Use perturbation method to approximate the solution to a problem that involves a small parameter, usually denoted by ϵ.

---

approximate the roots of

$$\epsilon\left(x^3+2\right) + x^2 + 1 = 0$$

approximate a solution to the nonlinear system

$$\epsilon x' = x + y$$

$$y' = x + \epsilon y^2$$

nonlinear oscillator

$$y'' + y + \epsilon f(y, y') = 0$$

Example problems

<u>Example</u>

$$x^2 + \epsilon x - 1 = 0$$

by the quadratic formula: $x = \dfrac{-\epsilon \pm \sqrt{\epsilon^2+4}}{2}$

this is the exact solution

let us expand the square root by Taylor series

$$x = -\frac{\epsilon}{2} \pm \sqrt{1+\left(\frac{\epsilon}{2}\right)^2} = -\frac{\epsilon}{2} \pm \left[1 + \frac{(\epsilon/2)^2}{2} + \ldots\right]$$

$$\rightarrow \quad x = \pm 1 - \frac{\epsilon}{2} \pm \frac{\epsilon^2}{8} + \ldots$$

$$\boxed{x^2 + \epsilon x - 1 = 0}$$

solution by perturbation method

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$$

determine x_0, x_1, x_2

$$\rightarrow \left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots\right)^2 + \epsilon\left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots\right) - 1 = 0$$

$$\left(x_0^2 + \epsilon^2 x_1^2 + \epsilon^4 x_2^2 + 2x_0x_1\epsilon + 2x_0x_2\epsilon^2 + \cdots\right) + \epsilon\left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots\right) - 1 = 0$$

$$\left(x_0^2 - 1\right) + \epsilon\left(2x_0x_1 + x_0\right) + \epsilon^2\left(x_1^2 + 2x_0x_2 + x_1\right) + \cdots = 0$$

$$x_0^2 - 1 = 0 \rightarrow \boxed{x_0 = \pm 1}$$

$$2x_0x_1 + x_0 = 0 \rightarrow x_0(2x_1 + 1) = 0 \rightarrow \boxed{x_1 = -\frac{1}{2}}$$

$$x_1^2 + 2x_0x_2 + x_1 = 0 \rightarrow \cancel{\frac{1}{4} + 2x_0x_2 - \frac{1}{2} = 0}$$

$$\rightarrow \cancel{2(x_0 \pm 1)x_2 = \frac{1}{4} \rightarrow x_2 = \frac{1}{8} \rightarrow \boxed{x_2 = \pm\frac{1}{8}}}$$

$$x_2 = \frac{-x_1(1 + x_1)}{2x_0}$$

$$x_1 = -\frac{1}{2} \rightarrow x_2 = \frac{1}{8x_0}$$

when $x_0 = 1$ $x_2 = +1/8$ and when $x_0 = -1$ $x_2 = -1/8$

so $\boxed{x_2 = \pm 1/8}$

$\cancel{\Rightarrow x =}$

$$\Rightarrow \boxed{x = \pm 1 - \frac{\epsilon}{2} \pm \frac{1}{8}\epsilon^2 + \cdots}$$

Example 2

$$(1-\epsilon)x^2 - 2x + 1 = 0$$

quadratic eqn so we have exact solution

$$x = \frac{2 \pm \sqrt{4-4(1-\epsilon)}}{2(1-\epsilon)} = \frac{1 \pm \sqrt{\epsilon}}{1-\epsilon}$$

Let us try the naive expansion

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$$

$$(1-\epsilon)(x_0^2 + \epsilon^2 x_1^2 + 2x_0x_1\epsilon + 2x_0x_2\epsilon^2) - 2(x_0 + \epsilon x_1 + \epsilon^2 x_2) + 1 = 0$$

$$(x_0^2 - 2x_0 + 1)\epsilon^0 + (-x_0^2 + 2x_0x_1 - 2x_1)\epsilon^1 + (x_1^2 + 2x_0x_2 - 2x_0x_1 - 2x_2) + \cdots = 0$$

leading order term

$O(1)$: $\quad x_0^2 - 2x_0 + 1 = (x_0-1)^2 = 0 \rightarrow \boxed{x_0 = 1, 1}$

$O(\epsilon)$: $\quad 2x_0x_1 - 2x_1 - x_0^2 = 0$

$$\cancel{2x_1 - 2x_1} - 1 = 0$$ this is an issue

so the naive expansion does not work

Try an expansion in terms of $\epsilon^{1/2}$

by the quadratic eqn, we found

$$x = \frac{1 \pm \sqrt{\epsilon}}{1-\epsilon} = (1 \pm \sqrt{\epsilon})(1 + \epsilon + \epsilon^2 + \cdots)$$

$$x = 1 + \epsilon + \epsilon^2 + \cdots \pm \epsilon^{1/2} \pm \epsilon^{3/2} \pm \epsilon^{5/2} + \cdots$$

$$\boxed{x = 1 \pm \epsilon^{1/2} + \epsilon + \cdots}$$

So a general expansion for x is

$$x = x_0 + \delta_1(\epsilon)\, x_1 + \delta_2(\epsilon)\, x_2 + \cdots$$

with the property that

$$\delta_i \to 0 \quad \text{as} \quad \epsilon \to 0$$

$$\text{and} \quad \delta_{i+1} << \delta_i \quad \text{as} \quad \epsilon \to 0$$

the functions δ_i are called Gauge functions, or scaling functions

So substitute expansion into $(1-\epsilon)\, x^2 - 2x + 1 = 0$

$$\Rightarrow (1-\epsilon)\left(x_0^2 + \delta_1^2 x_1^2 + \delta_2^2 x_2^2 + 2x_0 \delta_1 x_1 + 2x_0 x_2 \delta_2 + 2x_1 x_2 \delta_1 \delta_2 + \cdots\right)$$

$$- 2\left(x_0 + \delta_1 x_1 + \delta_2 x_2 + \cdots\right) + 1 = 0$$

We don't know magnitude of δ_1, δ_2, though we know they are small, but we can collect leading order term.

$$O(1) = x_0^2 - 2x_0 + 1 = 0 \quad \to \quad x_0 = 1$$

Substitute $x_0 = 1$ into expression

$$(1-\epsilon)\left(1 + \delta_1^2 x_1^2 + \delta_2^2 x_2^2 + 2\delta_1 x_1 + 2x_2\delta_2 + 2x_1x_2\delta_1\delta_2 + \dots\right)$$

$$-2\left(1 + \delta_1 x_1 + \delta_2 x_2 + \dots\right) + 1 = 0$$

$$\Rightarrow \delta_1^2 x_1^2 + \delta_2 x_2^2 + \cancel{2\delta_1 x_1} + 2\cancel{x_2\delta_2} + 2x_1x_2\delta_1 + \dots$$

$$-\epsilon - x_1^2\epsilon\delta_1^2 - x_2^2\epsilon\delta_2^2 - 2x_1\epsilon\delta_1 - 2x_2\epsilon\delta_2 - 2x_1x_2\epsilon\delta_1 + \dots$$

$$\cancel{-2x_1\delta_1} - \cancel{2x_2}\delta_2 = 0$$

consider terms

$$\delta_1^2 x_1^2 - \epsilon = 0$$

① ②

if ① dominates then $\delta_1^2 x_1^2 = 0 \rightarrow x_1 = 0$

if ② dominates then $-\epsilon = 0$

then ① and ② must balance so

$$\delta_1^2 x_1^2 = \epsilon$$

$$\Rightarrow x_1 = \pm 1, \quad \delta_1 = \epsilon^{1/2}$$

collect what may be the next higher order terms

$$2x_1x_2\delta_1\delta_2 - 2x_1\epsilon\delta_1 = 0$$

$$\Rightarrow x_2\delta_2 - \epsilon = 0$$

if one term dominates the other terms vanish, so these terms must balance and so

$$x_2\delta_2 = \epsilon \rightarrow x_2 = 1, \ \delta_2 = \epsilon$$

Example 3

$$\epsilon x^2 - 2x + 1 = 0$$

this problem is singular sice ϵ is the coefficient of the largest power of x.

leading order equation: $-2x + 1 = 0$

try the naive expansion

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$$

Keep terms up to $O(\epsilon^2)$

$$\epsilon (x_0^2 + 2x_0 x_1 \epsilon + \cdots) - 2(x_0 + \epsilon x_1 + \epsilon^2 x_2) + 1 = 0$$

$$(-2x_0 + 1)\epsilon^0 + (x_0^2 - 2x_1)\epsilon^1 + (2x_0 x_1 - 2x_2)\epsilon^2 + \cdots = 0$$

$$\rightarrow \boxed{x_0 = \frac{1}{2}}$$

$O(\epsilon)$: $x_0^2 - 2x_1 = 0$ $\quad \boxed{x_1 = \frac{1}{8}}$

$O(\epsilon^2)$: $2x_0 x_1 - 2x_2 = 0 \rightarrow \quad \boxed{x_2 = \frac{1}{16}}$

$$\Rightarrow \boxed{x_0 = \frac{1}{2} + \frac{x}{8} + \frac{x^2}{16} + \cdots}$$

we have computed solution though simple process, however we only have ONE root !!!

The problem with our expansion is that we assumed $\epsilon x^2 \to 0$ as $\epsilon \to 0$

but it could be that leading order term may be big such that $\epsilon x^2 \not\to 0$ as $\epsilon \to 0$

could go to a constant or infinity

Try a general expansion of the form:

$$x = \delta_0(\epsilon)\, x_0 + \delta_1(\epsilon)\, x_1 + \delta_2(\epsilon)\, x_2 + \cdots$$

substitute into eqn: $\epsilon x^2 - 2x + 1 = 0$

keep many terms because magnitudes unknown

$$\epsilon\left(\delta_0^2 x_0^2 + \delta_1^2 x_1^2 + \delta_2^2 x_2^2 + 2x_0x_1\delta_0\delta_1 + 2x_0x_2\delta_0\delta_2 + 2x_1x_2\delta_1\delta_2 + \cdots \right) - 2\left(\delta_0 x_0 + \delta_1 x_1 + \delta_2 x_2 + \cdots\right) + 1 = 0$$

must balance terms

so the possible leading order terms are:

$$\boxed{\underset{1}{\epsilon \delta_0^2 x_0^2} - \underset{2}{2\delta_0 x_0} + \underset{3}{1} = 0}$$

balance all three terms

1 and 2 must be $O(1)$

pick $\delta_0 = 1 \rightarrow \epsilon x_0^2 - 2x_0 + 1 = 0$

won't work

so all three terms can't balance

but two of the three terms balance

<u>case 1</u>: balance 1 and 3

$\epsilon \delta_0^2 x_0^2 + 1 = 0 \rightarrow \delta_0^2 \epsilon = 1 \rightarrow \delta_0 = \epsilon^{-1/2}$

such that $x_0 = \pm i$ <u>doesn't work</u>

<u>case 2</u>: balance 2 and 3

$-2\delta_0 x_0 + 1 = 0$

$\delta_0 = 1 \rightarrow x_0 = \frac{1}{2}$ results in naive expansion which results in 1 root only case 3 balance ① and ②

$$\epsilon\,\delta_0^2 x_0^2 - 2\delta_0 x_0 = 0$$

$$\epsilon\,\delta_0 x_0 - 2 = 0$$

$$\rightarrow \boxed{\delta_0 = \frac{1}{\epsilon} \rightarrow x_0 = 2}$$

So let us now write down the potential next highest order terms. Here is the expansion:

$$\epsilon\left(\delta_1^2 x_1^2 + \delta_2^2 x_2^2 + \underline{2x_0x_1\delta_0\delta_1} + 2x_0x_2\delta_0\delta_2 + 2x_1x_2\delta_1\delta_2 + \cdots\right)$$

$$-2(\underline{\delta_1 x_1} + \delta_2 x_2 + \cdots) + \underline{1} = 0$$

$$\Rightarrow \quad 2\epsilon\, x_0 x_1 \delta_0 \delta_1 - 2\delta_1 x_1 + 1 = 0$$

$$\rightarrow \quad 4\delta_1 x_1 - 2\delta_1 x_1 + 1 = 0$$

$$2\delta_1 x_1 + 1 = 0$$

pick $\boxed{\delta_1 = 1 \quad , x_1 = \frac{-1}{2}}$

so the expansion looks like

$$\epsilon\left(\underline{\delta_1^2 x_1^2} + \delta_2^2 x_2^2 + \underline{2x_0x_2\delta_0\delta_2} + 2x_1x_2\delta_1\delta_2 + \cdots\right)$$

$$-2(\underline{\delta_2 x_2} + \cdots) = 0$$

and the next highest terms are

$$\epsilon\,\delta_1^2 x_1^2 + 2\cancel{\epsilon}\,\cancel{\delta_0}\,\delta_2 x_0 x_2 - 2\delta_2 x_2 = 0$$

$$\frac{\epsilon}{4} + 4\delta_2 x_2 - 2\delta_2 x_2 = 0 \quad \rightarrow \quad 2\delta_2 x_2 + \frac{\epsilon}{4} = 0$$

$$\rightarrow \boxed{\delta_2 = \epsilon \quad , x_2 = \frac{-1}{8}}$$

therefore we have a second root

$$x = \delta_0(\epsilon)\, x_0 + \delta_1(\epsilon)\, x_1 + \delta_2(\epsilon)\, x_2 + \cdots$$

$$\boxed{x = \frac{2}{\epsilon} - \frac{1}{2} - \frac{\epsilon}{8} + \cdots}$$

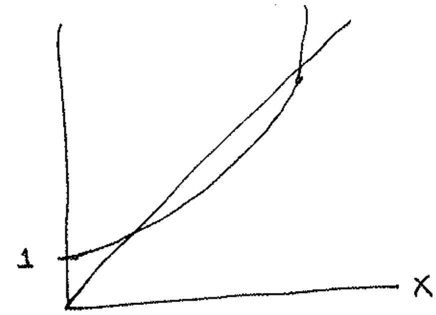
<u>Example</u>: $\epsilon^{3x} - x = 0$

Approximate the smallest of the two roots

$e^{\epsilon x} - x = 0$

Find smallest root

let $f(x) = e^{\epsilon x}$, $g(x) = x$



the smallest root is $x \sim 1$

then for $\epsilon \ll 1$ we have $\epsilon x \ll 1$

expand $e^{\epsilon x}$ in Taylor series

$$\left(1 + \epsilon x + \frac{(\epsilon x)^2}{2!} + \ldots\right) - x = 0$$

Blah

⋮

Theory / Definition / Notation

Order Symbols

$O(\cdot)$ - 'big oh'

$o(\cdot)$ - 'little oh'

---

we write $f(\epsilon) = O(g(\epsilon))$ as $\epsilon \to 0$
if there exists constants k and ϵ_0 s t

$$|f(\epsilon)| \le k |g(\epsilon)| \quad \text{for} \quad 0 < \epsilon < \epsilon_0$$

ie $\dfrac{f(\epsilon)}{g(\epsilon)}$ is bounded as $\epsilon \to 0$

~~We write $f(x) = o(g(x))$ as $x \to 0$~~ if for every $k > 0$,
~~if there is a constant~~ there exists an ϵ

we write $f(\epsilon) = o(g(\epsilon))$ as $\epsilon \to 0$ if for each k, there exists a number $\epsilon_0(k)$ such that

$$|f(\epsilon)| \leq k\,|g(\epsilon)| \quad \text{for} \quad 0 < \epsilon < \epsilon_0(k)$$

ie $\dfrac{f(\epsilon)}{g(\epsilon)} \to 0$ as $\epsilon \to 0$

Note

1) if $f(\epsilon) = o(g(\epsilon))$ then $f(\epsilon) << g(\epsilon)$

ie

2) $f(\epsilon) = o(g(\epsilon)) \Rightarrow f(\epsilon) = O(g(\epsilon))$

though the converse is not true

$$f(\epsilon) = O(g(\epsilon)) \not\Rightarrow f(\epsilon) = o(g(\epsilon))$$

Examples

1) $f(\epsilon) = O(f(\epsilon))$

obviously $\dfrac{f(\epsilon)}{f(\epsilon)} = 1$

$f(\epsilon) \neq o(f(\epsilon))$

2) $\epsilon^n = O(\epsilon^m)$ for $n \geq m$

$\epsilon^n = o(\epsilon^m)$ for $n > m$

$\dfrac{\epsilon^n}{\epsilon^m} \Rightarrow 1$ if $n = m$

3) $\cos \epsilon = 1 - \frac{\epsilon^2}{2} + \dots = O(1)$

$$= 1 + O(\epsilon^2)$$

$$= 1 + o(\epsilon)$$

$\cos \epsilon - 1 = O(\epsilon^2)$, etc . . .

4) $\sin \epsilon \sim \epsilon - \frac{\epsilon^3}{3!} + \dots = O(\epsilon)$

5) $\tan \epsilon = O(\epsilon)$

6) $\sin \epsilon = O(\tan \epsilon)$

7) $\sqrt{1-\epsilon^2} = 1 - \frac{\epsilon^2}{2} + \dots = O(1)$

$$= 1 + O(\epsilon^2)$$

Formally it is correct to write

$$\left.\begin{array}{l} \sin(\epsilon) = O(1) \\ \sin(\epsilon) = o(1) \\ \sin(\epsilon) = O(\sqrt{\epsilon}) \end{array}\right\} \text{ all these are correct}$$

but it is preferable to use the sharpest and most informative estimate, which in this case is $\sin \epsilon = O(\epsilon)$

Order operations

- $O(\cdot)$ and $o(\cdot)$ are insensitive to multiplicative constants

  examples, $\quad K\epsilon = O(\epsilon)$ for all K

  so $\quad 10^{100}\sin(\epsilon) = O(\epsilon)$

- addition and subtraction

$$O(F(\epsilon)) + O(g(\epsilon)) = O(g(\epsilon)) \text{ if } F(\epsilon) = O(g(\epsilon))$$

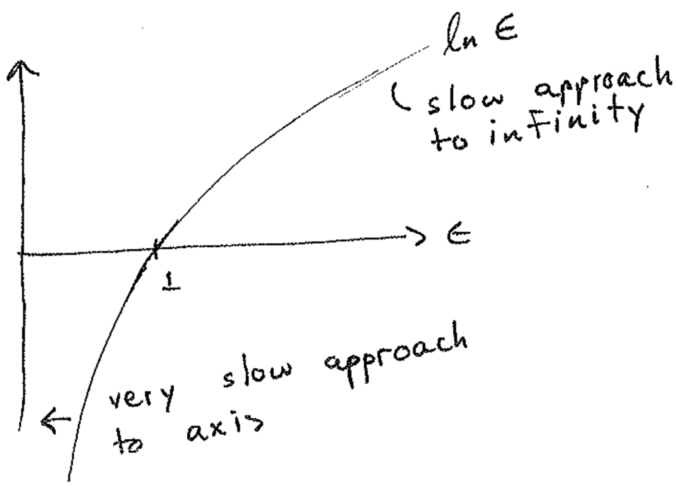
$$O(\epsilon^n) + O(\epsilon^m) = O(\epsilon^n) \text{ if } n \le m$$

- multiplication

$$O(F(\epsilon)) \cdot O(g(\epsilon)) = O(F(\epsilon)\cdot g(\epsilon))$$

$$O(\epsilon^n) \cdot O(\epsilon^m) = O(\epsilon^{n+m})$$

Logarithms



For $\epsilon \ll 1$

then

$$-\ln\epsilon = \ln\frac{1}{\epsilon} \gg 1$$

claim:

$$1 = o(-\ln\epsilon)$$

$$\frac{1}{-\ln\epsilon} \to 0$$

claim: $-\epsilon^\alpha \ln(\epsilon) = o(1)$ for any $\alpha > 0$

$$\frac{\epsilon^\alpha(-\ln(\epsilon))}{1} = \frac{-\ln\epsilon}{\epsilon^{-\alpha}} \xrightarrow{\text{L'hopitals}} \frac{-1/\epsilon}{-\alpha\epsilon^{-\alpha-1}} = \frac{1}{\alpha}\epsilon^\alpha \to 0 \quad \text{as } \epsilon \to 0$$

therefore we have

$$\epsilon^{\alpha}(-\ln\epsilon) \ll 1 \ll (-\ln\epsilon) \quad \text{for } \alpha > 0$$

alternatively

$$1 \ll (-\ln\epsilon) \ll \epsilon^{-\alpha}$$

Suppose we wish to order several terms of the form $\epsilon^{\alpha}(-\ln\epsilon)^{\beta}$ from <u>largest to smallest as $\epsilon \to 0$</u>

<u>Example:</u>

$$\epsilon\ln\epsilon,\ \epsilon^2\ln\epsilon,\ 1,\ \ln\epsilon,\ (\ln\epsilon)^2,\ \epsilon(\ln\epsilon)^5$$

The ordering is dominated by the power α of ϵ ~~and~~ whereas the power β of $\ln\epsilon$ has only a secondary effect

leading order terms: $\alpha = 0$

$$(\ln\epsilon)^2 \gg \ln\epsilon \gg 1$$

next order, $\alpha = 1$

$$\epsilon(\ln\epsilon)^5 \gg \epsilon\ln\epsilon$$

and so on ...

Gauge Functions

<u>Definition:</u> A sequence of <u>gauge functions</u> (scale functions, basis functions) is a sequence $\{g_n(\epsilon)\}_{n=1}^{\infty}$ such that

$$g_n(\epsilon) = o\left(g_{n-1}(\epsilon)\right)$$

<u>Examples</u>

1) "naive" expansion, $g_n(\epsilon) = \epsilon^{n-1}$, $n = 1, 2, \ldots$

2) $\{g_n(\epsilon)\} = \{1, \sin\epsilon, \sin\epsilon^2, \ldots\}$

3) $\{g_n(\epsilon)\} = \{(\ln\epsilon)^2, (\ln\epsilon), 1, \epsilon(\ln\epsilon)^3, \ldots\}$ From previous example

In the polynomial examples that we did last time, we expanded roots as

$$x \sim \delta_0(\epsilon)x_0 + \delta_1(\epsilon)x_1 + \cdots$$

where $\delta_i(\epsilon)$ is a gauge function and x_i is a constant

More generally, we want to expand functions $f(x;\epsilon)$ in terms of asymptotic series

$$f(x;\epsilon) = \underbrace{g_1(\epsilon)f_1(x) + g_2(\epsilon)f_2(x) + \cdots + g_N(\epsilon)f_N(x)}_{F_N(x;\epsilon)} + O\left(g_{N+1}(\epsilon)\right)$$

<u>Def:</u> The quantity $F_N(x;\epsilon)$ is called the N-term asymptotic expansion of $f(x;\epsilon)$ as $\epsilon \to 0$.

$f(x;\epsilon) \sim F_N(x;\epsilon)$ which says f is asymptotic to F_N.

1/16/04
4/5

Notes on Uniqueness

1) The choice of $\{g_n(\epsilon)\}$ is not unique, however, given $\{g_n(\epsilon)\}$, the coefficients are uniquely determined

Example: expand $\sin 2\epsilon$ with gauge functions
the natural thing to do is a Taylor series

$$\sin 2\epsilon \sim 2\epsilon - \frac{4}{3}\epsilon^3 + \frac{4}{15}\epsilon^5$$

there are other possibilities

$$\sin 2\epsilon \sim 2\tan\epsilon - 2\tan^3\epsilon - 2\tan^5\epsilon$$

$$\sin 2\epsilon \sim 2\ln(1+\epsilon) + \ln(1+\epsilon^2) - 2\ln(1+\epsilon^3)$$

$$\sin 2\epsilon \sim 6\left(\frac{\epsilon}{3+2\epsilon^2}\right) - \frac{378}{3}\left(\frac{\epsilon}{3+2\epsilon^2}\right)^5$$

2) An asymptotic expansion does not uniquely determine a function, ie, several functions may be represented by the same asymptotic series

$$f_1(\epsilon) = \frac{1}{1-\epsilon}, \quad f_2(\epsilon) = \frac{1}{1-\epsilon} + e^{-1/\epsilon}$$

$$f_3(\epsilon) = \frac{1}{1-\epsilon} + \frac{e^{-1/\epsilon}}{\epsilon^m}, \quad m \text{ real}$$

given the gauge functions $\{g_n(\epsilon)\} = \{1, \epsilon, \epsilon^2, \dots\}$ then all three functions are asymptotic to

$$f_1, f_2, f_3 \sim 1 + \epsilon + \epsilon^2 + \epsilon^3 + \cdots$$

this is because, for example

$$e^{-1/\epsilon} = o(\epsilon^n) \quad \text{for all } n$$

given that $g_n(\epsilon) = \epsilon^{n-1}$, then $e^{-1/\epsilon} \sim 0$

definition: **Transcendentally Small Terms**

Given a sequence $\{g_n(\epsilon)\}$, a transcendentally small term is a term which is much less than any function in the sequence, in the limit $\epsilon \to 0$, and therefore its expansion is zero in terms of these gauge functions.

- example: $e^{-1/\epsilon}$, $e^{-x/\epsilon}$
- example: suppose $\{g_n(\epsilon)\} = \{(\ln \epsilon)^{-n}\}$, then ϵ is a transcendentally small term, since $\epsilon \ll (-\ln \epsilon)^{-n}$ and so in terms of these gauge functions, $\epsilon \sim 0$.

1/16/04
5/5

Asymptotic Expansion of Functions

Recall: $f(x;\epsilon) \sim \underbrace{g_1(\epsilon) f_1(x) + \cdots + g_N(\epsilon) f_N(x)}_{F_N(x;\epsilon)} + O(g_{N+1}(\epsilon))$

Definition: For fixed x, we write $f(x;\epsilon) = O(g(\epsilon))$ if there exists numbers $K(x)$ and $\epsilon_0(x)$ such that

$$|f(x;\epsilon)| \leq K(x)|g(\epsilon)| \quad \text{for} \quad 0 < \epsilon < \epsilon_0(x)$$

Definition: If K and ϵ_0 can be chosen independently of x for all x in some interval I, then the ordering is uniform in I.

Example: If $f(x;\epsilon) = \frac{\epsilon}{x}$, then

$$f(x;\epsilon) = O(\epsilon) \quad \text{in} \quad 1 \leq x \leq 2$$

$$f = \frac{\epsilon}{x} \leq \epsilon \quad \text{for } x \in [1,2]$$

however this is not true for $x \in [0,2]$

uniform asymptotic expansions

For the asymptotic expansion

$$f(x;\epsilon) \sim g_1(\epsilon) f_1(x) + \cdots + g_N(\epsilon) f_N(x)$$

to be valid, each term must be much less than previous terms as $\epsilon \to 0$, ie $g_n(\epsilon) f_n(x) = o\big(g_{n-1}(\epsilon) f_{n-1}(x)\big)$. If this ordering is uniform in an interval I, then the asymptotic expansion is <u>uniformly valid</u> in I.

Differentiation of asymptotic expansion.

we often need to differentiate an asymptotic expansion with respect to x, though it is not always permissible.

<u>example:</u> $f(x) \sim \sin(x) + \epsilon \sin\left(\frac{x}{\epsilon^2}\right) + \epsilon^2 \sin\left(e^{-1/\epsilon} x\right)$

$$f'(x;\epsilon) \sim \cos x + \frac{1}{\epsilon}\cos\frac{x}{\epsilon} + \cancel{\epsilon \sin}\ \epsilon^2 e^{-1/\epsilon} \cos\left(e^{-1/\epsilon} x\right)$$

this expansion is not valid

Typically, we proceed under the assumption that differentiation is permissible, but recognizing that a violation of this assumption could lead to nonuniformities. This occurs in singular problems and it requires special treatment.

uniformly valid asymptotic expansions

consider $f(x;\epsilon) = \sum_{n=0}^{\infty} f_n(x;\epsilon)$

eg $$\frac{1}{1-\epsilon/x} = 1 + \frac{\epsilon}{x} + \left(\frac{\epsilon}{x}\right)^2 + \cdots \quad (1)$$

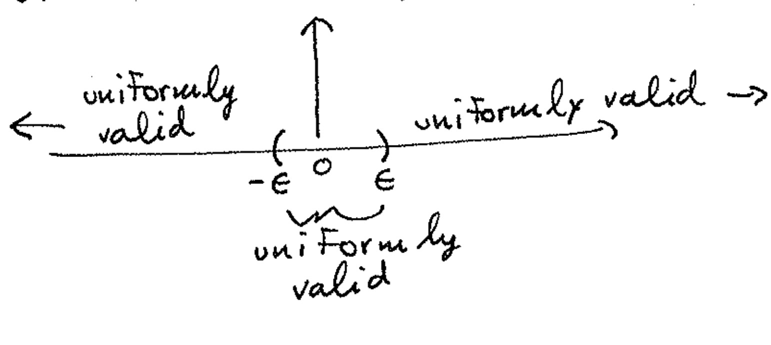
this series is uniformly valid on the interval $(1,2)$
however the series is not uniformly valid on $(0,2)$

Informally, an expansion is uniformly valid on an interval I, if for fixed ϵ,

$$|f_{n+1}| \ll |f_n| \quad \text{for each } x \in I$$

For the series (1)
when $\frac{\epsilon}{x} \sim 1$ then the terms are practically equal in value. In fact if $\epsilon = x$ then the series diverges.

region of nonuniformity for series (1) is $x = O(\epsilon)$



Example 2

$$f(x) = 1 + \frac{\epsilon^2}{1-x} + \frac{\epsilon^4}{(1-x)^2} + \cdots + \frac{\epsilon^{2n}}{(1-x)^n}$$

compare neighboring terms

$$\frac{\epsilon^2}{(1-x)} \sim \frac{\epsilon^4}{(1-x)^2}$$ then becomes nonuniform

$\rightarrow$ $1-x \sim \epsilon^2$

$\rightarrow$ region of nonuniformity

$$x = 1 + O(\epsilon^2)$$

or $|1-x| = O(\epsilon^2)$

Example 3

$$f(t;\epsilon) = 2 + \epsilon 2t \sin t + \frac{\epsilon^2 t^2}{2} \sin^2 t + \cdots$$

compare two terms

if $\epsilon t \sim O(1)$ then 2 and $2\epsilon t \sin t$

are ~~comp~~ same order

therefore if $t \sim \frac{1}{\epsilon}$ then nonuniform

so this is a valid expansion as long as t doesn't get too big

1/20/04
2/5

convergent series vs asymptotic series

Consider $f(\epsilon) = \sum_{n=0}^{\infty} f_n(\epsilon)$

The series is convergent if, for fixed ϵ

$$\lim_{n\to\infty} \left| \frac{f_{n+1}(\epsilon)}{f_n(\epsilon)} \right| = k < 1 \quad \text{(ratio test)}$$

and it is asymptotic, if for each n,

$$\lim_{\epsilon\to 0} \left| \frac{f_{n+1}(\epsilon)}{f_n(\epsilon)} \right| = 0 \quad \rightarrow \quad \left[f_{n+1}(\epsilon) = o\left(f_n(\epsilon)\right) \right]$$

Example

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent but not asymptotic

$$f(\epsilon) = \int_0^\infty \frac{e^{-t}}{1+\epsilon t}\, dt$$

integrate by parts, $u = \frac{1}{1+\epsilon t}$ $\qquad dv = e^{-t} dt$

$du = \frac{-\epsilon}{(1+\epsilon t)^2}$ $\qquad v = -e^{-t}$

$$\rightarrow f(\epsilon) = \int_0^\infty \frac{e^{-t}}{1+\epsilon t} = 1 - \epsilon \int_0^\infty \frac{e^{-t}}{(1+\epsilon t)^2}\, dt$$

integrate by parts

$$f(\epsilon) = 1 - \epsilon \int_0^\infty \frac{e^{-t}}{(1+\epsilon t)^2} dt = 1 - \epsilon + 2\epsilon^2 \int_0^\infty \frac{e^{-t}}{(1+\epsilon t)^3}\, dt$$

$$\rightarrow f(\epsilon) = 1 - \epsilon + 2!\,\epsilon^2 - 3!\,\epsilon^3 + \cdots = \sum_{n=0} (-1)^n \epsilon^n n!$$

so $f(\epsilon) = \int_0^\infty \frac{e^{-t}}{1+\epsilon t}\, dt$

$$= \sum_{n=0}^{\infty} (-1)^n \epsilon^n n!$$

$$\left| \frac{f_{n+1}}{f_n} \right| = \frac{(n+1)!\, \epsilon^{n+1}}{n!\, \epsilon^n} = (n+1)\epsilon$$

as $n \to \infty$, the ratio test shows that the series is not convergent

as $\epsilon \to 0$, the ratio goes to zero, so the series is asymptotic

Though the series is not convergent it may give a ~~reasonable~~ good asymptotic approximation with a few terms.

let $\epsilon = 0.1$

$f_4(\epsilon) = 1 - \epsilon + 2\epsilon^2 - 6\epsilon$ (4 terms)

$f(0.1) = 0.9156$ (exact solution)

$f_4(0.1) = 0.914$ (4 term series

1/20/04
3/3

Regular ODEs

example: $2yy' - y^2 + \epsilon\sqrt{y} = 0$, $y(0) = 1$

approximate the solution to $O(\epsilon)$

start off with "naive" expansion

$y(t;\epsilon) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \cdots$

$y' = y_0' + \epsilon y_1' + \epsilon^2 y_2 + \cdots$ assume we can differentiate and that derivative is well-ordered

$$\rightarrow \quad 2(y_0 + \epsilon y_1 + \cdots)(y_0' + \epsilon y_1' + \cdots) - (y_0 + \epsilon y_1 + \cdots)^2 + \epsilon\sqrt{y_0 + \epsilon y_1 + \cdots} = 0$$

$$\rightarrow \quad 2y_0 y_0' + 2\epsilon y_0 y_1' + 2\epsilon y_1 y_0' + O(\epsilon^2) - \left(y_0^2 + 2y_0 y_1 \epsilon + O(\epsilon^2)\right) + \epsilon\sqrt{y_0} + O(\epsilon^2) = 0$$

equate powers of ϵ:

$O(1)$: $2y_0 y_0' - y_0^2 = \rightarrow \quad y_0' = \frac{y_0}{2}$

(assume $y_0 \neq 0$ because $y(0) = 1$)

$O(\epsilon)$: $2y_0 y_1' + 2y_1 y_0' - 2y_0 y_1 + \sqrt{y_0} = 0$

initial condition:

$y_0(0) + \epsilon y_1(0) + \cdots = 1$

equate powers of zero

$y_0(0) = 1, \quad y_1(0) = 0$

$O(1)$: $y_0' = \frac{y_0}{2}$, $y_0(0) = 1$

$\rightarrow \quad y_0(t) = e^{t/2}$

$O(\epsilon)$ $\quad 2y_0 y_1' + 2y_0' y_1 - 2 y_0 y_1 + \sqrt{y_0} = 0$, $y_1(0) = 0$

$\cancel{2e^{t/2} y_1'(t) + e^{t/2} y_1 - 2e^{t/2} y_1 + e^{t/4} = 0}$

~~$y_1(t)$~~ divide by $2y_0$

$$y_1' + \left(\frac{y_0'}{y_0} - 1\right) y_1 = \frac{-\sqrt{y_0}}{2y_0}$$

but since $\frac{y_0'}{y_0} = \frac{1}{2}$ then

$$y_1' - \frac{1}{2} y_1 = \frac{-1}{2\sqrt{y_0}}$$

integrating factor $e^{-t/2}$

$$\frac{d}{dt}\left(e^{-t/2} y_1\right) = -\frac{1}{2} e^{-t/2} e^{-t/4} = -\frac{1}{2} e^{-3t/4}$$

$$e^{-t/2} y_1 = \frac{2}{3} e^{-3t/4} + k$$

$$y_1 = \frac{2}{3} e^{-t/4} + k e^{t/2}$$

$y_1(0) = 0 \rightarrow 0 = \frac{2}{3} + k \rightarrow k = -\frac{2}{3}$

$$y_1 = \frac{2}{3}\left(e^{-t/4} - e^{+t/2}\right)$$

$$\boxed{y(t) = e^{t/2} - \epsilon \frac{2}{3}\left(e^{t/2} - e^{-t/4}\right) + O(\epsilon^2)}$$

1/20/04
4/5

Example: Fire a projectile straight upwards from ground level with initial velocity v_0. Describe the motion of the projectile.

want to take into account that the gravitational force depends on altitude

$F = ma$, $\quad F_g = G\dfrac{m_1 m_2}{r^2}$

$$x''(t) = \frac{-g}{(1 + x/R)^2}$$

it is reasonable to assume that $\frac{x}{R} \ll 1$

let $\epsilon =$

after nondimensionalization

$$y''(\tau) = \frac{-2}{(1+\epsilon y)^2}, \quad y(0) = 0, \quad y'(0) = 2$$

expand nonlinearity in the ODE before expanding y

$$\frac{1}{(1+\epsilon y)^2} = \frac{1}{1 + 2\epsilon y + \epsilon^2 y^2} = 1 - (2\epsilon y + \epsilon^2 y) + (2\epsilon y + \epsilon^2 y)^2 + \cdots$$

$$\cancel{y''(\tau) = -2 + \epsilon(4y) + \epsilon^2(2y - 4y^2) +} \cdots$$

$$y''(\tau) = -2\left[1 - 2\epsilon y - \epsilon^2 y + 4\epsilon^2 y^2 + \cdots\right]$$

$$y''(\tau) = -2 + 4\epsilon y - 6\epsilon^2 y^2 + \cdots$$

$$y''(\tau) = -2 + 4\epsilon y - 6\epsilon^2 y^2 + \cdots$$

$$\rightarrow \cancel{y_0'' + y_1''\epsilon + \epsilon^2 y_2'' + \cdots = -2 + 4\epsilon y - 6\epsilon^2 y^2 + \cdots}$$

$\cancel{O(1): \quad y_0''(\tau) = -2}$

$\cancel{O(\epsilon): \quad y_1''(\tau) =}$

$$y_0'' + y_1''\epsilon + \epsilon^2 y_2'' = -2 + 4\epsilon(y_0 + \epsilon y_1 + \cdots) - 6\epsilon^2(y_0^2 + \cdots)$$

$$\boxed{\begin{aligned} O(1)&: \quad y_0''(\tau) = -2 \\ O(\epsilon)&: \quad y_1''(\tau) = 4y_0 \\ O(\epsilon^2)&: \quad y_2''(\tau) = 4y_1 - 6y_0^2 \end{aligned}}$$

initial conditions

$$y(0) = 0$$

$$\frac{dy}{d\tau}(0) = 2$$

$$y(0) = y_0(0) + \epsilon y_1(0) + \epsilon^2 y_2(0) + \cdots$$

$$\rightarrow \boxed{y_n(0) = 0, \quad n = 0, 1, 2, \ldots}$$

$$y'(0) = y_0'(0) + \epsilon y_1'(0) + \epsilon^2 y_2'(0) = 2$$

$$\boxed{y_0'(0) = 2, \quad y_n'(0) = 0, \quad n = 1, 2, \ldots}$$

1/20/04

5/5

$$y_0''(\tau) = -2 \longrightarrow y_0(\tau) = 2\tau - \tau^2$$

$$y_1''(\tau) = 4y_0 \longrightarrow y_1(\tau) = \frac{4}{3}\tau^3 - \frac{\tau^4}{3}$$

$$y_2''(\tau) = 4y_1 - 6y_0^2 \longrightarrow y_2(\tau) = \frac{-11}{45}\tau^6 + \frac{22}{15}\tau^5 - 2\tau^4$$

$$\Rightarrow \quad y(\tau;\epsilon) = (2\tau - \tau^2) + \epsilon\left(\frac{4}{3}\tau^3 - \frac{\tau^4}{3}\right) + \epsilon^2\left(\frac{-11}{45}\tau^6 + \frac{22}{15}\tau^5 - 2\tau^4\right) + O(\epsilon^3)$$

notice, when $\frac{1}{\tau} = O(\sqrt{\epsilon})$

$$\tau^2 \sim \epsilon\tau^4 \rightarrow \cancel{\tau}\, 1 \sim \epsilon\tau^2$$

Interior Layer

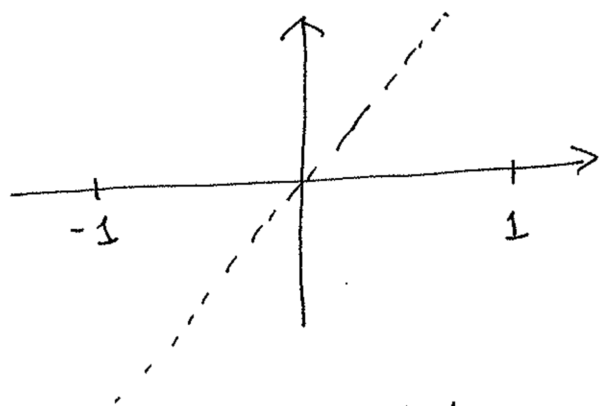
Example: $\epsilon y'' + x y' + x y = 0$

$y(-1) = e$

$y(1) = 2e^{-1}$, $\epsilon \ll 1$

Where might the boundary layers be?

expect an interior layer at $x \sim 0$.



outer solution

$$Y_{OUT} \sim Y_0(x) + \cdots$$

to leading order

$$x Y_{OUT}' + x Y_{OUT} = 0$$

for $x \neq 0$, $\rightarrow$ $Y_{OUT}' + Y_{OUT} = 0$

$\rightarrow$ $Y_0' + Y_0 = 0$ $\rightarrow$ $Y_0(x) = C_0 e^{-x}$

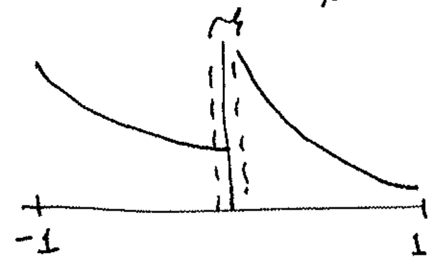
At $x = -1$, solution is e

so decay from there to $x \sim 0$

At $x = 1$, solution is $2e^{-1}$

so decay from $x \sim 0$ to $x = 1$

the solution will look something like



At the left side,

$y(-1) = e$

$y_0(x) = c_0 e^{-x}$

$y_0(-1) = c_0^- e^{+1} = e \rightarrow \quad c_0^- = 1$

at $y(1) = 2e^{-1}$

$y_0(1) = c_0^+ e^{-1} = 2e^{-1} \quad \longrightarrow \quad c_0^+ = 2$

So the outer solution is given by

$$\boxed{y_0(x) = \begin{cases} e^{-x}, & x < 0 \\ 2e^{-x}, & x > 0 \end{cases}}$$

Inner solution at $x = 0$:

need to stretch out the spatial component

$$\xi = \frac{x}{\delta(\epsilon)} \quad , \quad \epsilon y'' + xy' + xy = 0$$

$$\Rightarrow \quad \frac{d}{dx} y = \frac{d}{d\xi} y \frac{d\xi}{dx} = y_\xi \frac{1}{\delta} \quad , \quad \frac{d^2 y}{dx^2} = y_{\xi\xi} \frac{1}{\delta^2}$$

$$\Rightarrow \quad \frac{\epsilon}{\delta^2} y_{\xi\xi} + \xi\delta\, y_\xi \frac{1}{\delta} + \xi\delta\, y = 0$$

let $\delta = \sqrt{\epsilon} \quad \rightarrow \quad \boxed{y_{\xi\xi} + \xi y_\xi + \epsilon^{1/2} \xi y = 0}$

Expand using naive expansion

$$y_{IN}(\xi) \sim y_0(\xi) + \epsilon^{1/2} y_1(\xi) + \cdots$$

leading order eqn: $\boxed{(y_0)_{\xi\xi} + \xi (y_0)_\xi = 0}$

$$Y_{0\,\xi\xi} + \xi Y_{0\,\xi} = 0 \quad \rightarrow \quad \frac{Y_{0\,\xi\xi}}{Y_{0\,\xi}} = -\xi$$

$$\rightarrow \quad Y_{0\,\xi} = C_1 e^{-\xi^2/2}$$

integrate again

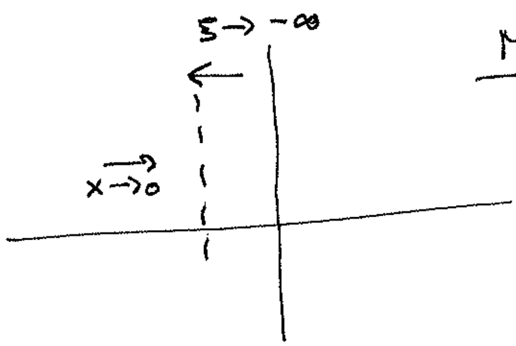
$$Y_0(\xi) - Y_0(0) = c_1 \int_0^{\xi} e^{-u^2/2}\, du$$

Recall, error function $\quad \mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$

let $\cancel{t} = \frac{u}{\sqrt{2}} \quad \rightarrow \quad I = \sqrt{2}\, \frac{c_1}{\cancel{\sqrt{\pi}}} \int_0^{\xi/\sqrt{2}} e^{-t^2} dt$

absorb $\sqrt{2}$ into c_1

$$\Rightarrow \quad \boxed{Y_0(\xi) = Y_0(0) + c_1\, \mathrm{erf}\left(\frac{\xi}{\sqrt{2}}\right)}$$



Matching to the left

$$\lim_{x \to 0^-} Y_0(x) = \lim_{\xi \to -\infty} Y_0(\xi)$$

$$\lim_{x \to 0^-} e^{-x} = \lim_{\xi \to -\infty} Y_0(0) + c_1\, \mathrm{erf}\left(\frac{\xi}{\sqrt{2}}\right)$$

$$\boxed{1 = Y_0(0) - c_1}$$

Matching to the right

$$\lim_{x\to 0^+} Y_0(x) = \lim_{\xi\to\infty} Y_0(\xi)$$

$$\lim_{x\to 0^+} 2e^{-x} = \lim_{\xi\to\infty} Y_0(0) + c_1 \operatorname{erf}\left(\frac{\xi}{\sqrt{2}}\right)$$

$$2 = Y_0(0) + c_1$$

2 equs, 2 unknowns

$$1 = Y_0(0) - c_1$$

$$2 = Y_0(0) + c_1$$

$$\Rightarrow \quad Y_0(0) = \frac{3}{2}, \quad c_1 = \frac{1}{2}$$

and therefore $\boxed{Y_0(\xi) = \frac{1}{2}\left(3 + \operatorname{erf}\left(\frac{\xi}{\sqrt{2}}\right)\right) \quad \text{inner solution}}$

the composite solution is

$$Y_c = Y_{OUT} - Y_{in} - CP, \qquad CP = \text{common part} = \begin{cases} 1, & x<0 \\ 2, & x>0 \end{cases}$$

to leading order

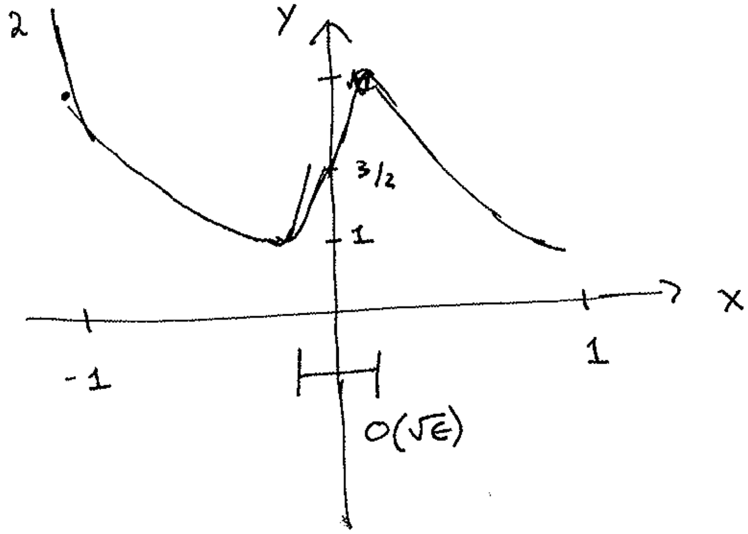
$$Y_c \sim Y_0(x) + y_0(\xi) - CP$$

$$\Rightarrow \quad Y_c \sim \begin{cases} e^{-x}, & x<0 \\ 2e^{-x}, & x>0 \end{cases} + \frac{1}{2}\left(3 + \operatorname{erf}\left(\frac{\xi}{\sqrt{2}}\right)\right) - \begin{cases} 1, & x<0 \\ 2, & x>0 \end{cases}$$

$$\Rightarrow \quad Y_c(x) \sim \begin{cases} e^{-x} + \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{x}{\sqrt{2\epsilon}}\right)\right), & x<0 \\ 2e^{-x} - \frac{1}{2}\left(1 - \operatorname{erf}\left(\frac{x}{\sqrt{2\epsilon}}\right)\right), & x>0 \end{cases}$$

2/3 04
3/

$$Y_c(x) \sim \begin{cases} e^{-x} + \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{x}{\sqrt{2\epsilon}}\right)\right), & x<0 \\ 2e^{-x} - \frac{1}{2}\left(1 - \operatorname{erf}\left(\frac{x}{\sqrt{2\epsilon}}\right)\right) & x>0 \end{cases}$$



continuity is satisfied at $x=0$

$Y_c(0^-) = Y_c(0^+) = \frac{3}{2}$

However the derivatives don't match up, so the solution is not continuously differentiable

$Y_c'(0^-) \neq Y_c'(0^+)$

$$Y_c'(x) = \begin{cases} -e^{-x} + \frac{1}{2}\frac{2}{\sqrt{\pi}} e^{-x^2/2\epsilon} \cdot \frac{1}{\sqrt{2\epsilon}} \\ -2e^{-x} + \frac{1}{2}\frac{2}{\sqrt{\pi}} e^{-x^2/2\epsilon} \cdot \frac{1}{\sqrt{2\epsilon}} \end{cases}$$

$$\operatorname{erf}\left(\frac{x}{\sqrt{2\epsilon}}\right) = \frac{2}{\sqrt{\pi}}\int_0^{x/\sqrt{2\epsilon}} e^{-t^2}\,dt$$

$$\frac{d}{dx}\left(\operatorname{erf}\left(\frac{x}{\sqrt{2\epsilon}}\right)\right) = \frac{2}{\sqrt{\pi}} e^{-x^2/2\epsilon} \frac{1}{\sqrt{2\epsilon}}$$

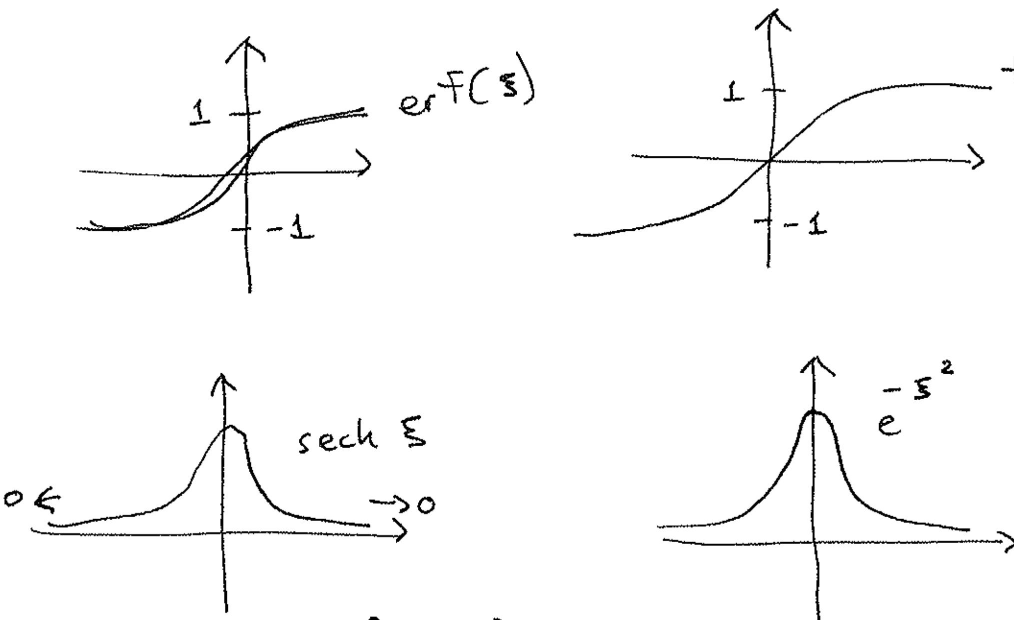
$$Y_c'(0) = \begin{cases} -1 + \frac{1}{\sqrt{2\pi\epsilon}} \\ -2 + \frac{1}{\sqrt{2\pi\epsilon}} \end{cases}$$

to leading order $(1/\sqrt{\epsilon})$ the derivative is continuous

if you zoomed in, you would see a kink in the derivative

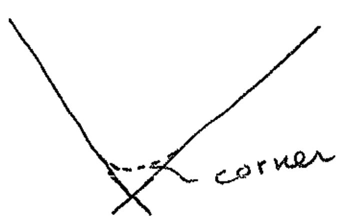
Note:

1) Interior inner solutions must match on both sides ($\xi \to \pm\infty$), so the inner solution must have limiting behavior as $\xi \to \pm\infty$



these two used to describe spikes, such as  , for example

Corner Layers



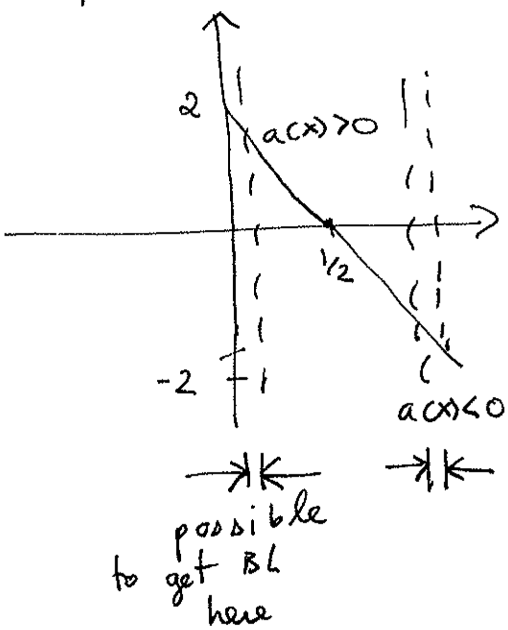
2/3/04
4/4

Boundary layer on both sides

$$\epsilon y'' - 2(2x-1)y' + 4y = 0$$

$$y(0) = 1 \qquad y(1) = 2$$

plot $a(x) = -2(2x-1)$



outer solution

$$Y_{OUT}(x) \sim Y_0(x) + \cdots$$

$$\Rightarrow \quad -2(2x-1)Y_0' + 4Y = 0$$

$$\frac{Y_0'}{Y_0} = \frac{2}{2x-1}$$

$\Rightarrow$ $Y_0(x) = c_0(2x-1)$ no boundary conditions for c_0, yet

Try a boundary layer at $x=0$
guess that the width of the BL is $O(\epsilon)$
rescale, let $\xi = \frac{x}{\epsilon}$

$$\Rightarrow \quad \frac{\epsilon}{\epsilon^2} y_{\xi\xi} - 2(2\xi\epsilon - 1)\frac{1}{\epsilon} y_\xi + 4y = 0$$

$$y_{\xi\xi} + 2(1 - 2\xi\epsilon)y_\xi + 4\epsilon y = 0 \quad \Big\} \text{ inner solution at } x = 0$$

must satisfy left BC, $y(0) = 1$

$$Y_{\xi\xi} + 2(1 - 2\epsilon\xi)\, Y_{\xi} + 4\epsilon Y = 0$$

let $Y_{in}(\xi) \sim y_0(\xi) + \cdots$

to leading order $(\epsilon = 0)$

$$y_0''(\xi) + 2y_0(\xi) = 0, \quad y_0(0) = 1$$

$$\to \boxed{y_0(\xi) = c_2 e^{-2\xi} + c_1}$$

$$y_0(0) = 1 \to \cancel{y_0(\xi)} \quad \boxed{c_1 + c_2 = 1}$$

$$\Rightarrow \boxed{y_0(\xi) = c_1 + (1 - c_1)e^{-2\xi}, \quad \xi = \frac{x}{\epsilon}}$$

This is enough to give a composite solution.
If the only BL is at $x = 0$ then $Y_0(1) = 2$

$$Y_0(x) = 2(2x - 1) \quad \text{where } c_0 = 2$$

match: $c_1 = -2$

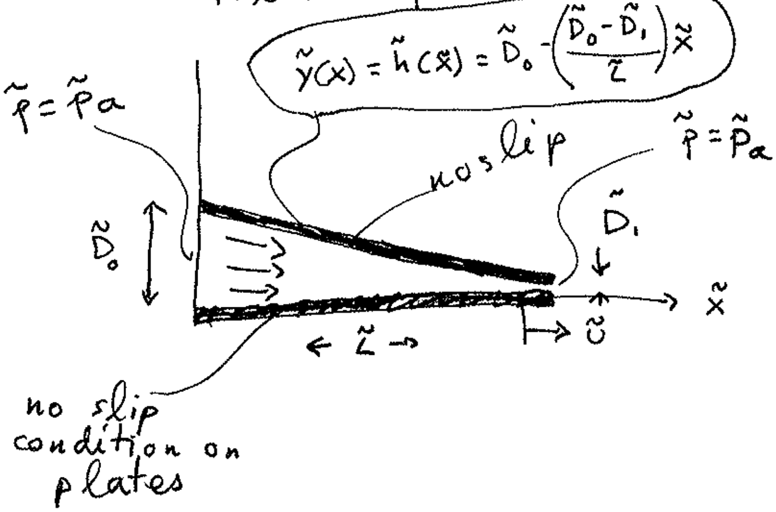
$$\Rightarrow \boxed{Y_{composite}(x) \sim 2(2x - 1) + 3e^{-2x/\epsilon}}$$

Though this solution seems to work, it is not the only possible composite solution. In fact, we can find a one-parameter family of composite solutions, but only one of these gives the correct results.

Example: Lubrication Theory / Slider Bearing

(see text by Holmes, pg 66 #3)

tilde quantities are dimensional



bottom plate moves at $\tilde{U}$ and pushes fluid along

Boundary condition pressure at inlet & outlet equals pressure of ambient $p = p_a$

viscosity, $\tilde{\nu}$

the goal is to calculate the total vertical force exerted by the fluid on the upward plate. The force measures the bearing effectiveness. The greater the force, the better the lubricant.

in two-dimensions, treat force as pressure per unit length: $\tilde{F} = \frac{\tilde{p} - \tilde{p}_a}{\tilde{L}} \Rightarrow \quad \tilde{F} = \int_0^{\tilde{L}} (\tilde{p} - \tilde{p}_a)\, d\tilde{x}$

we can write down Reynold's equation for pressure

$$\frac{d}{d\tilde{x}}\left(\tilde{h}^3(\tilde{x})\ \tilde{p}\ \frac{d\tilde{p}}{dx}\right) = 6\tilde{\nu}\,\tilde{U}\,\frac{d}{d\tilde{x}}\left(\tilde{h}(x)\,\tilde{p}\right)$$

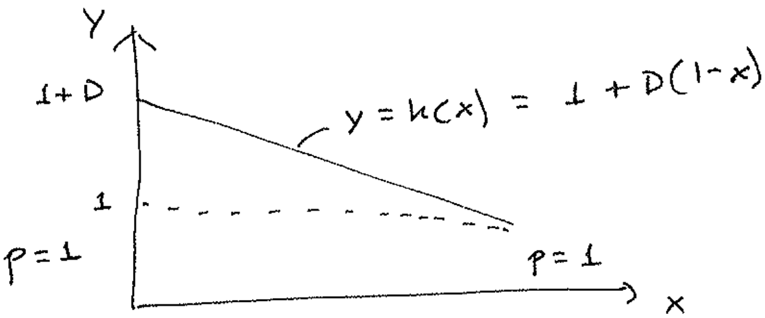
Nondimensionalize:

let $x = \frac{\tilde{x}}{\tilde{L}}$, $y = \frac{\tilde{y}}{\tilde{D}_1}$, $h = \frac{\tilde{h}}{\tilde{D}_1}$, $p = \frac{\tilde{p}}{\tilde{p}_a}$

$F = \tilde{p}_a \tilde{L}$

$$x = \frac{\tilde{x}}{\tilde{L}}, \quad y = \frac{\tilde{y}}{\tilde{D}_1}, \quad h = \frac{\tilde{h}}{\tilde{D}_1}, \quad p = \frac{\tilde{p}}{\tilde{p}_a}, \quad F = \tilde{p}_a \tilde{L}$$

$$\frac{d}{d\tilde{x}}\left(\tilde{h}^3(\tilde{x})\, \tilde{p}\, \frac{d\tilde{p}}{d\tilde{x}} \right) = 6\, \tilde{\mu}\, \tilde{U}\, \frac{d}{d\tilde{x}}\left(\tilde{h}(\tilde{x})\, \tilde{p} \right)$$



$$\Rightarrow \boxed{\begin{array}{l} \dfrac{d}{dx}\left(h^3 p\, p_x \right) = \Lambda \dfrac{d}{dx}(hp) \\ \text{with BCs :} \quad p(x=0) = p(1) = 1 \end{array}}$$

where $\Lambda = \dfrac{6\, \tilde{\mu}\, \tilde{L}\, \tilde{U}}{\tilde{p}_a\, \tilde{D}_1^2}$

"Bearing number"

usually Λ is large for lubricants with high viscosity

so assume

$$\boxed{\Lambda \gg 1}$$

this assumption →

gives us a small parameter, $\epsilon = \frac{1}{\Lambda} \ll 1$

such that Reynolds eqn becomes

$$\boxed{\begin{array}{l} \epsilon \left(h^3 p\, p_x \right)_x = (hp)_x \\ p(0) = p(1) = 1 \\ h = 1 + D(1-x) \end{array}}$$

and $F = \int_0^1 (p-1)\, dx$

2/10/04
2/4

Begin with outer expansion

$$p(x) \sim P_0(x) + \epsilon P_1(x) + \cdots$$

the $O(1)$ equation is $\left(P_0(x)h(x)\right)' = 0$

$$\rightarrow \quad P_0(x) = \frac{c_0}{h(x)} = \frac{c_0}{1+D(1-x)}$$

the value of c_0 depends on whether BL on left side or right side AND c_0 can't satisfy both BCs

Note, to leading order, the integral for F becomes

$$F \sim \int_0^1 \left(P_0(x) + \epsilon P_1(x) + \cdots - 1\right) dx$$

$$F \sim \int_0^1 \left(P_0(x) - 1\right) dx$$

Try boundary layer at $x=0$

if BL at $x=0$, outer solution must satisfy BC at $x=1$

$$\Rightarrow \quad P_0(1) = 1 \quad \Rightarrow \quad c_0 = 1$$

$$\Rightarrow \quad P_0(x) = \frac{1}{1+D(1-x)}$$

Stretch the inner variable and expand as usual

$$\xi = \frac{x}{\epsilon}$$

$$\Rightarrow \quad \frac{\epsilon}{\epsilon}\left(h^3 p \frac{1}{\epsilon} p_\xi\right)_\xi = \frac{1}{\epsilon}(hp)_\xi$$

$$\Rightarrow \left(h^3 p\, p_\xi\right)_\xi = (hp)_\xi \quad , \quad p(0) = 1$$

assume this is leading order, but lets expand anyways

$$\left(h^3 p p_\xi\right)_\xi = (h p)_\xi \quad , \quad p(0) = 1$$

Let $p_{IN} \sim p_0^I(\xi) + \epsilon p_1^I(\xi) + \ldots$

$$\cancel{O(1) \to \left(h^3 p_0^I (p_0^I)_\xi\right)_\xi = (h p_0^I)_\xi}$$

expand $h(x)$

$h(x) = 1 + D(1-x) = 1 + D(1 - \epsilon\xi)$

$\to h \sim 1 + D + O(\epsilon)$

$h^3 \sim (1+D)^3 + \ldots$

so that the $O(1)$ equ is:

$$\left((1+D)^3 p_0^I (p_0^I)_\xi\right)_\xi = \left((1+D) p_0^I\right)_\xi \quad , \quad p_0^I(0) = 1$$

Integrate

$$(1+D)^2 p_0^I (p_0^I)_\xi = p_0^I + c_1$$

integrate again, (separation variables)

$$(1+D)^2 \frac{p_0^I (p_0^I)_\xi}{p_0^I + c_1} = 1 \quad \Rightarrow \quad \underbrace{(1+D)^2\left[p_0^I(\xi) - c_1 \ln\left|p_0^I(\xi) + c_1\right|\right] = \xi + c_2}_{\text{implicit relation for } p_0^I(\xi)}$$

Need to match as $\xi \to \infty$

the RHS $\to \infty$

therefore LHS $\to \infty$ also

note, $p_0^I(\xi)$ is bounded, so it must be that

$-c_1 \ln\left|p_0^I(\xi) + c_1\right| \to \infty$

we require $c_1 > 0$ but then $p_0^I \to -c_1 < 0$

but pressure must be positive so no match is possible. $\Rightarrow$ $\boxed{\text{NO BL at } x = 0}$

2/10/04
3/4

Try BC at $x = 1$

outer soln satisfies BC at $x = 0$

$P_o(0) = 1$ so $c_o = 1 + D$ and

$$P_o(x) = \frac{1+D}{1+D(1-x)}$$

use stretching transformation, $\boxed{u = \frac{x-1}{\epsilon} \leftrightarrow x = 1 + \epsilon u}$

reminder, the eqn is $\epsilon(h^3 p p_x)_x = (hp)_x$, $\cancel{p(1)=1}$

$h = 1 + D(1-x)$ $\quad p(u=0) = 1$

Expand $h(x)$

$h(u) \sim 1 + D(\cancel{1} - \cancel{(1}+\epsilon u)) = 1 - \epsilon u D = 1 + O(\epsilon)$

so to leading order $h(u)$ is one

Expand $P_{IN2} \sim p_o^i(u) + \epsilon\, p_1^i(u)$

the order $O(1)$ equation is

$$(p_o^i(p_o^i)_u)_u = (p_o^i)_u$$

$$\rightarrow \quad p_o^i \frac{d}{du} p_o^i = p_o^i + c_3$$

$$\Rightarrow \quad \boxed{p_o^i - c_3 \ln|p_o^i(u) + c_3| = u + c_4}$$

this will match as $u \rightarrow -\infty$

as $u \rightarrow -\infty$, RHS $\rightarrow -\infty$

we require $c_3 \ln|p_o^i(u) + c_3| \rightarrow \infty$

so $c_3 < 0 \rightarrow p_o^i(u) \rightarrow -c_3 > 0$

this matches

Summary

constants: c_0, c_3, c_4

$$c_0 = 1 + D$$

obtain c_4 from BC at $\eta = 0$, $p_0^i(0) = 1$

$$\rightarrow \quad c_4 = p_0^i(0) - c_3 \ln \left| p_0^i(0) + c_3 \right|$$

$$\rightarrow \quad c_4 = 1 - c_3 \ln \left| 1 + c_3 \right|$$

by primitive matching

$$\lim_{x \to 1^-} P_0(x) = \lim_{\eta \to -\infty} p_0^i(\eta)$$

$$c_0 = -c_3$$

since $c_0 = 1 + D$, $c_3 = -\underbrace{(1 + D)}_{\text{this is the common part}}$

$$p_0^i(\eta) + (1+D) \ln \left| \frac{p_0^i(\eta) - (1+D)}{D} \right| = \eta + 1$$

composite solution

$$P_{comp} = P_{OUT} + P_{IN} - \underbrace{C.P.}_{\text{common part}}$$

$$P_{comp} \sim P_0(x) + p_0^i(\eta) - C.P.$$

$$\rightarrow P_{comp} \sim \frac{1 + D}{1 + D(1-x)} + p_0^i(\eta) - (1 + D)$$

$$\rightarrow \boxed{P_{comp} \sim -\frac{D(1+D)(1-x)}{1 + D(1-x)} + q_0\left(\frac{x-1}{\epsilon}\right)}$$

still have implicit relation, but at least got rid of diff equation

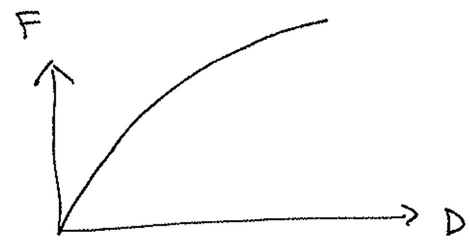
2/10/04
4/4

to the leading order, calculate Force

$$F \sim \int_0^1 \left(P_0(x) - 1\right) dx = \int_0^1 \left(\frac{1+D}{1+D(1-x)} - 1\right) dx$$

$$= \frac{1+D}{D} \ln(1+D) - 1$$

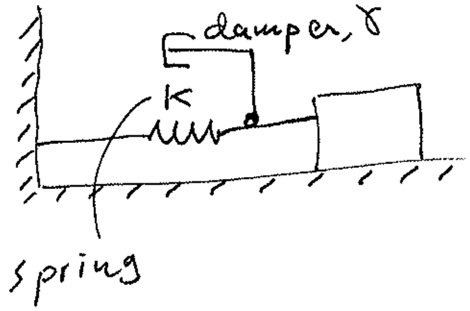
$\Rightarrow$



2/13/04

1/3

Example: Mass-Spring Damper system



Hooks Law

$$F_s = -kx$$

damper force, $F_D = \gamma \frac{dx}{dt}$

$$\sum F = F_s + F_D = ma$$

$$\rightarrow \quad -kx - \gamma \dot{x} = m\ddot{x} \quad \Rightarrow \quad \boxed{m\ddot{x} + \gamma\dot{x} + kx = 0}$$

Pick some initial conditions: $\boxed{x(0) = 0, \quad \dot{x}(0) = v_0}$

Nondimensionalize using the following choices

$$t = \frac{\tilde{t}}{t_*} \quad \text{where } t_* = \frac{\gamma}{k} \quad \text{and } \epsilon = \frac{mk}{\gamma^2}$$

$$y = \frac{x}{x_*} \quad \text{where } x_* = \frac{m v_0}{\gamma}$$

the resulting equation is $\boxed{\begin{array}{l} \epsilon y'' + y' + y = 0 \\ y(0) = 0, \quad y'(0) = \frac{1}{\epsilon} \end{array}}$

outer solution

$$y_{OUT}(t) \sim Y_0(t) + \epsilon Y_1(t) + \dots$$

$$\epsilon Y_0'' + \epsilon^2 Y_1''(t) + \dots + Y_0'(t) + \epsilon Y_1'(t) + \dots + Y_0 + \epsilon Y_1 + \dots = 0$$

to leading order: $Y_0' + Y_0 = 0 \rightarrow Y_0(t) = C_0 e^{-t}$

$O(\epsilon)$: $Y_0'' + \cancel{\epsilon} Y_1'(t) + Y_1 = 0 \rightarrow Y_1'(t) + Y_1(t) + C_0 e^{-t} = 0$

$$\rightarrow Y_1(t) = e^{-t}(-C_0 t + C_1)$$

so the outer expansion is

$$\boxed{Y_{out}(t) \sim C_0 e^{-t} + \epsilon(C_1 - C_0 t) e^{-t} + \dots}$$

$$\boxed{y_{OUT} \sim c_0 e^{-t} + \epsilon(c_1 - c_0 t) e^{-t} + \dots}$$

Notice
- y_{OUT} cannot satisfy the initial conditions so the expansion is not valid for small t
- y_{out} is not valid for large t

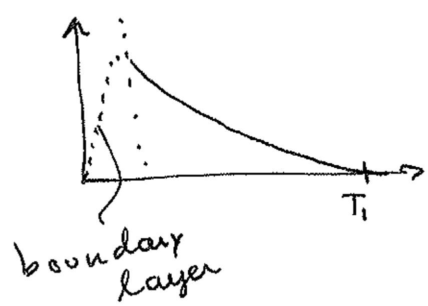
$$e^{-t} \sim \epsilon t e^{-t} \to t = O(1/\epsilon)$$

therefore y_{out} is valid for some $0 < T_0 < t < T_1 < \infty$ (T_1: $O(1/\epsilon)$)

If we let $t = 1/\epsilon$ then

$$y_{OUT} \sim c_0 e^{-1/\epsilon} + \epsilon\left(c_1 - c_0 \frac{1}{\epsilon}\right) e^{-1/\epsilon}$$

these terms become transcendentally small so the outer solution looks lice



so we're not going to worry about solution after T_1

Inner solution (BL at $x=0$)

let $\tau = \frac{t}{\delta}$, $\epsilon y'' + y' + y = 0$, $y(0) = 0$, $y'(0) = \epsilon^{-1}$

$$\to \underset{①}{\frac{\epsilon}{\delta^2} Y_{\tau\tau}} + \underset{②}{\frac{1}{\delta} Y_\tau} + \underset{③}{Y} = 0$$

balance ① and ② then $\epsilon \delta^{-2} = \delta^{-1} \to \delta = \epsilon$

$$\Rightarrow \boxed{Y_{\tau\tau} + Y_\tau + \epsilon Y = 0}$$

$$\boxed{Y(0) = 0}$$

$$\frac{dY}{dt}(t=0) = \frac{1}{\cancel{\epsilon}} \frac{dY}{d\tau}(0) = \frac{1}{\cancel{\epsilon}} \to \boxed{\frac{dY}{d\tau}(0) = 1}$$

<u>try naive expansion</u>

$$Y_{\tau\tau} + Y_\tau + \epsilon Y = 0 \quad , \; Y(0) = 0 \; , \; Y'(0) = 1$$

$$\boxed{Y_{IN}(\tau) \sim y_0(\tau) + \epsilon\, y_1(\tau) + \cdots}$$

$$y_0'' + \epsilon y_1'' + \cdots + y_0' + \epsilon y_1' + \cdots + \epsilon y_0 + \epsilon^2 y_1 + \cdots = 0$$

leading order, $O(1)$:

$$y_0'' + y_0' = 0 \quad , \; y_0(0) = 0 \quad , \; y_0'(0) = 1$$

$$y_0(\tau) = D_0 + D_1 e^{-\tau} \quad \longrightarrow \quad \boxed{y_0(\tau) = 1 - e^{-\tau}}$$

$O(\epsilon)$: $\quad y_1'' + y_1' + y_0 = 0 \;\longrightarrow\; y_1'' + y_1' = e^{-\tau} - 1$

the leading order satisfied ICs

so then $\quad y_1(0) = y_1'(0) = 0$

so by integrating factor, $\cancel{y_1(\tau) = e^{-\tau}(1-}$ $\quad \cancel{2(1+e^{-\tau}) - (1+e^{-\tau})}$

$$y_1(\tau) = 2(1 - e^{-\tau}) - \tau(1 + e^{-\tau})$$

therefore

$$\boxed{Y_{IN}(\tau) = 1 - e^{-\tau} + \epsilon\left[2(1-e^{-\tau}) - \tau(1 + e^{-\tau})\right] + \cdots}$$

Notice, the exact solution to the original eqn, $\epsilon y'' + y' + y = 0$, $y(0) = 0$, $y'(0) = \frac{1}{\epsilon}$ is given by

$$y_{exact}(t) = \frac{1}{\sqrt{1-4\epsilon}}\left[\exp\left(\frac{-1+\sqrt{1-4\epsilon}}{2\epsilon}\,t\right) - \exp\left(\frac{-1-\sqrt{1-4\epsilon}}{2\epsilon}\,t\right)\right]$$

~~substitute~~ expand in powers of ϵ

$$\sqrt{1-4\epsilon} \sim 1 - 2\epsilon - 2\epsilon^2 + \cdots$$

$$\frac{1}{\sqrt{1-4\epsilon}} \sim \frac{1}{1-2\epsilon+\cdots} = 1 + 2\epsilon + \cdots$$

$$\frac{-1\pm\sqrt{1-4\epsilon}}{2\epsilon} \sim \frac{-1 \pm (1-2\epsilon-2\epsilon^2+\cdots)}{2\epsilon} = \begin{cases} -\epsilon - \epsilon^2 + \cdots & (+) \\ -1 + \epsilon + \cdots & (-) \end{cases}$$

the expansion becomes

$$y_{exact}(t) \sim (1 + 2\epsilon + \cdots)\left[e^{(-\epsilon-\epsilon^2+\cdots)t/\epsilon} - e^{(-1+\epsilon+\cdots)t/\epsilon}\right]$$

transcendentally small

$$y_{exact}(t) \sim (1 + 2\epsilon + \cdots)\left[e^{-t}e^{-\epsilon t} - \cancel{e^{\left(-\frac{1}{\epsilon}+1\right)t}}\right]$$

$$y_{exact}(t) \sim (1 + 2\epsilon + \cdots)\, e^{-t}\left(1 - \epsilon t + \cdots\right)$$

$$y_{exact}(t) \sim e^{-t}\left(1 + \epsilon(2 - t) + \cdots\right)$$

the asymptotic solution is

$$y_{OUT} \sim c_0 e^{-t} + \epsilon(c_1 - c_0 t)e^{-t}$$

the comparison shows that $c_0 = 1$, $c_1 = 2$

For

$$y_{exact}(t) = \frac{1}{\sqrt{1-4\epsilon}}\left[\exp\left(\frac{-1+\sqrt{1-4\epsilon}}{2\epsilon}\,t\right) - \exp\left(\frac{-1-\sqrt{1-4\epsilon}}{2\epsilon}\,t\right)\right]$$

substitute $\tau = \frac{t}{\epsilon}$ then the inner ~~expansion~~ solution given by

$$y_{exact}(\tau) = \frac{1}{\sqrt{1-4\epsilon}}\left[\exp\left(\frac{-1+\sqrt{1+4\epsilon}}{2}\,\tau\right) - \exp\left(\frac{-1-\sqrt{1-4\epsilon}}{2\cancel{\epsilon}}\,\tau\right)\right]$$

the expansion becomes

$$y_{exact}(\tau) = (1 + 2\epsilon + \cdots)\left[e^{\tau(-\epsilon-\epsilon^2+\cdots)} - e^{\tau(-1+\epsilon+\cdots)}\right]$$

$$= (1+2\epsilon+\cdots)\left(e^{-\epsilon\tau} - e^{-\tau}e^{\epsilon\tau}\right)$$

$$= (1+2\epsilon+\cdots)\left(1 - \epsilon\tau - e^{-\tau}(1+\epsilon\tau+\cdots)\right)$$

$$= (1+2\epsilon+\cdots)\left(1 - \epsilon\tau - e^{-\tau} - \epsilon\tau e^{-\tau} + \cdots\right)$$

$$= 1 - e^{-\tau} + \epsilon\left[2(1-e^{-\tau}) - \tau(1+e^{-\tau})\right] + \cdots$$

this agrees with inner expansion

$$y_{IN}(\tau) = 1 - e^{-\tau} + \epsilon\left[2(1-e^{-\tau}) - \tau(1+e^{-\tau})\right]$$

2/24/03
1/3

$$(1+\epsilon)\, x^2 y' = \epsilon\left[(1-\epsilon)xy^2 - (1+\epsilon)x + y^3 + 2\epsilon y^2\right]$$

$$Y_{OUT}(x) = 1 + \epsilon\left(1-\frac{1}{x}\right) + \epsilon^2\left(\frac{3}{2x^2} - \frac{2}{x} + \frac{1}{2}\right), \quad n=3, \quad Y_{OUT}(1)=1$$

$$Y_{in}(\xi) = \sqrt{\frac{\xi}{2+\xi}} + \epsilon\,\xi^{1/2}\,\frac{1+c_1\xi}{(2+\xi)^{3/2}}, \quad m=2$$

$x = \epsilon\xi$
$\xi = \frac{x}{\epsilon}$

$\boxed{\text{3-2 Van Dyke matching}}$: the two term expansion of $Y_{OUT}(\epsilon\xi)$ = the three term expansion of $Y_{in}\left(\frac{x}{\xi}\right)$

$$Y_{OUT}(\epsilon\xi) \sim 1 + \epsilon\left(1 - \frac{1}{\epsilon\xi}\right) + \epsilon^2\left(\frac{3}{2\epsilon^2\xi^2} - \frac{2}{\epsilon\xi} + \frac{1}{2}\right) + \cdots$$

$$\sim 1 + \epsilon - \frac{1}{\xi} + \cdots + \frac{3}{2\xi^2} - \frac{2\epsilon}{\xi} + \frac{\epsilon^2}{2} + \cdots$$

$$\sim \left(1 - \frac{1}{\xi} + \frac{3}{2\xi^2}\right) + \epsilon\left(1 - \frac{2}{\xi}\right) + \cdots$$

$$Y_{in}\left(\frac{x}{\epsilon}\right) \sim \left(\frac{x/\epsilon}{2+\frac{x}{\epsilon}}\right)^{1/2} + \epsilon\left(\frac{x}{\epsilon}\right)^{1/2}\frac{1+c_1\frac{x}{\epsilon}}{\left(2+\frac{x}{\epsilon}\right)^{3/2}} \leftarrow \epsilon\left(\cancel{\frac{x}{\epsilon}}\right)^{1/2}\frac{\left(\frac{\epsilon}{x}+c_1\right)\left(\cancel{\frac{\epsilon}{x}}\right)^{1/2}}{\left(2\frac{\epsilon}{x}+1\right)^{3/2}}$$

$$\sim \left(\frac{1}{1+2\epsilon/x}\right)^{1/2} + \epsilon\,\frac{\frac{\epsilon}{x}+c_1}{\left(1+\frac{2\epsilon}{x}\right)^{3/2}} + \cdots$$

$$\sim \left(1 - \frac{2\epsilon}{x} + \frac{4\epsilon^2}{x^2} + \cdots\right)^{1/2} + \epsilon\left(\frac{\epsilon}{x}+c_1\right)\cdot\frac{1}{\left(1+\frac{3\epsilon}{x}+\cdots\right)}$$

$$\sim 1 + \frac{\left(-\frac{2\epsilon}{x} + \frac{4\epsilon^2}{x^2}\right)}{2} - \frac{4\epsilon^2}{8x^2} + \epsilon\left(\frac{\epsilon}{x}+c_1\right)\left(1 - \frac{3\epsilon}{x} + \cdots\right)$$

$$\sim 1 + \epsilon\left(c_1 - \frac{1}{x}\right) + \epsilon^2\left(\frac{3}{2x^2} + \frac{1-3c_1}{x} + \cdots\right)$$

$$Y_{OUT}(\epsilon\xi) = Y_{in}\left(\frac{x}{\epsilon}\right)$$ ← this is the common part of the expansion

$$1 - \frac{1}{\xi} + \frac{3}{2\xi^2} + \epsilon\left(1 - \frac{2\epsilon}{x}\right) = 1 + \epsilon\left(c_1 - \frac{1}{x}\right) + \epsilon^2\left(\frac{3}{2x^2} + \frac{1-3c_1}{x}\right),$$

let $\xi = \frac{x}{\epsilon}$

$$\cancel{1} - \frac{\epsilon}{x} + \frac{3\epsilon^2}{2x^2} + \epsilon\left(1 - \frac{2\epsilon}{x}\right) = \cancel{1} + \epsilon\left(c_1 - \frac{1}{x}\right) + \epsilon^2\left(\frac{3}{2x^2} + \frac{1-3c_1}{x}\right)$$

$O(\epsilon)$: $\quad -\frac{1}{x} + 1 = c_1 - \frac{1}{x} \quad \longrightarrow \quad \boxed{c_1 = 1}$

check $O(\epsilon^2)$:

$$\frac{3}{2x^2} - \frac{2}{x} = \frac{3}{2x^2} + \frac{1-3c_1}{x}$$ is satisfied for $c_1 = 1$

therefore this matching is satisfied

therefore the common part is: $1 + \epsilon\left(1 - \frac{1}{x}\right) + \epsilon^2\left(\frac{3}{2x^2} - \frac{2}{x}\right)$

and the composite solution is

$$y_c(x) \sim \left(\frac{x}{x+2\epsilon}\right)^{1/2} + \epsilon\left(\frac{x}{x+2\epsilon}\right)^{3/2} + \epsilon^2\left[\frac{1}{2} + \frac{x^{1/2}}{(x+2\epsilon)^{3/2}}\right] + O(\epsilon^3)$$

at $x = 1$, $$\frac{1}{\sqrt{1+2\epsilon}} + \frac{\epsilon}{(1+2\epsilon)^{3/2}} + \frac{\epsilon^2}{2} + \frac{\epsilon^2}{(\cancel{x}+2\epsilon)^{3/2}}$$

$$\frac{2(1+2\epsilon)}{2(1+2\epsilon)^{3/2}} + \frac{2\epsilon}{2(1+2\epsilon)^{3/2}} + \frac{\epsilon^2(1+2\epsilon)^{3/2}}{2(1+2\epsilon)^{3/2}} + \frac{2\epsilon^2}{2(1+2\epsilon)^{3/2}}$$

$$\frac{2 + 4\epsilon + 2\epsilon + \epsilon^2(1+2\epsilon)^{3/2} + 2\epsilon^2}{2(1+2\epsilon)^{3/2}}$$

$$\frac{2(\epsilon^2 + 3\epsilon + 1) + \epsilon^2(1+2\epsilon)^{3/2}}{2(1+2\epsilon)^{3/2}}$$

Modified Van-Dyke Principle

Example: $F(x) = 1 + \frac{\ln(x+\epsilon)}{\ln(\epsilon)}$

$O\left(\frac{1}{\ln(\epsilon)}\right)$

the outer expansion is given by

$$F(x) = 1 + \frac{\ln x + \ln(1+\epsilon/x)}{\ln \epsilon}$$

since $\ln(1+x) \sim x + \cdots$

$$F(x) \sim 1 + \frac{\ln x + \epsilon/x + \cdots}{\ln \epsilon}$$

$$\sim 1 + \frac{\ln x}{\ln \epsilon} + \frac{\epsilon}{x \ln(\epsilon)} + \cdots$$ ← well-ordered expansion

Inner expansion: let $\xi = \frac{x}{\epsilon}$

$$F(\epsilon\xi) = 1 + \frac{\ln(\epsilon\xi + \epsilon)}{\ln \epsilon} = 1 + \frac{\ln \epsilon + \ln(\xi+1)}{\ln \epsilon}$$

$$\sim 2 + \frac{\ln \xi}{\ln \epsilon} + \cdots$$

$$f(x) = 1 + \frac{\ln(x+\epsilon)}{\ln \epsilon}$$

$$f_{OUT}(x) \sim 1 + \frac{\ln x}{\ln \epsilon} + \frac{\epsilon}{\ln \epsilon} \cdot \frac{1}{x} + \cdots$$

$$f_{in}(\xi) \sim 2 + \frac{\ln(\xi+1)}{\ln \epsilon}$$

Note that the van Dyke method works for some m and n, but not all. Eg, pick $m = n = 1$ in this case only to obtain $1 \neq 2$, so the Van Dyke matching fails.

when logarithms are involved, all terms with the same power of ϵ should be counted as one term. In other words, ignore logarithms when counting terms.

$$\epsilon^{\alpha}(\ln \epsilon)^{\beta} \quad (\alpha \text{ fixed}, \beta = 0, \pm 1, \pm 2, \ldots)$$

are counted as one term

Therefore in the outer solution,

$$f_{OUT}(x) \sim \underbrace{1 + \epsilon^{0} \frac{\ln x}{\ln \epsilon}}_{\text{this is 1 term}} + \underbrace{\frac{\epsilon}{\ln \epsilon} \cdot \frac{1}{x}}_{\text{this is the next term}}$$

Van Dyke works with this counting scheme

Try $m = n = 1$

$$1 + \frac{\ln x}{\ln \epsilon} = 2 + \frac{\ln(\xi+1)}{\ln \epsilon}$$

$$= 2 + \frac{\ln(\frac{x}{\epsilon}+1)}{\ln \epsilon} = 2 + \frac{\ln \frac{x}{\epsilon} + \ln(1+\frac{\epsilon}{x})}{\ln \epsilon}$$

$$= 2 + \frac{\ln x - \ln \epsilon + \ln(1+\frac{\epsilon}{x})}{\ln \epsilon} = \boxed{1 + \frac{\ln x}{\ln \epsilon}} + \cancel{\frac{\epsilon}{x \ln \epsilon}}$$

neglect

there are other cases where counting terms is not so clear,

1) when logarithms are involved, which we just covered
2) when terms have a coefficient of 0, such as

$$y \sim y_0 + \epsilon y_1 + \epsilon^2 y_2 + \epsilon^4 y_4 + \cdots$$

what do we do about $O(\epsilon^3)$ term?

3) when Y_{OUT} and Y_{in} are scaled differently

$$Y_{OUT} \sim Y_0 + \epsilon Y_1 + \epsilon^2 Y_2 + \cdots$$

$$Y_{in} \sim \epsilon^{1/2} Y_1 + \epsilon^{3/2} Y_1 + \cdots$$

the Modified van Dyke Principle works for all these cases.

The inner expansion to order Δ_I of the outer expansion Δ_0 = The outer expansion to order Δ_0 of the inner expansion to Δ_I

Far Field / Switchback

2/26/0
1/4

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = -\epsilon u\frac{du}{dr} \qquad u(1)=0, \quad u\big|_{r\to\infty}=1$$

the Navie approch results in, $u \sim u_0 + \epsilon u_1 + \cdots$

$O(1)$: $\frac{d}{dr}\left(r^2\frac{du_0}{dr}\right)=0, \quad u_0(1)=0, \quad u_0\big|_{r\to\infty}=1$

$\rightarrow \quad u_0(r) = 1-\frac{1}{r}$

$O(\epsilon)$: $\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du_1}{dr}\right) = -u_0\frac{du_0}{dr}, \quad u_1(1)=0, \quad u_1\big|_{r\to\infty}=0$

this Fails

why?

expand the equation

$$u_{rr} + \left(\frac{2}{r}+\epsilon u\right)u_r = 0$$

substitute $u_0(r) = 1-\frac{1}{r}$, $(u_0)_r = \frac{1}{r^2}$, $(u_0)_{rr} = \frac{-2}{r^3}$

$$\rightarrow \quad -\frac{2}{r^3} + \left(\frac{2}{r}+\epsilon\left(1-\frac{1}{r}\right)\right)\frac{1}{r^2} = 0$$

$$O(1/r^3) \quad O(1/r^3) \quad O\left(\frac{\epsilon}{r^2}\right)$$

so naive expansion fails where $r = O(1/\epsilon)$

$$u_{rr} + \frac{2}{r}\frac{du}{dr} = -\epsilon u\frac{du}{dr} \longrightarrow u_{rr} + \frac{du}{dr}\left(\frac{1}{r}-\epsilon u\right) = 0$$

Far field solution

compress the spatial coordinate, which will make the derivatives bigger

$\rho = \epsilon r$ - where we picked ϵ after balancing more general term $\delta(\epsilon)$

$$\rightarrow \quad \frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2 \frac{du}{d\rho}\right) = -u\frac{du}{d\rho} \quad \text{with BC} \quad u\Big|_{\rho\to\infty} = 1$$

$$u_{far} \sim u_0(\rho) + \epsilon u_1(\rho) + \dots$$

to leading order, obtain the whole equation, however, since at the far field, $u \to 1$, by inspection, we find $u_0(\rho) \equiv 1$ (because derivatives are small, so $u_0 \sim$ const, but to satisfy BC, $u_0 = 1$)

$$O(\epsilon): \quad \frac{1}{\rho^2}\left(\rho^2 \frac{du_1}{d\rho}\right) = -u_1\frac{du_1}{d\rho}, \quad u_1\Big|_{\rho\to\infty} = 0$$

can solve this in integral form

$$\rightarrow \quad u_1(\rho) = -D_0\int_\rho^\infty \frac{e^{-s}}{s^2}\,ds$$

From 2-d Matching: ~~the~~ outer solution becomes

$$u_{OUT}(r) \sim c_0\left(1-\frac{1}{r}\right) + \epsilon\left[-c_0^2\left(1+\frac{1}{r}\right)\ln r + (c_0^2 + c_2)\left(1-\frac{1}{r}\right)\right]$$

$$u_{Far}(\rho) \sim 1 - \epsilon D_0 \underbrace{\int_\rho^\infty \frac{e^{-s}}{s^2}\,ds}$$

Need to expand this integral

$$\int_\rho^\infty \frac{e^{-s}}{s^2}\,ds \sim \frac{1}{\rho} + \ln\rho + (\gamma - 1) - \frac{1}{2}\rho + o(\rho), \quad \rho \ll 1$$

γ - Euler's constant $= 0.577\dots$

2/26/04
2/4

Van-Dyke matching

$$v_{OUT}(\rho) \sim c_0\left(1-\frac{\epsilon}{\rho}\right) + \epsilon\left[-c_0^2\left(1+\frac{\epsilon}{\rho}\right)(\ln\rho - \ln\epsilon) + (c_0^2+c_2)\left(1-\frac{\epsilon}{\rho}\right)\right]$$

$$\Rightarrow \boxed{v_{OUT}(\rho) \sim c_0 - \underbrace{c_0^2\,\epsilon\ln\frac{1}{\epsilon} - \epsilon\left[\frac{c_0}{\rho} + c_0^2\ln\rho - (c_0^2+c_2)\right]}_{\text{counts as one term in Van Dyke matching}} + \cdots} \quad (1)$$

$$v_{Far}(r) \sim 1 - \frac{D_0\epsilon}{\cancel{\epsilon\ln\epsilon}}\left(\frac{1}{\epsilon r} + \ln\epsilon r + (\gamma -) + \cdots\right)$$

$$\boxed{v_{Far}(r) \sim \left(1-\frac{D_0}{r}\right) + D_0\epsilon\ln\frac{1}{\epsilon} - \epsilon D_0(\ln r + \gamma - 1) + \cdots} \quad (2)$$

Equate (1) and (2)

$$\Rightarrow \left.\begin{array}{l} c_0 = D_0 = 1 \\ c_2 = \ln\frac{1}{\epsilon} - \gamma \end{array}\right\}$$ this is not asymptotically sound since c_2 must be a constant independent of ϵ

so the matching fails. ~~vsla~~

Suppose we do accept that $c_2 = \ln\frac{1}{\epsilon} - \gamma$

the outer expansion becomes

$$v_{OUT} \sim c_0\left(1-\frac{1}{r}\right) + \epsilon\left(-c_0^2\left(1+\frac{1}{r}\right) + (c_0^2+c_2)\left(1-\frac{1}{r}\right)\right) + \cdots$$

$$\sim \left(1-\frac{1}{r}\right) + \underbrace{\epsilon\ln\frac{1}{\epsilon}}_{\text{suspect}}\left(1-\frac{1}{r}\right) + \cdots$$

suspect, the gauge functions

<u>switch back:</u> Though the matching fails, it reveals the appropriate gauge functions for u_{OUT}

$$\Rightarrow \quad u_{OUT}(r) \sim u_0(r) + \left(\epsilon \ln \frac{1}{\epsilon}\right) \bar{u}_1 + \epsilon u_1 + \cdots$$

<u>Restart the problem</u> (~~outer~~ Far solution stays the same) ~~as does in~~

~~u_{OUT}~~ $$\frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{du}{dr}\right) = -\epsilon u \frac{du}{dr}, \quad u(1)=0, \quad u\big|_{r\to\infty} = 1$$

$O(1)$: $\quad u_0(r) = c_0\left(1 - \frac{1}{r}\right)$

$O(\epsilon \ln 1/\epsilon)$: $\quad \bar{u}_1(r) = \bar{c}_2\left(1 - \frac{1}{r}\right)$

$O(\epsilon)$: $\quad u_1(r) \sim -c_0^2\left(1 + \frac{1}{r}\right)\ln r + \left(c_0^2 + c_2\right)\left(1 - \frac{1}{r}\right) + \cdots$

$$\Rightarrow \quad u_{OUT}(r) \sim c_0\left(1 - \frac{1}{r}\right) + \underbrace{\left(\epsilon \ln \frac{1}{\epsilon}\right)\bar{c}_2\left(1 - \frac{1}{r}\right) + \epsilon\left[-c_0^2\left(1+\frac{1}{r}\right)\ln r + \left(c_0^2 + c_2\right)\left(1 - \frac{1}{r}\right)\right]}_{\text{these two terms are considered } O(\epsilon) \text{ for van-Dyke matching}}$$

As before

$$u_{FAR}(\rho) \sim 1 - \epsilon D_0 \int_\rho^\infty \frac{e^{-s}}{s^2}\, ds$$

<u>Use Van-Dyke matching</u> $(\rho = \epsilon r)$

(3) $$u_{OUT}(\rho) \sim c_0 + \epsilon \ln \frac{1}{\epsilon}\left(\bar{c}_2 - c_0^2\right) - \epsilon\left[\frac{c_0}{\rho} + c_0^2 \ln \rho - \left(c_0^2 + c_2\right)\right] + \cdots$$

(4) $$u_{FAR}(r) \sim 1 - \frac{D_0}{r} + \left(\epsilon \ln \frac{1}{\epsilon}\right) D_0 - \epsilon D_0\left(\ln r + \gamma - 1\right) + \cdots$$

Equate (3) and (4), to obtain

$$c_0 = D_0 = 1, \quad \bar{c}_2 = 1, \quad c_2 = -\gamma$$

2/26/04
3/4

composite solution

$$u_c = u_{OUT} + u_{FAR} - CP$$

$$\Rightarrow u_c(r) \sim \left(1 - \epsilon \underbrace{\int_{\epsilon r}^{\infty} \frac{e^{-s}}{s^2}\, ds}\right) + \left(\epsilon \ln \frac{1}{\epsilon}\right)\frac{1}{r} - \frac{\epsilon}{r}\left(\ln r + 1 - \gamma\right) + \cdots$$

included in leading order because $\int_{\epsilon r}^{\infty} \frac{e^{-s}}{s^2}\, ds \sim \frac{1}{\epsilon r} + \cdots$, to leading order

---

Classical Problem (Kevorkian & Cole) pg 88-95

Nonlinear

$$\epsilon y'' + y y' - y = 0, \quad y(0) = A, \quad y(1) = B, \quad \epsilon \ll 1$$

outer $\quad y_{OUT}(x) \sim y_0(x) + \epsilon y_1(x) + \cdots$

$O(1) \quad y_0 y_0' - y_0 = 0 \quad \rightarrow \quad y_0 = 0, \quad y_0 = x + c_0$

two solutions, which will depend on values of boundary conditions A and B

First, look for boundary layers of thickness $O(\epsilon)$ without knowing where BL is, use stretching transformation $\quad \xi = \frac{x - x_0}{\epsilon}$

$$\Rightarrow Y_{\xi\xi} + Y Y_\xi - \epsilon Y = 0$$

expand with $y_{in}(\xi) \sim Y_0(\xi) + \epsilon Y_1(\xi) + \cdots$

$O(1)$: $\quad Y_{0_{\xi\xi}} + Y_0 Y_{0_\xi} = 0 \quad \rightarrow \quad Y_{0_{\xi\xi}} + \frac{1}{2}\left(Y_0^2\right)_\xi = 0$

$$\rightarrow \; Y_{0_\xi} + \frac{1}{2} Y_0^2 = c_1 \qquad \rightarrow \qquad \frac{Y_{0_\xi}}{c_1 - \frac{1}{2} Y_0^2} = 1$$

integrate and obtain different solutions depending on sign of c_1

$c_1 < 0$: obtain tangents and cotangents, which aren't matchable because they blow up, so ignore this case

$c_1 = 0$: very special case that won't give much information, so just ignore for now

$c_1 > 0$: gives tanh and coth, which are matchable
integrate using hyperbolic trig substitution

$$\frac{dY_0}{1 - \left(\frac{Y_0}{\sqrt{2c_1}}\right)^2} = c_1 \, d\xi \qquad \text{let } \frac{Y_0}{\sqrt{2c_1}} = \tanh\Theta$$

$$\frac{1}{\sqrt{2c_1}} dY_0 = \operatorname{sech}^2\Theta \, d\Theta$$

$$\Rightarrow \frac{\sqrt{2c_1}\operatorname{sech}^2\Theta \, d\Theta}{\operatorname{sech}^2\Theta} = c_1 d\xi$$

$$\Rightarrow \sqrt{2c_1}\,\Theta = c_1 \xi + c_2 \quad \rightarrow \quad \sqrt{2c_1}\tanh^{-1}\left(\frac{Y_0}{\sqrt{2c_1}}\right) = c_1\xi + c_2$$

let $\beta = \sqrt{2c_1}$, $k = \frac{-c_2}{c_1}$

$$\boxed{Y_0(\xi) = \beta\tanh\left(\frac{\beta}{2}(\xi - k)\right)}$$

$$\frac{c_1}{\sqrt{2c_1}}\left(\xi + \frac{c_2}{c_1}\right)$$

$$\frac{\sqrt{2c_1}}{2}\left(\xi + \frac{c_2}{c_1}\right)$$

$$\frac{\beta}{2}(\xi - k)$$

Alternatively, let $\frac{Y_0}{\sqrt{2c_1}} = \coth\Theta$ to obtain

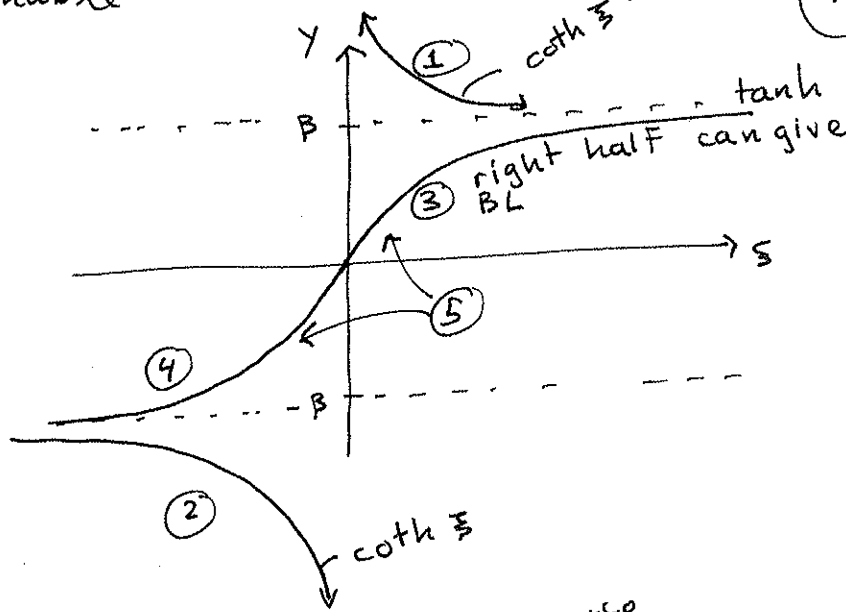
$$\boxed{Y_0(\xi) = \beta\coth\left(\frac{\beta}{2}(\xi - k)\right)}$$

We obtain two matchable inner solutions

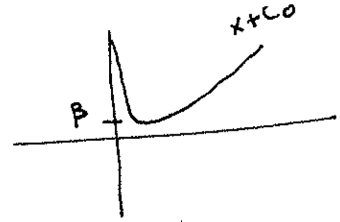
$$Y_0(\xi) = \beta \tanh\left(\frac{\beta}{2}(\xi - k)\right)$$

$$Y_0(\xi) = \beta \coth\left(\frac{\beta}{2}(\xi - k)\right)$$

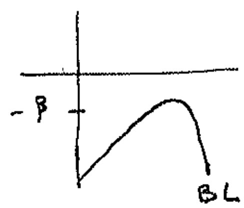
or tanh ξ can give interior layer



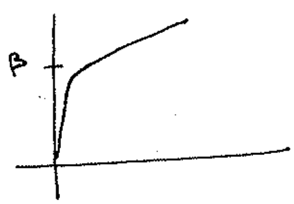
(1) decreases to β as $\xi \to \infty$
would correspond to this picture



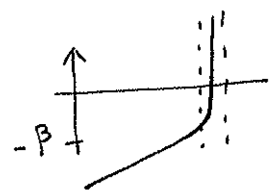
(2) increase to $-\beta$ as $\xi \to -\infty$
would correspond to this picture



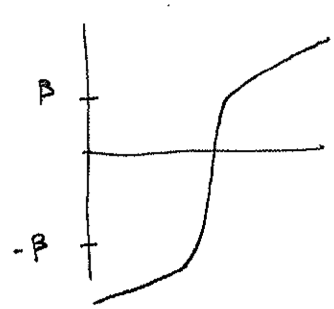
(3) increases to β as $\xi \to \infty$



(4) decrease to $-\beta$ as $\xi \to -\infty$



(5) increases to β as $\xi \to \infty$ and
decreases to $-\beta$ as $\xi \to -\infty$
gives us interior layer



Nonlinear Problem (continued)

$$\epsilon y'' + yy' - y = 0, \quad y(0) = A, \quad y(1) = B, \quad \epsilon << 1$$

$$Y_{OUT} \sim Y_0(x) + \epsilon Y_1(x) + \dots$$

$$O(1): \quad Y_0(Y_0' - 1) = 0 \quad \rightarrow \quad \boxed{\begin{array}{l} Y_0 = 1 \\ Y_0 = x + c \end{array}}$$

BLs of thickness $O(\epsilon)$

$$\xi = \frac{x - x_0}{\epsilon} \quad \Rightarrow \quad y'' + yy' - \epsilon y = 0$$

$$y_{in}(\xi) \sim y_0(\xi) + \epsilon y_1(\xi) + \dots$$

$$\Rightarrow \quad (y_0)_{\xi\xi} + y_0 (y_0)_\xi = 0 \quad \rightarrow \quad (y_0)_{\xi\xi} + \frac{1}{2}(y_0^2)_\xi = 0$$

integrate once: $\quad (y_0)_\xi + \frac{1}{2} y_0^2 = c_1 \quad \rightarrow \quad \dfrac{(y_0)_\xi}{c_1 - \frac{1}{2} y_0^2} = 1$

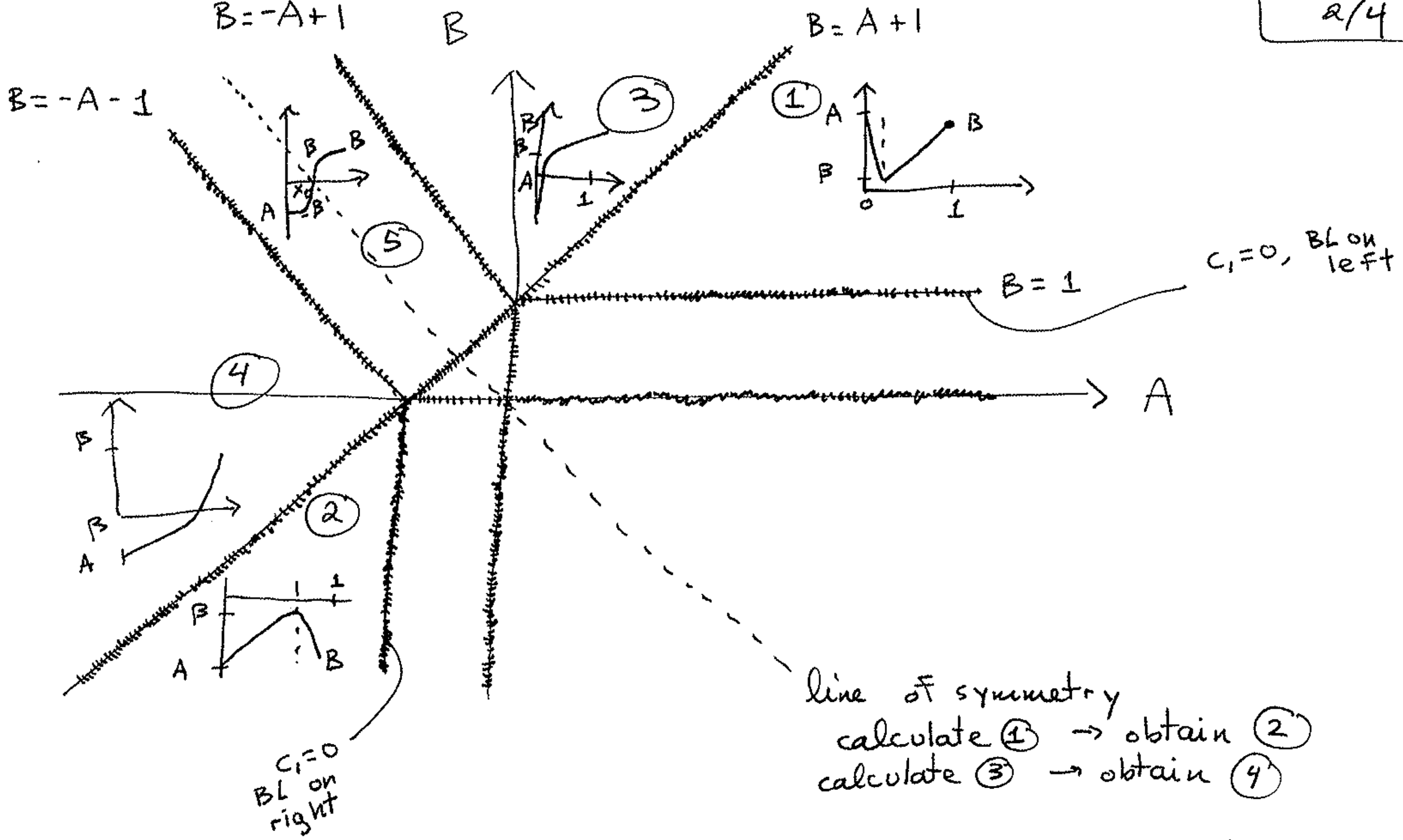
$$\rightarrow \quad \frac{(y_0)_\xi}{1 - \left(\frac{y_0}{\sqrt{2c_1}}\right)^2} = 1$$

recall there were 3 cases

$c_1 < 0 \Rightarrow$ results in tan and cot which don't match as $\xi \rightarrow \pm\infty$

$c_1 = 0 \Rightarrow$ special case, ignore for now

$c_1 > 0 \Rightarrow$ tanh and coth which do match



There is a symmetry in the~~se~~ equation

$$\epsilon y'' + x y' - y = 0, \qquad y(0) = A, \; y(1) = B$$

let $\quad w = 1 - x, \quad v = -y$

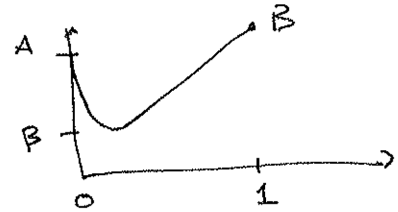
$$\rightarrow \quad \epsilon v_{ww} + w v_w - v = 0, \quad v(0) = -B, \quad v(1) = -A$$

$$\Rightarrow \quad y(x; A, B) = -y(1-x; -B, -A)$$

so there is a symmetry about line ~~y=x~~ $y = -x$

First, look ~~for~~ the coth layer: $A > B > 0$

this is case 1



$y_0(\xi) = \beta \coth\left(\frac{\beta}{2}(\xi - k)\right)$

so the solution looks like,

outer solution, $Y_0(x) = x + C_0$,
satisfies BC at $x = 1 \rightarrow C_0 = 1 - B$

matching: $\lim_{x\to 0} Y_0 = \lim_{\xi\to\infty} y_0$

$\Rightarrow B - 1 = \beta$ because $\lim_{\xi\to\infty} \coth \xi = 1$

the common part is C_0 so the composite solution is

$Y_c \sim x + \beta \coth\left(\frac{\beta}{2}(\xi - k)\right)$, $\beta = B - 1$

oops, forgot to incorporate BCs for k

$y_0(0) = A = \beta \coth\left(\frac{\beta}{2}(\xi - k)\right)$

$\longrightarrow k = -\frac{2}{\beta}\coth^{-1}\left(\frac{A}{\beta}\right)$

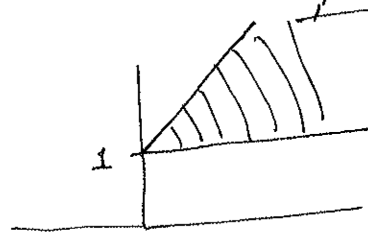
the conditions we obtain are

$A > \beta > 0 \longrightarrow \boxed{A > B - 1}$

$\Downarrow$

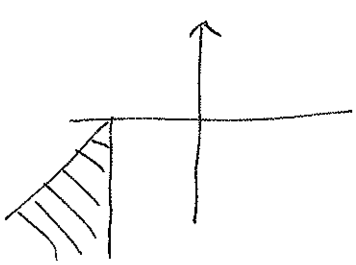
$B - 1 > 0 \longrightarrow B > 1 \quad \boxed{B < A + 1}$

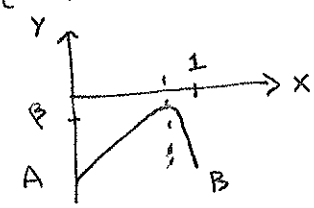
so the region is



By symmetry, we obtain solution in region

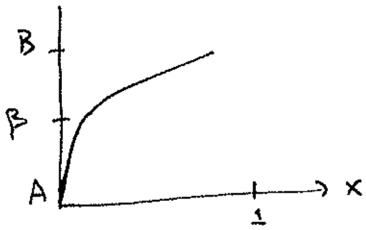
though solution is "flipped"





3/2/04
3/4

(3) y_0 increases to β as $\xi \to \infty$



$Y_0 = x + C_0$
satisfies BC at $x=1$ $\to$ $\boxed{C_0 = B-1}$

Match: $\lim_{x\to 0} x + B - 1 = \lim_{\xi \to 0} \beta \tanh\left(\frac{\beta}{2}(\xi - k)\right)$

$$\boxed{B - 1 = \beta}$$

from BCs: $y_0(\xi=0) = A \quad \to \quad k = \frac{-2}{\beta}\tanh^{-1}\left(\frac{A}{\beta}\right)$

composite soln: $Y_c \sim x + \beta \tanh\left(\frac{\beta}{2}(\xi - k)\right)$ with β, k known

conditions:

From plot of $\beta\tanh\xi$, the jump is a max of 2β

$B - 1 = \beta > 0$

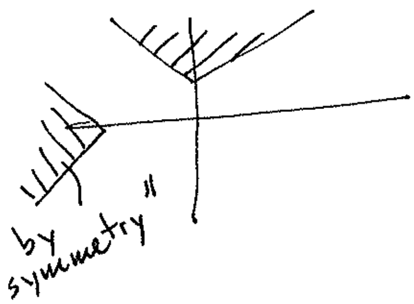
$A > -\beta = -(B-1) \quad \to \quad \boxed{B > -A+1}$

$A + 2B - 1 > B \quad \to \quad B > -A+1$

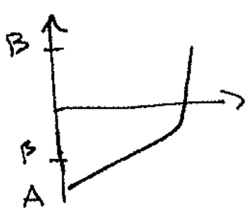
$\Rightarrow A + 2\beta + 1 > B \quad \to \quad \boxed{B > A+1}$

$A < \beta \quad \to \quad A < B - 1$

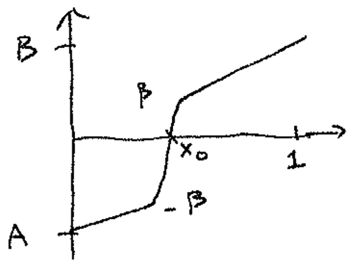
the region where these conditions is satisfied is, along with the region of symmetry is:



in region of symmetry, have



Region (5) Interior tanh layer



outer soln

$$Y_0 = x + c_0 = \begin{cases} x + A, & x < x_0 \\ x + B - 1, & x > x_0 \end{cases}$$

inner solution is hyperbolic tan

$$y_0 = \beta \tanh\left(\frac{\beta}{2}(\xi - k)\right)$$

pick $k = 0$ so that BL is centered at $x = x_0$

determine β and x_0 through matching

matching: $\lim_{x \to x_0^-} Y_0(x) = \lim_{\xi \to -\infty} y_0(\xi)$

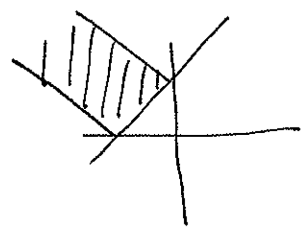
$$x_0 + A = -\beta$$

and $\lim_{x \to x_0^+} Y_0(x) = \lim_{\xi \to \infty} y_0(\xi)$

$$x_0 + B - 1 = \beta$$

solve for x_0 and B

$$x_0 = \frac{1 - A - B}{2}, \quad \beta = \frac{B - A - 1}{2}$$



since $0 < x_0 < 1 \implies -(A+1) < B < -A + 1 \implies$

and $\beta > 0 \implies B > A + 1$

composite solution,

$$y_c \sim x - x_0 + \beta \tanh\left(\frac{\beta}{2} \frac{x - x_0}{\epsilon}\right)$$

3/2/04

4/4

$\boxed{\text{Case where } c_1 = 0}$ $\Rightarrow$ $Y_{0\xi} + \frac{1}{2} Y_0^2 = 0$

$\Rightarrow$ $Y_0(\xi) = \dfrac{2}{\xi + c_2}$

$\dfrac{Y_{0\xi}}{Y_0^2} = \dfrac{-1}{2}$

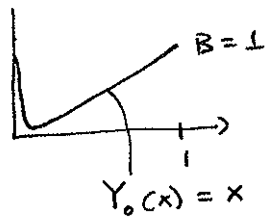
$-Y_0^{-1} = \dfrac{-\xi}{2} + c_2$

$Y_0 = \dfrac{1}{\frac{1}{2} + c_2} \to \dfrac{2}{\xi + c}$

notice, $y_0(\xi) \to 0$ as $\xi \to \pm\infty$

BL on left

$c_2 = \dfrac{2}{A}$



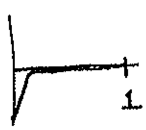
this only works for $\boxed{B = 1,\ A > 0}$

and by symmetry, $A = -1$, $B < 0$

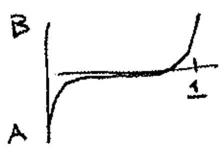
For regions (1) - (5), we have exhausted all possibilities for $\xi = \dfrac{x - x_0}{\epsilon}$

so we must find other balances

One possibility is an outer solution of zero, such as



or



Let $\eta = \dfrac{x - x_0}{\delta(\epsilon)}$ $\Rightarrow$ $\dfrac{\epsilon}{\delta^2} Y_{\eta\eta} + Y \dfrac{1}{\delta} Y_\eta - Y = 0$

multiply by δ $\rightarrow$ $\dfrac{\epsilon}{\delta} Y_{\eta\eta} + Y Y_\eta - \delta Y = 0$

Rescale y: ~~[illegible]~~ $y = \gamma(\epsilon)\, u$

$\rightarrow$ $\dfrac{\epsilon\gamma}{\delta} u_{\eta\eta} + \gamma u \gamma u_\eta - \delta\gamma u = 0$ $\rightarrow$ $\dfrac{\epsilon}{\delta} u_{\eta\eta} + \gamma u u_\eta - \delta u = 0$

class notes: ... /6620/notes/...

Weakly Nonlinear Oscillators

general form: $\ddot{x} + x + \epsilon h(x, \dot{x}) = 0$

to leading order, have oscillating functions like sine, cosine

Example 1: Duffing Equation

$$\ddot{x} + x + \epsilon x^3 = 0$$

a) describes a weakly nonlinear spring, $F_s = -kx - cx^3$ (Hooke's Law)

b) ~~using~~ describes a pendulum with small amplitude

$$\ddot{\theta} + \sin\theta = 0$$

→ replace $\sin\theta$ with Fourier series

Example 2: Linear spring with small damping

$$\ddot{x} + x + 2\epsilon\dot{x} = 0$$

Example 3: van der Pol equation

$$\ddot{x} + x + \epsilon(x^2 - 1)\dot{x} = 0$$

Example 4 Rayleigh's equation

$$\ddot{x} + x - \epsilon\left(\dot{x} - \frac{\dot{x}^3}{3}\right) = 0$$

Perturbation Methods for these Problems

1) Strained coordinates
2) Multiple scales

the naive expansion fails - you would think it should work but it doesn't

---

Example: $\boxed{\ddot{x} + (1+\epsilon)x = 0, \quad x(0) = \alpha, \quad \dot{x}(0) = 0}$

apply the naive approach to see why it fails

$x \sim x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$

the expanded equation becomes

$$\Rightarrow \ddot{x}_0 + \epsilon \ddot{x}_1 + \epsilon^2 \ddot{x}_2 + \cdots + (1+\epsilon)(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots) = 0$$

$O(1)$: $\ddot{x}_0 + x_0 = 0, \quad x_0(0) = \alpha, \quad \dot{x}_0(0) = 0$

$$\rightarrow \boxed{x_0(t) = \alpha \cos t}$$

no problem with naive expansion yet

$O(\epsilon)$: $\ddot{x}_1 + x_1 = -x_0, \quad x_1(0) = 0, \quad \dot{x}_1(0) = 0$

homogeneous solution, $x_{1_h} = c_1 \cos t + c_2 \sin t$

particular solution, $x_{1p} = [c_3 \cos t + c_4 \sin t]\, t$

$$\Rightarrow x_{1p}'' = -t[c_3 \cos t + c_4 \sin t] + 2[-c_3 \sin t + c_4 \cos t]$$

$$x_{1p}'' + x_{1p} = -2c_3 \sin t + 2c_4 \cos t \Rightarrow \boxed{c_3 = 0}, \quad 2c_4 = -\alpha \Rightarrow \boxed{c_4 = \frac{-\alpha}{2}}$$

$$x_{1p} = -\frac{\alpha t}{2} \sin t$$

$$\Rightarrow x_1 = c_1 \cos t + c_2 \sin t - \frac{\alpha t}{2} \sin t$$

Apply BCs to find that $c_1 = 0, \quad c_2 = 0$

$$\Rightarrow \boxed{x_1(t) = -\frac{\alpha t}{2} \sin t}$$

$$\boxed{(fg)'' = fg'' + 2f'g' + f''g}$$

$O(\epsilon^2)$: $\ddot{x}_2 + x_2 = -x_1$, $x_2(0) = \dot{x}_2(0) = 0$

$\rightarrow \ddot{x}_2 + x_2 = \frac{\alpha t}{2} \sin t$

homogeneous solution, $x_{2h} = c_5 \cos t + c_6 \sin t$

For the particular solution, take a ~~derivative~~ linear combination of the derivatives of the right hand side

$\sin t$, $\cos t$, $t\cos t$, $t\sin t$

but since $\cos t$ and $\sin t$ are homogeneous solution multiply by t to obtain

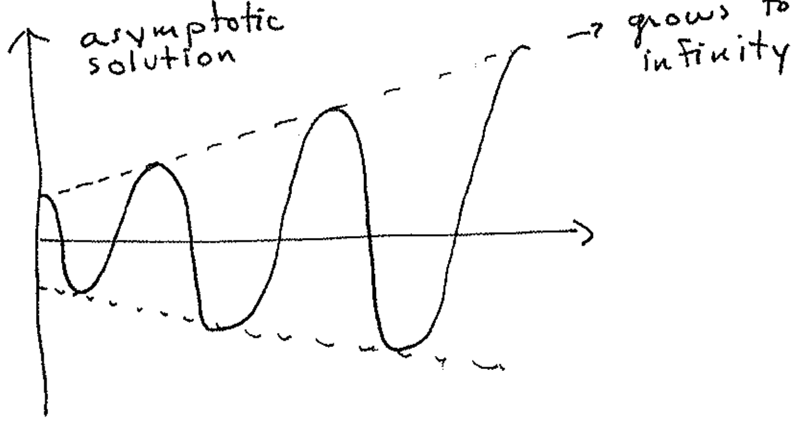
$x_{2p} = t\left((At+B)\cos t + (Ct+D)\sin t \right)$

$\Rightarrow \boxed{x_2(t) = -\frac{\alpha}{8}\left(t^2 \cos t - t\sin t\right)}$

and so the expansion becomes

$$\boxed{x(t) \sim \alpha\cos t - \epsilon\frac{\alpha t}{2}\sin t - \epsilon^2\frac{\alpha}{8}\left(t^2\cos t - t\sin t\right) + O(\epsilon^3)}$$

this expansion is not uniformly valid because x_2 has ϵt term and x_3 has $\epsilon^2 t^2$ term so the expansion Fails to be asymptotic.



in reality, the exact solution ~~de~~ oscillates but decays to infinity

At $O(\epsilon)$, we had

$$\ddot{x}_1 + x_1 = -x_0$$

this is a harmonic oscillator with a resonant Forcing term

For example, $\ddot{x} + \omega_0^2 x = 0 \rightarrow x = \cos(\omega_0 t)$. the Frequency is ω_0. If a Forcing term on the right hand side has same Frequency, resonance occurs. These homogeneous terms are called <u>secular</u> terms. These terms cause nonuniformities in the naive expansion.

so in $O(\epsilon)$ equation, $\ddot{x}_1 + x_1 = -\alpha \cos t$, the $-\alpha \cos t$ is a secular term, which causes the problem...

the exact solution to $\ddot{x} + x + \epsilon = 0$, $x(0) = \alpha$, $\dot{x}(0) = 0$ is $x_{exact}(t) = \alpha \cos(\sqrt{1+\epsilon}\, t)$ so the natural ~~solution~~ Frequency of the exact solution is $\sqrt{1+\epsilon}$. The exact solution looks like leading order term $x_0 = \alpha \cos t$, except with a perturbed Frequency. Due to this perturbed Frequency, x_0 and x_{exact} eventually go out of phase, more specifically when $t = O(1/\epsilon)$. This is not the problem so much as the secular terms are.

Expand the exact solution in powers of ϵ:

$$\sqrt{1+\epsilon} \sim 1 + \frac{\epsilon}{2} + \cdots$$

$$x_{ex} \sim \alpha \cos\left[\left(1 + \frac{\epsilon}{2} + \cdots\right)t\right] \sim \alpha \cos\left(t + \frac{\epsilon}{2}t + \cdots\right)$$

$$x_{ex} \sim \alpha \cos t \cos\left(\frac{\epsilon}{2}t + \cdots\right) - \alpha \sin t \sin\left(\frac{\epsilon}{2}t + \cdots\right)$$

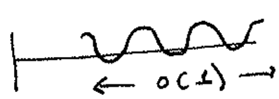
$$x_{ex} \sim \alpha \cos t \left(1 + O(\epsilon^2)\right) - \alpha \sin t \left(\frac{\epsilon t}{2} - O(\epsilon^3)\right)$$

$$x_{ex} \sim \alpha \cos t - \frac{\epsilon t}{2}\alpha \sin t + O(\epsilon^2)$$

this agrees with result for the naive expansion. This will be true at higher orders too. Therefore the infinite series converges to the exact solution for all t. The key is that you require infinitely many terms for convergence. But if you keep only a few terms, obtain nonuniform asymptotic expansion.

Notice, the exact solution involves two time scales

1) the function oscillates on a $O(1)$ time scale



2) the perturbed frequency acts on a $O(1/\epsilon)$ time scale - won't notice effect on frequency untill large times

Ideally,

The naive expansion fails to do so here because it does not allow for corrections in frequency

$$\omega_{ex} = \sqrt{1+\epsilon} \quad , \quad \omega_{naive} = 1$$

One fix is to rewrite the naive expansion as

$$x \sim \alpha\left(\cos t - \frac{\epsilon}{2} t \sin t + O(\epsilon^2)\right)$$

$$\sim \alpha \cos t \left(1 + O(\epsilon^2)\right) - \alpha \sin t \left(\frac{\epsilon t}{2} + O(\epsilon^3)\right) + O(\epsilon^2)$$

$$\sim \alpha \cos t \left(\cos \frac{\epsilon t}{2}\right) - \alpha \sin t \left(\sin \frac{\epsilon t}{2}\right) + O(\epsilon^2)$$

$$\sim \alpha \cos \left(\underbrace{\left(1 + \frac{\epsilon}{2}\right)}_{\sqrt{1+\epsilon}} t\right) + O(\epsilon^2)$$

valid for $t = O(1/\epsilon)$, but not for $t = O(1/\epsilon^2)$

$\cos\left(\left(1+\frac{\epsilon}{2}\right)t\right)$ eventually goes out of phase

but not until $t = O(1/\epsilon^2)$

3/16/04
4/5

Let's keep the naive expansion, but rescale time instead, in such a way that the exact solution has a frequency of exactly 1 (for this problem) or a period of 2π in the new coordinate system.

Let ω be the ~~new~~ frequency of the exact solution, which is unknown, and define $T = \omega t$.

Since ω is not known, we expand it as

$$\omega = \omega_0 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \dots$$

ω_0 is the natural frequency of the leading order equation

$$\rightarrow \quad T = (\omega_0 + \epsilon\omega_1 + \dots) t$$

substitute into $\ddot{x} + x + \epsilon = 0$, $\quad x(0) = \alpha$, $\quad \dot{x}(0) = 0$

$$\frac{dx}{dt} = \frac{dT}{dt}\frac{dx}{dT} = \omega \frac{dx}{dT} \sim (\omega_0 + \epsilon\omega_1 + \dots)\frac{dx}{dT}$$

$$\rightarrow \quad \frac{d^2x}{dt^2} = \omega^2 \frac{d^2x}{dT^2} \sim (\omega_0^2 + 2\epsilon\omega_0\omega_1 + \dots)\frac{d^2x}{dT^2}$$

Expand x as $x \sim X_0(t) + \epsilon X_1(t) + \dots$

$$\Rightarrow \quad (\omega_0^2 + 2\epsilon\omega_0\omega_1 + \dots)(X_{0_{TT}} + \epsilon X_{1_{TT}} + \dots)$$

$$+ \quad (1+\epsilon)(X_0 + \epsilon X_1 + \dots) = 0$$

since ω_0 is the natural frequency of the leading order

$$\omega_0 = 1$$

$$\left(\omega_0^2 + 2\epsilon\omega_0\omega_1 + \cdots\right)\left(X_{0_{TT}} + \epsilon X_{1_{TT}} + \cdots\right) + (1+\epsilon)\left(X_0 + \epsilon X_1 + \cdots\right) = 0$$

$\omega_0 = 1$

$O(1)$: $\quad X_{0_{TT}} + X_0 = 0 \quad , \quad X_0(0) = \alpha , \quad X_{0_T}(0) = 0$

$$\rightarrow \frac{dx}{dt} = (1 + \epsilon\omega_1 + \cdots)(X_{0_T} + \epsilon X_{1_T} + \cdots)$$

$$\rightarrow \frac{dx}{dt}(0) = (1 + \epsilon\omega_1 + \cdots)(X_{0_T}(0) + \epsilon X_{1_T}(0) + \cdots) = 0$$

$$\rightarrow \boxed{X_0(t) = \alpha\cos T}$$

$O(\epsilon)$: $\quad X_{1_{TT}} + X_1 = -2\omega_1 X_{0_{TT}} - X_0$

$$\rightarrow X_{1_{TT}} + X_1 = \alpha 2\omega_1 \cos T - \alpha\cos T$$

$$\rightarrow x_{1_{TT}} + x_1 = (2\omega_1 - 1)\alpha\cos T$$

$\rightarrow$ Pick ω_1 so that the coefficient of the secular term is zero, in order to kill secular terms

$$\rightarrow \omega_1 = \tfrac{1}{2} \Rightarrow T = \left(1 + \tfrac{\epsilon}{2} + \cdots\right)t$$

then $X_0(t) = \alpha\cos\left(1 + \frac{\epsilon}{2} + \cdots\right)t$

this is valid for $t = O(1/\epsilon)$ but it still becomes invalid at some point, $O(1/\epsilon^2)$

Perturbation methods

1) renormalization - not very current so will not use the basic idea is to algebraically manipulate the naive expansion to combine the nonuniform terms with lower order terms in the expansion

2) strained coordinates

a) Poincare'-Lindstedt (PL) Method

ω = frequency of exact solution

define $T = \omega t$ and expand ω as

$\omega \sim \omega_0 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \cdots$ and expand

$x \sim x_0(T) + \epsilon x_1(T) + \cdots$

pick ω_ns to kill secularities

works for oscillatory problems with constant amplitude

b) Poincare'-Lighthill-Kao (PLK) method

instead of perturbing the frequency, introduces a new time

$t \sim T + \epsilon s_1(T) + \epsilon^2 s_2(T) + \cdots$

$x \sim x_0(T) + \epsilon x_1(T) + \cdots$

pick $s_n(t)$ to kill secularities

this is a slight improvement

these strained coordinate methods predate multiple scale methods. Multiple scale methods are an improvement.

(3) Multiple Scales (various methods)

A. Define two time-scales

$$T = (1 + \epsilon^2 \omega_2 + \epsilon^3 \omega_3 + \cdots)\, t$$

$$\tau = \epsilon t$$

treat T and τ as independent variables

$$\rightarrow \quad x \sim x_0(T,\tau) + \epsilon x_1(T,\tau) + \epsilon^2 x_2(T,\tau) + \cdots$$

results in partial differential equation
gives freedom to kill secular terms

Example (~~strained coordinates~~ renormalization)

$$\ddot{x} + x + \epsilon x^3 = 0; \qquad x(0) = \alpha, \quad \dot{x}(0) = 0$$

expansion $\quad x \sim x_0 + \epsilon x_1 + \cdots$

$O(1)$: $\quad \ddot{x}_0 + x_0 = 0, \quad x_0(0) = \alpha, \ \dot{x}_0(0) = 0 \quad \rightarrow \boxed{x_0(t) = \alpha \cos t}$

$O(\epsilon)$: $\quad \ddot{x}_1 + x = -x_0^3 = -\alpha^3 \cos^3 t = -\frac{\alpha^3}{4}\left[3\cos t + \cos(3t)\right]$

$$\rightarrow \boxed{x_1(t) = \frac{-\alpha^3}{32}\left[\cos t + 12 t \sin t - \cos 3t\right]}$$

the expansion is

$$x(t) \sim \alpha \cos t - \frac{\epsilon \alpha^3}{32}\left[\cos t + 12 t \sin t - \cos 3t\right]$$

this expansion is nonuniform for $t = O(1/\epsilon)$

the error term is unbounded as $t \rightarrow \infty$

renormalize

$$x(t) \sim \alpha \cos t - \frac{3\alpha^3}{8}\epsilon\, t \sin t - \frac{\alpha^3}{32}\epsilon\left(\cos t - \cos 3t\right)$$

combine using Taylor series

$$\Rightarrow \quad x(t) \sim \underbrace{\alpha \cos\left[\left(1 + \frac{3\alpha^2}{8}\epsilon\right)t\right]}_{} - \frac{\alpha^3}{32}\epsilon\left(\cos t - \cos 3t\right)$$

this term is now valid for $t = O(1/\epsilon)$ but not for $t = O(1/\epsilon^2)$ - so we have extended our region of uniformity

Now use strained coordinates

$$\ddot{x} + x + \epsilon x^3 = 0, \quad x(0) = \alpha, \quad \dot{x}(0) = 0$$

A) PL method, $x \sim x_0(T) + \epsilon x_1(T) + \epsilon^2 x_2(T) + \dots$

let $T = \omega t$, $\omega \sim \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \dots$

$$\frac{d}{dt} \sim \frac{dT}{dt}\frac{d}{dT} = \omega\frac{d}{dT} \sim \left(\omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \dots\right)\frac{d}{dT}$$

$$\frac{d^2}{dt^2} \sim \left(\omega_0^2 + 2\epsilon\,\omega_0\omega_1 + \epsilon^2\omega_1^2 + 2\omega_0\omega_2\epsilon^2 + \dots\right)\frac{d^2}{dT^2}$$

initial conditions, $t = 0 \Rightarrow T = 0$

$$x(t=0) = x_0(0) + \epsilon x_1(0) + \epsilon^2 x_2(0) + \dots$$

$$\rightarrow \boxed{x_0(0) = \alpha, \quad x_i(0) = 0, \quad i > 0}$$

and $$\frac{dx}{dt}(t=0) = \left(\omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \dots\right)\frac{d}{dT}\left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots\right) = 0$$

$$\frac{dx}{dt}(t=0) = (\omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)\frac{d}{dT}\left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots\right) = 0$$

$$O(1): \quad \omega_0 \frac{dx_0(0)}{dT} = 0 \quad \rightarrow \quad \boxed{\frac{dx_0(0)}{dT} = 0}$$

$$O(\epsilon): \quad \omega_0 \frac{dx_1(0)}{dT} + \omega_1 \frac{dx_0(0)}{dT} = 0 \quad \rightarrow \quad \frac{dx_1(0)}{dT} = -\frac{\omega_1}{\omega_0}\frac{dx_0(0)}{dT} = 0$$

the differential equation becomes

$$\left(\omega_0^2 + 2\epsilon\omega_0\omega_1 + \epsilon^2\omega_1^2 + 2\epsilon^2\omega_0\omega_2 + \cdots\right)\left(x_{0_{TT}} + \epsilon x_{1_{TT}} + \epsilon^2 x_{2_{TT}} + \cdots\right)$$

$$+ \left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots\right) + \epsilon\left(x_0^3 + 3x_0 x_1^2 + \cdots\right) = 0$$

$$O(1): \quad \omega_0^2 x_{0_{TT}} + x_0 = 0, \quad x(0) = \alpha, \quad x_{0_T}(0) = 0$$

pick $\boxed{\omega_0 = 1, \Rightarrow x_0(T) = \alpha\cos T}$

$$\begin{aligned} O(\epsilon): \quad x_{1_{TT}} + x_1 &= -2\omega_1 x_{0_{TT}} - x_0^3 \\ &= -2\omega_1(-\alpha\cos T) - \alpha^3\cos^3 T \\ &= 2\omega_1\alpha\cos T - \frac{\alpha^3}{4}(3\cos T + \cos 3T) \\ &= \alpha\left(2\omega_1 \cancel{\cos T} - \frac{3\alpha^2}{4}\right)\cos T - \frac{\alpha^3}{4}\cos 3T \end{aligned}$$

kill secular term: pick $\boxed{\omega_1 = \frac{3\alpha^2}{8}}$

we can write the leading order ~~comp~~ expansion for T

$$T = \left(1 + \frac{3\alpha^2}{8}\epsilon\right)t$$

$$\rightarrow x(t) \sim \alpha\cos\left[\left(1 + \frac{3\alpha^2}{8}\epsilon\right)t\right] + \cdots \qquad \text{valid for } t = O(1/\epsilon)$$

3/19/04
2/2

so the $O(\epsilon)$ equation becomes

$$X_{1_{TT}} + X_1 = \frac{-\alpha^3}{4}\cos(3T)$$

$$\rightarrow \quad x_1(T) = \frac{-\alpha^3}{32}\left[\cos T - \cos(3T)\right]$$

$O(\epsilon^2)$:

$$X_{2_{TT}} + X_2 = -2\omega_1 X_{1_{TT}} - (\omega_1^2 + 2\omega_2) X_{0_{TT}} - 3x_0^2 x_1$$

$$= -2\omega_1\left[\frac{-\alpha^3}{32}\left(-\cos T + 9\cos(3T)\right)\right]$$

$$-(\omega_1^2 + 2\omega_2)(-\alpha\cos T)$$

$$-3\alpha^2\cos^2 T\left[-\frac{\alpha^3}{32}\left(\cos T - \cos 3T\right)\right]$$

we're only worried about secular terms to solve for ω_2

$$\rightarrow \quad X_{2_{TT}} + X_2 = \frac{-\omega_1\alpha^3}{16}\cos T + (\omega_1^2 + 2\omega_2)\alpha\cos T$$

$$+ \frac{3\alpha^5}{32}\left[\frac{1}{4}\left(3\cos T + \cos(3T)\right) + \frac{1}{2}\cos 3T - \frac{1}{4}\left(\cos 5t + \cos t\right)\right] + h.h$$

higher harmonics

$$\rightarrow \quad X_{2_{TT}} + X_2 = \Big(\underbrace{\frac{-\omega_1\alpha^3}{16} + \alpha\omega_1^2 + 2\alpha\omega_2 + \frac{9\alpha^5}{128} - \frac{3\alpha^5}{128}}_{\text{this vanishes}}\Big)\cos T + h.h$$

solve for ω_2, with $\omega_1 = \frac{3\alpha^2}{8}$, such that

$$-\frac{3\alpha^5}{128} + \frac{9\alpha^5}{64} + 2\alpha\omega_2 + \frac{\cancel{8}\,\alpha^5}{128} = 0 \quad \rightarrow \quad 2\alpha\omega_2 = \left(\frac{-3}{128} - \frac{18}{128}\right)\alpha^5$$

$$\rightarrow \quad \boxed{\omega_2 = \frac{-21\alpha^4}{256}}$$

In conclusion

$$T = \left(\omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots\right) t$$

$$\rightarrow T = \left(1 + \frac{3\alpha^2}{8}\epsilon - \frac{21\alpha^4}{256}\epsilon^2 + \cdots\right) t$$

$$\rightarrow \quad x \sim \alpha \cos\left[\left(1 + \frac{3\alpha^2}{8}\epsilon - \frac{21\alpha^4}{256}\epsilon^2\right) t\right]$$

$$- \epsilon\frac{\alpha^3}{32}\left[\cos\left\{\left(1 + \frac{3\alpha^2}{8}\epsilon - \frac{21}{256}\alpha^4\epsilon^2\right)t\right\} - \cos\left\{3\left(1 + \frac{3\alpha^2}{8}\epsilon - \frac{21}{256}\alpha^4\epsilon^2\right)t\right\}\right] + O(\epsilon^2)$$

this is valid for $t = O(1/\epsilon^2)$, but not for $t = O(1/\epsilon^3)$

Poincaré - Lighthill - Kuo (PLK)

$$t \sim T + \epsilon S_1(T) + \epsilon^2 S_2(T) + \cdots$$

Require that $S_n(0) = 0$

need $\frac{dT}{dt}$, so calculate $\frac{dt}{dT} \sim 1 + \epsilon S_1'(T) + \epsilon^2 S_2'(T) + \cdots$

$$\Rightarrow \quad \frac{dT}{dt} = \frac{1}{\left(\frac{dt}{dT}\right)} \approx \frac{1}{1 + (\epsilon S_1' + \epsilon^2 S_2' + \cdots)} \sim 1 - (\epsilon S_1' + \epsilon^2 S_2' + \cdots) + (\epsilon S_1' + \epsilon^2 S_2' + \cdots)^2$$

$$\Rightarrow \quad \frac{dT}{dt} \sim 1 - \epsilon S_1'(T) + \epsilon^2 \left[(S_1')^2 - S_2' \right] + \cdots$$

transform the derivatives

$$\boxed{\frac{d}{dt} \sim \frac{dT}{dt}\frac{d}{dT} \sim \left[1 - \epsilon S_1' + \epsilon^2\left((S_1')^2 - S_2'\right) + \cdots\right]\frac{d}{dT}}$$

Now calculate second derivative:

$$\frac{d^2}{dt^2} = \frac{d}{dt}\left(\frac{d}{dt}\right) = \frac{dT}{dt}\frac{d}{dT}\left\{\left[1 - \epsilon S_1' + \epsilon^2\left((S_1')^2 - S_2'\right) + \cdots\right]\frac{d}{dT}\right\}$$

$$\approx (1 - \epsilon S_1'(T))\left[(1 - \epsilon S_1')\frac{d^2}{dT^2} - \epsilon S_1''\frac{d}{dT} + \cdots\right]$$

$$\Rightarrow \quad \boxed{\frac{d^2}{dt^2} \sim \frac{d^2}{dT^2} - \epsilon\left[2S_1'\frac{d^2}{dT^2} + S_1''\frac{d}{dT} + \cdots\right]}$$

Example : Duffing equation

$$\ddot{x} + x + \epsilon x^3 = 0, \qquad x(0) = \alpha, \quad \dot{x}(0) = 0$$

we have already found that the answer is given by

$$x(t) \sim \alpha \cos\left(\left(1 + \frac{3\alpha^2}{8}\right) t \right) + \cdots$$

Using Poincaré - Lighthill method,

$$t \sim T + \epsilon S_1(T) + \epsilon^2 S_2(T) + \cdots$$

and $\quad x \sim x_0(T) + \epsilon x_1(T) + \epsilon^2 x_2(T) + \cdots$

to order ϵ :

$$\Rightarrow \left\{ \frac{d^2}{dT^2} - \epsilon\left[2S_1' \frac{d^2}{dT^2} + S_1'' \frac{d}{dT} \right] + \cdots \right\} \left(x_0 + x_1 \epsilon + \cdots \right) + \left(x_0 + x_1 \epsilon + \cdots \right) + \epsilon x_0^3 + \cdots = 0$$

$$\Rightarrow x_{0_{TT}} + \epsilon x_{1_{TT}} - \epsilon\left(2S_1' x_{0_{TT}} + S_1'' x_{0_T} \right) + \cdots + x_0 + x_1 \epsilon + \cdots + \epsilon x_0^3 + \cdots = 0$$

the initial conditions are the same (since going to leading order only) $\Rightarrow$ $x_0(0) = \alpha$ and $x_{0_T}(0) = 0$

$O(1)$: $\quad x_{0_{TT}} + x_0 = 0, \quad x_0(0) = \alpha, \quad x_{0_T}(0) = 0$

$$\Rightarrow x_0(T) = \alpha \cos(T)$$

Now go to higher order and ~~kill~~ supress the secularities

$O(\epsilon)$: $\quad x_{1_{TT}} + x_1 = 2S_1' x_{0_{TT}} + S_1'' x_{0_T} - x_0^3$

$$= -2S_1' \alpha \cos T - S_1'' \alpha \sin T - \alpha^3 \cos^3 T$$

$$\cos^3 T = \frac{1}{4}(3\cos T + \cos 3T)$$

regroup & kill secularities

$$x_{1_{TT}} + x_1 = \left(-2s_1'\alpha - \frac{3}{4}\alpha^3\right)\cos T - \alpha s_1'' \sin T$$

$$+ \text{ NST}$$ —— non secular terms

$$\Rightarrow \quad s_1' = -\frac{3}{8}\alpha^2 \quad \text{and} \quad s_1'' = 0$$

$$\Downarrow$$

$$s_1 = c_1 T + c_2$$

since we impose $s_1(0) = 0$ then $c_2 = 0$

$$s_1(T) = -\frac{3}{8}\alpha^2 T$$

Recall, $t \sim T + \epsilon s_1(T) + \cdots$

$$\Rightarrow \quad t \sim T - \frac{3}{8}\alpha^2 T \epsilon + \cdots$$

$$\Rightarrow \quad t \sim T\left(1 - \frac{3\alpha^2}{8}\epsilon\right) + \cdots$$

since we found $x_0(T) = \alpha \cos(T)$

we need to solve for T

$$\rightarrow \quad T \sim \frac{t}{1 - \frac{3\alpha^2}{8}\epsilon} \sim t\left(1 + \frac{3\alpha^2}{8}\epsilon\right) + \cdots$$

so that the leading order solution is given by

$$\boxed{x(t) \sim \alpha \cos\left(\left(1 + \frac{3\alpha^2}{8}\epsilon\right)t\right) + \cdots}$$

valid for $t = O(1/\epsilon)$ but not for $O(1/\epsilon^2)$

3/23/04
3/5

Multiple Scales

- Introduce 2 time scales

$$T = t\left(1 + \epsilon^2 \omega_2 + \epsilon^3 \omega_3 + \cdots\right)$$

$$\tau = \epsilon t$$

if include $\epsilon \omega_1$ term, for any problem ω_1 is completely arbitrary, so usually pick $\omega_1 = 0$

Treat T and τ as independent variables.
(This is clearly wrong, since T and τ are independent. The underlying assumption is wrong.)

$$\boxed{\frac{d}{dt} = \frac{dT}{dt}\frac{\partial}{\partial T} + \frac{d\tau}{dt}\frac{\partial}{\partial \tau} \sim \left(1 + \epsilon^2\omega_2 + \cdots\right)\frac{\partial}{\partial T} + \epsilon \frac{\partial}{\partial \tau} + \cdots}$$

$$\frac{d^2}{dt^2} = \frac{d}{dt}\left(\frac{d}{dt}\right) = \frac{dT}{dt}\frac{\partial}{\partial T}\left[\left(1+\epsilon^2\omega_2\right)\frac{\partial}{\partial T} + \epsilon\frac{\partial}{\partial \tau}\right] + \frac{d\tau}{dt}\frac{\partial}{\partial \tau}\left[\left(1+\epsilon^2\omega_2\right)\frac{\partial}{\partial T} + \epsilon\frac{\partial}{\partial \tau} + \cdots\right]$$

$$= \left(1 + \epsilon^2\omega_2 + \cdots\right)\left[\left(1+\epsilon^2\omega_2\right)\frac{\partial^2}{\partial T^2} + \epsilon\frac{\partial^2}{\partial T\partial \tau}\right]$$

$$+ \epsilon\left[\left(1+\epsilon^2\omega_2\right)\frac{\partial^2}{\partial T\partial\tau} + \epsilon\frac{\partial^2}{\partial \tau^2} + \cdots\right]$$

$$\Rightarrow \boxed{\frac{d^2}{dt^2} \sim \frac{\partial^2}{\partial T^2} + 2\epsilon\frac{\partial^2}{\partial T\partial\tau} + \epsilon^2\left[2\omega_2\frac{\partial^2}{\partial T^2} + \frac{\partial^2}{\partial \tau^2}\right] + \cdots}$$

Initial Conditions

suppose $x(0) = A$ and $\dot{x}(0) = B$

$$\boxed{x \sim x_0(T,\tau) + \epsilon x_1(T,\tau) + \epsilon^2 x_2(T,\tau) + \cdots}$$

$x(0) = A \quad , \qquad t=0 \quad \to \quad T=0, \ \tau=0$

$$\to \quad x(0) \sim x_0(0,0) + \epsilon x_1(0,0) + \cdots = A$$

$$\Rightarrow \quad \boxed{x_0(0,0) = A \quad , \quad x_i(0,0) = 0 \quad , \ i \geq 1}$$

unless of course initial condition is perturbed, like $x(0) = A + Q\epsilon$

$$x(t) \sim x_0(T,\tau) + \epsilon x_1(T,\tau) + \epsilon^2 x_2(T,\tau) + \cdots$$

$$\Rightarrow \quad \dot{x}(t) \sim \left(\frac{\partial}{\partial T} + \epsilon \frac{\partial}{\partial \tau}\right)\left(x_0 + \epsilon x_1 + \cdots\right)$$

$$\sim x_{0_T} + \epsilon\left(x_{1_T} + x_{0_\tau}\right) + \cdots$$

$$\Rightarrow \quad \dot{x}(t=0) \sim x_{0_T}(0,0) + \epsilon\left[x_{1_T}(0,0) + x_{0_\tau}(0,0)\right] + \cdots = B$$

$$\to \quad \boxed{\begin{array}{l} x_{0_T}(0,0) = B \\ x_{1_T}(0,0) = x_{0_\tau}(0,0) \qquad \text{and so on} \ldots \end{array}}$$

3/23/04
4/5

Example

$$\ddot{x} + x + \epsilon x^3 = 0, \quad x(0) = \alpha, \quad \dot{x}(0) = 0$$

$\left.\begin{array}{l} T = t + \cdots \\ \tau = \epsilon t \end{array}\right\}$ to leading order

$$\Rightarrow \quad \frac{d}{dt} = \frac{\partial}{\partial T} + \epsilon \frac{\partial}{\partial \tau}$$

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial T^2} + 2\epsilon \frac{\partial^2}{\partial T \partial \tau}$$

let $x \sim x_0(T,\tau) + \epsilon x_1(T,\tau) + \cdots$

$$\Rightarrow \quad x_{0_{TT}} + \epsilon x_{1_{TT}} + 2\epsilon x_{0_{T\tau}} + \cdots + x_0 + \epsilon x_1 + \cdots + \epsilon x_0^3 + \cdots = 0$$

$O(1)$: $x_{0_{TT}} + x_0 = 0$, $\quad x_0(0,0) = \alpha$

$x_{0_T}(0,0) = 0$ (to leading order)

$\rightarrow \quad x_0(T,\tau) = A(\tau)\cos(T) + B(\tau)\sin(T)$

determine A and B to supress the secular terms

$x_0(0,0) = \boxed{A(0) = \alpha}$

and $x_{0_T}(0,0) = \boxed{B(0) = 0}$

so far this is the best we can do to leading order, so go to next order for more information

$O(\epsilon)$: $\quad X_{1_{TT}} + X_1 = -2X_{0_{T\tau}} - X_0^3$

$$\Rightarrow \quad X_{1_{TT}} + X_1 = -2\left(-A'(\tau)\sin T + B'(\tau)\cos T\right)$$

$$- \left[A^3\cos^3 T + 3A^2B\cos^2 T\sin T + 3AB^2\cos T\sin^2 T + B^3\sin^3 T\right]$$

expand to obtain in terms of 1^{st} powers

$$\Rightarrow \quad X_{1_{TT}} + X_1 = 2A'\sin T - 2B'\cos T - \frac{A^3}{4}\left(3\cos T + \cos 3T\right) - \frac{3A^2B}{4}\left(\sin 3T \underbrace{- \sin T + 2\sin T}_{+\sin T}\right)$$

$$+ \frac{3AB^2}{4}\left(\cos 3T - \cos T\right) - \frac{B^3}{4}\left(3\sin T - \sin 3T\right)$$

$$\Rightarrow \quad X_{1_{TT}} + X_1 = \left(2A' - \frac{3A^2B}{4} - \frac{3B^3}{4}\right)\sin T - \left(2B' + \frac{3A^3}{4} + \frac{3AB^2}{4}\right)\cos T + NST$$

kill secular terms: $\quad A'(\tau) = \frac{3B}{8}\left(A^2 + B^2\right)$

$$B'(\tau) = \frac{-3A}{8}\left(A^2 + B^2\right)$$

<u>convert to polar coordinates</u>

let $R = \sqrt{A^2 + B^2}$, $\Theta = \tan^{-1}\left(\frac{B}{A}\right)$

$A = R\cos\Theta$, $B = R\sin\Theta$

$$\Rightarrow \quad R'\cos\Theta - R\Theta'\sin\Theta = \frac{3R\sin\Theta}{8}\cdot R^2$$

$$R'\sin\Theta + R\Theta'\cos\Theta = \frac{-3R\cos\Theta}{8}\cdot R^2$$

$$R' = \frac{3R^3}{8} \sin\Theta \cos\Theta - \frac{3R^3}{8} \sin\Theta \cos\Theta$$

$$\Rightarrow \quad R'(\tau) = 0 \quad , \quad R(0) = \alpha \quad \Rightarrow \quad R(\tau) = \alpha$$

the system of equations becomes

$$\left.\begin{aligned} -\alpha\,\Theta' \sin\Theta &= \frac{3\alpha^3}{8}\sin\Theta \\ \alpha\,\Theta' \cos\Theta &= -\frac{3\alpha^3}{8}\cos\Theta \end{aligned}\right\} \quad \Theta' = -\frac{3\alpha^2}{8}$$

$$\Theta(0) = 0 \quad \Rightarrow \quad \boxed{\Theta(\tau) = -\frac{3\alpha^2}{8}\tau}$$

so solve for $A(\tau)$ and $B(\tau)$

$$A(\tau) = \alpha \cos\left(\frac{3\alpha^2}{8}\tau\right)$$

$$B(\tau) = -\alpha \sin\left(\frac{3\alpha^2}{8}\tau\right)$$

we had

$$x_0(T,\tau) = A(\tau)\cos T + B(\tau)\sin T$$

$$x_0(T,\tau) = \alpha\cos\left(\frac{3\alpha^2}{8}\tau\right)\cos T - \alpha\sin\left(\frac{3\alpha^2}{8}\tau\right)\sin T$$

$$x_0(T,\tau) = \alpha\cos\left(\frac{3\alpha^2}{8}\tau + T\right) = \alpha\cos\left(t + \frac{3\alpha^2}{8}\epsilon t\right)$$

$$\Rightarrow \quad \boxed{x(t) \sim \alpha\cos\left(t\left(1 + \epsilon\frac{3\alpha^2}{8}\right)\right) + \cdots}$$

Alternative Forms of the Homogeneous Solution

consider: $\ddot{x} + \omega^2 x = 0$

1) $x(t) = c_1 \cos \omega t + c_2 \sin \omega t$ — usually leads to the most algebra possible, particularly if you have powers of x

2) $x(t) = R\cos(\omega t - \phi)$, where ϕ and R are arbitrary constants

$$= R\cdot\cos\omega t \cdot \cos\phi + R\cdot\sin\omega t\cdot\sin\phi$$

then $c_1 = R\cos\phi$ and $c_2 = +R\sin\phi$ } polar analogy

and of course $R^2 = c_1^2 + c_2^2$, $\theta = \tan^{-1}\left(\frac{c_2}{c_1}\right)$

3) $x(t) = A(t)\, e^{i\omega T} + \bar{A}\, e^{-i\omega T}$, A and $\bar{A}$ are constants of integration

$$= 2\,\mathrm{Re}\left\{ A e^{i\omega T} \right\}$$

$$= A(\cos\omega T + i\sin\omega T) + \bar{A}(\cos\omega T - i\sin\omega T)$$

$\Rightarrow A + \bar{A} = c_1$ and $Ai - \bar{A}i = c_2$

$\rightarrow A - \bar{A} = -ic_2$

Typically, 2) or 3) are always used and

3) is the most commonly used form

Example using homogeneous form

$$\boxed{A(\tau)\, e^{iT} + \overline{A(\tau)}\, e^{-iT}}$$

Duffing equation: $\ddot{x} + x + \epsilon x^3 = 0, \quad x(0) = \alpha, \quad \dot{x}(0) = 0$

introduce 2 time scales $T = t, \quad \tau = \epsilon t$

and expand the solution $x \sim x_0(T,\tau) + \epsilon x_1(T,\tau) + \cdots$

$$\Rightarrow \quad x_{0_{TT}} + \epsilon x_{1_{TT}} + 2\epsilon x_{0_{T\tau}} + x_0 + \epsilon x_1 + \epsilon x_0^2 + \cdots = 0$$

$O(1)$: $\quad x_{0_{TT}} + x_0 = 0, \quad x_0(0,0) = \alpha, \quad x_{0_T}(0,0) = 0$

$$\Rightarrow \boxed{x_0(T,\tau) = A(\tau) e^{iT} + \bar{A}(\tau) e^{-iT}}$$

apply initial conditions

$x_0(0,0) = A(0) + \bar{A}(0) = \alpha \quad \Rightarrow \quad 2\,\mathrm{Re}(A(0)) = \alpha$

$x_{0_T}(0,0) = iA(0) - i\bar{A}(0) = 0 \quad \Rightarrow \quad 2\,\mathrm{Im}(A(0)) = 0$

$O(\epsilon)$: $\quad x_{1_{TT}} + x_1 = -2x_{0_{T\tau}} - \underbrace{x_0^3}_{A^3 e^{3iT} + 3A^2\bar{A} e^{iT} + 3A\bar{A}^2 e^{-iT} + \bar{A}^3 e^{-3iT}}$

$$= -2\left[iA' e^{iT} - i\bar{A}' e^{-iT}\right] - \left[A e^{iT} + \bar{A} e^{-iT}\right]^3$$

$$= e^{iT}\left(-2iA' - 3A^2\bar{A}\right) + e^{-iT}\left(2i\bar{A}' - 3A\bar{A}^2\right) + NST$$

$$= e^{iT}\left(-2iA' - 3A|A|^2\right) + e^{-iT}\left(2i\bar{A}' - 3\bar{A}|A|^2\right) + NST$$

$\Rightarrow \quad -2iA' - 3A|A|^2 = 0, \quad 2i\bar{A}' - 3\bar{A}|A|^2 = 0$

these equations are complex conjugates, so only need to solve one of them - so in effect, only need to track coefficient of e^{iT} term (or e^{-iT} term), which is an advantage of using this form

$$A' = +\frac{3i}{2} A|A|^2$$

convert to polar coordinates

let $A(\tau) = R(\tau)\, e^{i\theta(\tau)}$, $\mathrm{Re}[A(0)] = \frac{\alpha}{2}$ $\Rightarrow$ $R(0)\cos(\theta(0)) = \frac{\alpha}{2}$, $R(0)\sin(\theta(0)) = 0$

$$\boxed{\begin{aligned}\theta(0) &= 0\\ R(0) &= \frac{\alpha}{2}\end{aligned}}$$

$$\Rightarrow \quad R' e^{i\theta} + iR\theta' e^{i\theta} = \frac{3i}{2} R e^{i\theta} \cdot R^2$$

divide by $e^{i\theta}$

$$\Rightarrow \quad R' + iR\theta' = \frac{3i}{2} R^3$$

real part: $R' = 0$, and $R(0) = \alpha$ $\Rightarrow$ $\boxed{R(\tau) = \frac{\alpha}{2}}$

imaginary part: $R\theta' = \frac{3}{2} R^3 \rightarrow \theta' = \frac{3}{2} R^2 = \frac{3}{8}\alpha^2$

$\rightarrow \cancel{\theta(\tau) = \frac{3\alpha^2}{8}\tau}$ $\quad \boxed{\theta(\tau) = \frac{3\alpha^2}{8}\tau}$

$$\Rightarrow \quad A(\tau) = \frac{\alpha}{2} e^{i\frac{3\alpha^2}{8}\tau}$$

$$\Rightarrow \quad x_0(T,\tau) = \cancel{\frac{\alpha}{2}\left(e^{i\frac{3\alpha^2}{8}\tau + iT} + e^{-i\frac{3\alpha^2}{8}\tau}\right) e^{iT} = \frac{\alpha}{2} e}$$

$x_0(T,\tau) = \cancel{\frac{\alpha}{2}\cos\left(\frac{3\alpha^2}{8}\tau\right)}$ $\quad x_0(T,\tau) = 2\,\mathrm{Re}\left(A e^{iT}\right)$

$$= 2\,\mathrm{Re}\left[\frac{\alpha}{2} e^{i\left(\frac{3\alpha^2}{8}\tau + T\right)}\right]$$

$$\Rightarrow \quad \cancel{x(t) \sim \frac{\alpha}{2}\cos\left(\frac{3\alpha^2}{8}\tau\right) + \ldots} \quad \boxed{x_0(T,\tau) = \alpha \cos\left(T + \frac{3\alpha^2}{8}\tau\right)}$$

$$\Rightarrow \quad \boxed{x(t) \sim \alpha \cos\left(\left(1 + \frac{3\alpha^2}{8}\epsilon\right) t\right)}$$

Example $\ddot{x} + 2\epsilon\dot{x} + x = 0$, $x(0) = 0$, $\dot{x}(0) = 1$

$\uparrow$ small damping

the exact solution is given by

$$x_{ex} = \frac{e^{-\epsilon t}}{\sqrt{1-\epsilon^2}} \sin\left(t\sqrt{1-\epsilon^2}\right)$$

expand the exact solution (you obtain the same expansion as if you substituted the naive expansion)

$$\rightarrow \quad x_{ex} \sim \sin t - \epsilon t \sin t + \frac{\epsilon^2}{2}\left(t^2 \sin t - t\cos t + \sin t\right) + \cdots$$

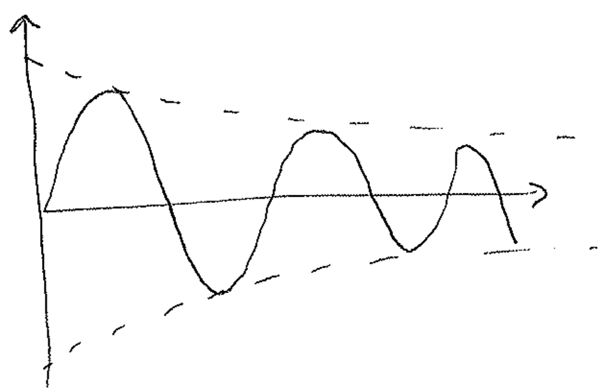
this expansion is nonuniform for $t = O(1/\epsilon)$ and renormalization is quite complicated. The amplitude varies so strained coordinates would fail; must use multiple scales

to plot the exact solution

$$\text{amplitude} = \frac{e^{-\epsilon t}}{\sqrt{1-\epsilon^2}} \sim 1 - \epsilon t + \cdots$$

$$\text{Frequency} = \sqrt{1-\epsilon^2} \sim 1 - \frac{\epsilon^2}{2} + \cdots$$

$$\text{period} = \frac{2\pi}{\sqrt{1-\epsilon^2}} \sim 2\pi\left(1 - \frac{\epsilon^2}{2} + \cdots\right)$$



The solution involves 3 distinct time scales

1) solution oscillates on an $O(1)$ time scale
2) amplitude decays on an $O(1/\epsilon)$ time scale
3) Frequency shift caused by damping term acts on an $O(1/\epsilon^2)$ time scale

We can capture all three time-scales with the leading order term using multiple scales:

$$T = t\left(1 + \epsilon^2\omega_2 + \epsilon^3\omega_3 + \cdots\right) \leftarrow \text{straining } T$$

$$\tau = \epsilon t$$

derivatives, using chain rule

$$\frac{d}{dt} = \left(1 + \epsilon^2\omega_2\right)\frac{\partial}{\partial T} + \epsilon\frac{\partial}{\partial \tau}$$

$$\frac{d^2}{dt^2} = \cancel{\frac{\partial^2}{\partial T^2}} \left(1 + 2\epsilon^2\omega_2\right)\frac{\partial^2}{\partial T^2} + 2\epsilon\left(1 + \epsilon^2\omega_2\right)\frac{\partial^2}{\partial T\partial\tau} + \epsilon^2\frac{\partial^2}{\partial\tau^2}$$

Expand x, $x \sim x_0(T,\tau) + \epsilon x_1(T,\tau) + \epsilon^2 x_2(T,\tau) + \cdots$

the equation $\ddot{x} + 2\epsilon\dot{x} + x = 0$ becomes

$$\Rightarrow \Big[\left(1 + 2\epsilon^2\omega_2\right)\left(x_{0TT} + \epsilon x_{1TT} + \epsilon^2 x_{2TT} + \cdots\right)$$

$$+ 2\epsilon\left(x_{0T\tau} + \epsilon x_{1T\tau} + \cdots\right) + \epsilon^2 x_{0\tau\tau} + \cdots\Big]$$

$$+ 2\epsilon\left[x_{0T} + \epsilon x_{1T} + \cdots + \epsilon x_{0\tau} + \cdots\right]$$

$$+ x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots = 0$$

$O(1)$: $\quad X_{0\,TT} + X_0 \quad , \quad X_0(0,0) = 0 \quad , \quad X_{0\,T}(0,0) = 1$

$$X_0(T,\tau) = R_0(\tau)\cos(T + \phi_0(\tau))$$

$$\begin{aligned} R_0(0)\cos(\phi_0(0)) &= 0 \\ -R_0(0)\sin(\phi_0(0)) &= 1 \end{aligned} \quad \Rightarrow \quad \boxed{\phi_0(0) = \frac{-\pi}{2}, \quad R_0(0) = 1}$$

$O(\epsilon)$: $\quad X_{1\,TT} + X_1 = -2X_{0\,T\tau} - 2X_{0\,T}$

$$= -2\left[-R_0'\sin(T+\phi_0) + R_0\phi_0'\cos(T+\phi_0)\right] + 2R_0\sin(T+\phi_0)$$

$$= 2(R_0 + R_0')\sin(T+\phi_0) + 2R_0\phi_0'\cos(T+\phi_0)$$

kill secularities

$$\begin{aligned} R_0' &= -R_0, \quad &\Rightarrow \\ R_0\phi_0' &= 0 &\Rightarrow \end{aligned} \quad \boxed{\begin{aligned} R_0(\tau) &= e^{-\tau} \\ \phi_0(\tau) &= \frac{-\pi}{2} \end{aligned}}$$

$$\Rightarrow \quad \boxed{X_0(T,\tau) = e^{-\tau}\sin T}$$

$O(\epsilon^2)$: $\quad X_{2\,TT} + X_2 = -2\omega_2 X_{0\,TT} - 2X_{1\,T\tau} - 2X_{1\,T} - 2X_{0\,\tau} - X_{0\,\tau\tau}$

there are three quantities to determine

$\rightarrow \quad \omega_2, \ R_1$, and ϕ_1

where pick R_1 and ϕ_1 to kill secularities in the terms $-2(X_{1\,T\tau} + X_{1\,T})$

and pick ω_2 to kill secularities in

$$-2(\omega_2 X_{0\,TT} + X_{0\,\tau}) - X_{0\,\tau\tau}$$

since $x_0(T, \tau) = e^{-\tau} \sin T$,

the right hand side becomes

$$+ 2w_2 e^{-\tau} \sin T + 2 e^{-\tau} \sin T - 2\left(x_{1T\tau} + x_{1T}\right) - e^{-\tau} \sin T$$

secolarites from x_0 term are $\left(2w_2 + 2 - 1\right) e^{-\tau} \sin T$

so pick $\omega_2 = \frac{-1}{2}$

$$\Rightarrow \quad T = \left(1 - \frac{\epsilon^2}{2}\right) t$$

$$\tau = \epsilon t$$

$$\rightarrow \quad x_0(T, \tau) = e^{-\tau} \sin\left(\left(1 - \frac{\epsilon^2}{2}\right) t\right)$$

$$\rightarrow \quad \boxed{x(t) \sim e^{-\epsilon t} \sin\left(\left(1 - \epsilon^2/2\right) t\right)}$$

Various Forms of Multiple Scales

1. Fast: $T = (1 + \epsilon^2 w_2 + \epsilon^3 w_3 + \dots)t$

   slow: $\tau = \epsilon t$

2. fast: $T = (1 + \epsilon w_1 + \epsilon^3 w_3 + \dots)t$

   slow: $\tau = \epsilon^2 t$

3. fast: $T = t$

   slow: $\tau = \epsilon^n t$, $n = 1, 2, \dots, N$

   $\Rightarrow$ PDE in $N+1$ variables

   $$\frac{d}{dt} = \frac{\partial}{\partial T} + \epsilon \frac{\partial}{\partial \tau_1} + \epsilon^2 \frac{\partial}{\partial \tau_2} + \dots + \epsilon^N \frac{\partial}{\partial \tau_N}$$

4. boundary layers

   Fast: $\xi = x/\epsilon$ (inner)

   slow: $x = x$ (outer)

5. fast: $T = \epsilon^a t$ (leading order)

   slow: $\tau = \epsilon^b t$ (strain fast time T for higher orders)

EXAMPLE

$$\epsilon \ddot{y} + \epsilon \gamma \dot{y} + y + \epsilon y^3 = 0, \quad y(0) = 0, \quad \dot{y}(0) = 1, \quad \gamma > 0$$

naive expansion: $y \sim y_0(t) + \epsilon y_1(t) + \dots$

yields $y \sim 0$ to all orders

try initial layer: let $T = \dfrac{t}{\delta(\epsilon)}$

$$\underset{①}{\frac{\epsilon}{\delta^2} Y_{TT}} + \underset{②}{\frac{\epsilon \gamma}{\delta} Y_T} + \underset{③}{y} + \underset{④}{\epsilon y^3} = 0$$

$$\frac{\epsilon}{\delta^2} Y_{TT} + \frac{\epsilon\gamma}{\delta} Y_T + Y + \epsilon Y^3 = 0 , \qquad \gamma > 0$$
$$\quad (1) \qquad\qquad (2) \qquad\quad (3) \quad (4)$$

balance	$\delta(\epsilon)$	(1)	(2)	(3)	(4)	comments
(1) ~ (2)	$\delta = 1$	$O(\epsilon)$	$O(\epsilon)$	$O(1)$	$O(\epsilon)$	NO
(1) ~ (3)	$\delta = \epsilon^{1/2}$	$O(1)$	$O(\epsilon^{1/2})$	$O(1)$	$O(\epsilon^{1/2})$	yes

$\Rightarrow \quad \delta(\epsilon) = \epsilon^{1/2}$ and $Y_{TT} + Y = 0$, to leading order

$\rightarrow \quad y(T) = A \sin T + B \cos T , \quad T = \dfrac{t}{\epsilon^{1/2}}$

apply initial conditions,

$$y(t=0) = y(T=0) = 0$$
$$\frac{dy(0)}{dt} = \frac{1}{\epsilon^{1/2}} \frac{dy(0)}{dT} = 1 \rightarrow \frac{dy(0)}{dT} = \epsilon^{1/2}$$

$y(0) = 0 \Rightarrow B = 0 \rightarrow y(T) = A \sin T$

$\dot{y}(0) = 1 \Rightarrow A = \epsilon^{1/2} \Rightarrow \boxed{y(t) \sim \epsilon^{1/2} \sin(t/\epsilon^{1/2})}$

where the fast time is $t/\epsilon^{1/2}$

<u>Let</u> $T = t/\epsilon^{1/2}$ and $\tau = \epsilon^a t$, $\boxed{a > -\frac{1}{2}}$

$$\Rightarrow \quad \frac{d}{dt} = \frac{1}{\epsilon^{1/2}} \frac{\partial}{\partial T} + \epsilon^a \frac{\partial}{\partial \tau}$$
$$\frac{d^2}{dt^2} = \frac{1}{\epsilon} \frac{\partial^2}{\partial T^2} + 2\epsilon^{a-1/2} \frac{\partial^2}{\partial T \partial \tau} + \epsilon^{2a} \frac{\partial^2}{\partial \tau^2}$$

with expansion $\quad y \sim \epsilon^{1/2} y_0(T, \tau) + \epsilon y_1(T, \tau) + \dots$

the equation $\epsilon \ddot{y} + \epsilon\gamma \dot{y} + y + \epsilon y^3 = 0$

$y(0) = 0 , \quad \dot{y}(0) = 1 \qquad$ becomes

$$\epsilon\left[\frac{1}{\epsilon}\frac{\partial^2}{\partial T^2} + 2\epsilon^{a-1/2}\frac{\partial^2}{\partial T\partial \tau} + \epsilon^{2a}\frac{\partial^2}{\partial \tau^2}\right]\left(\epsilon^{1/2}y_0 + \epsilon y_1 + \cdots\right)$$

$$+ \epsilon\gamma\left[\frac{1}{\epsilon^{1/2}}\frac{\partial}{\partial T} + \epsilon^{a}\frac{\partial}{\partial \tau}\right]\left(\epsilon^{1/2}y_0 + \epsilon y_1 + \cdots\right)$$

$$+ \epsilon^{1/2}y_0 + \epsilon y_1 + \cdots$$

$$+ \epsilon\underbrace{\left(\epsilon^{1/2}y_0 + \epsilon y_1 + \cdots\right)^3}_{} = 0$$

$$\Rightarrow \quad \epsilon^{1/2}y_{0_{TT}} + \epsilon y_{1_{TT}} + 2\epsilon^{a+1}y_{0_{T\tau}} + \cdots$$

$$+ \epsilon\gamma\left(y_{0_T} + \epsilon^{1/2}y_{1_T} + \epsilon^{a+1/2}y_{0_\tau} + \cdots\right)$$

remember $a > -\frac{1}{2}$

$$+ \epsilon^{1/2}y_0 + \epsilon y_1 + \cdots$$

$$+ \epsilon\left(\epsilon^{3/2}y_0^3 + \cdots\right) = 0$$

$$\Rightarrow \boxed{\epsilon^{1/2}\left(y_{0_{TT}} + y_0\right) + \epsilon y_{1_{TT}} + \epsilon\gamma y_{0_T} + 2\epsilon^{a+1}y_{0_{T\tau}} + \epsilon y_1 + \cdots = 0}$$

to leading order

$$y_{0_{TT}} + y_0 = 0, \quad \boxed{y_0(0,0) = 0}$$

$$\dot{y} \sim \frac{1}{\epsilon^{1/2}}\frac{\partial}{\partial T}\left(\epsilon^{1/2}y_0\right) = y_{0_T} \rightarrow \boxed{y_{0_T}(0,0) = 1}$$

$$\Rightarrow \boxed{y_0(T,\tau) = R_0(\tau)\sin\left(T + \phi_0(\tau)\right)}$$

$$y_0(0,0) = 0 \rightarrow R_0(0)\sin(\phi_0(0)) = 0$$

$$y_{0_T}(0,0) = 1 \Rightarrow R_0(0)\cos(\phi_0(0)) = 1$$

$$\Rightarrow \boxed{\phi_0(0) = 0, \quad R_0(0) = 1}$$

possible next order:

$$\epsilon \gamma Y_{0_T} + \epsilon Y_1 + \epsilon Y_{1_{TT}} + 2\epsilon^{a+1} Y_{0_{T\tau}} = 0$$

For balancing pick $a = 0 \Rightarrow \tau = \epsilon^a t = t$

$$\Rightarrow \quad Y_{1_{TT}} + Y_1 = -2Y_{0_{T\tau}} - \gamma Y_{0_T} \qquad , \; Y_0 = R_0(\tau)\sin(T + \phi_0(\tau))$$

$$= -2\left[R_0' \cos(T+\phi_0) - R_0\phi_0' \sin(T+\phi_0)\right] - \gamma R_0 \cos(T+\phi_0)$$

$$= \left(-2R_0' - \gamma R_0\right)\cos(T+\phi_0) + 2R_0\phi_0' \sin(T+\phi_0)$$

supress the secularities

$$-2R_0' = \gamma R_0 \; , \quad R_0(0) = 1 \quad \Rightarrow \quad \boxed{R_0(\tau) = e^{-\frac{\gamma}{2}\tau}}$$

$$2R_0\phi_0' = 0 \; , \quad \phi_0(0) = 0 \quad \Rightarrow \quad \boxed{\phi_0(\tau) = 0}$$

$$\Rightarrow \quad Y_0(T,\tau) = e^{-\frac{\gamma}{2}\tau}\sin(T) \qquad , \quad \tau = t \; , \; T = \frac{t}{\epsilon^{1/2}}$$

$$\Rightarrow \quad y(t) \sim \epsilon^{1/2} Y_0(T,\tau) + \cdots$$

$$\Rightarrow \quad \boxed{y(t) \sim \epsilon^{1/2} e^{-\frac{\gamma}{2}t}\sin\left(t/\epsilon^{1/2}\right) + \cdots}$$

Multiple Scales and Boundary Layer Problems 4/2/04

Multiple scales works for many problems in which the solution varies on different scales of the independent variable, including BL problems.

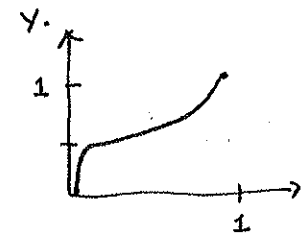
<u>Example</u>: $\epsilon y'' + y' - y^2 = 0 \quad , \; y(0) = 0, \; y(1) = 1$

outer solution , $Y_0' - Y_0^2 = 0$, $Y_0(1) = 1$ (assuming BL at $x=0$)

$$Y_0(x) = \frac{1}{2-x}$$

inner solution : let $\xi = \frac{x}{\epsilon} \Rightarrow y_0'' + y_0' = 0, \; y_0(0) = 0$

$$\longrightarrow y_0(\xi) = c_0(1 - e^{-\xi})$$



matching : $c_0 = \frac{1}{2}$

composite solution ; $\boxed{Y_c(x) \approx \frac{1}{2-x} - \frac{1}{2} e^{-x/\epsilon} + \cdots}$

The solution varies on 2 spatial scales
1) in the BL, y varies on the 'fast' scale ($\xi = \frac{x}{\epsilon}$)
2) in the outer region, y varies on the 'slow' scale

<u>Using multiple scales,</u>
- fast variable $\xi = x/\epsilon$
- slow variable $\eta = x$

$$\Rightarrow \frac{d}{dx} = \frac{d\xi}{dx}\frac{\partial}{\partial \xi} + \frac{d\eta}{dx}\frac{\partial}{\partial \eta} = \frac{1}{\epsilon}\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

$$\frac{d^2}{dx^2} = \frac{1}{\epsilon^2}\frac{\partial^2}{\partial \xi^2} + \frac{2}{\epsilon}\frac{\partial^2}{\partial \xi \partial \eta}$$

let $y \sim Y_0(\xi, \eta) + \epsilon Y_1(\xi, \eta) + \cdots$

$$\epsilon y'' + y' - y^2 = 0$$

$$\Rightarrow \epsilon\left(\frac{1}{\epsilon^2}\frac{\partial^2}{\partial\xi^2} + \frac{2}{\epsilon}\frac{\partial^2}{\partial\xi\partial x}\right)(y_0 + \epsilon y_1 + \cdots) + \left(\frac{1}{\epsilon}\frac{\partial}{\partial\xi} + \frac{\partial}{\partial x}\right)(y_0 + \epsilon y_1 + \cdots)$$

$$- (y_0 + \epsilon y_1 + \cdots)^2 = 0$$

$$\Rightarrow \frac{1}{\epsilon}\left(y_{0_{\xi\xi}} + \epsilon y_{1_{\xi\xi}}\right) + 2y_{0_{\xi x}} + \cdots + \frac{1}{\epsilon}\left(y_{0_\xi} + \epsilon y_{1_\xi}\right) + y_{0_x} + \cdots$$

$$- y_0^2 + \cdots = 0$$

<u>multiply by ϵ</u>:

$$\boxed{y_{0_{\xi\xi}} + y_{0_\xi} + \epsilon\left(y_{1_{\xi\xi}} + y_{1_\xi} + y_{0_x} + 2y_{0_{\xi x}} - y_0^2\right) + \cdots = 0}$$

boundary condition:

$$y(x=0) = y_0(0,0) + \epsilon y_1(0,0) + \cdots = 0 \quad \Rightarrow \quad \boxed{y_0(0,0) = 0}$$

$$y(x=1) = y_0\left(\frac{1}{\epsilon}, 1\right) + \epsilon y_1\left(\frac{1}{\epsilon}, 1\right) + \cdots = 1 \quad \Rightarrow \quad \boxed{y_0\left(\frac{1}{\epsilon}, 1\right) = 1}$$

<u>$O(1)$</u>: $\quad y_{0_{\xi\xi}} + y_{0_\xi} = 0, \quad y_0(0,0) = 0, \quad y_0(1/\epsilon, 1) = 1$

$$\rightarrow \boxed{y_0(\xi, x) = A(x) + B(x)e^{-\xi}}$$

$$y_0(0,0) = 0 \quad \Rightarrow \quad \boxed{A(0) + B(0) = 0}$$

$$y_0(1/\epsilon, 1) = 1 \quad \Rightarrow \quad A(1) + \underbrace{B(1)e^{-1/\epsilon}}_{\text{transcendentally small}} = 1 \quad \Rightarrow \quad \boxed{A(1) = 1}$$

$O(\epsilon)$: $Y_{1_{\xi\xi}} + Y_{1_\xi} = -2Y_{0_{\xi\eta}} - Y_{0_\eta} + Y_0^2$

substitute $Y_0(\xi, \eta) = A(\eta) + B(\eta) e^{-\xi}$

$\Rightarrow Y_{1_{\xi\xi}} + Y_{1_\xi} = -2(-B' e^{-\xi}) - (A' + B' e^{-\xi}) + (A^2 + 2ABe^{-\xi} + B^2 e^{-2\xi})$

$Y_{1_{\xi\xi}} + Y_{1_\xi} = (2B' - B' + 2AB) e^{-\xi} + (A^2 - A') + \underbrace{B^2 e^{-2\xi}}_{NST}$

supress secularities

$B' = -2AB$, $A' = A^2$, $A(0) + B(0) = 0$, $A(1) = 1$

$\rightarrow \frac{dA}{A^2} = d\eta \rightarrow \frac{-1}{A} = \eta + k \rightarrow A = \frac{-1}{\eta + k}$

$A(1) = 1 \Rightarrow 1 = \frac{-1}{1+k} \rightarrow k = -2 \Rightarrow A = \frac{-1}{-2+\eta} \Rightarrow \boxed{A(\eta) = \frac{1}{2-\eta}}$

$\Rightarrow B'(\eta) = \frac{-2}{2-\eta} B \Rightarrow \frac{B'}{B} = \frac{+2}{\eta - 2} \Rightarrow \ln B = 2\ln(\eta-2) + k$

$\ln \frac{B}{(\eta-2)^2} = k \Rightarrow B(\eta) = k(\eta-2)^2$

$A(0) + B(0) = 0 \rightarrow A(0) = \frac{1}{2} \Rightarrow B(0) = \frac{-1}{2} \Rightarrow -\frac{1}{2} = k(+4) \Rightarrow k = \frac{-1}{8}$

$\Rightarrow \boxed{B(\eta) = \frac{-1}{8}(\eta-2)^2}$

so that $Y_0(\xi, \eta) = \frac{1}{2-\eta} - \frac{1}{8}(\eta-2)^2 e^{-\xi}$, $\eta = x$, $\xi = x/\epsilon$

and $\boxed{Y_M(x) \sim \frac{1}{2-x} - \frac{1}{8}(x-2)^2 e^{-x/\epsilon} + \cdots}$

In conclusion

$$y_M(x) \sim \frac{1}{2-x} - \frac{1}{8}(x-2)^2 e^{-x/\epsilon} + \cdots$$

$$y_c(x) \sim \frac{1}{2-x} - \frac{1}{2} e^{-x/\epsilon} + \cdots$$

the solutions are asymptotically equivalent to leading order

OUTER: $x = O(1)$, $y_M, y_c \sim \frac{1}{2-x}$

INNER: $x = O(\epsilon)$: $y_M, y_c \sim \frac{1}{2} - \frac{1}{2} e^{-x/\epsilon}$

4/2/04

5/5

General weakly nonlinear oscillator

$$\ddot{x} + x + \epsilon h(x, \dot{x}) = 0 \quad , x(0) = a, \quad \dot{x}(0) = b$$

Multiple scales

$T = t$

$\tau = \epsilon t$

$x \sim x_0(T, \tau) + \epsilon x_1(T, \tau) + \cdots$

$$\Rightarrow \left(x_{0_{TT}} + \epsilon x_{1_{TT}} + 2\epsilon x_{0_{T\tau}} + \cdots\right) + (x_0 + \epsilon x_1 + \cdots) + \epsilon h(x_0, x_{0_T}) = 0$$

$O(1)$: $\quad x_{0_{TT}} + x_0 = 0 \quad , \quad x_0(0,0) = a, \quad x_{0_T}(0,0) = b$

let $\boxed{x_0(T, \tau) = R(\tau) \cos(T + \phi(\tau))}$

initial conditions $\Rightarrow$ $R(0) \cos(\phi(0)) = a$

$- R(0) \sin(\phi(0)) = b$

$\rightarrow \quad R(0)^2 = a^2 + b^2 \quad \rightarrow$

$\tan(\phi(0)) = \frac{-b}{a} \quad \rightarrow$

$$\boxed{\begin{aligned} R(0) &= \sqrt{a^2+b^2} \\ \phi(0) &= \tan^{-1}\left(\tfrac{-b}{a}\right) \end{aligned}}$$

$O(\epsilon)$: $\quad x_{1_{TT}} + x_1 = -2x_{0_{T\tau}} - h(x_0, x_{0_T})$

$$= -2\left(-R'(\tau)\sin(T+\phi) - R\phi'\cos(T+\phi)\right) - h\left(R\cos(T+\phi), -R\sin(T+\phi)\right)$$

supress the secularities

Note that $h(x_0, x_{0_T})$ is 2π-periodic in the T variable

$$\Rightarrow \quad h\big|_T = h\big|_{T+2\pi}$$

so we can express the function h by its Fourier series

Fourier series:

$$h(R\cos(T+\phi), -R\sin(T+\phi)) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n(T+\phi)) + B_n (\sin(n(T+\phi)))$$

where $A_n = \frac{1}{\pi}\int_{T_0}^{T_0+2\pi} h(R\cos(T+\phi), -R\sin(T+\phi)) \cos(n(T+\phi))\, dT$, $n=0,1,\ldots$

$B_n = \frac{1}{\pi}\int_{T_0}^{T_0+2\pi} h(R\cos(T+\phi), -R\sin(T+\phi)) \sin(n(T+\phi))\, dT$, $n = 1, 2, \ldots$

where T_0 is arbitrary and $A_n = A_n(z)$, $B_n = B_n(z)$

Then A_1 and B_1 are the coefficients of the secular terms of h

$$\Rightarrow \quad x_{1_{TT}} + x_1 = 2R'\sin(T+\phi) + 2R\phi'\cos(T+\phi) - A_1\cos(T+\phi) - B_1\sin(T+\phi) + \text{N.S.T.}$$

$$= (2R' - B_1)\sin(T+\phi) + (2R\phi' - A_1)\cos(T+\phi) + \text{N.S.T.}$$

solve for $R(z)$ and and $\phi(z)$ such that

$$\boxed{\begin{aligned} R'(z) &= \frac{B_1(z)}{2}, \quad R(0) = \sqrt{a^2+b^2} \\ \phi'(z) &= \frac{A_1(z)}{2R(z)}, \quad \phi(0) = \tan^{-1}\left(\frac{b}{a}\right) \end{aligned}}$$

1 800 456 6003

x 3645

Phase Plane / Limit Cycles

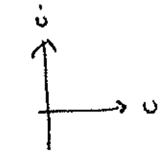
eg 1) Van der Pol equation

2) Rayleigh equation

Consider Rayleigh's equation

$$* \quad \ddot{u} - \epsilon\left(1 - \frac{\dot{u}^2}{3}\right)\dot{u} + u = 0 \quad , t > 0$$

the goal is to plot trajectories in the phase plane



write as a 1^{st} order system of ODEs

let $v = \dot{u}$ then
$$\dot{u} = v$$
$$\dot{v} = -u + \epsilon\left(1 - \frac{v^2}{3}\right)v$$

the critical points are $u = 0, v = 0$ - this is the only critical point

what type of critical point is this? (saddle, node, etc..)

and examine stability

linearize about $(0,0)$ to approximate in a neighborhood of $(0,0)$

$$\Rightarrow \quad \begin{aligned} \dot{u}_1 &= v_1 \\ \dot{v}_1 &= -u_1 + \epsilon v_1 \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} \dot{u}_1 \\ \dot{v}_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \epsilon \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$$

the type and stability depend on eigenvalues

$$\Rightarrow \quad -\lambda(\epsilon - \lambda) + 1 = 0 \quad \Rightarrow \quad \lambda^2 - \epsilon\lambda + 1 = 0$$

$$\rightarrow \quad \lambda = \frac{\epsilon \pm \sqrt{\epsilon^2 - 4}}{2} \quad \Rightarrow \quad \lambda = \frac{1}{2}\left[\epsilon \pm i\sqrt{4 - \epsilon^2}\right]$$

since λ is complex, $(0,0)$ is a spiral point

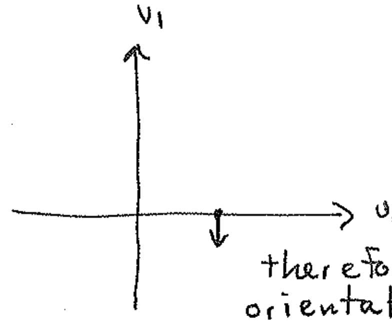
and since $\mathrm{Re}(\lambda) > 0$, the critical point is unstable

so it spirals outward, but does it spiral clockwise or counterclockwise?

For the direction, pick a point in phase plane and determine slope.

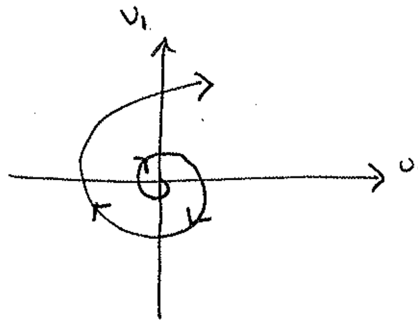
pick $(1,0)$ $\rightarrow$ $u_1 = 1$, $v_1 = 0$

$$\begin{pmatrix} \dot u_1 \\ \dot v_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \epsilon \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \Rightarrow$$



therefore orientation is clockwise

$\Rightarrow$ phase plane looks like



the linear analysis only determines behavior sufficiently close to ~~stable~~ critical point. The question is, what effect do the nonlinearities have on the phase plane?

Use Multiple scales Approach

- using complex form of homogeneous solution

$$\ddot u - \epsilon\left(1 - \frac{\dot u^2}{3}\right)\dot u + u = 0 \quad , \quad \text{assume } \epsilon \ll 1$$

to leading order, let $T = t$
$\tau = \epsilon t$

and $u \sim u_0(T, \tau) + \epsilon u_1(T, \tau) + \ldots$

The equation becomes

$$u_{0_{TT}} + \epsilon u_{1_{\tau\tau}} + 2\epsilon u_{0_{T\tau}} + \cdots$$

$$-\epsilon\left[1 - \frac{u_{0_T}^2}{3}\right]u_{0_T} + \cdots \quad + \quad u_0 + \epsilon u_1 + \cdots = 0$$

$\underline{O(1):}$ $\quad u_{0_{TT}} + u_0 = 0$

let $u_0(T,\tau) = A(\tau)e^{iT} + \bar{A}(\tau)e^{-iT}$ or $(Ae^{iT} + c.c.)$ ← complex conjugate

$\underline{O(\epsilon):}$ $\quad u_{1_{TT}} + u_1 = -2\cancel{\epsilon}u_{0_{T\tau}} + \left(1 - \frac{u_{0_T}^2}{3}\right)u_{0_T}$

substitute and kill secularities, $u_{0_T} = i(Ae^{iT} - \bar{A}e^{-iT})$

$$LHS = -2\left(iA'e^{iT} + c.c.\right) + \left[1 + \frac{1}{3}\left(A^2e^{2iT} - 2A\bar{A} + A^2e^{-2iT}\right)\right]$$
$$i\left(Ae^{iT} - \bar{A}e^{-iT}\right)$$

recall, the secular terms are e^{iT} and e^{-iT}

but since they are complex conjugates, need only pick coefficients of one of them

coefficients of e^{iT}: $\quad -2iA' + iA - \frac{i}{3}A^2\bar{A} - \frac{2}{3}iA^2\bar{A}$

supress coefficient: $\quad \frac{dA}{d\tau} = \frac{A}{2}\left(1 - |A|^2\right)$

write in polar form, let $A = R(\tau)e^{i\theta(\tau)}$, R, θ are real

$$A' = R'\cancel{e^{i\theta}} + i\theta' R\cancel{e^{i\theta}} = \frac{1}{2}R\cancel{e^{i\theta}}(1 - R^2)$$

$$\rightarrow \quad R' + i\theta' = \frac{1}{2}R(1 - R^2)$$

$\Rightarrow \quad \theta' = 0 \quad \rightarrow \quad \boxed{\theta(\tau) = \theta_0}$ (some constant that would be determined by initial conditions)

$$R' = \frac{1}{2}R(1 - R^2) \quad \Rightarrow \quad \boxed{R(\tau) = \frac{1}{\sqrt{1 + ce^{-\tau}}}}$$

$$\Theta = \Theta_0, \qquad R = \frac{1}{\sqrt{1+ce^{-\tau}}}$$

$$\Rightarrow \quad A = \frac{e^{i\Theta_0}}{\sqrt{1+ce^{-\tau}}} \qquad \Rightarrow \quad u_0(T,\tau) = 2\,\mathrm{Re}\left(A e^{iT}\right)$$

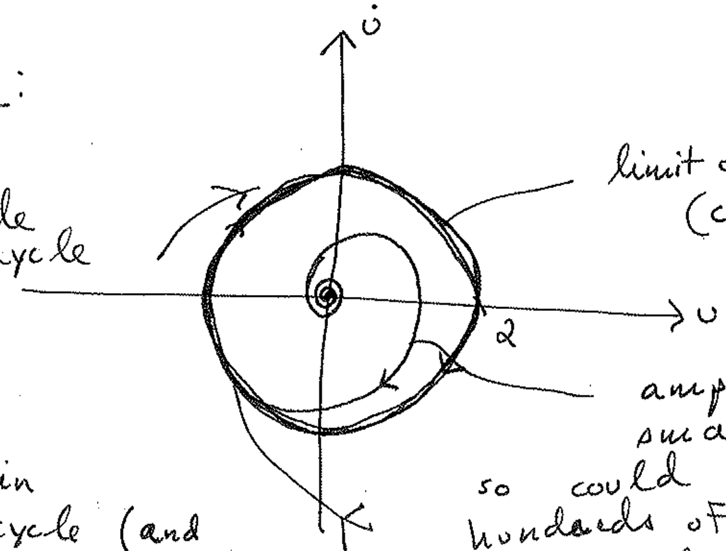
$$u_0(T,\tau) = 2\,\mathrm{Re}\left(\frac{e^{i(\Theta_0+T)}}{\sqrt{1+ce^{-\tau}}} \right)$$

$$\Rightarrow \quad u_0(T,\tau) = \frac{2\cos(\Theta_0+T)}{\sqrt{1+ce^{-\tau}}}$$

$$\Rightarrow \quad \boxed{u(t) \sim \frac{2\cos(\Theta_0+t)}{\sqrt{1+ce^{-\epsilon t}}} + \ldots}$$

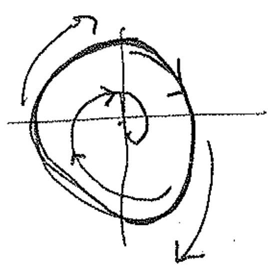
as $t \to \infty$, solution approaches $2\cos(\Theta_0+t)$ and $\dot{u} \to -2\sin(t+\Theta_0)$ and furthermore, $u^2+\dot{u}^2 = 4$

phase plane:



so could loop around hundreds of times before reaches limit cycle

For $\epsilon<0$, obtain unstable limit cycle (and stable critical point)



WKB(J) Approximation

WKB works for equations of the form

$$\frac{d^2y}{dx^2} - F(x;\lambda)\,y = 0$$

where $F(x;\lambda) = \lambda^2 f_0(x) + \lambda f_1(x) + f_2(x) + \dots$

where $\lambda \gg 1$ is a large parameter

Try $y \sim \phi(x)\, e^{\lambda w(x)}\left[1 + \frac{\psi_1(x)}{\lambda} + \frac{\psi_2(x)}{\lambda^2} + \dots\right]$

leading order (find ϕ and w)

substitute into equation

the second derivative is (with $\psi = 1 + \frac{\psi_1}{\lambda} + \frac{\psi_2}{\lambda^2} + \dots$

$$y'' = e^{\lambda w}\left[\lambda^2 \phi\psi (w')^2 + 2\lambda w'(\phi\psi)' + \lambda w''\phi\psi + (\phi\psi)''\right]$$

need to keep $O(\lambda^2)$ and $O(\lambda)$ for leading order

$$\Rightarrow \quad \lambda^2\phi(w')^2\left(1 + \frac{\psi_1}{\lambda}\right) + \lambda 2 w'\phi' + \lambda w''\phi - \lambda^2 f_0\left(1 + \frac{\psi_1}{\lambda}\right)\phi + \lambda f_1 \phi = 0$$

$O(\lambda^2)$: $\phi(w')^2 - f_0\phi = 0 \longrightarrow \quad \Rightarrow \boxed{w(x) = \pm\int f_0^{1/2}\,dx}$

(Eikonal eqn)

$O(\lambda)$: $\cancel{\phi\psi_1(w')^2} + 2w'\phi' + w''\phi - \cancel{f_0\psi_1\phi} + f_1\phi = 0$

the equation is then to be solved for ϕ, since w is known and f_1 is known

$$\phi' + \frac{w'' - f_1}{2w'}\phi = 0 \quad \text{(Transport eqn)}$$

the integrating factor is $\nu = \exp\left[\int \frac{w'' - f_1}{2w'}dx\right] = f_0^{-1/4} e^{\pm\frac{1}{2}\int\frac{f_1}{f_0^{1/2}}}$

$$\longrightarrow \quad \phi(x) = f_0^{-1/4}\, e^{\pm\int \frac{1}{2}\frac{f_1}{f_0^{1/2}}dx}$$

the leading order expansion is given by

$$y \sim f_0^{-1/4}(x)\, e^{\left[\pm \int \left(\lambda f_0^{1/2}(x) + \frac{1}{2} \frac{f_1}{f_0^{1/2}}\right) dx\right]}$$

Example, $f_0 = 1$, $f_1 = 0$, $f_2 = 0$, ...

$$\Rightarrow \quad y'' - \lambda^2 y = 0 \quad \longrightarrow \quad y \sim e^{\pm \lambda x}$$

Asymptotic Integration

- $\boxed{\text{small parameters:}}$ typically evaluate using Taylor ~~series~~ expansions

example: $\int_0^1 \frac{e^{\epsilon x}}{1+x^2}\,dx$

expand the integrand using $e^{\epsilon x} \sim 1 + \epsilon x + \frac{1}{2}\epsilon^2 x^2 + \cdots$

$$\Rightarrow \int_0^1 \frac{e^{\epsilon x}}{1+x^2}\,dx \sim \int_0^1 \left(\frac{1}{1+x^2} + \frac{\epsilon x}{1+x^2} + \frac{1}{2}\frac{\epsilon^2 x^2}{(1+x^2)} + \cdots \right) dx$$

$$\sim \tan^{-1}(1) + \frac{\epsilon}{2}\ln 2 + \frac{\epsilon^2}{2}(1 - \tan^{-1} 1) + \cdots$$

---

example: $I(\epsilon) = \int_0^{1-\epsilon} \frac{dx}{1+x^2}$

taylor expansion, $I(\epsilon) \sim I(0) + \epsilon I'(0) + \cdots$

$$I(0) = \int_0^1 \frac{dx}{1+x^2} = \tan^{-1}(1)$$

$$I'(\epsilon) = \frac{-1}{1+(1-\epsilon)^2} \rightarrow I'(0) = \frac{-1}{2}$$

$$\Rightarrow I(\epsilon) \sim \tan^{-1}(1) - \frac{\epsilon}{2} + \cdots$$

Consider general form, $I(\epsilon) = \int_{a(\epsilon)}^{b(\epsilon)} f(x;\epsilon)\,dx$

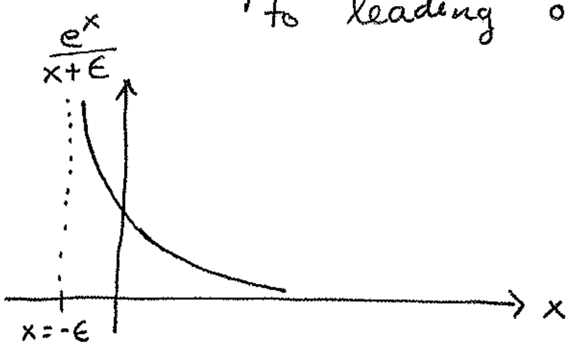
Leibnitz formula:

$$\frac{dI}{d\epsilon} = \int_{a(\epsilon)}^{b(\epsilon)} \frac{\partial f}{\partial \epsilon}(x;\epsilon)\,dx + b'(\epsilon) f(b(\epsilon);\epsilon) - a'(\epsilon) f(a(\epsilon);\epsilon)$$

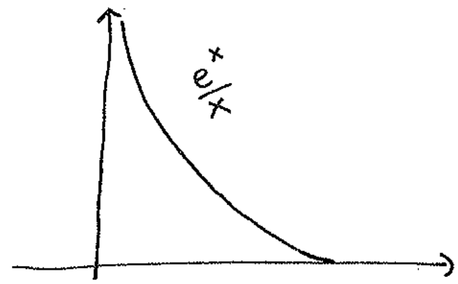
Example: $I(\epsilon) = \int_0^1 \frac{e^x}{x+\epsilon}\, dx$ → cannot evaluate at $\epsilon = 0$ so there will be problems when the integral is expanded

$$\to I(\epsilon) \sim \int_0^1 \frac{e^x}{x}\left(1 - \frac{\epsilon}{x} + \frac{\epsilon^2}{x^2} + \dots\right) dx$$

↑ to leading order, cannot evaluate this integral



and



Note, for more general case,

$$\int_0^1 \frac{f(x)}{x+\epsilon}\, dx \sim -f(0)\ln(\epsilon) + \cdots$$

Methods of Integration

1) Small parameters
   a) Taylor expansions
   b) singular perturbation method

2) Large parameters, $I(x) = \int_a^b f(t)\, e^{x\phi(t)}\, dt$, $x \gg 1$
   a) integration by parts ("naive approach")
   b) method of steepest descent

$$I(x) = \int_C f(t)\, e^{x\phi(t)}\, dt, \quad x \gg 1$$

, x is real and f, ϕ, t are complex, C is a contour in the complex plane

$\underbrace{\qquad}$ most general integral that we have a method for...

4/13/04

$\boxed{2/3}$

c) Method of stationary phase

(special case), $\int_a^b f(t)\, e^{ix\phi(t)}\,dt$, $x \gg 1$, where f, ϕ, x, t, a, b are real and the i makes exponent purely imaginary - the integral is like a Fourier transform.

d) Laplace method (will focus on this case)

(special case) $\int_a^b f(t)\, e^{x\phi(t)}\,dt$, $x \gg 1$, t, x, ϕ, f, a, b are real (everything is real)

Integration by Parts

Example: $I(x) = \int_a^b e^{xt^2}\,dt$, $0 < a < b < \infty$, $x \gg 1$

integrating by parts, $\int u\,dv = uv - \int v\,du$

this will not work if pick $u = e^{xt^2}$ and $dv = dt$, so pick $u = \frac{1}{t}$ and $dv = e^{xt^2}(2xt)\,dt$, from

$$I(x) = \int_a^b e^{xt^2}\,\frac{2xt}{2xt}\,dt = \frac{1}{2x}\int_a^b \frac{1}{t}\, e^{xt^2}(2xt)\,dt$$

$$\Rightarrow \quad du = \frac{-1}{t^2}\,dt, \qquad v = e^{xt^2}$$

$$\Rightarrow I(x) = \frac{1}{2x}\left[\frac{e^{xt^2}}{t}\right]_a^b + \frac{1}{2x}\int_a^b \frac{1}{t^2}\, e^{xt^2}\left(\frac{2xt}{2xt}\right)dt \Rightarrow \frac{1}{2x}\int_a^b \frac{1}{t^3}\, e^{xt^2}(2xt)\,dt$$

for another integration by parts

$u = \frac{1}{t^3}$, $dv = e^{xt^2}$

$$= \frac{1}{2x}\left[\frac{e^{xt^2}}{t}\right]_a^b + \cancel{\frac{1}{2x}\left[\frac{e^{xt^2}}{t^3}\cdot 2xt\right]_a^b}$$

$$= \frac{1}{2x}\left[\frac{e^{xt^2}}{t}\right]_a^b + \frac{1}{4x^2}\left[\frac{e^{xt^2}}{t^3}\right]_a^b + \frac{3}{4x^2}\int_a^b \frac{1}{t^4}\, e^{xt^2}\,dt + \cdots$$

$$\Rightarrow \quad I(x) \left\{ \frac{1}{2x}\left[\frac{e^{xt^2}}{t}\right]_a^b + \frac{1}{4x^2}\left[\frac{e^{xt^2}}{t^3}\right]_a^b + \frac{1\cdot 3}{(2x)^3}\left[\frac{e^{xt^2}}{t^5}\right]_a^b + \frac{1\cdot 3\cdot 5}{(2x)^4}\left[\frac{e^{xt^2}}{t^7}\right]_a^b + \cdots \right.$$

$$\cancel{\Rightarrow I(x) \sim \frac{e^{x^2 t}}{2xt}\left[1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2xt^2)^n}\right.}$$

$$I(x) = \left.\frac{e^{xt^2}}{2xt}\left[1 + \frac{1}{2xt^2} + \frac{1\cdot 3}{(2xt^2)^2} + \frac{1\cdot 3\cdot 5}{(2xt^2)^3} + \frac{1\cdot 3\cdot 5\cdot 7}{(2xt^2)^4} + \cdots\right]\right]_{t=a}^{t=b}$$

$$\Rightarrow \quad I(x) \sim \left.\frac{e^{xt^2}}{2xt}\left[1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2xt^2)^n}\right]\right|_{t=a}^{t=b}$$

$$\Rightarrow \boxed{I(x) \sim \frac{e^{xb^2}}{2xb}\left[1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2xb^2)^n}\right] - \frac{e^{xa^2}}{2xa}\left[1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2xa^2)^n}\right]}$$

Note that for $0 < a < b < \infty$, compare an arbitrary term from the first bracket, $O\left(\frac{e^{xb^2}}{x^{n+1}}\right)$, and the <u>leading order</u> term from the second bracket, $O\left(\frac{e^{xa^2}}{x}\right)$

Recall, the ratio of the orders is

$$\text{Ratio} = \frac{e^{xa^2}/x}{e^{xb^2}/x^{n+1}} \sim x^n e^{x(a^2-b^2)} \to 0 \quad \text{as } x \to \infty$$

by repeated applications of l'Hospital's rule.

Conclusion, $\frac{e^{xa^2}}{x} \ll \frac{e^{xb^2}}{x^{n+1}}$

Apply this idea to find an expansion for $I(x)$

$$I(x) \sim \frac{e^{xb^2}}{2xb}\left[1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot \ldots \cdot(2n-1)}{(2xb^2)^n}\right] + \cdots$$

Note that the "leading order" value of the integral is determined at the endpoint where the integrand is larger

---

Example: $I(x) = \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt, \quad -\infty < x < \infty$

- First consider small x, ie $|x| \ll 1$, then t is small too since $0 \le |t| \le |x| \ll 1$

$$\to I(x) \sim \frac{2}{\sqrt{\pi}}\int_0^x \left(1 - \frac{t^2}{1!} + \frac{1}{2!}t^4 - \frac{1}{3!}t^6 + \ldots\right)dt$$

$$\sim \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \ldots\right)$$

$$\boxed{I(x) \sim \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\,n!}, \quad |x| \ll 1}$$

- $x \gg 1$, $x > 0$

$$I(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt = 1 - \operatorname{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2}\,dt$$

$u = \frac{1}{t}$, $du = \frac{-1}{t^2}dt$; $dv = e^{-t^2}(-2t)\,dt$, $v = e^{-t^2}$

$$= 1 - \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2}\frac{(-2t)}{(-2t)}\,dt = 1 + \frac{1}{\sqrt{\pi}}\left[\frac{1}{t}e^{-t^2}\right]_x^{\infty} + \frac{1}{\sqrt{\pi}}\int_x^{\infty}\frac{1}{t^2}e^{-t^2}\,dt$$

$$\downarrow$$

$$I(x) = 1 + \frac{1}{\sqrt{\pi}}\left(0 - \frac{1}{x}e^{-x^2}\right) + O\left(\frac{1}{x^2}e^{-x^2}\right)$$

, etc...., by successive integrations by parts

$$\Rightarrow \boxed{I(x) \sim 1 - \frac{e^{-x^2}}{x\sqrt{\pi}}\left(1 + \sum_{n=1}^{\infty}(-1)^n \frac{1\cdot 3\cdot 5\ldots(2n-1)}{(2x^2)^n}\right) \quad x>0,\ x \gg 1}$$

$\boxed{-x \gg 1}$

• $x < 0$

$$I(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}} \left[\int_{-\infty}^{x} e^{-t^2}\,dt - \int_{-\infty}^{0} e^{-t^2}\,dt \right]$$

$$= \frac{2}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-t^2}\,dt - 1$$

so now, in the range of integration, everything is large in magnitude, and negative

$$= \frac{2}{\sqrt{\pi}} \int_{-x}^{\infty} e^{-t^2}\,dt - 1 = \operatorname{erf}(-x) - 1$$

so substitute $-x$ into formula for $\operatorname{erf}(x)$ and subtract 1

$$\Rightarrow \boxed{ I(x) \sim \frac{e^{-x^2}}{x\sqrt{\pi}} \left[1 + \sum_{n=0}^{\infty} \frac{(-1)^n\, 1\cdot 3\cdot 5 \cdots (2n-1)}{(2x^2)^n} \right] \quad -x \gg 1 \ (x<0) }$$

since $\operatorname{erf}(x)$ is odd,

then $\operatorname{erf}(-x) = -\operatorname{erf}(x)$

so to obtain expansion for $-x \gg 1$,

replace x by $-x$ in expansion for $\operatorname{erf}(x)$

$$\Rightarrow \quad I(x) \sim -1 - \frac{e^{-x^2}}{x\sqrt{\pi}} \left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{1\cdot 3 \cdots (2n-1)}{(2x^2)^n} \right]$$

~~xxxx~~ Example , $z \gg 1$, $0 < T_0 < T_1$

$$\int_{T_0}^{T_1} e^{z(1-1/T)}\, dT \sim e^z \int_{T_0}^{T_1} e^{-z/T} \cdot \frac{z}{T^2} \cdot \frac{T^2}{z}\, dT$$

$$= \frac{1}{z} e^z \int_{T_0}^{T_1} T^2 \, e^{-z/T} \frac{z}{T^2}\, dT = \frac{1}{z} e^z \left[T^2 e^{-z/T} \Big|_{T_0}^{T_1} - \int_{T_0}^{T_1} 2T e^{-z/T} dT \right]$$

$$= \frac{1}{z} e^z \left(\underbrace{T_1^2 e^{-z/T_1}}_{O(e^{-z/T_1})} - \underbrace{T_0^2 e^{-z/T_0}}_{O(e^{-z/T_0})} \right) + \frac{1}{z} e^z \underbrace{O\left(\frac{1}{z} e^{-z/T}\right)}_{\text{from integral}}$$

since $T_0 < T_1 \rightarrow \frac{1}{T_0} > \frac{1}{T_1} \rightarrow -\frac{1}{T_0} < -\frac{1}{T_1}$

and so ~~$O(e^{-z/T})$~~ $e^{-z/T_1} \gg e^{-z/T_0}$

so neglect $O(e^{-z/T_0})$ term because transcendentally small

$$\Rightarrow \boxed{\int_{T_0}^{T_1} e^{z(1-1/T)}\, dT \sim \frac{e^z}{z} T_1^2 e^{-z/T_1}} \rightarrow \frac{T_1^2}{z} e^{z(1-1/T_1)}$$

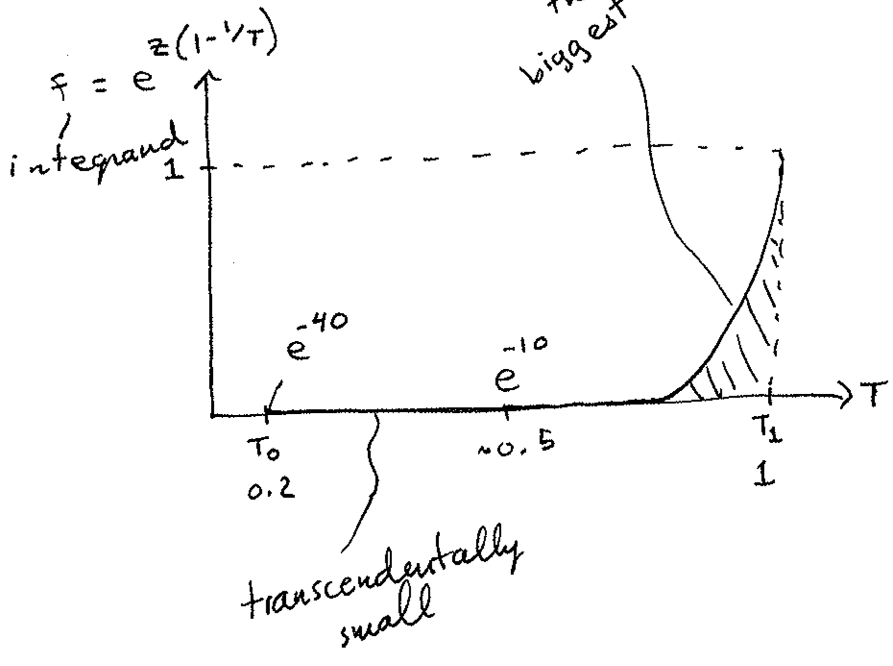
the integrand is biggest around T_1

some typical values are

$z = 10$

$T_0 = 0.2$

$T_1 = 1$



Laplace's Method

Consider an integral of the form

$$\int_a^b f(t)\, e^{x\phi(t)}\, dt \quad , \quad x \gg 1$$

where a and b may be $\pm\infty$

Note:
- When integration by parts works, it is equivalent to Laplace's Method
- The leading order approximation of the integral occurs where the integrand, and in particular, $\phi(t)$, is maximum (for $f(t) \neq 0$).

~~let $u = e^{x\phi}$, $dv = f(t)\,dt$, $du = x\phi' e^{x\phi}\,dt$, $v = \int f(t)\,dt$, [assuming $\phi' \neq 0$ in $[a,b]$]~~

$$I(x) = \int_a^b f(t)\, e^{x\phi(t)}\, dt$$

let $u = f(t)$
$du = f'(t)\, dt$

$\int u\,dv = uv - \int v\,du$

and $dv = x\phi'(t)\, e^{x\phi(t)}\, dt$
$v = e^{x\phi(t)}$

so multiply and divide by $x\phi'(t)$, so let $u = \dfrac{f(t)}{x\phi'(t)}$

$$I(x) = \int_a^b f(t)\, e^{x\phi(t)}\, dt = \int_a^b f(t)\, e^{x\phi(t)} \cdot \frac{x\phi'(t)}{x\phi'(t)}\, dt = \left[\frac{e^{x\phi(t)} f(t)}{x\phi'(t)}\right]_a^b - \underbrace{\int_a^b e^{x\phi(t)} \frac{d}{dt}\left(\frac{f}{\phi'}\right) dt}_{\text{SMALL}}$$

$$\Rightarrow I(x) \sim \frac{1}{x}\left[\underbrace{\frac{f(b)\, e^{x\phi(b)}}{\phi'(b)}}_{O(e^{x\phi(b)})} - \underbrace{\frac{f(a)\, e^{x\phi(a)}}{\phi'(a)}}_{O(e^{x\phi(a)})}\right] + \cdots$$

since $\phi'(t) \neq 0$, $\phi(t)$ is monotonic, so maximum of $e^{x\phi(b)}$ and $e^{x(\phi(a))}$ occurs at maximum of $\phi(a)$, $\phi(b)$

$\Rightarrow$ if $\phi(a) > \phi(b)$ $\Rightarrow$ $\boxed{I(x) \sim \frac{-1}{x} \frac{f(a)}{\phi'(a)} e^{x\phi(a)}}$

and if $\phi(b) > \phi(a)$ $\Rightarrow$ $\boxed{I(x) \sim \frac{e^{x\phi(b)}}{x} \frac{f(b)}{\phi'(b)}}$

provided $f(a)$, $f(b)$ not zero

<u>special cases</u>

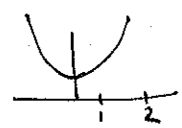
1) $\phi'(t) = 0$ in $[a, b]$

2) $f \neq 0$ at maximum of $\phi(t)$

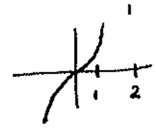
<u>Example</u>:

$$I(x) = \int_1^2 e^{x \cosh t}\, dt \quad , \; x \gg 1$$

In this case, $f(t) = 1$, $\phi(t) = \cosh t$



$\phi'(t) = \sinh t$



so $\phi'(t) \neq 0$ in $[1, 2]$

and $f \neq 0$ for all t

$\phi(t)$ is greater at $t = 2$, so

$$I(x) \sim \frac{1}{x} \frac{f(2)}{\phi'(2)} e^{x\phi(2)} \Rightarrow \boxed{I(x) \sim \frac{e^{x \cosh 2}}{x \sinh 2}}$$

<u>Special Cases</u>: $\phi'(t) = 0$ at an interior point $t = c \in (a, b)$

$\phi(c)$ is either a local min, local max, or an inflection point

• <u>$\phi(c)$ is a minimum</u>

$\phi''(c) > 0$

$\phi(a) \neq \phi(b)$

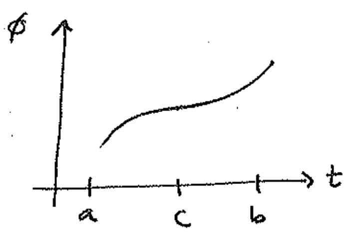
$\phi(c)$

so the main contribution to the integral is at an endpoint - use the above formulas

The exception occurs when $\phi(a) = \phi(b)$, so there will be a significant contribution at each endpoint

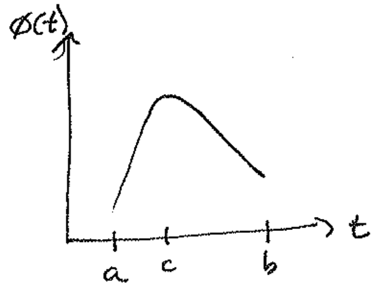
$$I(x) \sim \frac{1}{x}\left[\frac{f(b)}{\phi'(b)} e^{x\phi(b)} - \frac{f(a)}{\phi'(a)} e^{x\phi(a)}\right]$$

• <u>$\phi(c)$ is an inflection point</u>



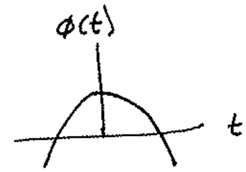
inflection point at $t = c$
still have a maximum at one endpoint
so use the above formulas

• <u>$\phi(c)$ is a maximum</u>



In this case, the idea is to expand $f(t)$ and $\phi(t)$ about $t = c$

Example: $\displaystyle\int_{-\infty}^{\infty} e^{-x\cosh t}\,dt$



$\phi(0)$ is a maximum

$f(t) = 1\ ,\quad \phi(t) = -\cosh t$

expand ~~the integral~~ $f(t)$ and $\phi(t)$ about the point $t=0$

$\Rightarrow$ $\cancel{f(t) = 1}$ $\cancel{\phi(t) =}$ $\quad f(t) \sim 1$

$\phi(t) \sim -\left(1 + \frac{t^2}{2} + \cdots\right)$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-x\cosh t}\,dt \sim \int_{-\infty}^{\infty} e^{-x\left(1 + t^2/2 + \cdots\right)}\,dt = e^{-x}\int_{-\infty}^{\infty} e^{-xt^2/2}\,dt$$

let $s^2 = \frac{xt^2}{2} \rightarrow s = \sqrt{\frac{x}{2}}\,t \qquad dt = \sqrt{\frac{2}{x}}\,ds$

$$\rightarrow \frac{\sqrt{2}}{\sqrt{x}}e^{-x}\int_{-\infty}^{\infty} e^{-s^2}\,ds \quad ,\ \text{given that} \int_{-\infty}^{\infty} e^{-s^2}\,ds = \sqrt{\pi}$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} e^{-x\cosh t}\,dt \sim \frac{e^{-x}}{\sqrt{x}}\sqrt{2\pi}}$$

Note: $\displaystyle\int_{-\infty}^{\infty} f(t)\,e^{-x\cosh t}\,dt \sim f(0)\,e^{-x}\sqrt{\frac{2\pi}{x}}$

because we would have expanded $f(t)$ about $t=0$

you can also conclude that

$$e^{x}\sqrt{\frac{x}{2\pi}}\,e^{-x\cosh t} = \delta(t) \quad \text{as} \quad x \to \infty$$

Fourier Integrals / Method of Stationary Phase

Fourier Integral: $I(x) = \int_a^b f(t)\, e^{ix\varphi(t)}\, dt$

where a, b, t, x, φ are all real
f may be complex
a and b may be infinite

Integrate by parts

$u = \frac{f(t)}{ix\varphi'}$ $\quad v = ix\varphi' e^{ix\varphi'}$

$$I(x) = \int_a^b f(t)\, e^{ix\varphi(t)} \cdot \frac{ix\varphi'(t)}{ix\varphi'(t)}\, dt$$

$$\boxed{I(x) \sim \frac{f(b)}{ix\varphi'(b)} e^{ix\varphi(b)} - \frac{f(a)}{ix\varphi'(a)} e^{ix\varphi(a)} + O\left(\frac{1}{x^2}\right)}$$

provided $\varphi' \neq 0$ in $[a,b]$

If $b \to \infty$ (or if $a \to -\infty$)
then $f(b) \to 0$ (or $f(a) \to \infty$)
for the integral to be defined
so set $f(b) = 0$ (or $f(a) = 0$) in the formula

What happens when $\varphi' = 0$ in $[a,b]$

$\Rightarrow$ Method of Stationary Phase

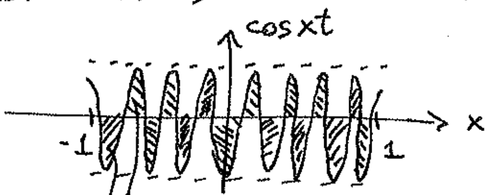
Example: $a = -1$, $b = 1$, $f(t) = 1$

$I(x) = \int_{-1}^{1} e^{ix\varphi(t)}\, dt \quad \Rightarrow \quad \mathrm{Re}[I(x)] = \int_{-1}^{1} \cos x\varphi(t)\, dt \quad , x \gg 1$

$$Re\left[I(x)\right] = \int_{-1}^{1} \cos x\varphi(t)\, dt \quad , \quad \underline{x >> 1}$$

<u>Case</u>: $\varphi(t) = t \rightarrow$ integrand is $\cos xt$

for $x >> 1$, high Frequency oscillation

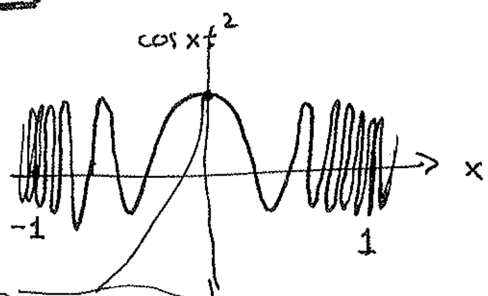


areas cancel $\Rightarrow$ all areas cancel except contributions at endpoints

$\Rightarrow$ the integral is determined near the endpoints

$\Rightarrow$ integration by parts works, because $\varphi' \neq 0$
$(\varphi = t \rightarrow \varphi' = 1)$

<u>**Case**</u>: $\varphi(t) = t^2 \rightarrow$ integrand is $\cos(xt^2)$



$t=0$ is a stationary point, because $\varphi'(0) = 0$

by <u>Riemann-Lebesgue Lemma</u>
the contributions away from the <u>stationary point</u> goes to zero as $x \rightarrow \infty$. (High frequency oscillations cancel out.)

<u>stationary points</u>: points where $\varphi'(t) = 0$. The dominant contributions to the integral occurs at these points.

<u>IF $c \in (a,b)$ is a stationary point</u> - then

$$\int_a^b f(t)\, e^{ix\varphi(t)}\, dt \sim \int_{c-\epsilon}^{c+\epsilon} f(t)\, e^{ix\varphi(t)}\, dt \quad , \quad x >> 1 \, , \text{ for fixed } \epsilon$$

Consider an interior stationary phase point $t = c \in (a,b)$. (then $\varphi'(c) = 0$)

- Expand f and φ about $t = c$
  to leading order, $f(t) \sim f(c)$
  and $\varphi(t) \sim \varphi(c) + \underbrace{\cancel{(t-c)\varphi'(c)}}_{\text{by definition of stationary point}} + \underbrace{\frac{(t-c)^2}{2}\varphi''(c)}_{\text{assume } \varphi''(c) \neq 0} + \cdots$

substitute into equation

$$I(x) \sim f(c)\, e^{ix\varphi(c)} \int_{-\infty}^{\infty} e^{i\frac{x}{2}(t-c)^2 \varphi''(c)}\, dt$$

by contour integration, find

$$\int_{-\infty}^{\infty} e^{i\alpha \xi^2}\, d\xi = \sqrt{\frac{\pi}{|\alpha|}}\, e^{i\pi/4\, \mathrm{sgn}(\alpha)}, \quad \mathrm{sgn}(\alpha) = \begin{cases} 1, & \alpha > 0 \\ -1, & \alpha < 0 \end{cases}$$

signum function

in our case, $\xi = t - c$ and $\alpha = \frac{x}{2}\varphi''(c)$

$$\Rightarrow \boxed{I(x) \sim \sqrt{\frac{2\pi}{x|\varphi''(c)|}}\, e^{ix\varphi(c)}\, e^{i\pi/4\, \mathrm{sgn}(\varphi''(c))} \quad \text{as } x \to \infty}$$

If $c = a$ (or b) then we obtain half this value,

$$I(x) \sim \sqrt{\frac{\pi}{x2|\varphi''(c)|}}\, e^{ix\varphi(c)}\, e^{i\pi/4\, \mathrm{sgn}(\varphi''(c))}$$

Multiple stationary parts $\Rightarrow$ add contribution from each

Consider an interior point $t=c$ where

$$\varphi'(c) = \varphi''(c) = \cdots = \varphi^{(p-1)}(c) = 0, \quad \varphi^{(p)}(c) = 0$$

the first nonzero derivative is the p^{th} derivative
still want to expand φ around c:

$$\varphi(t) \sim \varphi(c) + \frac{(t-c)^p \varphi^{(p)}(c)}{p!} + \cdots$$

and in this case, the solution becomes

$$I(x) \sim \frac{2\,\Gamma(1/p)}{p}\left[\frac{p!}{x\,|\varphi^{(p)}(c)|}\right]^{1/p} f(c)\, e^{ix\varphi(c)}\, e^{\frac{i\pi}{2p}\operatorname{sgn}(\varphi^{(p)}(c))}$$

Again, if c is an endpoint (a or b), divide by 2.

Example: Bessel function of order n

integral representation: $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin t - nt)\,dt$

$$\Rightarrow J_n(x) = \mathrm{Re}\left\{\frac{1}{\pi}\int_0^{\pi} e^{i(x\sin t - nt)}\,dt\right\} = \frac{1}{\pi}\mathrm{Re}\left\{\int_0^{\pi} e^{-int}\, e^{ix\sin t}\,dt\right\}$$

and so $f(t) = e^{-int}$, $\varphi(t) = \sin t$, $\varphi'(t) = \cos t$, $\varphi''(t) = -\sin t$

stationary point at $t = \pi/2$ $\quad \left\{ |\varphi''(\pi/2)| = |-1| = 1 \right.$

$$\Rightarrow I_n(x) = \frac{\mathrm{Re}}{\pi}\left\{\sqrt{\frac{2\pi}{x\,|\varphi''(c)|}}\, f(c)\, e^{ix\varphi(c)}\, e^{i\frac{\pi}{4}\operatorname{sgn}\varphi''(c)}\right\}$$

$$\Rightarrow \cancel{J_n(x) \sim \sqrt{\frac{2\pi}{x}}\, e^{-i}} \quad J_n(x) \sim \frac{1}{\pi}\mathrm{Re}\left\{\sqrt{\frac{2\pi}{x}}\, e^{-in\pi/2}\, e^{ix}\, e^{i\frac{\pi}{4}(-1)}\right\}$$

$$\Rightarrow \boxed{J_n(x) \sim \frac{1}{\pi}\sqrt{\frac{2\pi}{x}}\cos\left(x - n\frac{\pi}{2} - \frac{\pi}{4}\right) + \cdots, \quad x \gg 1}$$