} \mathbb{1}_{\{\nabla \widetilde{f}_t \in N_t M\}}.$$

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[4]Betti numbers are additional geometrical invariants of M of somewhat less straightforward interpretation than the Euler characteristic.

Proof. As usual, write $A_u = M \cap \widetilde{f}^{-1}[u, +\infty)$. If $-\widetilde{f}$ were a Morse function on A_u, and if we changed the condition $\widetilde{f}_t > u$ to $\widetilde{f}_t \geq u$ on the right hand side of (9.3.3), then the corollary would merely be a restatement of Morse's Theorem, and there would be nothing to prove. However, the change is not obvious, and $-f$ is not a Morse function on A_u, since it is constant on the strata of A_u that are subsets of $M \cap \widetilde{f}^{-1}\{u\}$.

The bulk of the proof involves finding a Morse function $\widehat{f}$ on A_u that agrees with $-\widetilde{f}$ on "most" of this set (thus solving the problem of the 'non-Morseness' of $\widetilde{f}$) and which, at the same time, has critical points in a one-to-one correspondence with the critical points of f on M above the level u. (This allows us to replace $\widetilde{f}_t \geq u$ with $\widetilde{f}_t > u$).

The space $M \cap f^{-1}[u, +\infty)$ can be decomposed into j-dimensional strata of the form $\partial_j M \cap f^{-1}(u, +\infty)$ and $\partial_{j+1} M \cap f^{-1}\{u\}$. More formally, consider $\widetilde{f}$ as a function on $\widetilde{M}$, which, recall, is assumed to have no boundary. Then

$$\partial_N \left(\widetilde{f}^{-1}[u, +\infty) \right) = \widetilde{f}^{-1}(u, +\infty),$$
$$\partial_{N-1} \left(\widetilde{f}^{-1}[u, +\infty) \right) = \widetilde{f}^{-1}\{u\},$$
$$\partial_j \left(\widetilde{f}^{-1}[u, +\infty) \right) = \emptyset, \qquad \text{for } j = 0, \dots, N - 2.$$

Since $\widetilde{f}$ is a Morse function and u is a regular point for $f_{|\partial_k M}$ for all k, it follows that M and $f^{-1}[u, \infty)$ intersect transversally as subsets of $\widetilde{M}$. Therefore A_u is a piecewise smooth subspace of $\widetilde{M}$ which can be decomposed as

$$(9.3.5) \quad A_u = \bigcup_{j=0}^{N} \left(\partial_j M \cap \widetilde{f}^{-1}(u, +\infty) \right) \cup \left(\partial_{j+1} M \cap \widetilde{f}^{-1}\{u\} \right)$$
$$= \left(\bigcup_{j=0}^{N} \partial_j M \cap \widetilde{f}^{-1}(u, +\infty) \right) \cup \left(\bigcup_{j=0}^{N-1} \partial_{j+1} M \cap \widetilde{f}^{-1}\{u\} \right)$$

We need to somehow get rid of the second term here.

Since f is a Morse function on M, it has only finitely many critical points inside a relatively compact neighbourhood V of M. Furthermore, again exploiting the fact that u is a regular value of $f_{|\partial_k M}$ for every k, there exists an $\varepsilon > 0$ such that

$$U_\varepsilon = \widetilde{f}^{-1}(u - \varepsilon, u + \varepsilon) \cap M \cap V$$

contains no critical points of $f_{|M}$. It is standard fare that there exists a $h \in C^2(\widetilde{M})$ which is a Morse function on $f^{-1}\{u\}$ and which h is zero outside of U_ε. Furthermore, since $\overline{V}$ is compact, there exist K_f and K_h

such that $|\nabla h| < K_h$ and

$$|P_{T_t \partial_j M} \nabla \widetilde{f}| \; > \; K_f,$$

for all $t \in \partial_j M \cap U_\varepsilon$, $1 \le j \le N$.

It then follows that the function

$$\widehat{f} \;\triangleq\; -\widetilde{f} + \frac{K_f}{3K_h} h$$

is a Morse function on A_u. By our choice of h, the critical points of $\widehat{f}_{|A_u}$ agree with those of $f = \widetilde{f}_{|M}$ on $M \cap U_\varepsilon^c$. Furthermore, $\nabla f_{|\partial_j M} \equiv \pi_{T_t \partial_j M} \nabla f$ can never be zero on $U_\varepsilon \cap \partial_k M$, and so there are no critical points of $\widehat{f}$ at all in this region. Consequently, the critical points of $\widehat{f}$ on $M \cap \widetilde{f}^{-1}[u, +\infty)$ are in one-to-one correspondence with those of $\widetilde{f}$ on $M \cap \widetilde{f}^{-1}(u, +\infty)$ from which the conclusion follows.   □

9.4   The Euclidean case

With basic Morse Theory for piecewise smooth spaces under our belt, it is now time to look at one rather important example for which everything becomes quite simple. The example is that of the N-dimensional cube $I^N = [0,1]^N$, and the ambient space is $\mathbb{R}^N$ with the usual Euclidean metric. In particular, we want to recover Theorem 6.2.4, which gave a point set representation for the Euler characteristic of the excursion set of smooth function over the square.

To recover Theorem 6.2.4 for the unit square we use Morse's Theorem 9.3.1 in its undiluted version. Reserving the notation f for the function of interest which generates the excursion sets, write the f of Morse's Theorem as f_m. We are interested in computing the Euler characteristic of the set

$$A_u \;=\; \left\{ t \in I^2 : f(t) \ge u \right\}$$

and will take as our Morse function the 'height function'

$$f_m(t) \;=\; f_m(t_1, t_2) \;=\; t_2.$$

Now assume that f is 'suitably regular' in the sense of Definition 6.2.1. This is almost enough to guarantee that f_m is a Morse function over I^2 for the ambient manifold $\mathbb{R}^2$. Unfortunately, however, all the points along the top and bottom boundaries of I^2 are degenerate critical points for f_m. We get around this by replacing I^2 with a tilted version, I_ε^2, obtained by rotating the square through ε degrees, as in Figure 9.4.1.
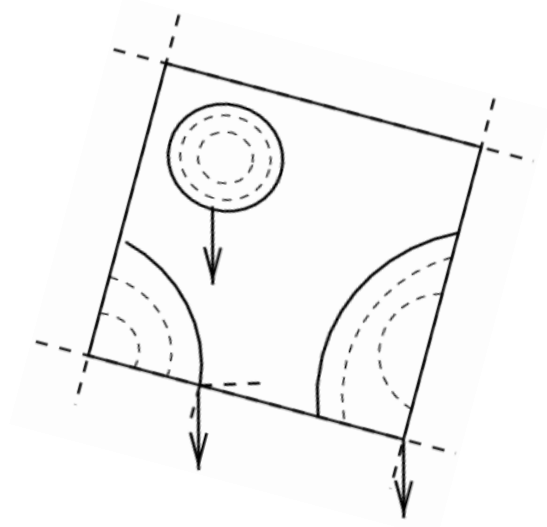
FIGURE 9.4.1. The Euler characteristic via Theorem 9.3.1

To compute $\varphi(A_u)$ we now apply (9.3.1), and so need to characterise the various critical points and their indices. The first fact to note is that, using the usual coordinate system, $\nabla f_m = (0, 1)$, and so there are no critical points of f_m in A_u°. Thus we can restrict interest to the boundary ∂A_u which we break into three parts:

(i) Points $t \in (I_\varepsilon^2)^\circ \cap \partial A_u$.

(ii) Points $t \in \partial I_\varepsilon^2 \cap A_u$, but not vertices of the square.

(iii) The four vertices of the square.

An example of each of these three classes appears in Figure 9.4.1, where the excursion set of f appear along with contour lines in the interiors of the various components.

At points of type (i), $f(t) = u$. Furthermore, since the normal cone $N_t(A_u)$ is then the one dimensional vector space normal to ∂A_u, $-\nabla f_m = (0, -1) \in N_t(A_u)$ at points for which $\partial f / \partial t_1 = 0$ and $\partial f / \partial t_2 > 0$. Such a point is at the base of the arrow coming out of the disk in Figure 9.4.1. Differentiating between points which contribute +1 and -1 to the Euler characteristic involves looking at $\partial^2 f / \partial t_1^2$. Comparing with Theorem 6.2.4, we see we have characterised the contributions of (6.2.17) and (6.2.18) to the Euler characteristic.

We now turn to points of type (ii). Again due constancy of ∇f_m, this time on ∂I_ε^2, there are no critical points to be counted on $(\partial I_\varepsilon^2 \cap A_u)^\circ$. We can therefore add the end points of the intervals making up $\partial I_\varepsilon^2 \cap A_u$ to those of type (iii). One of these appears as the base of the left most

arrow on the base of Figure 9.4.1. The rightmost arrow extends from such a vertex.

For points of these kinds, the normal cone is a closed wedge in $\mathbb{R}^2$, and it is left to you to check that that the contributions of these points correspond (on taking $\varepsilon \to 0$) to those of those described by (6.2.16).

This gives us Theorem 6.2.4, which trivially extends to any rectangle in $\mathbb{R}^2$. You should now check that Theorem 6.2.5, which computed the Euler characteristic for a subset of $\mathbb{R}^2$ with piecewise C^2 boundary, also follows from Morse's Theorem, using the same f_m.

The above argument was really unnecessarily complicated, since it did not use Corollary 9.3.4, which we built specifically for the purpose of handling excursion sets. Nevertheless, it did have the value of connecting the Integral Geometric and Differential Geometric approaches.

Now we apply Corollary 9.3.4. We again assume that f is suitably regular at the level u in the sense of Definition 6.2.1, which suffices to guarantee that it is a Morse function over the C^2 piecewise smooth space I^N for the ambient manifold $\mathbb{R}^N$ and that the conditions of the corollary apply.

Write $\mathcal{J}_k \equiv \partial_k I^N$ for the collection of faces of dimension k in I^N (cf. (6.2.2)). With this notation, we can rewrite the sum (9.3.3) as

$$(9.4.1) \qquad \varphi\left(A_u(f, I^N)\right) \;=\; \sum_{k=0}^{N} \sum_{J \in \mathcal{J}_k} \sum_{i=0}^{k} (-1)^i \, \mu_i(J),$$

where, for $i \leq \dim(J)$,

$$\mu_i(J) \;\triangleq\; \#\left\{t \in J : \nabla f_{|J}(t) = 0, \iota_{-f,J}(t) = i\right\}.$$

Recall that to each face $J \in \mathcal{J}_k$ there corresponds a subset $\sigma(J)$ of $\{1, \ldots, N\}$, of size k, and a sequence of $N - k$ zeroes and ones $\varepsilon(J) = \{\varepsilon_1, \ldots, \varepsilon_{N-k}\}$ so that

$$J \;=\; \left\{t \in I^N : t_j = \varepsilon_j, \text{ if } j \notin \sigma(J), \;\; 0 < t_j < 1, \text{ if } j \in \sigma(J)\right\}.$$

Set $\varepsilon_j^* = 2\varepsilon_j - 1$. Working with the definition of the C_i, it is then not hard to see that $\mu_i(J)$ is given by the number of points $t \in J$ satisfying the following four conditions:

$$(9.4.2) \qquad\qquad\qquad\qquad\qquad f(t) \;\geq\; u,$$
$$(9.4.3) \qquad\qquad\qquad\qquad\qquad f_j(t) \;=\; 0, \qquad j \in \sigma(J)$$
$$(9.4.4) \qquad\qquad\qquad\qquad\qquad \varepsilon_j^* f_j(t) \;>\; 0, \qquad j \notin \sigma(J)$$
$$(9.4.5) \qquad\qquad \text{index}\left(f_{mn}(t)\right)_{(m,n \in \sigma(J))} \;=\; k - i,$$

where, as usual, subscripts denote partial differentiation, and, consistent with the definition of the index of a critical point, we define the *index of a matrix* to be the number of its negative eigenvalues.
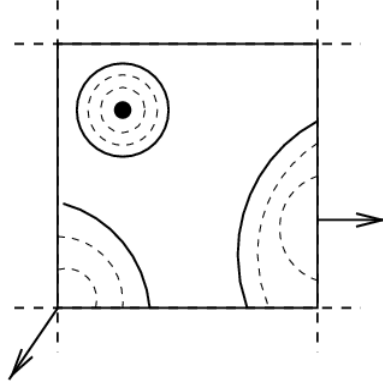
FIGURE 9.4.2. The Euler characteristic via Corollary 9.3.4

In Figure 9.4.2 there are three points which contribute to $\varphi(A_u(f, I^2))$. One, in the centre of the upper left disk, contributes via $J = (I^2)^\circ = \mathcal{J}_2$. That on the right side contributes via $J =$'right side'$\in \mathcal{J}_1$, and that on the lower left corner via $J = \{0\} \in \mathcal{J}_0$.

The representation (9.4.1) of the Euler characteristic of an excursion set, along with the prescription in (9.4.2)–(9.4.5) as to how to count the contributions of various points to the sum, is clearly a tidier way of writing things than that we obtained via Integral Geometric methods. Nevertheless, it is now clear that the two are essentially different versions of the same basic result. However, it is the compactness of (9.4.1) that will be of importance to us in the upcoming computations for random excursion sets in Part III of the book.

10
Volume of tubes

In Chapters 7 and 8 we invested a good deal of time and energy on developing the many results we need from Differential Geometry. The time has now come to begin to reap the benefits of our investment, while at the same time developing some themes a little further for later exploitation. This chapter focuses on the celebrated volume of tubes formula of Weyl [158, 69], which expresses the Lebesgue volume of a tube of radius ρ around a set M embedded in $\mathbb{R}^l$ or $S(\mathbb{R}^l)$ in terms of the radius of the tube[1] and the Lipschitz-Killing curvatures of M (see Theorem 10.5.6). It is an interesting fact, particularly in view of the fact that (despite current appearences) this is a book about Probability and with applications to Statistics, that despite the fact that Weyl's formula is today the basis of a large literature in Geometry, its origins were inspired by a statistical problem. This problem, and its solution, due to Hotelling [75] was related to regression analysis and involved the one-dimensional volume of tubes problem on a sphere, not unrelated to the computation we shall do in a moment.

Much of the work in using Weyl's tube formula and its generalisations comes from deriving explicit expressions for Lipschitz-Killing curvature measures. This being the case, even if you are comfortable with the material of Chapter 7 and are not interested in the volume of tubes approximation

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[1] A word on notation: If (T, d) is a metric space, then the sphere of radius ρ in T is denoted by $S_\rho(T)$, with $S(T) \equiv S_1(T)$. When $T = \mathbb{R}^N$ we continue, when convenient, to use the notations S_ρ^{N-1} and S^{N-1}.

per se, you will find it useful to read Sections 10.7 and 10.9 of this chapter. These describe the Lipschitz-Killing curvature measures of stratified Whitney spaces and the generalised Lipschitz-Killing curvature measures that will be needed in Chapter 14.

Beginning in the late 1980s and early 1990s, the usefulness of the tube formula in statistics was rediscovered, and there were a number of works applying these formulae to statistical problems, as in [1, 82]. A particularly interesting paper for us was a 1993 paper by Jiayang Sun [138], which came out of a thesis under David Siegmund. It had an early version (albeit not easily recognisable as such) of the simple[2] approximation

$$(10.0.1) \qquad \mathbb{P}\left\{\sup_{t \in M} f_t \geq u\right\} \simeq \sum_{j=0}^{N} \rho_j(u)\mathcal{L}_j(\psi(M)).$$

Here f is a sufficently nice, centred, unit variance Gaussian field over a sufficiently nice N-dimensional parameter space M. The functionals $\mathcal{L}_j$ are Lipschitz-Killing curvatures on a Euclidean sphere equipped with the Euclidean geodesic metric and are defined in (10.5.3). The functions ρ_j are defined as

$$\rho_j(u) \triangleq \begin{cases} 1 - \Phi(u) & j = 0 \\ (2\pi)^{-(j+1)/2}H_{j-1}(u)e^{-u^2/2} & j \geq 1 \end{cases}$$

where H_j is the j-th Hermite polynomial, which we shall meet in more detail later in Chapter 12. (cf. (11.5.8).) Finally, ψ is a mapping from the parameter space M to a unit sphere. Exactly which mapping and a sphere of which dimension will be clear after you have read Sections 10.2 and 10.6.

As we shall soon see, the above approximation, given enough side conditions, is actually rather simple to derive using volume of tube techniques. Furthermore, when both are defined, it agrees with the approximation that we shall derive in Chapter 13 which is based on the expected Euler characteristic and which is much harder to establish. Unfortunately, however, the volume of tubes approach has the disadvantage of being restricted to a somewhat small class of Gaussian processes, in that they are required to possess a finite Karhunen-Loève expansion. Specifically, these are processes defined on a smooth manifold M that can be expressed[3] as

$$(10.0.2) \qquad f(t, \omega) = \langle \psi_t, \xi(\omega) \rangle_{\mathbb{R}^l} = \sum_{j=1}^{l} \xi_j(\omega)\psi_j(t),$$

---

[2]This is 'simple' as long as you are also comfortable applying the same adjective to Lipschitz-Killing curvatures, defined below in (10.5.3).

[3]In [139], this approach was extended to piecewise smooth spaces M embedded in $S(\mathbb{R}^l)$ for some suitably large, but finite, l.

for some smooth mapping $\psi : M \rightarrow S(\mathbb{R}^l)$. While this expansion may arise from the $L^2(I^N)$ expansion of a covariance function on $I^N \times I^N$ as in Section 3.2, this is not necessary, the main issue being the existence of a finite orthogonal expansion. Consequently, these processes are more appropriately referred to as *finite orthogonal expansion processes*, although we shall retain the nomenclature 'finite Karhunen-Loève expansion processes' for historical reasons.

The volume of tubes based derivation of the approximation also has the disadvantage that all results on its accuracy are very distributional in nature and there is no known way to even begin thinking about how to extend them to non-Gaussian processes. In any case, we shall not address the accuracy of the approximation in this chapter, as it will follow as a corollary to the accuracy of the expected Euler characteristic approach in Chapter 13 and the equivalence of the two approaches [139] for finite Karhunen-Loève processes.

What we shall do in this chapter is describe the volume of tubes approximation to (10.0.1) in detail and also derive Weyl's tube formula for piecewise smooth subspaces of $\mathbb{R}^l$ and $S(\mathbb{R}^l)$. In our development of the this approximation, we also consider the volume of tubes problem in a slightly more general setting, necessary for the Gaussian Kinematic Formula of Chapter 14.

We begin with a description of the volume of tubes problem in a general sense, specializing later to the case of piecewise smooth spaces embedded in $\mathbb{R}^l$ or $S(\mathbb{R}^l)$. Before deriving Weyl's formula, we shall describe the connection between tubes and (10.0.1) so that you can see where we are heading, and why.

10.1   The volume of tubes problem

For a metric space (T, τ) the tube of radius ρ around $A \subset T$ is, as we have already seen in the Euclidean setting, defined as

$$(10.1.1) \qquad \mathrm{Tube}(A, \rho) \;\triangleq\; \{x \in T : \tau(x, A) \leq \rho\} \;=\; \bigcup_{y \in A} B_\tau(y, \rho)$$

where

$$(10.1.2) \qquad\qquad\qquad \tau(x, A) \;\triangleq\; \inf_{y \in A} \tau(x, y)$$

is the usual distance function of the set A.

The volume of tubes problem seeks to approximate

$$(10.1.3) \qquad\qquad\qquad V_{\mu, A}(\rho) \;\triangleq\; \mu\left(\mathrm{Tube}(A, \rho)\right)$$

for some measure μ, as a function of ρ.

Of course, the accuracy and validity of such an approximation will depend on the properties of the measure μ as well as the set A. The simplest spaces to work on are the metric spaces $\mathbb{R}^l$ with its natural metric structure and $S(\mathbb{R}^l)$ with the geodesic metric from the induced Riemannian structure on $S(\mathbb{R}^l)$. The natural measures to consider in these examples are Lebesgue measure on $\mathbb{R}^l$ and surface measure on $S(\mathbb{R}^l)$, which agree with the Riemannian measures induced by the canonical Riemannian structures. These are the two main examples that we shall consider. In addition, in Section 10.9 we shall also investigate the Gaussian volume of tubes, where the metric space is $\mathbb{R}^l$ with its standard metric and the measure $\mu = \gamma_{\mathbb{R}^l}$[4] the distribution of a random vector $W \sim N(0, I_{l \times l})$. This example is important in the Gaussian Kinematic Formula, presented in Chapter 14.

As for what type of sets A we consider, in order to get any useful results about $V_{\mu,A}$, we shall have to restrict ourselves to sets A that are embedded piecewise C^2 spaces of $\mathbb{R}^l$ or $S(\mathbb{R}^l)$.

10.2   Volume of tubes and Gaussian processes

We now start a discussion of the connection between the excursion probability (10.0.1) and the volume of tubes for finite Karhunen-Loève processes. We shall conclude it only in Section 10.6 by when we shall have more information on tube formulae. We therefore start with a Gaussian process f, which is a restriction to a piecewise smooth submanifold M of a process $\widetilde{f}$ on an ambient manifold $\widetilde{M}$ which has the representation[5]

$$(10.2.1) \qquad \widetilde{f}(t, \omega) \; = \; \langle \psi_t, \xi(\omega) \rangle_{\mathbb{R}^l} \; = \; \sum_{j=1}^{l} \xi_j(\omega) \psi_j(t).$$

Note that we have introduced a new parameter here. Whereas we still use N to denote $\dim(M) = \dim(\widetilde{M})$, the new parameter l denotes the order of the orthogonal expansion.

Note also that the vector $\xi/|\xi|$ is uniformly distributed on $S(\mathbb{R}^l)$, independently of $|\xi|$, which is distributed as the square root of a χ_l^2 random variable. We shall write η_l to denote the distribution of $\xi/|\xi|$; *viz.* the uniform measure over $S(\mathbb{R}^l)$.

---

[4]For a finite dimensional Hilbert space, H, we will use γ_H to denote the a canonical Gaussian random vector on H. That is, if $X \sim \gamma_H$, then for any orthonormal basis $\zeta_1, \ldots, \zeta_{\dim(H)}$ the sequence

$$\left(\langle \zeta_1, X \rangle_H, \ldots, \langle \zeta_{\dim(H)}, X \rangle_H \right)$$

are i.i.d. $N(0, 1)$ random variables.

[5]This representation is a little different to that of (10.0.2), but the reasons for the changes will become clear as we go along.

Finally, we shall assume that $\widetilde{f}$ has constant unit variance, so that (10.2.1) immediately implies

$$(10.2.2) \qquad |\xi(t)|^2 \;\triangleq\; \sum_{j=1}^{l} \xi_j^2(t) \;=\; 1,$$

for all $t \in \widetilde{M}$. This being the case, we can consider the map $t \to (\xi_i(t), \ldots, \xi_l(t))$ as defining an embedding $\xi : M \to S(\mathbb{R}^l)$.

Returning to tubes, when the metric space (T, τ) is $S(\mathbb{R}^l)$ with the geodesic metric, i.e. when

$$\tau(x, y) \;=\; \cos^{-1}\left(\langle x, y\rangle\right),$$

then a tube of radius ρ around a closed set A can be expressed as

$$(10.2.3) \qquad \begin{aligned} \mathrm{Tube}(A, \rho) \;&=\; \left\{x \in S(\mathbb{R}^l) : \tau(x, A) \le \rho\right\} \\ &=\; \left\{x \in S(\mathbb{R}^l) : \exists y \in A \text{ such that } \langle x, y\rangle \ge \cos(\rho)\right\} \\ &=\; \left\{x \in S(\mathbb{R}^l) : \sup_{y \in A} \langle x, y\rangle \ge \cos(\rho)\right\}. \end{aligned}$$

If we think of X being a random vector uniformly distributed on $S(\mathbb{R}^l)$ with distribution η_l, then (10.2.3) says that

$$\begin{aligned} \mathbb{P}\left\{\sup_{y \in A} \langle X, y\rangle \ge \cos(\rho)\right\} \;&=\; \eta_l\left(\left\{x \in S(\mathbb{R}^l) : \sup_{y \in A} \langle x, y\rangle \ge \cos(\rho)\right\}\right) \\ &=\; \eta_l\left(\mathrm{Tube}(A, \rho)\right). \end{aligned}$$

We now have enough to write out the basic equation in the volume of tubes approach for such processes:

(10.2.4)

$$
\begin{aligned}
\mathbb{P}\left\{\sup_{t\in M} f_t \geq u\right\} &= \int_0^\infty \mathbb{P}\left\{\sup_{t\in M} f_t \geq u \Big| |\xi| = r\right\} \mathbb{P}_{|\xi|}(dr) \\
&= \int_0^\infty \mathbb{P}\left\{\sup_{t\in M} \langle \psi_t, \xi\rangle \geq u \Big| |\xi| = r\right\} \mathbb{P}_{|\xi|}(dr) \\
&= \int_u^\infty \mathbb{P}\left\{\sup_{t\in M} \langle \psi_t, \xi\rangle \geq u \Big| |\xi| = r\right\} \mathbb{P}_{|\xi|}(dr) \\
&= \int_u^\infty \mathbb{P}\left\{\sup_{t\in M} \langle \psi_t, \xi/r\rangle \geq u/r \Big| |\xi| = r\right\} \mathbb{P}_{|\xi|}(dr) \\
&= \int_u^\infty \mathbb{P}\left\{\sup_{s\in \psi(M)} \langle s, \xi/r\rangle \geq u/r \Big| |\xi| = r\right\} \mathbb{P}_{|\xi|}(dr) \\
&= \int_u^\infty \eta_l\left(\mathrm{Tube}(\psi(M), \cos^{-1}(u/r))\right) \mathbb{P}_{|\xi|}(dr) \\
&= \frac{\Gamma\left(\frac{l}{2}\right)}{2\pi^{l/2}} \int_u^\infty \mathcal{H}_{l-1}\left(\mathrm{Tube}(\psi(M), \cos^{-1}(u/r))\right) \mathbb{P}_{|\xi|}(dr) \\
&= \frac{\Gamma\left(\frac{l}{2}\right)}{2\pi^{l/2}} \mathbb{E}\left\{\mathcal{H}_{l-1}\left(\mathrm{Tube}(\psi(M), \cos^{-1}(u/|\xi|))\right) \mathbb{1}_{\{|\xi|\geq u\}}\right\}.
\end{aligned}
$$

Therefore, the excursion probability (10.0.1) is a weighted average of the volume of tubes around $\psi(M)$ of varying radii, and being able to compute

$$\mathcal{H}_{l-1}\left(\mathrm{Tube}(\psi(M), \rho)\right)$$

will go a long way to computing (10.0.1). In particular, if it were true that

(10.2.5)
$$\eta_l\left(\mathrm{Tube}(\psi(M), \rho)\right) \simeq \sum_j \widetilde{G}_{j,l}(\rho)\, \mathcal{L}_j(\psi(M))$$

for functions $\widetilde{G}_{j,l}$ then the right-hand side of (10.2.4), and hence (10.0.1), would be approximately

(10.2.6)
$$\frac{\Gamma\left(\frac{l}{2}\right)}{2\pi^{l/2}} \sum_j \mathbb{E}\left\{\widetilde{G}_{j,l}(\cos^{-1}(u/|\xi|))\right\} \mathcal{L}_j(\psi(M)).$$

Much of the rest of this chapter is devoted to justifying this approximation[6] and obtaining explicit representations for $\widetilde{G}_{j,l}$. The justification is

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[6]Note that we can only really expect the approximation to work for for large u. The issue is that we only expect that tube expansion (10.2.5) to hold for small ρ. Since ρ is, effectively, replaced by $\cos^{-1}(u/|\xi|)$ in (10.2.6), we can only expect the tube approxiamtion to work well if $|\xi|$ is close to u (remember that $|\xi| \geq u$) with high probability. This is most likely to happen only if u is large.

based on Weyl's tube formula on $S_\lambda(\mathbb{R}^l)$, given below in Theorem 10.5.7. Weyl's formula states that (10.2.5) is exact for ρ small enough if $\psi(M)$ is a piecewise smooth space. Furthermore, it identifies the functions $\widetilde{G}_{j,l}(\rho)$.

Deriving Weyl's formula requires an understanding of the local properties of tubes, to which we devote the following section. At the end of the computation, you are invited to ask whether Weyl's classic comment on the details of a result which gave birth to an entire research area is still valid: "... we have hardly done more than what could have been accomplished by any student in a course of calculus"[7].

10.3   Local geometry of $\mathrm{Tube}(M, \rho)$

This section, as its title suggests, describes the local geometry of $\mathrm{Tube}(M, \rho)$ where, throughout, we shall assume that M is a piecewise smooth space. To formally define the local geometry of the tube, we first have to specify the ambient space in which M is assumed to be embedded and the metric space in which the tube lives. Although this sounds rather pedantic, it is important in understanding the *intrinsic* nature of the volume of tubes formula. As a piecewise smooth space M, by definition, needs an ambient space $\widetilde{M}$ in which to be embedded, but $\widetilde{M}$ itself might be embedded in a Riemannian manifold $(\widehat{M}, \widehat{g})$, so that we have the inclusion[8]

$$M \subset \widetilde{M} \subset \widehat{M}.$$

In this case,

(10.3.1) $\mathrm{Tube}(M, \rho) = \left\{ x \in \widehat{M} : d_{\widehat{M}}(x, M) \leq \rho \right\},$

where $d_{\widehat{M}}$ is geodesic distance on $\widehat{M}$.

One of the main conclusions of Weyl's formula is that, if $\widehat{M}$ is either $\mathbb{R}^l$ or $S(\mathbb{R}^l)$, then the volume of $\mathrm{Tube}(M, \rho)$ is intrinsic in the sense of the following definition.

Definition 10.3.1 *A quantity $Q(M)$ is intrinsic to a piecewise smooth space M embedded in $(\widetilde{M}, \widetilde{g})$ if it can be computed based solely on the Riemannian metric of $(\widetilde{M}, \widetilde{g})$. That is, if $\widetilde{M}$ is isometrically embedded in $(\widehat{M}, \widehat{g})$, and $i(M)$ is the image of M under the inclusion $i : \widetilde{M} \to \widehat{M}$, then $Q(M)$ is intrinsic if, and only if*

$$Q(M) = Q(i(M)).$$

---

[7] The issue here, presumably, is the precise meaning of the term "hardly"!

[8] To clarify things, think of $M = S^1 = S(\mathbb{R}^2)$, a one dimensional circle embedded in the two-dimensional plane $\widetilde{M} = \mathbb{R}^2$. If $\widehat{M} = \mathbb{R}^3$, then the tube around M, as defined by (10.3.1), is a solid ring.

A particularly important example of intrinsic quantities are the Lipschitz-Killing curvature measures $\mathcal{L}_i(M;\cdot)$, defined in (10.5.3).

The property of being intrinsic is arguably what has made Weyl's tube formula so useful, because it implies that it is enough to know only local properties of M as it sits in $\widetilde{M}$, without needing to know anything about how it sits in $\widehat{M}$.

With the triple $M \subset \widetilde{M} \subset \widehat{M}$ in mind, we establish the following convention about notation: In general, a $\widehat{\cdot}$ will denote an object when we are considering M as being embedded in $\widehat{M}$. For instance, $\widehat{\mathcal{S}_t M}$ is the support cone of M at $t \in \widehat{M}$, as opposed to $\mathcal{S}_t M$, the support cone of M at $t \in \widetilde{M}$. Sometimes, there is no distinction to be made, as in the case of support cones. It is easy to see that $\mathcal{S}_t M$ and $\widehat{\mathcal{S}_t M}$ are isomorphic. However, there *is* a difference between $N_t M$ and $\widehat{N_t M}$. Specifically,

$$
\begin{aligned}
\widehat{N_t M} &= N_t M \oplus T_t \widetilde{M}^\perp \\
&= \left\{ X_t \in T_t \widehat{M} : X_t = V_t + Y_t, V_t \in N_t M, Y_t \in T_t \widetilde{M}^\perp \right\}
\end{aligned}
$$

where $T_t \widetilde{M}^\perp$ is the orthogonal complement of $T_t \widetilde{M}$ in $T_t \widehat{M}$.

10.3.1 Basic structure of tubes

We now begin to describe the local structure of $\mathrm{Tube}(M, \rho)$, beginning with a linearization of $\widehat{M}$ near $t \in M$, which we use as motivation for an explicit description of the $\mathrm{Tube}(M, \rho)$. Our ultimate goal is to come up with an explicit parameterization of the tube which we can use to compute its volume essentially using multivariate calculus.

We then define various notions of *critical radius* for $\mathrm{Tube}(M, \rho)$. As we shall see, Weyl's tube formula is only valid for values of ρ smaller than the critical radius of M, because for larger ρ this explicit parameterization of the tube breaks down.

The linearization we shall use is essentially a linearization of the metric projection $\xi_M : \widehat{M} \to M$ given by

$$(10.3.2) \qquad \xi_M(s) \overset{\Delta}{=} \underset{t \in M}{\mathrm{argmin}} \, \widehat{d}(s, t),$$

which parameterizes the tube in a one-to-one fashion, as

$$\mathrm{Tube}(M, \rho) = \bigcup_{t \in M} \left\{ s \in \widehat{M} : \xi_M(s) = t, \, \widehat{d}(s, t) \leq \rho \right\}$$

and the union on the right is a *disjoint* union.

Actually, we shall need more than this, since we shall also need an explicit description of the level sets

$$\left\{ s \in \widehat{M} : \xi_M(s) = t \right\}$$

of ξ_M. To do this, we start by approximating ξ_M with the projection onto the convex cone $\widehat{\mathcal{S}_t M}$, defined on $T_t \widehat{M}$ by

$$(10.3.3) \qquad P_{\widehat{\mathcal{S}_t M}}(X_t) \;=\; \operatorname*{argmin}_{Y_t \in \widehat{\mathcal{S}_t M}} |X_t - Y_t| \;=\; \operatorname*{argmin}_{Y_t \in \mathcal{S}_t M} |P_{T_t \widetilde{M}} X_t - Y_t|.$$

A simple convexity argument then shows that X_t projects to the origin, if, and only if, $P_{T_t \widetilde{M}} X_t \in N_t M$. Alternatively, if $P_{T_t \widetilde{M}} X_t \notin N_t M$, then there exists some point $Y_t \neq 0 \in \mathcal{S}_t M$ such that $|P_{T_t \widetilde{M}} X_t - Y_t| < |X_t|$.

This argument is based on a local linearization of $\widehat{M}$ and M at $t \in M$, hence it holds only locally. However, using the fact that near t, M can be approximated by $\mathcal{S}_t M$ it is sufficient to establish that

$$\left\{ s \in \widehat{M} : \xi_M(s) = t \right\} = \exp_{\widehat{M}} \left(\left\{ X_t \in T_t \widehat{M} : P_{T_t \widetilde{M}} X_t \in N_t M, |X_t| \le \rho \right\} \right).$$

Therefore, for ρ small enough

$$(10.3.4) \qquad
\begin{aligned}
\mathrm{Tube}(M, \rho) &= \bigcup_{t \in M} \; \bigcup_{\{X_t \in T_t \widehat{M} : \, |X_t| \le \rho\}} \exp_{\widehat{M}}(t, X_t) \\
&= \bigcup_{t \in M} \; \bigcup_{\{X_t \in \widehat{N_t M} : \, |X_t| \le \rho\}} \exp_{\widehat{M}}(t, X_t).
\end{aligned}$$

The second equality is based on the local linearization described above, i.e. if $X_t \notin \widehat{N_t M}$ then there exists a point $s \in M$ such that

$$\widehat{d}(\exp_{\widehat{M}}(t, X_t), s) < \widehat{d}(\exp_{\widehat{M}}(t, X_t), t)$$

Therefore, we can leave out all vectors X_t such that $X_t \notin \widehat{N_t M}$.

At this point, we have not yet quantified how small ρ should be. First of all, ρ should be small enough so that

$$\exp_{\widehat{M}} : \; B(T_t \widehat{M}, \rho) \to \widehat{M}$$

is a diffeomorphism. This means ρ should be less than

$$(10.3.5) \quad \widehat{\rho}_c(\widehat{M}) \;\triangleq\; \sup \Big\{ r : \exp_{\widehat{M}} : B(T_t \widehat{M}, r) \to \widehat{M}$$

$$\text{is a diffeomorphism for all } t \in \widehat{M} \Big\},$$

the *radius of injectivity* of $\widehat{M}$.

In general, we shall have to take ρ even smaller, since we want the final expression in (10.3.4) to be a *disjoint* union. Suppose, then, that $\rho < \widehat{\rho}_c(\widehat{M})$. The geodesic with origin t and direction $X_t \in S(T_t \widehat{M})$ can be extended only so far without overlapping itself. We define $\rho_{c,l} : \; M \times (\widehat{N_t M} \cap S(T_t \widehat{M})) \to \mathbb{R}$, the *local critical radius of M in $\widehat{M}$ at t in the direction X_t*, as

$$(10.3.6) \qquad \rho_{c,l}(t, X_t) \;\triangleq\; \sup \left\{ r : \widehat{d}(\exp_{\widehat{M}}(t, rX_t), M) = r \right\}.$$

For $r < \rho_{c,l}(t, X_t)$ the point $\exp_{\widehat{M}}(t, rX_t)$ projects uniquely to the point t.

Taking the infimum over $N_t M$, we define $\rho_c : M \to \mathbb{R}$, the *local critical radius of M in $\widehat{M}$ at t* as

(10.3.7)
$$\rho_c(t) \;\triangleq\; \inf_{X_t \in \widehat{N_t M} \cap S(T_t \widehat{M})} \rho_{c,l}(t, X_t),$$

and, finally the *critical radius of M in $\widehat{M}$* as

(10.3.8)
$$\rho_c(M, \widehat{M}) \;\triangleq\; \inf_{t \in M} \rho_c(t).$$

Therefore, for $\rho \leq \min(\rho_c(M, \widehat{M}), \widehat{\rho}_c(\widehat{M}))$, the final expression in (10.3.4) is a disjoint union. Consequently, for such ρ, $\mathrm{Tube}(M, \rho)$ is the image of the region

$$\{(t, X_t, r) : \; X_t \in \widehat{N_t M} \cap S(T_t \widehat{M}), \; 0 \leq r < \rho\}$$

under the bijection (for such ρ) $\widehat{F}$ defined by

$$\widehat{F}(t, X_t, r) \;\triangleq\; \exp_{\widehat{M}}(t, rX_t).$$

For larger values of ρ, the map $\widehat{F}$ may not be one-to-one, however it is a one-to-one map when restricted to the region

(10.3.9) $\left\{(t, X_t, r) : \; X_t \in \widehat{N_t M} \cap S(T_t \widehat{M}), \; 0 \leq r < \min\left(\rho, \rho_{c,l}(t, X_t)\right)\right\}.$

10.3.2   Stratifying the tube

We now turn to describing the region (10.3.9) in more detail, essentially stratifying $\mathrm{Tube}(M, \rho)$. Once we have stratified $\mathrm{Tube}(M, \rho)$, computing its volume reduces to computing an integral over each stratum of $\mathrm{Tube}(M, \rho)$.

For each stratum $\partial_j M$ for $0 \leq j \leq l$ fix an orthonormal frame field $(\eta_1, \ldots, \eta_{l-j})$ for $\widehat{T_t \partial_j M}^{\perp}$ as t varies over $\partial_j M$. The region (10.3.9) can then be expressed as the image of the disjoint union of the regions

(10.3.10)
$$D_j(\rho) \;\triangleq\; \bigcup_{t \in \partial_j M} \bigcup_{s \in S(\mathbb{R}^{l-j})} \bigcup_{0 \leq r \leq \min\left(\rho, \rho_{c,l}\left(t, \sum_i s_i \eta_i(t)\right)\right)}$$
$$\left\{(t, s, r) : \sum_i s_i \eta_i(t) \in \widehat{N_t M}\right\}$$

under the maps

$$G_j : (t, s, r) \;\rightarrow\; \left(t, \sum_i s_i \eta_i(t), r\right).$$

Finally, $\mathrm{Tube}(M,\rho)$ is the disjoint union of the sets $D_j(\rho)$ under the maps $F_j \overset{\Delta}{=} \widehat{F} \circ G_j$, so that we have

$$(10.3.11) \qquad F_j : (t,s,r) \; \to \; \exp_{\widetilde{M}}\left(t, r\sum_i s_i\eta_i(t)\right).$$

10.4 Computing the volume of a tube

10.4.1 First steps

In the previous section, we decomposed $\mathrm{Tube}(M,\rho)$ into strata corresponding to the strata of M. The strata of $\mathrm{Tube}(M,\rho)$ are open sets in $\widehat{M}$, and to compute $\mu(\mathrm{Tube}(M,\rho))$ we are left with an integral over each stratum of $\mathrm{Tube}(M,\rho)$. For nice measures μ (the only ones about which we shall be able to say anything) what remains is just calculus, albeit carefully disguised behind all the geometric notation. Our requirement for μ is that it is the measure associated to an integrable l-form α_μ (see (7.4.5)).

From the previous section, we know that for each j, the maps F_j of (10.3.11) are one-to-one and differentiable on the interior of $D_j(\rho)$. Assuming that the measure μ is the measure associated to an integrable l-form α_μ, we can pull back α_μ (see (7.4.1)) to $D_j(\rho) \subset \partial_j M \times S(\mathbb{R}^{l-j}) \times \mathbb{R}^+$, so that

$$(10.4.1) \quad \mu(\mathrm{Tube}(M,\rho)) \;\; = \;\; \sum_j \mu(F_j(D_j(\rho)))$$

$$= \;\; \sum_j \int_{\widehat{M}} \mathbb{1}_{F_j(D_j(\rho))}\alpha_\mu$$

$$= \;\; \sum_j \int_{\partial_j M \times S(\mathbb{R}^{l-j})\times\mathbb{R}^+} \mathbb{1}_{D_j(\rho)}F_j^*(\alpha_\mu).$$

We can go somewhat further than this for a more detailed picture.

Since α_μ is an integrable l-form, it can be expressed as $f_\mu \cdot \mathrm{Vol}_{\widehat{M}}$ where $\mathrm{Vol}_{\widehat{M}}$ is the Riemannian volume form on $\widehat{M}$ for some bounded function f_μ. Applying the co-area formula (7.4.13), to the distance function of M (which is C^1 with Lipschitz gradient of norm 1) on $F_j(D_j(\rho))$ it follows from (10.4.1)

$$\mu(F_j(D_j(\rho))) = \int_0^\rho \int_{S_j(r)} \mathbb{1}_{F_j(D_j(\rho))}(x)f_\mu(x)\; d\mathcal{H}_{l-1}(x)dr,$$

where

$$(10.4.2) \qquad S_j(r) \overset{\Delta}{=} \{s \in \widehat{M} : d_{\widehat{M}}(s, \partial_j M) = r\}$$

and $\mathcal{H}_{l-1}$ is $(l-1)$-dimensional Hausdorff measure, which is, in the current setting, also the Riemannian measure on the level set $S_j(r)$.

Finally, let $\Omega_{j,r}$ be the volume form induced on $S_j(r)$ by α_μ. Then the contribution of $\partial_j M$ to $\mu(\text{Tube}(M,\rho))$ is

(10.4.3)
$$\mu(F_j(D_j(\rho))) = \int_0^\rho \int_{\partial_j M \times S(\mathbb{R}^{l-j})} \mathbb{1}_{D_j(\rho)}(t,s,r) f_\mu(F_{j,r}(t,s)) F_{j,r}^*(\Omega_{j,r}) \ dr$$

where $F_{j,r}$ is the partial map

$$F_{j,r}(t,s) \ \triangleq \ F_j(t,s,r).$$

We are left with computing the pull-back $F_{j,r}^*(\Omega_{j,r})$, as well as the integration over $D_j(\rho)$.

10.4.2   An intermediate computation

Although the details as to how best to compute $F_{j,r}^*(\Omega_{j,r})$ tend to be problem specific, there are some general preliminaries that will help for both cases that we wish to consider in detail, in which $\widehat{M}$ is either $\mathbb{R}^l$ or $S(\mathbb{R}^l)$.

We begin by recalling what it is we actually must compute. Fixing a point $(t,s) \in \partial_j M \times S(\mathbb{R}^{l-j})$, recall that, by the definition of the pull-back (7.4.1) and from (7.4.7),

$$
\begin{aligned}
(10.4.4) \quad F_{j,r}^*(\Omega_{j,r}) &\left(X_{1,(t,s)}, \ldots, X_{l-1,(t,s)} \right) \\
&= \Omega_{j,r} \left(F_{j,r\,*} X_{1,(t,s)}, \ldots, F_{j,r\,*} X_{l-1,(t,s)} \right) \\
&= \det \left(\langle F_{j,r\,*} X_{k,(t,s)}, W_{m,F_{j,r}(t,s)} \rangle \right)_{1 \le k,m \le l-1},
\end{aligned}
$$

for any orthonormal basis $(W_{1,F_{j,r}(t,s)}, \ldots, W_{l-1,F_{j,r}(t,s)})$ of the tangent space at $F_{j,r}(t,s)$ to the level set (10.4.2), a hypersurface in $\widehat{M}$. The inner product here is, of course, that of $\widehat{M}$.

Therefore, in order to evaluate $F_{j,r}^*(\Omega_{j,r})$ it will, in general, be necessary to choose an orthonormal basis $(W_{1,F_{j,r}(t,s)}, \ldots, W_{l-1,F_{j,r}(t,s)})$ of $T_{F_{j,r}(t,s)} S_j(r)$.

We now treat two specific cases, in both of which the exponential map has an explicit form that makes computations feasible[9].

_____

[9]Indeed, this is true of may spaces of constant curvature. In general, when $\widehat{M}$ is an arbitrary manifold, the Jacobean of F_j, which is needed to continue on from (10.4.3), is difficult to compute, since the map $\exp_{\widehat{M}}$ has no simple closed form. In [69] the Jacobean is expressed in terms of the solution to a Ricatti differential equation but we shall not pursue this approach here.

10.4.3   Subsets of $\mathbb{R}^l$

We now treat the case when the ambient space is $\widehat{M} = \mathbb{R}^l$. In this case, $F_{j,r}$ is particularly simple and is given by

$$(10.4.5) \qquad F_{j,r}(t,s) \;=\; t + r\sum_{i=1}^{l-j} s_i\eta_i(t).$$

The simple, explicit form of $F_{j,r}$ allows us to establish the following important lemma.

Lemma 10.4.1 *For any coordinate systems u on $\partial_j M$ and w on $S(\mathbb{R}^{l-j})$ and any orthonormal frames $(\eta_i)_{1\leq i\leq l-j}$ normal to $\partial_j M$, the following relations hold:*

$$(10.4.6) \qquad F_{j,r\,*}\left(\frac{\partial}{\partial u_k}\right) \;=\; \frac{\partial}{\partial u_k} + r\sum_{i=1}^{l-j} s_i\nabla_{\partial/\partial u_k}\eta_i(t(u))$$

$$(10.4.7) \qquad F_{j,r\,*}\left(\frac{\partial}{\partial w_k}\right) \;=\; 0 + r\sum_{i=1}^{l-j}\frac{\partial s_i}{\partial w_k}\eta_i(t(u))$$

$$(10.4.8) \qquad \left\langle F_{j,r\,*}\left(\frac{\partial}{\partial u_k}\right),\; \sum_{i=1}^{l-j} s_i(w)\eta_i(t(u))\right\rangle \;=\; 0$$

$$(10.4.9) \qquad \left\langle F_{j,r\,*}\left(\frac{\partial}{\partial w_k}\right),\; \sum_{i=1}^{l-j} s_i(w)\eta_i(t(u))\right\rangle \;=\; 0.$$

Proof.   To verify (10.4.6) and (10.4.7), let (U,ϕ) be a chart on M, in which case we can rewrite (10.4.5) locally as a mapping from $U \times S(\mathbb{R}^{l-j})$ by defining

$$(10.4.10) \qquad \widetilde{F}_{j,r}(u,s) \;=\; \varphi(u) + r\sum_{i}^{l-j} s_i\,\eta_i(\varphi(u)).$$

With the usual abuse of notation that identifies $\partial/\partial u_k$ and $\varphi_*(\partial/\partial u_k)$ we can compute $F_{j,r\,*}(\partial/\partial u_k)$ from $\widetilde{F}_{j,r\,*}(\partial/\partial u_k)$. To see how to do this, take the standard basis $(e_1,\ldots,e_l)$ on $\mathbb{R}^l$ and start by noting that, for

$1 \leq m \leq l,\ 1 \leq k \leq l - j$,

$$\left(\frac{\partial}{\partial u_k} \widetilde{F}_{j,r}(u, s) \right)_m$$

$$= \frac{\partial \varphi_m(u)}{\partial u_k} + r \sum_i^{l-j} s_i \frac{\partial \langle \eta_i(\varphi(u)), e_m \rangle}{\partial u_k}$$

$$= \varphi_* \left(\frac{\partial}{\partial u_k} \right) + r \sum_i^{l-j} s_i \left\langle \left(\left(\varphi_* \left(\frac{\partial}{\partial u_k} \right) \eta_i \right) (\varphi(u)), e_m \right\rangle$$

$$= \frac{\partial}{\partial u_k} + r \sum_i^{l-j} s_i \frac{\partial}{\partial u_k} \langle \eta_i(\varphi(u)), e_m \rangle,$$

where the notational abuse is in the passage between the last two lines.

This leads to the relation

$$F_{j,r\,*} \left(\frac{\partial}{\partial u_k} \right) = \widetilde{F}_{j,r\,*} \left(\frac{\partial}{\partial u_k} \right)$$

$$= \sum_{m=1}^{l} \left(\frac{\partial}{\partial u_k} \widetilde{F}_{j,r}(u, s) \right)_m \frac{\partial}{\partial e_m}$$

$$= \partial/\partial u_k + r \sum_i s_i \sum_m \frac{\partial}{\partial u_k} \langle \eta_i(\varphi(u)), e_m \rangle \frac{\partial}{\partial e_m}$$

$$= \partial/\partial u_k + r \sum_i s_i \sum_m \left(\langle \nabla_{\partial/\partial u_k} \eta_i(\varphi(u)), e_m \rangle \right.$$

$$\left. + \langle \eta_i(\varphi(u)), \nabla_{\partial/\partial u_k} e_m \rangle \right) \frac{\partial}{\partial e_m}$$

$$= \partial/\partial u_k + r \sum_i s_i \sum_m \left(\langle \nabla_{\partial/\partial u_k} \eta_i(\varphi(u)), e_m \rangle + 0 \right) \frac{\partial}{\partial e_m}$$

$$= \partial/\partial u_k + r \sum_i s_i \nabla_{\partial/\partial u_k} \eta_j(\varphi(u)),$$

which gives (10.4.6). (The third line from the end here comes from the 'compatability' property of the Levi-Civita connection – cf. (7.3.7).) A similar argument gives (10.4.7).

We now turn to (10.4.8) and (10.4.9).

To verify (10.4.8), first note that

$$F_{j,r\,*} \left(\frac{\partial}{\partial u_k} \right) \in T_{F_{j,r}(t,s)} S_j(r),$$

while

$$\sum_{i=1}^{l-j} s_i \eta_i(t(u)) \in T_{F_{j,r}(t,s)}^{\perp} S_j(r).$$

To verify (10.4.9) note that we can express the left hand side as

$$F_{j,r\,*}\left(\frac{\partial}{\partial w_k}\right) \;=\; \sum_{i=1}^{l-j} a_{ki}(s(w))\eta_i(t(u)),$$

where the a_{ki}'s satisfy

$$\sum_{i=1}^{l-j} a_{ki}(s(w))s_i(w) = 0, \qquad \forall\, 1 \le k \le l-j-1.$$

Therefore,

$$\left\langle F_{j,r\,*}\left(\frac{\partial}{\partial w_k}\right), \sum_{i=1}^{l-j} s_i\eta_i(t(u))\right\rangle$$

$$= \left\langle \sum_{n=1}^{l-j} a_{k,n}(s(w))\eta_n(t(u)), \sum_{i=1}^{l-j} s_i(w)\eta_i(t(u))\right\rangle$$

$$= \sum_{i=1}^{l-j} a_{ki}(s(w))s_i(w)$$

$$= 0.$$

$\square$

We now return to our main task, the computation of the determinant in (10.4.4).

By Lemma 10.4.1, it is clear that the orthonormal basis $(W_{1,F_{j,r}(t,s)},\ldots,$ $W_{l-1,F_{j,r}(t,s)})$ appearing in (10.4.4) is an orthonormal basis for the orthogonal complement of

$$\eta(t,s) \;=\; \sum_{i=1}^{l-j} s_i\eta_i(t)$$

in $T_{F_{j,r}(t,s)}\mathbb{R}^l$.

We now construct such a basis. For $1 \le m \le j$, let

$$W_{m,F_{j,r}(t,s)} \;=\; X_{m,t}$$

where $(X_{1,t},\ldots,X_{j,t})$ is an orthonormal basis of $T_t\partial_j M$. For $1 \le m \le l-j-1$, let

$$\widetilde{X}_m(s) \;=\; \sum_{i=1}^{l-j} a_{mi}(s)\frac{\partial}{\partial s_i}$$

be an orthonormal basis[10] of $T_sS(\mathbb{R}^{l-j})$ and define

$$W_{j+m,F_{j,r}(t,s)} = \sum_{i=1}^{l-j} a_{m,i}(s)\eta_i(t).$$

It is not difficult to see that the set $\{W_{1,F_{j,r}(t,s)}, \ldots, W_{l-1,F_{j,r}(t,s)}\}$ is an orthonormal basis of $T_{F_{j,r}(t,s)}S_j(r)$ as it is an orthonormal set and all elements are orthogonal to $\eta(t,s)$.

It follows from the above that the determinant in (10.4.4) has the form

(10.4.11) $\det \begin{pmatrix} I_{j\times j} - r\{S_\eta(X_{k,t}, X_{m,t})\}_{k,m=1,\ldots,j} & \cdots \\ 0 & rI_{(l-j-1)\times(l-j-1)} \end{pmatrix},$

where $\eta = \eta(s,t)$ is as above and S is the scalar second fundamental form of $\partial_j M$ in $\mathbb{R}^l$ (see (7.5.10)).

There are two things to note when computing this determinant. The first is the following familiar formula

$$\det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det(A)\det(C)$$

where both A and C are square. Consequently, (10.4.11) simplifies to

(10.4.12) $r^{l-j-1}\det\left(I_{j\times j} - r\{S_\eta(X_{k,t}, X_{m,t})\}\right).$

To evaluate the remaining determinant, we can either use standard, matrix theoretic, formulae for evaluating $\det(A + \lambda I)$, or, more formally, we can view S as a double form and use (7.2.10). As we shall use the latter approach for more difficult calculations later, we use it here as well. Therefore,

$$\det \begin{pmatrix} I_{j\times j} - r\{S_\eta(X_{k,t}, X_{m,t})\} & \cdots \\ 0 & rI_{(l-j-1)\times(l-j-1)} \end{pmatrix}$$

$$= r^{l-j-1}\frac{1}{j!}\mathrm{Tr}^{T_t\partial_j M}\left((I - rS_\eta)^j\right)$$

$$= \sum_{i=0}^{j} r^{l-j+i-1}\frac{1}{i!(j-i)!}\mathrm{Tr}^{T_t\partial_j M}\left(I^{j-i}S_{-\eta}^i\right)$$

$$= \sum_{i=0}^{j} r^{l-j+i-1}\frac{1}{i!}\mathrm{Tr}^{T_t\partial_j M}\left(S_{-\eta}^i\right),$$

where we have used (7.2.10) in going from the third to the fourth line.

Putting the above together, we have proven the following theorem, which is the main result of this section.

---

[10]The a_{mi} can be taken to be the same as in the proof of Lemma 10.4.1, and the $\partial/\partial s_i$ are a basis of $\mathbb{R}^l$.

Theorem 10.4.2 *If $\widehat{M} = \mathbb{R}^l$, then*

$$F_{j,r}^*(\Omega_{j,r})\left(X_{1,t},\ldots,X_{j,t},\widetilde{X}_{1,s},\ldots,\widetilde{X}_{l-j-1,s}\right)$$

$$= \sum_{i=0}^{j} r^{l-j-1+i}\frac{1}{i!}\mathrm{Tr}^{T_t\partial_j M}(S_{-\eta}^i)\mathrm{Vol}_{\partial_j M \times S(\mathbb{R}^{l-j})},$$

where S is the scalar second fundamental form of $\partial_j M$ in $\mathbb{R}^l$.

In particular, if μ is the measure associated to an integrable $l-1$-form α_μ on $S(\mathbb{R}^l)$, then

(10.4.13)

$$\mu\left(\mathrm{Tube}(M,\rho)\right) = \sum_{j=0}^{l}\sum_{i=0}^{j}\int_{\partial_j M}\int_{S(\mathbb{R}^{l-j})}\int_0^\rho \mathbb{1}_{D_j(\rho)}(t,s,r)f_\mu\left(F_{j,r}(t,s)\right)$$

$$\times r^{l-j-1+i}\frac{1}{i!}\mathrm{Tr}^{T_t\partial_j M}(S_{-\eta}^i)\,drdsdt.$$

10.4.4   Subsets of spheres

In this section, we treat the case $\widehat{M} = S_\lambda(\mathbb{R}^l)$, the sphere of radius λ in $\mathbb{R}^l$. This case is very similar to the previous one, so we need only go over the major differences.

The geodesic metric on $S_\lambda(\mathbb{R}^l)$ is given by

$$\widehat{d}_\lambda(t,s) = \lambda\cos^{-1}\left(\lambda^{-2}\langle t,s\rangle\right).$$

The exponential mapping $\exp_{S_\lambda(\mathbb{R}^l)}$ is also explicitly computable in this case. For a unit vector $X_t \in T_t S_\lambda(\mathbb{R}^l)$ and $r > 0$

$$\exp_{S_\lambda(\mathbb{R}^l)}(t,rX_t) = \cos(r/\lambda)t + \lambda\sin(r/\lambda)X_t.$$

The maps $F_{j,r}$ are of the form

$$F_{j,r}(t,s) = \cos(r/\lambda)t + \lambda\sin(r/\lambda)\sum_{i=1}^{l-1-j} s_i\eta_i(t),$$

for some orthonormal frame $(\eta_1(t),\ldots,\eta_{l-1-j}(t))$ spanning $T_t\widehat{\partial_j M}^\perp$ the orthogonal complement of $T_t\partial_j M$ in $T_t\widehat{M} = T_t S_\lambda(\mathbb{R}^l)$.

The orthogonality relations (10.4.8) and (10.4.9) still hold in this case, with the minor change that the W_i of the previous subsection now give a coordinate system on $S(\mathbb{R}^{l-1-j})$ instead of $S(\mathbb{R}^{l-j})$.

To be more specific, to construct an orthonormal basis on $T_{F_{j,r}(t,s)}S_j(r)$, we set

$$W_{m,F_{j,r}(t,s)} = X_{j,t}$$

as before, for $1 \le m \le j$, and for $1 \le m \le l-2-j$ we set

$$W_{j+m,F_{j,r}(t,s)} \ = \ \sum_{i=1}^{l-j-1} a_{mi}(s)\eta_i(t),$$

where

$$\widetilde{X}_m(s) \ = \ \sum_{i=1}^{l-j-1} a_{mi}(s)\frac{\partial}{\partial s_i}, \qquad 1 \le m \le l-2-j,$$

is an orthonormal basis of $T_s S(\mathbb{R}^{l-1-j})$.

Turning to the determinant (10.4.4) in $F_{j,r}^*(\Omega_{j,r})$, we have that it is now given by

(10.4.14)

$$\det \begin{pmatrix} \cos(r/\lambda)I + \lambda\sin(r/\lambda)S_{-\eta}(X_{k,t},X_{m,t}) & \cdots \\ 0 & \lambda\sin(r/\lambda)I \end{pmatrix}$$

$$= \sum_{i=0}^{j} \lambda^{l-2-j+i}\cos(r/\lambda)^{j-i}\sin(r/\lambda)^{l-j-2+i}\mathrm{Tr}^{T_t\partial_j M}(S_{-\eta}^i)$$

where the first unit matrix is of size j and the second of size $l-j-1$. The vector $\eta = \eta(s,t)$ is as in the previous subsection and S is now the scalar second fundamental form of $\partial_j M$ in $S_\lambda(\mathbb{R}^l)$ (cf. (7.5.10)).

We thus have

Theorem 10.4.3 *If* $\widehat{M} \ = \ S_\lambda(\mathbb{R}^l)$, *then*

$$F_{j,r}^*(\Omega_{j,r})\left(X_{1,t},\ldots,X_{j,t},\widetilde{X}_{1,s},\ldots,\widetilde{X}_{l-j-1,s}\right)$$

$$= \sum_{i=0}^{j} \lambda^{l-2-j+i}\cos(r/\lambda)^{j-i}\sin(r/\lambda)^{l-j-2+i}\mathrm{Tr}^{T_t\partial_j M}(S_{-\eta}^i)\mathrm{Vol}_{\partial_j M\times S(\mathbb{R}^{l-j-1})}$$

where $S_{-\eta}$ *is the scalar second fundamental form of* $\partial_j M$ *in* $S(\mathbb{R}^l)$.

In particular, if μ *is the measure associated to an integrable* $l-1$-*form* α_μ *on* $S(\mathbb{R}^l)$, *then*

(10.4.15)

$$\mu\left(\mathrm{Tube}(M,\rho)\right)$$

$$= \sum_{j=0}^{l-1}\sum_{i=0}^{j} \int_{\partial_j M}\int_{S(\mathbb{R}^{l-1-j})}\int_0^\rho \mathbb{1}_{D_j(\rho)}(t,s,r)f_\mu\left(F_{j,r}(t,s)\right)$$

$$\times \lambda^{l-2-j+i}\cos(r/\lambda)^{j-i}\sin(r/\lambda)^{l-j-2+i}\frac{1}{i!}\mathrm{Tr}^{T_t\partial_j M}(S_{-\eta}^i)\ drdsdt.$$

10.5   Weyl's tube formula

As we have written them, Theorems 10.4.2 and 10.4.3 are not exactly Weyl's tube formula [158] as the remaining integral is not generally not intrinsic. Weyl's tube formula applies specifically to the case when μ is Lebesgue measure on $\mathbb{R}^l$, in which case $f_\mu \equiv 1$, or when μ is surface measure on $S(\mathbb{R}^l)$. The last step in deriving Weyl's tube formula involves proving that only intrinsic quantities arise when the integrals in Theorems 10.4.2 and 10.4.3 are evaluated. This is the *deep* part of Weyl's tube formula, and it is one of the first appearances of the Lipschitz-Killing curvatures.

The derivation that follows is a little unusual, since it involves expectations of random Gaussian forms, and so is not what you will find in a standard textbook. Nevertheless, it does fit in nicely with the theme of this book, which attempts to integrate both Geometry and Probability.

The drawback of this approach is that the following argument depends on Lemma 12.3.1, which is still two chapters into the future. With only minor effort, we could rearrange things so that lemma appeared here rather than there, but it is in Chapter 12 where it logically belongs. To make things a little easier for you, we shall precall one definition now.

Let V be a vector space and $\mu \in \Lambda^{1,1}(V)$ a double form on V. Furthermore, let $\mathrm{Cov} : (V \otimes V) \times (V \otimes V) \to \mathbb{R}$ be bilinear, symmetric and non-negative definite. We think of μ as a mean function and Cov as a covariance function, and call W a random, Gaussian, 2-form on $V \otimes V$ with mean function μ and covariance function Cov, if

$$W(v_1, v_2) \ \sim \ N\big(\mu(v_1, v_2), \mathrm{Cov}((v_1, v_2), (v_1, v_2))\big)$$

for all $v_1, v_2 \in V$.

Lemma 12.3.1 is about computing expectations of powers of Gaussian forms.

To return to our main task, what remains in order to derive Weyl's tube formula is proving that, for ρ small enough,

$$\int_{\partial_j M} \int_{S(T_t \partial_j M^\perp)} \int_0^\rho \mathbb{1}_{D_j(\rho)}(t, s, r) \mathrm{Tr}^{\partial_j M}(S_{-\eta}^i) \, drdsdt \ = \ \widetilde{C}(j, i, l) \, Q(\partial_j M),$$

for some constant $\widetilde{C}$ and some intrinsic quantity $Q(\partial_j M)$, which will turn out to be related to the intrinsic volumes of M. Recall that, in the Euclidean case, we have $S(T_t \partial_j M^\perp) = S(\mathbb{R}^{l-j})$ while for the spherical case $S(T_t \partial_j M^\perp) = S(\mathbb{R}^{l-j-1})$.

As a first step, define the constants

$$C(m, i) \ \triangleq \ \begin{cases} \frac{(2\pi)^{i/2}}{s_{m+i}} & m + i > 0 \\ 1 & m = 0 \end{cases}$$

where, as usual,

$$s_m = \frac{2\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right)}$$

is the surface area of the unit sphere in $\mathbb{R}^m$.

Our first result relating the quantities in Theorem 10.4.2 to intrinsic quantities is the following.

Lemma 10.5.1 *Suppose that $M \subset \widetilde{M}$ is a piecewise smooth space embedded in the N-manifold $(\widetilde{M}, \widetilde{g})$ which is isometrically embedded in $\mathbb{R}^l$. Then, for $j < l$ the following quantities are intrinsic, for each $t \in M$.*

(10.5.1)

$$C(l-j,i) \int_{S(\mathbb{R}^{l-j})} \mathbb{1}_{\widehat{N_t M}}(\eta) \frac{1}{i!} \mathrm{Tr}^{T_t \partial_j M}(S_{-\eta}^i)\, \mathcal{H}_{l-j-1}(d\eta)$$

$$= \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} C(N-j, i-2m)(-1)^i \frac{(-1)^m}{m!\,(i-2m)!}$$

$$\times \int_{S(T_t \partial_j M^\perp)} \mathrm{Tr}^{T_t \partial_j M}\left(\widetilde{R}^m \widetilde{S}_{-\nu_{N-j}}^{i-2m}\right) \mathbb{1}_{N_t M}(\nu_{N-j})\, \mathcal{H}_{N-j-1}(d\nu_{N-j}),$$

where S is the scalar second fundamental form of $\partial_j M$ as it sits in $\mathbb{R}^l$, $T_t \partial_j M^\perp$ is the orthogonal complement of $T_t \partial_j M$ in $T_t \widetilde{M}$, $\widetilde{R}$ is the curvature tensor of $(\widetilde{M}, \widetilde{g})$, $\widetilde{S}$ is the scalar second fundamental form $\partial_j M$ as it sits in $\widetilde{M}$, and

$$\eta = \eta(s,t) = \sum_i s_i \eta_i(t),$$

with $\{\eta_i(t)\}_{1 \le i \le l-j}$ an orthonormal basis of $\widehat{T_t \partial_j M}^\perp$, the orthogonal complement of $T_t \partial_j M$ in $\mathbb{R}^l$.

Remark 10.5.2 *Above, if $j = l$, then $\widetilde{M}$ is essentially an open subset of $\mathbb{R}^l$ so that $N = l$ and the left and right hand side of (10.5.1) are not defined. This of course plays havoc with our notation, as we would be integrating over the sphere in the zero subspace of $T_t \widetilde{M}$. We therefore adopt the following somewhat strange but convenient convention: for any zero space $\mathbb{0}$ and any function h on $\mathbb{0}$*

$$\int_{S(\mathbb{0})} h(\nu)\, d\nu \triangleq h(\mathbb{0}).$$

This is well-defined since $\mathbb{0}$ has only one element and is effectively equivalent to defining $S(\mathbb{0}) = \mathbb{0}$. We also adopt the convention that

$$S_{\mathbb{0}}^j = \begin{cases} 1 & j = 0 \\ 0 & otherwise. \end{cases}$$

Finally, we define the volume of the unit ball in $\mathbb{0}$ to be 1. i.e.

$$\omega_0 \; \overset{\triangle}{=} \; 1.$$

These issues will thankfully only arise when defining the Lipschitz-Killing curvatures. These conventions are pedantic, but are important to ensure that the Lipschitz-Killing curvature measures for piecewise smooth spaces, defined in (10.5.3), agree with the Lipschitz-Killing curvature measures for smooth manifolds defined in (7.6.1).

The corresponding version of Lemma 10.5.1 for subsets of a hypersphere is as follows.

Lemma 10.5.3 *Suppose that $M \subset \widetilde{M}$ is a piecewise smooth space embedded in the N-manifold $(\widetilde{M}, \widetilde{g})$ which is isometrically embedded in $S_\lambda(\mathbb{R}^l)$. Then, for $j < l$, the following quantities are intrinsic, for each $t \in M$.*

$$(10.5.2) \quad C(l-1-j,i) \int_{S(\mathbb{R}^{l-1-j})} \mathbb{1}_{\widehat{N_t M}}(\eta) \frac{1}{i!} \mathrm{Tr}^{T_t \partial_j M}(S_{-\eta}^i) \, ds$$

$$= \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} C(N-j, i-2m)(-1)^i \frac{(-1)^m}{m!\,(i-2m)!}$$

$$\times \int_{S(T_t \partial_j M^\perp)} \mathrm{Tr}^{T_t \partial_j M} \left(\left(\frac{\lambda^{-2}}{2} I^2 + \widetilde{R} \right)^m \widetilde{S}_{-\nu_{N-j}}^{i-2m} \right)$$

$$\times \mathbb{1}_{N_t M}(\nu_{N-j}) \, \mathcal{H}_{N-j-1}(d\nu_{N-j}),$$

where S is the scalar second fundamental form of $\partial_j M$ as it sits in $S_\lambda(\mathbb{R}^l)$, $T_t \partial_j M^\perp$ is the orthogonal complement of $T_t \partial_j M$ in $T_t \widetilde{M}$, $\widetilde{R}$ is the curvature tensor of $(\widetilde{M}, \widetilde{g})$, $\widetilde{S}$ is the scalar second fundamental form $\partial_j M$ as it sits in $\widetilde{M}$, and

$$\eta \;=\; \eta(s,t) \;=\; \sum_i s_i \eta_i(t)$$

with $\{\eta_i(t)\}_{1 \leq i \leq l-j}$ an orthonormal basis of $\widehat{T_t \partial_j M}^{\perp}$, the orthogonal complement of $T_t \partial_j M$ in $\mathbb{R}^l$.

Remark 10.5.4 *Above, if $j = l-1$, then $\widetilde{M}$ is essentially an open subset of $S_\lambda(\mathbb{R}^l)$, so that $N = l-1$ and the left and right hand side of (10.5.2) are not defined. We therefore again adopt the conventions of Remark 10.5.2.*

The proofs of Lemmas 10.5.1 and 10.5.3 are virtually identical and are based on the following lemma, for which we recall that $\gamma_{\mathbb{R}^m}$ denotes Gauss measure on $\mathbb{R}^m$ and η_m denotes uniform measure over $S(\mathbb{R}^m)$.

Lemma 10.5.5 *Let $W \sim \gamma_{\mathbb{R}^m}$ and let P_i be a homogenous polynomial of order i in the standard basis on $\mathbb{R}^m$, and $A \subset \mathbb{R}^m$ a cone. Then*

$$\mathbb{E}\left\{P_i(W)\mathbb{1}_A(W)\right\} \; = \; C(m,i)\int_{S(\mathbb{R}^m)} P_i(\nu)\mathbb{1}_A(\nu)\,\mathcal{H}_{m-1}(d\nu).$$

Proof.

$$\mathbb{E}\left\{P_i(W)\mathbb{1}_A(W)\right\}$$
$$= \; \mathbb{E}\left\{|W|^i P_i(W/|W|)\mathbb{1}_A(W/|W|)\right\}$$
$$= \; \int_0^\infty r^i \int_{S(\mathbb{R}^m)} P_i(\nu)\mathbb{1}_A(\nu)\,\eta_m(d\nu)\,\mathbb{P}_{|W|}(dr)$$
$$= \; \frac{\Gamma\left(\frac{m}{2}\right)}{2\pi^{m/2}}\int_0^\infty r^i \mathbb{P}_{|W|}(dr)\int_{S(\mathbb{R}^m)} P_i(\nu)\mathbb{1}_A(\nu)\,\mathcal{H}_{m-1}(d\nu)$$
$$= \; \frac{1}{2}(2\pi)^{-m/2}2^{(i+m)/2}\Gamma\left(\frac{m+i}{2}\right)\int_{S(\mathbb{R}^m)} P_i(\nu)\mathbb{1}_A(\nu)\,\mathcal{H}_{m-1}(d\nu).$$

$\square$

Proof of Lemma 10.5.1 Our proof is somewhat indirect and, as mentioned above, proceeds via Gaussian random forms, which are not generally found in geometric arguments. Rather than taking one expression in (10.5.1) and showing that it is equal to the other, we shall show that both are given by a certain expectation.

One of the nice aspects of this approach is that it immediately shows that both sides of (10.5.1) are truly intrinsic[11], since the expectation will, *a priori*, involve only intrinsic quantities.

We start with the case $j < N$, leaving the easier case of $j = N$ for later.

Therefore, let $X_{l-j} \sim \gamma_{\widehat{T_t\partial_j M}^\perp}$ be a canonically distributed Gaussian random vector on $\widehat{T_t\partial_j M}^\perp$, the orthogonal complement of $T_t\partial_j M$ in $T_t\widehat{M}$. For a fixed orthonormal basis of $\widehat{T_t\partial_j M}^\perp$, the random variable

$$\frac{1}{i!}\text{Tr}^{T_t\partial_j M}\left(S^i_{-X_{l-j}}\right)$$

---

[11]Of course, this would also follow from *any* proof that (10.5.1) held, since while the left hand side - which seems to depend on the non-intrinsic parameter l - is not obviously intrinsic, the right hand side depends only on intrinsic quantities.

is a homogeneous polynomial of degree i in the components of X_{l-j} in this basis. Hence, by Lemma 10.5.5,

$$\frac{1}{i!}\mathbb{E}\left\{\mathrm{Tr}^{T_t\partial_j M}(S^i_{-X_{l-j}})\mathbb{1}_{\widehat{N_t M}}(X_{l-j})\right\}$$

$$= C(l-j,i)\int_{S(\widehat{T_t\partial_j M})}\mathbb{1}_{\widehat{N_t M}}(\nu_{l-j})\frac{1}{i!}\mathrm{Tr}^{T_t\partial_j M}(S^i_{-\nu_{l-j}})\,\mathcal{H}_{l-j-1}(\nu_{l-j})$$

$$= C(l-j,i)\int_{S(\mathbb{R}^{l-j})}\mathbb{1}_{\widehat{N_t M}}(\eta)\frac{1}{i!}\mathrm{Tr}^{T_t\partial_j M}(S^i_{-\eta})\,ds,$$

which is the left hand side of (10.5.1).

On the other hand, X_{l-j} can be decomposed as

$$X_{l-j} = P_{T_t\widetilde{M}}X_{l-j}+P^{\perp}_{T_t\widetilde{M}}X_{l-j} \stackrel{\Delta}{=} X_{N-j}+X_{l-N},$$

With this notation, we can write

$$\frac{1}{i!}\mathrm{Tr}^{T_t\partial_j M}(S^i_{-X_{l-j}}) = \sum_{m=0}^{i}\frac{1}{m!\,(i-m)!}\mathrm{Tr}^{T_t\partial_j M}\left(S^m_{-X_{l-N}}S^{i-m}_{-X_{N-j}}\right)$$

where the double forms $S_{-X_{N-j}}$ and $S_{-X_{l-N}}$ are independent.

Furthermore,

$$\mathbb{1}_{\widehat{N_t M}}(X_{l-j}) = \mathbb{1}_{N_t M}(X_{N-j}).$$

Consequently, we have

$$\mathbb{E}\left\{\frac{1}{i!}\mathrm{Tr}^{T_t\partial_j M}(S^i_{-X_{l-j}})\mathbb{1}_{\widehat{N_t M}}(X_{l-j})\right\}$$

$$= \sum_{m=0}^{i}\frac{1}{m!\,(i-m)!}\mathrm{Tr}^{T_t\partial_j M}\left(\mathbb{E}\left\{S^m_{-X_{l-N}}S^{i-m}_{-X_{N-j}}\mathbb{1}_{N_t M}(X_{N-j})\right\}\right)$$

$$= \sum_{m=0}^{i}\frac{1}{m!\,(i-m)!}\mathrm{Tr}^{T_t\partial_j M}\left(\mathbb{E}\left\{S^m_{-X_{l-N}}\right\}\mathbb{E}\left\{\widetilde{S}^{i-m}_{-X_{N-j}}\mathbb{1}_{N_t M}(X_{N-j})\right\}\right).$$

The second expectation here is easily computed, as above, via Lemma 10.5.5. For the first, we shall apply Lemma 12.3.1, but to do this we need to know the 'means' and 'covariances' of S. From symmetry arguments, the expected value is clearly 0. That is, for any $X_t, Y_t, Z_t, W_t \in T_t\widetilde{M}$,

$$\mathbb{E}\left\{S_{-X_{l-N}}(X_t, Y_t)\right\} = 0.$$

As for the 'variance' term, by (7.5.8)

$$
\begin{aligned}
\mathbb{E}&\left\{\frac{1}{2}S^2_{-X_{l-N}}\left((X_t,Y_t),(Z_t,W_t)\right)\right\} \\
&= \mathbb{E}\left\{\widetilde{S}_{-X_{l-N}}(X_t,Z_t)\widetilde{S}_{-X_{l-N}}(Y_t,W_t)\right. \\
&\qquad\qquad\left. - \widetilde{S}_{-X_{l-N}}(X_t,W_t)\widetilde{S}_{-X_{l-N}}(Y_t,Z_t)\right\} \\
&= \mathbb{E}\left\{\left\langle P_{T_t\widetilde{M}^\perp}\widehat{\nabla}_{X_t}Z_t,X_{l-N}\right\rangle\left\langle P_{T_t\widetilde{M}^\perp}\widehat{\nabla}_{Y_t}W_t,X_{l-N}\right\rangle -\right. \\
&\qquad\qquad\left. - \left\langle P_{T_t\widetilde{M}^\perp}\widehat{\nabla}_{X_t}W_t,X_{l-N}\right\rangle\left\langle P_{T_t\widetilde{M}^\perp}\widehat{\nabla}_{Y_t}Z_t,X_{l-N}\right\rangle\right\} \\
&= \left\langle P_{T_t\widetilde{M}^\perp}\widehat{\nabla}_{X_t}Z_t,P_{T_t\widetilde{M}^\perp}\widehat{\nabla}_{Y_t}W_t\right\rangle - \left\langle P_{T_t\widetilde{M}^\perp}\widehat{\nabla}_{X_t}W_t,P_{T_t\widetilde{M}^\perp}\widehat{\nabla}_{Y_t}Z_t\right\rangle \\
&= \widehat{R}(X_t,Y_t,Z_t,W_t) - \widetilde{R}(X_t,Y_t,Z_t,W_t) \\
&= -\widetilde{R}(X_t,Y_t,Z_t,W_t),
\end{aligned}
$$

where the second last line follows from the Gauss equation (7.5.8) and the last from the fact that $\widehat{M} = \mathbb{R}^l$, and $\mathbb{R}^l$ is flat.

Consequently, applying Lemmas 10.5.5 and 12.3.1, we have

$$
\begin{aligned}
\mathbb{E}&\left\{\frac{1}{i!}\mathrm{Tr}^{T_t\partial_j M}(S^i_{-X_{l-j}})\mathbb{1}_{\widehat{N_tM}}(X_{l-j})\right\} \\
&= \sum_{m=0}^{\lfloor\frac{i}{2}\rfloor}\frac{(-1)^m}{m!\,(i-2m)!}\mathrm{Tr}^{T_t\partial_j M}\left(\widetilde{R}^m\mathbb{E}\left\{\widetilde{S}^{i-2m}_{-X_{N-j}}\mathbb{1}_{N_tM}(X_{N-j})\right\}\right) \\
&= \sum_{m=0}^{\lfloor\frac{i}{2}\rfloor}\frac{(-1)^m}{m!\,(i-2m)!}\mathbb{E}\left\{\mathrm{Tr}^{T_t\partial_j M}\left(\widetilde{R}^m\widetilde{S}^{i-2m}_{-X_{N-j}}\right)\mathbb{1}_{N_tM}(X_{N-j})\right\} \\
&= \sum_{m=0}^{\lfloor\frac{i}{2}\rfloor}C(N-j,i-2m)\frac{(-1)^m}{m!\,(i-2m)!} \\
&\qquad\times \int_{S(T_t\partial_j M^\perp)}\mathrm{Tr}^{T_t\partial_j M}\left(\widetilde{R}^m\widetilde{S}^{i-2m}_{-\nu_{N-j}}\right)\mathbb{1}_{N_tM}(\nu_{N-j})\,\mathcal{H}_{N-j-1}(d\nu_{N-j}),
\end{aligned}
$$

which is the right hand side of (10.5.1), and so we are done when $j < N$.

When $j = N$ we need the conventions that we adopted in Remark 10.5.2 to sort things out. If $j = N < l$ then $N_tM = \mathbb{0} \subset T_t\widetilde{M}$ and, by our

conventions,

$$\sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} C(N-j, i-2m) \frac{(-1)^m}{m!\,(i-2m)!}$$

$$\times \int_{S(T_t\partial_j M^\perp)} \mathrm{Tr}^{T_t\partial_j M}\left(\widetilde{R}^m \widetilde{S}^{i-2m}_{-\nu_{N-j}}\right) \mathbb{1}_{N_t M}(\nu_{N-j})\,\mathcal{H}_{N-j-1}(d\nu_{N-j})$$

$$= \begin{cases} \mathrm{Tr}^{T_t \widetilde{M}}(\widetilde{R}^k) & i = 2k \text{ is even} \\ 0 & i \text{ is odd.} \end{cases}$$

On the other hand, if $j = N = l$ then, again by our conventions,

$$\mathbb{E}\left\{ \frac{1}{i!} \mathrm{Tr}^{T_t\partial_j M}(S^i_{-X_{l-j}}) \mathbb{1}_{\widehat{N_t M}}(X_{l-j}) \right\} = \begin{cases} 1 & i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The right hand side of (10.5.1) is also 0 in this case, by our convention and the fact that $\widetilde{R} = 0$, and so we are done.

$\square$

The proof of Lemma 10.5.3 is virtually identical and so we omit it. As in the previous proof, there are some special cases ($j = N = l - 1$ and $j < N = l$) to be checked, but the conventions established in Remark 10.5.4 again suffice to handle these.

We are now almost ready to complete the derivation of Weyl's tube formula on $\mathbb{R}^l$ and $S_\lambda(\mathbb{R}^l)$. For $0 \le j \le k$ and $l - k \le i \le l$ define the (signed) Lipschitz-Killing curvature measures [61] of a piecewise smooth space M for $0 \le i \le N$, by

(10.5.3)

$$\mathcal{L}_i(M, A) = \sum_{j=i}^{N} (2\pi)^{-(j-i)/2} \sum_{m=0}^{\lfloor \frac{j-i}{2} \rfloor} C(N-j, j-i-2m) \frac{(-1)^m}{m!\,(j-i-2m)!}$$

$$\times \int_{\partial_j M \cap A} \int_{S(T_t\partial_j M^\perp)} \mathrm{Tr}^{T_t\partial_j M}\left(\widetilde{R}^m \widetilde{S}^{j-i-2m}_{\nu_{N-j}}\right)$$

$$\times \mathbb{1}_{N_t M}(-\nu_{N-j})\,\mathcal{H}_{N-j-1}(d\nu_{N-j})\mathcal{H}_j(dt),$$

where, as usual, N is the dimension of the ambient manifold. We define $\mathcal{L}_i(M;)$ to be 0 if $i > \dim(M)$.

The above formula is somewhat simpler if M is embedded in $\mathbb{R}^l$ and endowed with the canonical Riemannian structure on $\mathbb{R}^l$. Then, by Lemma 10.5.1 and the flatness of $\mathbb{R}^l$, the curvature measures (10.5.3) can be written

as

(10.5.4)

$$
\begin{aligned}
\mathcal{L}_i(M, A) \;\;=\;\; &\sum_{j=i}^{N} (2\pi)^{-(j-i)/2} C(l-j, j-i) \\
&\times \int_{\partial_j M \cap A} \int_{S(\mathbb{R}^{l-j})} \frac{1}{(j-i)!} \operatorname{Tr}^{T_t \partial_j M}(S_\eta^{j-i}) \\
&\times \mathbb{1}_{\widehat{N_t M}}(-\eta) \mathcal{H}_{l-j-1}(d\eta) \mathcal{H}_j(dt),
\end{aligned}
$$

where S is the scalar second fundamental form of $\partial_j M$ as it sits in $\mathbb{R}^l$ and $\widehat{N_t M}$ is the normal cone at $t \in \partial_j M$ as it sits in $T_t \mathbb{R}^l$.

As before, the total masses of the curvature measures give the *intrinsic volumes*.

(10.5.5) $$\mathcal{L}_i(M) \;\overset{\triangle}{=}\; \mathcal{L}_i(M, M).$$

Theorem 10.5.6 (*Weyl's tube formula on $\mathbb{R}^l$*) *Suppose $M \subset \mathbb{R}^l$ is a piecewise smooth manifold. For $\rho < \rho_c(M, \mathbb{R}^l)$,*

(10.5.6) $$\mathcal{H}_l\left(\text{Tube}(M, \rho)\right) \;=\; \sum_{i=0}^{N} \rho^{l-i} \omega_{l-i} \mathcal{L}_i(M).$$

The upper limit of the sum can be taken as infinity, since $\mathcal{L}_i(M,) = 0$ for $i > N$.

Proof. Changing the second index of summation in (10.4.13) and setting $f_\mu \equiv 1$ we have that

$$
\begin{aligned}
&\mathcal{H}_l\left(\text{Tube}(M, r)\right) \\
&= \sum_{j=0}^{N} \sum_{i=0}^{j} \int_{\partial_j M} \int_{S(\mathbb{R}^{l-j})} \int_0^\rho \mathbb{1}_{D_j(\rho)}(t, s, r) \frac{r^{l-i-1}}{(j-i)!} \operatorname{Tr}^{T_t \partial_j M}(S_{-\eta}^{j-i}) dr \, ds \, dt.
\end{aligned}
$$

By assumption $\rho < \rho_c(M)$, so that

$$
D_j(\rho) \;=\; \bigcup_{t \in \partial_j M} \bigcup_{s \in S(\mathbb{R}^{l-j})} \bigcup_{0 \le r \le \rho} \left\{ (t, s, r) : \sum_i s_i \eta_i(t) \in N_t M, r < \rho \right\}.
$$

Applying Lemma 10.5.1 to each term in the above summation, followed by integration with respect to r on $[0, \rho]$, yields the desired conclusion.   □

In preparation for Weyl's tube formula on spheres, we need to extend the Lipschitz-Killing measures slightly to obtain a one parameter family of

measures. As we shall see in a moment, the κ-th such measure is particularly useful for generating a comparatively simple formula for tube volumes on a sphere of radius $1/\sqrt{\kappa}$. The definition, for $\kappa \geq 0$, is given by

(10.5.7)

$$
\mathcal{L}_i^\kappa(M, A) \;\triangleq\; \sum_{j=i}^{N} (2\pi)^{-(j-i)/2} \sum_{m=0}^{\lfloor \frac{j-i}{2} \rfloor} \frac{(-1)^m C(N-j, j-i-2m)}{m!(j-i-2m)!}
$$

$$
\times \int_{\partial_j M \cap A} \int_{S(T_t \partial_j M^\perp)} \mathrm{Tr}^{T_t \partial_j M}\left(\left(\widetilde{R} + \frac{\kappa}{2} I^2\right)^m \widetilde{S}_{\nu_{N-j}}^{j-i-2m} \right)
$$

$$
\times \mathbb{1}_{N_t M}(-\nu_{N-j})\, \mathcal{H}_{N-j-1}(d\nu_{N-j})\, \mathcal{H}_j(dt).
$$

Again, the above apparently infinite sum is in fact a finite sum. Note that $\mathcal{L}_i^0(M, \cdot) \equiv \mathcal{L}_i(M, \cdot)$.

As for $\mathcal{L}_i(M)$, we define the one parameter family of intrinsic volumes

(10.5.8) $$\mathcal{L}_i^\kappa(M) \;\triangleq\; \mathcal{L}_i^\kappa(M, M).$$

Furthermore, if $\kappa > 0$ and $M \subset S_{\kappa^{-1/2}}(\mathbb{R}^l)$ then, as in (10.5.4),

(10.5.9)

$$
\mathcal{L}_i^\kappa(M, A) \;=\; \sum_{j=i}^{N} (2\pi)^{-(j-i)/2} C(l-1-j, j-i)
$$

$$
\times \int_{\partial_j M \cap A} \int_{S(T_t \partial_j M^\perp)} \frac{1}{(j-i)!} \mathrm{Tr}^{T_t \partial_j M}(S_\eta^{j-i})
$$

$$
\times \mathbb{1}_{\widehat{N_t M}}(-\eta) \mathcal{H}_{l-j-2}(d\eta),
$$

where S is the scalar second fundamental form of $\partial_j M$ as it sits in $S_{\kappa^{-1/2}}(\mathbb{R}^l)$, $T_t \partial_j M^\perp$ is the orthogonal complement of $T_t \partial_j M$ in $T_t S_{\kappa^{-1/2}}(\mathbb{R}^l)$ and $\widehat{N_t M}$ is the normal cone at $t \in \partial_j M$ as it sits in $T_t S_{\kappa^{-1/2}}(\mathbb{R}^l)$.

A slight modification of the proof of Theorem 10.5.6 is enough to complete the derivation of Weyl's tube formula on $S_\lambda(\mathbb{R}^l)$.

Theorem 10.5.7 (*Weyl's tube formula on $S_\lambda(\mathbb{R}^l)$*) *Suppose M is a piecewise smooth manifold of $S_\lambda(\mathbb{R}^l)$. For $\rho < \rho_c(M, S_\lambda(\mathbb{R}^l))$,*

$$
\mathcal{H}_{l-1}\left(\mathrm{Tube}(M, \rho)\right)
$$

$$
= \sum_{i=0}^{\infty} \lambda^{l-1-i} G_{i, l-1-i}(\rho/\lambda) \mathcal{L}_i^{\lambda^{-2}}(M)
$$

$$
= \sum_{j=0}^{\infty} \left(\sum_{n=0}^{\lfloor \frac{j}{2} \rfloor} (-4\pi)^{-n} \frac{\lambda^{l-1+j}}{n!} \frac{j!}{(j-2n)!} G_{j-2n, l-1+2n-j}(\rho/\lambda) \right) \mathcal{L}_j(M),
$$

where

$$G_{a,b}(\rho) \quad \triangleq \quad \frac{\pi^{b/2}}{b\Gamma\left(\frac{b}{2}+1\right)} \int_0^\rho \cos^a(r)\sin^{b-1}(r)\, dr$$

$$= \quad \frac{\pi^{b/2}}{\Gamma\left(\frac{b}{2}\right)} \overline{IB}_{(a+1)/2,b/2}(\cos^2\rho),$$

with

$$\overline{IB}_{(a+1)/2,b/2}(x) \quad \triangleq \quad \int_x^1 x^{(a-1)/2}(1-x)^{(b-2)/2}\, dx$$

the tail of the incomplete beta function.

Finally we note the following result for use in Chapter 14.

Lemma 10.5.8 *The following equivalences hold between the standard and extended Lipschitz-Killing curvatures:*

$$(10.5.10) \qquad \mathcal{L}_i^\kappa(\cdot) \quad = \quad \sum_{n=0}^\infty \frac{(-\kappa)^n}{(4\pi)^n} \frac{(i+2n)!}{n!i!} \mathcal{L}_{i+2n}(\cdot),$$

and

$$(10.5.11) \qquad \mathcal{L}_i(\cdot) \quad = \quad \sum_{n=0}^\infty \frac{\kappa^n}{(4\pi)^n} \frac{(i+2n)!}{n!i!} \mathcal{L}_{i+2n}^\kappa(\cdot).$$

Proof. We shall prove only (10.5.10) since (10.5.11) follows from an almost identical argument. In both cases the argument is primarily algebraic and rather elementary, but since keeping track of combinatorial coefficients is not always trivial, and this lemma will be rather important later, we give the details.

By the definition (10.5.7), $\mathcal{L}_i^\kappa(M, A)$ is given by

$$\sum_{j=i}^{N} (2\pi)^{-(j-i)/2} \sum_{m=0}^{\lfloor \frac{j-i}{2} \rfloor} C(N-j, j-i-2m) \frac{(-1)^m}{m! \, (j-i-2m)!}$$

$$\times \int_{\partial_j M \cap A} \int_{S(T_t \partial_j M^\perp)} \mathrm{Tr}^{T_t \partial_j M} \left(\left(\widetilde{R} + \frac{\kappa}{2} I^2 \right)^m \widetilde{S}_{\nu_{N-j}}^{j-i-2m} \right)$$

$$\times \mathbb{1}_{N_t M}(-\nu_{N-j}) \, \mathcal{H}_{N-j-1}(d\nu_{N-j}) \, \mathcal{H}_j(dt)$$

$$= \sum_{j=i}^{N} (2\pi)^{-(j-i)/2} \sum_{m=0}^{\lfloor \frac{j-i}{2} \rfloor} \sum_{n=0}^{m} (-\kappa/2)^n \binom{m}{n} \frac{C(N-j, j-i-2m)}{m! \, (j-i-2m)!}$$

$$\times \iint \mathrm{Tr}^{T_t \partial_j M} \left(I^{2n} (-\widetilde{R})^{m-n} \widetilde{S}_{\nu_{N-j}}^{j-i-2m} \mathbb{1}_{N_t M}(\nu_{N-j}) \right)$$

$$\times \mathcal{H}_{N-j-1}(d\nu_{N-j}) \, \mathcal{H}_j(dt)$$

$$= \sum_{j=i}^{N} (2\pi)^{-(j-i)/2} \sum_{n=0}^{\lfloor \frac{j-i}{2} \rfloor} \sum_{m=n}^{\lfloor \frac{j-i}{2} \rfloor} (-\kappa/2)^n \binom{m}{n} \frac{C(N-j, j-i-2m)}{m! \, (j-i-2m)!}$$

$$\times \iint \mathrm{Tr}^{T_t \partial_j M} \left(I^{2n} (-\widetilde{R})^{m-n} \widetilde{S}_{\nu_{N-j}}^{j-i-2m} \mathbb{1}_{N_t M}(\nu_{N-j}) \right)$$

$$\times \mathcal{H}_{N-j-1}(d\nu_{N-j}) \, \mathcal{H}_j(dt)$$

$$= \sum_{j=i}^{N} (2\pi)^{-(j-i)/2} \sum_{n=0}^{\lfloor \frac{j-i}{2} \rfloor} \sum_{m=0}^{\lfloor \frac{j-i-2n}{2} \rfloor} \frac{(-\kappa/2)^n}{n!} \frac{C(N-j, j-i-2m-2n)}{m! \, (j-i-2m-2n)!}$$

$$\times \iint \mathrm{Tr}^{T_t \partial_j M} \left(I^{2n} (-\widetilde{R})^m \widetilde{S}_{\nu_{N-j}}^{j-i-2m-2n} \mathbb{1}_{N_t M}(\nu_{N-j}) \right)$$

$$\times \mathcal{H}_{N-j-1}(d\nu_{N-j}) \, \mathcal{H}_j(dt),$$

where we have used (7.2.10) in obtaining the last line.

Continuing, the last line can be rewritten as

$$
\sum_{n=0}^{\lfloor \frac{N-i}{2} \rfloor} \frac{(4\pi)^{-n}(-\kappa)^n}{n!} \frac{(i+2n)!}{i!} \sum_{j=i+2n}^{k} (2\pi)^{-(j-i-2n)/2}
$$

$$
\times \sum_{m=0}^{\lfloor \frac{j-i-2n}{2} \rfloor} \frac{C(N-j, j-i-2m-2n)}{m!\,(j-i-2m-2n)!}
$$

$$
\times \iint \mathrm{Tr}^{T_t \partial_j M} \left((-\widetilde{R})^m \widetilde{S}_{\nu_{N-j}}^{j-i-2m-2n} \mathbb{1}_{N_t M}(\nu_{N-j}) \right)
$$

$$
\times \mathcal{H}_{N-j-1}(d\nu_{N-j})\,\mathcal{H}_j(dt)
$$

$$
= \sum_{n=0}^{\lfloor \frac{N-i}{2} \rfloor} \frac{(4\pi)^{-n}(-\kappa)^n}{n!} \frac{(i+2n)!}{i!} \mathcal{L}_{i+2n}(M, A)
$$

$$
= \sum_{n=0}^{\infty} \frac{(-4\pi/\kappa)^{-n}}{n!} \frac{(i+2n)!}{i!} \mathcal{L}_{i+2n}(M, A),
$$

which is what we require.   □

10.6   Volume of tubes and Gaussian processes, continued

With Weyl's tube formula established for submanifolds of $S_\lambda(\mathbb{R}^l)$ we are now able to continue with the volume of tubes approximation (10.0.1) to excursion probabilities for Gaussian processes. In (10.2.4), we established that, for a Gaussian process f with an orthogonal expansion of order l,

$$
\mathbb{P}\left\{ \sup_{t \in M} f_t \geq u \right\} = \frac{\Gamma\left(\frac{l}{2}\right)}{2\pi^{l/2}} \mathbb{E}\left\{ \mathcal{H}_{l-1}\left(\mathrm{Tube}(\psi(M), \cos^{-1}(u/|\xi|)) \right) \mathbb{1}_{\{|\xi| \geq u\}} \right\},
$$

where $\xi^2 \sim \chi_l^2$. On the other hand, Theorem 10.5.7 gives an explicit expression for

$$
\mathcal{H}_{l-1}\left(\mathrm{Tube}(\psi(M), \cos^{-1}(u/|\xi|)) \right)
$$

for $\cos^{-1}(u/|\xi|) < \rho_c(M, S(\mathbb{R}^l))$. However, as $|\xi| \to \infty$, the radius of the tube above grows and may surpass $\rho_c(M, S(\mathbb{R}^l))$, in which case Weyl's formula is not exact. *Formally ignoring this fact*, we arrive at the result:

Theorem 10.6.1 *Let f be a finite Karhunen-Loève process, such that $\psi(M)$ is a piecewise smooth submanifold of $S(\mathbb{R}^l)$, $l \geq N$. The formal tube approximation is given by*

$$
(10.6.1) \qquad \mathbb{P}\left\{ \sup_{t \in M} f_t \geq u \right\} \simeq \sum_{j=0}^{N} \rho_j(u)\mathcal{L}_j(\psi(M)),
$$

where

$$\rho_j(u) \;\triangleq\; \begin{cases} 1 - \Phi(u) & j = 0 \\ (2\pi)^{-(j+1)/2} H_{j-1}(u) e^{-u^2/2} & j \geq 1 \end{cases}$$

and H_j is the j-th Hermite polynomial (11.5.8).

Remark 10.6.2 *We re-emphasise what is the meaning of* (10.6.1). *The '$\simeq$' does* not *indicate approximate equality, asymptotic equality, or any other relationship that, at this stage, equates the left and right hand sides. Rather, it indicates that the right hand side comes from taking an expression that is eqivalent to the left hand side, and substituting into it expressions that, while sometimes equivalent, are sometimes definitely wrong. Only in Chapter 13 shall we show, via completely different methods, that the purely formal equivalence in* (10.6.1) *can be replaced by asymptotic, in u, equivalence, with specific bounds on errors.*

Proof. As the approximation formally ignores the condition $\rho < \rho_c(M)$ in Theorem 10.5.7, we have

$$\mathbb{P}\left\{ \sup_{t \in T} f_t \geq u \right\}$$

$$\simeq \frac{\Gamma\left(\frac{l}{2}\right)}{2\pi^{l/2}} \sum_{j=0}^{N} \mathcal{L}_j(\psi(M))$$

$$\times \mathbb{E}\left\{ \sum_{n=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(-4\pi)^{-n}}{n!} \frac{j!}{(j-2n)!} G_{j-2n,l-1+2n-j}(\cos^{-1}(u/|\xi|)) \mathbb{1}_{\{|\xi| \geq u\}} \right\}$$

and all that remains is to prove that $\rho_j(u)$ is equal to

$$\frac{\Gamma\left(\frac{l}{2}\right)}{2\pi^{l/2}} \sum_{n=0}^{\lfloor \frac{j}{2} \rfloor} \mathbb{E}\left\{ \frac{(-4\pi)^{-n}}{n!} \frac{j!}{(j-2n)!} G_{j-2n,l-1+2n-j}(\cos^{-1}(u/|\xi|)) \mathbb{1}_{\{|\xi| \geq u\}} \right\}.$$

We artificially decompose ξ as

$$|\xi|^2 \;=\; X_1 + X_2,$$

where $X_1 \sim \chi^2_{j-2n+1}$, $X_2 \sim \chi^2_{l-1+2n-j}$ are independent, so that

$$\frac{X_1}{X_1 + X_2} \;\sim\; \text{Beta}\left((j - 2n + 1)/2, (l - 1 + 2n - j)/2\right),$$

independently of $|\xi|^2$. It follows that

$$G_{j-2n,l-1+2n-j}(\cos^{-1}(u/|\xi|^2))$$

$$= \frac{\Gamma\left(\frac{j-2n+1}{2}\right)}{\Gamma\left(\frac{l}{2}\right)} \pi^{(l-1+2n-j)/2} \mathbb{P}\left(\frac{X_1}{X_1 + X_2} \geq \frac{u^2}{|\xi|^2} \,\middle|\, |\xi|^2 \right).$$

Continuing with the calculation,

$$\frac{(-4\pi)^{-n}\Gamma\left(\frac{l}{2}\right)}{2\pi^{l/2}}\mathbb{E}\left\{G_{j-2n,l-1+2n-j}(\cos^{-1}(u/|\xi|))\mathbb{1}_{\{|\xi|\geq u\}}\right\}$$

$$= \frac{(-4\pi)^{-n}\Gamma\left(\frac{j-2n+1}{2}\right)}{2}\pi^{-(j-2n+1)/2}$$

$$\times\mathbb{E}\left\{\mathbb{P}\left\{\frac{X_1}{X_1+X_2}\geq\frac{u^2}{|\xi|^2}\bigg||\xi|^2\right\}\mathbb{1}_{\{|\xi|\geq u\}}\right\}$$

$$= \frac{(-4\pi)^{-n}\Gamma\left(\frac{j-2n+1}{2}\right)}{2}\pi^{-(j-2n+1)/2}\mathbb{P}\left\{X_1\geq u^2\right\}$$

$$= \frac{(-4\pi)^{-n}\Gamma\left(\frac{j-2n+1}{2}\right)}{2}\pi^{-(j-2n+1)/2}\int_{u^2}^{\infty}\frac{x^{(j-2n-1)/2}e^{-x/2}}{2^{(j-2n+1)/2}\Gamma\left(\frac{j-2n+1}{2}\right)}\,dx$$

$$= (2\pi)^{-(j+1)/2}\frac{(-1)^n}{2^n}\int_u^{\infty}x^{j-2n}e^{-x^2/2}\,dx.$$

Finally

$$\frac{\Gamma\left(\frac{l}{2}\right)}{2\pi^{l/2}}\sum_{n=0}^{\lfloor\frac{j}{2}\rfloor}\mathbb{E}\left\{\frac{(-4\pi)^{-n}}{n!}\frac{j!}{(j-2n)!}G_{j-2n,l-1+2n-j}(\cos^{-1}(u/|\xi|))\mathbb{1}_{\{|\xi|\geq u\}}\right\}$$

$$= (2\pi)^{-(j+1)/2}\sum_{n=0}^{\lfloor\frac{j}{2}\rfloor}\frac{j!\,(-1)^n}{n!\,(j-2n)!\,2^n}\int_u^{\infty}x^{j-2n}e^{-x^2/2}\,dx$$

$$= (2\pi)^{-(j+1)/2}\int_u^{\infty}H_j(x)e^{-x^2/2}\,dx$$

$$= \rho_j(u),$$

as required, with the last line following from the basic properties of Hermite polynomials. (cf. (11.5.11).) $\qquad\square$

10.7   Intrinsic volumes for Whitney stratified spaces

So far, we have defined intrinsic volumes for the convex sets of Section 6.3[12] the Riemannian manifolds of Section 7.6 and in this chapter for piecewise smooth spaces, which possess a local convexity property. For our computations in Part III we shall also need intrinsic volumes of Whitney stratified spaces, for which no convexity of any kind need hold. In this section, after describing the basic conditions under which the intrinsic volumes can

---

[12]We also gave a a possible, but not very promising, extension to basic complexes via kinematic integrals at (6.3.11).

be defined, we shall define intrinsic volumes in this more general setting. The defining expression will be a natural generalisation of (10.5.3), which covered the piecewise smooth scenario. To now, much of this chapter was devoted to the relationship between Weyl's formula and intrinsic volumes, a relationship which strongly motivated accepting the inherent clumsiness of the integrals defining the Lipschitz-Killing curvatures. As we shall soon see in Section 10.8 Weyl's formula need no longer hold in the more general setting of Whitney stratified spaces, and so we need alternative motivation for the new curvatures. You can find this, among other details, either in the geometry literature (e.g. [24, 32]) or wait for Chapter 12 where they will arise when computing the expected Euler characteristic of excursion sets.

As in Chapter 8, we assume that we have an N-dimensional stratified Whitney space $(M, \mathcal{Z})$, decomposed as a disjoint union

$$M \;=\; \bigcup_{j=0}^{N} \partial_k M,$$

where the manifolds $\partial_j M$ are (possibly empty) j-dimensional C^3 submanifolds of a Riemannian N-manifold $(\widetilde{M}, \widetilde{g})$. Recall that each $\partial_j M$ will in general contain many connected components, namely, all those sets in the partition $\mathcal{Z}$ of M of dimension j.

The Riemannian metric $\widetilde{g}$ determines a volume form on both $\partial_j M$ and $S(T_t \partial_j M^\perp)$ where $T_t \partial_j M^\perp$ is the orthogonal complement of $T_t \partial_j M$ in $T_t \widetilde{M}$. We shall use $\mathcal{H}_j$ or $\mathrm{Vol}_{\partial_j M, g}$ to denote the first and $\mathcal{H}_{N-j-1}$ for the second. (Of course $\mathcal{H}_{N-j-1}$ really depends on t, but we have more than enough subscripts already.)

We now have almost all that we need to define the Lipschitz-Killing curvature measures of M. What is missing is the issue that arose in Section 9.2, that α may not be well defined at vectors ν_t which annihilate a generalised tangent space at t. Therefore, some restriction on $(M, \mathcal{Z})$ is needed which guarantees that for every $t \in M$ the partial map $\alpha_t : T_t^\perp \partial_k M \to \mathbb{Z}$, when restricted to $S(T_t^\perp \partial_k M)$ is well defined on a set of full $\mathcal{H}_{N-k-1}$ measure. Alternatively, some restrictions are needed to guarantee that the set of vectors where α_t is not well defined has $\mathcal{H}_{N-k}$ measure 0. We shall make this assumption without justification. These restrictions have to do with the set of degenerate tangent vectors, as defined in (9.2.1). For wide classes of Whitney stratified spaces, this assumption is easily justified (cf. Section 1.8 of [68]).

Another issue, which is particulary important to keep in mind if you wish to compare the following definition of the Lipschitz-Killing curvatures to that at (7.6.1) for manifolds without boundary, we summarise as

Remark 10.7.1 *Referring to the convention adopted in Remark 10.5.2, we define*

$$\alpha(\mathbb{0}) = 1.$$

Definition 10.7.2 *Suppose that the set* (9.2.1) *has Hausdorff dimension less than or equal to* $\dim(\widetilde{M}) - 1$, *and that the conventions of Remark 10.5.2 are in place. Then the Lipschitz-Killing curvature measures of the Whitney stratified space* $(M, \mathcal{Z})$ *are defined by*

(10.7.1)

$$
\begin{aligned}
\mathcal{L}_i(M, A) &= \sum_{j=i}^{N} (2\pi)^{-(j-i)/2} \sum_{m=0}^{\lfloor \frac{j-i}{2} \rfloor} \frac{(-1)^m C(N-j, j-i-2m)}{m!\,(j-i-2m)!} \\
&\qquad \times \int_{\partial_j M \cap A} \int_{S(T_t \partial_j M^\perp)} \mathrm{Tr}^{T_t \partial_j M}\left(\widetilde{R}^m \widetilde{S}_{\nu_{N-j}}^{j-i-2m} \right) \\
&\qquad \qquad \times \alpha(\nu_{N-j})\, \mathcal{H}_{N-j-1}(d\nu_{N-j})\, \mathcal{H}_j(dt).
\end{aligned}
$$

We shall assume that all the signed measures defined here are finite[13].

As we did before (cf. Lemma 10.5.8) we can again extend these measures to a one parameter family of Lipschitz-Killing curvature measures, which we do by setting

(10.7.2) $$\mathcal{L}_i^\kappa(M, A) \triangleq \sum_{n=0}^{\infty} \frac{(-\kappa)^{-n}}{(4\pi)^n} \frac{(i+2n)!}{n!\,i!} \mathcal{L}_{i+2n}(M, A).$$

Furthermore, we can define a one parameter family of *Lipschitz-Killing curvatures*, or *intrinsic volumes*, of M by

(10.7.3) $$\mathcal{L}_j^\kappa(M) \triangleq \mathcal{L}_j^\kappa(M, M).$$

Although it is not at all obvious from the definition (10.7.1), it can be shown that the measures $\mathcal{L}_j^\kappa(M;)$ are independent of the stratification of M. Furthermore, $\mathcal{L}_j^\kappa$ is a finitely additive valuation in its first variable and a (signed) measure in the second one. (cf. [24].)

The Lipschitz-Killing curvature $\mathcal{L}_0(M)$ is actually the Euler-Poincaré characteristic of M, as defined via (8.1.1) and a triangulation. The fact that the curvature integral and the Euler characteristic are equivalent is a celebrated result, known as the *Chern-Gauss-Bonnet Theorem*.

---

[13] *A priori*, there is no reason for the $\mathcal{L}_j(M, \cdot)$ be finite. However, for all the examples we consider in this section and, indeed, all situations of interest in this book, that will in fact be the case. We therefore invoke this a standing assumption for the remainder of the book.

To get a feel for the fact that (10.7.1) is not quite as forbidding as it seems, you might want to look at the simple example[14] of the unit cube in $\mathbb{R}^N$ now to see how it works for familiar sets.

The formulae simplify considerably if M is a C^2 domain of $\mathbb{R}^N$ with the induced Riemannian structure, since the ambient curvature tensor is identically zero. In that case $\mathcal{L}_N(M, U)$ is the Lebesgue measure of U and

$$(10.7.5) \quad \mathcal{L}_j(M, U) \;=\; \int_{\partial M \cap U} \frac{1}{s_{N-j}(N-1-j)!}\, \mathrm{Tr}(S_{\nu_t}^{N-1-j})\, \mathrm{Vol}_{\partial M, g},$$

for $0 \le j \le N-1$, where s_j is given by (5.7.5) and ν_t is the inward normal at t. Using this formula, you might want to check how to compute the Lipschitz-Killing curvatures for an N-dimensional ball[15], although it would be really be easier to wait for (10.7.6).

---

[14]Consider, for our first example, the unit cube in $\mathbb{R}^N$ equipped with the standard Euclidean metric. Thus, we want to recover (6.3.5), *viz.*

$$(10.7.4) \qquad\qquad \mathcal{L}_j\left([0, T]^N\right) \;=\; \binom{N}{j} T^j.$$

Since we are in the Euclidean scenario, both the Riemannian curvature and second fundamental form are identically zero, and so the only terms for which the final integral in (10.7.1) is non-zero are those for which $l = k - j = 0$. Thus all of the sums collapse and (10.7.1) simplifies to

$$\pi^{-(N-j)/2}\Gamma\left(\frac{N-j}{2}\right) 2^{-1} \int_{\partial_j M} \mathcal{H}_j(dt) \int_{S(T_t^\perp \partial_j M)} \alpha(\nu)\, \mathcal{H}_{N-j-1}(d\nu).$$

Note the following:

(i) Referring to the convex polytope discussion in Section 9.2, $\alpha(\nu) = \mathbb{1}_{N_t \partial_j M}(-\nu)$ is zero except on a $(1/2^{N-j})$-th part of the sphere $S(T^\perp \partial_j M)$ and the measure $\mathcal{H}_{N-j-1}$ is surface measure on $S(T^\perp \partial_j M)$. Consequently

$$\int_{S(T_t^\perp \partial_j M)} \alpha(\nu)\, \mathcal{H}_{N-j-1}(d\nu) \;=\; \frac{2\pi^{(N-j)/2} 2^{-(N-j)}}{\Gamma\left(\frac{N-j}{2}\right)}.$$

(ii) There are $2^{N-j}\binom{N}{j}$ disjoint components to $\partial_j M$, each one a j dimensional cube of edge length T.

(iii) The volume form on $\partial_j M$ is Lebesgue measure, so that each of the cubes in (ii) has volume T^j.

Now combine all of the above, and (10.7.4) follows.

[15]For the ball $B^N(T)$ we want to recover (6.3.7). It is easy to check that the second fundamental form of $\partial B^N(T) = S^{N-1}(T)$, with respect to the inner normal, is a constant T^{-1} over the sphere. Thus the term involving its trace can be taken out of the integral in (10.7.5) leaving only the volume of $S^{N-1}(T)$, given by $s_N T^{N-1}$. To compute the trace we use (7.2.10), to see that $\mathrm{Tr} S^{N-1-j} = T^{-(N-j-1)}(N-1)!/j!$, so that

$$\mathcal{L}_j\left(B^N(T)\right) \;=\; \frac{s_N T^{N-1}\, T^{-(N-j-1)}(N-1)!}{s_{N-j}(N-1-j)!j!} \;=\; T^j\binom{N-1}{j}\frac{s_N}{s_{N-j}} \;=\; T^j\binom{N}{j}\frac{\omega_N}{\omega_{N-j}},$$

which is (6.3.7).

In this setting (10.7.5) can also be written in another form that is often more conducive to computation. If we choose an orthonormal frame field $(E_1 \ldots, E_{N-1})$ on ∂M, and then extend this to one on M in such a way that E_N is the inward normal, then it follows from (7.2.6) that

$$(10.7.6) \; \mathcal{L}_j(M, U) \;=\; \frac{1}{s_{N-j}} \int_{\partial M \cap U} \mathrm{detr}_{N-1-j}(\mathrm{Curv}) \, \mathrm{Vol}_{\partial M, g},$$

where detr_j is given by (7.2.7) and the curvature matrix Curv is given by

$$(10.7.7) \qquad\qquad \mathrm{Curv}(i, j) \;\overset{\Delta}{=}\; S_{E_N}(E_i, E_j).$$

It is important to note that while the elements of the curvature matrix may depend on the choice of basis, $\mathrm{detr}_{N-1-j}(\mathrm{Curv})$ is independent of the choice, as will be $\mathcal{L}_j(M, U)$.

Since all of the simplifictions (and examples in the footnotes) that we have just looked at are locally convex, in the sense that they have convex support cones, we now describe how to compute the Lipschitz-Killing curvatures for one highly non-convex example, a many armed star.

By a star, we mean the union of a finite collection line segments $L_1, \ldots, L_r$ emanating from a point v_0 with endpoints $(v_j)_{1 \leq j \leq r}$. The ambient space is $\mathbb{R}^2$. The space $M = \bigcup_{j=1} L_r$ has a natural stratification

$$\partial_1 M \;=\; \bigcup_{j=1}^r L_r^\circ, \qquad \partial_0 M \;=\; \bigcup_{j=0}^r v_j.$$

First, we note that

$$\mathcal{L}_j(M,) \;=\; 0, \qquad j \geq 2$$

and

$$\mathcal{L}_1(M, A) \;=\; \sum_{j=1}^r \mathcal{H}_1(L_j \cap A),$$

so it remains to compute the measure $\mathcal{L}_0(M,)$. Clearly, $\mathcal{L}_0(M,)$ concentrates on the vertices of M and, considering the case of one segment alone it follows easily from (10.7.1) that

$$\mathcal{L}_0(M, v_j) \;=\; \frac{1}{2}, \qquad 1 \leq j \leq r.$$

As for $\mathcal{L}_0\left(\bigcup_{j=1}^r, v_0\right)$ note that, for ν a unit vector emanating from v_0,

$$\alpha(\nu) \;=\; 1 - \sum_{j=1}^r \mathbb{1}_{\{\langle \nu, L_j \rangle < 0\}},$$

so that it again follows easily from (10.7.1) that

$$\mathcal{L}_0(M, v_0) \; = \; 1 - \frac{r}{2}.$$

Here is one more example that is simple, but informative. You should find it easy to check, using the fact that Lipschitz-Killing curvature measures are additive in M. For $0 \leq \phi \leq 2\pi$, let

$$M_\phi \; = \; \big\{(x,y) \in \mathbb{R}^2 : 0 \leq r(x,y) \leq 1, 0 \leq \theta(x,y) \leq \phi\big\}$$

be a solid angle in the plane of angle ϕ. Let $v_0 = (0,0), v_1 = (1,0), v_2 = (\cos\phi, \sin\phi)$ be the three vertices of M_ϕ. Then,

$$
\begin{aligned}
\mathcal{L}_2(M_\phi, U) &= \mathcal{H}_2(U \cap M_\phi), \\
\mathcal{L}_1(M_\phi, U) &= \frac{1}{2}\mathcal{H}_1(U \cap \partial M_\phi), \\
\mathcal{L}_0(M_\phi, U) &= \frac{1}{2\pi}\mathcal{H}_1(U \cap M_\phi \cap S(\mathbb{R}^2)) \\
&\quad -\frac{\phi - \pi}{2\pi}\mathbb{1}_U(v_0) + \frac{1}{4}\mathbb{1}_U(v_1) + \frac{1}{4}\mathbb{1}_U(v_2).
\end{aligned}
$$

10.7.1 Alternative representation of the curvature measures

We conclude this section by noting an alternative representation of the curvature measures of subsets of $\mathbb{R}^l$ and $S(\mathbb{R}^l)$. This representation is sometimes easier to work with and we shall use it in Chapter 14. Its proof follows from a straightforward application of Lemma 10.5.5.

Lemma 10.7.3 *Let $M \subset \mathbb{R}^l$ be a piecewise smooth space. Then,*

(10.7.8)

$$
\begin{aligned}
\mathcal{L}_i(M, A) \; = \; &\sum_{j=i}^{l}(2\pi)^{-(j-i)/2} \\
&\times \int_{\partial_j M \cap A} \mathbb{E}\left\{\mathrm{Tr}^{T_t \partial_j M}(S_{-X_{l-j,t}})^{j-i}\mathbb{1}_{N_t M}(X_{l-j,t})\right\}\mathcal{H}_j(dt)
\end{aligned}
$$

where, for each $t \in \partial_j M$, $X_{l-j,t} \sim \gamma_{T_t \partial_j M^\perp}$ and $N_t M$ is the normal cone at $t \in M$ as it sits in $\mathbb{R}^l$.

If M is a piecewise smooth space in $S_{\sqrt{\kappa}}(\mathbb{R}^l)$, then

(10.7.9)

$$
\begin{aligned}
\mathcal{L}_i^\kappa(M, A) \; = \; &\sum_{j=i}^{l}(2\pi)^{-(j-i)/2}\int_{\partial_j M \cap A} \\
&\mathbb{E}\left\{\mathrm{Tr}^{T_t \partial_j M}(S_{-X_{l-1-j,t}})^{j-i}\mathbb{1}_{\widehat{N_t M}}(X_{l-1-j,t})\right\}\mathcal{H}_j(dt)
\end{aligned}
$$

where, for each $t \in \partial_j M$, $X_{l-1-j,t} \sim \gamma_{T_t \partial_j M^\perp}$ *where* $T_t \partial_j M^\perp$ *is now the orthogonal complement of* $T_t \partial_j M$ *in* $T_t S_{\sqrt{\kappa}} S(\mathbb{R}^l)$, *and* $\widehat{N_t M}$ *is the normal cone at* $t \in M$ *as it sits in* $S_{\sqrt{\kappa}}(\mathbb{R}^l)$.

10.8   Breakdown of Weyl's tube formula

While the next three paragraphs may seem to be too brief to demand their own numbered section, the observation that they make is important enough that the honour is deserved.

From the definition of the Lipschitz-Killing curvatures for Whitney stratified spaces, one may be tempted to extend Weyl's tube formula to compute the volume of a tube around a Whitney stratified space $(M, \mathcal{Z})$ embedded in $\mathbb{R}^l$. Specifically, one may be tempted to conclude that

$$\mathcal{H}_l(\text{Tube}(M, \rho)) \;=\; \sum_{j=0}^{l} \rho^{l-j} \omega_{l-j} \mathcal{L}_j(M)$$

even if M does not have convex support cones.

However, this is not true in general, and the same example that appeared in Figure 6.3.2 suffices to provide a counter-example.

It is essentially for this reason that, in Chapter 13, we shall have to restrict our attention to piecewise smooth spaces to obtain strong results about the accuracy of the volume of tubes approach to computing excursion probabilities, as well as that of the approximation via expected Euler characteristics.

10.9   Generalised Lipschitz-Killing curvature measures

As we saw in the earlier sections, the measures $\mathcal{L}_j(M;)$ are signed measures on M that determine the coefficients in a (Lebesgue) volume of tubes expansion around M when M is embedded in $\mathbb{R}^l$. These measures are adequate to derive Weyl's tube formula, but they will need to be generalised to continue our general study of the volume of tubes problem, that is, computing

$$\mu(\text{Tube}(M, \rho))$$

for measures other than $\mathcal{H}_l$ or Lebesgue measure.

Our interest in general measures μ is not purely academic, for the main result in Chapter 14, which treats non-Gaussian random fields, relies heavily on the coefficients in the tube expansion

$$\gamma_{\mathbb{R}^l}(\text{Tube}(M, \rho)).$$

Since this is the main expansion that we shall need, for the remainder of this section we shall assume that the ambient space in the definition of M is $\mathbb{R}^l$, rather than an arbitrary ambient manifold $\widetilde{M}$.

The generalisation of the measures $\mathcal{L}_j(M, \cdot)$ is needed to deal with the term $f_\mu(F_{j,r}(t,s))$ in Theorem 10.4.2, which was, conveniently, identically 1 in Weyl's tube formula. The general strategy is to assume that f_μ is smooth enough so that there is a Taylor series expansion for $f_\mu(F_{j,r}(t,s))$ of the form

$$(10.9.1) \quad f_\mu(F_{j,r}(t,s)) \;=\; f_\mu\left(t + r\eta(s)\right) \;=\; \sum_{m=0}^{\infty} \frac{r^m}{m!} \frac{d^m}{d\eta^m} f_\mu(t),$$

where $\eta(s)$ is the unit vector from t to s, and by $d^m f/d\eta^m$ we mean the m-th order derivative of f in the direction $\eta(s)$.

Using this expansion, we proceed to integrate

$$\frac{d^m}{d\eta^m} f_\mu(t)$$

over $S_j(r)$, the hypersurface at distance r from $\partial_j M$ for $0 \le j \le l$. After integrating over $S_j(r)$, we integrate r over $[0, \rho]$, much as we did in deriving Weyl's tube formula (10.5.6).

We begin by defining generalised Lipschitz-Killing curvature measures on $SB(\mathbb{R}^l)$, the sphere bundle of $\mathbb{R}^l$. These measures have finer information than the basic Lipschitz-Killing curvature measures in that, up to a constant, they encode surface measure on the hypersurfaces at distance r from M.

10.9.1   The generalised curvature measures

Before we explicitly define these measures, we note that they *should be* measures on $SB(\mathbb{R}^l)$, the sphere bundle of $\mathbb{R}^l$, which is canonically isomorphic to the product $S(\mathbb{R}^l) \times \mathbb{R}^l$. For simplicity, however, we choose to define the generalised Lipschitz-Killing measures on $S(\mathbb{R}^l) \times \mathbb{R}^l$ rather than on $SB(\mathbb{R}^l)$, keeping this canonical isomorphism in mind. We also note that, when convenient, we shall think of them as measures on $SB(\mathbb{R}^l)$.

Definition 10.9.1 *Let $(M, \mathcal{Z}) \subset \mathbb{R}^l$ be a stratified Whitney space satisfying the requirements of Definition 10.7.2. The generalised Lipschitz-Killing curvature measures of M, defined on Borel sets $A \subset \mathbb{R}^l, B \subset S(\mathbb{R}^l)$ and*

supported on $M \times S(\mathbb{R}^l)$, are defined, for $0 \le i \le l-1$, by

(10.9.2)

$$\widetilde{\mathcal{L}}_i(M, A \times B)$$

$$\triangleq \sum_{j=i}^{l} (2\pi)^{-(j-i)/2} \sum_{m=0}^{\lfloor \frac{j-i}{2} \rfloor} \frac{(-1)^m C(l-j, j-i-2m)}{m! \, (j-i-2m)!}$$

$$\times \int_{\partial_j M \cap A} \int_{S(T^\perp \partial_j M) \cap B} \mathrm{Tr}^{T_t \partial_j M} \left(\widetilde{R}^m \widetilde{S}^{j-i-2m}_{\nu_{l-j}} \right)$$

$$\times \alpha(\nu_{l-j}) \, \mathcal{H}_{l-j-1}(d\nu_{l-j}) \, \mathcal{H}_j(dt),$$

where α is the normal Morse index of Section 9.2.

For any Borel function $f : \mathbb{R}^l \times S(\mathbb{R}^l) \to \mathbb{R}$ we use the standard notation

$$\widetilde{\mathcal{L}}_i(M, f)$$

to denote the integral of f with respect to $\widetilde{\mathcal{L}}_i(M, \cdot)$. If $i = l$, we define $\widetilde{\mathcal{L}}_l(M, \cdot)$ only on Borel functions on $\mathbb{R}^l \times S(\mathbb{R}^l)$ that are constant over $S(\mathbb{R}^l)$. Specifically,

(10.9.3) $\qquad \widetilde{\mathcal{L}}_l(M, f) \triangleq \begin{cases} \int_M f(t, \nu) \, \mathcal{H}_l(dt) & f \in \mathcal{B}(\mathbb{R}^l) \times S(\mathbb{R}^l) \\ 0 & \text{otherwise,} \end{cases}$

for some fixed but arbitrary $\nu \in S(\mathbb{R}^l)$ where

$$\mathcal{B}(\mathbb{R}^l) \times S(\mathbb{R}^l) = \left\{ A \times S(\mathbb{R}^l) : A \in \mathcal{B}(\mathbb{R}^l) \right\}$$

is the sub-sigma algebra of $\mathcal{B}(\mathbb{R}^l) \otimes \mathcal{B}(S(\mathbb{R}^l))$ generated by functions that are constant over $S(\mathbb{R}^l)$.

The generalized curvature measures $\widetilde{\mathcal{L}}_j(M, \cdot)$ can also be represented non-intrinsically as in (10.5.4) and (10.5.9), though we leave this generalization to the reader. The following two results are immediate corollaries of this definition.

Corollary 10.9.2 *Let $(M, \mathcal{Z}) \subset \mathbb{R}^l$ be a stratified Whitney space satisfying the requirements of Definition 10.7.2. Then, for any Borel $A \subset \mathbb{R}^l$,*

$$\mathcal{L}_j(M, A) \ = \ \widetilde{\mathcal{L}}_j(M, A \times S(\mathbb{R}^l)).$$

Corollary 10.9.3 *If $(M, \mathcal{Z})$ is a piecewise smooth space, then (10.9.2) holds with $\alpha(\nu_{l-j})$ replaced by $\mathbb{1}_{N_t M}(-\nu_{l-j})$.*

10.9.2   Surface measure on the boundary of a tube

Having defined the measures $\{\widetilde{\mathcal{L}}_j(M,)\}_{0 \leq j \leq l}$, we now proceed to describe how they can be used to integrate over $\partial\mathrm{Tube}(M,r)$, the hypersurface of distance r from a piecewise smooth space M.

Before we attack this task in earnest, we revisit the *Minkowski functionals*, which we defined, in the context of Integral Geometry, in Section 6.3. Recall that they were defined as

$$\mathcal{M}_j(M) \;=\; (j!\,\omega_j)\,\mathcal{L}_{l-j}(M),$$

although in Chapter 6 we only treated the case $\dim(M) = N = l = \dim(\widetilde{M})$.

Steiner's formula (6.3.10), extended to the case $N \leq l$, (or, equivalently, Weyl's tube formula (10.5.6)) gives

$$\mathcal{H}_l(\mathrm{Tube}(M,r)) \;=\; \sum_{j=0}^{l} \frac{r^j}{j!} \mathcal{M}_j(M).$$

That is, Weyl's tube formula can be expressed as a (finite) power series with the $\mathcal{M}_j(M)$ as coefficients. This motivates our definition of *Minkowski curvature measures* as

(10.9.4) $$\mathcal{M}_j(M, A) \;\triangleq\; (j!\,\omega_j)\,\mathcal{L}_{l-j}(M, A),$$

and *generalised Minkowski curvature measures* as

(10.9.5) $$\widetilde{\mathcal{M}}_j(M, A \times B) \;\triangleq\; (j!\,\omega_j)\,\widetilde{\mathcal{L}}_{l-j}(M, A \times B).$$

The reason we turn to (generalised) Minkowski curvature measures now, instead of (generalised) Lipschitz-Killing curvature measures is primarily one of notational convenience and will mesh in well with the power series expansions of f_μ to come. Ultimately, this will also turn out to be the most convenient approach for the results of Chapter 14.

We now return to describing surface measure on $\partial\mathrm{Tube}(M,r)$. In Section 10.4, we invested a substantial amount of effort in computing $F_{j,r}^*(\Omega_{j,r})$, the pull-back of the volume form on $S_j(r)$, the hypersurface of distance r from $\partial_j M$. In Section 10.5 we combined this formula with our explicit description of the regions $D_j(\rho)$ to derive (10.5.6), Weyl's formula for piecewise smooth manifolds in $\mathbb{R}^l$. In describing regions we used the maps $F_{j,r}$ which, when taken together and when r is small enough, form a bijection

$$\bigcup_{j=0}^{l} \bigcup_{t \in \partial_j M} \{t\} \times S(N_t M) \;\overset{F_r}{\leftrightarrows}\; \partial\mathrm{Tube}(M,r),$$

which can explicitly be expressed as

$$
\begin{aligned}
F_r(t, \nu) &= t + r\nu \\
F_r^{-1}(s) &= \left(\xi_M(s),\, (s - \xi_M(s))/r\right),
\end{aligned}
$$

where ξ_M is the metric projection of M, defined in (10.3.2).

Pulling pieces from Sections 10.4 and 10.5, it is now not difficult to prove the following theorem.

Theorem 10.9.4 *Let $(M, \mathcal{Z}) \subset \mathbb{R}^l$ be a piecewise smooth space. For any Borel $A \subset \mathbb{R}^l$ such that*

$$\mathcal{H}_l(A \cap M) \;=\; 0,$$

and for any $r < \rho_c(M, \mathbb{R}^l)$, the surface area of the intersection of A with $\partial \mathrm{Tube}(M, r)$ can be written as

(10.9.6)

$$\mathcal{H}_{l-1}\left(A \cap \partial \mathrm{Tube}(M, r)\right) \;=\; \sum_{j=0}^{l-1} \frac{r^j}{j!} \widetilde{\mathcal{M}}_{j+1}(M, F_{-r}^{-1}(A \cap \partial \mathrm{Tube}(M, r))).$$

Remark 10.9.5 *Above, F_{-r} appears instead of F_r because we have used $S_{\nu_{l-j}}$ and $\alpha(\nu_{l-j})$ in Definition 10.9.1 rather than $S_{-\nu_{l-j}}$ and $\alpha_{-\nu_{l-j}}$. We have used this choice of signs to agree with [?]* COMMENT ME LATER: CHECK THEIR SIGNS LATER.

Proof. We do not provide a detailed proof, as nearly all of the work needed has already appeared in Sections 10.4 and 10.5. In fact, the present case is even simpler, since the integration over r is no longer necessary.

Essentially, all we need to do is to decompose $A \cap \partial \mathrm{Tube}(M, r)$ into cross sections of $\partial \mathrm{Tube}(M, r)$. That is, we first find the image of A under the metric projection ξ_M. Next, for each t in $\xi_M(A)$, we find all points in $A \cap \partial \mathrm{Tube}(M, r)$ that projected to t. Then, we sum the contribution of all these points, resulting in the integral against the generalised Minkowski curvature measures $\widetilde{\mathcal{M}}_{j+1}(M, \cdot)$. $\qquad\square$

10.9.3   Series expansions for the Gaussian measure of tubes

We are now ready to prove the main result of this section, in which we derive a power series expansion for

$$\mu(\mathrm{Tube}(M, \rho))$$

when μ has a bounded, analytic density with respect to Lebesgue measure. We can replace analytic functions with the Schwartz class of functions, although the power series expansion is only formal in this case (cf. [148]).

Theorem 10.9.6 *Let $(M, \mathcal{Z}) \subset \mathbb{R}^l$ be a piecewise smooth space and μ a measure on $\mathbb{R}^l$ which has a bounded density f_μ with respect to Lebesgue*

measure. Assume that

$$\mu(\text{Tube}(M,\rho)) \;=\; \int_{\text{Tube}(M,\rho)} f_\mu(x)\,d\mathcal{H}_l(x) \;<\; \infty.$$

Then, for $\rho < \rho_c(M,\mathbb{R}^l)$,

$$(10.9.7)\ \mu(\text{Tube}(M,\rho)) \;=\; \int_0^\rho \sum_{j=0}^{l-1} \frac{r^j}{j!}\,\widetilde{\mathcal{M}}_{j+1}(M, f_\mu \circ F_{-r})\,dr.$$

Suppose, in addition, that f_μ is analytic and for every $\varepsilon > 0$ there exists a compact $K(\varepsilon) \subset \mathbb{R}^l$ such that

(10.9.8)

$$\int_0^\rho \sum_{j=0}^{l-1} \frac{r^j}{j!}\,|\widetilde{\mathcal{M}}_{j+1}|\left(M, \mathbb{1}_{K(\varepsilon)^c} \circ F_{-r} \times \left|\sum_{m=0}^{n} \frac{(-r)^m}{m!}\frac{d^m}{d\eta^m}f_\mu\right|\right)\,dr < \varepsilon,$$

uniformly in n, where $|\widetilde{\mathcal{M}}_j|$ is the (positive) measure corresponding to the signed measure $\widetilde{\mathcal{M}}_j$ and we interpret $\frac{d^m}{d\eta^m}$ as in (10.9.1). Then

(10.9.9)

$$\int_{\text{Tube}(M,\rho)} f_\mu(x)\,d\mathcal{H}_l(x) \;=\; \int_0^\rho \sum_{j=0}^{l-1} \frac{r^j}{j!}\widetilde{\mathcal{M}}_{j+1}\left(M, \sum_{m=0}^{\infty} \frac{(-r)^m}{m!}\frac{d^m}{d\eta^m}f_\mu\right)\,dr$$

$$=\; \mu(M) + \sum_{j=1}^{\infty} \frac{\rho^j}{j!}\mathcal{M}_j^\mu(M),$$

where

$$\mathcal{M}_j^\mu(M) \;\triangleq\; \sum_{m=0}^{j-1} \binom{j-1}{m}(-1)^{j-1-m}\widetilde{\mathcal{M}}_{m+1}\left(M, \frac{d^{j-1-m}}{d\eta^{j-1-m}}f_\mu\right).$$

Proof. The equality in (10.9.7) is a straightforward application of the co-area formula (7.4.13) with the Lipschitz map over whose level sets we integrate being the distance function $d_{\mathbb{R}^l}(M,\cdot)$, along with (10.9.6).

As far as (10.9.9) is concerned, there are two things to show. The first is that the first inequality is valid, and the second that the final expression is equal to that on the line above it. The first inequality arises by formally replacing $f_\mu \circ F_{-r}$ in (10.9.7) by its Taylor series expansion as in (10.9.1), *viz.*

$$f_\mu(F_{-r}(t,s)) \;=\; \sum_{m=0}^{\infty} \frac{(-r)^m}{m!}\frac{d^m}{d\eta^m}f_\mu(t).$$

We do, however, need to check that the resulting series expansion in (10.9.9) converges nicely. Fix $\varepsilon > 0$ and choose $K(\varepsilon)$ satisfying (10.9.8). Since we long ago agreed that we only consider tubes of finite measure, we can, without loss of generality, assume that

$$\mu(\text{Tube}(M,\rho) \cap K(\varepsilon)^c) \; < \; \varepsilon,$$

so that we need only focus our concentration on what happens on $K(\varepsilon)$. On $F_{-r}^{-1}(K(\varepsilon))$, the convergence

$$\sum_{m=0}^{n} \frac{(-r)^m}{m!} \frac{d^m}{d\eta^m} f_\mu(t) \overset{n\to\infty}{\to} f_\mu(t - r\eta)$$

is uniform in n. Alternatively, on $K(\varepsilon)$ the convergence

$$f_\mu^n(s) \; = \; \sum_{m=0}^{n} \frac{r(s)^m}{m!} \frac{d^m}{d\eta(s)^m} f_\mu(t(s)) \overset{n\to\infty}{\to} f_\mu(s)$$

holds uniformly in s. By the boundedness of f_μ and compactness of $K(\varepsilon)$ we can therefore choose $n(\varepsilon)$ large enough so that

$$\left| \mu(\text{Tube}(M \cap K(\varepsilon), \rho)) \right.$$

$$\left. - \int_0^\rho \sum_{j=0}^{l-1} \frac{r^j}{j!} \widetilde{\mathcal{M}}_{j+1} \left(M \cap K(\varepsilon), \sum_{m=0}^{n(\varepsilon)} \frac{(-r^m}{m!} \frac{d^m}{d\eta^m} f_\mu \right) dr \right|$$

$$= \left| \int_{\text{Tube}(M,\rho) \cap K(\varepsilon)} \sum_{m=n(\varepsilon)+1}^{\infty} \frac{(-r(s))^m}{m!} \frac{d^m}{d\eta(s)^m} f_\mu(t(s)) \, d\mathcal{H}_l(s) \right|$$

$$< \; \varepsilon.$$

Thus there are no problems with the convergence in (10.9.9). As for the equality between the two expressions there, note that

$$\int_{\text{Tube}(M,\rho)} f_\mu(x)\, d\mathcal{H}_l(x)$$

$$= \mu(M) + \int_0^\rho \sum_{j=0}^{l-1} \frac{r^j}{j!} \widetilde{\mathcal{M}}_{j+1}\left(M, \sum_{m=0}^\infty \frac{(-r)^m}{m!} \frac{d^m}{d\eta^m} f_\mu \right)\, dr$$

$$= \mu(M) + \sum_{j=0}^{l-1} \sum_{m=0}^\infty (-1)^m \frac{\rho^{j+1+m}}{j!m!(j+1+m)} \widetilde{\mathcal{M}}_{j+1}\left(M, \frac{d^m}{d\eta^m} f_\mu \right)$$

$$= \mu(M) + \sum_{j=1}^{l} \sum_{m=j}^\infty \frac{\rho^m}{m!} \binom{m-1}{j-1} (-1)^{m-j} \widetilde{\mathcal{M}}_{j+1}\left(M, \frac{d^{m-j}}{d\eta^{m-j}} f_\mu \right)$$

$$= \mu(M) + \sum_{m=1}^\infty \frac{\rho^m}{m!} \sum_{j=0}^{m-1} \binom{m-1}{j} (-1)^{m-j-1} \widetilde{\mathcal{M}}_j\left(M, \frac{d^{m-j-1}}{d\eta^{m-j-1}} f_\mu \right)$$

which, expect for the fact that m and j have switched rôles, is precisely what we need. $\qquad\square$

We conclude with a corollary to Theorem 10.9.6, which gives an explicit power series representation of

$$\gamma_{\mathbb{R}^l}(\text{Tube}(M,\rho))$$

for suitable piecewise smooth spaces $(M, \mathcal{Z}) \subset \mathbb{R}^l$. This expansion will be crucial for our analysis of non-Gaussian processes in Chapter 14.

Corollary 10.9.7 *Suppose $(M, \mathcal{Z}) \subset \mathbb{R}^l$ is a piecewise smooth space and for every $\varepsilon > 0$ there exists $K(\varepsilon) \subset \mathbb{R}^l$ compact such that*

$$\int_0^\rho \sum_{j=0}^{l-1} \frac{r^j}{j!} |\widetilde{\mathcal{M}}_{j+1}| \left(M, \mathbb{1}_{K(\varepsilon)^c} \circ F_{-r} \left| \sum_{m=0}^n \frac{r^m}{m!} H_m\left(\langle \eta, t \rangle\right) e^{-|t|^2/2} \right| \right) dr$$

is less than ε, uniformly in n, where H_m is the m-th Hermite polynomial (11.5.8). Then,

$$(10.9.10) \quad \gamma_{\mathbb{R}^l}(\text{Tube}(M,\rho)) = \gamma_{\mathbb{R}^l}(M) + \sum_{j=1}^\infty \frac{\rho^j}{j!} \mathcal{M}_j^{\gamma_{\mathbb{R}^l}}(M),$$

where

$$\mathcal{M}_j^{\gamma_{\mathbb{R}^l}}(M)$$
$$\triangleq (2\pi)^{-l/2} \sum_{m=0}^{j-1} \binom{j-1}{m} \widetilde{\mathcal{M}}_{m+1}\left(M, H_{j-1-m}\left(\langle \eta, t \rangle\right) e^{-|t|^2/2} \right).$$

Proof. Comparing the above with Theorem (10.9.6), it is clear (cf. (10.9.10)) that all we need show is that

$$\frac{d^{j-1-m}}{d\eta^{j-1-m}} f_\mu(t) \;=\; (-1)^{j-1-m} H_{j-1-m}\left(\langle \eta, t \rangle\right) \mathrm{e}^{-|t|^2/2}.$$

However, this is easy to check from the generating function definition of Hermite polynomials at (11.5.10).   □

To see how this expansion works in the simplest of cases, take $M = [u, \infty) \subset \mathbb{R}$. Since all the $\widetilde{\mathcal{M}}_k$ are then identically zero for $k \geq 2$, there is only one term appearing in the sum defining $\mathcal{M}_j^{\gamma_{\mathbb{R}^l}}(M)$. Furthermore, since Tube$([u, \infty), \rho) = [u - \rho, \infty))$, we have that $\widetilde{\mathcal{M}}_1$ is supported on $\{u\} \times \{+1\}$ (+1 being the right pointing unit vector at u) and so

$$\mathcal{M}_j^{\gamma_{\mathbb{R}^l}}([u, \infty)) \;=\; H_{j-1}(u) \frac{\mathrm{e}^{-u^2/2}}{\sqrt{2\pi}}.$$

It is elementary calculus to check that this gives the same expansion, when substituted into (10.9.10), that comes from a simple Taylor series expansion of the Gaussian tail probability Ψ of (1.2.1).

We close with the following simple, but rather important comment.

Remark 10.9.8 *Unlike the usual Minkowski functionals, the functionals $\mathcal{M}_j^{\gamma_{\mathbb{R}^l}}$ are actually normalized independently of the dimension $\mathbb{R}^l$, in the sense that, for all integers $k > 0$,*

$$\mathcal{M}_j^{\gamma_{\mathbb{R}^l}}(\cdot) \;=\; \mathcal{M}_j^{\gamma_{\mathbb{R}^{l+k}}}(\cdot \times \mathbb{R}^k).$$

Hence, from now on, we shall drop the $\mathbb{R}^l$ in the definition of $\mathcal{M}_j^{\gamma_{\mathbb{R}^l}}$, using the simpler notation $\mathcal{M}_j^\gamma$. Further, although they were derived under the assumption that D is of positive reach, they are for non locally convex sets, as long as the integrals with respect to the measures $\widetilde{\mathcal{M}}_j(M, \cdot)$ are finite.

We defer the examples of the applications of the results of this last section to Chapter 14, where you will have a chance to see *why* we care about these power series expansions.

11
PART III: THE GEOMETRY OF RANDOM FIELDS

You could not possibly have got this far in our book without having read the Preface, so you already know that you have finally reached the main part of this book. It is here where the important results lie and it is here where, after 250 pages of preparation, there will also be new results.

With the general theory of Gaussian processes behind us in Part I, and the geometry of Part II now established, we return to the stochastic setting.

There are three main (classes of) results in this Part. The first is an explicit formula for the expected Euler characteristic of the excursion sets of smooth Gaussian random fields. In the same way that we divided the treatment of the geometry into two parts. Chapter 11 will cover the theory of random fields defined over simple Euclidean domains and Chapter 12 will cover fields defined over Whitney stratified manifolds. Unlike the case in Chapter 6, however, even if you are primarily interested in the manifold scenario you will need to read the Euclidean case first, since some of the manifold computations will be lifted from this case via atlas based arguments.

As an aside, in the final Section 12.7 of Chapter 12 we shall return to a purely deterministic setting and use our Gausian field results to provide a probabilistic proof of the classical Chern-Gauss-Bonnet Theorem of Differential Geometry using nothing[1] but Gaussian processes. This really has

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[1] Of course, this cannot really be true. Establishing the Chern-Gauss-Bonnet Theorem without any recourse to algebraic geometry would have been a mathematical coup that might even have made Probability Theory a respectable topic within Pure Mathematics. What will be hidden in the small print is that everything relies on the Morse theory of

nothing to do with anything else in the book, but we like it too much not to include it.

The second main result appears in Chapter 13, where we shall finally prove the result promised long ago, that not only is the difference

$$\left| \mathbb{P}\left\{ \sup_{t \in M} f(t) \geq u \right\} \ - \ \mathbb{E}\left\{ \varphi\left(A_{[u,\infty)} \right) \right\} \right|$$

extremely small for large u, but it can even be bounded in a rigorous fashion. This will not only justify our claims that the mean Euler characteristic of excursion sets yields an excellent approximation to excursion probabilities, but, *en passant*, we shall also show that the volume of tubes approximation of Chapter 10 can be made rigorous as well.

In the closing Chapter 14 we finally leave the Gaussian scenario and develop the third main result, an explicit formula for the expected Euler characteristic of excursion sets for a wide range of non-Gaussian random fields, working in the general environment of Differential, rather than Integral, Geometry. The proof here is entirely new.

Three results may seem too few to justify an investment of over 100 pages, but their importance along with the many interesting side issues that appear on the way should justify it.

Section 9.3 and this, in turn, uses algebraic geometry. However, our approach will save myopic probabilists from having to read the small print.

11
Random fields on Euclidean spaces

The main result of this chapter is Theorem 11.6.2 and its corollaries, which give explicit formulae for the mean Euler characteristic of the excursion sets of smooth Gaussian fields over rectangles in $\mathbb{R}^N$.

The chapter develops in a number of distinct stages. Initially, we shall develop rather general results which give integral formulae for the expected number of points at which a vector valued random field takes specific values. These are Theorem 11.1.1 and its corollaries. Aside from their basic importance in the current setting, these results will also form the basis of corresponding results for the manifold setting of Chapter 12. In view of the results of Chapter 6, which relate the global topology of excursion sets of a function to its local behaviour, it should be clear what this has to do with Euler characteristic computations. However, the results are important beyond this setting and indeed beyond the setting of this book, so we shall develop them slowly and carefully.

11.1   An expectation meta-theorem

We start with a meta-theorem about the expected number of points at which a vector-valued random field takes values in some set. For the moment, we gain nothing by assuming that our fields are Gaussian and so do not do so. Here is the setting:

For some $N, K \geq 1$ let $f = (f^1, \ldots, f^N)$ and $g = (g^1, \ldots, g^K)$, respectively, be $\mathbb{R}^N$ and $\mathbb{R}^K$ valued N-parameter random fields. We need two sets,

$T \subset \mathbb{R}^N$ and $B \subset \mathbb{R}^K$. T, as usual, is a compact parameter set, but now we add the assumption that its boundary ∂T has finite $\mathcal{H}_{N-1}$-measure. (cf. Footnote 25 in Chapter 6.) As for B, we shall assume that it is open and that its boundary $\partial B = \bar{B} \setminus B$ has Hausdorff dimension $K - 1$.

As usual ∇f denotes the gradient of f. Since f takes values in $\mathbb{R}^N$ this is now a $N \times N$ matrix of first-order partial derivatives of f; i.e.

$$(\nabla f)(t) \ \equiv \ \nabla f(t) \ \equiv \ \left(f_j^i(t)\right)_{i,j=1,\ldots,N} \ \equiv \ \left(\frac{\partial f^i(t)}{\partial t_j}\right)_{i,j=1,\ldots,N}.$$

All the derivatives here are assumed to exist in an almost sure sense[1].

Theorem 11.1.1 *Let f, g, T and B be as above. Assume that the following conditions are satisfied for some $u \in \mathbb{R}^N$:*

(a) *All components of f, ∇f, and g are a.s. continuous and have finite variances (over T).*

(b) *For all $t \in T$, the marginal densities $p_t(x)$ of $f(t)$ (implicitly assumed to exist) are continuous at $x = u$.*

(c) *The conditional densities[2] $p_t(x|\nabla f(t), g(t))$ of $f(t)$ given $g(t)$ and $\nabla f(t)$ (implicitly assumed to exist) are bounded above and continuous at $x = u$, uniformly in $t \in T$.*

(d) *The conditional densities $p_t(z|f(t) = x)$ of $\det \nabla f(t)$ given $f(t) = x$, are continuous for z and x in neighbourhoods of 0 and u, respectively, uniformly in $t \in T$.*

(e) *The conditional densities $p_t(z|f(t) = x)$ of $g(t)$ given $f(t) = x$, are continuous for all z and for x in a neighbourhood u, uniformly in $t \in T$.*

(f) *The following moment condition holds:*

$$(11.1.1) \qquad \sup_{t \in T} \ \max_{1 \leq i,j \leq N} \ \mathbb{E}\left\{\left|f_j^i(t)\right|^N\right\} \ < \ \infty.$$

---

[1] This is probably too strong an assumption, since, at least in one dimension, Theorem 11.1.1 is known to hold under the assumption that f is absolutely continuous, and so has only a weak sense derivative. (cf. [102].) However, since we shall need a continuous sample path derivative later for other things, we assume it now.

[2] Standard notation would demand that we replace the matrix ∇y by the $N(N+1)/2$ dimensional vector $\text{vech}(\nabla y)$ both here and, even moreso, in (11.1.3) following, where the differential $d(\nabla y)$, which we shall generally write in even greater shorthand as $d\nabla y$, is somewhat incongruous. Nevertheless, on the basis that it is clear what we mean, and that it is useful to be able to easily distinguish between reals, vectors and matrices, we shall always work with this slightly unconventionial notation.

(g) *The moduli of continuity with respect to the usual Euclidean norm (cf.* (1.3.6)*) of each of the components of* f, ∇f, *and* g *satisfy*

(11.1.2) $\mathbb{P}\{\omega(\eta) > \varepsilon\} = o(\eta^N),$ *as* $\eta \downarrow 0,$

for any $\varepsilon > 0$.

Then, if

$$N_u \equiv N_u(T) \equiv N_u(f, g : T, B)$$

denotes the number of points in T *for which*

$$f(t) = u \in \mathbb{R}^N \quad and \quad g(t) \in B \subset \mathbb{R}^K,$$

and $p_t(x, \nabla y, v)$ *denotes the joint density of* $(f_t, \nabla f_t, g_t)$, *we have, with* $D = N(N+1)/2 + K$,

(11.1.3) $\mathbb{E}\{N_u\} = \int_T \int_{\mathbb{R}^D} |\det \nabla y|\, \mathbb{1}_B(v)\, p_t(u, \nabla y, v)\, d(\nabla y)\, dv\, dt.$

It is sometimes more convenient to write this as

(11.1.4) $\mathbb{E}\{N_u\} = \int_T \mathbb{E}\left\{|\det \nabla f(t)|\, \mathbb{1}_B(g(t)) \,\middle|\, f(t) = u\right\} p_t(u)\, dt,$

where the p_t *here is the density of* $f(t)$.

While conditions (a)–(g) arise naturally in the proof of Theorem 11.1.1, they all but disappear in one of the cases of central interest to us, when the random fields f and g are Gaussian. In these cases all the marginal and conditional densities appearing in the conditions of Theorem 11.1.1 are also Gaussian and so their boundedness and continuity is immediate, as long as all the associated covariance matrices are non-degenerate, which is what we need to assume in this case. This will also imply that all variances are finite and that the moment condition (11.1.1) holds. Thus the only remaining conditions are the a.s. continuity of f, ∇f and g, and condition (11.1.2) on the moduli of continuity.

Note first, without reference to normality, that if ∇f is continuous then so must[3] be f. Thus we have only the continuity of ∇f and g to worry about. However, we spent a lot of time in Chapter 1 finding conditions which will guarantee this. For example, we can apply Theorem 1.4.1.

Write $C_f^i = C_f^i(s, t)$ for the covariance function of f^i, so that $C_{f_j}^i = \partial^2 C_f^i / \partial s_j \partial t_j$ is the covariance function of $f_j^i = \partial f^i / \partial t_j$. Similarly, C_g^i is the

---

[3]The derivative can hardly exist, let alone be continuous, if f is not continuous! In fact, this condition was vacuous all along and was only included for 'symmetry' considerations.

covariance function of g^i. Then, by (1.4.4), ∇f and g will be a.s. continuous if

$$(11.1.5) \qquad \max\left(\max_{i,j}\left|C^i_{f_j}(t) - C^i_{f_j}(s)\right|, \quad \max_i\left|C^i_g(t) - C^i_g(s)\right|\right)$$
$$\leq \ K\left|\ln|t-s|\right|^{-(1+\alpha)},$$

for some finite $K > 0$, some $\alpha > 0$ and all $|t-s|$ small enough.

All that now remains to check is condition (11.1.2) on the moduli of continuity. Here the Borell-TIS Inequality – Theorem 2.1.1 – comes into play. Write h for any of the components of ∇f or g and H for the random field on $T \times T$ defined by $H(s,t) = h(t) - h(s)$. Then, writing

$$\omega(\eta) \ = \ \sup_{s,t:\,|t-s|\leq\eta} |H(s,t)|,$$

we need to show, under (11.1.5), that for all $\varepsilon > 0$,

$$(11.1.6) \qquad\qquad \mathbb{P}\left\{\omega(\eta) \ > \ \varepsilon\right\} \ \leq \ o\left(\eta^N\right).$$

The Borell-TIS Inequality gives us that

$$(11.1.7) \ \ \mathbb{P}\left\{\omega(\eta) \ > \ \varepsilon\right\} \ \leq \ 2\exp\left\{-\left(\varepsilon - \mathbb{E}\{\omega(\eta)\}\right)^2\big/2\sigma_\eta^2\right\},$$

where $\sigma_\eta^2 = \sup_{s,t:\,|t-s|\leq\eta}\mathbb{E}\{(H(s,t))^2\}$. But (11.1.5) immediately implies that $\sigma_\eta^2 \leq K|\ln\eta|^{-(1+\alpha)}$, while together with Theorem 1.4.1 it implies a bound of similar order for $\mathbb{E}\{\omega(\eta)\}$. Substituting this into (11.1.7) gives an upper bound of the form $C_\varepsilon\eta^{|\ln\eta|^\alpha}$, and so (11.1.6) holds with room to spare, in that holds for any N and not just $N = \dim(T)$.

Putting all of the above together, we have that Theorem 11.1.1 takes the following much more user friendly form in the Gaussian case.

Corollary 11.1.2 *Let f and g be centered Gaussian fields over a T which satisfies the conditions of Theorem 11.1.1. If, for each $t \in T$, the joint distributions of $(f(t), \nabla f(t), g(t))$ are non-degenerate, and if (11.1.5) holds, then so do (11.1.3) and (11.1.4).*

We now turn to the proof of Theorem 11.1.1. We shall prove it in a number of stages, firstly by setting up a result that rewrites the random variable N_u in an integral form, more conducive to computing expectations. Indeed, we could virtually end the proof here, if we were prepared to work at a level of rigour which would allow us to treat the Dirac delta function with gay abandon and exchange delicate orders of integration without justification. Since it is informative to see how such an argument works, we shall give it, before turning to a fully rigorous proof.

The rigorous proof comes in a number of steps. In the first, we use the integral representation to derive an upper bound to $\mathbb{E}\{N_u\}$, which actually

gives the correct result. (The upper bound actually involves little more than replacing the 'gay abandon' mentioned above with Fatou's Lemma.) The second step involves showing that this upper bound is also a lower bound. The argument here is far more delicate, and involves locally linear approximations of the fields f and g. Both of these steps will involve adding conditions to the already long list of Theorem 11.1.1 and so the third and final step involves showing that these additional conditions can be lifted under the conditions of the theorem.

To state the first result, let $\delta_\varepsilon : \mathbb{R}^N \to \mathbb{R}$ be constant on the N-ball $B(\varepsilon) = \{r \in \mathbb{R}^N : |t| < \varepsilon\}$, zero elsewhere, and normalized so that

$$(11.1.8) \qquad \int_{B(\varepsilon)} \delta_\varepsilon(t)\, dt \; = \; 1.$$

Theorem 11.1.3 *Let $f : \mathbb{R}^N \to \mathbb{R}^N$, $g : \mathbb{R}^N \to \mathbb{R}^K$ be deterministic, and $T \subset \mathbb{R}^N$ and $B \subset \mathbb{R}^K$ open. Suppose, furthermore, that the following conditions are all satisfied for $u \in \mathbb{R}^N$:*

(a) *The components of f, g and ∇f are all continuous.*

(b) *There are no points $t \in T$ satisfying both $f(t) = u$ and either $g(t) \in \partial B$ or $\det \nabla f(t) = 0$.*

(c) *There are no points $t \in \partial T$ satisfying $f(t) = u$.*

(d) *There are only a finite number of points $t \in T$ satisfying $f(t) = u$.*

Then

$$(11.1.9) \quad N_u(f, g; T, B) \; = \; \lim_{\varepsilon \to 0} \int_T \delta_\varepsilon(f(t) - u)\, \mathbb{1}_B(g(t))\, |\det \nabla f(t)|\, dt.$$

Proof. To save on notation, and without any loss of generality, we take $u = 0$. Consider those $t \in T$ for which $f(t) = 0$. Since there are only finitely many such points, and none lie in ∂T, each one can be surrounded by an open sphere, of radius η, say, in such a way that the spheres neither overlap nor intersect ∂S. Furthermore, because of (b), we can ensure η is small enough so that within each sphere $g(t)$ always lies in either B or the interior of its complement, but never both.

Let $\sigma(\varepsilon)$ be the sphere $|f| < \varepsilon$ in the image space of f. From what we have just established we claim that we can now choose ε small enough for the inverse image of $\sigma(\varepsilon)$ in T to be contained within the union of the η spheres. (In fact, if this were not so, we could choose a sequence of points t_n in T not belonging to any η sphere, and a sequence ε_n tending to zero such that $f(t_n)$ would belong to $\sigma(\varepsilon_n)$ for each n. Since T is compact the sequence t_n would have a limit point t^* in T, for which we would have $f(t^*) = 0$. Since $t^* \notin \partial T$ by (c), we must have $t^* \in T$. Thus t^* is contained

in the inverse image of $\sigma(\varepsilon)$ for any ε, as must be infinitely many of the t_n. This contradiction establishes our claim.)

Furthermore, by (b) and the inverse mapping theorem (cf. Footnote 5 of Chapter 6) we can choose ε, η so small that, for each η sphere in T, $\sigma(\varepsilon)$ is contained in the f image of the η sphere, so that the restriction of f to such a sphere will be one-one. Since the Jacobian of the mapping of each η sphere by f is $|\det\nabla f(t)|$ it follows that we can choose ε small enough so that

$$N_0 \;=\; \int_T \delta_\varepsilon(f(t))\, \mathbb{1}_B(g(t))\, |\det\nabla f(t)|\, dt.$$

This follows since each η sphere in T over which $g(t) \in B$ will contribute exactly one unit to the integral, while all points outside the η spheres will not be mapped onto $\sigma(\varepsilon)$. Since the left-hand side of this expression is independent of ε we can take the limit as $\varepsilon \to 0$ to obtain (11.1.9) and thus the theorem. $\qquad\square$

Theorem 11.1.3 does not tell us anything about expectations. Ideally, it would be nice simply to take expectations on both sides of (11.1.9) and then, hopefully, find an easy way to evaluate the right-hand side of the resulting equation. While this requires justification and further assumptions, let us nevertheless proceed in this fashion, just to see what happens. We then have

$$\mathbb{E}\{N_0\} \;=\; \lim_{\varepsilon\to 0} \mathbb{E} \int_T \delta_\varepsilon(f(t))\, \mathbb{1}_B(g(t))\, |\det\nabla f(t)|\, dt$$

$$=\; \int_T \int_{\mathbb{R}^{N(N+1)/2}} \int_{\mathbb{R}^K} \mathbb{1}_B(v)\, |\det\nabla y|$$
$$\times \left\{ \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \delta_\varepsilon(x)\, p_t(x, \nabla y, v)\, dx \right\} d\nabla y\, dv\, dt,$$

where the p_t are the obvious densities. Taking the limit in the innermost integral yields

$$(11.1.10)\quad \mathbb{E}\{N_0\} \;=\; \int_T \int \int \mathbb{1}_B(v)\, |\det\nabla y|\, p_t(0, \nabla y, v)\, d\nabla y\, dv\, dt$$

$$=\; \int_T \mathbb{E}\{|\det\nabla f(t)|\, \mathbb{1}_B(g(t))\,\big|\, f(t) = 0\}\, p_t(0)\, dt.$$

Of course, interchanging the order of integration and the limiting procedure requires justification. Nevertheless, at this point we can state the following tongue-in-cheek 'corollary' to Theorem 11.1.3.

Corollary 11.1.4 *If the conditions of Theorem 11.1.3 hold, as well as 'adequate' regularity conditions, then*

(11.1.11)

$$
\begin{aligned}
E\{N_u\} &= \int_T \int_{\mathbb{R}^{N(N+1)/2}} \int_{\mathbb{R}^K} |\det\nabla x|\, \mathbb{1}_B(v)\, p_t(u, \nabla x, v)\, d\nabla x\, dv\, dt \\
&= \int_T \mathbb{E}\{\, |\det\nabla f(t)|\, \mathbb{1}_B(g(t))\,\big|\, f(t) = u\}\, p_t(u)\, dt.
\end{aligned}
$$

As will be seen below, 'adequate' regularity conditions generally require no more than that the density p_t above be well behaved (i.e. continuous, bounded) and that enough continuous derivatives of f and g exist, with enough finite moments.

At this point we suggest that if you are not the kind of person who cares too much about rigour then you move directly to Section 11.5, where we begin the preparations for using the above results. Indeed, there is probably much to be said in favour of even the mathematically inclined doing the same on the first reading.

We now turn to the rigorous upper bound, and start with a useful lemma. It is easily proven via induction started from Hölder's inequality.

Lemma 11.1.5 *Let $X_1, \ldots, X_n$ be any real valued random variables. Then*

(11.1.12)
$$
\mathbb{E}\{|X_1 \cdots X_n|\} \ \leq \ \prod_{i=1}^n \left[\mathbb{E}\{|X_i|^n\}\right]^{1/n}.
$$

Theorem 11.1.6 *Let f, g, B and T be as in Theorem 11.1.3, but with f and g random and conditions (a)–(d) there holding in an almost sure sense. Assume, furthermore, that Conditions (b)–(g) of Theorem 11.1.1 hold, with the notation adopted there. Then*

(11.1.13)
$$
\mathbb{E}\{N_u(f, g; T, B)\} \ \leq \ \int_T dt \int_B dv \int_{\mathbb{R}^{N(N+1)/2}} |\det\nabla y|\, p_t(u, \nabla y, v)\, d\nabla y.
$$

Proof. Again assume that $u = 0$. Start by setting, for $\varepsilon > 0$,

$$
N^\varepsilon \ = \ \int_T \delta_\varepsilon(f(t))\, \mathbb{1}_B(g(t))\, |\det\nabla f(t)|\, dt,
$$

where δ_ε is as in (11.1.8). By expanding the determinant, applying (11.1.12) and recalling the moment assumption (11.1.1), we have that everything is

nicely finite, and so Fubini's theorem gives us that

$$
\begin{aligned}
\mathbb{E}\{N^{\varepsilon}(T)\} &= \int_T dt \int_{\mathbb{R}^N \times \mathbb{R}^{N(N+1)/2} \times B} \delta_{\varepsilon}(x)\, |\det\nabla y|\, p_t(x, \nabla y, v)\, dx\, d\nabla y\, dv \\
&= \int_T dt \int_{\mathbb{R}^{N(N+1)/2} \times B} |\det\nabla y|\, p_t(\nabla y, v)\, d\nabla y\, dv \\
&\quad \times \int_{\mathbb{R}^N} \delta_{\varepsilon}(x)\, p_t(x|\nabla y, v)\, dx
\end{aligned}
$$

Since all densities are assumed continuous and bounded, the innermost integral clearly converges to

$$
p_t(0|\nabla y, v)
$$

as $\varepsilon \to 0$. Furthermore,

$$
\begin{aligned}
\int_{\mathbb{R}^N} \delta_{\varepsilon}(x)\, p_t(x|\nabla y, v)\, dx &\leq \sup_x p_t(x|\nabla y, v) \int_{\mathbb{R}^N} \delta_{\varepsilon}(x)\, dx \\
&= \sup_x p_t(x|\nabla y, v),
\end{aligned}
$$

which, by assumption, is bounded.

Again noting the boundedness of $\mathbb{E}\{|\det\nabla f_t|\}$, it follows from (11.1.9), dominated convergence and Fatou's Lemma, that

$$
\begin{aligned}
E\{N_0\} &\leq \lim_{\varepsilon \to 0} E\{N^{\varepsilon}\} \\
&= \int_T dt \int_B dv \int_{\mathbb{R}^{N(N+1)/2}} |\det\nabla y|\, p_t(0, \nabla y, v)\, d\nabla y.
\end{aligned}
$$

This, of course, proves the lemma.   □

We can now turn to the more difficult part of our problem: showing that the upper bound for $\mathbb{E}\{N(T)\}$ obtained in the preceding theorem also serves as a lower bound under reasonable conditions. We shall derive the following result.

Theorem 11.1.7 *Assume the setup and assumptions of Theorem 11.1.6, along with (11.1.2) of Theorem 11.1.1; viz. that the moduli of continuity of each of the components of f, ∇f, and g satisfy*

$$
(11.1.14) \qquad \mathbb{P}\{\omega(\eta) > \varepsilon\} = o(\eta^N), \qquad as\ h \downarrow 0,
$$

for all $\varepsilon > 0$. Then (11.1.13) holds with the inequality sign reversed, and so is an equality.

Since the proof of this theorem is rather involved, we shall start by first describing the principles underlying it. Essentially, the proof is based on

constructing a pathwise approximation to the vector-valued process f and then studying the zeros of the approximating process. The approximation is based on partitioning T and replacing f within each cell of the partition by a hyperplane tangential to f at the cell's midpoint. We then argue that if the approximating process has a zero within a certain subset of a given cell then f has a zero somewhere in the full cell. Thus the number of zeros of the approximating process will give a lower bound to the number of zeros of f.

In one dimension, for example, we replace the real valued function f on $T = [0, 1]$ by a series of approximations $f^{(n)}$ given by

$$f^{(n)}(t) \;\; = \;\; f\left((j + \tfrac{1}{2})2^{-n}\right) \;+\; \left[t - (j + \tfrac{1}{2})2^{-n}\right] f'\left((j + \tfrac{1}{2})2^{-n}\right),$$

for $j2^{-n} \leq t < (j+1)2^{-n}$, and study the zeros of $f^{(n)}$ as $n \to \infty$. Although this is perhaps not the most natural approximation to use in one dimension, it generalizes easily to higher dimensions.

Proof of Theorem 11.1.7 As usual, we take the level $u = 0$, and start with some notation. For each $n \geq 1$ let $\mathbb{Z}_n$ denote the lattice of points in $\mathbb{R}^N$ whose components are integer multiples of 2^{-n}; i.e.

$$\mathbb{Z}_n = \{t \in \mathbb{R}^N : t_j = i\, 2^{-n}, \; j = 1, \ldots, N, \; t \in \mathbb{Z}\}.$$

Now fix $\varepsilon > 0$, and for each $n \geq 1$ define two half-open hypercubes centred on an arbitrary point $t \in \mathbb{R}^N$ by

$$\begin{aligned}
\Delta_n(t) &= \{s \in \mathbb{R}^N : -2^{-(n+1)} \leq s_j - t_j < 2^{-(n+1)}\}, \\
\Delta_n^\varepsilon(t) &= \{s \in \mathbb{R}^N : -(1 - \varepsilon)2^{-(n+1)} \leq s_j - t_j < (1 - \varepsilon)2^{-(n+1)}\}.
\end{aligned}$$

Set

$$I_{nt} \;\; = \;\; \begin{cases} 1, & \text{if } N_0(f, g; \Delta_n(t), B) \geq 1 \text{ and } \Delta_n(t) \subset T, \\ 0, & \text{otherwise}, \end{cases}$$

and define approximations N^n to $N_0(f, g; T, B)$ by

$$N^n \;\; = \;\; \sum_{t \in \mathbb{Z}_n} I_{nt}.$$

Note (since it will be important later) that only those $\Delta_n(t)$ which lie wholly within T contribute to the approximations. However, since the points being counted are, by assumption, almost surely isolated, and none lie on ∂T, it follows that

$$N^n \;\; \overset{a.s.}{\to} \;\; N_0(f, g; T, B) \qquad \text{as } n \to \infty.$$

Since the sequence N^n is non-decreasing in n, monotone convergence yields

$$(11.1.15) \qquad \begin{aligned}
\mathbb{E}\{N_0\} &= \lim_{n \to \infty} \mathbb{E}\{N_n\} \\
&\geq \lim_{n \to \infty} \sum \mathbb{P}\{N(\Delta_n(t)) > 0\},
\end{aligned}$$

where the summation is over all $t \in \mathbb{Z}_n$ for which $\Delta_n(t) \subset T$.

The remainder of the proof involves determining the limiting value of $\mathbb{P}\{N(\Delta_n(t)) > 0\}$. Fix $\delta > 0$ small, and $K > 0$ large. For given realizations of f and g define the function $\omega_f^*(n)$ by

$$\omega_f^*(n) \;=\; \max\left[\max_i \omega_{f^j}(2^{-n}),\; \max_{ij} \omega_{(\nabla f)_{ij}}(2^{-n}),\; \max_i \omega_{g^j}(2^{-n})\right]$$

where the moduli of continuity are all taken over T. Furthermore, define M_f by

$$M_f \;=\; \max\left[\max_{1 \le j \le N} \sup_{t \in T} |\,f^j(t)\,|,\; \max_{1 \le i,j \le N} \sup_{t \in T} \left|\left(\partial f^i/\partial t_j\right)(t)\right|\right].$$

Finally, set

$$\eta = \frac{\delta^2 \varepsilon}{2N!\,(K+1)^{N-1}}.$$

Then the conditions of the theorem imply that, as $n \to \infty$,

(11.1.16) $$\mathbb{P}\{\omega^*(n) > \eta\} \;=\; o\left(2^{-Nn}\right).$$

Choose now a fixed n and $t \in \mathbb{Z}_n$ for which $\Delta_n(t) \subset T$. Assume that

(11.1.17) $$\omega_X^*(n) \;<\; \eta$$

and that the event $G_{\delta K}$ occurs, where

$$G_{\delta K} \;=\; \left\{\, |\det \nabla f(t)| > \delta,\; M_f < K/N,\right.$$
$$\left. g(t) \in \{x \in B:\; \inf_{y \in \partial B} \|x - y\|_1 > \delta\}\right\},$$

where $\|x\|_1 = \sum_i |x^i|$ is the usual ℓ^1 norm. Finally, define t^* to be the solution of the following equation, when a unique solution in fact exists:

(11.1.18) $$f(t) \;=\; (t - t^*) \cdot \nabla f(t).$$

We claim that if both $\omega^*(n) < \eta$ and $G_{\delta K}$ occurs, then, for n large enough, $t^* \in \Delta_n^\varepsilon(t)$ implies

(11.1.19) $$N_0(\Delta_n(t)) \;>\; 0$$

and

(11.1.20) $$\|f(t)\|_1 \;\le\; 2^{-n}\, K.$$

These two facts, which we shall establish in a moment, are enough to make the remainder of the proof quite simple. From (11.1.19), it follows that by choosing n large enough for (11.1.16) to be satisfied we have

$$\mathbb{P}\left\{N(\Delta_n(t)) > 0\right\} \;\ge\; \mathbb{P}\left\{G_{\delta K} \cap [t^* \in \Delta_n^\varepsilon(t)]\right\} \;+\; o\left(2^{-nN}\right).$$

Using this, making the transformation $f(t) \to t^*$ given by (11.1.18), and using the notation of the theorem, we obtain

$$\mathbb{P}\{N_0(\Delta_n(t)) > 0\}$$
$$\geq \int_{G_{\delta K} \cap \{t^* \in \Delta_n^\varepsilon(t)\}} |\det \nabla y| \, p_t((t - t^*)\nabla y, \nabla y, v) \, dt^* d\nabla y dv + o\left(2^{-nN}\right).$$

Noting (11.1.20), the continuity and boundedness assumptions on p_t, and the boundedness assumptions on the moments of the f_j^i, it follows that, as $n \to \infty$, the last expression, summed as in (11.1.15), converges to

$$(1 - \varepsilon)^N \int_T dt \int_{G_{\delta K}} |\det \nabla y| \, p_t(0, \nabla y, v) \, d\nabla y \, dv.$$

Letting $\varepsilon \to 0$, $\delta \to 0$, $K \to \infty$ and applying monotone convergence to the above expression we obtain, from (11.1.15), that

$$E\{N(T)\} \geq \int_T dt \int_B dv \int_{\mathbb{R}^{N(N+1)/2}} |\det \nabla y| \, p_t(0, \nabla y, v) \, d\nabla y.$$

This, of course, completes the proof, bar the issue of establishing that (11.1.19) and (11.1.20) hold under the conditions preceeding them.

Thus, assume that $\omega^*(n) < \eta$, $G_{\delta K}$ occurs, and the t^* defined by (11.1.18) satisfies $t^* \in \Delta_n^\varepsilon(t)$. Then (11.1.20) is immediate. The hard part is to establish (11.1.19). To this end, note that (11.1.18) can be rewritten as

(11.1.21) $$t - f(t)[\nabla f(t)]^{-1} \in \Delta_n^\varepsilon(t).$$

Let τ be any other point $\Delta_n(t)$. It is easy to check that

(11.1.22) $$|\det \nabla f(\tau) - \det \nabla f(t)| < \frac{\varepsilon \delta^2}{2},$$

under the conditions we require. Thus, since $\det \nabla f(t) > \delta$, it follows that $\det \nabla f(\tau) \neq 0$ for any $\tau \in \Delta_n(t)$ and so the matrix $\det \nabla f(\tau)$ is invertible throughout $\Delta_n(t)$. Similarly, one can check that $g(\tau) \in B$ for all $\tau \in \Delta_n(t)$.

Consider first (11.1.19). We need to show that $t^* \in \Delta_n^\varepsilon(t)$ implies the existence of at least one $\tau \in \Delta_n(t)$ at which $f(\tau) = 0$.

The mean value theorem[4] allows us to write

(11.1.24) $$f(\tau) - f(t) = (\tau - t) \cdot \nabla f(t^1, \ldots, t^N)$$

---

[4]In the form we need it, here is the mean value theorem for Euclidean spaces. A proof can be found, for example, in [13].

Lemma 11.1.8 (*Mean Value Theorem for* $\mathbb{R}^N$.) *Let T be a bounded open set in $\mathbb{R}^N$ and let $f : T \to \mathbb{R}^N$ have first-order partial derivatives at each point in T. Let s and t be two points in T such that the line segment*

$$L(s, t) = \{u : u = \theta s + (1 - \theta)t, \, 0 < \theta < 1\}.$$

for some points $t^1, \ldots, t^N$ lying on the line segment $L(t, \tau)$, for any t and τ, where $\nabla f(t^1, \ldots, t^N)$ is the matrix function ∇f with the elements in the k-th column evaluated at the point t^k. Using similar arguments to those used to establish (11.1.22) and the invertibility of $\det \nabla f(\tau)$ throughout $\Delta_n(t)$, invertibility can be shown for $\nabla f(t^1, \ldots, t^N)$ as well. Hence we can rewrite (11.1.24) as

(11.1.25)
$$f(\tau)[\nabla f(t^1, \ldots, t^N)]^{-1} \;=\; f(t)[\nabla f(t^1, \ldots, t^N)]^{-1} \;+\; (\tau - t).$$

Suppose we could show $f(t)[\nabla f(t^1, \ldots, t^N)]^{-1} \in \Delta_n(0)$ if $t^* \in \Delta_n^\varepsilon(t)$. Then by the Brouwer fixed point theorem[5] it would follow that the continuous mapping of $\Delta_n(t)$ into $\Delta_n(t)$ given by

$$\tau \;\to\; t - f(t)[\nabla f(t^1, \ldots, t^N)]^{-1}, \qquad \tau \in \Delta_n(t),$$

has at least one fixed point. Thus, by (11.1.25), there would be at least one $\tau \in \Delta_n(t)$ for which $f(\tau) = 0$. In other words,

$$\{\omega_X^*(n) < \eta, \; G_{\delta K}, \; t^* \in \Delta_n^\varepsilon(t), n \text{ large}\} \;\Rightarrow\; N(T \cap \Delta_n(t)) > 0.$$

But $f(t)[\nabla f(t^1, \ldots, t^N)]^{-1} \in \Delta_n(0)$ is easily seen to be a consequence of $t^* \in \Delta_n^\varepsilon(t)$ and $G_{\delta K}$ simply by writing

$$
\begin{aligned}
&f(t)[\nabla f(t^1, \ldots, t^N)]^{-1}\\
&= \; f(t)[\nabla f(t)]^{-1} \, \nabla f(t) \, \nabla f(t) \, [\nabla f(t^1, \ldots, t^N)]^{-1}\\
&= \; f(t)[\nabla f(t)]^{-1} \left(I + \left(\nabla f(t) - \nabla f(t^1, \ldots, t^N)\right)[\nabla f(t^1, \ldots, t^N)]^{-1}\right)
\end{aligned}
$$

noting (11.1.21) and bounding the rightmost expression using basically the same argument employed for (11.1.22). This completes the proof.   □

We now complete the task of this section – i.e. the proof of Theorem 11.1.1 – which gave the expression appearing in both Theorems 11.1.6 and 11.1.7 for $\mathbb{E}\{N_u\}$, but under seemingly weaker conditions than those we

---

is wholly contained in T. Then there exist points $t^1, \ldots, t^N$ on $L(s, t)$ such that

(11.1.23) $f(t) - f(s) \;=\; \nabla f(t^1, \ldots, t^N) \cdot (t - s),$

where by $\nabla f(t^1, \ldots, t^N)$ we mean the matrix valued function ∇f with the elements in the k-th column evaluated at the point t^k.

[5] In the form we need it, the Brouwer fixed point theorem is as follows. Proofs of this result are easy to find (e.g. [137]).

Lemma 11.1.9 *Let T be a compact, convex subset of $\mathbb{R}^N$ and $f : T \to T$ a continuous mapping of T into itself. Then f has at least one fixed point; i.e. there exists at least one point $t \in T$ for which $f(t) = t$.*

have assumed. What remains to show is that Conditions (b)–(d) of Theorem 11.1.3 are satisfied under the conditions of Theorem 11.1.1. Condition (c) follows immediately from the following rather intuitive result, taking $h = f$, $n = N - 1$, and identifying the T of the lemma with ∂T in the theorems. The claims in (b) and (d) are covered in the three remaining lemmas of the section. Lemma 11.1.10 has roots going back to Bulinskaya [34].

Lemma 11.1.10 *Let T be a compact set of Hausdorff dimension n. Let $h : T \to \mathbb{R}^{n+1}$ be C^1, with a.s. bounded partial derivatives. Furthermore, assume that the univariate probability densities of h are bounded in a neighbourhood of $u \in \mathbb{R}^{n+1}$, uniformly in t. Then, for such u, there are a.s. no $t \in T$ with $h(t) = u$. In the notation of inverses and excursion sets:*

$$(11.1.26) \quad \mathbb{P}\left\{h^{-1}(u) = \emptyset\right\} \;=\; \mathbb{P}\left\{\partial A_u(h, T) \;=\; \emptyset\right\} \;=\; 1.$$

Proof. We start with the observation that under the conditions of the lemma, for any $\varepsilon > 0$, there exists a finite C_ε so that

$$\mathbb{P}\{\max_{ij} \sup_T \partial h^i(t)/\partial t_j > C_\varepsilon\} \;>\; 1 - \varepsilon.$$

Writing ω_h for the modulus of continuity of h, it follows from the Mean Value Theorem that

$$(11.1.27) \qquad\qquad \mathbb{P}\{E_\varepsilon\} \;>\; 1 - \varepsilon,$$

where

$$E_\varepsilon \;\triangleq\; \left\{\omega_h(\eta) \le \sqrt{n} C_\varepsilon \eta, \text{ for small enough } \eta\right\}.$$

Take now a sequence $\{\delta_m\}$ of positive numbers converging to 0 as $m \to \infty$. The fact that T has Hausdorff dimension n implies that for any any $\delta > 0$, and m large enough, there there exists a collection of Euclidean balls $\{B_{mj}\}$ covering T such that

$$(11.1.28) \qquad\qquad \sum_j \left(\mathrm{diam}(B_{mj})\right)^{n+\delta} \;<\; \delta_m.$$

Define the events

$$A \;=\; \left\{\exists t \in T : h(t) = u\right\}, \qquad A_{mj} \;=\; \left\{\exists t \in B_{mj} : h(t) = u\right\}.$$

Fix $\varepsilon > 0$ and note that

$$(11.1.29) \qquad \mathbb{P}\{A\} \;\le\; \sum_j \mathbb{P}\left\{A_{mj} \cap E_\varepsilon\right\} \;+\; \mathbb{P}\left\{E_\varepsilon^c\right\}.$$

In view of (11.1.27) it suffices to show that the sum here can be made arbitrarily small.

To see this, let t_{mj} be the centrepoint of B_{mj}. Then, if both A_{mj} and E_ε occur, it follows that, for large enough m,

$$|h(t_{mj}) - u| \leq \sqrt{n} C_\varepsilon \operatorname{diam}(B_{mj}).$$

Since h_t has a bounded density, it thus follows that

$$\mathbb{P}\{A_{mj} \cap E_\varepsilon\} \leq M(\sqrt{n} C_\varepsilon \operatorname{diam}(B_{mj}))^{n+1},$$

where M is a bound on the densities. Substituting into (11.1.29) and noting (11.1.28) we are done. $\square$

We now turn to the second part of Condition (b) of Theorem 11.1.3, relating to the points $t \in T$ satisfying $f(t) - u = \det \nabla f(t) = 0$. Note firstly that this would follow easily from Lemma 11.1.10 if we were prepared to assume that $f \in C^3(T)$. This, however, is more than we are prepared to assume. That the conditions of Theorem 11.1.1 contain all we need is the content of the following lemma.

Lemma 11.1.11 *Let f and T be as in Theorem 11.1.1, with Conditions (a), (b), (d) and (g) of that theorem in force. Then, with probability one, there are no points $t \in T$ satisfying $f(t) - u = \det \nabla f(t) = 0$.*

Proof. As for the proof of the previous lemma, we start with an observation. In particular, for any $\varepsilon > 0$, there exists a continuous function ω_ε for which $\omega_\varepsilon(\eta) \downarrow 0$ as $\eta \downarrow 0$, and a finite positive constant C_ε, such that $\mathbb{P}\{E_\varepsilon\} > 1 - \varepsilon$, where now

$$E_\varepsilon \triangleq \left\{ \max_{ij} \sup_{t \in T} |f^i_j(t)| < C_\varepsilon, \ \max_i \omega_{f^i_j}(\eta) \leq \omega_\varepsilon(\eta), \text{ for } 0 < \eta \leq 1 \right\}.$$

To see this, the following simple argument[6] suffices: For ease of notation, set

$$\omega^*(\eta) = \max_{ij} \omega_{f^i_j}(\eta).$$

It then follows from (11.1.2) that there are sequences $\{c_n\}$ and $\{\eta_n\}$, both decreasing to zero, such that

$$\mathbb{P}\{\omega^*(\eta_n) < c_n\} > 1 - 2^{-n}\varepsilon.$$

Defining $\omega_\varepsilon(\eta) = c_n$ for $\eta_{n+1} \leq \eta < \eta_n$, Borel-Cantelli then gives, for some $\eta_1 > 0$, that

$$\mathbb{P}\{\omega^*(\eta) < \omega_\varepsilon(\eta), 0 < \eta \leq \eta_1\} > 1 - \varepsilon/2.$$

---

[6] Note that this event tells us less about the $\omega_{f^i_j}$ than in the corresponding event in the previous lemma, and so we shall have to work harder to show that $\mathbb{P}\{E_\varepsilon\} > 1 - \varepsilon$. The problem is that, this time, we are not assuming that the f^i_j, unlike the h^i of the previous lemma, are C^1.

If $\eta_1 < 1$, set $\omega_\varepsilon = c_0$ for $\eta_1 < \eta \le 1$, where c_0 is large enough so that

$$\mathbb{P}\left\{\omega^*(\eta) < c_0, \ \eta_1 < \eta \le 1\right\} \ > \ 1 - \varepsilon/2.$$

Now choose C_ε large enough so that

$$\mathbb{P}\left\{\sup_{t \in T}|f(t)| \ < \ C_\varepsilon\right\} \ > \ 1 - \varepsilon/2,$$

and we are done.

We now continue in a similar vein to that of the previous proof. Let the B_{mj} and t_{mj} be as defined there, although (11.1.28) is now replaced with

$$(11.1.30) \qquad \sum_j (\mathrm{diam}B_{mj})^N \ \to \ c_N \lambda_N(T) \ < \ \infty$$

as $m \to \infty$ for some dimension dependent constant c_N.

Define the events

$$A_{mj} \ = \ \left\{\exists t \in B_{mj} : f(t) - u = \det\nabla f(t) = 0\right\}.$$

Fixing $\varepsilon > 0$, as before, we now need only show that $\sum_j \mathbb{P}\left\{A_{mj} \cap E_\varepsilon\right\}$ can be made arbitrarily small.

Allowing C_N to be a dimension dependent constant that may change from line to line, if A_{mj} and E_ε occur, then, by expanding the determinant, it is easy to see that

$$|\det\nabla f(t_{mk})| \ \le \ C_N\left(\max(1, C_\varepsilon)\right)^{N-1}\omega_\varepsilon(\sqrt{N}\mathrm{diam}(B_{mj})),$$

and that, as before,

$$|f(t_{mj}) - u| \ \le \ C_N C_\varepsilon \mathrm{diam}(B_{mj}).$$

It therefore follows that

$$\mathbb{P}\{A_{mj} \cap E_\varepsilon\}$$
$$\le \ \int \mathbb{P}\left\{|\det\nabla f(t_{mj})| \le (C_N\left(\max(1, C_\varepsilon)\right)^{N-1}\omega_\varepsilon(\sqrt{N}\mathrm{diam}(B_{mj}))\right.$$
$$\left.\big|\ f(t_{mj}) = x\right\}p_{t_{mj}}(x)\,dx,$$

where p_t is the density of $f(t)$ and the integral is over a region in $\mathbb{R}^N$ of volume no greater than $C_N \mathrm{diam}(B_{mj})^N$.

By assumption *(g)* of Theorem 11.1.1 the integrand here tends to zero as $m \to \infty$, and so can be made arbitrarily small. Furthermore, p_t is uniformly bounded. Putting all of this together with (11.1.30) proves the result. □

We now turn to the first part of Condition (b) of Theorem 11.1.3, relating to the points $t \in T$ satisfying satisfying $f(t) = 0$ and $g(t) \in \partial B$. These are covered by the following lemma, whose proof is almost identical to that of Lemma 11.1.11 and so is left to you.

Lemma 11.1.12 *Let f and T be as in Theorem 11.1.1, with Conditions (a), (b), (e) and (g) of that theorem in force. Then, with probability one, there are no points $t \in T$ satisfying $f(t) - u = 0$ and $g(t) \in \partial B$.*

We have one more result to establish before completing this section and the proof of Theorem 11.1.1 – that Condition (d) of Theorem 11.1.3 holds. This is an immediate consequence of the following lemma.

Lemma 11.1.13 *Let the assumptions of Lemma 11.1.11 hold. Then, with probability one, there are only a finite number of points $t \in T$ satisfying $f(t) = u$.*

Proof. Given the *conclusion* of Lemma 11.1.11, this is actually a reasonably standard deterministic result related to the Morse Theory of Chapter 6 (cf. the proof of Lemma 6.2.3). Thus we give only an outline of the proof.

Suppose that t is such that $f(t) = u$. (If there are no such t, there is nothing more to prove.) The fact that $f \in C^1$ implies that there is a neighbourhood of t in which f is locally linear; *viz.* it can be approximated by its tangent plane. Furthermore, since $\det \nabla f(t) \neq 0$ (Lemma 11.1.11) not all the partial derivatives can be zero, and so the tangent plane cannot be at a constant 'height'. Consequently, throughout this neighbourhood there is no other point at which $f = u$. Since T is compact, there can therefore be no more than a finite number of t satisfying $f(t) = u$, and we are done. $\square$

11.2 Suitable regularity and Morse functions

Back in Chapter 6 we laid down a number of regularity conditions on deterministic functions f for the geometric analyses developed there to be valid. In the Integral Geometric setting they were summarised under the 'suitable regularity' of Definition 6.2.1. In the manifold setting, we moved to the 'Morse functions' of Section 9.3.

We shall soon (in Section 11.6) need to know when a random f is, with probability one, either suitably regular or a Morse function over a finite rectangle $T \subset \mathbb{R}^N$ and later (in Section 12.4) when it is a Morse function over a piecewise smooth manifold.

Despite the fact that this is (logically) not the right place to be addressing these issues, we shall nevertheless do so, while the details of the previous section are still fresh in your mind. In particular, without much further effort, Lemmas 11.1.10 and 11.1.11 will give us what we need.

We consider Morse functions over rectangles first, since here the conditions are tidiest. We start with a reminder (and slight extension) of notation: As usual, $\partial_k T$ is the k-dimensional boundary, or skeleton, of T.

Since T is a rectangle, each $\partial_k T$ is made up of $2^{N-k}\binom{N}{k}$ open rectangles, or 'faces', of dimension k.

Morse functions can then be defined via the following three characteristics:

(i) f is C^2 on an open neighbourhood of T.

(ii) The critical points of $f_{|\partial_k T}$ are non-degenerate for all $k = 0, \ldots, N$.

(iii) $f_{|\partial_k T}$ has no critical points on $\bigcup_{j=0}^{k-1} \partial_j T$ for all $k = 1, \ldots, N$.

Recall that, in the current Euclidean setting, critical points are those t for which $\nabla f(t) = 0$, and 'non-degeneracy' means that the Hessian $\nabla^2 f(t)$ has non-zero determinant. Note also that in both conditions (ii) and (iii) there is a strong dependence on dimension. In particular, in (ii) the requirement is that the $\mathbb{R}^{k+1}$-valued function $(\nabla(f_{|\partial_k T}), \det\nabla^2(f_{|\partial_k T}))$ not have zeroes over a k-dimensional parameter set. Regarding (iii) the requirement is that the $\mathbb{R}^k$-valued function $\nabla(f_{|\partial_k T})$ defined on a k-dimensional set not have zeroes on a subset of dimension $k-1$.

In this light (ii) and (iii) are clearly related to Lemma 11.1.11 and we leave it to you to check the details that give us

Theorem 11.2.1 *Let f be a real valued random field over a bounded rectangle T of $\mathbb{R}^N$ with first and second order partial derivatives f_i and f_{ij}. Then f is, with probability one, a Morse function over T if the following conditions hold for each face J of T:*

(a) *f is, with probability one, C^2 on an open neighbourhood of T, and all second derivatives have finite variance.*

(b) *For all $t \in J$, the marginal densities $p_t(x)$ of $\nabla f_{|J}(t)$ are continuous at 0, uniformly in t.*

(c) *The conditional densities $p_t(z|x)$ of $\det\nabla^2 f_{|J}(t)$ given $\nabla f_{|J}(t) = x$ are continuous for (z, x) in a neighbourhood of 0, uniformly in $t \in T$.*

(d) *On J, the moduli of continuity of f and its first and second order partial derivatives all satisfy*

$$\mathbb{P}\{\omega(\eta) > \varepsilon\} = o\left(\eta^{\dim(J)}\right), \qquad \text{as } \eta \downarrow 0.$$

As usual, we also have a Gaussian corollary (cf. Corollary 11.1.2). It is even simpler than usual, since Gaussianity allows us drop the references to the specific faces J that so cluttered the conditions of the theorem.

Corollary 11.2.2 *Let f be a centered Gaussian field over a finite rectangle T. If, for each $t \in T$, the joint distributions of $(f_i(t), f_{ij}(t))_{i,j=1,\ldots,N}$ are non-degenerate, and if, for some finite K,*

(11.2.1) $\displaystyle\sup_{s,t \in T} \max_{i,j} \left| C_{f_{ij}}(t) - C_{f_{ij}}(s) \right| \leq K \left| \ln|t-s| \right|^{-(1+\alpha)},$

then the sample functions of f are, with probability one, Morse functions over T.

The next issue is to determine when sample funtions are, with probability one, suitably regular in the sense of Definition 6.2.1. This is somewhat less elegant because of the asymmetry in the conditions of suitable regularity and their dependence on a particular coordinate system. Nevertheless, the same arguments as above work here as well and it is easy (albeit a little time consuming) to see that the following suffices to do the job.

Theorem 11.2.3 *Under the conditions of Corollary 11.2.2 the sample functions of f are, with probability one, suitably regular over bounded rectangles.*

A little thought will show that the assumptions of this theorem would seem to give more than is required. Consider the case $N = 2$. In that case, Condition (6.2.6) of suitable regularity requires that there are no $t \in T$ for which $f(t) - u = f_1(t) = f_{11}(t) = 0$. This is clearly implied by Theorem 11.2.3. However, the theorem also implies that $f(t) - u = f_2(t) = f_{22}(t) = 0$ which is not something which we require. Rather, it is a consequence of a desire to write the conditions of the theorem in a compact form.

In fact, Theorem 11.2.3 goes even further, in that it implies that for every *fixed* choice of coordinate system, the sample functions of f are, with probability one, suitably regular over bounded rectangles[7].

We now turn the what is really the most important case, that of determining sufficient conditions for a random field to be, almost surely, a Morse function over a piecewise C^2 Riemannian manifold (M, g). Writing our manifold as usual as

$$(11.2.2) \qquad M \; = \; \bigcup_{j=0}^{N} \partial_j M,$$

(cf. (9.3.5)) Conditions (i)–(iii) still characterise whether or not f will be a Morse function. The problem is how to generalise Theorem 11.2.1 to this scenario, since its proof was based on the results of the previous section, all of which were established in a Euclidean setting. The trick, of course, is to recall that each of the three required properties is of a local nature. We can then argue as follows:

Choose a chart (U, φ) from a countable atlas covering M. Let $t^* \in U$ be a critical point of f, and note that this property is independent of the choice of local coordinates. Working therefore with the natural basis for $T_t M$ it is easy to see that $\varphi(t^*)$ is also a critical point of $\varphi(U)$. Furthermore, the covariant Hessian $\nabla^2 f(t^*)$ will be zero if and only if the same is true

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[7]Recall that throughout our discussion of Integral Geometry there was a fixed coordinate system and that suitable regularity was defined relative to this system.

of the regular Hessian of $f \circ \varphi^{-1}$, and since φ is a diffeomorphism this implies that t^* will be a degenerate critical point of f if and only if t^* is for $f \circ \varphi^{-1}$. It therefore follows that, even in the manifold case, we can manage, with purely Euclidean proofs, to establish the following result. The straightforward but sometimes messy details are left to you.

Theorem 11.2.4 *Let f be a real valued random field over a piecewise C^2, compact, Riemannian submanifold (M, g) of a C^3 manifold $\widetilde{M}$. Assume that M has a countable atlas. Then f is, with probability one, a Morse function over M if the following conditions hold for each submanifold $\partial_j M$ in (11.2.2). Throughout ∇ and ∇^2 are to be understood in their Riemannian formulations and implicitly assumed to have the dimension of the $\partial_j M$ where they are being applied.*

(a) *f is, with probability one, C^2 on an open neighbourhood of M in $\widetilde{M}$, and $\mathbb{E}\{(XYf)^2\} < \infty$ for $X, Y \in T_t M$, $t \in M$.*

(b) *For each $\partial_j M$, the marginal densities $p_t(x)$ of $\nabla f_{|\partial_j M}(t)$ are continuous at 0, uniformly in t.*

(c) *The densities $p_t(z|x)$ of $\mathrm{Tr}^{\partial_j M} \nabla^2 f_{|\partial_j M}(t)$ given $\nabla f_{|\partial_j M}(t) = x$ are continuous for (z, x) in a neighbourhood of 0, uniformly in $t \in M$,*

(d) *On $\partial_j M$, the modulus of continuity of $\nabla^2 f_{|\partial_j M}(t)(X, Y)$ satisfies*

$$(11.2.3) \quad \mathbb{P}\{\omega(\eta) > \varepsilon\} = o\left(\eta^{\dim(\partial_j M)}\right), \qquad as\ \eta \downarrow 0,$$

for all $X, Y \in S(M) \cap T_t \partial_j M$, where $S(M)$ is the sphere bundle of M and the modulus of continuity is taken with respect to the distance induced by the Riemannian metric g.

As usual, there is a Gaussian corollary to Theorem 11.2.4 which requires only non-degeneracies and Condition (d). There are two ways that we could go about attacking Condition (d). The more geometric of the two would be to return to the entropy conditions of Section 1.3 and find a natural entropy condition for (11.2.3) to hold. This, however, involves adding a notion of 'canonical distance' (under the canonical metric d on M) to the already existing distance corresponding to the Riemannian metric induced by f. The details of carrying this out, while not terribly difficult, would involve some work. Perhaps more importantly, the resulting conditions would not be in a form that would be easy to check in practice.

Thus, we take another route, given conditions that are less elegant but easier to establish and generally far easier to check in practice. As for Theorem 11.2.4 itself we leave it to you to check the details of the (straighforward) proof of the Corollary.

Corollary 11.2.5 *Take the setup of Theorem 11.2.4 and let f be a centered Gaussian field over M. Let $\mathcal{A} = (U_\alpha, \varphi_\alpha)_{\alpha \in I}$ be a countable atlas for M such that for every α the Gaussian field $f_\alpha = f \circ \varphi_\alpha^{-1}$ on $\varphi_\alpha(U_a) \subset \mathbb{R}^N$ satisfies the conditions of Corollary 11.2.2 with $T = \varphi_\alpha(U_\alpha)$, $f = f_\alpha$ and some $K_\alpha > 0$. Then the sample functions of f are, with probability one, Morse functions over M.*

11.3   An alternate proof of the meta-theorem

The proof of the 'meta-theorem' Theorem 11.1.1 given in the previous section is but the latest tale in a long history.

The first proof of this kind is probably due to Rice [122] in 1945, who worked in the setting of real valued processes on the line. His proof was made rigorous by Itô [77] and Ylvisaker [171] in the mid 1960's. Meanwhile, in 1957, Longuet-Higgins [99, 100] was the first to extend it to the setting of random fields and used it to compute the expectations of such variables as the mean number of local maxima of two- and three-dimensional Gaussian fields. Rigorous versions of Theorem 11.1.1, at various levels of generality and with various assumptions, appeared only in the early 1970's as in [2, 8, 16, 101] with the proof of Section 11.1 being essentially that of [2].

Recently, however, Azais and Wschebor [15] have developed a new proof, based on Federer's coarea formula, in the form (7.4.15). In the notation of the previous section, this can be rewritten as

$$\int_{\mathbb{R}^N} \left(\sum_{t:\, f(t)=u} \alpha(t) \right) du \;\; = \;\; \int_{\mathbb{R}^N} \alpha(t)\, |\det \nabla f(t)|\, dt$$

assuming that f and $\alpha : \mathbb{R}^N \to \mathbb{R}^N$ are sufficiently smooth.

Take $\alpha(t) = \varphi(f(t)) \mathbb{1}_T(t)$, where φ is a smooth (test) function. (Of course, α is now no longer smooth, but we shall ignore this for the moment.) The above then becomes

$$\int_{\mathbb{R}^N} \varphi(u)\, N_u(f:T)\, du \;\; = \;\; \int_T \varphi(f(t))\, |\det \nabla f(t)|\, dt.$$

Now take expectations (assuming this is allowed) of both sides to obtain

$$\int_{\mathbb{R}^N} \varphi(u)\, \mathbb{E}\{N_u(f:T)\}\, du$$
$$= \int_T \mathbb{E}\left\{ \varphi(f(t))\, |\det \nabla f(t)| \right\} dt$$
$$= \int_{\mathbb{R}^N} \varphi(u) \int_T \mathbb{E}\left\{ |\det \nabla f(t)| \,\big|\, f(t) = u \right\} p_t(u)\, dt du.$$

Since φ was arbitrary, this implies that for (Lebesgue) almost every u,

$$(11.3.1) \quad \mathbb{E}\{N_u(f:T)\} \;=\; \int_T \mathbb{E}\left\{|\det\nabla f(t)|\big|f(t)=u\right\} p_t(u)\, dt,$$

which is precisely (11.1.4) of Theorem 11.1.1 with the g there identically equal to 1. Modulo this restriction on g, which is simple to remove, this is the result we have worked so hard to prove. The problem, however, is that since it is true only for almost every u one cannot be certain that it is true for a specific value of u.

To complete the proof, we need only show that both sides of (11.3.1) are continuous functions of u and that the assumptions of convenience made above are no more than that. This, of course, is not as trivial as it may sound. Going through the arguments actually leads to repeating many of the technical points we went through in the previous section, and eventually Theorem 11.1.1 reappears with the same long list of conditions. However (and this is the big gain) the details have no need of the construction, in the proof of Theorem 11.1.7, of the linear approximation to f.

You can find all the details in [15] and decide for yourself which proof you prefer.

11.4   Higher moments

While not at all obvious at first sight, hidden away in Theorem 11.1.1 is another result, about higher moments of the random variable $N_u(f, g : T, B)$. To state it, we need, for integral $k \geq 1$, the k-th partial factorial of a real x defined by

$$(x)_k \;\triangleq\; x(x-1)\ldots(x-k+1).$$

Then we have

Theorem 11.4.1 *Let f, g, T and B be as in Theorem 11.1.1, and assume that Conditions (a), (f) and (g) there still hold. For $k \geq 1$, write*

$$\begin{aligned}
T^k &= \{\widetilde{t} = (t_1, \ldots, t_k) : t_j \in T, \ \forall 1 \leq j \leq k\}, \\
\widetilde{f}(\widetilde{t}) &= (f(t_1), \ldots, f(t_k)) : T^k \to \mathbb{R}^{Nk}, \\
\widetilde{g}(\widetilde{t}) &= (g(t_1), \ldots, g(t_k)) : T^k \to \mathbb{R}^{Nk}.
\end{aligned}$$

Replace assumptions (b), (c) and (d) of Theorem 11.1.1 by the following, assumed to hold for each $t \in T^k \setminus \{t \in T^k : t_i = t_j \text{ for some } i \neq j\}$:

(b′) *The marginal densities $p_{\widetilde{t}}$ of $\widetilde{f}(\widetilde{t})$ are continuous at $\tilde{u} \triangleq (u, \ldots, u)$.*

(c′) *The conditional densities of $\widetilde{f}(\widetilde{t})$ given $\widetilde{g}(\widetilde{t})$ and $\nabla\widetilde{f}(\widetilde{t})$ are bounded above and continuous at $\tilde{u}$.*

(d′) The conditional densities $p_{\tilde{t}}(z|\tilde{f}(\tilde{t}) = x)$ of $\det\nabla\tilde{f}(\tilde{t})$ given $\tilde{f}(\tilde{t}) = x$ are continuous for z and x in neighbourhoods of 0 and $\tilde{u}$, respectively.

Then,

(11.4.1)

$$
\mathbb{E}\{(N_u)_k\} = \int_{T^k} \mathbb{E}\left\{ \prod_{j=1}^{k} |\det\nabla f(t_j)| \, \mathbb{1}_B(g(t_j)) \,\Big|\, \tilde{f}(\tilde{t}) = \tilde{u} \right\} p_{\tilde{t}}(\tilde{u}) \, d\tilde{t}
$$

$$
= \int_{T^k} \prod_{j=1}^{k} \int_{\mathbb{R}^{kD}} |\det D_j| \, \mathbb{1}_B(v_j) \, p_{\tilde{t}}(\tilde{u}, \tilde{D}, \tilde{v}) \, d\tilde{D} \, d\tilde{v} \, d\tilde{t}
$$

where

(i) $p_{\tilde{t}}(\tilde{x})$ *is the density of $\tilde{f}(\tilde{t})$.*

(ii) $p_{\tilde{t}}(\tilde{x}, \tilde{D}, \tilde{v})$ *is the joint density of $\tilde{f}(\tilde{t}), \tilde{D}(\tilde{t})$ and $\tilde{g}(\tilde{t})$.*

(iii) $\tilde{D}(\tilde{t})$ *represents the $Nk \times Nk$ matrix $\nabla\tilde{f}(\tilde{t})$. Note that $\tilde{D}(\tilde{t})$ is a diagonal block matrix, with the j-th block $D_j(\tilde{t})$ containing the matrix $\nabla f(t_j)$, where $t_j \in T$ is the j-th component of $\tilde{t}$.*

(iv) $D = N(N+1)/2 + K$.

Proof. For each $\delta > 0$ define the domain

$$
T_\delta^k \triangleq \{\tilde{t} \in T^k : |t_i - t_j| \geq \delta, \text{ for all } 1 \leq i < j \leq k\}.
$$

Then the field $\tilde{f}$ satisfies all the assumptions of Theorem 11.1.1, *uniformly* over the parameter set T_δ^k.

It therefore follows from (11.1.3) and (11.1.4) that $\mathbb{E}\{N_{\tilde{u}}(\tilde{f}, \tilde{g} : T_\delta^k, B^k)\}$ is given by either of the integrals in (11.1.1), with the outer integrals taken over T_δ^k rather than T^k.

Let $\delta \downarrow 0$. Then, using the fact that $f = u$ only finitely often (cf. Lemma 11.1.13) it is easy to see that, with probability one,

$$
N_{\tilde{u}}(\tilde{f}, \tilde{g} : T_\delta^k, B^k) \ \uparrow\ (N_u(f, g : T, B))_k.
$$

The monotonicity of the convergence then implies covergence of the expectations $\mathbb{E}\{N_{\tilde{u}}(\tilde{f}, \tilde{g} : T_\delta^k, B^k)\}$.

On the other hand, the integrals in (11.4.1) are trivially the limit of the same expressions with the outer integrals taken over T_δ^k rather than T^k, and so we are done. □

As for the basic expectation result, Theorem 11.4.1 takes a far simpler form if f is Gaussian, and we have

Corollary 11.4.2 *Let f and g be centered Gaussian fields over a T which satisfies the conditions of Theorem 11.1.1. If, for each $t \in T$, the joint distributions of $\{(f(t_j), \nabla f(t_j), g(t_j))\}_{1 \leq j \leq k}$ are non-degenerate for all choices of distinct $t_j \in T$, and if (11.1.5) holds, then so does (11.1.1).*

11.5 Preliminary Gaussian computations

In the following Section 11.6 we shall begin our computations of expectations for the Euler characteristics of excursion sets of Gaussian fields. In preparation, we need to collect a few facts about Gaussian integrals. Some are standard, others particularly tailored to our specific needs. However, since all are crucial components for the proofs of the main results of this chapter, we shall give proofs for all of them.

The first, in particular, is standard fare:

Lemma 11.5.1 *Let $X_1, X_2, \ldots, X_n$ be a set of real-valued random variables having a joint Gaussian distribution and zero means. Then for any integer m*

$$(11.5.1) \quad \mathbb{E}\{X_1 X_2 \cdots X_{2m+1}\} \;=\; 0,$$

$$(11.5.2) \quad \mathbb{E}\{X_1 X_2 \cdots X_{2m}\} \;=\; \sum \mathbb{E}\{X_{i_1} X_{i_2}\} \cdots \mathbb{E}\{X_{i_{2m-1}} X_{i_{2m}}\},$$

where the sum is taken over the $(2m)! \, / m! \, 2^m$ different ways of grouping $X_1, \ldots, X_{2m}$ into m pairs.

Note that this result continues to hold even if some of the X_j are identical, so that the lemma also applies to compute joint moments of the form $\mathbb{E}\{X_1^{i_1} \cdots X_k^{i_k}\}$.

Proof. Recall from (1.2.4) that the joint characteristic function of the X_i is

$$\phi(\theta) \;=\; \mathbb{E}\left\{e^{i \sum_i \theta_i X_i}\right\} \;=\; e^Q,$$

where

$$Q \;=\; Q(\theta) \;=\; -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \theta_i \mathbb{E}\{X_i X_j\} \theta_j.$$

Following our usual convention of denoting partial derivatives by subscripting, we have, for all l, k, j.

$$(11.5.3) \qquad\qquad Q_j \;=\; -\sum_{k=1}^{n} \mathbb{E}\{X_j X_k\} \theta_k,$$

$$Q_{kj} \;=\; -\mathbb{E}\{X_j X_k\},$$

$$Q_{lkj} \;=\; 0.$$

Successive differentiations of (11.5.3) yield

$$
\begin{aligned}
\phi_j &= \phi Q_j, \\
\phi_{kj} &= \phi_k Q_j + \phi Q_{kj}, \\
\phi_{lkj} &= \phi_{lk} Q_j + \phi_k Q_{lj} + \phi_l Q_{kj}, \\
&\vdots \qquad \vdots \\
\phi_{12\cdots n} &= \phi_{12\cdots(j-1)(j+1)\cdots n} Q_j + \sum_{k \neq j} \phi_{r_1 \cdots r_{n-2}} Q_{kj},
\end{aligned}
$$

(11.5.4)

where, in the last equation, the sequence $r_1, \dots, r_{n-2}$ does not include the two numbers k and j.

The moments of various orders can now be obtained by setting $\theta = 0$ in the equations of (11.5.4). Since from (11.5.3) we have $Q_j(0) = 0$ for all j, the last (and most general) equation in (11.5.4) thus leads to

$$
\mathbb{E}\{X_1 \cdots X_n\} = \sum_{k \neq j} \mathbb{E}\{X_{r_1} \cdots X_{r_{n-2}}\} \mathbb{E}\{X_j X_k\}.
$$

From this relationship and the fact that the X_j all have zero mean it is easy to deduce the validity of (11.5.1) and (11.5.2). It remains only to determine exactly the number, M say, of terms in the summation (11.5.2). We note first that while there are $(2m)!$ permutations of $X_1, \dots, X_{2m}$, since the sum does not include identical terms, $M < (2m)!$. Secondly, for each term in the sum, permutations of the m factors result in identical ways of breaking up the $2m$ elements. Thirdly, since $\mathbb{E}\{X_j X_k\} = \mathbb{E}\{X_k X_j\}$, an interchange of the order in such a pair does not yield a new pair. Thus

$$
M(m!)(2^m) = (2m)!
$$

implying

$$
M = \frac{(2m)!}{m! \, 2^m}
$$

as stated in the lemma. $\square$

For the next lemma we need some notation. Let Δ_N be a symmetric $N \times N$ matrix with elements Δ_{ij}, such that each Δ_{ij} is a zero mean normal variable with arbitrary variance but such that the following relationship holds:

(11.5.5) $\mathbb{E}\{\Delta_{ij}\Delta_{kl}\} = \mathcal{E}(i, j, k, l) - \delta_{ij}\delta_{kl},$

where $\mathcal{E}$ is a symmetric function of i, j, k, l and δ_{ij} is the Kronecker delta. Write $|\Delta_N|$ for the determinant of Δ_N.

Lemma 11.5.2 *Let m be a positive integer. Then, under (11.5.5),*

(11.5.6) $$\mathbb{E}\{|\Delta_{2m+1}|\} = 0,$$

(11.5.7) $$\mathbb{E}\{|\Delta_{2m}|\} = \frac{(-1)^m (2m)!}{m!\, 2^m}.$$

Proof. Relation (11.5.6) is immediate from (11.5.1). Now

$$|\Delta_{2m}| = \sum_P \eta(p)\Delta_{1 i_1}\ldots\Delta_{2m,\, i_{2m}}$$

where $p = (i_1, i_2, \ldots, i_{2m})$ is a permutation of $(1, 2, \ldots, 2m)$, P is the set of the $(2m)!$ such permutations, and $\eta(p)$ equals $+1$ or -1 depending on the order of the permutation p. Thus by (11.5.2) we have

$$\mathbb{E}\{|\Delta_{2m}|\} = \sum_P \eta(p) \sum_Q \mathbb{E}\{\Delta_{1 i_1}\Delta_{2 i_2}\}\cdots\mathbb{E}\{\Delta_{2m-1,\, i_{2m-1}}\Delta_{2m,\, i_{2m}}\},$$

where Q is the set of the $(2m)!\,/m!\,2^m$ ways of grouping $(i_1, i_2, \ldots, i_{2m})$ into pairs without regard to order, keeping them paired with the first index. Thus, by (11.5.5),

$$\mathbb{E}\{|\Delta_{2m}|\} = \sum_P \eta(p) \sum_Q \{\mathcal{E}(1, i_1, 2, i_2) - \delta_{1 i_1}\delta_{2 i_2}\} \times \cdots$$
$$\times \{\mathcal{E}(2m-1, i_{2m-1}, 2m, i_{2m}) - \delta_{2m-1,\, i_{2m-1}}\delta_{2m,\, i_{2m}}\}.$$

It is easily seen that all products involving at least one $\mathcal{E}$ term will cancel out because of their symmetry property. Hence

$$\mathbb{E}\{|\Delta_{2m}|\} = \sum_P \eta(p) \sum_Q (-1)^m \left(\delta_{1 i_1}\delta_{2 i_2}\right)\cdots\left(\delta_{2m-1,\, i_{2m-1}}\delta_{2m,\, i_{2m}}\right)$$
$$= \frac{(-1)^m (2m)!}{(m!)2^m},$$

the last line coming from changing the order of summation and then noting that for only one permutation in P is the product of delta functions non-zero. This completes the proof. $\square$

Note that (11.5.7) in no way depends on the specific (co)variances among the elements of Δ_N. These all disappear in the final result due to the symmetry of $\mathcal{E}$.

Before stating the next result we need to introduce the family of *Hermite polynomials*. The n-th Hermite polynomial is the function

(11.5.8) $$H_n(x) = n! \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j x^{n-2j}}{j!\,(n-2j)!\,2^j}, \qquad n \geq 0, \quad x \in \mathbb{R},$$

where $\lfloor a \rfloor$ is the largest integer less than or equal to a. For convenience later, we also define

(11.5.9) $$H_{-1}(x) \;=\; \sqrt{2\pi}\,\Psi(x)\,e^{x^2/2}, \qquad x \in \mathbb{R},$$

where Ψ is the tail probability function for a standard Gaussian variable (cf. (1.2.1)). With the normalisation inherent in (11.5.8) the Hermite polynomials form an orthogonal (but not orthonormal) system with respect to standard Gaussian measure on $\mathbb{R}$, in that

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} H_n(x)\,H_m(x)e^{-x^2/2}\,dx \;=\; \begin{cases} n! & m = n, \\ 0 & m \neq n. \end{cases}$$

An alternative definition of the Hermite polynomials is via a generating function approach, which gives

(11.5.10) $$H_n(x) \;=\; (-1)^n e^{x^2/2} \frac{d^n}{dx^n}\left(e^{-x^2/2}\right), \qquad n \geq 0.$$

From this it immediately follows that

(11.5.11) $$\int_u^\infty H_n(x)\,e^{-x^2/2}\,dx \;=\; H_{n-1}(u)\,e^{-u^2/2}, \qquad n \geq 0.$$

The centrality of Hermite polynomials for us lies in the following corollary to Lemma 11.5.2.

Corollary 11.5.3 *Let Δ_N be as in Lemma 11.5.2, with the same assumptions in force. Let I be the $N \times N$ unit matrix, and $x \in \mathbb{R}$. Then*

$$\mathbb{E}\{\det(\Delta_N - xI)\} \;=\; (-1)^N\,H_N(x).$$

Proof. It follows from the usual Laplace expansion of the determinant that

(11.5.12)
$$\begin{aligned}
\det(\Delta_N - xI) \;=\;\; & (-1)^N\left[x^N \;-\; S_1(\Delta_N)x^{N-1}\right. \\
& \left. + S_2(\Delta_N)x^{N-2} + \cdots + (-1)^N S_N(\Delta_N)\right],
\end{aligned}$$

where $S_k(\Delta_N)$ is the sum of the $\binom{N}{k}$ principal minors of order k in $|\Delta_N|$. The result now follows trivially from (11.5.6) and (11.5.7).   □

11.6   The mean Euler characteristic

We now assume f to be a centered, stationary, Gaussian process on a rectangle $T \subset \mathbb{R}^N$ and satisfying the conditions of Corollary 11.2.2. As usual,

$C : \mathbb{R}^N \to \mathbb{R}$ denotes the covariance function, and ν the spectral measure. Then f has variance $\sigma^2 = C(0) = \nu(\mathbb{R}^N)$. We change notation a little from Section 5.4, and introduce the *second-order spectral moments*indexspectral moment!second order

$$(11.6.1) \qquad \lambda_{ij} \;=\; \int_{\mathbb{R}^N} \lambda_i \lambda_j \, \nu(d\lambda).$$

Since we shall use it often, denote the $N \times N$ matrixindexspectral moment!matrix of these moments by Λ.

Denoting also differentiation via subscripts, so that $f_i = \partial f / \partial t_i$, $f_{ij} = \partial^2 f / \partial t_i \partial t_j$, etc. we have, from (5.5.7), that

$$(11.6.2) \qquad \mathbb{E}\{f_i(t) f_j(t)\} \;=\; \lambda_{ij} \;=\; -C_{ij}(0).$$

Thus Λ is also the variance-covariance matrix of ∇f. We could, of course, define the components of Λ using only derivatives of R, without ever referring to the spectrum.

The covariances among the second order derivatives can be similarly defined. However all we shall need is that

$$(11.6.3) \quad \mathcal{E}_{ijk\ell} \;\triangleq\; \mathbb{E}\{f_{ij}(t) f_{kl}(t)\} \;=\; \int_{\mathbb{R}^N} \lambda_i \lambda_j \lambda_k \lambda_\ell \, \nu(d\lambda)$$

is a symmetric function of i, j, k, ℓ.

Finally, note, as shown in Section 5.4, that f and its first order derivatives are independent (at any fixed point t) as are the first and second order derivatives (from one another). The field and its second order derivatives are, however, correlated, and

$$(11.6.4) \qquad \mathbb{E}\{f(t) f_{ij}(t)\} \;=\; -\lambda_{ij}.$$

Finally, denote the $N \times N$ Hessian matrix (f_{ij}) by $\nabla^2 f$.

Lemma 11.6.1 *Let f and T be as described above, and set*

$$(11.6.5) \; \mu_k \;=\; \#\{t \in T : f(t) \geq u, \; \nabla f(t) = 0, \; \text{index}(\nabla^2 f) = k\}.$$

Then, for all $N \geq 1$,

$$(11.6.6) \; \mathbb{E}\left\{ \sum_{k=0}^{N} (-1)^k \mu_k \right\} \;=\; \frac{(-1)^N |T| \, |\Lambda|^{1/2}}{(2\pi)^{(N+1)/2} \sigma^N} \, H_{N-1}\left(\frac{u}{\sigma}\right) e^{-u^2/2\sigma^2}.$$

Before turning to the proof of the lemma, there are some crucial points worth noting. The first is the rather surprising fact that the result depends on the covariance of f only through some of its derivatives at zero; *viz.* only through the variance and second order spectral moments This is particularly surprising in view of the fact that the definition of the μ_k depends

quite strongly on the f_{ij}, whose distribution involves fourth order spectral moments.

As will become clear from the proof, the disappearance of the fourth order spectral moments has a lot to do with the fact that we compute the mean of the alternating sum in (11.6.6) and do not attempt to evaluate the expectations of the individual μ_k. Doing so would indeed involve fourth order spectral moments. As we shall see in later chapters, the fact that this is all we need is extremely fortunate, for it is actually *impossible* to obtain closed expressions for any of the $\mathbb{E}\{\mu_k\}$.

Proof. We start with the notationally simplifying assumption that $\mathbb{E}\{f_t^2\} = \sigma^2 = 1$. It is clear that if we succeed in establishing (11.6.6) for this case, then the general case follows by scaling. (Note that scaling f also implies scaling ∇f, which, since Λ contains the variances of the elements of ∇f, gives the factor of σ^{-N} in (11.6.6).)

Our second step is to simplify the covariance structure among the elements of ∇f. Let Q be the orthogonal matrix which diagonalises Λ to the unit matix I, so that

(11.6.7) $$Q'\Lambda Q \;=\; \mathrm{diag}\,(1,\ldots,1)\,.$$

Note that $\det Q = (\det\Lambda)^{-1/2}$. Now take the transformation of $\mathbb{R}^N$ given by $t \to tQ$, under which $T \to T^Q = \{\tau : \tau = tQ^{-1} \text{ for some } t \in T\}$ and define $f^Q : T^Q \to \mathbb{R}$ by

$$f^Q(t) \;\overset{\Delta}{=}\; f\,(tQ)\,.$$

The new process f^Q has covariance function

$$C^Q(s,t) \;=\; C(sQ,tQ) \;=\; C((t-s)Q)$$

and so is still stationary, with constant, unit variance. Furthermore, simple differentiation shows that $\nabla f^Q = (\nabla f)Q$, from which it follows that

(11.6.8) $$\begin{aligned} \Lambda^Q \;&\overset{\Delta}{=}\; \mathbb{E}\left\{((\nabla f^Q)(t))'((\nabla f^Q)(t))\right\} \\ &=\; Q'\Lambda Q \\ &=\; I\,. \end{aligned}$$

That is, the first order derivatives of the transformed process are now uncorrelated and of unit variance. We now show that it is sufficient to work with this, much simpler, transformed process.

Firstly, it is crucial to note that the μ_k of (11.6.5) for f over T are *identical* to those for f^Q over T^Q. Clearly, there is trivial a one-one correspondence between those points of T at which $f(t) \geq u$ and those of T^Q at which $f^Q(t) \geq u$. We do, however, need to check more carefully what happens with the conditions on ∇f and $\nabla^2 f$.

Since $\nabla f^Q = (\nabla f)Q$, we have that $(\nabla f^Q)(t) = 0$ if, and only if, $\nabla f(tQ)$ $= 0$. In other words, there is also a simple one-one correspondence between critical points. Furthermore, since $\nabla^2 f^Q = Q'\nabla^2 Q$ and Q is a positive definite matrix $\nabla^2 f^Q(t)$ and $\nabla^2 f(tQ)$ have the same index.

Consequently, we can work now work with f^Q rather than f, so that by (11.1.4) the expectation (11.6.6) is given by

(11.6.9)

$$\int_{T^Q} p_t(0)\, dt \sum_{k=0}^{N}(-1)^k$$
$$\times \mathbb{E}\left\{ \left|\det\nabla^2 f^Q(t)\right| \mathbb{1}_{D_k}\left(\nabla^2 f^Q(t)\right)\mathbb{1}_{[u,\infty)}(f^Q(t)) \,\middle|\, \nabla f^Q(t)=0\right\},$$

where p_t is the density of ∇f^Q and D_k is set of square matrices of index k. Now note the following:

(i) Since f, and so f^Q, are stationary, the integrand does not depend on t, and we can ignore the t's throughout. The remaining integral then gives the Lebesgue volume of T^Q, which is simply $|\det Q|^{-1}|T| = |\Lambda|^{1/2}|T|$.

(ii) The term $p_t(0)$ is simply $(2\pi)^{-N/2}$, and so can be placed to the side for the moment.

(iii) Most importantly, on the event D_k, the matrix $\nabla^2 f^Q(t)$ has k negative eigenvalues, and so has sign $(-1)^k$. We can combine this with the factor $(-1)^k$ coming immediately after the summation sign, and so remove both it and the absolute value sign on the determinant.

(iv) Recall that from the discussion on stationarity in Section 5.4 (esp. (5.5.7)–(5.5.9)) we have the following relationships between the various derivatives of f^Q, for all $i,j,k \in \{1,\dots,N\}$.

$$\mathbb{E}\{f^Q(t)f_i^Q(t)\} = 0, \quad \mathbb{E}\{f_i^Q(t)f_{jk}^Q(t)\} = 0, \quad \mathbb{E}\{f^Q(t)f_{ij}^Q(t)\} = -\delta_{ij},$$

where δ_{ij} is the Kronecker delta. The independence of the second derivatives from all the others means that the conditioning on ∇f^Q in (11.6.9) can be ignored, and so all that remains is to evaluate

(11.6.10) $$\int_u^\infty \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,\mathbb{E}\{\det\Delta_x\}\,dx,$$

where, by (1.2.7) and (1.2.8) Δ_x is a matrix of Gaussian random variables whose elements Δ_{ij} have means $-x\delta_{ij}$ and covariances

$$\mathbb{E}\{\Delta_{ij}\Delta_{k\ell}\} = \mathbb{E}\{f_{ij}^Q(t)f_{k\ell}^Q(t)\} - \delta_{ij}\delta_{k\ell}$$
$$= \mathcal{E}^Q(i,j,k,\ell) - \delta_{ij}\delta_{k\ell}.$$

By (11.6.3), $\mathcal{E}^Q$ is a symmetric function of its parameters.

This puts us directly into the setting of Corollary 11.5.3, from which it now follows that (11.6.10) is equivalent to

$$\frac{(-1)^N}{\sqrt{2\pi}} \int_u^\infty H_N(x)\, e^{-x^2/2}\, dx \;=\; \frac{(-1)^N}{\sqrt{2\pi}}\, H_{N-1}(u)\, e^{-u^2/2},$$

by (11.5.11).

Recalling (i) (ii) and (iv) above, substituting into (11.6.9) and lifting the restriction that $\sigma^2 = 1$ gives us (11.6.6) and we are done.   □

With the proof behind us, it should now be clear why it is essentially impossible to evaluate the individual $\mathbb{E}\{\mu_k\}$. In doing so, we would have had to integrate over the various subsets $D_k \subset \mathbb{R}^{N(N+1)/2}$ of (11.6.9), and, with only the most rare exceptions, such integrals do not have explicit forms.

A carefull reading of the proof also shows that we never really used the full power of stationarity, but rather only the existence of the matrix Q of (11.6.7) and a number of relationships between f and its first and second order derivatives. Actually, given the simplicity of the final result, which depends only on σ^2 and Λ, one is tempted to conjecture that in the non-stationary case one could replace Q by a family Q_t of such matrices. The final result might then be much the same, with the term $|T|\,|\Lambda|^{1/2}$ perhaps being replaced by an integral of the form $\int_T |\Lambda_t|^{1/2} dt$ where Λ_t would be a local version of Λ, with elements $(\Lambda_t)_{ij} = \mathbb{E}\{f_i(t)f_j(t)\}$. That this is *not* the case will be shown later, when we do tackle the (far more complicated) non-stationary scenario as a corollary of results related to fields on manifolds in the following chapter. (cf. the discussion of non-stationary fields in Section 12.5.)

We can now turn to our first mean Euler characteristic computation, for which we need to set up a little notation, much of it close to that of Section 9.4. Nevertheless, since there are some slight changes, we shall write it all out again. We start with

$$T = \prod_{i=1}^{N}[0, T_i],$$

a rectangle in $\mathbb{R}^N$. As in Section 9.4, write $\mathcal{J}_k = \mathcal{J}_k(T)$ for the collection of the $2^{N-k}\binom{N}{k}$ faces of dimension k in T. As opposed to our previous conventions, we take these faces as closed. Thus, all faces in $\mathcal{J}_k$ are subsets of some face in $\mathcal{J}_{k'}$ for all $k' > k$. (For example, $\mathcal{J}_N$ contains only T while $\mathcal{J}_0$ contains the 2^N vertices of T.) Let $\mathcal{O}_k$ denote the $\binom{N}{k}$ elements of $\mathcal{J}_k$ which include the origin.

We need one more piece of notation. Take $J \in \mathcal{J}_k$. With the λ_{ij} being the spectral moments of (11.6.1), write Λ_J for the $k \times k$ matrix with elements λ_{ij}, $i, j \in \sigma(J)$.

This is enough to state the following result.

Theorem 11.6.2 *Let f be as described at the beginning of this section and T as above. For real u, let $A_u = A_u(f, T) = \{t \in T : f(t) \geq u\}$, be an excursion set, and let φ be the Euler characteristic. Then*

(11.6.11)

$$\mathbb{E}\left\{\varphi\left(A_u\right)\right\} = e^{-u^2/2\sigma^2} \sum_{k=1}^{N} \sum_{J \in \mathcal{O}_k} \frac{|J| |\Lambda_J|^{1/2}}{(2\pi)^{(k+1)/2}\sigma^k} H_{k-1}\left(\frac{u}{\sigma}\right) + \Psi\left(\frac{u}{\sigma}\right).$$

Note that the k-dimensional volume $|J|$ of any $J \in \mathcal{J}_k$ is given by $|J| = \prod_{i \in \sigma(J)} T_i$.

Since (11.6.11) and its extension to manifolds in Section 12.4 is, for our purposes, probably the single most important equation in this book, we shall take a little time to investigate some of its consequences, before turning to a proof. To do so, we first note that it simplifies somewhat if f is isotropic. In that case we have

Corollary 11.6.3 *In addition to the conditions of Theorem 11.6.2, let f be isotropic and T the cube $[0, T]^N$. If λ_2 denotes the variance of f_i (independent of i by isotropy) then*

(11.6.12)

$$\mathbb{E}\left\{\varphi\left(A_u\right)\right\} = e^{-u^2/2\sigma^2} \sum_{k=1}^{N} \frac{\binom{N}{k}T^k \lambda_2^{k/2}}{(2\pi)^{(k+1)/2}\sigma^k} H_{k-1}\left(\frac{u}{\sigma}\right) + \Psi\left(\frac{u}{\sigma}\right).$$

The simplification follows immediately from the spherical symmetry of the spectral measure in this case, which (cf. (5.7.3)) implies that each matrix $\Lambda_J = \lambda_2 I$. In fact, looking back into the proof of Lemma 11.6.1, which is where most of the calculation occurs, you can see that the transformation to the process f^Q is now rather trivial, since $Q = \lambda^{-1/2}I$ (cf. (11.6.7)). Looked at in this light, it is clear that one of the key points of the proof was a transformation that made the first derivatives of f behave as if they were those of an isotropic process. We shall see this again, but at a far more sophisticated level, when we turn to the manifold setting.

Now consider the case $N = 1$, so that T is simply the interval $[0, T]$. Then, using the definition of the Hermite polynomials given by (11.5.8), it is trivial to check that

(11.6.13) $\mathbb{E}\left\{\varphi\left(A_u(f, [0, T])\right)\right\} = \Psi(u/\sigma) + \dfrac{T\lambda_2^{1/2}}{2\pi\sigma}e^{-u^2/2\sigma^2}.$

Figure 11.6.1 gives two examples, with $\sigma^2 = 1$, $\lambda_2 = 200$ and $\lambda_2 = 1,000$.

There are at least two interesting facts that you should note from (11.6.13):

Firstly, as $u \to -\infty$, $\mathbb{E}\left\{\varphi\left(A_u\right)\right\} \to 1$. The excursion set geometry behind this is simple. Once $u < \inf_T f(t)$ we have $A_u \equiv T$, and so $\varphi\left(A_u\right) = \varphi(T)$
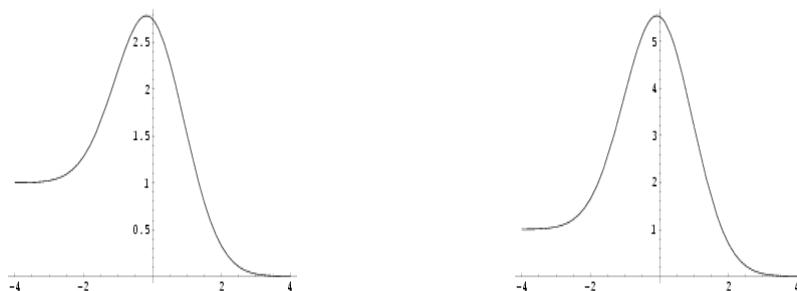
FIGURE 11.6.1. $\mathbb{E}\left\{\varphi\left(A_u\right)\right\}:\ N=1$

which, in the current case, is 1. This is, of course, a general phenomenon, independent of dimension or the topology of T.

To see this analytically, simply look at the expression (11.6.13), or even (11.6.11) for general rectangles. In both cases it is trivial that as $u \to -\infty$ all terms other than $\Psi(u/\sigma)$ disappear, while $\Psi(u/\sigma) \to 1$.

It thus seems not unreasonable to expect that when we turn to a more general theory (i.e. for T a piecewise smooth manifold) the term corresponding to the last term in (11.6.11) might be $\varphi(T)\Psi(u/\sigma)$. That this is in fact the case can be seen from the far more general results of Section 12.4 below.

Secondly, it is trivial to see that, still in dimension one,

$$\varphi\left(A_u\right)\ \equiv\ \mathbb{1}_{f_0 \geq u}\ +\ N_u^+(0, T)$$

where

$$
\begin{aligned}
N_u^+(0, T)\ &\overset{\Delta}{=}\ \#\{t \in [0, t]:\ f(t) = u,\ f'(t) > 0\} \\
&=\ \text{the number of } \textit{upcrossings} \text{ of the level } u \text{ in } [0, T].
\end{aligned}
$$

Putting this together with (11.6.13), we have recovered the famous *Rice formula*[8] for the expected number of upcrossings of a zero mean stationary process, *viz.*

$$(11.6.14)\qquad \mathbb{E}\left\{N_u^+(0, T)\right\}\ =\ \frac{T\lambda_2^{1/2}}{2\pi\sigma}e^{-u^2/2\sigma^2}.$$

---

[8] Rice's formula is really due to both Rice [122] and Kac [84] in the early 1940's, albeit in slightly different settings. Over the years it underwent significant improvement, being proven under weaker and weaker conditions. In its final form, the only requirement on f (in the current stationary, zero mean Gaussian scenario) is that its sample paths be, almost surely, absolutely continuous with respect to Lebesgue measure. This is far less than we demand, since it does even require a continuous sample path derivative, let alone a continuous second derivative. Furthermore (11.6.14) turns out to hold whether or not λ_2 is finite, whereas we have assumed $\lambda_2 < \infty$. For details see, for example, [90].

It is, by the way, hard to believe that the general 'random field version of Rice's formula' given by (11.6.11) could ever be derived without the additional levels of smoothness that we have found it necessary to assume.

Written in terms of level crossings, the rôle of the second spectral moment λ_2 becomes rather clear. For fixed variance σ^2, it follows from the spectral theory of Section 5.4 that increasing λ_2 increases the 'high frequency components' of f. In other words, the sample paths become locally much rougher. For example, Figure 11.6.2 shows realisations of the very simple Gaussian process with discrete spectrum

$$(11.6.15) \qquad \frac{a}{\sqrt{2}}\, \delta_{\pm\sqrt{200}}(\lambda) \ + \ \frac{\sqrt{1-a^2}}{\sqrt{2}}\, \delta_{\pm\sqrt{1,000}}(\lambda),$$

for $a = 0$ and $a = 5/13$, which gives second spectral moments of $\lambda_2 = 1,000$ and $\lambda_2 = 200$, respectively.

From the figure it should be reasonably clear what this means in sample path terms: higher second spectral moments correspond to an increased number of level crossings generated by local fluctuations. Similar phenomena occur also in higher dimensions, as we shall see soon.
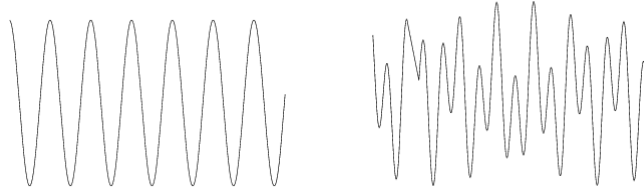

FIGURE 11.6.2. Realisations of Gaussian sample paths with the spectrum (11.6.15) and second spectral moments $\lambda_2 = 200$ (left) and $\lambda_2 = 1,000$ (right).

We now turn to two dimensions, in which case the right hand side of (11.6.12) becomes, for $\sigma^2 = 1$,

$$(11.6.16) \qquad \left[\frac{T^2\lambda_2}{(2\pi)^{3/2}}\, u \ + \ \frac{2T\lambda_2^{1/2}}{2\pi}\right] e^{-u^2/2} \ + \ \Psi(u).$$

Figure 11.6.3 gives two examples, again with $\lambda_2 = 200$ and $\lambda_2 = 1,000$.

Many of the comments that we made for the one dimensional case have similar analogues here and we leave them to you. Nevertheless, we emphasise three points:

(i) You should note, for later reference, how the expression before the exponential term can be thought of as one of a number of different power series; one in T, one in u, and one in $\sqrt{\lambda_2}$.

(ii) The geometric meaning of the negative values of (11.6.16) are worth understanding. They are due to the excursion sets having, in the mean, more holes than connected components for (most) negative values of u.
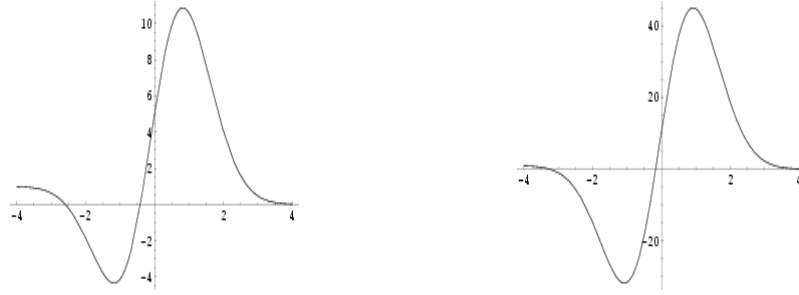
FIGURE 11.6.3. $\mathbb{E}\left\{\varphi\left(A_u\right)\right\}: \ N = 2$

(iii) The impact of the spectral moments is not quite as clear in higher
dimensions as it is in one. Nevertheless, to get a feel for what is
happening, look back at the simulation of a Brownian sheet in Fig-
ure 1.4.1. The Brownian sheet is, of course, both non-stationary and
non-differentiable, and so hardly belongs in our current setting. Nev-
ertheless, in a finite simulation, it is impossible to 'see' the difference
between non-differentiability and large second spectral moments[9], so
consider the simulation in the latter light. You can then see what is
happening. Large spectral moments again lead to local fluctuations
generating large numbers of small islands (or lakes, depending on the
level at which the excursion set is taken) and this leads to larger
variation in the values of $\mathbb{E}\{\varphi(A_u)\}$.

In three dimensions, the last case that we write out, (11.6.12) becomes,
for $\sigma^2 = 1$,

$$\left[\frac{T^3\lambda_2^{3/2}}{(2\pi)^2}\, u^2 \ + \ \frac{3T^2\lambda_2}{(2\pi)^{3/2}}\, u \ + \ \frac{3T\lambda_2^{1/2}}{2\pi} \ - \ \frac{T^3\lambda_2^{3/2}}{(2\pi)^2}\right] e^{-u^2/2} \ + \ \Psi(u).$$

Figure 11.6.4 gives the examples with $\lambda_2 = 200$ and $\lambda_2 = 1,000$.

Note that once again there are a number of different power series ap-
pearing here, although now, as opposed to the two dimensional case, there
is no longer a simple correspondence between the powers of T, $\sqrt{\lambda_2}$ and u.

The two positive peaks of the curve are due to A_u being primarily com-
posed of a number of simply connected components for large u and primar-
ily of simple holes for negative u. (Recall that in the three dimensional case
the Euler characteristic of a set is given by the number of components mi-
nus the number of handles plus the number of holes.) The negative values
of $\mathbb{E}\{\varphi(A_u)\}$ for u near zero are due to the fact that A_u, at those levels, is
composed mainly of a number of interconnected, tubular like regions; i.e.
of handles.

---

[9]Ignore the non-stationarity of the Brownian sheet, since this has no qualitative
impact on the discussion.
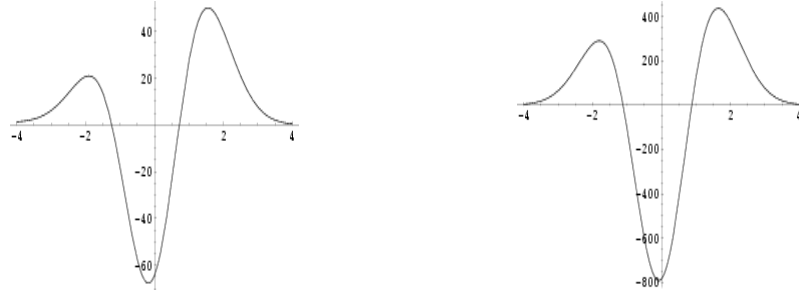
FIGURE 11.6.4. $\mathbb{E}\left\{\varphi\left(A_u\right)\right\}:\ N=3$

An example[10] is given in Figure 11.6.5 which shows a impression of typical excursion sets of a function on I^3 above high, medium and low levels.
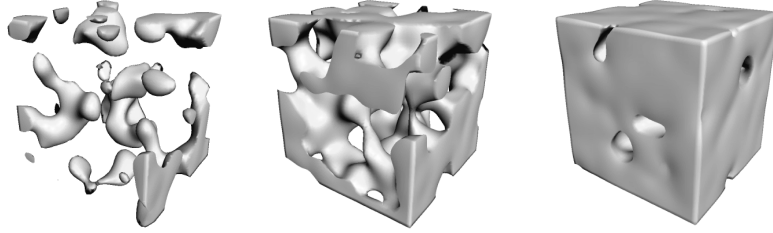
FIGURE 11.6.5. Three dimensional excursion sets above high, medium and low levels. For obvious reasons, astrophysicists refer to these three cases (from left to right) as 'meatball', 'sponge' and 'bubble' topologies.

We have one more – extremely important – observation to make before we turn to the proof of Theorem 11.6.2. Recall (6.3.5), which gave that the intrinsic volumes, or Lipschitz-Killing curvatures, of $[0,T]^N$ were given by $\binom{N}{j}T^j$. With this in mind, simplifying to the case $\sigma^2 = \lambda_2 = 1$, (11.6.12) can be written far more tidily as

$$(11.6.17) \qquad \mathbb{E}\left\{\varphi\left(A_u(f,T)\right)\right\} \;=\; \sum_{k=0}^{N} \mathcal{L}_k(T)\,\rho_k(u),$$

where

$$\rho_k(u) \;=\; (2\pi)^{-(k+1)/2}\,H_{k-1}(u)\,e^{-\frac{u^2}{2}} \quad k \ge 0,$$

since $H_{-1}(u) \;=\; \Psi(u)$, (cf.(11.5.9))

---

[10]Taken, with permission, from Keith Worsley's entertaining and illuminating *Chance* article [166].

In fact, the above holds also when T is a N-dimensional, piecewise smooth manifold, and f has constant variance. (i.e. There are no assumptions of isotropy, or even stationarity.) The Lipschitz-Killing curvatures, however, will be somewhat more complex, depending on a Riemannian metric related to f. This will be the content of Sections 12.2 and 12.4 below. Now, however, we (finally) turn to the

Proof of Theorem 11.6.2. We start by recalling that each k-dimensional face $J \in \mathcal{J}_k$ is determined by a subset $\sigma(J)$ of $\{1, \ldots, N\}$, of size k, and a sequence of $N-k$ zeroes and ones, which we write as $\varepsilon(J) = \{\varepsilon_1, \ldots, \varepsilon_{N-k}\}$, so that

$$J = \left\{ t \in \mathbb{R}^N : t_j = \varepsilon_j T_j, \text{ if } j \notin \sigma(J), \ \ 0 \le t_j \le T_j, \text{ if } j \in \sigma(J) \right\}.$$

Corresponding to each set $\varepsilon(J)$ we define a set $\varepsilon^*(J)$ of ± 1's, according to the rule $\varepsilon_j^* = 2\varepsilon_j - 1$.

Now recall[11] from Section 9.4 (cf. (9.4.1)–(9.4.5)) that the Euler characteristic characteristic of A_u is given by

$$(11.6.18) \qquad \varphi\left(A_u(f, I^N)\right) = \sum_{k=0}^{N} \sum_{i=0}^{k} (-1)^i \sum_{J \in \mathcal{J}_k} \mu_i(J),$$

where, for $i \le \dim(J)$, $\mu_i(J)$ is the number of $t \in J$ for which

$$(11.6.19) \qquad f(t) \ \ge \ u,$$
$$(11.6.20) \qquad f_j(t) \ = \ 0, \qquad\qquad j \in \sigma(J)$$
$$(11.6.21) \qquad \varepsilon_j^* f_j(t) \ \ge \ 0, \qquad\qquad j \notin \sigma(J)$$
$$(11.6.22) \qquad I(t) \ \overset{\Delta}{=} \ \text{Index}\, (f_{mn}(t))_{(m,n \in \sigma(J))} \ = \ k - i,$$

where, as usual, the index of a matrix is the number of its negative eigenvalues.

Our first and main step will hinge on stationarity, which we exploit to replace the expectation of the sum over $J \in \mathcal{J}_k$ in (11.6.18) by something simpler. Fix a particular $J \in \mathcal{O}_k$ – i.e. a face containing the origin – and let $\mathcal{P}(J)$ denote all faces in $\mathcal{J}_k$ (including J itself) which are affine translates of (i.e. parallel to) J. There are 2^{N-k} faces in each such $\mathcal{P}(J)$. We can then rewrite the right hand side of (11.6.18) as

$$(11.6.23) \qquad \sum_{k=0}^{N} \sum_{i=0}^{k} (-1)^i \sum_{J \in \mathcal{O}_k} \sum_{J' \in \mathcal{P}(J)} \mu_i(J').$$

---

[11] 'Recall' includes extending the results there from cubes to rectangles and changing the order of summation, but both these steps are trivial.

Consider the expectation of the innermost sum. By Theorem 11.1.1 (cf. (11.1.4)) we can rewrite this as

$$\sum_{J' \in \mathcal{P}(J)} \int_{J'} \mathbb{E} \left\{ \left| \det \nabla^2 f_{|J'}(t) \right| \mathbb{1}_{I(t)=k-i} \mathbb{1}_{\mathcal{E}_{J'}(t)} \Big| \nabla f_{|J'}(t) = 0 \right\} p_{\nabla f_{|J'}}(0) \, dt,$$

where $\mathcal{E}_J(t)$ denotes the event that (11.6.19) and (11.6.21) hold.

Further simplification requires one more item of notation. For $J \in \mathcal{O}_k$, let $\mathcal{P}^\varepsilon(J)$ denote the collections of all sequences of ± 1's of length $N - k$. Note that $\mathcal{P}^\varepsilon(J)$ is made up of the 2^{N-k} sequences $\varepsilon^*(J')$ with $J' \in \mathcal{P}(J)$.

With this notation, and calling on stationarity, we can replace the last expression by

$$\sum_{\varepsilon^* \in \mathcal{P}^\varepsilon(J)} \int_{J} \mathbb{E} \left\{ \left| \det \nabla^2 f_{|J}(t) \right| \mathbb{1}_{I(t)=k-i} \mathbb{1}_{\mathcal{E}_{\varepsilon^*}(t)} \Big| \nabla f_{|J}(t) = 0 \right\} p_{\nabla f_{|J}}(0) \, dt,$$

where $\mathcal{E}_{\varepsilon^*}(t)$ denotes the event that (11.6.19) and (11.6.21) hold for ε^*.

Now note the trivial fact that

$$\bigcup_{\varepsilon^* \in \mathcal{P}^\varepsilon(J)} \left\{ \varepsilon_j^* f_j(t) \ \geq \ 0 \right\}$$

is the sure event. Applying this to the last sum, we see that it simplifies considerably to

$$\int_{J} \mathbb{E} \left\{ \left| \det \nabla^2 f_{|J}(t) \right| \mathbb{1}_{I(t)=k-i} \mathbb{1}_{f(t) \geq u} \Big| \nabla f_{|J}(t) = 0 \right\} p_{\nabla f_{|J'}}(0) \, dt.$$

Going back to Theorem 11.1.1, we have that this is no more that the expected number of points in J for which

$$f(t) \geq u, \quad \nabla f(t) = 0, \quad \text{Index}(\nabla^2 f) = k - i.$$

If we call the number of points satisfying these conditions $\mu'_{k-i}(J)$, then putting all of the above together and substituting into (11.6.18) we see that we need the expectation of

$$\sum_{k=0}^{N} \sum_{J \in \mathcal{O}_k} (-1)^k \sum_{i=0}^{k} (-1)^{k-i} \mu'_{k-i}(J).$$

Lemma 11.6.1 gives us a precise expression for the expectation of the innermost sum, at least for $k \geq 1$; *viz.*

$$\frac{(-1)^k |J| \, |\Lambda_J|^{1/2}}{(2\pi)^{(k+1)/2} \sigma^k} \, H_{k-1} \left(\frac{u}{\sigma} \right) e^{-u^2/2\sigma^2}.$$

It is left to you to check that for the case $k = 0$ (i.e. when $\mathcal{J}_k$ contains the 2^N vertices of T) that the remaining term is given by $\Psi(u/\sigma)$. Putting all this together immediately gives (11.6.11) and so the proof is complete. $\square$

11.7   Mean intrinsic volumes

So far, this entire chapter has concentrated on finding the mean value of the Euler characteristic of Gaussian excursion sets. However, the Euler characteristic is only one of $N+1$ intrinsic volumes, and it would be useful to know how the remaining N behave. We shall prove, in the setting of manifolds, in Section 12.6 that

$$(11.7.1) \qquad \mathbb{E}\left\{\mathcal{L}_j(A_u(f,T))\right\} = \sum_{l=0}^{N-j} \begin{bmatrix} j+l \\ l \end{bmatrix} \rho_l(u)\mathcal{L}_{j+l}(T).$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ is a combinatorial factor defined by (12.6.5) and (12.6.6).

Although, as we noted, the proof appears only after we have moved to the manifold setting, there is essentially no difference between this and our current Euclidean one. Consequently, even if you do not want to read the rest of Chapter 12, you can still turn now to in Section 12.6 to read the proof of (11.7.1).

11.8   On the importance of stationarity

Before leaving the setting of random fields defined over N-dimensional rectangles, it is worthwhile, once again, to recall how crucial stationarity was for our ability to carry out the precise computations that lead to theorem 11.6.2.

In fact, stationarity appeared twice. We first used some consequences of stationarity (although not its full force) in making the crucial transformation to the process f^Q (cf. (11.6.7)) in the proof of Lemma 11.6.1, which is where the really detailed Gaussian calculations were made. The second time, we exploited the full power of stationarity in the proof of Theorem 11.6.2 for handling the expectation of the awkward summation of (11.6.18). At this stage, our argument also relied rather heavily on the assumption that our parameter space was a rectangle.

It should be reasonably clear that handling non-stationary fields and/or non-rectangular domains is going to require new tools. The tools for non-rectangular domains are, not surprisingly, those that will arise in the following chapter, where we turn to the setting of (Riemannian) manifolds. What is a perhaps more than a little surprising, however, are that these will be precisely the tools needed for handling non-stationary fields as well.

12
Random fields on manifolds

In essence, this chapter will repeat, for random fields on manifolds, what we have already achieved in the Euclidean setting.

As there, our first step will be to set up a 'meta-theorem' for computing the mean number of points at which a random field takes a certain value under specific side conditions. This turns out to be rather easy to do, involving little more than taking the Euclidean result and applying it, chart by chart, to the manifold. This is the content of the first section.

Actually computing the resulting expression for special cases – such as finding the mean Euler characteristic of excursion sets over Whitney stratified manifolds – turns out to be somewhat more complicated. We start the computation in Section 12.2, where, for the second time[1], we shall begin to see why we need our heavy investment in the Riemannian Geometry of Chapter 7. The main contribution of this section will be to take the parameter space on which our Gaussian process is defined, endow it with a Riemmanian metric induced by the process, and study this metric a little.

Section 12.3 is devoted to some general Gaussian computations, including the moment formulae for random Gaussian forms that we used when looking at volume of tube formulae. The real work is done in Section 12.4, which puts all the preceeding material together to devlop mean Euler characteristic formulae.

---

[1] The first was in the volume of tubes approximation of Chapter 10, which was something of an aside. It is in the current chapter that we shall see the real need for Riemannian geometry in a general setting, and not just geometry of $\mathbb{R}^N$ or $S(\mathbb{R}^N)$ with the usual Euclidean metrics.

The strength of the general approach is shown by the examples of Section 12.5 and, as we have already mentioned, we indulge ourselves somewhat with a fun proof of the Chern-Gauss-Bonnet Theorem in Section 12.7. Our proof of the Chern-Gauss-Bonnet Theorem is not completely intrinsic, as we appeal to Nash's embedding theorem [111]. This trick has uses beyond a proof of the Chern-Gauss-Bonnet Theorem, and we actually use it first in Section 12.6 to compute *all* of the mean intrinsic volumes of the excursion sets of smooth, centered, unit variance Gaussian processes.

12.1 The meta-theorem on manifolds

To formulate the meta-theorem for manifolds we need one small piece of notation.

Let (M, g) be an N-dimensional Riemannian manifold, and $f : M \to \mathbb{R}^N$ be C^1. Fix an orthonormal frame field E. Then ∇f_E denotes the vector field whose coordinates are given by

$$(12.1.1) \qquad (\nabla f_E)_i \equiv \nabla f_{Ei} \stackrel{\Delta}{=} (\nabla f)(E_i) \equiv E_i f,$$

where ∇ is the gradient operator defined at (7.3.2). If $f = (f^1, \ldots, f^N)$ takes values in $\mathbb{R}^N$, then ∇f_E denotes the $N \times N$ matrix with elements ∇f_{Ei}^j.

Theorem 12.1.1 *Let M be a compact, oriented, N-dimensional C^1 manifold with a C^1 Riemannian metric g. Let $f = (f^1, \ldots, f^N) : M \to \mathbb{R}^N$ and $h = (h^1, \ldots, h^K) : M \to \mathbb{R}^K$ be random fields on M. For an open set $B \subset \mathbb{R}^K$ for which ∂B has dimension $K - 1$ and a point $u \in \mathbb{R}^N$, let*

$$N_u \equiv N_u(M) \equiv N_u(f, h; M, B)$$

denote the number of points $t \in M$ for which

$$f(t) = u \quad and \quad h(t) \in B.$$

Assume that the following conditions are satisfied for some orthonormal frame field E:

(a) *All components of f, ∇f_E, and h are a.s. continuous and have finite variances (over M).*

(b) *For all $t \in M$, the marginal densities $p_t(x)$ of $f(t)$ (implicitly assumed to exist) are continuous at $x = u$.*

(c) *The conditional densities $p_t(x|\nabla f_E(t), h(t))$ of $f(t)$ given $h(t)$ and $(\nabla f_E)(t)$ (implicitly assumed to exist) are bounded above and continuous at $x = u$, uniformly in $t \in M$.*

(d) *The conditional densities $p_t(z|f(t) = x)$ of $\det(\nabla f^i_{Ej}(t))$ given $f(t) = x$ are continuous for z and x in neighbourhoods of 0 and u, respectively, uniformly in $t \in M$.*

(e) *The conditional densities $p_t(z|f(t) = x)$ of $g(t)$ given $f(t) = x$, are continuous for all z and for x in a neighbourhood u, uniformly in $t \in M$.*

(f) *The following moment condition holds:*

$$(12.1.2) \qquad \sup_{t \in M} \max_{1 \le i,j \le N} \mathbb{E}\left\{ \left| \nabla f^j_{Ei}(t) \right|^N \right\} < \infty.$$

(g) *The moduli of continuity with respect to metric induced by g (cf. (7.3.1)) of each component of h, each component of f and each ∇f^j_{Ei} all satisfy, for any $\varepsilon > 0$,*

$$(12.1.3) \qquad \mathbb{P}\left\{ \omega(\eta) > \varepsilon \right\} = o\left(\eta^N \right), \qquad as\ \eta \downarrow 0,$$

Then

$$(12.1.4) \quad \mathbb{E}\{N_u\} = \int_M \mathbb{E}\left\{ \left| \det\left(\nabla f_E\right) \right| \mathbb{1}_B(h) \,\middle|\, f = u \right\} p(u)\, \mathrm{Vol}_g,$$

where p is the density[2] of f and Vol_g the volume element on M induced by the metric g.

Before turning to the proof of the theorem, there are a few points worth noting. The first is that the conditions of the theorem do not depend on the choice of orthonormal frame field. Indeed, as soon as they hold for one such choice, not only will they hold for all orthonormal frame fields but also for any bounded vector field X. In the latter case the notation will change slightly, and ∇f^j_{Ei} needs to be replaced by $(Xf^j)_i$.

Once this is noted, you should note that the only place that the metric g appears in the conditions is in the definition of the neighbourhoods $B_\tau(t, h)$ in the final condition. A quick check of the proof to come will show that any neighbourhood system will in fact suffice. Thus the metric does not really play a rôle in the conditions beyond convenience.

---

[2]Of course, what is implicit here is that, for each $t \in M$, we should really write p as p_t, since it is the density of f_t. There are also a number of additional places in (12.1.4) where we could append a t, but since it has been our habit to drop the subscript when working in the setting of manifolds, we leave it out here as well.

Note that it is implicitly assumed that the integrand in (12.1.4) is a well defined N-form on M, or, equivalently, that the expectation term is a well defined Radon-Nikodym derivative. That this is the case will follow from the proof.

Furthermore, the definition of the random variable N_u is totally unrelated to the metric. From this it follows that the same must be true of its expectation. Consequently, although we require a metric to be able to define the integration in (12.1.4), the final expression must actually yield a result that is independent of the choice of g and so be a function *only* of the 'physical' manifold and the distribution of f. However, choosing an appropriate g can greatly simplify the calculation.

Proof. Since M is compact it has a finite atlas. Let (U, φ) be one of its charts and consider the random fields $\bar{f} : \varphi(U) \subset \mathbb{R}^N \to \mathbb{R}^N$ and $\bar{h} : \varphi(U) \subset \mathbb{R}^N \to \mathbb{R}^K$ defined by

$$\bar{f} \triangleq f \circ \varphi^{-1}, \qquad \bar{h} \triangleq h \circ \varphi^{-1}.$$

It is immediate from the definition of N_u that

$$N_u(f, h; U, B) \equiv N_u(\bar{f}, \bar{h}; \varphi(U), B),$$

and so the expectations of both of these random variables are also identical.

Recall the comments made prior to the proof: All conditions in the theorem that involve the orthonormal frame field E also hold for any other bounded vector field on $U \subset M$. In particular, they hold for the natural coordinate vector field $\{\partial/\partial x_i\}_{1 \leq i \leq N}$ determined by φ.

Comparing conditions (a)–(g) of the current theorem with those in Theorem 11.1.1, it is clear that $\bar{f}$ and $\bar{h}$ satisfy[3] the conditions of Theorem 11.1.1. Consequently,

$$\mathbb{E}\{N_u(f, h; U, B)\} = \int_{\varphi(U)} \mathbb{E}\left\{ \left|\det\nabla\bar{f}(x)\right| \mathbb{1}_B(\bar{h}(x)) \,\Big|\, \bar{f}(x) = u \right\} \bar{p}_x(u)\, dx,$$

where $\bar{p}_x$ is the density of $\bar{f}(x)$.

All that remains is to show that this is equivalent to (12.1.4) with the domain of integration restricted to U, and that we can replace U by M throughout. Consider the right hand side above, and rewrite it in terms of an integral over U. To this end, note that

$$\left(\nabla\bar{f}^j(x)\right)_i \;=\; \left(\frac{\partial}{\partial x_i} f^j\right)_{t=\varphi^{-1}(x)}$$

where $\partial/\partial x_i$ is the push-forward under φ^{-1} of the natural basis on $\varphi(U)$. Together with the definition of integration of differential forms in Section

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[3] The only condition that needs any checking is (11.1.2) on the moduli of continuity. It is here that the requirement that g be C^1 over M comes into play. The details are left to you.

7.4 this gives us that

$$\mathbb{E}\{N_u(f, h; U, B)\}$$
$$= \int_U \mathbb{E}\left\{ \left|\det\left(\partial/\partial x_i f^j\right)_t\right| \mathbb{1}_B(h(t)) \,\Big|\, \bar{f}(t) = u \right\} p_t(u)\,\partial x_1 \wedge \cdots \wedge \partial x_N.$$

The next step involves moving from the natural basis on U to the basis given by the orthonormal frame field E. Doing so generates two multiplicative factors, which fortunately cancel. The first comes from the move from the form $\partial x_1 \wedge \cdots \wedge \partial x_N$ to the volume form Vol_g, and generates a factor of $(\det(g_{ij}))^{-1/2}$, where $g_{ij}(t) = g_t(\partial/\partial x_i, \partial/\partial x_j)$. (cf. (7.4.9).)

The second factor comes from noting that

$$\frac{\partial}{\partial x_i} f^j = \sum_k g\left(\frac{\partial}{\partial x_i}, E_k\right) E_k f^j$$
$$= \sum_k g_{ik}^{1/2} E_k f^j,$$

where $g^{1/2} = (g(E_i, \partial x_j))_{1 \le i, j \le N}$ is a square root of the matrix $g = (g_{ij})_{1 \le i, j \le N}$. Consequently,

$$\det\left(\partial/\partial x_i f^j\right)_t = \sqrt{\det(g_{ij})}\,\det\left(\nabla f_E\right).$$

Putting the pieces together gives us

$$(12.1.5) \quad \mathbb{E}\{N_u(U)\} = \int_U \mathbb{E}\left\{ \left|\det\left(\nabla f_E\right)\right| \mathbb{1}_B(h) \,\Big|\, f = u \right\} p(u)\,\mathrm{Vol}_g,$$

for each chart (U, φ).

To finish the proof, note that for each chart (U, φ) the conditions of the theorem imply that there are only a finite number of points in $\varphi(U)$ at which $\bar{f} = u$ (cf. Lemma 11.1.13) and that there are no points of this kind on $\partial\varphi(U)$. (cf. Lemma 11.1.10.)

Consequently, the same is true of f over U. In particular, this means that we can refine a given atlas so that each point for which $f = u$ appears in only one chart and no chart contains more than one point of this kind. If this is the case, the integrals in (12.1.5) are either zero or one, and so it is trivial to combine them to obtain a single integral over M and so the theorem. □

As usual, we have the following corollary for the Gaussian case:

Corollary 12.1.2 *Let (M, g) be a Riemannian manifold satisfying the conditions of Theorem 12.1.1. Let f and h be centered Gaussian fields over M. Then if f, h and ∇f_E are a.s. continuous over M, and if, for each $t \in M$, the joint distributions of $(f(t), \nabla f_E(t), h(t))$ are non-degenerate, then (12.1.4) holds.*

Ultimately, we shall apply the above corollary to obtain, among other things, an expression for the expected Euler characteristic of Gaussian excursion sets over manifolds. Firstly, however, we need to set up some machinery.

12.2 Riemannian structure induced by Gaussian fields

Up until now, all our work with Riemannian manifolds has involved a general Riemannian metric g. Using this, back in Section 7.5 we developed a number of concepts, starting with connections and leading up to curvature tensors and shape operators, in corresponding generality.

For our purposes, however, it will turn out that, for each random field f on a piecewise C^2 manifold M, there is only one Riemannian metric that we shall need. It is induced by the random field f, which we shall assume has zero mean and, with probability one, is C^2 over M. It is defined by

$$(12.2.1) \qquad g_t\left(X_t, Y_t\right) \ \triangleq \ \mathbb{E}\left\{(X_t f) \cdot (Y_t f)\right\},$$

where $X_t, Y_t \in T_t M$, the tangent manifold to M at t.

Since the notation of (12.2.1) is rather heavy, we shall in what follows generally drop the dependence on t. Thus (12.2.1) becomes

$$(12.2.2) \qquad g(X, Y) \ = \ \mathbb{E}\{Xf\,Yf\}.$$

We shall call g the *metric induced by the random field*[4] f. The fact that this definition actually gives a Riemannian metric follows immediately from the positive semi-definiteness of covariance functions.

Note that, at this stage, there is nothing in the definition of the induced metric that relies on f being Gaussian[5]. The definition holds for any C^2 random field. Furthermore, there are no demands related to stationarity, isotropy, etc.

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[4]A note for the theoretician: Recall that a Gaussian process has associated with it a natural L^2 space which we denoted by $\mathcal{H}$ in Section 3.1. The inner product between two random variables in $\mathcal{H}$ is given by their covariance. There is also a natural geometric structure on $\mathcal{H}$, perhaps seen most clearly through orthogonal expansions of the form (3.1.6). In our current scenario, in which f is indexed by a manifold M, it is easy to see that the Riemannian structure induced on M by f (i.e. via the associated metric (12.2.2)) is no more than the pull-back of the natural structure on $\mathcal{H}$.

[5]Or even on f being C^2. The induced metric is well defined for f which is merely C^1. However, it is not possible to go much further – e.g. to a treatment of curvature – without more derivatives.

One way to develop some intuition for this metric is via the geodesic metric τ that it induces on M. Since τ is given by

$$(12.2.3) \qquad \tau(s,t) \;=\; \inf_{c \in D^1([0,1];M)_{(s,t)}} \int_{[0,1]} \sqrt{g_t(c',c')(t)}\, dt$$

(cf. (7.3.1)) it follows that the geodesic between two points on M is the curve along which the expected variance of the derivative of f is minimised.

It is obvious that g is closely related to the covariance function $C(s,t) = \mathbb{E}(f_s f_t)$ of f. In particular, it follows from (12.2.1) that

$$(12.2.4) \qquad\qquad g_t(X_t, Y_t) \;=\; X_s X_t C(s,t)\big|_{s=t}$$

Consequently, it is also obvious that the tools of Riemannian manifolds – connections, curvatures, etc. – can be expressed in terms of covariances. In particular, in the Gaussian case, to which we shall soon restrict ourselves, all of these tools also have interpretations in terms of conditional means and variances. Since these interpretations will play a crucial rôle in the extension of the results of Sections 11.5 and 11.6 to Gaussian fields over manifolds we shall now spend some time developing them.

12.2.1 Connections and curvatures

Our first step is to describe the Levi-Civita connection ∇ determined by the induced metric g. Recall from Chapter 6 that the connection is uniquely determined by Koszul's formula,

$$(12.2.5) \quad 2g(\nabla_X Y, Z) \;=\; Xg(Y,Z) + Yg(X,Z) - Zg(X,Y)$$
$$+ g(Z,[X,Y]) + g(Y,[Z,X]) + g(X,[Z,Y]).$$

where X, Y, Z are C^1 vector fields. (cf. (7.3.8).)

Since $g(X,Y) = \mathbb{E}\{Xf \cdot Yf\}$, it follows that

$$\begin{aligned} Zg(X,Y) &=& Z\mathbb{E}\{Xf \cdot Yf\} \\ &=& \mathbb{E}\{ZXf \cdot Yf + Xf \cdot ZYf\} \\ &=& g(ZX,Y) \;+\; g(X,ZY). \end{aligned}$$

Substituting this into (12.2.5) yields

$$(12.2.6) \qquad g(\nabla_X Y, Z) \;=\; \mathbb{E}\left\{(\nabla_X Y f)\,(Zf)\right\} \;=\; \mathbb{E}\left\{(XYf)\,(Zf)\right\},$$

and so we have a characterisation of the connection in terms of covariances. We shall see how to exploit this important relationship to obtain more explicit representations of ∇ when we turn to specific examples in a moment.

We now turn to the curvature tensor R of (7.5.2), given by

$$R(X,Y,Z,W) \;=\; g\left(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,\ W\right).$$

In order to also write R in terms of covariances, we recall (cf. (7.3.17) the covariant Hessian of a C^2 function f, *viz.*

(12.2.7)
$$\nabla^2 f(X,Y) \;=\; XYf \;-\; \nabla_X Yf.$$

It follows from the fact that ∇ is torsion free (cf. (7.3.6)) that $\nabla^2 f(X,Y) = \nabla^2 f(Y,X)$, and so ∇^2 is a symmetric form.

With this definition, we now prove the following useful result, which relates the curvature tensor R to covariances[6] and is crucial for later computations.

Lemma 12.2.1 *If f is a zero mean, C^2 random field on a C^3 Riemannian manifold equipped with the metric induced by f then the curvature tensor R on M is given by*

(12.2.8)
$$-2R \;=\; \mathbb{E}\left\{\left(\nabla^2 f\right)^2\right\},$$

where the square of the Hessian is to be understood in terms of the dot product of tensors developed at (7.2.3).

Proof. Note that for C^1 vector fields it follows from the definition[7] (12.2.7) that

$$\left(\nabla^2 f\right)^2 ((X,Y),(Z,W))$$
$$= 2\left[\nabla^2 f(X,Z)\nabla^2 f(Y,W) - \nabla^2 f(X,W)\nabla^2 f(Y,Z)\right]$$
$$= 2\left[(XZf - \nabla_X Zf)(YWf - \nabla_Y Wf)\right.$$
$$\left. -\ (XWf - \nabla_X Wf)(YZf - \nabla_Y Zf)\right].$$

Take expectations of this expression and exploit (12.2.6) to check (after a little algebra) that

$$\mathbb{E}\left\{\left(\nabla^2 f\right)^2 ((X,Y),(Z,W))\right\} \;=\; 2\left(\mathbb{E}[XZf \cdot YWf] - g(\nabla_X Z,\nabla_Y W)\right.$$
$$\left. -\ \mathbb{E}[XWf \cdot YZf] - g(\nabla_X W,\nabla_Y Z)\right).$$

---

[6]Keep in mind, however, that while (12.2.8) looks like it has only geometry on the left hand side and covariances on the right, the truth is a little more complicated, since ∇^2 involves the connection which depends on the metric which depends on covariances.

[7]Alternatively, apply (7.2.4), treating ∇^2 as a $(1,1)$ rather than $(2,0)$ tensor.

Now apply (7.3.7) along with (12.2.6) to see that the last expression is equal to

$$
\begin{aligned}
2 \big(X\mathbb{E}[Zf \cdot YWf] &- \mathbb{E}[Zf \cdot XYWf] - g(\nabla_X Z, \nabla_Y W) \\
&- Y\mathbb{E}\big[XWf \cdot Zf\big] + \mathbb{E}[Zf \cdot YXWf] + g(\nabla_X W, \nabla_Y Z)\big) \\
=\; 2 \big(Xg(Z, \nabla_Y W) &- g(\nabla_X Z, \nabla_Y W) - g(Z, \nabla_{[X,Y]}W) \\
&- Yg(\nabla_X W, Z) - g(\nabla_X W, \nabla_Y Z)\big) \\
=\; 2 \big(g(Z, \nabla_X \nabla_Y W) &- g(\nabla_Y \nabla_X W, Z) - g(Z, \nabla_{[X,Y]}W)\big) \\
=\; 2R\big((X,Y),(W,Z)\big) & \\
=\; -2R\big((X,Y),(Z,W)\big), &
\end{aligned}
$$

the first equality following from the definition of the Lie bracket, the second from (7.3.7), the third from the definition of the curvature tensor R and the last is trivial.

This establishes[8] (12.2.8) which is what we were after. □

12.2.2   Some covariances

Many of the Eulcidean computations of Section 11.6 were made possible as a result of convenient independence relationships between f and its first and second order derivatives. The independence of f and ∇f followed from the fact that f had constant variance, while that of ∇f and the matrix $\nabla^2 f$ followed from stationarity. Computations were further simplified by a global transformation (cf. (11.6.7)) that transformed f to isotropic.

While we shall continue to assume that f has constant variance, we no longer can assume stationarity nor find easy transformations to isotropy. However, we have invested considerable effort in setting up the geometry of our parameter space with the metric induced by f, and now we are about to start profiting from this. We start with some general computations, which require no specific assumptions.

We start with the variance function

$$
\sigma_t^2 \;=\; \mathbb{E}\big\{f_t^2\big\}.
$$

Assuming, as usual, that $f \in \mathcal{C}^2(M)$, we also have that $\sigma^2 \in \mathcal{C}^2(M)$, in which case there are no problems in changing the order of differentiation and expectation to see that, for C^1 vector fields X and Y,

$$
(12.2.9) \qquad X\sigma^2 \;=\; X\mathbb{E}\big\{f^2\big\} \;=\; 2\mathbb{E}\{f \cdot Xf\}.
$$

---

[8]If you are a stickler for detail, you may have noticed that since our assumptions only require that f is C^2, it is not at all clear that the terms $XYWf$ and $YXWf$ appearing in the derivation make sense. However, their difference, $[X,Y]Wf$, is well defined, and that is all we have really used.

Note that an important consequence of this is that if σ^2 is constant, so that $X\sigma^2 = 0$ for all X, then $f(t)$ and $Xf(t)$ are independent for all t.

Continuing in this vien, but for general σ^2, we have

$$XY\sigma^2 \;=\; 2\left(\mathbb{E}\left\{XYf \cdot f\right\} + \mathbb{E}\left\{Xf \cdot Yf\right\}\right)$$

and

$$XY\sigma^2 - \nabla_X Y\sigma^2 \;=\; 2\left(\mathbb{E}\left\{XYf \cdot f\right\} + \mathbb{E}\left\{Xf \cdot Yf\right\} - \mathbb{E}\left\{\nabla_X Yf \cdot f\right\}\right).$$

Rearranging the last line yields

$$(12.2.10) \quad \mathbb{E}\left\{\nabla^2 f(X,Y) \cdot f\right\} \;=\; -\mathbb{E}\left\{Xf \cdot Yf\right\} + \tfrac{1}{2}\nabla^2\sigma^2(X,Y)$$
$$=\; -g(X,Y) + \tfrac{1}{2}\nabla^2\sigma^2(X,Y)$$

Now note that Xf and $\nabla^2 f(Y,Z)$ are uncorrelated (and so independent in our Gaussian scenario) since

$$(12.2.11) \quad \mathbb{E}\left\{Xf \cdot \nabla^2 f(Y,Z)\right\} \;=\; \mathbb{E}\left\{Xf \cdot (YZf - \nabla_Y Zf)\right\}$$
$$=\; 0$$

by (12.2.6). You should note that this result requires no assumptions whatsoever. It is an immediate consequence of the geometry that f induces on M via the induced metric and the fact that the covariant Hessian ∇^2 incorporates this metric in its definition.

Putting all the above together gives that

$$\mathbb{E}\left\{\nabla^2 f_t \;\middle|\; f_t = x, \nabla f_t = v\right\} \;=\; -\frac{x}{\sigma_t^2}I + \frac{x}{2\sigma_t^2}\nabla^2\sigma_t^2,$$

where where I is the identity double form determined by g, defined at (7.2.9).

Assume now that f has constant variance, which we take to be 1. Then $X\sigma^2 \equiv 0$ and the last equality simplifies to give

$$(12.2.12) \qquad \mathbb{E}\left\{\nabla^2 f_t \;\middle|\; \nabla f_t = v, f_t = x\right\} \;=\; -xI.$$

The conditional variance of $\nabla^2 f$ is also easy to compute, since combining (12.2.8) and (12.2.10) gives what is perhaps the most crucial equality for the detailed computations that we shall carry out in Section 12.4, *viz.*

$$(12.2.13)$$
$$\mathbb{E}\left\{\left(\nabla^2 f - \mathbb{E}\left\{\left(\nabla^2 f\right) \;\middle|\; \nabla f = v, f = x\right\}\right)^2 \;\middle|\; \nabla f = v, f = x\right\}$$
$$=\; -(2R + I^2)$$

The above correlations change somewhat if, instead of concentrating on the covariant Hessian $\nabla^2 f$, we look at simple second derivatives. For example, it follows from (12.2.11) that

$$(12.2.14) \qquad \mathbb{E}\left\{XYf \cdot Zf\right\} \;=\; \mathbb{E}\left\{\nabla_X Yf \cdot Zf\right\}$$
$$=\; g(\nabla_X Y, Z).$$

Continuing to assume that f has unit variance, let E be an orthonormal frame field for the induced metric g. It then immediately follows from the above and the fact that $g(E_i, E_j) = \delta_{ij}$ that

$$(12.2.15) \qquad \mathbb{E}\left\{XYf_t \,\Big|\, E_k f_t = v_k, k = 1, \ldots, N,\ f_t = x\right\}$$

$$=\; -xI \;+\; \sum_{k=1}^{N} v_k g(\nabla_X Y, E_k)$$

$$=\; -xI \;+\; g(\nabla_X Y, v)$$

Now might be a good time to take some time off to look at a few examples.

12.2.3 *Gaussian fields on* $\mathbb{R}^N$

An extremely important example, which can be treated in detail without too much pain is given by the differential structure induced on a compact domain M in $\mathbb{R}^N$ by a zero mean, C^2, Gaussian field. For the moment we shall assume that M has a C^2 boundary, although at the end of the discussion we shall also treat the piecewise C^2 case.

We shall show how to explicitly compute both the curvature tensor R and the shape operator S in terms of the covariance function C, as well as traces of their powers. We shall also discuss the volume form Vol_g.

We shall not, in general, assume that f is either stationary or isotropic. In fact, one of the strengths of the manifold approach is that it handles the non-stationary case almost as easily as the stationary one.

The basis for our computations is Section 7.7, where we saw how to compute what we need after starting with a convenient basis. Not surprisingly, we start with $\{E_i\}_{1 \le i \le N}$, the standard coordinate vector fields on $\mathbb{R}^N$. This also gives the natural basis in the global chart $(\mathbb{R}^N, i)$, where i is the inclusion map. We give $\mathbb{R}^N$ the metric g induced by f.

From Section 7.7 we know that, as far as the curvature operator is concerned, everything depends on two sets of functions, the covariances

$$(12.2.16) \qquad g_{ij}(t) \;=\; g(E_{ti}, E_{tj}) \;=\; \left.\frac{\partial^2 C(r,s)}{\partial r_i \partial r_j}\right|_{(t,t)}$$

and the Christoffel symbols of the first kind,

$$(12.2.17) \qquad \Gamma_{ijk} \;\stackrel{\triangle}{=}\; g\left(\nabla_{E_i} E_j, E_k\right) \;=\; \left.\frac{\partial^3 C(r,s)}{\partial r_i \partial r_j \partial s_k}\right|_{(t,t)},$$

where the last equality is a trivial consequence of (12.2.6) and (5.5.7).

All other terms appearing in Section 7.7 are derivable from these two via either simple algebraic operations or by taking inverses and square roots of matrices. In other words, there is nothing that cannot be computed directly from the first three derivatives of the covariance function. Of course, for each example, considerable perseverance or, even better, computer algebra might come in handy to actually carry out the computations.

Nevertheless, there is one rather important case in which the expressions simplify considerably. If f is stationary, then the $g_{ij}(t)$ are actually independent of t. Consequently, it follows from (5.5.7) and the symmetry of the spectral measure that

$$(12.2.18) \qquad\qquad \Gamma_{ijk} \;\equiv\; 0, \qquad \text{for all } i, j, k$$

and so the curvature tensor, its powers and their traces are identically zero. As a consequence, most of the complicated formulae of Section 7.7 simply disappear. The isotropic situation is, of course, simpler still, since then

$$(12.2.19) \qquad\qquad g_{ij}(t) \;\equiv\; \lambda_2\, \delta_{ij},$$

where λ_2 is the variance of any first order directional derivative of f. (cf. (5.7.3).)

The computation of the shape operator S also follows from the considerations of Section 7.7. For a specific example, assume that ∂M is a C^2 manifold of co-dimension one in M. Thus the normal space to ∂M in M consists of a one dimensional vector field, which we take to be generated by an inward pointing unit normal vector field ν on ∂M.

As we saw in Section 7.7, the natural choice of basis in this setting is an orthonormal frame field $E^* = \{E_i^*\}_{1 \leq i \leq N}$ chosen so that, on ∂M, $E_N^* = \nu$. In this case, we only need to know how to compute the Γ_{jiN}^* of (7.7.13). While this is not always easy, if f is stationary and isotropic then as for the curvature tensor things do simplify somewhat. In particular, if it is possible to explicitly determine functions a_{ij} so that

$$E_{it}^* \;=\; \sum_{k=1}^{N} a_{ik}(t) E_{kt},$$

then, as in Footnote 34 of Chapter 6, we have that

$$(12.2.20) \qquad \Gamma_{jiN}^*(t) \;=\; \lambda_2 \sum_{k,l=1}^{N} a_{jk}(t)\, \frac{\partial}{\partial t_k}\big(a_{Nl}(t)\, a_{il}(t)\big),$$

so that the summation has dropped one dimension. Far more significant, however, are the facts that the information about the Riemannian structure of (M, g) is now summarized in the single parameter λ_2 and that this

information has been isolated from the 'physical' structure of ∂M inherent in the functions a_{ik}.

In fact, this can also be seen directly from the definition of the shape operator. From (12.2.19) it is also easy to check that, for any vectors X, Y,

$$g(X, Y) = \lambda_2 \langle X, Y \rangle,$$

where the right hand side denotes the Euclidean inner product of X and Y. Consequently, writing S^g for the shape operator under the induced Gaussian metric and S^E for the standard Euclidean one, we have

(12.2.21) $$S^g(X, Y) = \lambda_2 S^E(X, Y),$$

a result that will be useful for us later.

There is another scenario in which significant simplification occurs. Suppose that $A \subset \partial M$ is locally flat, in the sense that A is a subset of a $N-1$ dimensional hyperplane. In that case the $a_{jk}(t)$ are constant over A and so it follows from (12.2.20) $\Gamma^{*}_{jiN}(t) = 0$ for all $t \in A$.

The last issue that we need to look at for this class of examples is that of the form of the volume element Vol_g corresponding to the metric induced by the Gaussian field. Since our parameter space is a compact domain $M \subset \mathbb{R}^N$ we can take an atlas consisting of the single chart (M, i), where i is the identity mapping on $\mathbb{R}^N$. This being the case, it is immediate from (7.4.9) that, for any $A \subset \mathbb{R}^N$

(12.2.22) $$\int_A \text{Vol}_g = \int_A |\det \Lambda_t|^{1/2} \, dt,$$

where Λ_t is the matrix with entries

$$\lambda_t(i, j) = \mathbb{E}\left\{ \frac{\partial f}{\partial t_i} \frac{\partial f}{\partial t_j} \right\} = \left. \frac{\partial^2 C(r, s)}{\partial r_i \partial s_j} \right|_{(r,s)=(t,t)}.$$

If f is stationary, then Λ_t is independent of t and is simply the matrix of second order spectral moments that we met at (11.6.1) and (11.6.2).

If f is also isotropic, then $\Lambda_t = \lambda_2 I$, where I is the identity matrix.

12.3   Another Gaussian computation

At the core of the calculation of the expected Euler characteristic in the Euclidean case were the results of Lemma 11.5.2 and Corollary 11.5.3 about mean values of determinants of Gaussian matrices. In the manifold case we shall need a somewhat more general result.

To start, recall the discussion following (7.2.5). If we view a $N \times N$ matrix Δ as representing a linear mapping T_Δ from $\mathbb{R}^N$ to $\mathbb{R}^N$, with $\Delta_{ij} =$

$\langle e_i, T_\Delta e_j \rangle$, then Δ can also be represented as a double form $\gamma_\Delta \in \Lambda^{1,1}(\mathbb{R}^N)$, and from the discussion in Section 7.1

$$(12.3.1) \qquad \det\!\Delta \ = \ \frac{1}{N!}\,\mathrm{Tr}\left((\gamma_\Delta)^N\right).$$

Thus it should not be surprising that we now turn our attention the expectations of traces of random double forms, for which we need a little notation.

Let V be a vector space and $\mu \in \Lambda^{1,1}(V)$ a double form on V. Furthermore, let $\mathrm{Cov} : (V \otimes V) \times (V \otimes V) \to \mathbb{R}$ be bilinear, symmetric and non-negative definite. If we think of μ as a mean function and Cov as a covariance function, then we can define a random, Gaussian, 2-form W on V with mean function μ and covariance function Cov, so that

$$(12.3.2) \quad W(v_1, v_2) \ \sim \ N\big(\mu(v_1, v_2), \mathrm{Cov}((v_1, v_2), (v_1, v_2))\big)$$

for all $v_1, v_2 \in V$.

Lemma 12.3.1 *With the notation and conditions described above, and understanding all powers and products of double forms as being with respect to the double wedge product of* (7.2.3),

$$(12.3.3) \qquad \mathbb{E}\left\{W^k\right\} \ = \ \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!}{(k-2j)!j!2^j}\, \mu^{k-2j}\, C^j$$

in the sense that, for all $v_1, \ldots, v_k, v_1', \ldots, v_k' \in V$,

$$\mathbb{E}\left\{W^k\big((v_1, \ldots, v_k), (v_1', \ldots, v_k')\big)\right\}$$
$$= \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!}{(k-2j)!j!2^j}\left(\mu^{k-2j}C^j\right)\big((v_1, \ldots, v_k), (v_1', \ldots, v_k')\big),$$

where $C \in \Lambda^{2,2}(V)$ is defined by

$$C\big((v_1, v_2), (v_1', v_2')\big)$$
$$= \mathbb{E}\left\{\left(W - \mathbb{E}\{W\}\right)^2\big((v_1, v_2), (v_1', v_2')\big)\right\}$$
$$= 2\left(\mathrm{Cov}\big((v_1, v_1'), (v_2, v_2')\big) - \mathrm{Cov}\big((v_1, v_2'), (v_2, v_1')\big)\right).$$

Proof. Since it is easy to check that the standard binomial expansion works also for dot products, the general form of (12.3.3) will follow from the special case $\mu = 0$, once we show that for this case

$$(12.3.4) \qquad \mathbb{E}\left\{W^k\right\} \ = \ \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \frac{(2j)!}{j!2^j}C^j & \text{if } k = 2j. \end{cases}$$

Thus, assume now that $\mu = 0$. The case of odd k in (12.3.4) follows immediately from (11.5.1) and so we have only the even case to consider.

Recalling the definition (7.2.3) of the dot product of double forms we have that

$$W^{2j}\big((v_1,\ldots,v_{2j}),(v'_1,\ldots,v'_{2j})\big) = \sum_{\pi,\sigma \in S(2j)} \varepsilon_\pi \varepsilon_\sigma \prod_{k=1}^{2j} W(v_{\pi(k)}, v'_{\sigma(k)}),$$

where, as usual, $S(2j)$ is the symmetric group of permutations of $2j$ letters and ε_σ is the sign of the permutation σ. It then follows immediately from (11.5.2) that the expectation on the right hand side is given by

$$\frac{(2j)!}{j!} \sum_{\pi,\sigma \in S(2j)} \varepsilon_\pi \varepsilon_\sigma \prod_{k=1}^{j} \mathbb{E}\left\{ W(v_{\pi(2k-1)}, v'_{\sigma(2k-1)}) W(v_{\pi(2k)}, v'_{\sigma(2k)}) \right\},$$

where the combinatorial factor comes from the different ways of grouping the vectors $(v_{\pi(k)}, v'_{\sigma(k)})$, $1 \le k \le 2j$, into ordered pairs[9].

The last expression can be rewritten as

$$\frac{(2j)!}{j!2^j} \sum_{\pi,\sigma \in S(2j)} \varepsilon_\pi \varepsilon_\sigma \prod_{k=1}^{j} \mathbb{E}\left\{ W(v_{\pi(2k-1)}, v'_{\sigma(2k-1)}) W(v_{\pi(2k)}, v'_{\sigma(2k)}) \right.$$

$$\left. - W(v_{\pi(2k)}, v'_{\sigma(2k-1)}) W(v_{\pi(2k-1)}, v'_{\sigma(2k)}) \right\}$$

$$= \frac{(2j)!}{j!2^j} \sum_{\pi,\sigma \in S(2j)} \varepsilon_\pi \varepsilon_\sigma \prod_{k=1}^{j} C\big((v_{\pi(2k-1)}, v_{\pi(2k)}), (v'_{\sigma(2k-1)}, v'_{\sigma(2k)})\big)$$

$$= \frac{(2j)!}{j!2^j} C^j\big((v_1,\ldots,v_{2j}),(v'_1,\ldots,v'_{2j})\big),$$

which completes the proof. $\qquad\qquad\square$

The following corollary is immediate from Lemma 12.3.1 and the definition (7.2.5) of the trace operator.

Corollary 12.3.2 *With the notation and conditions of Lemma 12.3.1,*

$$(12.3.5) \quad \mathbb{E}\left\{ \mathrm{Tr}(W^k) \right\} = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!}{(k-2j)!\,j!\,2^j} \, \mathrm{Tr}\left(\mu^{k-2j} C^j \right).$$

---

[9]In comparing this with (11.5.2), note that there we had an extra summation over the groupings into *unordered* pairs rather than a simple multiplicative factor. We already have each possible grouping due to the summation over π and σ in $S(2j)$, and since we are keeping pairs ordered we also lose the factor of 2^{-j} there.

12.4   The mean Euler characteristic

We now have everything we need to undertake the task of developing an explicit formula for $\mathbb{E}\{\varphi(A_u(f,M))$ for a smooth, Gaussian random field f over a manifold M. We shall treat the cases in which M does and does not have a boundary separately, even though the first scenario is a special case of the second. Nevertheless, it is best to see the calculations, for the first time, in the simpler scenario. When we get around to adding in the all the structure of piecewise smooth manifolds the notation will become somewhat heavier, although the main result, Theorem 12.4.2, will not look very different to the non-boundary case of Theorem 12.4.1.

12.4.1   Manifolds without boundary

Throughout this section, we shall assume that M is a C^2 submanifold of a C^3 manifold. Here is the main result:

Theorem 12.4.1 *Let f be a centered, unit variance Gaussian field on an N-dimensional manifold M, and satisfying the conditions of Corollary 11.2.5. Then*

$$(12.4.1) \qquad \mathbb{E}\left\{\varphi\left(A_u\right)\right\} = \sum_{j=0}^{N} \mathcal{L}_j(M)\rho_j(u),$$

where the ρ_j are given by

$$(12.4.2) \qquad \rho_j(u) = (2\pi)^{-(j+1)/2}H_{j-1}(u)\mathrm{e}^{-\frac{u^2}{2}}, \qquad j \geq 0,$$

H_j is the j-th Hermite polynomial, given by (11.5.8) and (11.5.9), and the $\mathcal{L}_j(M)$ are the Lipschitz-Killing curvatures (7.6.1) of M, calculated with respect to the metric (12.2.2) induced by f; viz.

$$(12.4.3)\mathcal{L}_j(M,U)$$
$$= \begin{cases} \frac{(-2\pi)^{-(N-j)/2}}{(N-j)!} \int_U \mathrm{Tr}^M\left(R^{(N-j)/2}\right)\mathrm{Vol}_g & \text{if } N-j \text{ is even,} \\ 0 & \text{if } N-j \text{ is odd.} \end{cases}$$

Proof. The first consequence of the assumptions on f is that the sample functions of f are almost surely Morse functions over M. Thus Corollary 9.3.4 gives us that

$$\varphi\left(A_u\right) = \sum_{\{t\in M:\nabla f_t=0\}} (-1)^{\iota-f,M(t)}$$
$$= \sum_{k=0}^{N}(-1)^k \,\#\left\{t \in M : f_t \geq u,\, \nabla f_t = 0,\, \mathrm{index}(-\nabla^2 f_t) = k\right\}.$$

To compute the expectation here first choose an orthonormal frame field $E = (E_1, \ldots, E_N)$ for M and apply Theorem 12.1.1 $N + 1$ times. If we write the f there as f^*, then we set $f^* = \nabla f_E$ and h to be $(f, -(E_i E_j)f)$, where, to save on subscripts, we shall write $(E_i E_j f)$ to denote the matrix $(E_i E_j f)_{i,j=1,\ldots,N}$.

We read off the components of $(E_i E_j f)$ (and later of $\nabla^2 f_E$) in lexicographic order to get a vector. To avoid confusion, we write everything out once more in full:

$$\nabla f_{E,k} = E_k f, \quad (E_i E_j f)_{kl} = E_k E_l f, \quad \nabla^2 f_{E,kl} = \nabla^2 f(E_k, E_l),$$

Finally, for the k-th application of Theorem 12.1.1 we set

$$B = (u, \infty) \times A_k \triangleq (u, \infty) \times \{H : \text{index}(-H) = k\} \subset \mathbb{R} \times \text{Sym}_{N \times N},$$

where $\text{Sym}_{N \times N}$ is the space of symmetric $N \times N$ matrices. Then the second consequence of our assumptions on f is that Theorem 12.1.1 is applicable in this setting. Applying it gives

$$\mathbb{E}\{\varphi(A_u)\} = \sum_{k=0}^{N} \int_M (-1)^k \mathbb{E}\{|\det(E_i E_j f)|$$
$$\times \mathbb{1}_{A_k}(\nabla^2 f_E) \, \mathbb{1}_{(u,\infty)}(f) \,\Big|\, \nabla f = 0\} \, p_{\nabla f}(0) \, \text{Vol}_g,$$

Recall now that $\nabla^2 f(E_i, E_j) = E_i E_j f - \nabla_{E_i} E_j f$. However, conditioning on $\nabla f_E = 0$ gives $\nabla^2 f(E_i, E_j) \equiv E_i E_j f$, so that we can replace $(E_i E_j f)$ by $\nabla^2 f_E$ in the last equation above to obtain

$$\sum_{k=0}^{N} \int_M (-1)^k \mathbb{E}\left\{\left|\det\nabla^2 f_E\right| \mathbb{1}_{A_k}(\nabla^2 f_E) \, \mathbb{1}_{(u,\infty)}(f) \,\Big|\, \nabla f = 0\right\} \, p_{\nabla f}(0) \, \text{Vol}_g.$$

If we now interchange summation and integration and bracket the factor of $(-1)^k$ together with $\left|\det\nabla^2 f_E\right|$ then we can drop the absolute value sign on the latter, although there is a remaining factor of -1. This allows us to exchange expectation with summation, and the factor of $\mathbb{1}_{A_k}(\nabla^2 f_E)$ and the sum over k disappear completely.

Now recall that f has constant variance and note that since E is an orthonormal frame field (with respect to the induced metric g) the components of ∇f_E at any $t \in M$ are all independent standard Gaussians. Furthermore, as we saw in Section 12.2.2, the constant variance of f implies that they are also independent of $f(t)$. Consequently, the joint probability density of $(f, \nabla f_E)$ at the point $(x, 0)$ is simply

$$\frac{e^{-x^2/2}}{(2\pi)^{(N+1)/2}}.$$

Thus, not only is it known, but it is constant over M.

Noting all this, conditioning on f and integrating out the conditioning allows us to rewrite the above in the much simpler format

(12.4.4)

$$
\begin{aligned}
\mathbb{E}\left\{\varphi\left(A_u\right)\right\} &= (2\pi)^{-(N+1)/2} \int_u^\infty \mathrm{e}^{-x^2/2}\, dx \\
&\quad \times \int_M \mathbb{E}\left\{\det\left(-\nabla^2 f_E\right) \big| \nabla f_E = 0, f = x\right\} \mathrm{Vol}_g.
\end{aligned}
$$

Recalling the definition of the trace (cf.(12.3.1)), the innermost integrand can be written as

$$
\frac{1}{N!}\mathbb{E}\left\{\mathrm{Tr}\left((-\nabla^2 f)^N\right) \big| \nabla f_E = 0, f = x\right\}.
$$

Since $\nabla^2 f$ is a Gaussian double form, we can use Corollary 12.3.2 to compute the above expectation, once we recall (12.2.12) and (12.2.13) to give us the conditional mean and covariance of $\nabla^2 f$. These give

$$
\begin{aligned}
\frac{(-1)^N}{N!}&\mathbb{E}\left\{\mathrm{Tr}^M\left(\nabla^2 f\right)^N \big| \nabla f_E = 0, f = x\right\} \\
&= \sum_{j=0}^{\lfloor\frac{N}{2}\rfloor} \frac{(-1)^j}{(N-2j)!j!2^j}\, \mathrm{Tr}^M\left((xI)^{N-2j}(I^2 + 2R)^j\right) \\
&= \sum_{j=0}^{\lfloor\frac{N}{2}\rfloor}\sum_{l=0}^{j} \frac{(-1)^j}{j!2^j(N-2j)!}x^{N-2j}\,\mathrm{Tr}^M\left(I^{N-2l}\binom{j}{l}(2R)^l\right) \\
&= \sum_{l=0}^{\lfloor\frac{N}{2}\rfloor} \frac{(-1)^l}{l!}\mathrm{Tr}^M\left(R^l \sum_{k=0}^{\lfloor\frac{N-2l}{2}\rfloor} \frac{(-1)^k}{2^k(N-2k-2l)!k!}x^{N-2k-2l}I^{N-2l}\right) \\
&= \sum_{l=0}^{\lfloor\frac{N}{2}\rfloor} \frac{(-1)^l}{l!}\mathrm{Tr}^M(R^l)\,H_{N-2l}(x),
\end{aligned}
$$

where, in the last line, we have used (7.2.10) and the definition (11.5.8) of the Hermite polynomials.

Substituting back into (12.4.4) we conclude that $\mathbb{E}\left\{\varphi\left(A_u\right)\right\}$ is given by

$$
\begin{aligned}
\sum_{l=0}^{\lfloor\frac{N}{2}\rfloor} &\left[\int_u^\infty (2\pi)^{-(N+1)/2}H_{N-2l}(x)\mathrm{e}^{-x^2/2}\, dx\right] \frac{(-1)^l}{l!}\int_M \mathrm{Tr}^M(R^l)\mathrm{Vol}_g \\
&= \sum_{l=0}^{\lfloor\frac{N}{2}\rfloor} (2\pi)^{-(N+1)/2}H_{N-2l-1}(x)\mathrm{e}^{-u^2/2}\frac{(-1)^l}{l!}\int_M \mathrm{Tr}^M(R^l)\mathrm{Vol}_g \\
&= \sum_{j=0}^{N} \mathcal{L}_j(M)\rho_j(u),
\end{aligned}
$$

where the first equality follows from (11.5.11) and the second from the definitions (12.4.2) of the ρ_j and (12.4.3) for the $\mathcal{L}_j$, along with a little algebra.

That is, we have (12.4.1) and so the theorem.   □

12.4.2   Manifolds with boundary

We now turn to the case of manifolds with boundaries and shall incorporate, in one main result, both the results of the previous section where the manifold had no boundary, and the results of Section 11.6. There, as you will recall, the parameter space was a N-dimensional rectangle, and the Gaussian process was required to be stationary.

Thus, we return to the setting of Sections 8.1 and 9.3, and take our manifold M be a C^2, N-dimensional, Whitney stratified space embedded in a C^3 ambient manifold $\widetilde{M}$.

Theorem 12.4.2 *Let M be as above, and f, as in Theorem 12.4.1, a centered, unit variance Gaussian field on M satisfying the conditions of Corollary 11.2.5. Further, suppose that the set (9.2.1) of degenerate tangent vectors has Hausdorff dimension less than or equal to $\dim(\widetilde{M}) - 1$ and that α, the normal Morse index of M, satisfies*

$$|\alpha(\nu)| \; \leq \; K,$$

for some finite K and all $\nu \in T^\perp(M)$.

Then $\mathbb{E}\{\varphi(A_u)\}$ is again given by (12.4.1), so that

$$(12.4.5) \qquad \mathbb{E}\left\{\varphi\left(A_u\right)\right\} \;=\; \sum_{i=0}^{\dim M} \mathcal{L}_i(M)\rho_i(u),$$

with the single change that the Lipschitz-Killing curvatures $\mathcal{L}_j$ are now defined by (10.7.1), viz.

(12.4.6)

$$\mathcal{L}_k(M) \;=\; \sum_{j=k}^{\dim M} (2\pi)^{-(j-k)/2} \sum_{l=0}^{\lfloor \frac{j-k}{2} \rfloor} C(N-j, j-k-2l) \frac{(-1)^l}{l!\,(j-k-2l)!}$$

$$\times \int_{\partial_j M} \int_{S(T_t \partial_j M^\perp)} \mathrm{Tr}^{T_t \partial_j M}\left(R^l\, S^{j-k-2l}_{\nu_{N-j}}\right)$$

$$\times \alpha(\nu_{N-j})\,\mathcal{H}_{N-j-1}(d\nu_{N-j})\,\mathcal{H}_j(dt).$$

Remark 12.4.3 *If we replace the assumption in Theorem 12.4.2 that M is a Whitney stratified manifold by the assumption that it is a piecewise*

smooth space in the sense of Section 8.2, then $\alpha(\nu) \equiv \mathbb{1}_{N_t M}(-\nu)$ and so α disappears from both the conditions and the conclusion of the theorem in the sense that the integral over $S(T_t \partial_j M^\perp)$ becomes an integral over $S(N_t M)$.

Before turning to the proof of Theorem 12.4.2 we need to establish a slightly extended version of the meta-theorem Theorem 12.1.1 and its Gaussian corollary, Corollary 12.1.2. The extension is needed due to the fact that for, general Whitney stratified manifolds, Morse theory 'counts' critical points with multiplicities, given by their normal Morse index α. Since $\alpha \equiv 1$ for piecewise smooth spaces, this issue does not arise for these spaces. Consequently, if these are the cases that you care about, you can skip directly to the proof of Theorem 12.4.2 without worrying about the following result. However, if you want to see a more complete proof of Theorem 12.4.2 in its full generality, then you will have to bear with a little notation and some technicalities.

We start by adopting all the notation of Theorem 12.1.1 and adding to it a new random field $F : M \to \mathbb{R}^L$ for some finite L. Next, we take a (deterministic) function

$$\alpha : M \times \mathbb{R}^L \to \mathbb{Z}$$

and define, for $u \in \mathbb{R}^N$,

$$N_u^\alpha(f, h, F; M, B) \ \equiv \ N_u^\alpha(M) \ \equiv \ N_u^\alpha$$

by

$$N_u^\alpha(f, h, F; M, B) \ \overset{\Delta}{=} \ \sum_{t \in M:\, f(t) = u \text{ and } h(t) \in B} \alpha\left(F(t)\right),$$

assuming, of course, that the sum is over a countable number of points.

We can now state

Theorem 12.4.4 *Retaining the notation and all conditions of Theorem 12.1.1 and adopting the notation above, assume also that*

(i) *F is almost surely continuous over M.*

(ii) *For each $t \in M$, the marginal densities $p_t(x)$ of $F(t)$ (implicitly assumed to exist) are continuous in x.*

(iii) *The conditional densities $p_t(x | \nabla f_E(t), h(t), F(t))$ of $f(t)$ given $h(t)$, $(\nabla f_E)(t)$ and $F(t)$ (implicitly assumed to exist) are bounded above and continuous at $x = u$, uniformly in $t \in M$.*

(iv) *The conditional densities $p_t(z | f(t) = x)$ of $F(t)$ given $f(t) = x$ are continuous for all z and for x in a neighbourhood of u, uniformly in $t \in M$.*

(v) *The modulus of continuity of F satisfies* (12.1.3).

(vi) $|\alpha|$ *is bounded and α is piecewise constant on $M \times \mathbb{R}^L$. Furthermore, for each $t \in M$, the set of discontinuities of $\alpha(t, \cdot)$ has dimension no greater than $L - 1$ in $\mathbb{R}^L$, while for each $x \in \mathbb{R}^L$ the set of discontinuities of $\alpha(\cdot, x)$ has dimension no greater than $N - 1$ in $\mathbb{R}^N$.*

Then

$$(12.4.7) \quad \mathbb{E}\{N_u^\alpha\} = \int_M \mathbb{E}\left\{ |\det(\nabla f_E)| \, \mathbb{1}_B(h) \, \alpha(F) \, \Big| \, f = u \right\} p(u) \, \mathrm{Vol}_g.$$

Proof. The proof of the theorem is really no different from that of the proof of Theorem 12.1.1, but we are missing an analogue of its Euclidean precursor, Theorem 11.1.1, with the extra complication of the α term. However, if you had the patience to follow the details of the proof of Theorem 11.1.1, you will agree that it *really* is easy to see that under the conditions we have added the proof that we had of that theorem requires only minor changes to cover the current scenario.

Further (unnecessary) details are left to you. □

The Gaussian corollary to Theorem 12.4.4 is

Corollary 12.4.5 *Let (M, g) be a Riemannian manifold satisfying the conditions of Theorem 12.4.4. Let f, h and F be centered Gaussian fields over M and α as in the theorem. Then if f, h, ∇f_E and F are a.s. continuous over M, and if, for each $t \in M$, the joint distributions of $(f(t), \nabla f_E(t), h(t), F(t))$ are non-degenerate, then (12.4.7) holds.*

Proof of Theorem 12.4.2. In essence, although Theorem 12.4.2 subsumes many others that have gone before, it will not be hard to prove. We have already done most of the hard work in proving some of the earlier cases, and so we now face a proof that is more concerned with keeping track of notation than needing any serious new computations. Nevertheless, there is something new here, and that is the way the Morse index and integration over normal cones are handled. These two points actually take up most of the proof.

We start by recalling the setup, which implies that M has the unique decomposition

$$M = \bigcup_{j=0}^{\dim M} \partial_j M,$$

as in (9.3.5). Morse's Theorem, as given in Corollary 9.3.4 and slightly rewritten, states that

$$(12.4.8) \qquad \varphi(A_u) = \sum_{j=0}^{\dim M} \sum_{k=-K}^{K} \sum_{n=0}^{j} (-1)^n k \mu_{jkn},$$

where

$$\mu_{jkn} \triangleq \#\Big\{t \in \partial_j M : f(t) \geq u, \ \nabla f(t) \in N_t M,$$

$$\alpha(-P_{T_t \partial_j M}^\perp \nabla f(t)) = k, \ \iota_{-f, \partial_j M}(t) = n\Big\}.$$

Since for $t \in M^\circ$ the normal cone is $\{0\}$, the expectation of the $j = N$ term in (12.4.8) has already been computed, since this is the computation of the previous proof. Nevertheless, we shall rederive it, *en passant*, in what follows. For this case, however, it will be important that you recall the conventions set out in Remark 10.5.2, which we shall use freely.

Fix a j, $0 \leq j \leq N$ and choose an orthonormal frame field E so that $E_{j+1}, \ldots, E_N$ are normal to $\partial_j M$, from which it follows that $E_1 f(t), \ldots, E_N f(t)$ are independent standard Gaussians for each t.

To compute the expectation of (12.4.8) we argue much as we did in the proof of Theorem 12.4.1, applying Corollary 12.4.5 rather than Corollary 12.1.2 to allow for the new factor of α. (cf. the argument leading up to (12.4.4)).

This gives us the following, in which ν is a $(N-j)$-dimensional vector which, for convenience, we write as $(\nu_{j+1}, \ldots, \nu_N)$.

$$\mathbb{E}\left\{\sum_{j=0}^{\dim M} \sum_{k=-K}^{K} \sum_{n=0}^{j} (-1)^n k \mu_{jkn}\right\}$$

$$= (2\pi)^{-(N+1)/2} \int_u^\infty e^{-x^2/2}\, dx \int_{\partial_j M} \mathcal{H}_j(dt)$$

$$\times \int_{T_t \partial_j M^\perp} e^{-|\nu|^2/2} \theta_t^j(\nu, x) \alpha(-\nu)\, \mathcal{H}_{N-j}(d\nu),$$

where $\theta_t^j(\nu, x)$ is given by

$$\mathbb{E}\left\{\det\left(-\big(\nabla^2 f(E_m, E_n)\big)_{1 \leq n, m \leq j}\right)\right.$$

$$\left. \Big| f = x, \ E_m f = 0, 1 \leq m \leq j, \ E_m f = \nu_m, j+1 \leq m \leq N\right\}.$$

As usual, $\mathcal{H}_j$ denotes the volume (Hausdorff) measures that g induces on the $\partial_j M$ and $\mathcal{H}_{N-j}$ denotes the corresponding measure on $T_t \partial_j M^\perp$.

We need to compute a few basic expectations before we can continue. The conditional covariances of the Hessian remain the same as in the previous section, as we are still conditioning on the vector $(f, \nabla f_E)$. Specifically,

$$\mathrm{Var}\left(\nabla^2 f_{|\partial_k M} \,\Big|\, f = x, \nabla f_E = (0, \nu)\right) = -(2R + I^2),$$

where the zero vector here is of length j.

Conditional means, however, do change, and from (12.2.15) we have, for $X, Y \in C^2(T(\partial_j M))$,

$$
\mathbb{E}\left\{XYf \,\middle|\, f = x, \nabla f_E = (0, \nu)\right\}
$$
$$
= \mathbb{E}\left\{\nabla^2 f_{|\partial_j M}(X, Y) \,\middle|\, f = x, \nabla f_E = (0, \nu)\right\}
$$
$$
= -xg(X, Y) + g\left(\nabla_X Y, (0, \nu)\right)
$$
$$
= -xg(X, Y) + S_\nu(X, Y),
$$

where S is the scalar second fundamental form, given by (7.5.7) and (7.5.11). Equivalently,

$$
\mathbb{E}\left\{\nabla^2 f_{|\partial_j M} \,\middle|\, f = x, \nabla f_E = (0, \nu)\right\} = -xI + S_\nu = -(xI + S_{-\nu}).
$$

We now have all we need to evaluate $\theta_t^j(\nu, x)$, which, following the argument of the preceding section and using the above conditional expectations and variance is equal to

$$
\frac{(-1)^j}{j!}\mathbb{E}\left\{\mathrm{Tr}^{\partial_j M}\left(\nabla^2 f_{|\partial_j M}\right)^j \,\middle|\, f = x, \nabla f_E = (0, \nu)\right\}
$$
$$
= \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(-1)^k}{(j-2k)!k!2^k} \mathrm{Tr}^{\partial_j M}\left((xI + S_{-\nu})^{j-2k}\left(I^2 + 2R\right)^k\right)
$$

by Lemma 12.3.1. Rearranging somewhat, this is equal to

$$
\sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \sum_{l=0}^{k} \frac{(-1)^k}{(j-2k)!} \frac{1}{2^{k-l}l!(k-l)!} \mathrm{Tr}^{\partial_j M}\left((xI + S_{-\nu})^{j-2k}I^{2k-2l}R^l\right)
$$
$$
= \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \sum_{m=0}^{j-2k} \sum_{l=0}^{k} \frac{(-1)^k(j-2l-m)!}{(j-2k-m)!m!} \frac{1}{2^{k-l}l!(j-l)!}
$$
$$
\times x^{j-2k-m}\mathrm{Tr}^{\partial_j M}\left(S_{-\nu}^m R^l\right)
$$

by (7.2.10). Further rearrangement gives that this is the same as

$$
\sum_{m=0}^{j} \sum_{k=0}^{\lfloor \frac{j-m}{2} \rfloor} \sum_{l=0}^{k} \frac{(-1)^k (j-2l-m)!}{(j-2k-m)!m!} \frac{1}{2^{k-l}l!(k-l)!}
$$
$$
\times x^{j-2k-m} \mathrm{Tr}^{\partial_j M} \left(S_{-\nu}^m R^l \right)
$$
$$
= \sum_{m=0}^{j} \sum_{l=0}^{\lfloor \frac{j-m}{2} \rfloor} \sum_{k=l}^{\lfloor \frac{j-m}{2} \rfloor} \frac{(-1)^k (j-2l-m)!}{(j-2k-m)!m!} \frac{1}{2^{k-l}l!(k-l)!}
$$
$$
\times x^{j-2k-m} \mathrm{Tr}^{\partial_j M} \left(S_{-\nu}^m R^l \right)
$$
$$
= \sum_{m=0}^{j} \sum_{l=0}^{\lfloor \frac{j-m}{2} \rfloor} \frac{(-1)^l \mathrm{Tr}^{\partial_j M} \left(S_{-\nu}^m R^l \right)}{m!l!}
$$
$$
\times \sum_{i=0}^{\lfloor \frac{j-2l-m}{2} \rfloor} \frac{(-1)^i (j-2l-m)!}{(j-2l-m-2i)!} \frac{1}{2^i i!} x^{j-2l-m-2i}
$$
$$
= \sum_{m=0}^{j} \sum_{l=0}^{\lfloor \frac{j-m}{2} \rfloor} \frac{(-1)^l}{l!m!} \mathrm{Tr}^{\partial_j M} (S_{-\nu}^m R^l) H_{j-2l-m}(x)
$$
$$
= \sum_{m=0}^{j} \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^l}{l!(m-2l)!} \mathrm{Tr}^{\partial_j M} (S_{-\nu}^{m-2l} R^l) H_{j-m}(x)
$$
$$
= \sum_{k=0}^{j} H_k(x) \sum_{l=0}^{\lfloor \frac{j-k}{2} \rfloor} \frac{(-1)^l}{l!(j-k-2l)!} \mathrm{Tr}^{\partial_j M} (S_{-\nu}^{j-k-2l} R^l).
$$

We now fix a $t \in \partial_j M$ and concentrate on computing the expectation

$$
(2\pi)^{-N/2} \int_{T_t \partial_j M^\perp} e^{-|\nu|^2/2} \mathrm{Tr}^{\partial_j M} (S_{-\nu}^{j-k-2l} R^l) \, \alpha(-\nu) \mathcal{H}_{N-j}(d\nu)
$$
$$
= (2\pi)^{-N/2} \int_{T_t \partial_j M^\perp} e^{-|\nu|^2/2} \mathrm{Tr}^{\partial_j M} (S_\nu^{j-k-2l} R^l) \, \alpha(\nu) \mathcal{H}_{N-j}(d\nu).
$$

We essentially computed this integral in proving Lemmas 10.5.1 and 10.5.3. Specifically, by a simple generalization of Lemma 10.5.5, using the fact that α is constant along rays, i.e. that for $c > 0$

$$
\alpha(c\nu) = \alpha(\nu),
$$

the above integral can be evaluated as

$$
(2\pi)^{-N/2} \int_{T_t \partial_j M^\perp} e^{-|\nu|^2/2} \mathrm{Tr}^{\partial_j M} (S_\nu^{j-k-2l} R^l) \, \alpha(\nu) \mathcal{H}_{N-j}(d\nu)
$$
$$
= C(N-j, j-k-2l) \int_{S(T_t \partial_j M^\perp)} \mathrm{Tr}^{\partial_j M} (S_\nu^{j-k-2l} R^l) \alpha(\nu) \mathcal{H}_{N-j-1}(d\nu).
$$

Collecting all the above, we finally have that

$$
\begin{aligned}
\mathbb{E}\{\varphi(A_u)\} \;=\;& \sum_{j=0}^{\dim M} (2\pi)^{-(N+1)/2} \int_u^\infty \mathrm{e}^{-x^2/2} \sum_{k=0}^{j} H_k(x) \\
& \times \sum_{l=0}^{\lfloor \frac{j-k}{2} \rfloor} \frac{(-1)^l}{l!(j-k-2l)!} C(N-j, j-k-2l) \\
& \int_{S(T_t\partial_j M^\perp)} \mathrm{Tr}^{\partial_j M}(S_\nu^{j-k-2l} R^l)\, \alpha(\nu) \mathcal{H}_{N-j-1}(d\nu) \\
\;=\;& \sum_{k=0}^{\dim M} \rho_k(u) \times \Bigg(\sum_{j=k}^{N} \sum_{l=0}^{\lfloor \frac{j-k}{2} \rfloor} \frac{(-1)^l}{l!(j-k-2l)!} C(N-j, j-k-2l) \\
& \int_{S(T_t\partial_j M^\perp)} \mathrm{Tr}^{\partial_j M}(S_\nu^{j-k-2l} R^l)\, \alpha(\nu) \mathcal{H}_{N-j-1}(d\nu) \Bigg)
\end{aligned}
$$

after integrating out x (via (11.5.11)) and changing the order of summation.

Comparing the last expression with the definitions (12.4.2) of the ρ_k and (12.4.6) of the $\mathcal{L}_k$, the proof is complete. $\qquad\square$

12.5 Examples

With the hard work behind us, we can now look at some applications of Theorems 12.4.1, and 12.4.2. One of the most powerful implications of the formula

$$
(12.5.1) \qquad \mathbb{E}\left\{\varphi\left(A_u\right)\right\} \;=\; \sum_{j=0}^{\dim M} \mathcal{L}_j(M)\rho_j(u),
$$

is that, for any example, all that needs to be computed are the Lipschitz-Killing curvatures $\mathcal{L}_j(M)$, since the ρ_j are well defined by (12.4.2) and dependent neither on the geometry of M nor the covariance structure of f.

Nevertheless, this is not always easy and there is no guarantee that explicit forms for the $\mathcal{L}_j(M)$ exist. In fact, more often than not, this will unfortunately be the case, and one needs to turn to a computer for assistance, performing either (or often both) symbolic or numeric evaluations.

However, there are some cases that are not too hard and so we shall look at these.

Stationary fields over rectangles. This is the example that we treated in Theorem 11.6.2 via the techniques of Integral rather than Differential Geometry.

Nevertheless, since N-dimensional rectangles are definitely piecewise C^2 manifolds, we should be able to recover Theorem 11.6.2 from Theorem 12.4.2. Doing so is in fact quite easy, so we set $M = \prod_{i=1}^{N}[0, T_i]$ and, to make life even easier, assume that f has unit variance and is isotropic with the variance of its derivatives given by λ_2. Thus, what we are trying to recover is (11.6.12) with $\Lambda_J = \lambda_2 I$, where I is the identity matrix.

The first point to note is that induced Riemannian g is given by

$$g(X, Y) \;=\; \mathbb{E}\{Xf\, Yf\} \;=\; \lambda_2\langle X, Y\rangle,$$

where the last term is the simple Euclidean inner product. Thus g changes the usual flat Euclidean geometry of $\mathbb{R}^N$ only by scaling, and so the geometry remains flat.

This being the case (6.3.6) gives us the necessary Lipschitz-Killing curvatures, although each $\mathcal{L}_j$ of (6.3.6) needs to be multiplied by a factor of $\lambda_2^{j/2}$. Substituting this into (12.5.1) gives the required result; *viz.* (11.6.12).

The few lines of algebra needed along the way are left to you.

Also left to you is the non-isotropic case, which is not much harder, although you will need a slightly adapted version of (6.3.6) which allows for a metric that is a constant times the Euclidean metric on hyperplanes in $\mathbb{R}^N$, but for which the constant depends on the hyperplane. This will give Theorem 11.6.2 in its full generality.

Isotropic fields over smooth domains. In the previous example, we assumed isotropy only for convenience, and leaving the argument for the general stationary case to you was simply to save us having to do more complicated algebra.

However, once one leaves the setting of rectangles it is almost impossible to obtain simple closed form expressions for the $\mathcal{L}_j$ if f is not isotropic. To see how isotropy helps, we now take f isotropic over a compact, piecewise C^2 domain in $\mathbb{R}^N$, and also assume that $\mathbb{E}\{f^2\} = \mathbb{E}\{(\partial f/\partial t_k)^2\} = 1$, so as to save carrying through an awkward constant.

What makes the Lipschitz-Killing curvatures simpler in this case is not that their defining formula (12.4.6) changes in any appreciable way, but that the symbols appearing in it have much simpler meanings. In particular, $\mathcal{H}_k$ is no more that standard Hausdorff measure over $\partial_k M$, while $\mathcal{H}_{N-k-1}$ becomes surface measure on the unit sphere S^{N-k-1}. In other words, the $\mathcal{L}_k$ no longer carry any information related to f.

The Reimannian curvature R is now zero, and the second fundamental form S simplifies considerably. To see how this works in an example, assume that M is a C^2 domain, so that there is only one C^2 component to its boundary, of dimension $N-1$. Then $\mathcal{L}_N(M) = \mathcal{H}_N(M)$ and from (10.7.6) we have that, for $0 \leq j \leq N-1$,

$$(12.5.2) \qquad \mathcal{L}_j(M) \;=\; \frac{1}{s_{N-j}}\int_{\partial M}\mathrm{detr}_{N-1-j}(\mathrm{Curv})\,\mathcal{H}_{N-1}.$$

where $\det\mathrm{r}_j$ is given by (7.2.7) and the curvature matrix Curv is given by (10.7.7)

A simple example was given in Figure 6.2.3, for which we discussed finding, via Integral Geometric techniques, the Euler characteristic of $A_u(f, M)$, where M was a C^2 domain in $\mathbb{R}^2$. Although we found a point process representation for the $\varphi(A_u)$ we never actually managed to use Integral Geometric techniques to find its expectation. The reason is that Differential Geometric techniques work much better, since applying (12.5.2) with $N = 2$ immediately gives us the very simple result that

$$\mathcal{L}_2(M) = \mathrm{Area}(M), \qquad \mathcal{L}_1(M) = \frac{\mathrm{length}(\partial M)}{2}, \qquad \mathcal{L}_0(M) = 1,$$

which, when substituted into (12.5.1), gives the required expectation.

An historical note is appropriate here: It was the example of isotropic fields on C^2 domains which, in Keith Worsley's paper [163], was really the genesis of the manifold approach to Gaussian geometry that has been the central point of this chapter and the reason for writing this book.

Under isotropy, you should now be able to handle other examples yourself. All reduce to calculations of Lipschitz-Killing curvatures under a constant multiple of the standard Euclidean metric, and for many simple cases, such as balls and spheres, these have already been computed in Chapter 6. What is somewhat harder is the non-isotropic case.

Stationary and non-stationary fields on $\mathbb{R}^N$. It would be nice to be able to find nice formulae for non-stationary Gaussian fields over smooth domains, and even for stationary but non-isotropic fields, as we have just done for isotropic fields over smooth domains and stationary processes over rectangles. Unfortunately, although Theorems 12.4.1 and 12.4.2 allow us to do this in principle, it is not so simple to do in practice.

The basic reason for this can already be seen in the stationary scenario of Theorem 11.6.2, from which one can see that the Lipschitz-Killing curvatures for a stationary process over a rectangle $T = \left(\prod_{i=1}^{N}[0, T_i]\right)$ are given by

$$(12.5.3) \qquad \mathcal{L}_j(T) \ = \ \sum_{J \in \mathcal{O}_j} |J|\,|\Lambda_J|^{1/2}$$

the sum being over faces of dimension j in T containing the origin and the rest of the notation as in Theorem 11.6.2. What (12.5.3) shows is that there is no simple averaging over the boundary of T as there is in the isotropic case. Each piece of the boundary has its own contribution to make, with its own curvature and second fundamental form. In the case of a rectangle this is not too difficult to work with. In the case of a general domain it is not so simple.

In Section 7.7 we saw how to compute curvatures and second fundamental forms on Euclidean surfaces in terms of Christoffel symbols. In Section 12.2.3 we saw how to compute Christoffel symbols for the metric induced by f in terms of its covariance function. For any given example, these two computations need to be coordinated and then fed into definitions such as (12.4.3) and (12.4.6) of Lipschitz-Killing curvatures. From here to a final answer is a long path, often leading through computer algebra packages.

There is, however, one negative result that is worth mentioning here, since it is sufficiently anti-intuitive that it has lead many to an incorrect conjecture. As we mentioned following the proof of Lemma 11.6.1, a first guess at extending (12.5.3) to the non-stationary case would be to replace the terms $|J| \, ||\Lambda_J|^{1/2}$ there by integrals of the form $\int_J |\Lambda_t|^{1/2} \, dt$ where the elements of Λ_t are the covariances $\mathbb{E}\{f_i(t)f_j(t)\}$. Without quoting references, it is a fact that this has been done more than once in the past. However, it is clear from the computations of the Christoffel symbols in Section 12.2.3 that this does not work, and additional terms involving third order derivatives of the covariance function enter into the computation. The fact that these terms are all identically zero in the stationary case is probably what lead to the errors[10].

Stationary fields over Lie groups. Lie groups provde an example which, while perhaps a little abstract, still yields a simple form for the expected Euler characteristic of excursion sets. We first met Lie groups back in Section 5.3, where we discussed them in the framework of stationarity.

Recall that a Lie group G is a group that is also a C^∞ manifold, such that the map taking g to g^{-1} is C^∞ and the map taking (g_1, g_2) to $g_1 g_2$ is also C^∞. We need a little more notation than we did in Section 5.3. We denote the identity element of G by e, the left and right multiplication maps by L_g and R_g, and the inner automorphism of G induced by g by $I_g = L_g \circ R_g^{-1}$.

Recall that a vector field X on G is said to be *left invariant* if for all $g, g' \in G$, $(L_g)_* X_{g'} = X_{gg'}$. Similarly, a covariant tensor field ω is said to be left invariant (right invariant) if, for every g_0, g in G, $L_{g_0}^* \omega_{g_0 g} = \omega_g$ ($R_{g_0}^* \omega_{g g_0} = \Phi_g$). As usual, ω is said to be *bi-invariant* if it is both left and right invariant. If h is a (left, right, bi-)invariant Riemannian metric on G, then it is clear that, for every g, the map (L_g, R_g, I_g) is an isometry[11] of (G, h). In particular, the curvature tensor R of h, is (left, right, bi-)invariant. This means that for Gaussian random fields that induce such

---

[10]On the other hand, if one thinks of the expression for $\mathbb{E}\{\varphi(A_u)\}$ as $e^{-u^2/2\sigma^2}$ multiplied by a power series in the level u, then it *is* correct to say that this conjecture gives the correct coefficient for the leading term of the power series.

[11]An *isometry* between two C^k Riemannian manifolds (M, g) and $(\hat{M}, \hat{g})$ is a C^{k+1} diffeomorphism $F : M \to \hat{M}$ for which $F^* \hat{g} = g$.

Riemannian metrics, the integrals needed to evaluate $\mathbb{E}\{\varphi(A_u(f,M))\}$ are significantly easier to calculate.

Theorem 12.5.1 *Let G be a compact N-dimensional Lie group and f a centered, unit variance Gaussian field on G satisfying the conditions of Corollary 11.2.5 for G. Suppose that the Riemannian metric g induced by f is (left, right, bi)-invariant. Then*

(12.5.4)

$$\mathbb{E}\{\,\varphi(A_u(f,M))\} \;=\; \mathrm{Vol}_g(G)\sum_{k=0}^{\lfloor\frac{N}{2}\rfloor}\frac{(-1)^k\rho_{N-2k}(u)}{(2\pi)^k k!}\,\mathrm{Tr}^{T_eG}(R_e^k),$$

where T_eG is the tangent space to G at e.

Proof. This is really a corollary of Theorem 12.4.1, since G has no boundary. Applying that result, and comparing (12.5.4) with (12.4.1), it is clear that all we need to show is that

$$\int_G \mathrm{Tr}^G\left(R^l\right)_{g'}\,\mathrm{Vol}_g(dg') \;=\; \mathrm{Tr}^{T_eG}\left(R_e^l\right)\,\mathrm{Vol}_g(G).$$

Suppose X,Y,Z,W are left-invariant vector fields. Since $g'\mapsto L_gg'$ is an isometry for every g, we have,

$$
\begin{aligned}
R_g\left((X_g,Y_g),(Z_g,W_g)\right)& \\
&= R_e\left((L_{g^{-1}*}X_g,L_{g^{-1}*}Y_g),(L_{g^{-1}*}Z_g,L_{g^{-1}*}W_g)\right)\\
&= R_e\left((L_{g*})^{-1}X_g,(L_{g*})^{-1}Y_g,(L_{g*})^{-1}Z_g,(L_{g*})^{-1}W_g\right)\\
&= R_e\left(X_e,Y_e,Z_e,W_e\right).
\end{aligned}
$$

Therefore, if $(X_i)_{1\leq i\leq n}$ is an orthonormal set of left-invariant vector fields,

$$
\begin{aligned}
(R_g)^l\left((X_{i_1g},\dots,X_{i_lg}),(X_{j_1g},\dots,X_{j_lg})\right)& \\
&= (R_e)^l\left((X_{i_1e},\dots,X_{i_le}),(X_{j_1e},\dots,X_{j_le})\right),
\end{aligned}
$$

from which it follows that,

$$\mathrm{Tr}^{T_gG}\left((R_g)^l\right) = \mathrm{Tr}^{T_eG}\left((R_e)^l\right),$$

which completes the proof. □

12.6   Mean intrinsic volumes

In this section, we shall use a simple embedding technique in order to derive a formula for all of the mean intrinsic volumes of the excursion

sets of a smooth process. For an alternative approach to computing all mean intrinsic volumes see [167]. The proof will rely on two very useful formulae from classical integral geometry: Hadwiger's formula (12.6.7) and its generalization, Crofton's formula (12.6.8) [86, 130]. Before spelling out these formulae, we review the idea of the *affine* Grassmanian manifold $\mathrm{Graff}(n, k)$.

In Chapters 7 and 8 we referred briefly to the Grassmanian manifolds $\mathrm{Gr}(n, k)$: the set of k-dimensional subspaces of $\mathbb{R}^n$. The affine Grassmanian $\mathrm{Graff}(n, k)$ is the set of all k-dimensional flats in $\mathbb{R}^n$: that is, all linear spaces in $\mathbb{R}^n$ that do not necessarily pass through the origin. The affine Grassmanian is diffeomorphic to $\mathrm{Gr}(n, k) \times \mathbb{R}^n$: for $V \in \mathrm{Graff}(n, k)$, let $p(V)$ be the unique point in V closest to the origin and let $\tilde{V}(V) = V - p(V) \in \mathrm{Gr}(n, k)$ be the translation of V to a subspace of $\mathbb{R}^n$. The space $\mathrm{Graff}(n, k)$ is a homogeneous space, being a quotient space of G_n, the set of rigid motions on $\mathbb{R}^n$. Hence it inherits a measure λ_k^n invariant under G_n. This measure factors as the invariant measure, say ν_k^n, on $\mathrm{Gr}(n, k)$ and Lebesgue measure on $\mathbb{R}^n$ under the diffeomorphism

$$V \;\mapsto\; (\tilde{V}(V), p(V)).$$

There are of course many ways to normalize ν_k^n, and hence λ_k^n. One useful normalization (useful because of Hadwiger's (12.6.7) and Crofton's (12.6.7) formulae below) is the following

$$(12.6.5) \qquad \nu_k^n(\mathrm{Gr}(n, k)) \;=\; \begin{bmatrix} n \\ k \end{bmatrix} \;\triangleq\; \frac{[n]!}{[k]!\,[n-k]!},$$

where

$$(12.6.6) \qquad\qquad [n]! \;=\; n!\,\omega_n.$$

The numbers $\begin{bmatrix} n \\ k \end{bmatrix}$ are referred to as the *flag coefficients* [86].

With this normalization, we are now ready to state Hadwiger's formula[12]

$$(12.6.7) \qquad \mathcal{L}_k(M) \;=\; \int_{\mathrm{Graff}(n, n-k)} \mathcal{L}_0(M \cap V)\, d\lambda_k^n(V),$$

and Crofton's formula [86, 130]

$$(12.6.8) \qquad \int_{\mathrm{Graff}(n, n-k)} \mathcal{L}_j(K \cap V)\, d\lambda_k^n(V) \;=\; \begin{bmatrix} k+j \\ j \end{bmatrix} \mathcal{L}_{k+j}(K).$$

We are also now ready to prove the following

---

[12] Actually, in [86], the relation (12.6.7) is only derived for the convex ring. However, by additivity, (12.6.7) holds for more than just sets K in the convex ring. It holds over any lattice for which the intrinsic volumes are valuations.

Theorem 12.6.1 *Suppose that $(M, \mathcal{Z})$ is a stratified Whitney space and there exists an isometric embedding $i : \widehat{M} \to \mathbb{R}^n$ of $\widehat{M}$ such that, for every $0 \leq k \leq n$ the following holds: for λ_k^n-almost every $V \in \mathrm{Graff}(n,k)$ the set $i(M) \cap V$ is a stratified Whitney space with*

$$\mathcal{L}_j(i(M) \cap V) < \infty, \qquad 0 \leq j \leq k.$$

Then, for every $0 \leq j \leq \dim(M)$

$$(12.6.9) \qquad \mathbb{E}\left\{\mathcal{L}_j(A_u(f, M))\right\} \; = \; \sum_{l=0}^{\dim M - j} \begin{bmatrix} j+l \\ l \end{bmatrix} \rho_l(u) \mathcal{L}_{j+l}(M).$$

Proof. By Hadwiger's formula

$$
\begin{aligned}
\mathcal{L}_j(A_u(f, M)) \; &= \; \mathcal{L}_j\big(i(A_u(f, M))\big) \\
&= \; \int_{\mathrm{Graff}(n, n-j)} \mathcal{L}_0\big(i(A_u(f, M)) \cap V\big) \, d\lambda_{n-j}^n(V)
\end{aligned}
$$

Hence, by Fubini's and Crofton's formulae and Theorem 12.4.2

$$
\begin{aligned}
\mathbb{E}&\left\{\mathcal{L}_j(A_u(f, M))\right\} \\
&= \int_{\mathrm{Graff}(n, n-j)} \mathbb{E}\left\{\mathcal{L}_0\big(i(A_u(f, M)) \cap V\big)\right\} \, d\lambda_{n-j}^n(V) \\
&= \int_{\mathrm{Graff}(n, n-j)} \mathbb{E}\left\{\mathcal{L}_0\big((i(M) \cap V) \cap A_u(f \circ i^{-1}, i(M))\big)\right\} \, d\lambda_{n-j}^n(V) \\
&= \sum_{l=0}^{n-j} \rho_l(u) \int_{\mathrm{Graff}(n, n-j)} \mathcal{L}_l(i(M) \cap V) \, d\lambda_{n-j}^n(V) \\
&= \sum_{l=0}^{n-j} \begin{bmatrix} j+l \\ l \end{bmatrix} \rho_l(u) \mathcal{L}_{j+l}(i(M)) \\
&= \sum_{l=0}^{\dim M - j} \begin{bmatrix} j+l \\ l \end{bmatrix} \rho_l(u) \mathcal{L}_{j+l}(M).
\end{aligned}
$$

$\square$

12.6.1 Hadwiger / Crofton formulae for processes

Theorem 12.6.2 *Let $y^1, \ldots, y^k$ be IID centered, unit variance, suitably regular Gaussian process on $\widetilde{M}$. For $u \in \mathbb{R}^k$ define*

$$D_u = \left\{ t \in \widetilde{M} : y_{1,t} = u_1, \ldots, y_{k,t} = u_k \right\}$$

and suppose $Z_k \sim \gamma_{\mathbb{R}^k}$ independent of the field $y = (y^1, \ldots, y^k)$. Define a functional Ψ_k

$$M \overset{\Psi_k}{\mapsto} \mathbb{E}\left\{\chi(M \cap D_{Z_k})\right\}.$$

Then,

$$\Psi_k(\cdot) = [k]! \mathcal{L}_k(\cdot).$$

Further, define the functionals

$$M \overset{\Psi_k^j}{\mapsto} \mathbb{E}\left\{\mathcal{L}_j(M \cap D_{Z_k})\right\}.$$

Then,

$$\Psi_k^j(\cdot) = \frac{[k]!}{[j]!} \mathcal{L}_{k+j}(\cdot).$$

Proof: We start with the evaluation of $\Psi_k(\widetilde{M})$. We must evaluate

$$\int_{\mathbb{R}^k} \mathbb{E}\left(\chi(\widetilde{M} \cap D_u)\right) \gamma_{\mathbb{R}^k}(du).$$

Define $\tilde{y}$, another copy of y_1, independent of $y_1, \ldots, y_k$. We will compute the expected Euler characteristic using $\tilde{y}$ as a Morse function on $\widetilde{M} \cap D_u$. The meta-theorem, and similar calculations to those in Theorem 12.4.1 allow us to conclude that

$$\mathbb{E}\left\{\chi\left(\widetilde{M} \cap D_u\right)\right\}$$

$$= \int_{\widetilde{M}} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n \omega_n} \frac{1}{(n-k)!} \mathbb{E}\left\{\mathrm{Tr}^{L_t^{\perp}}(\nabla^2 \tilde{y}_{|D_u,t}^{n-k}) J_t \mathbb{1}_{\{(y_t, P_{L_t}^{\perp} \nabla \tilde{y}_t) \in B_n(0,\varepsilon)\}}\right\} \mathcal{H}_n(dt)$$

$$= \int_{\widetilde{M}} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n \omega_{n-k} \omega_k} \frac{1}{(n-k)!} \mathbb{E}\left\{\mathrm{Tr}^{L_t^{\perp}}(\nabla^2 \tilde{y}_{|D_u,t}^{n-k}) J_t\right.$$

$$\left. \mathbb{1}_{\{y_t \in B_k(u,\varepsilon)\}} \mathbb{1}_{\{P_{L_t}^{\perp} \nabla \tilde{y}_t \in B_{n-k}(0,\varepsilon)\}}\right\} \mathcal{H}_n(dt)$$

where

$$J_t = \sqrt{\det\left(\langle \nabla y_{i,t}, \nabla y_{j,t}\rangle\right)_{1 \leq i,j \leq k}}$$

$$\overset{\Delta}{=} \sqrt{g_{ij,t}}$$

$$L_t = \mathrm{span}\left(\nabla y_{i,t}, 1 \leq i \leq k\right)$$

$$\nabla^2 \tilde{y}_{|D_u,t} \overset{\Delta}{=} \nabla^2 \tilde{y}_t + \sum_{i,j=1}^{k} \langle \nabla \tilde{y}_t, \nabla y_{i,t}\rangle g^{ij,t} \nabla^2 y_{j,t}$$

$$\overset{\Delta}{=} \nabla^2 \tilde{y}_t + \sum_{j=1}^{k} a_{j,t} \nabla^2 y_{j,t}$$

We proceed by evaluating the above expectation by conditioning on nested sigma algebras. We first condition on

$$\mathcal{F}_t = \sigma(y_t, y_{*,t}, \tilde{y}_t, P_{L_t}\nabla\tilde{y}_t).$$

Conditional on $\mathcal{F}_t$, $\nabla^2\tilde{y}_{|D_u,t}$ is a Gaussian double form on $T_t\widetilde{M}$. Further, it is conditionally independent of $P_{L_t}^\perp\nabla\tilde{y}_t$ and has conditional mean and variance

$$\mathbb{E}\left\{\nabla^2\tilde{y}_{|D_u,t}\big|\mathcal{F}_t\right\} = \left(\tilde{y}_t - \sum_{j=1}^k a_{j,t}y_{j,t}\right)\cdot I$$

$$\stackrel{\Delta}{=} I\cdot A_t$$

$$\mathbb{E}\left\{\left(\nabla^2\tilde{y}_{|D_u,t} - \mathbb{E}\left\{\nabla^2\tilde{y}_{|D_u,t}\big|\mathcal{F}_t\right\}\right)^2\big|\mathcal{F}_t\right\} = (-2R + I^2)\cdot\left(1 + \sum_{j=1}^k a_{j,t}^2\right)$$

$$\stackrel{\Delta}{=} (-2R + I^2)\cdot B_t.$$

Putting these facts together,

$$\mathbb{E}\left\{\chi\left(\widetilde{M}\cap D_u\right)\right\}$$

$$= \sum_{i=0}^{\lfloor\frac{n-k}{2}\rfloor} \frac{1}{(n-k-2i)!i!2^i} \int_{\widetilde{M}} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^n\omega_{n-k}\omega_k}\mathbb{E}\left(\mathrm{Tr}^{L_t^\perp}((-2R + I^2)^i I^{n-k-2i})\right.$$

$$\left.\times A_t^{n-k-2i}B_t^i J_t\mathbb{1}_{\{y_t\in B_k(u,\varepsilon)\}}\gamma_{\mathbb{R}^{n-k}}(B_{n-k}(0,\varepsilon))\right)\mathcal{H}_n(dt).$$

where we have used the conditional independence of $P_{L_t}^\perp\nabla\tilde{y}_t$ and $\nabla^2\tilde{y}_{|D_u,t}$ as well as the fact that

$$\mathbb{E}\left(\mathbb{1}_{\{P_{L_t}^\perp\nabla\tilde{y}_t\in B_{n-k}(0,\varepsilon)\}}\Big|\mathcal{F}_t\right) = \gamma_{\mathbb{R}^{n-k}}(B_{n-k}(0,\varepsilon)).$$

This follows from the fact that, conditional on the subspace $L_t\in\mathcal{F}_t$

$$P_{L_t}^\perp\nabla\tilde{y}_t \sim \gamma_{L_t^\perp}.$$

The next sigma algebra we condition on is

$$\mathcal{G}_t = \sigma\left(\tilde{y}_t, y_t, g_{ij,t}, 1\le i,j\le k\right).$$

Applying Lemma 12.6.3, proved below, to the above expression, using the fact that the subspace $L_t^\perp$ has dimension $n - k$ and is independent of $\mathcal{G}_t$. We see that

$$\mathbb{E}\left(\mathrm{Tr}^{L_t^\perp}((-2R + I^2)^i I^{n-k-2i})A_t^{n-k-2i}B_t^i J_t\mathbb{1}_{\{y_t\in B_k(u,\varepsilon)\}}\Big|\mathcal{G}_t\right)$$

$$= \binom{n}{k}^{-1}\mathrm{Tr}^{T_t\widetilde{M}}((-2R + I^2)^i I^{n-k-2i})A_t^{n-k-2i}B_t^i J_t\mathbb{1}_{\{y_t\in B_k(u,\varepsilon)\}}.$$

Using this information, the expression reduces to

$$
\mathbb{E}\left\{\chi\left(\widetilde{M}\cap D_u\right)\right\}
$$

$$
= \sum_{i=0}^{\lfloor\frac{n-k}{2}\rfloor} \frac{1}{(n-k-2i)!i!2^i} \binom{n}{k}^{-1} \left(\int_{\widetilde{M}} \mathrm{Tr}^{T_t\widetilde{M}}((-2R+I^2)^i I^{n-k-2i})\,\mathcal{H}_n(dt)\right)
$$

$$
\times \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^n \omega_{n-k}\omega_k}\mathbb{E}\left\{A_t^{n-k-2i}B_t^i J_t \mathbb{1}_{\{y_t\in B_k(u,\varepsilon)\}}\gamma_{\mathbb{R}^{n-k}}(B_{n-k}(0,\varepsilon))\right\}
$$

Further, sending $\varepsilon \to 0$ as well as applying (7.2.10), (7.6.1) and (10.5.10)

$$
\int_{\mathbb{R}^k}\mathbb{E}\left\{\chi\left(\widetilde{M}\cap D_u\right)\right\}\gamma_{\mathbb{R}^k}(du)
$$

$$
= (2\pi)^{-(n-k)/2}\sum_{i=0}^{\lfloor\frac{n-k}{2}\rfloor} \frac{(n-2i)!}{k!(n-k-2i)!i!2^i}\binom{n}{k}^{-1}\left(\int_{\widetilde{M}}\mathrm{Tr}^{T_t\widetilde{M}}((-2R+I^2)^i)\,\mathcal{H}_n(dt)\right)
$$

$$
\times\ \mathbb{E}\left\{A_t^{n-k-2i}B_t^i J_t\right\}
$$

$$
= \sum_{i=0}^{\lfloor\frac{n-k}{2}\rfloor}\mathcal{L}_{n-2i}^1(\widetilde{M})(2\pi)^{-(n-k-2i)/2}\frac{(n-2i)!}{k!(n-k-2i)!}\binom{n}{k}^{-1}
$$

$$
\times\ \mathbb{E}\left\{A_t^{n-k-2i}B_t^i J_t\right\}.
$$

We have also use the fact that the joint distribution of (A_t, B_t, J_t) does not depend on t.

The final step in the conditioning argument is to condition on

$$
\mathcal{J}_t = \sigma(g_{ij,t}).
$$

From the definitions of A_t, B_t

$$
A_t\big|\mathcal{J}_t \sim N(0, B_t),
$$

it follows that

$$
\mathbb{E}\left\{A_t^{n-k-2i}B_t^i J_t\right\} = \begin{cases} \frac{(2j)!}{j!2^j}\mathbb{E}\left\{B_t^{(n-k)/2}J_t\right\} & n-k-2i=2j \text{ is even} \\ 0 & n-k-2i \text{ is odd.} \end{cases}
$$

Assume, then that $n - k$ is even. It follows that

$$\int_{\mathbb{R}^k} \mathbb{E}\left\{ \chi\left(\widetilde{M} \cap D_u \right) \right\} \gamma_{\mathbb{R}^k}(du)$$

$$= \frac{(n-k)!}{n!} \mathbb{E}\left\{ B_t^{(n-k)/2} J_t \right\} \sum_{i=0}^{\lfloor \frac{n-k}{2} \rfloor} \mathcal{L}_{n-2i}^1(\widetilde{M})(4\pi)^{-(n-k-2i)/2} \frac{(n-2i)!}{\left(\frac{n-k-2i}{2}\right)!}$$

$$= \frac{(n-k)!}{n!} \mathbb{E}\left\{ B_t^{(n-k)/2} J_t \right\} \sum_{i=0}^{\lfloor \frac{n-k}{2} \rfloor} \mathcal{L}_{k+2i}^1(\widetilde{M})(4\pi)^{-i} \frac{(k+2i)!}{i!}$$

$$= \frac{(n-k)!}{n!} \mathbb{E}\left\{ B_t^{(n-k)/2} J_t \right\} \mathcal{L}_k(\widetilde{M}).$$

We have yet to prove that

$$\frac{(n-k)!}{n!} \mathbb{E}\left\{ B_t^{(n-k)/2} J_t \right\} = [k]!.$$

Before doing so, we describe the general case for M a stratified Whitney subspace of $\widetilde{M}$, for which very little changes above. One of the more important things to note below is that the normal Morse index is independent of $(y_t, y_{*,t})$. That said, similar arguments to those appearing in Theorem 12.4.2, along with the meta-theorem show that

$$\mathbb{E}\left\{ \chi\left(M \cap D_u \right) \right\}$$

$$= \sum_{l=k}^{n} \int_{\partial_l M} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^l \omega_{l-k} \omega_k} \frac{1}{(l-k)!} \mathbb{E}\left\{ \alpha\left(\nu_t \right) \mathrm{Tr}^{L_t^\perp} (\nabla^2 \tilde{y}_{|D_u,t}^{l-k}) J_t \right.$$

$$\left. \mathbb{1}_{\{y_t \in B_k(u,\varepsilon)\}} \mathbb{1}_{\{P_{L_t}^\perp \nabla \tilde{y}_t \in B_{l-k}(0,\varepsilon)\}} \right\} \mathcal{H}_l(dt)$$

where

$$J_t = \sqrt{\det\left(\langle P_{T_t \partial_l M} \nabla y_{i,t}, P_{T_t \partial_l M} \nabla y_{j,t} \rangle \right)_{1 \le i,j \le k}}$$

$$\stackrel{\Delta}{=} \sqrt{g_{ij,t}}$$

$$P_{L_t}^\perp = P_{T_t \partial_l M} - P_{L_t}$$

$$\nu_t = P_{T_t \partial_l M}^\perp \nabla \tilde{y}_t$$

$$L_t = \mathrm{span}\left(P_{T_t \partial_l M} \nabla y_{i,t}, 1 \le i \le k \right)$$

$$\nabla^2 \tilde{y}_{|D_u,t} \stackrel{\Delta}{=} \nabla^2 \tilde{y}_t + \sum_{i,j=1}^{k} \langle \nabla \tilde{y}_t, \nabla y_{i,t} \rangle g^{ij,t} \nabla^2 y_{j|\partial_l M,t} + S_{\nu_t}$$

$$\stackrel{\Delta}{=} \nabla^2 \tilde{y}_t + \sum_{j=1}^{k} a_{j,t} \nabla^2 y_{j,t} + S_{\nu_t}.$$

Above, S. is, as usual, the second fundamental form of $\partial_l M$ as it sits in $\widetilde{M}$.
We proceed to condition on

$$\mathcal{F}_t = \sigma\left(y_t, y_{*,t|T_t\partial_l M}, \tilde{y}_t, P_{T_t\partial_l M}\nabla\tilde{y}_t, P_{L_t}\nabla\tilde{y}_t\right).$$

INCOMPLETE

Lemma 12.6.3 *Let $(V, \langle , \rangle)$ be an inner product space with $\dim(V) = n$. Suppose that $L \subset V$ is a uniformly distributed random subspace of V of dimension k, i.e. $gL \stackrel{D}{=} L$ for all orthogonal transformations g of V. Then, for $\alpha \in \bigwedge^{j,j}(V)$*

$$\mathbb{E}\left\{\operatorname{Tr}^L(\alpha)\right\} = \begin{cases} \frac{\binom{k}{j}}{\binom{n}{j}}\operatorname{Tr}^V(\alpha) & j \leq k \\ 0 & otherwise. \end{cases}$$

Proof: By "invariance" it is clear that

$$\mathbb{E}\left\{\operatorname{Tr}^L(\alpha)\right\} = c_j\operatorname{Tr}^V(\alpha),$$

and it is enough to determine c_j. As $\bigwedge^{*,*}(L) = \oplus_{l=0}^k \bigwedge^{l,l}(L)$, if $j > k$ it is clear that $\operatorname{Tr}^L(\alpha) = 0$. For $j \leq k$, it is sufficient to use the special case $\alpha = I^j$, for which

$$\operatorname{Tr}^V(I^j) = \frac{n!}{(n-j)!}, \qquad \operatorname{Tr}^L(I^j) = \frac{k!}{(k-j)!}.$$

$\square$

12.7 Chern-Gauss-Bonnet Theorem

As promised at the beginning of the chapter, we now give a purely probabilistic proof of the classical Chern-Gauss-Bonnet Theorem. Of course, 'purely' is somewhat of an overstatement, since the results on which our proof is based were themselves based on Morse's critical point theory.

The Chern-Gauss-Bonnet Theorem is one of the most fundamental and important results of Differential Geometry, and gives a representation of the Euler characteristic of a deterministic manifold in terms of curvature integrals. It has a long and impressive history, starting in the early nineteenth century with simple Euclidean domains. Names were added to the result as the setting became more and more general. While it has nothing to do with Probability, it can be obtained as a simply corollary to Theorems 12.4.1 and 12.4.2. Here is a version very close to that originally proven in 1943 by Allendoerfer and Weil [12], for what they called "Riemannian polyhedra".

Theorem 12.7.1 (Chern-Gauss-Bonnet Theorem) *Let (M,g) be a C^3 compact, orientable, N dimensional, Riemannian manifold, or piecewise C^2 and isometrically embedded in some C^3 Riemannian manifold $(\widetilde{M}, \widetilde{g})$ of dimension N. Then, if M has no boundary,*

$$(12.7.1) \quad \varphi(M) \;\equiv\; \mathcal{L}_0(M) \;=\; \frac{(-1)^{N/2}}{(2\pi)^{N/2} N!} \int_M \mathrm{Tr}^M(R^{N/2}) \, \mathrm{Vol}_g,$$

if N is even, and 0 if N is odd. In the piecewise smooth case,

$$\varphi(M) \;=\; \sum_{j=0}^{\dim M} (2\pi)^{-j/2} \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} C(N-j, j-2m) \frac{(-1)^m}{m! \, (j-2m)!}$$

$$\times \int_{\partial_j M} \int_{S(T_t \partial_j M^\perp)} \mathrm{Tr}^{T_t \partial_j M} \left(R^m S_{\nu_{N-j}}^{j-2m} \right)$$

$$\times \alpha(\nu_{N-j}) \, \mathcal{H}_{N-j-1}(d\nu_{N-j}) \, \mathcal{H}_j(dt).$$

where we adopt the notation of (10.7.1) and the convention of Remark 10.5.2.

Proof. Suppose f is a Gaussian random field on M, such that f induces[13] the metric g. Suppose, furthermore, that f satisfies the side all the conditions of either Theorem 12.4.1 or Theorem 12.4.2, depending in whether M does, or does not have, a boundary.

To save on notation, assume now that M does not have a boundary. Recall that in computing $\mathbb{E}\{\varphi(A_u(f,M))\}$ we first wrote $\varphi(A_u(f,M)))$ as an alternating sum of N different terms, each one of the form

$$\mu_k(u) \;\stackrel{\Delta}{=}\; \# \left\{ t \in M : f_t > u, \; \nabla f_t = 0, \; \mathrm{index}(-\nabla^2 f_t) = k \right\}.$$

Clearly, as $u \to -\infty$, these numbers increase to

$$\mu_k \;\stackrel{\Delta}{=}\; \# \left\{ t \in M : \nabla f_t = 0, \; \mathrm{index}(-\nabla^2 f_t) = k \right\},$$

and $\varphi(M)$ is given by an alternating sum of the μ_k. Since $\mu_k(u)$ is bounded by the total number of critical points of f, which is an integrable random variable, dominated convergence gives us that

$$\varphi(M) \;=\; \lim_{u \to -\infty} \mathbb{E}\left\{ \varphi\left(A_u(f,M)\right) \right\},$$

and the statement of the theorem then follows by first using Theorem 12.4.1 to evaluate $\mathbb{E}\left\{ \varphi\left(A_u(f,M)\right) \right\}$ and then checking that the right hand side of (12.7.1) is, in fact, the above limit.

---

[13]Note, this is the opposite situation to that which we have faced until now. We have always started with the field f and used it to define the metric g. Now, however, g is given and we are assuming that we can find an appropriate f.

If we do not know that g is induced by an appropriate Gaussian field, then we need to adopt a non-intrinsic approach via Nash's embedding theorem in order to construct one.

Nash's embedding theorem [111] states that for any C^3, N-dimensional Riemannian manifold (M, g), there is an isometry $i_g : M \to i_g(M) \subset \mathbb{R}^{N'}$ for some finite N' depending only on N. More importantly, $\mathbb{R}^{N'}$ is to be taken with the usual Euclidean metric.

The importance of this embedding is that it is trivial to find an appropriate f when the space is Euclidean with the standard metric. Any unit zero-mean unit variance isotropic Gaussian random field, f' say, on $\mathbb{R}^{N'}$, whose first partial derivatives have unit variance, and which satisfies the non-degeneracy conditions of Theorem 12.4.1, will do. If we now define

$$f \;=\; f'_{|i_g(M)} \circ i_g^{-1}$$

on M then it is easy to see that f induces the metric g on M, and so our construction is complete

Finally, we note that the case of piecewise smooth M follows exactly the same argument, simply appealing to Theorem 12.4.2 rather than Theorem 12.4.1. □

13

Excursion probabilities for smooth fields

As we have mentioned more than once before, one of the oldest and most important problems in the study of stochastic proceses of any kind is to evaluate the *excursion probabilities*

(13.0.1) $$\mathbb{P}\left\{\sup_{t \in T} f(t) \geq u\right\},$$

where f is a random process over some parameter set T. As usual, we shall restrict ourselves to the case in which f is centered and Gaussian and T is compact for the canonical metric of (1.3.1).

In this chapter we shall restrict ourselves to the case of parameter sets which are piecewise smooth manifolds in the sense of Definition 8.2.1, and so shall write M rather than T for the parameter space. The reason why we work with these spaces rather than with (non-locally convex) Whitney stratified manifolds will become clear from a counter-example in Section 13.4.4. For later needs, we shall, as usual, assume that M is orientable.

In particular, we shall show that, for such M and for smooth Gaussian random fields f,

(13.0.2) $$\left|\mathbb{P}\left\{\sup_{t \in M} f(t) \geq u\right\} - \mathbb{E}\left\{\varphi\left(A_u(f, M)\right)\right\}\right| < O\left(e^{-\alpha u^2/(2\sigma^2)}\right),$$

where φ is the Euler characteristic, $A_u(f, M)$ is the excursion set of f over M, σ^2 is the variance of f (assumed constant) and $\alpha > 1$ is a constant that we shall be able to identify. Since we already have explicit expressions for

$\mathbb{E}\left\{\varphi\left(A_u(f, M)\right)\right\}$ in Chapters 11 and 12, we can rewrite (13.0.2) as

$$(13.0.3) \quad \mathbb{P}\left\{\sup_{t \in M} f(t) \geq u\right\} = C_0 \Psi\left(\frac{u}{\sigma}\right) + u^N e^{-u^2/2\sigma^2} \sum_{j=1}^{N} C_j u^{-j}$$
$$+ o\left(e^{-\alpha u^2/2\sigma^2}\right),$$

where the C_j are constants depending on σ^2 and the Lipschitz-Killing curvatures of M with the Riemannian structure induced by f, and $N = \dim(M)$.

A little thought shows that (13.0.3) is really a remarkable result. If we think of the right hand side as an expansion of the form

$$C_0 \Psi\left(\frac{u}{\sigma}\right) + e^{-u^2/2\sigma^2} \sum_{j=1}^{N} C_j u^{N-j} + \text{error},$$

it would be natural to expect that the error term here would be the 'next' term of what seems like the beginning of an infinite expansion for the excursion probability, and so of order $u^{-1} e^{-u^2/2\sigma^2}$. However (13.0.3) indicates that this is *not* the case. Since $\alpha > 1$, the error is actually *exponentially smaller* than this.

Our main result, which formalises the above, will be given in Theorem 13.3.3. It has a number of precursors, the closest being a recent paper [149] with Akimichi Takemura, from which most of this chapter is taken[1]. Takemura and Kuriki [139] had a version of Theorem 13.3.3 for Gaussian fields with finite Karhunen-Loève expansions. Their approach was somewhat different to the one which is adopted here, in that they used the tube formulae of Chapter 10 to compute the excursion probability, and showed, via a kinematic formula on the sphere, that this was equivalent (up to a super-exponentially small error) to computing the expected Euler characteristic. The above papers all recognised the fact that the expansion in (13.0.3) was actually the expected Euler characteristic of an excursion set.

An important precursor to all of these results was an approximation of Piterbarg [117, 118], who obtained a result like (13.0.3), over a class of subsets $\mathbb{R}^N$, although without realising that the expected Euler characteristic of an excursion set was involved. We shall return to Piterbarg's seminal results at the end of the chapter, when we can appreciate them fully with the benefits of hindsight.

Finally, before starting the real work, we note that there is a rich literature of results, dating back over half a century to Rice's formula (11.6.14) on $\mathbb{R}^1$ and a quarter of a century to [3] for the first calculation of an expected

---

[1] In [149] piecewise smooth manifolds were defined in a somewhat different fashion, but this leads to no significant differences in either the results or their proofs.

Euler characteristic, which uses these formulae to approximate excursion probabilities. Beyond the theory, these are approximations that have been heavily used by practicioners in a wide variety of areas (cf. [9], our planned next volume). The literature of applications, naturally enough, had little, if anything, to say about either the validity or the accuracy of the approximation. The theoretical papers (which generally treated only stationary processes over smooth Euclidean domains) were able to give various justifications for the validity of the approximation, but were never able to identify its precise level of accuracy. Doing so, particularly for the wide class of parameter spaces with which we are working, is the central aim of this chapter.

The proof of the main result of the this chapter – Theorem 13.3.3 – is surprisingly technical, although the idea behind it is actually quite simple. In principle, since we do not know much about how to compute excursion probabilities, but we do know a lot about computing expectations, we shall argue as follows:

(i) Suppose f has, almost surely, a unique global supremum.

(ii) Let M_u be the number of global suprema above the level u.
(In view of (i), M_u is either 0 or 1.)

(iii) Note that

$$\mathbb{P}\left\{\sup_{t \in M} f(t) \geq u\right\} = \mathbb{E}\left\{M_u\right\}.$$

(iv) Find a (useful) point process representation for M_u that allows the application of the meta-theorem, Theorem 12.1.1, to obtain an integral expression for $\mathbb{E}\left\{M_u\right\}$.

(v) Relate this expression to an integral expression for $\mathbb{E}\left\{\varphi\left(A_u(f, M)\right)\right\}$ of the type we developed in Chapter 12.

(vi) Evaluate, or at least bound, the difference between the expressions in (iv) and (v).

In this six-step procedure, (i)–(iii) are trivial. Step (iv) is surprisingly technical, the problem being caused by the seemingly innocuous adjective "useful". What is interesting to note, however, is that since in steps (iv) and (v) we are only looking for "integral expressions" for $\mathbb{E}\left\{M_u\right\}$ and $\mathbb{E}\left\{\varphi\left(A_u(f, M)\right)\right\}$, and not seeking to evaluate them, while the first five steps will require certain regularity properties for f and M they will not require that f be Gaussian. Hence, for much of this chapter we shall not make this assumption. As usual, however, the closing step of the argument, in which we need to evaluate something, will require an assumption of Gaussianity.

13.1   On global suprema

Our setting will be that of Chapter 12, so that (M, g) will be a C^2, N-dimensional, piecewise smooth (stratified) manifold $(M = \bigcup_{j=0}^{N} \partial_j M)$ embedded in a C^3 manifold $(\widetilde{M}, \widetilde{g})$. The reason for working with piecewise smooth manifolds, rather than in the full generality of Whitney stratified spaces, will become clear in Section 13.4.4 below, where we shall give an example indicating that the main results of the chapter fail to hold in the more general scenario.

Furthermore, we take a random process $\widetilde{f} \in C^2(\widetilde{M})$, write f for the restriction of $\widetilde{f}$ to M and assume that f is a Morse function on M.

Of course, since f is random, each of the above requirements on $\widetilde{f}$ and f is assumed to hold with probability one.

However, as noted above, our argument starts in a purely deterministic setting, searching for a way to characterise global suprema of smooth functions. Thus, so as to make it clear which parts of the argument are deterministic and which stochastic, we start with (a realisation of $\widetilde{f}$, denoted by) $\widetilde{h}$, which satisfies

$$\widetilde{h} \in C^2(\widetilde{M}), \qquad h \triangleq \widetilde{h}\big|_M \text{ is a Morse function on } M.$$

13.1.1   A first representation

Our first representation of global suprema lies in the following lemma, whose main contribution lies in its third set of conditions.

Lemma 13.1.1 *With M, $\widetilde{h}$ and h as above, fix $t \in \partial_j M$, for some $0 \leq j \leq N$. Choose $\alpha^t \in C^2(M, (-\infty, 1])$ such that*

$$\alpha^t(s) = 1 \quad \Rightarrow \quad h(t) = h(s), \qquad \alpha^t(t) = 1.$$

Then t is a maximizer of h above the level u if, and only if, the following three conditions hold:

(i) *$h(t) \geq u$.*

(ii) *$\nabla \widetilde{h}(t) \in N_t M$, where $N_t M$ is the normal cone to M at t. Thus t is an extended outward critical point of h. (cf. Definition 9.3.3.)*

(iii) *For all $t \in M$,*

$$h(t) \geq \sup_{s \in M \setminus \{t\}} h^t(s),$$

where, for all $s \in M$

$$h^t(s) \triangleq \begin{cases} \frac{h(s) - \alpha^t(s) h(t)}{1 - \alpha^t(s)} & \text{if } \alpha^t(s) \neq 1 \\ h(s) & \text{if } \alpha^t(s) = 1. \end{cases}$$

Furthermore, if $\nabla^2 \alpha^t(t)$ is non-degenerate, and t is a critical point of $h_{|\partial_j M}$, then, for any C^2 curve $c : (-\delta, \delta) \to \partial_j M$ with $c(0) = t$, $\dot{c}(0) = X_t \in T_t \partial_j M$,

(13.1.1)

$$\lim_{u \to 0} h^t(c(u)) = \frac{\nabla^2 h_{|\partial_j M}(t)(X_t, X_t) - \nabla^2 \alpha^t_{|\partial_j M}(t)(X_t, X_t) h(t)}{-\nabla^2 \alpha^t_{|\partial_j M}(t)(X_t, X_t)}.$$

Remark: The condition that $\alpha^t(t) = 1$ ensures that, for each $t \in M$, t is a critical point of α^t.

Proof. The condition $h(t) \geq u$ is self-evident. Suppose, then, that $t \in \partial_i M$, $0 \leq i \leq k$, is a maximizer of h. Then, $\nabla \widetilde{h}(t) \in N_t M$, otherwise there exists a direction $X_t \in \mathcal{S}_t M$ such that $\widetilde{g}_t(X_t, \nabla \widetilde{h}(t)) > 0$ and t cannot be a maximizer.

Because t is a maximizer, for all s such that $\alpha^t(s) < 1$ it follows that

$$\frac{h(s) - \alpha^t(s) h(t)}{1 - \alpha^t(s)} < h(t).$$

On the other hand, if $\alpha^t(s) = 1$, then, by choice of α^t, $h(s) = h(t)$ which proves that

$$h(t) \geq \sup_{s \in M \setminus \{t\}} h^t(s).$$

To prove the reverse implication, assume that t is an extended outward critical point of $h_{|\partial_i M}$ and

$$h(t) \geq \sup_{s \in M \setminus \{t\}} h^t(s).$$

Now suppose that t is not a maximizer of h. Then there exists $s \in M \setminus \{t\}$ such that

$$h(t) < h(s).$$

In particular, for such a s, our choice of α^t implies that $\alpha^t(s) < 1$. It follows that

$$h(t) < \frac{h(s) - \alpha^t(s) h(t)}{1 - \alpha^t(s)},$$

which yields a contradiction.

The limit (13.1.1) follows from two applications of l'Hopital's rule. Specifically, we note that t is a critical point of $h_{|\partial_i M}$ by assumption and the

properties of α^t imply that t must also be a non-degenerate critical point of α^t. Therefore,

$$
\begin{aligned}
\lim_{u \to 0} \frac{h(c(u)) - \alpha^t(c(u))h(t)}{1 - \alpha^t(c(u))} &= \lim_{u \to 0} \frac{\frac{d}{du}\left(h(c(u)) - \alpha^t(c(u))h(t)\right)}{\frac{d}{du}\left(1 - \alpha^t(c(u))\right)} \\
&= \lim_{u \to 0} \frac{\frac{d^2}{du^2}\left(h(c(u)) - \alpha^t(c(u))h(t)\right)}{\frac{d^2}{du^2}\left(1 - \alpha^t(c(u))\right)}.
\end{aligned}
$$

To compute the numerator and denominator here note the following: Fix a $t \in \partial_i M$ and take $\beta \in C^2(\partial_i M)$ for which t is a critical point of β. Let $c : (-\delta, \delta) \to \partial_i M$ be a C^2 curve with $c(0) = t$ and $\dot{c}(0) = X_t$. Then

$$
\lim_{u \to 0} \frac{d^2}{du^2} \beta(c(u)) = \nabla^2 \beta(t)(X_t, X_t).
$$

Apply this fact to each term in the last ratio above and use the assumed properties of α^t to complete the proof.   □

Although we are currently in a deterministic setting, it is nevertheless worthwhile to 'think stochastically' for a moment, to see what Lemma 13.1.1 has gained us. A much simpler and, indeed, trivial result would be to say that t is a maximiser of h if, and only if, conditions (i) and and (ii) of the lemma hold, and if (iii) is replaced by

$$
h(t) \geq \sup_{s \in M \setminus \{t\}} h(s).
$$

(Of course, this replacement for (iii) means that (i) and (ii) must automatically hold.) This seems far simpler than the original (iii) and does not require the introduction of the seemingly unnecessary function α^t.

However, now think of $h \equiv f$ as a mean zero, unit variance Gaussian field, and let $\alpha^t(s) = \rho(t, s)$ where ρ is the correlation function of f. It is trivial to check that, for each $t \neq s$ in M,

$$
\mathbb{E}\left\{f(t)f^t(s)\right\} \equiv 0,
$$

so that $f(t)$ and $\sup_{s \in M \setminus \{t\}} f^t(s)$ are independent, something which is not true of $f(t)$ and $\sup_{s \in M \setminus \{t\}} f(s)$. It is to gain this independence that we have introduced α^t and h^t. In fact, if life were good, we would stop here and count ourselves lucky at having already found a useful point process representation for global suprema, as required in part (iv) of our six-part plan. However, life is not always good, as we shall soon see that the function $(s, t) \mapsto h^t(s)$ can have some unpleasant properties.

13.1.2   The problem with the first representation

Despite all that we have said so far about the functions h^t, we have not looked too closely at their properties. From (13.1.1) we know that $h^t(s)$ is

continuous as $s \to t$ in $\partial_j M$, but we have not looked at other directions. In fact, in other directions h^t is badly behaved. It is straightforward to check, again by two applications of l'Hopital's rule, that

$$(13.1.2) \qquad \nabla \widetilde{h}(t) \in N_t M \quad \Longleftrightarrow \quad \sup_{s \in M \setminus \{t\}} h^t(s) < \infty$$

$$\nabla \widetilde{h}(t) \in N_t M \quad \Rightarrow \quad \inf_{s \in M \setminus \{t\}} h^t(s) = -\infty$$

In other words, there are paths of approach to t, in M but not in $\partial_j M$, along which the behaviour of h^t is highly singular. We now begin attempting to resolve this singularity, with the solution finally appearing in Corollary 13.1.5, which gives a useable point process representation for global maximizers. The stochastic version of the solution appears in Corollary 13.1.6.

13.1.3   A second representation

Our aim in this section is to replace the function h^t we studied above by something smoother, although, unfortunately, it will be somewhat more involved. In order to profit from the characterisation of global maxima as certain types of extended outward critical points (cf. Lemma 13.1.1) we shall ensure that the replacement function will be identical to our original h^t at critical points. Furthermore, we shall show that for this new function, which will shall continue to denote by h^t, the function

$$(13.1.3) \qquad\qquad W(t) \;\stackrel{\Delta}{=}\; \sup_{s \in M \setminus \{t\}} h^t(s)$$

is continuous as a function of $t \in M$ if additional smoothness conditions are placed on the functions α^t. This will be the main result of this section.

   Our strategy will to augment M around a neighbourhood of $t \in \partial_i M$ with additional information that allows us to examine functions on M in finer scales. This will lead to a 'correction term' to h^t that will compensate for the singular behaviour inherent in (13.1.2).

   To start, we build the augmented space. Recall that we have assumed that M is a piecewise smooth space, and so at any $t \in M$ it is locally homeomorphic to the support cone $\mathcal{S}_t M$. Furthermore, from Section 8.2, we have that if $t \in \partial_i M$ then we can decompose the support cone by writing

$$\mathcal{S}_t M \;=\; T_t \partial_i M \times K_t,$$

where K_t is a convex cone that contains no subspace.

   For ease of exposition, for the moment we move to a simple Euclidean setting and replace $\mathcal{S}_t M$ by

$$\widehat{K} \;=\; L \times K \;\subset\; \mathbb{R}^q = \widetilde{M},$$

with L a subspace and K a convex cone containing no subspaces. We describe the construction in this scenario now and return to piecewise smooth spaces later.

In this scenario, the augmented space $B(\widehat{K})$ of $\widehat{K}$ along L is taken to be the disjoint union of the spaces

$$
\begin{aligned}
\mathcal{X} &= L \times (\widehat{K} \setminus \{0\}) \\
\partial\mathcal{X} &= L \times (\widehat{K} \cap S(\mathbb{R}^q)).
\end{aligned}
$$

You should think of $\mathcal{X}$ as the image of

$$(L \times \widehat{K}) \setminus \{(t, s) \in L \times \widehat{K} : t = s\}$$

under the map Ψ given by

(13.1.4) $\Psi(t, s) = (t, s - t)$

and the second space, $\partial\mathcal{X}$ as the 'boundary' of $\mathcal{X}$. The boundary is attached as follows: a sequence of points $(t_n, s_n) \in \mathcal{X}$ converges to $(t_0, s_0) \in \partial\mathcal{X}$ if, and only if

(13.1.5) $t_n \to t_0, \quad |s_n| \to 0, \quad \dfrac{s_n}{|s_n|} \to s_0.$

This notion of convergence corresponds to a sequence $\Psi^{-1}(t_n, s_n) = (t_n, t_n + s_n)$ converging to the diagonal

$$\left\{ (t, s) \in L \times \widehat{K} : s = t \right\}$$

along a well-defined direction.

Note that by identifying the tangent bundle $T(\mathbb{R}^q)$ with $\mathbb{R}^q \times \mathbb{R}^q$ we can think of $B(\widehat{K})$ as a subset of $T(\mathbb{R}^q)$, in which the t's above are replaced with $X_t \in \mathcal{S}_t M \subset T_t \mathbb{R}^q$. When convenient, we will either consider $B(\widehat{K})$ as a subset of $\mathbb{R}^{2q}$ or a subset of $T(\mathbb{R}^q)$.

With an augmented space at hand, we now take our first steps towards a new definition of h^t. Note firstly, however, that the $\{\alpha^t\}_{t \in M}$ of Lemma 13.1.1 were a family of functions which did not necessarily arise as the partial map of a function $\alpha : \widetilde{M} \times \widetilde{M} \to (-\infty, 1]$. In some cases, particularly in the stochastic setting below, this is a natural assumption to make. Doing so allows us to prove the following.

Lemma 13.1.2 *Suppose* $\alpha \in C^2(\mathbb{R}^q \times \mathbb{R}^q)$ *is such that the partial map*

$$\alpha^t(s) = \alpha(t, s)$$

satisfies the conditions of Lemma 13.1.1 at every $t \in L$ *and such that the Hessian of the partial map* α^t *is non-degenerate at every* $t \in L$. *Then, any* $\widetilde{h} \in C^2(\mathbb{R}^q)$ *maps to a continuous function,* $\widetilde{h}^{\alpha,\widehat{K}}$ *on* $B(\widehat{K})$ *as follows:*

$$\widetilde{h}^{\alpha,\widehat{K}}(t, s) \triangleq \frac{\widetilde{h}(t + s) - \alpha(t, t + s)\widetilde{h}(t) - \langle \nabla \widetilde{h}_t, s \rangle_{\mathbb{R}^q}}{1 - \alpha(t, t + s)}$$

if either t or s is not in $\partial\mathcal{X}$, and

$$\lim_{(u,v)\to(t,s)} \widetilde{h}^{\alpha,\widehat{K}}(u,v) \;\triangleq\; \widetilde{h}(t) - \frac{\nabla^2\widetilde{h}(t)(s,s)}{\nabla^2\alpha(t)(s,s)}$$

for $(t,s)\in\partial\mathcal{X}$.

Proof. All we need to prove is the continuity of $\widetilde{h}^{\alpha,\widehat{K}}$. This, however, follows from two applications of l'Hôpital's Rule.   □

The term

$$\frac{\langle\nabla\widetilde{h}_t, s\rangle_{\mathbb{R}^q}}{1-\alpha(t,t+s)}$$

above 'resolves' the singularity along the diagonal in some sense. In effect, it forces every $t\in L$ to be a critical point of the map

$$s \;\mapsto\; \frac{\widetilde{h}(t+s) - \alpha(t,t+s)\widetilde{h}(t) - \langle\nabla\widetilde{h}(t), s\rangle_{\mathbb{R}^q}}{1-\alpha(t,t+s)}.$$

To see how to exploit this fact, recall that our motivation for introducing $\widetilde{h}^{\alpha,\widehat{K}}$ was to describe the singularities in the function $h^t(s)$ at critical points t of $h_{|L}$. (Recall also that L takes the place of $T_t\partial_i M$ in our general cone $\widehat{K}$). We are therefore interested in critical points t of $h_{|L} = \widetilde{h}_{|L}$. Note that if t is a critical point of $h_{|L}$, then, for all $s\in\mathbb{R}^q$,

(13.1.6)

$$
\begin{aligned}
h^t(s) &= \frac{h(s) - \alpha(t,s)h(t) - \langle\nabla h(t), s-t\rangle_{\mathbb{R}^q}}{1-\alpha(t,s)} + \frac{\langle P_L^\perp\nabla h(t), s-t\rangle_{\mathbb{R}^q}}{1-\alpha(t,s)} \\
&= \widetilde{h}^{\alpha,\widehat{K}}(t,s-t) + \frac{\langle P_L^\perp\nabla h(t), s-t\rangle_{\mathbb{R}^q}}{1-\alpha(t,s)},
\end{aligned}
$$

where $P_L^\perp$ represents orthogonal projection onto $L^\perp$, the orthogonal complement of L in $\mathbb{R}^q$. The expression (13.1.6) indicates that, at critical points t of $h_{|L}$, $h^t(s)$ is the sum of a well-behaved term, $\widetilde{h}^{\alpha,\widehat{K}}$, and a singular term.

This representation holds for all critical points of $h_{|L}$. However, for later arguments we would like to have something of this form for all $t\in L$ and therefore redefine h^t by setting

(13.1.7) $\qquad h^t(s) \;\triangleq\; \begin{cases} \frac{h(s) - \alpha^t(s)h(t) - \langle P_L\nabla h(t), s\rangle_{\mathbb{R}^q}}{1-\alpha^t(s)} & \text{if } \alpha^t(s) \neq 1 \\ h(s) & \text{if } \alpha^t(s) = 1. \end{cases}$

With this (re)definition of h, (13.1.6) holds for all $t\in L$ and, furthermore, for each critical point t of $h_{|L}$ the two definitions of h^t coincide. For the

remainder of this section, as long as we remain in the Euclidean settting, we use the definition (13.1.7). We shall, however, need to adjust it somewhat when we return to the piecewise smooth scenario.

We now derive some properties of our new h^t.

Lemma 13.1.3 $P_L^\perp \nabla h(t) \in K^*$, the dual of $K \subset L^\perp$, if, and only if, for any bounded neighbourhood O_t of t,

$$\sup_{s \in O_t \backslash \{t\}} h^t(s) < +\infty.$$

Proof. If $P_L^\perp \nabla h(t) \in K^*$ then (13.1.6) implies that, for all s,

$$h^t(s) \leq \widetilde{h}^{\alpha, \widehat{K}}(t, s - t).$$

However, by Lemma 13.1.2, $\widetilde{h}^{\alpha, \widehat{K}}$ is continuous in s, and therefore bounded on bounded sets. This proves one direction of the lemma.

For the other, if $P_L^\perp \nabla h(t) \notin K^*$, then there exists a unit vector $v \in \widehat{K}$ such that $\langle P_L^\perp \nabla h(t), v \rangle_{\mathbb{R}^q} > 0$ and $t + \delta v \in O_t$ for sufficiently small δ. Equation (13.1.6) then implies that

$$\lim_{\delta \downarrow 0} h^t(t + \delta v) = +\infty,$$

since the numerator on the right of (13.1.6) is strictly positive of order $O(\delta)$ for δ small, while the denominator is of order $O(\delta^2)$. $\square$

Lemma 13.1.4 If $P_L^\perp \nabla h(t) \in (K^*)^\circ$, then, for any bounded neighbourhood O of the origin in $\mathbb{R}^q$,

$$W_O(t) \triangleq \sup_{s \in \widehat{K} \cap (\{t\} \oplus O) \backslash \{t\}} h^t(s)$$

$$= \sup_{v \in \widehat{K} \cap (O \backslash \{0\})} \left(\widetilde{h}^{\alpha, \widehat{K}}(x, v) + \frac{\langle P_L^\perp \nabla h(t), v \rangle_{\mathbb{R}^q}}{1 - \alpha(t, t + v)} \right)$$

is continuous in t.

Proof. We first note that by (13.1.6) the two suprema above are equal, and so it suffices to consider the second one.

Consider a convergent sequence

$$\{(t, v_n(t)\}_{n \geq 0} \rightarrow (x, v^*(t))$$

in $B(\widehat{K})$ along which the supremum $W_O(t)$ is approached. Then either $\|v_n(t)\|_{\mathbb{R}^q} > 0$ for all n sufficiently large or $\|v_n(t)\|_{\mathbb{R}^q} \to 0$. In the first case

it is immediate that $(t, v^*(t))$ is in $\mathcal{X}$ and, as we will show in a moment, in the second case $(t, v^*(t)) \in \partial\mathcal{X} \cap L \times S(L)$, where $S(L)$ is the unit sphere in L. In other words, the limiting direction $v^*(t)$ is in $S(L)$. Furthermore, if $\|v_n(t)\|_{\mathbb{R}^q} \to 0$ then it is easy to see that the sequence $(t, P_L v_n(t))$ also achieves the supremum $W_O(t)$.

To see why $v^*(t)$ must be in $S(L)$ in the second case, suppose that $(t, v_n(t))$ converges to $(x, v^*(t))$ with $v^*(t) \in \widehat{K} \setminus L$. As $P_L^\perp \nabla h(t) \in (K^*)^\circ$ it follows that

$$\langle P_L^\perp \nabla h(t), v^*(t) \rangle_{\mathbb{R}^q} < 0.$$

Therefore, applying the same argument as in (13.1.6),

$$\lim_{n\to\infty} \frac{\langle P_L^\perp \nabla h(t), v_n(t) \rangle_{\mathbb{R}^q}}{1 - \alpha(t, t + v_n(t))} = -\infty,$$

which contradicts the assumption that

$$\lim_{n\to\infty} h^t(t + v_n(t)) = W_O(t).$$

The above implies that, for any $t \in L$, and any convergent sequence $(x, v_n(t))$ achieving $W_O(t)$, we can either assume that $|v_n(t)|$ is bounded uniformly below, or that $v_n(t) \in L$ for all n. Continuity of $W_O(t)$ now follows, as for such sequences both

$$\sup_{s \in B(t, \delta(\varepsilon))} \left| \widetilde{h}^{\alpha, \widehat{K}}(t, v_n(t)) - \widetilde{h}^{\alpha, \widehat{K}}(s, v_n(t)) \right|$$

and

$$\sup_{s \in B(t, \delta(\varepsilon))} \left| \frac{\langle P_L^\perp \nabla h(t), v_n(t) \rangle_{\mathbb{R}^q}}{1 - \alpha(t, t + v_n(t))} - \frac{\langle P_L^\perp \nabla h(s), v_n(t) \rangle_{\mathbb{R}^q}}{1 - \alpha(s, s + v_n(t))} \right|$$

go to 0 as $n \to \infty$. This completes the proof. □

Lemmas 13.1.3 and 13.1.4 give us what we need for the Euclidean scenario, *viz.* a candidate for h^t that is well behaved and that we can plug into our characterisation for global suprema. The same arguments also work for piecewise smooth spaces, after some slight modifications.

Specifically, the map Ψ, defined in (13.1.4), has no natural replacement for a piecewise smooth space. However, we can think of $B(\widehat{K})$ as a subset of the tangent bundle $T(\mathbb{R}^q)$, so that we can think of Ψ as a map from $\mathbb{R}^q \times \mathbb{R}^q$ to $T(\mathbb{R}^q)$. Translating this to the piecewise smooth setting, we should therefore look for a replacement map, say H, from $\widetilde{M} \times \widetilde{M}$ to $T(\widetilde{M})$, such that, for each $t \in \widetilde{M}$,

$$(13.1.8) \qquad\qquad H\left(\{t\} \times \widetilde{M}\right) \subset T_t \widetilde{M}.$$

One of the key properties of Ψ was that for sequences $(t_n, s_n) \in \mathcal{X}$ converging to $(t, 0) \in \partial \mathcal{X}$ for which the unit vector

$$\frac{t_n - s_n}{\|t_n - s_n\|_{\mathbb{R}^q}}$$

converged in $S(\mathbb{R}^q)$ (cf. (13.1.5)), $\Psi(t_n, s_n)$ converged to a point in $\mathcal{X}$. We can replace this property of Ψ by asking the following of H. For any C^1 curve $c : (-\delta, \delta) \to \widetilde{M} \times \widetilde{M}$ with $c(0) = (c_1(0), c_2(0)) = (t, t)$ and $\dot{c}_2(0) = X_t$ we require that

$$(13.1.9) \qquad \lim_{u \to 0} \frac{H(t, c_2(u)) - H(t, t)}{u} = X_t.$$

Given such an H, we can, one again, redefine h^t for a C^2 function on a piecewise smooth space M. For $t \in \partial_i M$, we redefine h^t to be

(13.1.10)

$$h^t(s) \triangleq \begin{cases} \frac{h(s) - \alpha^t(s)h(t) - \widetilde{g}(\nabla h_{|\partial_i M}(t), H(t,s) - \alpha^t(s)H(t,t))}{1 - \alpha^t(s)} & \text{if } \alpha^t(s) \neq 1 \\ h(s) & \text{if } \alpha^t(s) = 1. \end{cases}$$

Note that at critical points of $h_{|\partial_i M}(t)$ we always have that $\nabla h_{|\partial_i M} = 0$, so that the term in $\widetilde{g}$ in (13.1.10) disappears, as it does for (i) in Lemma 13.1.5 below. Having made this observation, by working in suitably chosen charts, it is not difficult to prove the following lemma.

Lemma 13.1.5 *With M and h as above, suppose that $\alpha \in C^2(\widetilde{M} \times \widetilde{M}, (-\infty, 1])$ is such that the partial maps $\alpha^t(s) = \alpha(t, s)$ satisfy*

$$\alpha^t(s) = 1 \quad \Rightarrow \quad h(t) = h(s), \qquad \alpha^t(t) = 1$$

for every $t \in M$ and such that the Hessian of the partial map α^t is non-degenerate at every $t \in M$. Furthermore, suppose $H : \widetilde{M} \times \widetilde{M} \to T(M)$ satisfies (13.1.8) and (13.1.9), and let h^t be defined as in (13.1.10).
Then t is a maximizer of h above the level u if, and only if, the following three conditions hold:

(i) $h(t) - \widetilde{g}\left(H(t, t), \nabla h_{|\partial_i M}(t)\right) \geq u.$

(ii) $\nabla \widetilde{h}(t) \in N_t M.$

(iii) *For all $t \in M$,*

$$h(t) \geq W(t) \triangleq \sup_{s \in M \setminus \{t\}} h^t(s).$$

Furthermore, for every $t \in \partial_i M$ *and any* C^2 *curve* $c : (-\delta, \delta) \to \partial_i M$ *with* $c(0) = t$ *and* $\dot{c}(0) = X_t$

$$(13.1.11)\ \lim_{t \to 0} h^t(c(t)) \quad = \quad \frac{\nabla^2 \left(h(\cdot) - \widetilde{g}\left(H^t(\cdot), \nabla h_{|\partial_i M}(t)\right)\right)(X_t, X_t)}{-\nabla^2 \alpha^t_{|\partial_i M}(t)(X_t, X_t)}$$
$$- \left(h(t) - \widetilde{g}(H^t(t), \nabla h_{|\partial_i M}(t))\right),$$

where H^t *is the partial map* $H^t(s) = H(t, s)$.
In addition, the function

$$W(t) \quad \triangleq \quad \sup_{s \in M \setminus \{t\}} h^t(s)$$

is, for each $0 \leq i \leq k$, *continuous on the set*

$$\left\{t \in \partial_i M : P^{\perp}_{T_t \partial_i M} \nabla h(t) \in (P^{\perp}_{T_t \partial_i M} N_t M)^{\circ}\right\},$$

where $P^{\perp}_{T_t \partial_i M}$ *represents projection onto the orthogonal complement of* $T_t \partial_i M$ *in* $T_t \widetilde{M}$.

To appreciate why all of this hard work has been necessary to characterise something as simple as a supremum, we need first to translate it all to a stochastic setting.

13.1.4   Random fields

We now turn to a stochastic version of the previous section, with M a C^2 a piecewise smooth space M embedded in a C^3 manifold $\widetilde{M}$ and $\widetilde{f}$ a C^2 process on $\widetilde{M}$ so that $f = \widetilde{f}_{|M}$ is a Morse function on M.

We assume that f is L^2, with correlation function

$$\rho(s, t) \quad = \quad \frac{\mathbb{E}\left\{f(s)f(t)\right\}}{\sigma(t)\sigma(s)},$$

where $\sigma^2(t) = \mathbb{E}\{f^2(t)\}$.

Although we are not, at this stage, assuming that $\widetilde{f}$ is Gaussian, we nevertheless endow $\widetilde{M}$ with the usual Riemannian metric induced by $\widetilde{f}$, so that, for $X_t, Y_t \in T_t \widetilde{M}$

$$\widetilde{g}(X_t, Y_t) \quad \triangleq \quad \mathbb{E}\left\{X_t \widetilde{f} \cdot Y_t \widetilde{f}\right\}.$$

As usual, M is given the metric induced by $\widetilde{g}$, and we also assume that $\rho \in C^2(\widetilde{M} \times \widetilde{M})$, so that both metrics are also C^2.

We also place a new condition on ρ, that we have not met before. It ensures that the map $t \mapsto \widetilde{f}(t)$ is an embedding of $\widetilde{M}$ into $L^2(\Omega, \mathcal{F}, \mathbb{P})$ and

rules out 'global' singularities' in the process, a term whose meaning will become clearer later. The condition is

$$(13.1.12) \qquad\qquad \rho(t,s) = 1 \iff t = s.$$

This condition rules out, for example, periodic processes on $\mathbb{R}$. However, as we shall see in Section 13.4.4, there is a good reason why this should be the case.

With the above definitions, we can now choose candidates for our functions α and H of the previous subsection. Specifically, we take an orthonormal (with respect to $\widetilde{g}$) frame field $(X_{1,t}, \ldots, X_{\dim \widetilde{M},t})$ on $T_t\widetilde{M}$ and set

$$(13.1.13) \qquad\qquad \alpha(t,s) \;\triangleq\; \rho(t,s)$$

$$(13.1.14) \quad H(t,s) \;=\; F(t,s) \;\triangleq\; \sum_{j=1}^{\dim \widetilde{M}} \mathrm{Cov}(\widetilde{f}(s), X_j \widetilde{f}(t)) X_{j,t}.$$

It is easy to check that the definition of F is independent of the choice of frame field.

We can then state the stochastic analogy of Lemma 13.1.5 – i.e. a point process representation for the maximizers of the random field f.

Corollary 13.1.6 *In addition to the above assumptions, assume that the maximizers of $\widetilde{f}$ are almost surely isolated and that there are no critical points of $f_{|\partial_i M}$ such that $P^\perp_{T_t \partial_i M} \nabla \widehat{f}(t) \in \partial N_t M \subset T_t \partial_i M^\perp$. Then the maximizers of f are the points $t \in \partial_i M$, $0 \leq i \leq N$, satisfying the following four conditions:*

(i) $f(t) - \widetilde{g}\left(F(t,t), \nabla f_{|\partial_i M}(t) \right) \;\geq\; u.$

(ii) $\nabla f_{|\partial_i M}(t) \;=\; 0.$

(iii) $P^\perp_{T_t \partial_i M} \nabla \widetilde{f}(t) \;\in\; (P^\perp_{T_t \partial_i M} N_t M)^\circ.$

(iv) $f(t) \;\geq\; \sup_{s \in M \setminus \{t\}} f^t(s),$

where

$(13.1.15)$

$$f^t(s) \;\triangleq\; \begin{cases} \dfrac{f(s) - \rho(t,s)f(t) - \widetilde{g}\left(F(t,s) - \rho(t,s)F(t,t), \nabla f_{|\partial_i M}(t) \right)}{1 - \rho(t,s)} & \text{if } \rho(t,s) \neq 1, \\[2mm] f(s) & \text{if } \rho(t,s) = 1. \end{cases}$$

Remark 13.1.7 *Note that, for later convenience, we have split the condition that t be an extended outward critical point into two parts – (ii) and (iii).*

Proof. Note firstly that since there are no critical points of $f_{|\partial_i M}$ such that $P^{\perp}_{T_t \partial_i M} \nabla \widehat{f}(t) \in \partial N_t M \subset T_t \partial_i M^{\perp}$, all global maximizers will be such that $P^{\perp}_{T_t \partial_i M} \nabla \widehat{f}(t)$ is in the relative interior of $N_t M$ in $T_t \partial_i M^{\perp}$. Thus (ii) and (iii) are equivalent to the requirement that t be an extended outward critical point.

Next, note that while α and ρ have been assumed to have almost identical properties, they differ in one important aspect: While we assumed that $\alpha^t(s) = 1$ implied $h(t) = h(s)$, the same cannot be said to hold when $\rho(t, s) = 1$ unless we also assume that $\mathbb{E}\{f(t)\}$ is constant and $\mathbb{E}\{f^2(t)\} = 1$ for all t. Nevertheless, a quick check through the previous proofs will confirm the fact that, given the definition of f^t, this difference affects none of the arguments.

Having noted these two facts, we can apply Lemma 13.1.5 on an almost sure basis, with h^t replaced by f^t, α by ρ and H by F. □

13.1.5   Suprema and Euler characteristics

We now have enough to finally see what the link is between $\varphi(A_u)$, the Euler characteristic of the excursion set $\{t \in M : f(t) \geq u\}$ and the global supremum of f.

To do this, we need one more assumption, that the global maximizer of the process f is almost surely unique. In this case, it follows from Corollary 13.1.6, under the assumptions made there along with uniqueness of the global supremum, that[2]

(13.1.16)

$$\mathbb{1}_{\left\{\sup_{t \in M} f(t) \geq u\right\}} \;=\; \#\Big\{\text{extended outward critical points } t \text{ of } f \text{ for which}$$
$$f(t) \geq u,\; \sup_{s \in M \setminus \{t\}} f^t(s) \leq f(t)\Big\}.$$

On the other hand, if we return to Corollary 9.3.4, which gives a point set characterisation for the Euler characteristic for excursion sets over piecewise smooth spaces, and compare this to the above, we immediately obtain that

$$\left|\mathbb{1}_{\left\{\sup_{t \in M} f(t) \geq u\right\}} - \varphi(A_u(f, M))\right|$$
$$\leq\; \#\Big\{\text{extended outward critical points } t \text{ of } f \text{ for which}$$
$$f(t) \geq u,\; \sup_{s \in M \setminus \{t\}} f^t(s) > f(t)\Big\}.$$

---

[2]Recall, once again, that at extended outward critical points $\nabla f_{|\partial_i M} = 0$, so that all the expressions in Corollary 13.1.6 involving $\nabla f_{|\partial_i M}$ disappear.

While correct, this relationship is not terribly useful, since we have no way of computing the right hand side of the inequality. What turns out to be a lot more accessible is to begin evaluating the expectations of each term on the left, and then compute with the difference. To do this, we need additional regularity conditions on f, and adopt those of Theorem 11.2.4 along with the assumption[3] that for all $t \in M$,

$$(13.1.17) \qquad \mathbb{P}\left\{W(t) = f(t)\right\} \;=\; \mathbb{P}\left\{W(t) = u\right\} \;=\; 0,$$

where

$$W(t) \;\triangleq\; \sup_{s \in M\backslash\{t\}} f^t(s),$$

and f^t is as defined at (13.1.15). Note that none of these additional assumptions require that f be Gaussian.

Noting (13.1.16), it is then a straightforward[4] application of Theorem 12.1.1, to see, in the notation of Chapter 12, that

$$\mathbb{P}\left\{\sup_{t \in M} f(t) \geq u\right\}$$

$$= \sum_{i=0}^{N} \int_{\partial_i M} \mathbb{E}\left\{\det\left(-\nabla^2 f_{|\partial_i M, E_i}(t)\right) \mathbb{1}_{A_t^{SUP}} \,\Big|\, \nabla f_{|\partial_i M, E_i}(t) = 0\right\}$$

$$\times \; \varphi_{\nabla f_{|\partial_i M, E_i}(t)}(0)\, d\mathcal{H}_i(t),$$

where

$$A_t^{SUP} \;\triangleq\; \left\{f(t) \geq u \wedge W(t),\; \nabla \widetilde{f}(t) \in N_t M\right\},$$

---

[3]Sufficient conditions to guarantee that (13.1.17) hold are that for all pairs $1 \leq i, j \leq N$ and all pairs $\{(t, s) : t \in \partial_i M, s \in \partial_j M\}$ the random vector

$$V(t, s) \;\triangleq\; \left(f(t) - f(s), \nabla f_{|\partial_i M, E_i}(t), \nabla f_{|\partial_j M, E_j}(s)\right)$$

has a density, bounded by some constant $K(t, s)$. Then it is an easy consequence of Lemma 11.1.10 that

$$\mathbb{P}\left\{\exists (t, s) : t \in \partial_i M, y \in \partial_j M, V(t, s) = 0 \in \mathbb{R}^{i+j+1}\right\} \;=\; 0.$$

which implies (13.1.17).

[4]Well ... almost straightforward. A trivial application of Theorem 12.1.1 would require that W be continuous, which is not the case. However, (an almost sure version of) Corollary 13.1.5 does give us that W is continuous on the open set

$$\left\{t \in \partial_i M : P_{T_t \partial_i M}^{\perp} \nabla \widehat{f}(t) \in (N_t M)^{\circ}\right\},$$

and, if you check through the proofs leading to Theorem 12.1.1 – which is still 'straightforward' but which would take some time – you will see that this is all we actually need.

and where $\varphi_{\nabla f_{|\partial_i M, E_i}(t)}$ is the density of $\nabla f_{|\partial_i M, E_i}(t)$ and $\mathcal{H}_i$ is the volume measure induced by the Riemannian metric g.

Although the following formula does not explicitly appear there, we have already seen in Chapter 12 that

$$\mathbb{E}\left\{\varphi\left(A_u(f, M)\right)\right\}$$
$$= \sum_{i=0}^{N} \int_{\partial_i M} \mathbb{E}\left\{\det\left(-\nabla^2 f_{|\partial_i M, E_i}(t)\right) \mathbb{1}_{A_t^{EC}} \Big| \nabla f_{|\partial_i M, E_i}(t) = 0\right\}$$
$$\times \varphi_{\nabla f_{|\partial_i M, E_i}(t)}(0) \, d\mathcal{H}_i(t),$$

where

$$A_t^{EC} \triangleq \{f(t) \geq u, \ \nabla \widetilde{f}(t) \in N_t M\}.$$

The discrepancy between the mean Euler characteristic of an excursion set and the supremum distribution is therefore

(13.1.18)
$$\mathrm{Diff}_{f,M}(u)$$
$$\triangleq \mathbb{E}\left\{\varphi\left(A_u(f, M)\right)\right\} - \mathbb{P}\left\{\sup_{t \in M} f(t) \geq u\right\}$$
$$= \sum_{i=0}^{N} \int_{\partial_i M} \mathbb{E}\left\{\det\left(-\nabla^2 f_{|\partial_i M, E_i}(t)\right) \mathbb{1}_{A_t^{ERR}} \Big| \nabla f_{|\partial_i M, E_i}(t) = 0\right\}$$
$$\times \varphi_{\nabla f_{|\partial_i M, E_i}(t)}(0) \, d\mathcal{H}_i(t)$$

where

$$A_t^{ERR} \triangleq \left\{u \leq f(t) \leq W(t), \ \nabla \widetilde{f}(t) \in N_t M\right\}.$$

There are three remarkable facts to note about (13.1.18). The first is that it is an identity, and not an approximation or a bound. The second is that it holds for a wide class of smooth random fields over a large class of piecewise smooth parameter spaces. The last is that these smooth random fields were never assumed to be Gaussian.

Given (13.1.18), it is not difficult to guess how the rest of the proof should proceed. The key point is that the expectation is taken only over the event A_t^{ERR}, which is included in the event $\{u \leq f(t) \leq W(t)\}$. Now, if you remember our brief stochastic diversion at the end of Section 13.1.1, you will remember that, if f was Gaussian with constant mean and unit variance, $f(t)$ and $W(t)$ are independent. Thus,

$$(13.1.19) \qquad \mathbb{P}\left\{A_t^{ERR}\right\} \quad \leq \quad \mathbb{P}\{u \leq f(t) \leq W(t)\}$$
$$\leq \quad \mathbb{P}\{u \leq f(t)\} \times \mathbb{P}\{u \leq W(t)\}.$$

We know from the very general theory of Chapter 4 that for the smooth random fields that we are considering it is likely that

$$\mathbb{P}\left\{\sup_{t\in M} f(t) \geq u\right\} \;\simeq\; u^{\alpha} \sup_{t\in M} \mathbb{P}\left\{u \leq f(t)\right\},$$

for some α, and where, since the current argument is heuristic, we make no attempt to give a precise mathematical meaning to the $\simeq$ here. Furthermore, we should also be able to bound the rightmost probability in (13.1.19) by the general theory of Chapter 4, since it is just a Gaussian excursion probability. If so, it too is going to be exponentially small. Putting these two facts into (13.1.19) makes $\mathbb{P}\left\{A_t^{ERR}\right\}$ *super*-exponentially small (when compared to $\mathbb{P}\left\{\sup_{t\in M} f(t) \geq u\right\}$) and, modulo technicalities, we basically have the result promised at the beginning of the chapter. However, since that result also promised explicit bounds on the rate at which $\mathrm{Diff}_{f,M}(u) \to 0$ as $u \to \infty$ we are also going to have to do some more detailed computation.

Of course, what is clear from the above description is that to truly exploit (13.1.18) we are going to have to make distributional assumptions. Above, we have assumed Gaussianity. Perhaps not surprisingly, despite the universality of (13.1.18), we do not know how to handle non-Gaussian processes.

This is a shame, since in Chapter 14 we shall obtain explicit formulae for $\mathbb{E}\left\{\varphi\left(A_u(f, M)\right)\right\}$ for a wide class of non-Gaussian random fields f, and it would be nice to be able to exploit (13.1.18) to also approximate their excursion probabilities. We do hope/believe that results akin to those that we are about to encounter for Gaussian processes will also one day be proven in non-Gaussian scenarios but, for moment, we are stumped by the details.

13.2   Some fine tuning

While (13.1.18) is an explicit formula for the error in the expected Euler characteristic approximation, it is still a little hard to compute with. The problem lies in the fact that while we finally managed to find a version of $W(t) = \sup_{s\in M\setminus\{t\}} f^t(s)$ that was continuous in t (at least where it counts) f^t itself is still rather badly behaved. In particular, it is generally singular near t, with infinite variance[5]. This, of course, means that the approach outlined at the end of the last subsection for the Gaussian case – of using

---

[5] To see why this should be so, it suffices to look at one example. Take f to the restriction of an isotropic field to $[0,1]^2$. Fix a point $t = (x, 0)$. We compute the variance of f^t along the curve

$$c(u) \;=\; (x, 0) - u(1, 0).$$

the general theory of Chapter 4 to handle the distribution of W – is doomed to failure, unless something is done.

The key to circumventing this problem relies on two facts. Firstly, we are really only concerned with large *positive* values of f^t, and, secondly, we only care about the behaviour of f^t on the set

$$(13.2.1) \qquad \left\{ (t, \omega) : P^{\perp}_{T_t \partial_i M} \nabla \widetilde{f}(t)(\omega) \in (P^{\perp}_{T_t \partial_i M} N_t M)^{\circ} \right\}.$$

We exploit these facts and introduce a process $\widehat{f}^t$ in this section which has, under appropriate conditions, finite variance while dominating[6] f^t on the set (13.2.1). The good news is that this is the last time we shall have to 'adjust' f^t. The final adjustment is important, for not only does it make the proofs work, but the variance of $\widehat{f}^t$ will appear as an important parameter in our final result, related to the α in (13.0.3).

The process $\widehat{f}^t$ is defined by

$$(13.2.2) \;\; \widehat{f}^t(s) \;\triangleq\; \begin{cases} \dfrac{f(s) - \rho(t,s)f(t) - \widetilde{g}\left(\widehat{F}^t(s) - P_{N_t M}\widehat{F}^t(s), \nabla \widetilde{f}(t)\right)}{1 - \rho(t,s)} & \text{if } \rho(t,s) \neq 1 \\[2ex] f(s) - \widetilde{g}\left(\widehat{F}^t(s) - P_{N_t M}\widehat{F}^t(s), \nabla \widetilde{f}(t)\right) & \text{if } \rho(t,s) = 1 \end{cases}$$

where $P_{N_t M} : T_t \widetilde{M} \to N_t M$ represents orthogonal projection onto $N_t M$ and $\widehat{F} : \widetilde{M} \times \widetilde{M} \to T(\widetilde{M})$ is given by

$$(13.2.3) \qquad \widehat{F}^t(s) \;=\; \begin{cases} F(t,s) - \rho(t,s)F(t,t) & \text{if } \rho(t,s) \neq 1 \\ F(t,s) & \text{if } \rho(t,s) = 1, \end{cases}$$

Straightforward calculations show that

$$F(t, c(u)) \;=\; 0, \qquad F(t,t) \;=\; 0,$$

so that

$$\widetilde{f}^t(c(u)) \;=\; \frac{f(c(u)) - \rho(t, (c(u))f(t)}{1 - \rho(t, c(u))}.$$

The variance of $f^t(c(u))$ is

$$\mathrm{Var}\left(f^t(c(u))\right) \;=\; \frac{1 + \rho(t, c(u))}{1 - \rho(t, c(u))}$$

and

$$\lim_{u \to 0} \mathrm{Var}\left(f^t(c(u))\right) \;=\; \lim_{u \to 0} \frac{1 + \rho(t, c(u))}{1 - \rho(t, c(u))} \;=\; +\infty.$$

In general, if $t \in \partial_i M$ then, along a curve $c : (-\delta, 0] \to M$ with $\dot{c}(0) = -X_t \in \mathcal{S}_t M \setminus T_t \partial_i M$ then

$$\lim_{u \uparrow 0} \mathrm{Var}\left(f^t(c(u))\right) \;=\; +\infty.$$

---

[6]Obviously, $\widehat{f}^t$ does not dominate the *absolute* value of f^t. Were this true, $\widehat{f}^t$ would also have infinite variance and we would have gained nothing.

where F is defined in (13.1.14).

Lemma 13.2.1 *We adopt the setting of Section 13.1.4 and the assumptions of Corollary 13.1.6 Then, for every t in the set (13.2.1) of extended outward critical points and for every $s \in M$,*

$$\widehat{f}^t(s) \;\geq\; f^t(s).$$

If $t \in \partial_N M = M°$, then equality holds above.

Proof. First, we note that

$$(1 - \rho(t, s)) \cdot (f^t(s) - \widehat{f}^t(s)) \;=\; \widetilde{g}\left(\widehat{F}^t(s) - P_{N_t M} \widehat{F}^t(s),\, P^{\perp}_{T_t \partial_i M} \nabla \widetilde{f}(t) \right).$$

As $N_t M$ is a convex cone, it follows that for any $Y_t \in T_t \widetilde{M}$

$$Y_t - P_{N_t M} Y_t \;\in\; N_t M^*,$$

where $N_t M^*$ is the dual cone of $N_t M$, which is just the convex hull of $\mathcal{S}_t M$. By duality,

$$\widetilde{g}\left(Y_t - P_{N_t M} Y_t,\, V_t \right) \;\leq\; 0$$

for every $V_t \in N_t M$. Consequently, on the set (13.2.1),

$$\widetilde{g}\left(Y_t - P_{N_t M} Y_t,\, P^{\perp}_{T_t \partial_i M} \nabla \widetilde{f}(t) \right) \;\leq\; 0$$

for every $Y_t \in T_t \widetilde{M}$. As $\widehat{F}^t(s) \in T_t \widetilde{M}$ for each s, the first claim holds.

As for the second, if $t \in \partial_N M$, then $N_t M = T_t \partial_N M^{\perp}$ and $P_{N_t M} V_t = 0$ for all $V_t \in T_t \partial_N M$. Similarly,

$$\widetilde{g}\left(V_t,\, P^{\perp}_{T_t \partial_i M} \nabla \widetilde{f}(t) \right) \;=\; 0.$$

Therefore, on this set

$$\widetilde{g}\left(\widehat{F}^t(s) - P_{N_t M} \widehat{F}^t(s),\, \nabla \widetilde{f}(t) \right) \;=\; 0.$$

$\square$

As far as the continuity (in t) of

$$(13.2.4) \qquad\qquad \widehat{W}(t) \;\overset{\Delta}{=}\; \sup_{s \in M \setminus \{t\}} \widehat{f}^t(s)$$

is concerned, it is not difficult to show that, almost surely, Lemma 13.1.5 holds with f^t replaced by $\widehat{f}^t$, i.e. that $\widehat{W}(t)$ is continuous on the set

$$\left\{ t \in \partial_i M : P^{\perp}_{T_t \partial_i M} \nabla \widetilde{f}(t) \in (P^{\perp}_{T_t \partial_i M} N_t M)° \right\}.$$

As we will see in a moment in the proof of Theorem 13.2.2, Lemma 13.2.1 provides the basic bounds for $\mathrm{Diff}_{f,M}(u)$.

Lemma 13.2.1 now easily implies the following theorem for which, since is the main result of this section, we collect all the conditions and notations.

Theorem 13.2.2 *Suppose M is a C^2 compact, orientable, piecewise smooth manifold, embedded in a C^3 manifold $\widetilde{M}$. Let $\widetilde{f}$ be a C^2 random field on $\widetilde{M}$, and f its restriction to M. Let ρ be the correlation function of f, and assume that $\rho \in C^2(\widetilde{M} \times \widetilde{M})$ and that (13.1.12) holds. Furthermore, assume that for each $t \in \widetilde{M}$ the partial map $\rho(t,\cdot)$ has a non-degenerate Hessian.*

Assume that the maximiser of $\widetilde{f}$ is unique and that there are no no critical points of $f_{|\partial_i M}$ such that $P^\perp_{T_t \partial_i M} \nabla \widehat{f}(t) \in \partial N_t M \subset T_t \partial_i M^\perp$.

Finally, assume that all the conditions of Theorem 11.2.4 are in force and that, for every $t \in M$

$$\mathbb{P}\left\{\widehat{W}(t) = f(t)\right\} \;=\; \mathbb{P}\left\{\widehat{W}(t) = u\right\} \;=\; 0.$$

where $\widehat{W}$ is defined by (13.2.4), where $\widehat{f}^t$ there is defined by (13.2.2).

Then, with $\mathrm{Diff}_{f,M}(u)$ denoting the difference (13.1.18),

(13.2.5)

$$
\begin{aligned}
&|\mathrm{Diff}_{f,M}(u)| \\
&\leq \sum_{i=0}^{N} \int_{\partial_i M} \mathbb{E}\left\{\left|\det\left(-\nabla^2 f_{|\partial_i M, E_i}(t)\right)\right| \mathbb{1}_{B_t^{ERR}} \Big| \nabla f_{|\partial_i M, E_i}(t) = 0\right\} \\
&\qquad\qquad \times \varphi_{\nabla f_{|\partial_i M, E_i}(t)}(0)\, d\mathcal{H}_i(t)
\end{aligned}
$$

where

$$B_t^{ERR} \;=\; \{u \leq f(t) \leq \widehat{W}(t)\}.$$

Of course, there is the usual simple Gaussian corollary to the theorem.

Corollary 13.2.3 *With M as in Theorem 13.2.2, assume that $\widetilde{f}$ is a zero mean Gaussian field satisfying the conditions of Corollary 11.2.5 and that (13.1.12) holds. Then (13.2.5) holds.*

13.3   Gaussian fields with constant variance

We now come to what is the main section of this chapter, and the last part of the six-stage program we described at its beginning. That is, we shall now actually obtain an explicit bound for the difference between excursion probabilities and $\mathbb{E}\{\varphi(A_u)\}$.

Our first step will be to show that, for each $t \in M$, the process $\widehat{f}^t$ of (13.2.2) is uncorrelated with the random vector $\nabla f_{|\partial_i M}(t)$. Hence, in

the Gaussian case, $\widehat{f}^t(s)$ is independent of $\nabla f_{|\partial_i M}(t)$. The following lemma requires no side conditions beyond the basic ones on M and the requirement that f and ρ are smooth enough for all terms to be well defined. Since this is much weaker than the conditions we will leave more details for later. It also does not require that f be Gaussian, nor that it have constant variance.

Lemma 13.3.1 *For every $t \in \partial_i M$, $0 \leq i \leq k$, and every $s \in M \setminus \{t\}$,*

$$\text{Cov}\left(\widehat{f}^t(s), X_t f\right) = 0$$

for every $X_t \in T_t \partial_i M$.

Proof. Noting the definition (13.2.2) of $\widehat{f}^t$, if $\rho(t, s) \neq 1$, then

$$(1 - \rho(t, s)) \,\text{Cov}\left(\widehat{f}^t(s), X_t f\right)$$
$$= \text{Cov}\left(f(s) - \rho(t, s) f(t), X_t f\right)$$
$$\quad - \text{Cov}\left(\widetilde{g}\left(\widehat{F}^t(s) - P_{N_t M} \widehat{F}^t(s), \nabla \widetilde{f}(t)\right), X_t f\right)$$
$$= \text{Cov}\left(f(s) - \rho(t, s) f(t), X_t f\right) - \widetilde{g}\left(\widehat{F}^t(s) - P_{N_t M} \widehat{F}^t(s), X_t\right),$$

where probably the best way to check the last line is to expand all the vectors in the middle expression and then compute.

If, on the other hand, $\rho(t, s) = 1$, then a similar computation shows that

$$\text{Cov}\left(\widehat{f}^t(s), X_t f\right) = \text{Cov}\left(f(s), X_t f\right)$$
$$\quad - \text{Cov}\left(\widetilde{g}\left(\widehat{F}^t(s) - P_{N_t M} \widehat{F}^t(s), \nabla \widetilde{f}(t)\right), X_t f\right)$$
$$= \text{Cov}\left(f(s), X_t f\right) - \widetilde{g}\left(\widehat{F}^t(s) - P_{N_t M} \widehat{F}^t(s), X_t\right).$$

The conclusion will therefore follow once we prove that, for every $t \in M$,

$$\text{Cov}\left(f(s), X_t f\right) = \widetilde{g}\left(\widehat{F}^t(s) - P_{N_t M} \widehat{F}^t(s), X_t\right),$$
$$\text{Cov}\left(f(s) - \rho(t, s) f(t), X_t f\right) = \widetilde{g}\left(\widehat{F}^t(s) - P_{N_t M} \widehat{F}^t(s), X_t\right).$$

As the two arguments are similar, we prove only the first equality. The map $\widehat{F}^t$ can be decomposed as

$$\widehat{F}^t(s) = P_{T_t \partial_i M} \widehat{F}^t(s) + P^{\perp}_{T_t \partial_i M} \widehat{F}^t(s),$$

where

$$P_{T_t \partial_i M} \widehat{F}^t(s) = \sum_{j=1}^{i} \text{Cov}\left(f(s), X_j f(t)\right) X_{j,t},$$

$$P^{\perp}_{T_t \partial_i M} \widehat{F}^t(s) = \sum_{j=i+1}^{q} \text{Cov}\left(f(s), X_j f(t)\right) X_{j,t},$$

and $(X_{1,t}, \ldots, X_{q,t})$ is chosen so that $(X_{1,t}, \ldots, X_{i,t})$ forms an orthonormal basis for $T_t \partial_i M$ while $(X_{i+1,t}, \ldots, X_{q,t})$ forms an orthonormal basis for $T_t \partial_i M^\perp$, the orthogonal complement of $T_t \partial_i M$ in $T_t \widetilde{M}$.

Furthermore, since $\widetilde{g}(X_t, V_t) = 0$ for every $X_t \in T_t \partial_i M$ and $V_t \in N_t M$, it follows that

$$P_{N_t M} \widehat{F}^t(s) \;=\; P_{N_t M} P_{T_t \partial_i M}^\perp \widehat{F}^t(s),$$

and, for every $X_t \in T_t \partial_i M$,

$$
\begin{aligned}
\widetilde{g}\left(\widehat{F}^t(s) - P_{N_t M} \widehat{F}^t(s), X_t \right) & \\
&= \widetilde{g}\left(P_{T_t \partial_i M} \widehat{F}^t(s), X_t \right) \\
&\quad + \widetilde{g}\left(P_{T_t \partial_i M}^\perp \widehat{F}^t(s) - P_{N_t M} P_{T_t \partial_i M}^\perp \widehat{F}^t(s), X_t \right) \\
&= \widetilde{g}\left(P_{T_t \partial_i M} \widehat{F}^t(s), X_t \right) \\
&= \sum_{j=1}^i \mathrm{Cov}\left(f(s), X_j f(t) \right) \widetilde{g}\left(X_{j,t}, X_t \right) \\
&= \mathrm{Cov}\left(f(s), X_t f \right),
\end{aligned}
$$

and we are done. $\qquad\qquad\square$

If we now assume that f is Gaussian, then the lack of correlation of Lemma 13.3.1 becomes independence, and the independence between the process f^t and $\nabla f_{|\partial_i M}(t)$ allows us to remove the conditioning on $\nabla f_{|\partial_i M}(t)$ in the expression for $\mathrm{Diff}_{f,M}(u)$, regardless of whether f has constant variance or not.

Corollary 13.3.2 *Suppose f is a Gaussian process satisfying the conditions Corollary 13.2.3. Then, with the notation of Theorem 13.2.2,*

(13.3.1)

$$
\begin{aligned}
|\mathrm{Diff}_{f,M}(u)| \;\leq\; \sum_{i=0}^N \int_{\partial_i M} \mathbb{E}\left\{ \left| \det\left(-\nabla^2 f_{|\partial_i M, E_i}(t) \right) \right| \mathbb{1}_{C_t^{ERR}} \right\} \\
\times \varphi_{\nabla f_{|\partial_i M, E_i}(t)}(0) \, d\mathcal{H}_i(t),
\end{aligned}
$$

where

$$C_t^{ERR} \;=\; \left\{ u \leq f(t) - \widetilde{g}\left(P_{T_t \partial_i M} \widehat{F}^t(t), \nabla \widetilde{f}(t) \right) \leq \widehat{W}(t) \right\}.$$

If f has constant variance, then $\widehat{F}^t(t) = 0 \in T_t M$ and

$$C_t^{ERR} \;=\; \{ u \leq f(t) \leq \widehat{W}(t) \}.$$

Proof. Comparing this result with Theorem 13.2.2, it is clear that the only thing we need prove is that, if the event C_t^{ERR} occurs, then the condition $\{u \leq f(t) \leq W(t)\}$ can be replaced with

$$\left\{ u \leq f(t) - \widetilde{g}\left(P_{T_t \partial_i M} \widehat{F}^t(t), \nabla \widetilde{f}(t) \right) \leq \widehat{W}(t) \right\}$$

and the conditioning can be removed.

The replacement is justified by the fact that, on the event $\{\nabla f_{|\partial_i M}(t) = 0\}$, we trivially have

$$f(t) = f(t) - \widetilde{g}\left(P_{T_t \partial_i M} \widehat{F}^t(t), \nabla \widetilde{f}(t) \right).$$

Furthermore, $f(t) - \widetilde{g}\left(P_{T_t \partial_i M} \widehat{F}^t(t), \nabla \widetilde{f}(t) \right)$ is independent of $\nabla f_{|\partial_i M}(t)$, and Lemma 13.3.1 implies that $\widehat{W}(t)$ is also independent of $\nabla f_{|\partial_i M}(t)$. Therefore, the conditioning on $\nabla f_{\partial_i M}(t)$ can be removed. $\square$

We are now ready to prove the main result of this chapter, and one of the main ones of the book. The remainder of the chapter is about special cases, examples and counter-examples.

Theorem 13.3.3 *Let $\widetilde{f}$ be a Gaussian process on a piecewise continuous manifold M, with the assumptions of Corollary 13.2.3 in force. Assume, furthermore, that $\widetilde{f}$ has constant variance 1. Then,*

$$(13.3.2) \quad \liminf_{u \to \infty} -u^{-2} \log |\mathrm{Diff}_{f,M}(u)| \ \geq \ \frac{1}{2}\left(1 + \frac{1}{\sigma_c^2(f)} \right),$$

where

$$(13.3.3) \qquad \sigma_c^2(f,t) \ \triangleq \ \sup_{s \in M \setminus \{t\}} \mathrm{Var}\left(\widehat{f}^t(s) \right).$$

and

$$(13.3.4) \qquad \sigma_c^2(f) \ \triangleq \ \sup_{t \in M} \sigma_c^2(f,t).$$

Proof. We must find an upper bound for (13.3.1). Write

$$\begin{aligned}
\nabla^2 f_{|\partial_i M, E_i}(t) &= \nabla^2 f_{|\partial_i M, E_i}(t) - \mathbb{E}\left\{ \nabla^2 f_{|\partial_i M, E_i}(t) \Big| f(t) \right\} \\
&\qquad\qquad + \mathbb{E}\left\{ \nabla^2 f_{|\partial_i M, E_i}(t) \Big| f(t) \right\} \\
&= \nabla^2 f_{|\partial_i M, E_i}(t) + f(t)I - f(t)I.
\end{aligned}$$

(cf. (12.2.12).). Now expand the determinant in (13.3.1) via a standard Laplace expansion as we did in (11.5.12) and apply Hölder's inequality for

conjugate exponents p and q to see that

$$
\begin{aligned}
|\mathrm{Diff}_{f,M}(u)| \;\leq\; & \sum_{i=0}^{N} \int_{\partial_i M} \sum_{j=0}^{i} \mathbb{E}\left\{f(t)^j \mathbb{1}_{\{f(t)\geq u\}}\right\} \\
& \times \left(\mathbb{E}\left\{\left|\mathrm{detr}_{i-j}\left(-\nabla^2 f_{|\partial_i M, E_i}(t) - f(t)I\right)\right|^p\right\}\right)^{1/p} \\
& \times \left(\mathbb{P}\left\{\widehat{W}(t) \geq u\right\}\right)^{1/q} d\mathcal{H}_i(t)
\end{aligned}
$$

where detr_k is the 'det-trace' defined at (7.2.7). Define

$$
\mu(t) \;\overset{\Delta}{=}\; \mathbb{E}\left\{\sup_{s\in M\setminus t}\widehat{f}^t(s)\right\}, \qquad \mu^+ \;\overset{\Delta}{=}\; \sup_{t\in M}\mu(t)\mathbb{1}_{\{\mu(t)\geq 0\}}.
$$

For

$$
u \;\geq\; \sup_{t\in M}\mathbb{E}\left\{\sup_{s\in M\setminus\{t\}}\left(\widehat{f}^t(s) - \mathbb{E}(\widehat{f}^t(s))\right)\right\} + \mu^+
$$

the Borell-TIS inequality (Theorem 2.1.1) then implies that

$$
\mathbb{P}\left\{\widehat{W}(t) \geq u\right\} \;\leq\; 2\mathrm{e}^{-(u-\mu^+)^2/2\sigma_c^2(f,x)}.
$$

Recalling that $\widehat{f}^t(t) = f(t)$, for such u it also follows that

$$
\mathbb{E}\left\{f(t)^j \mathbb{1}_{\{f(t)\geq u\}}\right\} \;\leq\; C_j u^{j-1}\mathrm{e}^{-(u-\mu^+)^2/2}.
$$

Putting these facts together, for any conjugate exponents p, q

$$
\begin{aligned}
|\mathrm{Diff}&_{f,M}(u)| \\
&\leq\; Cu^{N-1}\mathrm{e}^{-\frac{(u-\mu^+)^2}{2}\left(1+\frac{1}{q\sigma_c^2(f)}\right)} \sum_{i=0}^{N}\int_{\partial_i M}\sum_{j=0}^{i} \\
&\qquad \left(\mathbb{E}\left\{\left|\mathrm{detr}_{i-j}\left(-\nabla^2 f_{|\partial_i M, E_i}(t) - f(t)I\right)\right|^p\right\}\right)^{1/p} d\mathcal{H}_i(t).
\end{aligned}
$$

The result now follows after noting that we can choose q arbitrarily close to 1, and $u(q)$ so that, for $u \geq u(q)$, the remaining terms are arbitrarily small logarithmically when compared to u^2. □

There is, of course, a glaring weakness in Theorem 13.3.3, in that (13.3.2) contains a liminf rather a lim, and it contains an inequality rather than an equality.

Fortunately, the theorem provides a lower bound on the exponential decay of $\mathrm{Diff}_{f,M}(u)$ and this is almost always the important one in any practical situation. We believe that this lower bound is generally tight when a

maximizer of $\sigma_c^2(f)$ occurs in the interior of M, in the sense that then the term corresponding to $\partial_N M$ in the sum defining $\mathrm{Diff}_{f,M}(u)$ in (13.1.18) is exponentially of the same order as the upper bound. However, we have not been able to round off the desired proof, since it seems difficult to determine the sign of the lower order terms in the error. This opens the (unlikely) possibility that some terms in the sum (13.1.18) cancel each other out, leading to a faster rate of exponential decay.

Thus, in the piecewise smooth setting, it is therefore still open as to whether (13.3.2) can be improved on and, if so, under what conditions.

13.4   Examples

We now look at a number of applications of Theorem 13.3.3. The theorem contains two 'unknowns'. One is the expected Euler characteristic and the other is the parameter $\sigma_c^2(f)$. The expected Euler characteristic is the material of Chapters 11 and 12 so we have no need to treat it again here. Consequently, looking at examples means computing, or trying to understand, $\sigma_c^2(f)$, which, for reasons that will become clear below, we now refer to as the *critical variance*. We shall start with the simplest possible example – stationary processes on $\mathbb{R}$ – and then jump to a rather general scenario. We shall then close the chapter with a number of rather instructive special cases.

It is the general scenario which is actually the most instructive, for the simpler ones are so special that they give little or no insight as to what is really going on. Hopefully, after reading the general case, with its elegant geometric interpretation of $\sigma_c^2(f)$, it will be clearer to you from where the constructions of the previous sections of this chapter, particularly $\widehat{f}^t$, came.

The general case also allows us to investigate some very special cases where the approximation of Theorem 13.3.3 is either perfect $(\sigma_c^2(f) = \infty)$ or meaningless $(\sigma_c^2(f) = 0)$. Along the way, we shall also spend a little time with the 'cosine random field' (which actually provides the only example for which the approximation is perfect) and see how it provides both examples and counter-examples for various approximation techniques.

13.4.1   Stationary processes on $[0, T]$

A simple but illustrative example is given by a stationary Gaussian $f(t), t \in [0, T]$ satisfying the conditions of Theorem 13.3.3. The ambient manifold will be $\mathbb{R}$. Thus f has variance 1, and we write ρ for its correlation (and covariance) function. To save on notation, we shall also assume that the time scale has been chosen so that $\mathrm{Var}(\dot{f}(t)) = -\ddot{\rho}(0) = 1$. This makes the Riemannian metric $\widetilde{g}$ to be the usual Euclidean metric on $\mathbb{R}$.

We now need to find the supremum of $\sigma_c^2(f,t)$, which we shall do by dividing the argument into two cases, t in the open set $(0,T)$ and t one of the boundary points $\{0,T\}$. This division into separate cases will be a theme in virtually all of the (more complicated) examples to come. For this we introduce the notation

$$(13.4.1) \qquad\qquad \sigma_c^2(f,A) \overset{\Delta}{=} \sup_{t\in A}\sigma_c^2(f,t).$$

For $t \in (0,T)$ the process $\widehat{f}^t$ is given by

$$\widehat{f}^t(s) \;=\; \frac{f(s) - \rho(t-s)f(t) - \dot\rho(t-s)\dot f(t)}{1 - \rho(t-s)},$$

where we apply (5.5.7) and the fact that the normal cone at interior points is empty so that the projection $P_{N_t M}$ in the definition (13.2.2) disappears.

To compute the critical variance at such t we use the relationships (5.5.7)–(5.5.9). These easily give

$$(13.4.2) \qquad \sigma_c^2(f,t) \;=\; \sup_{\substack{-t\le s\le T-t\\ s\ne 0}} \frac{1 - \rho(s)^2 - \dot\rho(s)^2}{(1 - \rho(s))^2}$$

$$= \sup_{s\in[0,T]\setminus\{t\}} \frac{\mathrm{Var}(f(s)\mid f(t),\dot f(t))}{(1 - \rho(t-s))^2},$$

where the second line also uses (1.2.8).

By (13.4.2) the critical variance in the interior of $[0,T]$ is

$$(13.4.3) \qquad \sigma_c^2(f,(0,T)) \;=\; \sup_{0<t<T} \frac{1 - \rho(t)^2 - \dot\rho(t)^2}{(1 - \rho(t))^2}.$$

The critical variance at the end points $\{0,T\}$ needs a little more care. We consider only the point $t = 0$, since by stationarity the critical variance at T is the same. In the notation of Section 7.1, the normal cone at $t = 0$ is

$$N_0[0,T] \;=\; \bigcup_{c\le 0} c\frac{\partial}{\partial t} \subset T_0\mathbb{R}.$$

For ease of notation, we write N_0 for $N_0[0,T]$. The projection onto N_0 is

$$P_{N_0}\left(a\frac{\partial}{\partial t}\right) \;=\; \mathbb{1}_{\{a\le 0\}}\left(a\frac{\partial}{\partial t}\right)$$

and so the process $\widehat{f}^0(s)$ is given by

$$\widehat{f}^0(s) \;=\; \frac{f(s) - \rho(s)f(0) - \mathbb{1}_{\{\dot\rho(s)\le 0\}}\dot\rho(s)\dot f(0)}{(1 - \rho(s))^2}.$$

The critical variance at $t = 0$ is therefore given by

$$(13.4.4) \quad \sigma_c^2(f, 0) \;=\; \sup_{0 \le t \le T} \frac{1 - \rho(t)^2 - \dot\rho(t)^2 + \max(\dot\rho(t), 0)^2}{(1 - \rho(t))^2}$$

$$\ge \; \sup_{0 \le t \le T} \frac{1 - \rho(t)^2 - \dot\rho(t)^2}{(1 - \rho(t))^2}.$$

This implies that $\sigma_c^2(f, 0) \ge \sigma_c^2(f, t), \; 0 \le t \le T$. Therefore the critical variance $\sigma_c^2(f)$ is attained at the end points $t = 0, T$.

We can summarise the above discussion as follows.

Example 13.4.1 *Suppose f is a centered, unit variance C^2 stationary Gaussian process on $\mathbb{R}$ satisfying the conditions of Theorem 13.3.3 and such that $\mathrm{Var}(\dot f(t)) = -\ddot\rho(0) = 1$. Then, for all $T > 0$,*

$$\sigma_c^2(f, [0, T]) \;=\; \sup_{0 \le t \le T} \frac{1 - \rho(t)^2 - \dot\rho(t)^2 + \max(\dot\rho(t), 0)^2}{(1 - \rho(t))^2}.$$

While this is a reasonably simple expression to compute, it is actually quite trivial if ρ is monotone over $[0, T]$, since then $\dot\rho(t) \le 0$ for all $t > 0$ and the max term disappears. The remaining supremum then turns out to be easy to compute, as we shall see in the next section.

There are a number of results akin to Example 13.4.1 to be found in the literature, some even slightly more general. (cf. [14, 88, 117] and the references therein.) All, however, have proofs which rely on the fact that the random process under consideration is defined over $\mathbb{R}$, proofs which do not generalise to more general scenarios. As we are about to see, the approach we take generalises easily.

13.4.2   Isotropic fields with monotone covariance

Carrying on in the spirit of the previous example, we now consider isotropic fields over convex sets M in $\mathbb{R}^N$. We add a new condition related to a monoticity property for the covariance function. Thus, while this example is more general than Example 13.4.1 in that it treats more general parameter spaces, it is more specific in that it requires more of the covariance function. Nevertheless, it is quite general, and gives a very simple and computable formula for the critical variance σ_c^2.

Example 13.4.2 *Suppose f is a centered, unit variance, C^2 isotropic Gaussian process on $\mathbb{R}^N$ satisfying the conditions of Theorem 13.3.3 and such that*

$$\mathrm{Var}\left(\frac{\partial f(t)}{\partial t_j)}\right) \;=\; -\frac{\partial^2 \rho(t)}{\partial t_j^2}\Big|_{t=0} \;=\; 1,$$

for all j. Suppose that M is compact, piecewise smooth and convex, and set $T = diam(M)$, where the diameter is computed in the standard Euclidean sense.

Suppose also that

(13.4.5)
$$\frac{\partial \rho(t)}{\partial t_j} \leq 0,$$

for all $t \in [0, T]^N$. Then,

(13.4.6) $\sigma_c^2(f, M) \ = \ \mathrm{Var}\left(\frac{\partial^2 f(t)}{\partial t_1^2}\Big| f(t)\right) \ = \ \frac{\partial^4 \rho(t)}{\partial t_1^4}\Big|_{t=0} \ - \ 1.$

Proof. We start by noting that since f is isotropic with unit second spectral moment, the metric induced by f is the standard Euclidean one. Furthermore, we can take $\widetilde{M} = \mathbb{R}^N$.

We need to compute the maximal variance of the random field $\widehat{f}^t(s)$ given by (13.2.2). Consider the term $P_{N_t M}\widehat{F}^t(s)$ which appears there, where

(13.4.7) $\widehat{F}^t(s) \ = \ \sum_{i=1}^{N} \mathrm{Cov}\left(f(s), X_i f(t)\right) X_i$

and the X_i are the usual orthonormal basis elements of $\mathbb{R}^N$.

We claim that under the above assumptions $P_{N_t M}\widehat{F}^t(s)$ is identically zero, in which case $\widehat{f}^t(s)$ trivially simplifies to

(13.4.8)
$$
\begin{aligned}
\widehat{f}^t(s) \ &= \ \frac{f(s) - \rho(t,s)f(t) - \widetilde{g}(\widehat{F}^t(s), \nabla \widetilde{f}(t))}{1 - \rho(t,s)} \\
&= \ \frac{f(s) - \rho(t,s)f(t) - \sum_{i=1}^{N} \mathrm{Cov}(f(s), X_i f(t)) X_i f(t)}{1 - \rho(t,s)}.
\end{aligned}
$$

To prove this claim, we start with $t \in \partial_N M$, the highest order stratum of M. In this case $N_t M = \{0\}$, so that the projection $P_{N_t M}$ is trivially zero and there is nothing to prove.

Thus, take $t \in \partial_k M$ with $k < N$ and $s \in M$, $s \neq t$. Let X_t^s be the unit vector in $T_t \mathbb{R}^N$ in the direction $s - t$. Since M is convex, $X_t^s \in \mathcal{S}_t M$ and, as we shall see in a moment, X_t^s is parallel to $\widehat{F}^t(s)$. Consequently, $X_t^s \in \mathcal{S}_t M$ implies that $P_{N_t M}\widehat{F}^t(s) = 0$, as required

To see that X_t^s is parallel to $\widehat{F}^t(s)$ we use isotropy twice. To make the calculations easier, we use it first to note that isotropy implies that there is no loss of generality involved by taking $t = 0$ and $s = (\tau, 0, \dots, 0)$, where $\tau = |s - t|$. In this case X_t^s is simply X_1.

Secondly, it is easy to check from (5.5.7) that, again because of isotropy,

(13.4.9)

$$
\text{Cov}\left(f(\tau,0,\ldots,0), X_i f(0)\right) \;=\; \begin{cases} -\dot\rho(t) \stackrel{\Delta}{=} -\frac{\partial\rho(t)}{\partial t_1}\big|_{(\tau,0,\ldots,0)} & \text{if } i = 1 \\ 0 & \text{if } i \neq 1. \end{cases}
$$

Substituting into (13.4.7) and using the fact that $\dot\rho(s) \leq 0$ gives us that $\widehat{F}^t(s)$ is proportional to X_1, so that X_t^s is parallel to $\widehat{F}^t(s)$.

We turn now to computing the variance of (13.4.8) for given s, t. Once again, we can choose $t = 0$ and $s = (\tau, 0, \ldots, 0)$, for $\tau = |s - t|$. If we then substitute (13.4.9) into (13.4.8), a little algebra easily leads to the fact that

$$
\text{Var}(\widehat{f}^t(s)) \;=\; \frac{1 - \rho^2(|s - t|) - \dot\rho^2(|s - t|)}{(1 - \rho(|s - t|))^2}
$$

and so

(13.4.10) $\quad \sigma_c^2(f, M) \;=\; \sup_{s \neq t \in M} \dfrac{1 - \rho^2(|s - t|) - \dot\rho^2(|s - t|)}{(1 - \rho(|s - t|))^2}.$

By isotropy, we need not take the supremum over t here, and so fix t, taking $t = 0$. We now claim is that the supremum over s is achieved as $s \to 0$. If this is true, then isotropy implies that we can take $s \to 0$ along any Euclidean axis. This being the case, two applications of l'Hopital's rule show that

$$
\sigma_c^2(f, M) \;=\; \frac{\partial^4 \rho(t)}{\partial t_1^4}\bigg|_{t=0} \;-\; 1,
$$

while (5.5.7) and the basic properties of Gaussian distributions give that this, in turn, is the same as $\text{Var}\left(\frac{\partial^2 f(t)}{\partial t_1^2}\big| f(t)\right)$.

Consequently, modulo the issue of where the supremum in (13.4.10) is achieved, we have completed the proof. We shall return to this issue at the end of the next section. □

13.4.3   A geometric approach

In this section we want to look a little more carefully at the critical variance

(13.4.11) $\qquad\qquad \sigma_c^2(f) \;=\; \sup_{t \in M} \sigma_c^2(f, t)$

where

(13.4.12) $\qquad\qquad \sigma_c^2(f, t) \stackrel{\Delta}{=} \sup_{s \in M \setminus \{t\}} \text{Var}\left(\widehat{f}^t(s)\right).$

and $\widehat{f}^t$ is given by (13.2.2). (cf. (13.3.3) and (13.3.4).) Our aim is to give a geometrical interpretation of $\sigma_c^2(f)$ in a quite general setting. With this, we shall return to complete the proof of Example 13.4.2.

We shall start by looking closely at (13.4.12), for points[7] $t \in \partial_N M = M^\circ$. In later examples we shall see that these are often the most important points to consider. They are also the points for which a geometric intepretation of (13.4.12) is most accessible.

The geometry which comes into play here is that of H, the reproducing kernel Hilbert space (RKHS) of Section 3.1. In particular, since we shall retain the assumption that f has constant unit variance, we shall be concerned with the spherical geometry of the unit sphere $S(H)$ in H.

The unit variance assumption also implies that the H inner product between $f(t)$ and $f(s)$ is given by

$$(f(t), f(s))_H \;=\; \rho(s, t),$$

where ρ is the correlation function of f.

As a first step, we define a map $\Psi : M \to S(H)$ by

$$\Psi(t) \;=\; f(t).$$

Under the assumptions of Theorem 13.3.3 it is immediate that Ψ is a piecewise C^2 embedding of M in $S(H)$. Further, it is not hard to see that

$$\Psi_*(X_t) \;=\; X_t f,$$

so that the tangent space $T_{f(t)}\Psi(M)$ is spanned by $(X_{1,t}f, \ldots, X_{N,t}f)$ for any basis $\{X_{1,t}, \ldots, X_{N,t}\}$ of $T_t M$. We denote the orthogonal complement of $T_{f(t)}\Psi(M)$ in $T_{f(t)}H$ by $T_{f(t)}^\perp \Psi(M)$, and the orthogonal complement of $T_{f(t)}\Psi(M)$ in $T_{f(t)}S(H)$ by $N_{f(t)}\Psi(M)$.

Given a point $f(t)$ and a unit normal vector $v_{f(t)} \in N_{f(t)}\Psi(M)$ we denote the geodesic, in $S(H)$, originating at $f(t)$ in the direction $v_{f(t)}$ by $c_{f(t),v_{f(t)}}$. It is easy to check that this is given by

$$(13.4.13) \quad c_{f(t),v_{f(t)}}(r) \;=\; \cos r \cdot f(t) + \sin r \cdot v_{f(t)}, \quad 0 \le r < \pi.$$

As discussed in Section 10.3.1, when we looked at tube formulae, up to a certain point along $c_{f(t),v_{f(t)}}$ the points on this geodesic project[8] uniquely to t. The largest r for which this is true is the local critical radius at $f(t)$

---

[7]Actually, the arguments will work in slightly greater generality. For example, if M is composed of a number of disjoint piecewise smooth manifolds, then they will also hold for all t in the top dimensional stratum of the separate piecewise smooth manifolds. For example, if $M = [0,1]^2 \cup (\{3\} \times [0,1])$ then it will hold for all $t \in (0,1)^2 \cup (\{3\} \times (0,1))$.

[8]That is, for r small enough, the unique point on $\Psi(M)$ closest to $c_{f(t),v_{f(t)}}(r)$, in the geodesic metric, is $f(t)$. This is generally referred to as *metric projection*.

in the direction $v_{f(t)}$ and we denote it by $\theta(f(t), v_{f(t)})$. The local critical radius at $f(t)$ is then

$$(13.4.14) \qquad \theta_c(f(t)) \;=\; \inf_{v_{f(t)} \in S(N_{f(t)}\Psi(M))} \theta(f(t), v_{f(t)})$$

and the global critical radius is

$$(13.4.15) \qquad \theta_c \;=\; \theta_c(\Psi(M)) \;=\; \inf_{t \in M} \theta_c(f(t)).$$

The following lemma links these critical radii of $\Psi(M)$ to the critical variances (13.4.11) and (13.4.12).

Lemma 13.4.3 *Under the setup and assumptions of Theorem 13.3.3,*

$$(13.4.16) \quad \sigma_c^2(f,t) \;=\; \begin{cases} \cot^2(\theta_c(f(t))) & \text{if } 0 \le \theta_c < \pi/2 \\ \infty & \text{if } \pi/2 \le \theta_c < \pi, \end{cases}$$

for all $t \in \partial_N M$.

Proof. Much as in the proof of Lemma 13.2.1 and Example 13.4.2, we start by noting that if $t \in \partial_N M$, then $N_t M = T_t \partial_N M^\perp$ and $P_{N_t M} V_t = 0$ for all $V_t \in T_t \partial_N M$. Now choose an orthonormal frame field for $T(\widetilde{M})$ so that, at every $t \in \partial_N M$, the first N vectors in the frame field generate $T_t M$ while the remaining vectors generate $N_t M$ (as a subspace of $T_t \widetilde{M}$). Then $\widehat{f}^t$ simplifies somewhat, and for $s \ne t$, $t \in \partial_N M$, it is easy to check that

$$\begin{aligned}
\widehat{f}^t(s) &= \frac{f(s) - \rho(t,s)f(t) - \widetilde{g}(\widehat{F}^t(s), \nabla \widetilde{f}(t))}{1 - \rho(t,s)} \\
&= \frac{f(s) - \rho(t,s)f(t) - \sum_{i=1}^N \mathrm{Cov}(f(s), X_i f(t)) X_i f(t)}{1 - \rho(t,s)}.
\end{aligned}$$

We now compute the variance of $\widehat{f}^t(s)$. Note that

$$\rho(t,s)f(t) + \sum_{i=1}^N \mathrm{Cov}(f(s), X_i f(t)) X_i f(t)$$

$$= (f(t), f(s))_H f(t) + \sum_{i=1}^N (f(s), X_i f(t))_H X_i f(t)$$

$$= P_{L_t} f(s),$$

where $L_t = \mathrm{span}\{f(t), X_1 f(t), \ldots, X_N f(t)\}$.

However, $N_{f(t)}\Psi(M)$ is the orthogonal complement (in $T_{f(t)}H$) of the subspace L_t. Consequently,

$$\begin{aligned}
\mathrm{Var}\left(\widehat{f}^t(s)\right) &= \mathrm{Var}\left(f(s) - \rho(t,s)f(t) - \sum_{i=1}^N \mathrm{Cov}(f(s), X_i f(t)) X_i f(t)\right) \\
&= \|P_{N_{f(t)}\Psi(M)} f(s)\|_H^2
\end{aligned}$$

where $P_{N_{f(t)}\Psi(M)}$ represents orthogonal projection onto $N_{f(t)}\Psi(M)$.

Finally, we have that

(13.4.17) $$\sigma_c^2(f,t) \; = \; \sup_{s \in M\setminus\{t\}} \frac{\|P_{N_{f(t)}\Psi(M)}f(s)\|_H^2}{(1-\rho(t,s))^2}.$$

We now need to show that the same expression holds for $\cot^2(\theta_c(f(t)))$. Turning to the picture in terms of geodesics, fix $f(t)$ and $v_{f(t)}$. Suppose that, for a certain r, the point $c_{f(t),v_{f(t)}}(r)$ does not metrically project to $f(t)$. This and a little basic Hilbert sphere geometry implies there is a point $f(s) \in \Psi(M)$ such that

(13.4.18)
$$
\begin{aligned}
\cos^{-1}\left(\left(c_{f(t),v_{f(t)}}(r), f(s)\right)_H\right) \; &= \; \tau\left(c_{f(t),v_{f(t)}}(r), f(s)\right) \\
&< \; \tau\left(c_{f(t),v_{f(t)}}(r), f(t)\right) \\
&= \; \cos^{-1}\left(\left(c_{f(t),v_{f(t)}}(r), f(t)\right)_H\right),
\end{aligned}
$$

where τ is geodesic distance on $S(H)$.

Alternatively, applying (13.4.13),

$$
\begin{aligned}
\left(c_{f(t),v_{f(t)}}(r), f(s)\right)_H \; &= \; \cos r \cdot (f(t),f(s))_H + \sin r \cdot \left(v_{f(t)}, f(s)\right)_H \\
&= \; \cos r \cdot \rho(t,s) + \sin r \cdot \left(v_{f(t)}, f(s)\right)_H \\
&> \; \left(c_{f(t),v_{f(t)}}(r), f(t)\right)_H \\
&= \; \cos r.
\end{aligned}
$$

A little rearranging shows that this is true if and only if

$$\cot t \; < \; \frac{\left(v_{f(t)}, f(s)\right)_H}{1-\rho(t,s)}.$$

Therefore,

$$\cot\left(\theta(f(t), v_{f(t)})\right) \; = \; \sup_{s \in M\setminus\{t\}} \frac{\left(v_{f(t)}, f(s)\right)_H}{1-\rho(t,s)}.$$

Taking the supremum over all $v_{f(t)} \in S(N_{f(t)}\Psi(M))$ we find

$$
\begin{aligned}
\cot\left(\theta_c(f(t))\right) \; &= \; \sup_{v_{f(t)} \in S(N_{f(t)}\Psi(M))} \sup_{s \in M\setminus\{t\}} \frac{\left(v_{f(t)}, f(s)\right)_H}{1-\rho(t,s)} \\
&= \; \sup_{s \in M\setminus\{t\}} \frac{\|P_{N_{f(t)}\Psi(M)}f(s)\|_H}{1-\rho(t,s)}.
\end{aligned}
$$

Comparing this to (13.4.17) completes the proof.   □

Lemma 13.4.3 gives a very succinct representation for the critical variance $\sigma_c^2(f)$ that requires nothing about the rather complicated process $\widehat{f}^t$ that led to its original definition. Here is another result, part of which relates to the geometric structure of H and part of which has a rather down to earth implication for the correlation function ρ.

Lemma 13.4.4 *Retain the assumptions of Lemma 13.4.3, and suppose that the pair $(t^*, s^*) \in \partial_N M \times \partial_N M$ achieves the supremum in*

$$(13.4.19) \qquad \sup_{t \in M} \sup_{s \in M \setminus \{t\}} \frac{\|P_{N_{f(t)}\Psi(M)} f(s)\|_H^2}{(1 - \rho(t,s))^2}.$$

Then,

$$(13.4.20) \qquad f(s^*) \;=\; \cos(2\theta_c) \cdot f(t^*) + \sin(2\theta_c) \cdot v^*_{f(t^*)}$$

where

$$v^*_{f(t^*)} \;=\; \frac{P_{N_{f(t)}\Psi(M)} f(s^*)}{\|P_{N_{f(t)}\Psi(M)} f(s^*)\|_H}$$

is the direction of the geodesic between $f(t^*)$ and $f(s^*)$. For any $X_{t^*} \in T_{t^*}M$ and any $X_{s^*} \in T_{s^*}M$,

$$(13.4.21) \qquad \mathrm{Cov}(X_{t^*} f, f(s^*)) \;=\; (\Psi_*(X_{t^*}), f(s^*))_H \;=\; 0,$$
$$(13.4.22) \qquad \mathrm{Cov}(X_{s^*} f, f(t^*)) \;=\; (\Psi_*(X_{s^*}), f(t^*))_H \;=\; 0.$$

Furthermore, the partial map

$$\rho^{t^*}(s) \;\triangleq\; \rho(t^*, s)$$

has a local maximum at s^*, and the partial map

$$\rho^{s^*}(t) \;\triangleq\; \rho(t, s^*)$$

has a local maximum at t^*.

In order to appreciate the geometry behind (13.4.20) you might think as follows: Given a pair of points in $\partial_N M$ which maximise (13.4.19), the three equations (13.4.20), (13.4.21) and (13.4.22) imply that the tube of radius θ_c around $\Psi(M)$ should self-intersect itself along a geodesic from t^* to s^* in such a way that the tube, viewed locally from the point t^*, shares a hyperplane with the tube viewed locally from the point s^*. Alternatively, at the point of self-intersection the outward pointing unit normal vectors should be pointing in opposite directions.

Proof. Our first requirement is to establish (13.4.20). Looking back through the proof of Lemma 13.4.3 (cf. (13.4.18)) it is immediate that if a pair (t^*, s^*) achieves the supremum in (13.4.19) then there exists a point $z \in H$ equidistant from $f(t^*)$ and $f(s^*)$ and unit vectors $v_{f(t^*)} \in N_{f(t^*)}\Psi(M)$ and $w_{f(s^*)} \in N_{f(s^*)}\Psi(M)$ such that

$$z \;=\; \cos\theta_c \cdot f(t^*) + \sin\theta_c \cdot v_{f(t^*)} \;=\; \cos\theta_c \cdot f(s^*) + \sin\theta_c \cdot w_{f(s^*)}.$$

It is not *a priori* obvious that $w_{f(s^*)} \in N_{f(s^*)}\Psi(M)$. However, if this were not the case, then there exists $t \in M$, $t \neq s$ such that $d(t, z) < \theta_c$. This would contradict the assumption that (t^*, s^*) achieve the supremum in (13.4.19) and the critical radius is θ_c. This proves (13.4.20).

To prove the (13.4.21) and (13.4.22) it is enough to show that the unit tangent vectors $\dot{c}_{f(t^*),v_{f(t^*)}}(\theta_c)$ and $\dot{c}_{f(s^*),w_{f(s^*)}}(\theta_c)$ at z satisfy

$$(13.4.23) \quad u \;=\; \dot{c}_{f(t^*),w_{f(t^*)}}(\theta_c) + \dot{c}_{f(s^*),w_{f(s^*)}}(\theta_c) \;=\; 0 \in T_z H.$$

Suppose that $u \neq 0$. A simple calculation shows that

$$(u, f(t^*))_H \;=\; (u, f(s^*))_H \;=\; \frac{\cos(2\theta_c) - (f(t^*), f(s^*))_H}{\sin\theta_c}.$$

If $u \neq 0$, then $f(t^*), f(s^*)$ and z are not on the same geodesic and the triangle inequality implies that

$$\cos(2\theta_c) \;<\; (f(t^*), f(s^*))_H,$$

which implies that

$$(u, f(t^*))_H \;=\; (u, f(s^*))_H \;<\; 0.$$

Consider the geodesic originating at z in the direction $u^* = u/\|u\|_H$ and given by

$$c_{z,u^*}(t) \;=\; \cos t \cdot z + \sin t \cdot u^*.$$

Assuming that $u^* \neq 0$, we have

$$(\dot{c}_{z,u^*}(0), f(s^*))_H \;=\; (\dot{c}_{z,u^*}(0), f(t^*))_H \;=\; (u^*, f(t^*)) \;<\; 0.$$

This implies that, for $r < 0$ and sufficiently small $|r|$,

$$(f(t^*), c_{z,u^*}(r))_H \;=\; (f(s^*), c_{z,u^*}(r))_H \;>\; \cos\theta_c.$$

For such an r, there exist distinct points $\widehat{t}(r)$ and $\widehat{s}(r)$ such that

$$d(c_{z,u^*}(r), \widehat{t}(r)) \;<\; \theta_c, \qquad d(c_{z,u^*}(r), \widehat{s}(r)) \;<\; \theta_c,$$

such that the geodesic connecting $\widehat{t}(r)$ and $c_{z,u^*}(r)$ is normal to M at $\widehat{t}(r)$, and the geodesic connecting $\widehat{s}(r)$ and $c_{z,u^*}(r)$ is normal to M at $\widehat{s}(r)$.

Without loss of generality we assume that

$$d(c_{z,u^*}(r), \widehat{t}(r)) \leq d(c_{z,u^*}(r), \widehat{s}(r)).$$

In this case, the geodesic connecting $\widehat{s}$ to $c_{z,u^*}(r)$ is no longer a minimizer of distance once it passes the point $c_{z,u^*}(r)$, which is of distance strictly less than θ_c from $\widehat{s}(r)$. That is, for some point $\widehat{z}(r)$ along the geodesic connecting $\widehat{s}(r)$ and $c_{z,u^*}(r)$, beyond $c_{z,u^*}(r)$ but of distance strictly less than θ_c from $\widehat{s}(r)$, there exist points in M strictly closer to $\widehat{z}(r)$ than $\widehat{s}(r)$. We therefore have a contradiction, as the critical radius of $\Psi(M)$ is θ_c.

Consequently we have that the u of (13.4.23) is indeed zero, and so (13.4.21) and (13.4.22) are proven.

To prove claims about the partial maps $\rho^{t^*}(s)$ and $\rho^{s^*}(t)$ note first that $f(s^*)$ is a linear combination of $f(t^*)$ and $v_{f(t^*)}$ which are perpendicular to every vector in $T_{f(t^*)}\Psi(M)$. Similarly, $f(t^*)$ is a linear combination of $f(s^*)$ and $w_{f(s^*)}$ which are perpendicular to every vector in $T_{f(t^*)}\Psi(M)$. The fact that the partial maps $\rho^{t^*}(s)$ and $\rho^{s^*}(t)$ have local maxima then follows from the same contradiction argument used above.   □

Conclusion of the proof of Example 13.4.2.   Recall that what remained was to show that the supremum in

$$(13.4.24) \quad \sup_{s \neq t \in M} \frac{1 - \rho^2(|s-t|) - \dot\rho^2(|s-t|)}{(1 - \rho(|s-t|))^2} \; = \; \sup_{t \to 0} \frac{1 - \rho^2(t) - \dot\rho^2(t)}{(1 - \rho^2(t))}.$$

(cf. (13.4.10).)

It is trivial that the right hand side of (13.4.24) serves as a lower bound to the left hand side. It is also trivial that left hand side is less than

$$(13.4.25) \quad \sup_{t \in [0, \text{diam}(M)]^N} \frac{1 - \rho^2(t) - \dot\rho^2(t)}{(1 - \rho(t))^2}.$$

Thus if we can show that this is equal to the right hand side of (13.4.24) we shall be done.

Let t^* be one of the points achieving the supremum in (13.4.25) with $N = 1$. Then Lemma 13.4.4 implies that $\dot\rho(t^*) = 0$, and that t^* is a local maximum of $\rho(t)$. But under the monotonicity assumption of the example, there is only one such point, given by $t^* = 0$.   □

13.4.4   The cosine field

We now turn the what is perhaps the grandfather of all Gaussian random fields, the so-called *cosine random field* on $\mathbb{R}^N$. It is defined as

$$(13.4.26) \quad f(t) \overset{\triangle}{=} N^{-1/2} \sum_{k=1}^{N} f_k(\omega_k t_k),$$

where each f_k is the process on $\mathbb{R}$ given by

$$f_k(t) \;\triangleq\; \xi_k \cos t + \xi_k' \sin t.$$

The ξ_k and ξ_k' are independent, standard Gaussians and the ω_k are positive constants.

It is a trivial computation that f is a stationary, centered Gaussian process on $\mathbb{R}^N$ with covariance and correlation functions

$$C(t) \;=\; \rho(t) \;=\; N^{-1} \sum_{k=1}^{N} \cos(\omega_k t_k).$$

We shall concentrate on the case in which $\omega_k = 1$ for all k, in which case it is easy to check that

$$\mathbb{E}\{f^2(t)\} \;=\; \mathbb{E}\{(X_j f(t))^2\} \;=\; \mathbb{E}\{(X_j X_j f(t))^2\} \;=\; 1,$$

for all j, where $X_1, \ldots, X_j$ is the usual orthonormal basis of $\mathbb{R}^N$. Similarly,

$$\mathbb{E}\{X_i f(t) X_j f(t)\} \;=\; 0$$

if $i \neq j$.

The cosine field is important for a number of reasons, perhaps the most significant of which is the fact that it is simple enough to enable explicit computations. For example, set $N = 1$ and consider the excursion probability

(13.4.27)
$$\mathbb{P}\left\{ \sup_{0 \le t \le T} f(t) \ge u \right\}$$

for $u > 0$. Note that

(13.4.28)
$$\xi \cos t + \xi' \sin t \;=\; R \cos(t - \theta),$$

where $R^2 = \xi^2 + (\xi')^2$ has an exponential distribution with mean $1/2$, $\theta = \arctan(\xi/\xi')$ has a uniform distribution on $[0, 2\pi]$, and R and θ are independent. You can use this information to compute the excursion probability (13.4.27) directly, but here is an easier way:

Assume that $T < \pi$ and write N_u for the number of upcrossings of u in $(0, T]$. Then

$$\mathbb{P}\left\{ \sup_{0 \le t \le T} f(t) \ge u \right\} \;=\; \mathbb{P}\{f(0) \ge u\} \;+\; \mathbb{P}\{f(0) < u, N_u \ge 1\}.$$

But since $T < \pi$, the event $\{f(0) \ge u, N_u \ge 1\}$ is empty, implying that

$$\mathbb{P}\{f(0) < u, N_u \ge 1\} \;=\; \mathbb{P}\{N_u \ge 1\}.$$

Again using the fact that $T < \pi$, we note that N_u is either 0 or 1, so that

$$\mathbb{P}\left\{\sup_{0 \le t \le T} f(t) \ge u\right\} = \Psi(u) + \mathbb{E}\{N_u\}.$$

Looking back at Chapter 11, in particular (11.6.13), what we have just proven is that

$$(13.4.29) \qquad \mathbb{P}\left\{\sup_{0 \le t \le T} f(t) \ge u\right\} \equiv \mathbb{E}\left\{\varphi\left(A_u(f, [0, T])\right)\right\}.$$

In other words, the approximation that has been at the center of this chapter is *exact* for the one-dimensional cosine process on an interval of length less than π.

On the other hand, if $\varphi < T \le 2\pi$ then much of the above argument breaks down and (13.4.29) no longer holds. If $T > 2\pi$, then, since the cosine process is periodic, increasing T has no affect whatsoever on $\sup_{[0,T]} f(t)$ and the left hand side of (13.4.29) is independent of T. The right hand side, however, contains the term $E\{N_u\}$, which grows linearly with T. Hence we see that the assumption in the central Theorem 13.3.3 that $\rho(t, s) = 1 \iff t = s$, which does not hold for periodic processes, was necessary.

But more than this can be seen from this very simple example. For example, if we take an interval of length greater than 2π, so that we get at least a full cycle of the process, the excursion probability is simply

$$\mathbb{P}\{R \ge u\} = e^{-u^2/2},$$

where R is defined at (13.4.28). Note that this does not depend on T. On the other hand, $\mathbb{E}\left\{\varphi\left(A_u(f, [0, T])\right)\right\}$ contains a term growing linearly in T, and so here the approximation breaks down. To relate these phenomena to the rest of this chapter, we need to look at the critical variance σ_c^2 for this example.

Unfortunately, we do not have among the results we have already proven one which immediately yields σ_c^2. However, we do have a technique. The technique is that of the proof of Example 13.4.2. Although the cosine process over $\mathbb{R}^N$ is not isotropic, as that example requires, if we restrict f to a unit cube of the form $[0, T]^N$ with $T \le \pi/2$ then the same argument as given there also works now. The conclusion is again that

$$(13.4.30) \qquad \sigma_c^2\left(f, [0, T]^N\right) = \mathrm{Var}\left(\frac{\partial^2 f(t)}{\partial t_1^2} \Big| f(t)\right) = N - 1,$$

the rightmost expression resulting from a straightforward computation.

Taking $N = 1$ in (13.4.30) we immediately see why (13.4.29) is true. In this case the critical variance is zero, and so Theorem 13.3.3 tells us that the mean Euler characteristic approximation to the excursion probability is, in fact, exact. In higher dimensions, however, this is no longer the case.

The cosine process can also be exploited to show why the basic approximation of Theorem 13.3.3 works only for piecewise smooth manifolds, and not for Whitney stratified manifolds[9]. Consider a two-dimensional cosine process restricted to the set L_α depicted in Figure 13.4.1. We take the angle α (measured from the horizontal line) in $[-\pi/2, \pi/2]$ and we take each edge to be of length $T < \pi/4$. (Our convention is that if $\alpha = -\pi/2$ then the resulting vertical edge is *above* the horizontal one.)
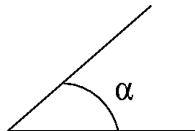
FIGURE 13.4.1. The stratified manifold L_α.

It is easy to check that, regardless of the value of α, when restricted to an edge the cosine field is stationary with variance one and second spectral moment $\lambda_2 = 1/2$. Furthermore, the expected Euler characteristic of the excursion set $A_u(f, L_\alpha)$ is independent of α and is given by

$$\mathbb{E}\left\{\varphi\left(A_u(f, L_\alpha)\right)\right\} \;=\; \Psi(u) \;+\; \frac{T}{\sqrt{2\pi}} e^{-u^2/2}.$$

The excursion probability, however, is most definitely not independent on α. For example, if $\alpha = \pi/2$, so that L_α is simply a straight line of length $2T$, then the argument above for the one dimensional cosine process gives that the excursion probability is identical to $\mathbb{E}\left\{\varphi\left(A_u(f, L_\alpha)\right)\right\}$. On the other other hand, if $\alpha = -\pi/2$ then L_α is a straight line of length T and so

$$\mathbb{P}\left\{\sup_{L_T} f(t) \geq u\right\} \;=\; \Psi(u) \;+\; \frac{T}{2\sqrt{2\pi}} e^{-u^2/2}.$$

This differs from $\mathbb{E}\left\{\varphi\left(A_u(f, L_\alpha)\right)\right\}$ by a factor equal to the last term here, and not by a lower order expression as we might have hoped for from Theorem 13.3.3. In fact, in the notation of that theorem we have, for this case, that the critical variance σ_c is infinite, or, in the notation of of Lemma 13.4.3, that the critical radius θ_c is zero[10].

All this comes from the non-(local)-convexity of L_α.

Before leaving the cosine field, we note that in actually deserves far more credit than we have given it so far. To this point, it has been a curiousity

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[9]Recall that the main difference between these two classes of objects is that piecewise smooth manifolds have convex support cones. Back in Chapter 12 we were able to obtain expected Euler characteristics of excursion sets in both settings. In the current chapter, however, we have been restricted to the locally convex scenario.

[10]For more sophisticated examples of what can, and usually does, go wrong when the critical radius is zero see [140].

useful for providing examples and counter-examples. In truth, however, it was the cosine process that gave birth to this entire chapter and also had much to do with the Lipschitz-Killing curvatures that appeared in Chapter 12. The history, which we describe only briefly, is as follows.

Firstly, we restrict our attention to the cosine process on a rectangle of the form $T = \prod_{k=1}^{N}[0, T_k]$, where $T_k \in (0, \pi/2]$. Then, given the explicit form of the cosine process (we are assuming still that $\omega_k = 1$ for all k), note that

$$
\begin{aligned}
M_N &= \sup_{t \in T} \sum_{k=1}^{N} \frac{1}{\sqrt{N}} \sup_{0 \leq t_k \leq T_k} \{\xi_k \cos t_k + \xi'_k \sin t_k\} \\
&\triangleq \frac{1}{\sqrt{N}} \sum_{k=1}^{N} m_k,
\end{aligned}
$$

where the random variables m_k are independent.

However, we know the precise distribution of each of the m_k by (13.4.29), so that, in principle at least, we should now be able to find the distribution of M_N.

It turns out that this computation is a little too involved to be done exactly, and some approximations need to be made along the way. This is what Piterbarg did in [117], and what he derived was basically Theorem 13.3.3 for this case. You can find all the details of the calculations in [117].

Theorem 13.3.3 is much richer, however, since it treats quite general processes over piecewise smooth spaces. To go from the cosine process to a wider class of processes, Piterbarg used Slepian's Inequality (Theorem 2.2.1). While Slepian's inequality allowed an extension beyond the cosine process, it did restrict the result to random fields for which the matrix of second order spectral moments was diagonal. After all we have seen, we now know this to be a form of local isotropy and that the way to remove this assumption is via the Riemannian metric induced by the field.

To go from rectangles to a wider class of parameter sets made up of locally convex cells, not unlike the way we built up the basic complexes of Chapter 6, Piterbarg used a result not unlike our Theorem 6.3.1. As a consequence, he obtained an expression for the excursion probability involving Minkowski functionals, which we now know to be a Euclidean version of Lipschitz-Killing curvatures.

What Piterbarg did not realise was that the approximation that he obtained for the excursion probability, involving second order spectral moments and Minkowski functionals, was in fact the expected Euler characteristic of excursion sets. This was realised only by one of us in [5]. In the meantime Keith Worsely had worked out some expected Euler characteristics for isotropic random fields on two dimensional surfaces (e.g. [164]) and Lipschitz-Killing curvatures (although not yet recognised as such) began to appear.

All this came together in the thesis [147] where the induced Riemannian metric appeared for the first time. As we have seen, this allowed the development of a theory that was no longer restricted to stationary (and often also isotropic) random fields.

The rest is history, and the (currently) final result is what you have been reading for the best part of 400 pages.

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