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$x \in \mathbb{R}^n$

$U \subset \mathbb{R}^n$, U open, bounded

$U_T = \{(x,t) \mid x \in U, t \in (0,T)\}$

A - $n \times n$ matrix, symmetric

anistropy $A = \left(a^{ij}(x)\right)_{i,j=1,\dots,n}$

Wave Equation

$$\boxed{\begin{aligned} u_{tt} &= \nabla \cdot (A \nabla u) + f \\ u &= 0, \quad x \in \partial U, \quad 0 < t \leq T \\ u(x,0) &= g(x), \quad t = 0, \quad x \in U \\ u_t(x,0) &= h(x), \quad t = 0, \quad x \in U \end{aligned}}$$

Linear Elasticity

$\underset{\sim}{u}$ - displacement, $\underset{\sim}{u} = (u_1, u_2, u_3)$

notation $u_{i,j} = \dfrac{\partial u_i}{\partial x_j}$

$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ - symmetric matrix (or 2 tensor)

Suppose C is a 4-tensor (4 subindeces)

$C\varepsilon = \sigma \rightarrow$ stress

$$\boxed{\begin{aligned} \rho \underset{\sim}{u}_{tt} &= \nabla \cdot (C\varepsilon) + \underset{\sim}{f} \qquad (x,t) \in U_T \\ u &= 0, \quad x \in \partial U, \quad 0 < t \leq T \\ u(x,0) &= g(x), \quad x \in U, \quad t = 0 \\ u_t(x,0) &= h(x), \quad x \in U, \quad t = 0 \end{aligned}}$$

isotropic cases

for the wave eqn: $A = \sigma(x) I$

for linear elasticity

$$\rho \underset{\sim}{u}_{tt} = \nabla(\lambda \nabla \cdot \underset{\sim}{u}) + \nabla \cdot \left(\mu \left(\nabla \underset{\sim}{u} + (\nabla \underset{\sim}{u})^T \right) \right)$$

where λ, μ are Lamé parameters

add boundary and initial conditions

---

constant coefficient cases

and Isotropic case

without forcing

- the wave eqn becomes:

$$u_{tt} = \nabla \cdot (\sigma I \nabla u) = \sigma \Delta u$$

where speed of propagation of disturbances is $\sqrt{\sigma}$

- the linear elasticity becomes

notice

$$\nabla \underset{\sim}{u} + (\nabla \underset{\sim}{u})^T = \left((u_{i,j}) + (u_{j,i}) \right)_{i,j=1,\dots,3}$$

$$= \begin{pmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{pmatrix} + \begin{pmatrix} u_{1,1} & u_{2,1} & u_{3,1} \\ u_{1,2} & u_{2,2} & u_{3,2} \\ u_{1,3} & u_{2,3} & u_{3,3} \end{pmatrix}$$

Now take divergence of this

operate with $(\partial_{x_1}, \partial_{x_2}, \partial_{x_3}) \cdot$

$$\nabla \cdot \left(\nabla \underset{\sim}{u} + (\nabla \underset{\sim}{u})^T \right) = \nabla (\nabla \cdot \underset{\sim}{u}) + \Delta \underset{\sim}{u}$$

$$\Rightarrow \quad \rho \underset{\sim}{u}_{tt} = \lambda \nabla (\nabla \cdot \underset{\sim}{u}) + \mu \left[\nabla (\nabla \cdot \underset{\sim}{u}) + \Delta \underset{\sim}{u} \right]$$

note that $\nabla \cdot \underset{\sim}{u}$ is a volumetric change

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take divergence of this equation

$$\rho (\nabla\cdot u)_{tt} = \lambda \Delta(\nabla \cdot u) + \mu \Delta(\nabla\cdot u) + \mu\Delta(\nabla\cdot u)$$

$$\rightarrow \boxed{\rho(\nabla\cdot \underline{u})_{tt} = (\lambda + 2\mu)\Delta(\nabla\cdot\underline{u})}$$

wave equation of the divergence of $\underline{u}$

speed for volumetric change is $\sqrt{\frac{\lambda+2\mu}{\rho}}$

---

aside $\nabla \times \nabla\phi = \begin{vmatrix} i & j & k \\ \partial_{x_1} & \partial_{x_2} & \partial_{x_3} \\ \phi_{x_1} & \phi_{x_2} & \phi_{x_3} \end{vmatrix} = 0$

so that curl of the gradient vanishes

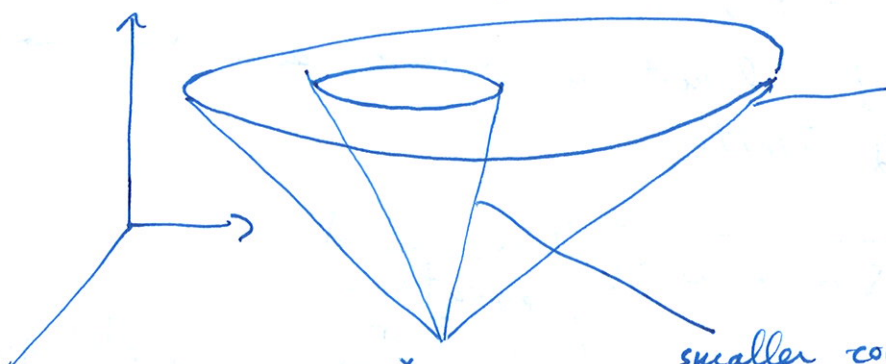
Instead of divergence, take curl of

$$\rho\, \underline{u}_{tt} = \lambda \nabla(\nabla\cdot\underline{u}) + \mu\left[\nabla(\nabla\cdot u) + \Delta u\right]$$

$$\boxed{\rho(\nabla\times\underline{u})_{tt} = \mu\Delta(\nabla\times u)}$$

wave eqn for the curl

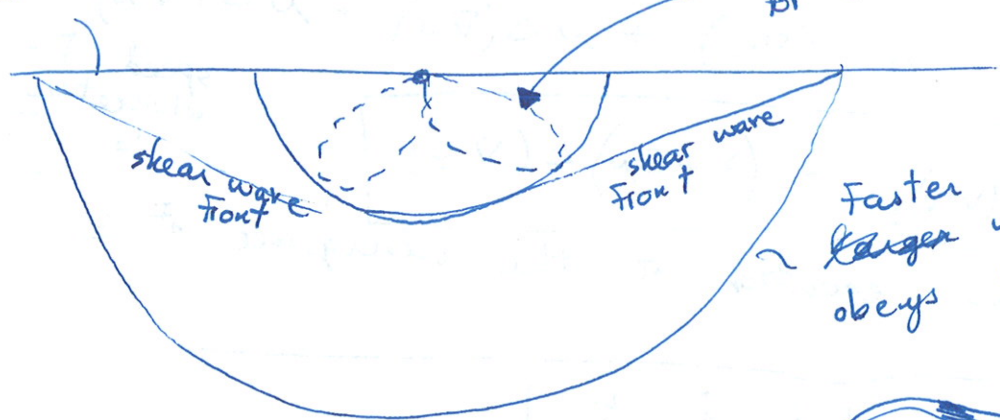
speed of disturbance is $\sqrt{\frac{\mu}{\rho}}$



x_0

bigger cone corresponds to volumetric change with radious $t_0\sqrt{\frac{\lambda+2\mu}{\rho}}$

smaller cone corresponds to shear wave with radious $t_0\sqrt{\frac{\mu}{\rho}}$



Free surface condition: normal derivative is zero

traction is:

$$\left[\lambda I \nabla\cdot\underline{u} + 2\mu\,\epsilon\right]\cdot\underline{n}$$

<u>incompressible case</u> - no volumetric change

$\nabla\cdot\underline{u} = 0$

define $\lambda \nabla u = p$, pressure

Now look at limit as $\lambda \to \infty$ and $\nabla\cdot\underline{u} \to 0$ with μ held finite

$$\Rightarrow \boxed{\begin{array}{c} \rho \underline{u}_{tt} = \nabla p + \mu \Delta \underline{u} \\ \nabla\cdot u = 0 \end{array}}$$

system of equations in four unknowns

take divergence, obtain: $\Delta p = 0$ (elliptic equation)

take curl, obtain: $\rho(\nabla\times u)_{tt} = \mu\Delta(\nabla\times u)$

(remember $\nabla\times\nabla\phi = 0$)

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Electromagnetic Equations (constant coeff - case)

electric Field $\underline{E} = \underline{E}(x,t)$

magnetic Field $\underline{B} = \underline{B}(x,t)$

$$\underline{E}_t = \frac{1}{\varepsilon_0} \nabla \times \underline{B}$$

$$\underline{B}_t = \frac{-1}{\mu_0} \nabla \times \underline{E}$$

where ε_0 - electric permeability
μ_0 - magnetic permeability

$$\left.\begin{aligned} \nabla \cdot \underline{E} &= 0 \\ \nabla \cdot \underline{B} &= 0 \end{aligned}\right\} \text{ for constant coeff. case}$$

$$\Rightarrow \quad \underline{E}_{tt} = \frac{1}{\varepsilon_0} \nabla \times \underline{B}_t = \frac{-1}{\varepsilon_0 \mu_0} \nabla \times \nabla \times \underline{E}$$

what is curl of curl?

$$\begin{vmatrix} i & j & k \\ \partial x_1 & \partial x_2 & \partial x_3 \\ E_{3,x_2} - E_{2,x_3} & E_{1,x_3} - E_{3,x_1} & E_{2,x_1} - E_{1,x_2} \end{vmatrix} = \underline{i}\left[\left(E_{2,x_1} - E_{1,x_2}\right)_{x_2} - \left(E_{1,x_3} - E_{3,x_1}\right)_{x_3}\right] + [\cdot]j + [\;]\underline{k}$$

$$= \underbrace{-E_{1,x_2x_2} - E_{1,x_3x_3} - E_{1,x_1x_1}}_{-\Delta E_1} + \cancel{E_{1,x_1x_1}}$$
$$+ \underbrace{E_{1,x_1x_1} + E_{2,x_2x_2} + E_{3,x_3x_3}}_{\cancel{(\nabla \cdot \underline{E})_{x_1}} \to 0}$$

$$\Rightarrow \quad \boxed{\underline{E}_{tt} = \frac{1}{\mu_0 \varepsilon_0} \Delta \underline{E}}$$

for $\underline{B}$, we have

$$\underline{B}_{tt} = -\frac{1}{\mu_0} \nabla \times \left(\frac{1}{\varepsilon_0} \nabla \times \underline{B}\right) \Rightarrow \boxed{\underline{B}_{tt} = \frac{+1}{\mu_0 \varepsilon_0} \Delta \underline{B}}$$

the electromagnetic equns have one wave speed, so obtain one cone.

. Hodge decomposition

$\underline{u} = \nabla \phi + \nabla \times U$ see Chorin & Marsden

Weak Solutions

$$-\nabla \cdot (A \nabla u) = f , \quad x \in U \qquad (\text{time-independent})$$

$$u = 0 , \quad x \in \partial U$$

assumptions on A

A - symmetric, positive definite

$A = \left(a^{ij}(x) \right)$ and each $|a^{ij}(x)| \le C \quad \forall x \in U$

positive definite:

$$(\xi, A\xi) = \xi^T A \xi \geq \theta |\xi|^2 , \quad \xi \in \mathbb{R}^n, \ \theta > 0$$

consider any function $v \in C_0^\infty(U)$

Multiply equation by v

$$-v \nabla \cdot (A \nabla u) = v f$$

$$\int_U -\nabla \cdot [v (A \nabla u)] + \int_U (\nabla u)^T A \nabla u = \int_U f v$$

$$\downarrow$$

$$\int_{\partial U} (v A \nabla u) \cdot n = 0$$ because v has compact support

$$\Rightarrow \int_U (\nabla u)^T A \nabla u = \int_U f v$$

def u is a weak solution if u satisfies

$$\int_U (Dv)^T A\, Du = \int_U fv$$

For v satisfying what condition ???

if we pick $v \in C_0^\infty(U)$, this is too stringent

we could pick H^1 norm

$$\|u\|_{H^1(U)} = \left[\int_U \left[|Du|^2 + u^2\right]\right]^{1/2}$$

<u>wave eqn</u> (scalar case)

$U \subset \mathbb{R}^n$, $n = 2, 3, \ldots$, U open bounded

$U_T = \{(x,t) \mid x \in U,\ 0 < t \le T\}$

$$\boxed{u_{tt} = \nabla\cdot(A\nabla u) + F \qquad \begin{array}{l} u = 0,\ x \in \partial U,\ 0 < t \le T \\ u(x,0) = g,\ x \in U,\ t = 0 \\ u_t(x,0) = h,\ x \in U,\ t = 0 \end{array}}$$

assumptions on A:

A is $n \times n$ symmetric matrix with bounded coefficients

$A = \left(a^{ij}(x)\right)_{i,j=1}^{N}$ $\exists\, C > 0 \ni |a^{ij}| \le C$, $i,j = 1,\ldots,n$

} the matrix A could be very rough

and assume uniform positive definite

$$\xi^T A \xi \ge \theta|\xi|^2, \quad \theta > 0 \quad \forall \xi \in \mathbb{R}^n$$

Thinking formally, consider a smooth function v that vanishes on boundary

$v \in C_0^\infty(U)$, $\operatorname{supp} v = \overline{\{x \in U \mid v \ne 0\}} \subset U$

the formal part of the procedure is the following integration by parts, since we have not spoken of differentiability

$$\int_U v u_{tt} - \int_U v \nabla\cdot(A\nabla u) = \int_U vF$$

note $\nabla\cdot(vA\nabla u) = v\nabla\cdot(A\nabla u) + \nabla v A \nabla u$

apply divergence theorem:

$$\int_U v u_{tt} - \underbrace{\int_U \nabla\cdot(vA\nabla u)}_{\text{vanishes since } v \text{ has compact support}} + \int_U \nabla v A \nabla u = \int_U vF$$

} accomplished goal of avoiding divergence of A, since A could be very rough

$$\int_{\mathcal{V}} v u_{tt}\, dx + \int \nabla v A \nabla u \, dx = \int v f \, dx$$ WEAK FORMULATION

this term will motivate choice of method of measuring size, i.e., picking a norm

here is a possible choice:

$$\left\{ \int_{\mathcal{V}} u^2 + |\nabla u|^2 \right\}^{1/2} = \|u\|_{H^1(\mathcal{V})}$$ "H-one norm"

Consider the sequence $\{v_k\}_{k=1}^{\infty} \subset C_0^{\infty}(\mathcal{V})$. This is a Cauchy Sequence iff given $\varepsilon > 0$ $\exists N > 0$ with $\|v^k - v^l\|_{H^1(\mathcal{V})} < \varepsilon$ when $k, l > N$.

<u>definition</u>: $H_0^1(\mathcal{V})$ is the set of all (limits of) Cauchy sequence wrt $\|\cdot\|_{H^1(\mathcal{V})}$

we know the following terms have bounded norm:

$\int_{\mathcal{V}} |\nabla v|^2 < \infty$, $\int_{\mathcal{V}} v^2\, dx < \infty$, $\int_{\mathcal{V}} |\nabla u|^2 < \infty$, $\int |u|^2 < \infty$

The Poincaré Inequality tells us that if $u \in H_0^1(\mathcal{V})$ then $c_1 \int_{\mathcal{V}} u^2 \le \int_{\mathcal{V}} |\nabla u|^2$, $c_1 > 0$

this tells us that if $\nabla u = 0$ then since $u = 0$ on $\partial\mathcal{V}$, (the boundary) then $u = 0$ everywhere. Consequently, we can change definition of our norm

by the properties of norms: (triangle inequality)

$$\int |\nabla u|^2\, dx \le \int u^2 + |\nabla u|^2 \le \left(1 + \frac{1}{c_1}\right) \int |\nabla u|^2$$

This means that $\|\cdot\|_{H_0^1(\mho)} = \left\{\int_\mho |\nabla u|^2 dx\right\}^{1/2}$

is equivalent to $\|\cdot\|_{H^1(\mho)}$ for $H_0^1(\mho)$

The whole point of discussing size measure is to discuss convergence

$$\int_\mho |\nabla(v^k - v^\ell)|^2 \le \int_\mho |v^k - v^\ell|^2 + |\nabla(v^k - v^\ell)|^2 \le \left(1 + \frac{1}{c_1}\right)\int_\mho |\nabla(v^k - v^\ell)|^2$$

Let us turn attention to a few assumptions on g, h, the initial conditions

$g \in H_0^1(\mho)$ - need

$h \in L^2(\mho) \rightarrow$ we allow h to be a lot rougher

The method for showing we have a solution:

Suppose we have a set of $\{w_k\}_{k=1}^\infty \subset H_0^1(\mho)$, orthonormal with respect to $L^2(\Omega)$, $\int_\mho w_k w_\ell \, dx = \begin{cases} 0, & k \ne \ell \\ 1, & k = \ell \end{cases}$

Let us create an approximation to the solution u

$$u^N = \sum_{k=1}^N d_k^N(t)\, w_k(x)$$

We also assume that we can approximate f (the forcing function) in this ~~matter~~ manner

$$f^N = \sum_{k=1}^N f_k(t)\, w_k(x)$$

$$\int_{\mho} v u_{tt}\, dx + \int_{\mho} \nabla v A \nabla u \, dx = \int_{\mho} v F \, dx$$ ← u satisfies this formula for all $v \in H_0^1(\Omega)$

substitute expansions

$$\sum_{k=1}^{N} d_{k,tt}^{N}(t) \int_{\mho} v\, w_k \, dx + \sum_{k=1}^{N} d_k^N(t) \int_{\mho} \nabla v \, A \, \nabla w_k \, dx = \sum_{k=1}^{N} f_k(t) \int_{\mho} w_k(x) v(x)\, dx$$

Now replace v by $w_\ell(x)$, so that we obtain

$$d_{\ell,tt}^N + \sum_{k=1}^{N} d_k^N(t)\, e_{\ell k}^N = f_\ell(t)$$ ⇐ system of ODEs 2^{nd} order

what are the initial conditions? $d_\ell(0) = ?$, $d_{\ell,t}(0) = ?$

Expand original initial conditions g, h !!

$$g^N(x) = \sum_{k=1}^{N} g_k(0) w_k(x) = \sum_{k=1}^{N} g_k w_k$$

$$h^N(x) = \sum_{k=1}^{N} d_{k,t}(0)\, w_k(x) = \sum_{k=1}^{N} h_k w_k$$

} NOTE, these are approximations (finite) to g and h

so that we have

$$d_\ell(0) = g_\ell, \quad \ell = 1, \dots, N$$

$$d_\ell'(0) = h_\ell$$

this has a C^2 solution if $f_\ell(t)$, $\ell = 1, \dots, N$ are continuous

But now we want to show that $u^N \to u$, in $H_0^1(\Omega)$.

<u>So we in fact pick:</u>

$$F \in L^2(0,T; L^2(\mho))$$ (fixed time)

mapping $F : [0,T] \to L^2(\mho)$

$$\| F \|_{L^2(0,T; L^2(\mho))} = \left[\int_0^T \left(\| F(t,\cdot) \|_{L^2(\mho)} \right)^2 \right]^{1/2}$$

similar notation

$v: [0,T] \to H_0^1(\mho)$

$$\|v\|_{L^2(0,T;H_0^1(\mho))} = \left[\int_0^T \|v(t,\cdot)\|^2_{H_0^1(\mho)} dt\right]^{1/2}$$

$v \in L^2(0,T;H_0^1(\mho))$

and similarly

$v_t : [0,T] \to L^2(\mho)$, $v_t \in L^2(0,T;L^2(\mho))$

$$\|v_t\|_{L^2(0,T;L^2(\mho))} = \left[\int_0^T \|v_t(t,\cdot)\|^2_{L^2(\mho)} dt\right]^{1/2}$$

the range of map gets rougher: $H_0^1(\mho)$ then $L^2(\mho)$ what is next space?

and now define v_{tt}

The question is, where does v_{tt} live?

go back and consider v_{tt} fixed and v changing

Think of the integral $\int v v_{tt}$ as an operator,

$\Lambda v = \int_\mho v v_{tt}$ so that $\Lambda : H_0^1(\mho) \to \mathbb{R}$

this is a linear operator, $\Lambda(av) = a\Lambda v$

in fact this is a linear functional

But what we want is $|\Lambda v| \le c_2 \|v\|_{H_0^1(\mho)}$, $c_2 > 0$ i.e.

~~there~~ whate we have is a bounded linear functional.

what are the functions v that satisfy this? There is a

space of functions that satisfies this, called $H^{-1}(\mho)$.

Def: $H^{-1}(\mho)$ is the set of operators $\Lambda : H_0^1(\mho) \to \mathbb{R}$

and satisfies $\|\Lambda v\| \le c_2\|v\|_{H_0^1(\mho)}$ for some $c_2 > 0$.

By convention, we represent Λ as

$$\Lambda v = \int_\mho m(x)\, v(x)\, dx , \quad m \in H^{-1}(\mho)$$

we need a notion of size.

$$\|\Lambda\|_{H^{-1}} = \|m\|_{H^{-1}} = \sup_{\substack{v \in H_0(\mho) \\ \|v\|_{H_0^1(\mho)} \le 1}} |\Lambda v|$$

In conclusion, for v_{TT}

$v_{tt} : (0,T] \to H^{-1}(\mho)$

$v_{tt} \in L^2(0,T;H^{-1}(\mho))$

$$\|v_{tt}\|_{L^2[0,T;H^{-1}(\mho)]} = \left[\int_0^T \|v_{tt}(t,\cdot)\|^2_{H^{-1}(\mho)} dt\right]^{1/2}$$

we're going to need energy estimates to show convergence ...

<u>Energy Estimates</u>

$$\max_{0 \le t \le T} \left(\| u^N \|_{H_0^1(\mho)} + \| u_t^N \|_{L^2(\mho)} \right) + \cancel{\| u_{tt}^N \|} + \| u_{tt}^N \|_{L^2(0,T;H^{-1}(\mho))}$$

$$\le C_4 \left[\| f^N \|_{L^2(0,T;L^2(\mho))} + \| g^N \|_{H_0^1(\mho)} + \| h^N \|_{L^2(\mho)} \right]$$

we replace each term by the difference of two terms in a sequence and we'll show that the right hand side vanishes

hmwk for 1/

$\boxed{\text{prove triangle inequality in } H_0^1}$

<u>outline</u>

$u = u(x,t)$, $\mho \in R^n$

(*)
$$\boxed{\begin{array}{ll} u_{tt} = \nabla\cdot(A\nabla u) + f , & \\ u(x,t) = 0 , & x \in \partial\mho , 0 \le t \le T \\ u(x,0) = g & x \in \mho , t = 0 \\ u_t(x,0) = h , & x \in \mho , t = 0 \end{array}}$$

where A is an $n\times n$ symmetric matrix satisfying
$$\Theta_1 |\xi|^2 \ge \xi^T A \xi \ge \Theta|\xi|^2$$
$$\Theta, \Theta_1 > 0 , \quad \xi \in R^n$$

- we defined some spaces

  $H_0^1(\mho)$ – completion of $C_0^\infty(\mho)$

  wrt $\left[\int_\mho |\nabla u|^2\right]^{1/2} = \|u\|_{H_0^1(\mho)}$

- as well, we have

  $$L^2(0,T;H_0^1(\mho)) = \{ u(x,t) \mid u(\cdot,t) \in H_0^1(\mho) \}$$

  $$\|u\|_{L^2(0,T;H_0^1(\mho))} = \left[\int_0^T \|u\|^2_{H_0^1(\mho)} dt\right]^{1/2} < \infty$$

- then we have the mapping

  $u(t,\cdot) : [0,T] \longrightarrow H_0^1(\mho)$

  $f \in L^2(0,T;L^2(\mho))$

  $g \in H_0^1(\mho)$

  $h \in L^2(\mho)$

  and we have

  $u_t(t,\cdot) : [0,T] \rightarrow L^2(\mho)$

  $u_t \in L^2(0,T;L^2(\mho))$

$$u_{tt} \in L^2(0,T;\underbrace{H^{-1}(\mho)}_{\text{set of bounded linear functionals defined on } H_0^1(\mho)})$$

$$\|u_{tt}\|_{L^2(0,T;H^{-1}(\mho))} = \left[\int_0^T \left\{\|\Lambda^{u_{tt}}(t)\|_{H^{-1}(\mho)}\right\}^2 dt\right]^{1/2}$$

where $\Lambda^{v} v = \int_{\mho} F v \, dx \,, \quad v \in H_0(\mho)$

and $\Lambda^{v} : H_0^1(\mho) \to \mathbb{R}$

linear, bounded

$$\|\Lambda\| = \sup_{\substack{w \in H_0^1(\mho) \\ \|w\|_{H_0^1(\mho)} \le 1}} \{|\Lambda w|\}$$

This gives us all the information we need to define a weak solution

<u>Definition</u> Suppose A, g, h, f satisfy our conditions then u is a weak solution of eq $(*)$ iff $u \in L^2(0,T;H_0^1(\mho))$, $u_t \in L^2(0,T;L^2(\mho))$, $u(x,0) = g$, $u_t(x,0) = h$, $u_{tt} \in L^2(0,T,H^{-1}(\mho))$ and

$$\underbrace{\langle u_{tt}, v\rangle}_{\Lambda^{u_{tt}}(v)} + \int_{\mho} \nabla u \cdot A \nabla v = \int_{\mho} f v \, dx \quad \forall\, v \in H_0^1(\Omega)$$

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Let us look at some Finite dimensional approximations.

$$\{w_k\}_{k=1}^{\infty} \subset H_0^1(\mho) , \quad \int_{\mho} w_k w_l = \delta_{kl}$$

$$u^N(x,t) = \sum_{k=1}^{N} d_k^N(t)\, w_k(x)$$

substitute into weak formulation to find that $\{d_k^N\}_{k=1}^{\infty}$ satisfies a system of N o.d.e. with initial conditions $d_k^N(0) = \int_{\mho} g(x)\, w_k(x)\, dx$ and $d^N_{k,t}(0) = \int_{\mho} h(x)\, w_k(x)\, dx$. The solution exists, it is a C^2 solution.

We want to show that the u^N converge to a weak solution u. To show this we need an energy inequality.

~~Theorem Suppose g satisfies the criteria for a weak solution. Then ∃ c > 0~~

~~$\max_{0<t<T}$~~

Theorem Suppose u^N is our approximate solution, $N = 1, 2, \ldots$ Then $\exists\ c > 0$

$$\max_{0 \le t \le T} \left(\|u^N(t,\cdot)\|^2_{H_0^1(\mho)} + \|u_t^N\|^2_{L^2(\mho)} \right) + \|u_{tt}^N\|^2_{L^2(0,T,H_0^1(\mho))}$$

$$\le C \left[\|g\|^2_{H_0^1(\mho)} + \|h\|^2_{L^2(\mho)} + \|F\|^2_{L^2(0,T;L(\mho))} \right]$$

The standard way to derive energy estimates is to multiply $u_{tt} = \nabla\cdot(A\nabla u) + F$ by u_t. But we are dealing with a weak formulation, such that we have

$$\langle u^N_{tt}, v\rangle + \int_{\mho} \nabla u^N A \nabla v = \int_{\mho} f v \quad , \quad v \in \operatorname{span}\{w_k\}$$

this equation is parametrized by t, for instance, last integral is $\int_{\mho} f(x,t)\, v(x)\, dx$

Remember, $u^N = \sum_{k=1}^{N} d_k^N(t)\, w_k(x)$, $u^N_t(x,t) = \sum_{k=1}^{N} d_{k,t}^N(t)\, w_k(x)$

and $\langle u^N_{tt}, v\rangle = \int_{\mho} u^N_{tt}\, v\, dx$

So we replace v with u^N_t to obtain

$$\int_{\mho} u^N_{tt}\, u^N_t\, dx + \int_{\mho} \nabla u^N A \nabla u^N_t = \int_{\mho} f(x,t)\, u^N_t(x)$$

$$\Rightarrow \frac{d}{dt}\frac{1}{2}\int_{\mho} (u^N_t)^2\, dx + \frac{d}{dt}\frac{1}{2}\int_{\mho} (\nabla u^N)^T A \nabla u^N = \int_{\mho} f(x,t)\, u^N_t(x)\, dx$$

integrate this entire expression from 0 to t

$$\int_0^t \left[\frac{d}{dt}\frac{1}{2}\int_{\mho} (u^N_t)^2\, dx + \frac{d}{dt}\frac{1}{2}\int (\nabla u^N)^T A \nabla u^N\right] dt = \int_0^t \left[\int_{\mho} f\, u^N_t\, dx\right] dt$$

$$\frac{1}{2}\left[\int_{\Omega}\left(u_t^N(x,t)\right)^2 dx + \int_{\Omega}\left(\nabla u^N(x,t)\right)^T A(x)\nabla u^N(x,t)\, dx\right]$$

$$= \frac{1}{2}\left[\int_{\Omega} u_t^N(x,0)^2\, dx + \int_{\Omega}\nabla u^N(x,0)\, A\, \nabla u^N(x,0)\, dx + \int_0^t\int_{\Omega} F(x,t)\, u_t^N(x,t)\, dx\, dt\right.$$

using boundedness assumption on A

$\le \|h\|_{L^2}^2$

$\le \Theta_1 \int_{\Omega} |\nabla u^N|^2 dx$ (at $t=0$) $= \Theta_1 \|u^N(x,0)\|_{H_0^1(\Omega)}$

where Θ_1 corresponds to largest eigenvalue of A

Aside

If $g^N(x) = \sum_{k=1}^{N} d_k(0)\, w_k(x)$

then $\|g^N\|_{H_0^1(\Omega)} \le \|g\|_{H_0^1(\Omega)}$

since $\int_{\Omega} g^{N2}(x)\, dx = \int \sum_{k=1}^{N} g_k^N w_k \sum_{l=1}^{N} g_l^N w_l = \sum_{k=1}^{N} g_k^2$

Note, the L^2 inner product for 2 functions v, w in $L^2(\Omega)$

$\int_{\Omega} vw\, dx$, $v, w \in L^2(\Omega)$

the Cauchy-Schwartz inequality

note $(a \pm b)^2 \ge 0 \rightarrow a^2 \pm 2ab + b^2 \ge 0$

$a^2 + b^2 \ge \mp 2ab$

so this last integral

$$\int_0^T \int_\mho F(x,t)\, u_t(x,t)\, dx\, dt$$

consider integrand - like L^2 norm

$$\langle f, u_t^N \rangle_{L^2(\mho)} \le \| f \|_{L^2(\mho)} \| u_t^N \|_{L^2(\mho)} \le \frac{1}{2}\left[\| f \|^2_{L^2(\mho)} + \| u_t^N \|^2_{L^2(\mho)} \right]$$

Back up $\rightarrow$ complicating life, start over

$$\frac{d}{dt}\frac{1}{2}\int_\mho (u_t^N)^2\, dx + \frac{d}{dt}\frac{1}{2}\int_\mho (Du^N)^T A\, Du^N\, dx = \int_\mho f(x,t)\, u_t^N(x)\, dx$$

$$\Rightarrow \frac{d}{dt}\left[\frac{1}{2}\int_\mho (u_t^N)^2\, dx + \int_\mho (Du^N)^T A(x) Du^N\, dx \le \frac{1}{2}\left[\| f \|^2_{L^2(\mho)} + \| u_t^N \|^2_{L^2(\mho)} \right] + \int_\mho (Du^N)^T A\, Du^N\, dx\right.$$

just add to the right hand of inequality

Let $\eta(t) = \int_\mho (u_t^N)^2 dx + \int_\mho (Du^N)^T A(x) Du^N\, dx$

$$\Rightarrow \frac{d}{dt}\eta(t) \le F(t) + \eta(t), \quad \text{where } F(t) = \| f \|^2_{L^2(\mho)}$$

and $\eta(0) = \int_\mho u_t^N(x,0)^2\, dx + \int_\mho (Du^N(x,0))^T A(x)\, Du^N(x,0)\, dx$

Now we want to use Gronwall's inequality

$$\frac{d}{dt}\eta(t) - \eta(t) \le F(t)$$

solve using integrating factor, which is e^{-t}

$$\rightarrow \frac{d}{dt}\left[e^{-t} n(t)\right] \leq e^{-t} F(t)$$

integrate from 0 to s to obtain

$$e^{-s} n(s) - n(0) \leq \int_0^s F(t)\, e^{-t}\, dt \leq \int_0^s F(t)\, dt$$

because e^{-t} bounded above by 1, for $t > 0$

So what we have is an upper bound on $n(s)$

$$n(s) \leq e^{s}\left[n(0) + \int_0^s F(t)\, dt\right]$$

this is the exponent e^s

$$\Rightarrow \int_U (u^N_t(x,t))^2\, dx + \int_U (\nabla u^N)^T A(x)\, \nabla u^N\, dx \leq C\left[\|h\|^2_{L^2(U)} + \|g\|^2_{H^1_0(U)} + \|f\|^2_{L^2(0,T;L^2(U))}\right]$$

remember, $A = A(x)$ and

$$\theta_1 |\xi|^2 \geq \xi^T A \xi \geq \theta |\xi|^2, \quad \theta_1, \theta > 0, \quad \xi \in \mathbb{R}^n$$

so that from previous result

$$\boxed{\begin{aligned} &\int_U (u^N_t)^2\, dx + \theta \int_U |\nabla u^N(x,t)|^2\, dx \leq \\ &\int_U (u^N_t(x,t))^2\, dx + \int_U (\nabla u^N)^T A(x)\, \nabla u^N\, dx \leq \\ &C\left[\|h\|^2_{L^2(U)} + \|g\|^2_{H^1_0(U)} + \|f\|^2_{L^2(0,T;L^2(U))}\right]\end{aligned}}$$

we're almost done - missing $\|u_{tt}\|$ term

since this is for every time, we can take a max for left-hand inequality

Let us turn our attention to ~~the~~ weak form

$$\int_{\mho} u_{tt}^N v = -\int_{\mho} (\nabla u^N)^T A \nabla v + \int_{\mho} F v \, dx, \quad v \in \text{span}\{w_k\}_{k=1}^N$$

Look at $\nabla u^N A \nabla v$ term. what do we know?

$$|(\nabla u^N)^T A \nabla v| \le |\nabla u^N| \, |A \nabla v| \le \begin{pmatrix}\text{max eigenvalue}\\ \text{of } A\end{pmatrix} \underbrace{|\nabla u^N| \, |\nabla v|}_{\text{just the product of two numbers, as written down previously}}$$

$$\to |(\nabla u^N)^T A \nabla v| \le \underbrace{\frac{1}{2}\begin{pmatrix}\text{max eigenvalue}\\ \text{of } A\end{pmatrix}}_{\text{this is some constant}} \left[|\nabla u^N|^2 + |\nabla v|^2 \right]$$

$$\Rightarrow \left| \int_{\mho} u_{tt}^N v \, dx \right| \le C \left[\|u^N\|_{H_0^1(\mho)} + \|F\|_{L^2(\mho)} \right] \|v\|_{H_0^1(\mho)}$$

since $\int_{\mho} F v \, dx \le \|F\|_{L^2} \|v\|_{L^2} \le \phi \|F\|_{L^2} \|v\|_{H_0^1(\mho)}$

divide by $\|v\|_{H_0^1(\mho)}$ to obtain

$$\underbrace{\sup_v \frac{\left| \int_{\mho} u_{tt}^N v \, dx \right|}{\|v\|_{H_0^1(\mho)}}}_{\|u_{tt}^N\|_{H^{-1}(\mho)}} \le C \left(\|u^N\|_{H_0^1(\mho)} + \|F\|_{L^2(\mho)} \right)$$

Now square both sides and then integrate

$$\int_0^T \|u_{tt}^N\|_{H^{-1}(\mho)} \, dt \le C \int_0^T \left[\|u^N\|_{H_0^1(\mho)}^2 + \underbrace{2\|u^N\|_{H_0^1(\mho)} \|F\|_{L^2(\mho)}}_{\text{replace this product by sum of squares}} + \|F\|_{L^2(\mho)}^2 \right] dt$$

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$$\int_0^T \| u_{tt}^N \|_{H^{-1}(U)} dt \leq 2C \int_0^T \left(\| u^N \|^2_{H_0^1(U)} + \| f \|^2_{L^2(U)} \right) dt$$

the last term is perfect, i.e.,

$$\int_0^T \| f \|^2_{L^2(U)} dt = \| f \|_{L^2(0,T;L^2(U))}$$

but the second to last term isn't, yet...

From previous estimates

$$\rightarrow \int_0^T \| u_{tt}^N \|_{H^{-1}(U)} dt \leq \left[\| h \|^2_{L^2(U)} + \| g \|^2_{H_0^1(U)} + \| f \|_{L^2(0,T,L^2(U))} \right] C$$

$$+ 2C \| f \|_{L^2(0,T;L^2(U))}$$

From current estimate

So we have the inequality

$$\|u^N_{tt}\|^2_{L^2(0,T,H^{-1}(\mho))} + \max_{0\le t\le T}\left[\|u^N\|^2_{H^1_0(\mho)} + \|u^N_t\|^2_{L^2(\mho)}\right]$$

inequality

$$\le C\left[\|g\|^2_{H^1_0(\mho)} + \|h\|^2_{L^2(\mho)} + \|F\|^2_{L^2(0,T,L^2(\mho))}\right]$$

Note - in the text, the same expression has each term NOT squared - the expressions can be made equivalent by employing inequalities

$$2ab \le a^2+b^2$$

and

$$a^2+b^2 = \tfrac{1}{2}(a^2+b^2) + \tfrac{1}{2}(a^2+b^2) \ge \tfrac{1}{2}(a^2+b^2) + ab = \tfrac{1}{2}(a+b)^2$$

rem. remember - multiply $v_{tt} = \nabla\cdot(A\nabla v)$ by v_t to obtain

$$\frac{1}{2}\frac{d}{dt}(v_t^2) = \nabla\cdot(v_t A\nabla v) - (\nabla v_t)^T A\nabla v$$

but there is no source term

Now integrate, etc....

We defined weak solution: u is a weak solution if

$$\langle u_{tt}, v\rangle + \int_{\mho}(\nabla v)^T A\nabla u = \int_{\mho} vF \qquad \forall v\in H^1_0(\mho)$$

and all other conditions are met.

We want to show the existance of a unique solution. Not going to show uniqueness - see text book.

what the inequality tells us is that

$\|u_{tt}\|_{L^2(0,T,H^{-1}(\Omega))}$, $\|u_t\|_{L^2(0,T,L^2(\Omega))}$, $\|u\|_{L^2(0,T,H_0^1(\Omega))}$ are Bounded

we want to use this fact to say that we have a convergent subsequence. $\{u^{N_j}\}_{j=1}^{\infty}$ for

$$u^N = \sum_{k=1}^{N} d_k^N(t)\, w_k(x) \quad , \quad \{w_k\}_{k=1}^{\infty} \text{ orthonormal}$$

$\boxed{\text{WEAK CONVERGENCE}}$

<u>Definition</u> "<u>converges weakly</u>"

$u^{N_j} \longrightarrow u$ in $L^2(0,T,H_0^1(\Omega))$ if

$$\int_0^T \int_\Omega \nabla v \cdot (\nabla u^{N_j} - \nabla u)\, dx\,dt \to 0 \quad \forall\, v \in L^2(0,T;H_0^1(\Omega))$$

similarly, for our purposes

$u_t^{N_j} \longrightarrow u_t$ in $L^2(0,T;L^2(\Omega))$ if

$$\int_0^T \int_\Omega \nabla v \cdot (\nabla u_t^{N_j} - \nabla u_t)\, dx\,dt \to 0 \quad \forall\, v \in L^2(0,T;L^2(\Omega))$$

and similarly

$u_{tt}^{N_j} \longrightarrow u_{tt}$ in $L^2(0,T;H^{-1}(\Omega))$ if

$$\int_0^T \langle u_{tt}^{N_j} - u_{tt}, v\rangle\, dt \to 0 \quad \forall\, v \in L^2(0,T;H_0^1(\Omega))$$

Typical example concerning weak convergence makes use of cosine functions, for example

$$\{\cos nx\}_{n=1}^{\infty}, \; -\pi < x < \pi \;,\; \int_{-\pi}^{\pi} f \cdot \cos(nx)\, dx \longrightarrow 0 \quad \forall\, f \in L^2(-\pi,\pi)$$

$$\cos(nx) \longrightarrow 0 \quad \text{as } n \to \infty$$

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Theorem (From functional analysis)

Every bounded sequence has a weakly convergent subsequence

Now we want to show we have a weak solution

$$\langle u_{tt}^{N_j}, v\rangle + \int_{\Omega} (\nabla u^{N_j})^T A \nabla v = \int_{\Omega} f v$$

where $v \in H_0^1(\Omega)$, but really $v \in \text{span}\{w_k\}_{k=1}^{N_j}$

Integrate from 0 to T and ~~...~~ take the limit as $N_j \to \infty$

$$\int_0^T \langle u_{tt}^{N_j}, v\rangle\, dt + \int_0^T \int_{\Omega} (\nabla u^{N_j})^T A \nabla v \, dx\, dt = \int_0^T \int_{\Omega} f v \, dx\, dt$$

~~this~~ which results in

$$\int_0^T \langle u_{tt}, v\rangle + \int_0^T \int_{\Omega} (\nabla u)^T A \nabla v = \int_0^T \int_{\Omega} f v \, dx\, dt$$

we want to allow v to have coefficients in time, $v = l(t) k(x)$, $k \in H_0^1(\Omega)$, $l \in C^1[0,T]$

~~$v \in L^2[0,T; H_0^1(\Omega)]$~~

$\boxed{\text{FACT}}$ $\int_0^T f g = 0$ for fixed f and $\forall g \in L^2(0,T) \Rightarrow f = 0$

we want to use that fact in our formula, which is

$$\int_0^T l(t) \langle u_{tt}, k(x)\rangle\, dt + \int_0^T l(t) \int_{\Omega} (\nabla u)^T A \nabla k \, dx\, dt = \int_0^T l \int_{\Omega} f k \, dx\, dt$$

$$\Rightarrow \int_0^T l(t) \left[\langle u_{tt}, k\rangle + \int_{\Omega} (\nabla u)^T A \nabla k \, dx - \int_{\Omega} f k \, dx\right] dt = 0$$

$\forall l \in C^1[0,T]$,

$\forall l \in L^2[0,T]$,

using our Fact, we Find

$$\langle u_{tt}, k(x) \rangle + \int_{\mho} (\nabla u)^T A \nabla k \, dx = \int_{\mho} k F \, dx \quad \forall \; k \in H_0^1(\mho)$$

Are we done? What about initial conditions?

We need to establish initial conditions:

$$\langle u_{tt}^{N_j}, v \rangle + \int_{\mho} \nabla v A \nabla u^{N_j} = \int_{\mho} F v \quad , \; v \in \text{span}\{w_k\}_{k=1}^{N_j}$$

and we specify $v \in C^2[0, T, H_0^1(\mho)]$

remember, $\langle u_{tt}^{N_j}, v \rangle = \int_{\mho} u_{tt}^{N_j} v \, dx$

integrate entire expression from 0 to T

$$\int_0^T \int_{\mho} u_{tt}^{N_j} v \, dx \, dt + \int_0^T \int_{\mho} (\nabla u^{N_j})^T A v \, dx \, dt = \int_0^T \int_{\mho} F v \, dx \, dt$$

going to use integration by parts on left hand term to re-introduce initial conditions, ~~specific~~ with $v(T) = 0$, $v_t(T) = 0$

$$\int_0^T u_{tt}^{N_j} v \, dt = u_t^{N_j} v \Big|_{t=0}^{t=T} - \int_0^T u_t^{N_j} v_t \, dt$$

$$= u_t^{N_j} v \Big|_{t=0}^{t=T} - u^{N_j} v_t \Big|_{t=0}^{t=T} + \int_0^T u^{N_j} v_{tt} \, dt$$

$$\Rightarrow \int_0^T \int_U u^{N_j} v_{tt} \, dx\, dt + \int_0^T \int_U Dv A Du^{N_j}$$

$$+ \int_U v_t(0)\, u^{N_j} \, dx - \int_U v(0)\, u_t^{N_j} = \int_0^T \int_U f v \, dx\, dt$$

we want $u^{N_j} \to u(0)$ and $u_t^{N_j} \to u_t(0)$

Rember $u^{N_j} = \sum_{k=1}^{N_j} d_k^{N_j}(t)\, w_k(x)$, $d_k^{N_j}(0) = \int_U g\, w_k \, dx$

$$(u^{N_j}, w_k) = d_k^{N_j}(t)$$

$$u_t^{N_j} = \sum_{k=1}^{N_j} d_{k,t}^{N_j} w_k(x) \quad , \quad d_{k,t}^{N_j}(0) = \int_U h\, w_k \, dx$$

$$u^{N_j}(x,0) = \sum_{k=1}^{N_j} d_k(0)\, w_k(x) \to g \quad \text{as } N_j \to \infty$$

$$u_t^{N_j}(x,0) = \sum_{k=1}^{N_j} d_{k,t}(0)\, w_k(x) \to h \quad \text{as } N_j \to \infty$$

Let us now turn to the linear equations of elasticity for the isotropic case

$$\underline{u}_{tt} = \nabla(\lambda \nabla\cdot\underline{u}) + \frac{1}{2}\nabla\cdot\left[\mu\left(\nabla\underline{u} + (\nabla\underline{u})^T\right)\right]$$

$$\boxed{\begin{matrix}\lambda > 0\\ \mu > 0\end{matrix}}$$

$\underline{u}(x,t) = 0$, $x \in \partial U$, $0 < t < T$,

$\underline{u}(x,0) = \underline{g}$

$\underline{u}_t(x,0) = \underline{h}$

These are the underlying equations

---

we want to define a weak solution so multiply by a test function $\underline{v}$

$$\underline{v}\cdot\underline{u}_{tt} = \underline{v}\cdot\nabla(\lambda\nabla\cdot\underline{u}) + \underline{v}\cdot\nabla\cdot\left[\mu\left(\nabla\underline{u} + (\nabla\underline{u})^T\right)\right]$$

$$\underline{v}\cdot\underline{u}_{tt} = \nabla\cdot(\underline{v}\,\lambda\nabla\cdot\underline{u}) - \lambda(\nabla\cdot\underline{v})(\nabla\cdot\underline{u}) \cdots$$

Exercise:

define weak solution analogous to weak soln to wave eqn and define app^x solutions

come up with correct basis functions to do the expansion

due Friday Feb 4th

Now back to wave eqn

$u_{tt} = \nabla \cdot (A \nabla u) + F \quad , (x,t) \in \mho_T$

$u(x,t) = k(x,t) \quad , x \in \partial \mho \, , t = 0 \quad \leftarrow$ GENERAL BOUNDARY CONDITION

$u(x,0) = g \quad , x \in \mho \, , t = 0$

$u_t(x,0) = h \quad , x \in \mho \, , t = 0$

usual way to deal with this is to extend $k(x,t)$ into the interior

$v = u - k$, the BC returns to zero, so that test functions lie in $H_0^1(\mho)$

Alternatively, ~~set~~ consider flux type BC

$\underline{n} \cdot (A \nabla u) = \ell(x,t)$

where $\underline{n}$ is unit normal; this requires some smoothness of the boundary to have the normal defined. The weak formulation

$v u_{tt} = v \nabla \cdot (A \nabla u) + vF$

$$\int_{\mho} v u_{tt} = \underbrace{\int_{\mho} \nabla \cdot (v A \nabla u)}_{\substack{= \int_{\partial \mho} n \cdot v A \nabla u \\ \int_{\partial \mho} v (n \cdot A \nabla u) \\ \int_{\partial \mho} v \, \ell(x,t)}} - \int_{\mho} \nabla v \cdot A \nabla u + \int_{\mho} F v$$

two ~~pr~~ issues

- can no longer use Poincaré inequality to ~~dif~~ modify H^1 norm as we did in previous case
- additional boundary term

$$\|v\|^2 = \int_{\mho} v^2 + |\nabla v|^2$$

For the wave eqn $u_{tt} = \nabla\cdot(A\nabla u) + f$

defined an approximation $u^N = \sum_{k=1}^{N} d_k^N(t)\, w_k(x)$

where $\{w_k\}_{k=1}^{N}$, orthonormal in $L^2(\mho)$, $w_k \in H_0^1(\mho)$

showed weak convergence

$u^{N_j} \rightharpoonup u \qquad L^2(0,T; H_0^1(\mho))$

$u_t^{N_j} \rightharpoonup u_t \qquad L^2(0,T; L^2(\mho))$

$u_{tt}^{N_j} \rightharpoonup u_{tt} \qquad L^2(0,T; H^{-1}(\mho))$

we also want to show that

$u(x,0) = g(x)$ and $u_t(x,0) = h(x)$

---

we know $u^{N_j}(x,0) = \sum_{k=1}^{N_j} d_k^{N_j}(0)\, w_k(x)$

where $d_k^{N_j}(0) = \int_{\mathbb{R}^n} g w_k \, dx$ so that $u^{N_j}(x,0) \xrightarrow[L^2(\mho)]{} g(x)$

and we also know that $u_t^{N_j}(x,t)\Big|_{t=0} \xrightarrow[L^2(\mho)]{} h(x)$

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pg 287, Theorem 3: Suppose $u \in L^2(0,T; H_0^1(\mho))$

and $u_t \in L^2(0,T; H^{-1}(\mho))$. Then $u \in C([0,T], L^2(\mho))$.

(The space for u has been "weakened")

- what this says is that for fixed $\{t_j\}_{j=1}^{\infty}$ with $t_j \to \hat{t}$, then $u(t_j, x) \to u(\hat{t}, x)$ in $L^2(\mho)$

$\lim_{t_j \to \hat{t}} \int_{\mho} |u(t_j,x) - u(\hat{t},x)|\, dx = 0$

Not necessarily the theorem we want - but it is similar

Theorem A: Suppose $u \in L^2(0,T; H_0^1(\mho))$ and

$u_t \in L^2(0,T; L^2(\mho))$ THEN $u \in C(0,T; L^2(\mho))$.

Theorem B: Suppose $u \in L^2(0,T; H_0^1(\mho))$ and $u_t \in L^2(0,T; L^2(\mho))$ and $u_{tt} \in L^2(0,T; H^{-1}(\mho))$

then $u_t \in C(0,T; L^2(\mho))$.

Before we use these theorems, or prove them, we need to establish the relationship between our weak limits u, u_t, u_{tt}.

$$\left.\begin{array}{lll} u^{N_j} \rightharpoonup u, & L^2(0,T; H_0^1(\mho)) \\ u_t^{N_j} \rightharpoonup u_t & L^2(0,T; L^2(\mho)) \\ u_{tt}^{N_j} \rightharpoonup u_{tt} & L^2(0,T; H^{-1}(\mho)) \end{array}\right\} \text{how are these weak limits related?}$$

What do we know?

$$\int_0^T \int_\mho (\nabla u^{N_j} - \nabla u) \cdot \nabla v \, dx\, dt \to 0 \text{ as } N_j \to \infty \quad \forall v \in L^2(0,T; H_0^1(\mho))$$

claim: this is equivalent to

$$\int_0^T \int_\mho \left[(\nabla u^{N_j} - \nabla u) \cdot \nabla v + (u^{N_j} - u) v\right] \to 0 \text{ as } N_j \to \infty \quad \forall v \in L^2(0,T; H_0^1(\mho))$$

we can replace the limits of integral from 0 to T, to from 0 to t, $0<t<T$. Change the order of integration

$$\int_\mho \int_0^t \left(u_t^{N_j} - u_t\right) v \, dt\, dx$$

Now pick v independent of time

$$\int_\mho \left[u^{N_j}(t,x) - u^{N_j}(0,x) - \int_0^t u_t(s,x)\, ds\right] v(x)\, dx$$

multiply by some function of time, $e(t)$, and integrate over time

$$\int_0^T \int_\mho \left[u^{N_j}(t,x) - u^{N_j}(0,x) - \int_0^t u_t(s,x)\, ds\right] v(x)\, e(t)\, dx\, dt$$

to construct all possible functions $\in L^2(0,T; L^2(\mho))$

$$\int_0^T \int_\mho \left[u(t,x) - g(x) - \int_0^t u_t(s,x)\, ds\right] e(t)\, v(x)\, dx\, dt$$

so this tells us that

$$u(t,x) - g(x) - \int_0^t u_t(s,x)\,ds = 0 \quad \text{a.e.}\ t \ \text{in } L^2(\mho)$$

and likewise we can also show that

$$u_t(t,x) - h(x) - \int_0^t u_{tt}(s,x)\,ds = 0 \quad \text{a.e.}\ t \ \text{in } L^2(\mho)$$

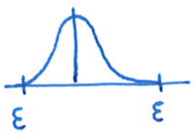
Demonstration of Theorem A:

Look at $u(t,x)$, $0<t<T$, $x \in \mho$

This Function is rough, so we want to smoothen it out by multiplying by a mollifier, but to do that, we must extend u to the interval $-\sigma < t < T+\sigma$, $\sigma > 0$ and $u(t,x) = 0$, $-\sigma < t < 0$, $T < t < T+\sigma$. So we have

$$u^\varepsilon(t,x) = \int_0^T u(s,x)\, f_\varepsilon(t-s)\,ds$$

where f looks like



$f \in C_0^\infty$ and $\int_{-\infty}^{\infty} f(s)\,ds = 1$ where

and f is very smooth

$u^\varepsilon(t,x)$ is very smooth in t.

$$\int_0^T \| u^\varepsilon(t,x) - u(t,s)\|_{H_0^1(\mho)}\,dx \to 0 \quad \text{as } \varepsilon \to 0$$

and $\| u^\varepsilon(t,x) - u(t,x)\|^2_{L^2(\mho)} \to$ a.e.t (almost every time)

Reminder: $\left| \|u^\varepsilon\|_{L^2(\mho)} - \|u\|_{L^2(\mho)} \right| \le \| u^\varepsilon(t,x) - u(t,x)\|_{L^2(\mho)} \le \|u^\varepsilon(t,x)\|_{L^2(\mho)} + \|u(t,x)\|_{L^2(\mho)}$

$\Rightarrow \quad \|u^\varepsilon(t,x)\| \to \|u(t,x)\|$ a.e.t.

Great

$\cancel{\frac{d}{dt}\|u^\varepsilon(t)\|}$

$$\frac{d}{dt}\| u^\varepsilon(t,\cdot) - u^\delta(t,\cdot)\|^2_{L^2(\mho)} = \frac{d}{dt}\int_{\mho} \left[u^\varepsilon(t,x) - u^\delta(t,x)\right]^2 dx$$

$$= 2\int_{\mho} \frac{\partial}{\partial t}\left[u^\varepsilon(t,x) - u^\delta(t,x)\right]\left(u^\varepsilon - u^\delta\right) dx$$

$$\leq \int_{\mho} \left(u^\varepsilon_t - u^\delta_t\right)^2 + \left(u^\varepsilon - u^\delta\right)^2 dx$$

Now integrate from s to t

$$\int_s^t \frac{d}{dt}\| u^\varepsilon(t,x) - u^\delta(t,x)\|^2_{L^2(\mho)}\, dt \leq \int_s^t \int_{\mho} \left(u^\varepsilon_t - u^\delta_t\right)^2 + \left(u^\varepsilon - u^\delta\right)^2 dx\, dt$$

replace bounds, from 0 to T

$$\| u^\varepsilon(t,\cdot) - u^\delta(t,\cdot)\|^2_{L^2(\mho)} - \| u^\varepsilon(s,\cdot) - u^\delta(s,\cdot)\|^2_{L^2(\mho)} \leq \int_0^T \int_{\mho} \left(u^\varepsilon_t - u^\delta_t\right)^2 + \left(u^\varepsilon - u^\delta\right)^2 dx\, dt$$

select s such that

$$\| u^\varepsilon - u^\delta\|^2_{L^2(\mho)} \to 0$$

~~this implies~~ the inequality holds for any t, since right hand side is time-independent, so we can take the sup then take limit as $\varepsilon, \delta \to 0$

$$\Rightarrow \lim_{\varepsilon,\delta\to 0} \sup_{0\leq t\leq T} \underbrace{\| u^\varepsilon(t,x) - u^\delta(t,x)\|^2_{L^2(\mho)}}_{\text{uniform convergence in time}} = 0$$

this implies $\| u^\varepsilon(t,x)\|_{L^2(\mho)}$ converges uniformly as $\varepsilon \to 0$

$\Rightarrow \| u(t,x)\|_{L^2(\mho)}$ is continuous in time

Show that $\int_0^T \int_0^x \left|\left(u^\varepsilon(t,x) - u^\delta(t,x)\right)'\right|^2 dx\, dt \to 0$ as $\varepsilon, \delta \to 0$

$$\boxed{\text{END OF PROOF OF THEOREM A}}$$

Exercise

given

① $u(t,x) \in C(0,T; L^2(\mathcal{V})) \cap L^2(0,T; H_0^1(\mathcal{V}))$

② $\int_0^T \int_{\mathcal{V}} \left(u^{N_j}(x,t) - u(x,t)\right) v(x)\, e(t)\, dx\, dt \xrightarrow[\text{weakly}]{} 0 \quad \forall\, e \in L^2(0,T),\ v \in \cancel{L^2(\mathcal{V})}$
$L^2(0,T; H_0^1(\mathcal{V}))$

③ $\int_{\mathcal{V}} \left[u^{N_j}(x,0) - g(x)\right]^2 dx \to 0$ as $N_j \to \infty$

$$\boxed{\text{Show } u(x,0) = g(x) \text{ in } L^2(\mathcal{V})}$$

$$\boxed{\text{due February } 11^{th}}$$

using

$$\int_0^T \int_{\mathcal{V}} \left(u(x,t) - u(x,0)\right)^2 dx\, dt$$

add $-u^{N_j}(x,t) + u^{N_j}(x,t) - g(x) + g(x)$

use triangle inequality

$$\leq \int_0^T \int_0^{\mathcal{V}} \underbrace{\left(u - u_j^{N_j}\right)^2}_{\text{use}} + \underbrace{\left(u^{N_j} - g\right)^2}_{\text{use}} + \left(u(x,0) - g(x)\right)^2 dx\, dt$$

use ② with $v(x)\, e(t) = \left(u^{N_j} - u(x,t)\right)$ to ~~eliminate~~ bound?

In addition, we know

Theorem There exists a unique weak solution to our problem.

$$u_{tt} = \nabla\cdot(A\nabla u) + F, \quad (x,t)\in U_t$$

$$u(x,t) = 0 \quad , x\in\partial U, 0<t<T$$

$$u(x,0) = g$$

$$u_t(x,0) = h$$

A, symmetric, uniform positive definite, bounded, e

$$\langle u_{tt}, v\rangle + \int_U (\nabla u)^T A\nabla v = \int f v \quad \forall\, v\in L^2(0,T;H_0^1(U))$$

Suppose there exists two solutions u_1, u_2

Let $u = u_1 - u_2$

$$\langle u_{1,tt}, v\rangle + \int_U \nabla u_1 A\nabla v = \int_U f v$$

$$\langle u_{2,tt}, v\rangle + \int_U \nabla u_2 A\nabla v = \int_U f v$$

subtract to find

$$\langle u_{tt}, v\rangle + \int_U \nabla u A\nabla v = 0, \quad u(x,0) = 0, \quad u_t(x,0) = 0$$

The wave eqn

$$\left.\begin{aligned} u_{tt} &= \nabla(A\nabla u) + F \\ u(x,t) &= 0, \quad x\in\partial U, \ 0<t<T \\ u(x,0) &= g \in H_0^1(U) \\ u_t(x,0) &= h \in L^2(U) \end{aligned}\right\} (WV)$$

A is uniform, positive definite, symmetric

$$u \in L^2(0,T; H_0^1(U))$$
$$u_t \in L^2(0,T; L^2(U))$$
$$u_{tt} \in L^2(0,T; H^{-1}(U))$$

and we (sort of) showed that

$$u \in C[0,T; L^2(U)]$$
$$u_t \in C[0,T; L^2(U)]$$

and we had the variational formulation

$$\underbrace{\langle u_{tt}, v\rangle}_{\text{linear Functional}} + \int_U \nabla u\cdot A\nabla v = \int_U F v \qquad \forall v\in H_0^1(U) \text{ a.e. t.}$$

we also showed $u(t,x) - u(t,s) - \int_s^t u_t(s',x)\,ds' = 0$

suppose $v \in C^2([0,T], H_0^1(U))$

$$\underbrace{\int_0^s \langle u_{tt}, v\rangle}_{\text{integrate by parts}} + \int_0^s\int_U \nabla u\cdot A\nabla v = \int_0^s\int_U F v$$

$$\int_U \left\{ \Big[u_t v\Big]_{t=0}^{t=s} - \int_0^s u_t v_t \,dt \right\} dx$$

$\int_0^s \langle u_{tt}, v \rangle$ this is a linear functional
why can we integrate by parts

$$\int_0^s \int_U u_{tt}\, v \, dx\, dt = \int_U \int_0^s u_{tt}\, v \, dt\, dx$$

$\int p\, dq = pq - \int q\, dp$

$p = v \quad dq = u_{tt}\, d\cancel{t}$

$dp = \frac{dv}{dt}\, dt \quad q = u_t$

$$= \int_U \left\{ u_t v \Big]_{t=0}^{t=s} - \int_0^s u_t v_t \, dt \right\} dx$$

$p = v_t \quad dq = u_t\, dt$

$dp = v_{tt}\, dt \quad q = u$

$$= \int_U u_t v \Big]_{t=0}^{t=s} dx - \int_U u v_t \Big]_{t=0}^{t=s} dx + \int_0^s \int_U u v_{tt} \, dx\, dt$$

<u>Theorem</u> : there exists a unique weak solution

demonstration: suppose we have two solutions u_1, u_2
that satisfies wave eqn (WV) and

$$u_1(x,0) = u_2(x,0) = g(x)$$

$$u_{1,t}(x,0) = u_{2,t}(x,0) = h(x)$$

Let $u = u_1 - u_2$, $u(x,0) = 0 = u_t(x,0)$.

Then u satisfies homogeneous weak formula:

$$\langle u_{tt}, v \rangle + \int_U \nabla u \cdot A \cdot \nabla v \, dx = 0$$

Fix s, $0 < s \le T$

$$\tilde{v}(x,t) = \begin{cases} \int_t^s u(x,s')\, ds' , & 0 < t < s \\ 0 , & s < t < T \end{cases}$$

For fixed t, $\tilde{v}(x,t) \in H_0^1(U)$

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$\Rightarrow \quad \tilde{v}_t(x,t) = -u(x,t)$ by integration by parts

Now replace v by $\tilde{v}$ in weak formulation and integrate from 0 to s

$$\int_0^s \langle u_{tt}, \tilde{v} \rangle + \int_0^s \int_{\mho} \nabla \tilde{u} A \nabla \tilde{v} = 0$$

replace ∇u with $-\nabla \tilde{v}_t$, integrate by parts

$$\underbrace{\int_{\mho} u_t \tilde{v} \Big]_{t=0}^{t=s} dx}_{\text{these terms drop out because}} - \int_{\mho} \int_0^s u_t \underset{\substack{\uparrow \\ \text{replace} \\ \text{with } -u}}{\tilde{v}_t}\, dt\, dx - \int_0^s \int_{\mho} \underbrace{\nabla \tilde{v}_t A \nabla \tilde{v}}_{\frac{1}{2}\frac{d}{dt} \nabla \tilde{v} A \nabla \tilde{v}}\, dx\, dt$$

these terms drop out because

$u_t(t=0) = 0$ and $\tilde{v}(t=s) = 0$

So now rewrite

$$\frac{1}{2}\int_{\mho}\int_0^s \frac{d}{dt}\left(u^2 - \nabla \tilde{v} A \nabla \tilde{v} \right) dt\, dx = 0$$

$$\frac{1}{2}\int_{\mho} \Big[u^2 - \nabla \tilde{v} A \nabla \tilde{v} \Big]_{t=0}^{t=s} dx = 0$$

$$\rightarrow \frac{1}{2}\int_{\mho} u^2(x,s)\, dx + \frac{1}{2}\int_{\mho} \nabla \tilde{v}(x,0) A \nabla \tilde{v}(x,0)\, dx = 0$$

$\Rightarrow u(x,s) = 0$, fixed s, a.e. x.

Improved Regularity

Recall the inequality we derived

$$\max_{0 \le t \le T} \left\{ \| u^{N_j} \|^2_{H_0^1(\mathcal{U})}(t) + \| u_t^{N_j} \|^2_{L^2(\mathcal{U})} \right\} + \| u_{tt}^{N_j} \|^2_{L^2(0,T;\, H_0^1(\mathcal{U}))}$$

$$\le C \left\{ \| g \|^2_{H_0^1(\mathcal{U})} + \| h \|^2_{L^2(\mathcal{U})} + \| f \|^2_{L^2(0,T;\, L^2(\mathcal{U}))} \right.$$

And now consider

$A = (a^{ij})_{i,j=1,\dots,n}$ Lipschitz continuous

$g \in H^2(\mathcal{U}) \cap H_0^1(\mathcal{U})$

$h \in H_0^1(\mathcal{U})$

$f_t \in L^2(0,T;\, L^2(\mathcal{U}))$

The inequality changes

$$\max_{0 \le t \le T} \left\{ \| u^{N_j} \|^2_{H^2(\mathcal{U})} + \| u_t^{N_j} \|^2_{H_0^1(\mathcal{U})} + \| u_{tt}^{N_j} \|^2_{L^2(\mathcal{U})} \right\} + \| u_{ttt}^{N_j} \|_{L^2(0,T;\, H_0^1(\mathcal{U}))}$$

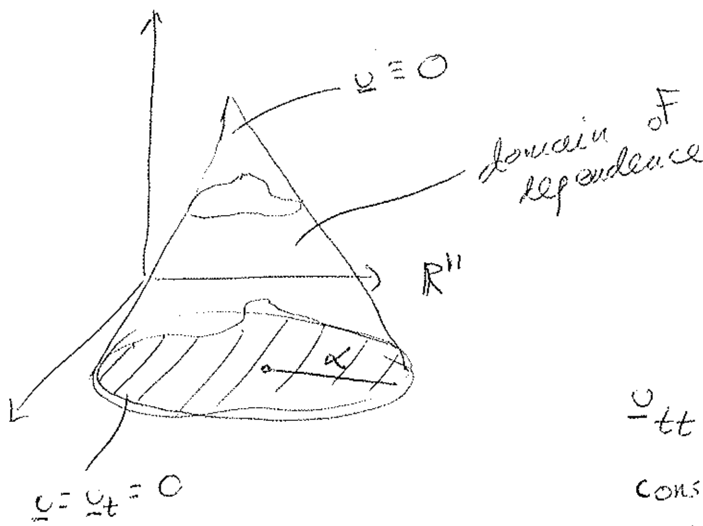
$$\le C \left\{ \| g \|^2_{H^2(\mathcal{U})} + \| h \|^2_{H_0^1(\mathcal{U})} + \| f \|^2_{H_0^1(\mathcal{U})} \right.$$

Discussion in text book

Not going to do anymore.

Propagation of Disturbances

(Finite propagation schemes)



isotropic, linear elastic equations

$$\underline{u}_{tt} = \nabla(\lambda \nabla \cdot \underline{u}) + \nabla \cdot \left[\mu\left(\nabla \underline{u} + (\nabla \underline{u})^T\right)\right] \quad \text{(LEQ)}$$

constant coefficient, λ, μ

$\underline{u} = 0$, $t = 0$ when $|x - x_0| \leq \alpha$

now let us define our cone

$$C = \{ (x,t) \mid |x - x_0| \leq \alpha - \gamma t , \ 0 < t < s, \ \alpha - \gamma s > 0 \}$$

which looks like this

we also define ~~on~~ some slices

$$C_t = \{ x \mid |x - x_0| \leq \alpha - \gamma t \}$$

this is the surface of the cone

$$\Lambda(s) = \{ (x,t) \mid |x - x_0| = \alpha - \gamma t , \ 0 < t < s \}$$

Assume that we have smooth solutions

Take inner product of (LEQ) with $\underline{u}_t$

Note, $\underline{u}_t \cdot \underline{u}_{tt} = \frac{1}{2} \frac{d}{dt}(\underline{u}_t \cdot \underline{u}_t)$

and

$$\underline{u}_t \cdot \nabla(\lambda \nabla \cdot \underline{u}) = \nabla \cdot (\underline{u}_t \lambda \nabla \cdot \underline{u}) - (\nabla \cdot \underline{u}_t)\lambda(\nabla \cdot \underline{u})$$

$$= \nabla \cdot (\underline{u}_t \lambda \nabla \cdot \underline{u}) - \frac{1}{2} \frac{d}{dt}\left[\lambda (\nabla \cdot \underline{u})^2\right]$$

and lastly

$$\underline{u}_t \nabla\cdot\left[\nu\left(\nabla\underline{u} + (\nabla\underline{u})^T\right)\right] = \nabla\cdot\left[\underline{u}_t \nu\cdot\left(\nabla\underline{u} + (\nabla\underline{u})^T\right)\right]$$

$$- \nu \nabla\underline{u}_t : \left(\nabla\underline{u} + (\nabla\underline{u})^T\right)$$

so we are left with a bunch of time derivative terms and divergence terms

term by term multiplication of matrix components

We also want to write $-\nu\nabla\underline{u}_t : (\nabla\underline{u} + (\nabla\underline{u})^T)$ in time derivative form.

$$-\nu\nabla\underline{u}_t : \left(\nabla\underline{u} + (\nabla\underline{u})^T\right) = -\frac{\nu}{2}\left(\nabla\underline{u}_t + (\nabla\underline{u}_t)^T\right) : \left(\nabla\underline{u} + (\nabla\underline{u})^T\right)$$

$$= -\frac{\nu}{4}\frac{d}{dt}\left[\left(\nabla u + (\nabla u)^T\right) : \left(\nabla u + (\nabla u)^T\right)\right]$$

$$= -\frac{\nu}{4}\frac{d}{dt}\left|\nabla u + (\nabla u)^T\right|^2$$

$$\Rightarrow \boxed{\begin{aligned} 0 &= \frac{d}{dt}\left[\frac{1}{2}|\underline{u}_t|^2 + \frac{1}{2}\lambda(\nabla\cdot u)^2 + \frac{\nu}{4}\left|\nabla u + (\nabla u)^T\right|^2\right] \\ &\quad - \nabla\cdot\left[\underline{u}_t\lambda\,\nabla\cdot\underline{u} + \nu\,\underline{u}_t\cdot\left(\nabla\underline{u} + (\nabla\underline{u})^T\right)\right]\end{aligned}}$$

we want to integrate over the cone C ((in space and time)

but it is also in divergence form (in space and time)

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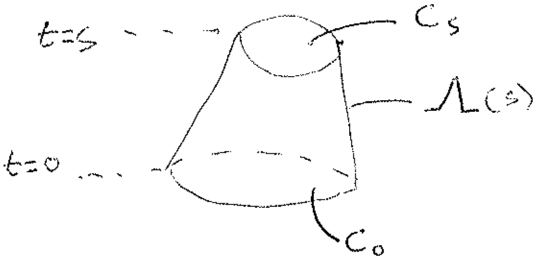
Apply the divergence theorem in space and time, to obtain a few integrals over the boundary



what are the normals?

the unit normal to slice C_0 is $(0,0,0,-1)$ and the unit normal to slice C_s is $(0,0,0,1)$

the function defining the surface $\Lambda(s)$ is

$$|x-x_0| + \gamma t = \alpha$$

the normal is then $\left(\frac{x-x_0}{|x-x_0|}, \gamma\right)$

and the unit normal is $\frac{1}{\sqrt{1+\gamma^2}}\left(\frac{x-x_0}{|x-x_0|}, \gamma\right)$

The integral over the boundary (surface C_0) is

$$-\int_{C_0}\left[\frac{1}{2}|u_t|^2 + \frac{1}{2}\lambda(\nabla\cdot u)^2 + \frac{\mu}{4}|\nabla u + (\nabla u)^T|^2\right](x,0)\,dx$$

but this vanishes since $\underline{u}=0$, $\underline{u}_t=0$ on surface C_0

The integral over the surface C_s is

$$\int_{C_s}\left[\frac{1}{2}|u_t|^2 + \frac{1}{2}\lambda(\nabla\cdot\underline{u})^2 + \frac{\mu}{4}|\nabla u + (\nabla u)^T|^2\right](x,s)\,ds$$

The last and most complicated ~~term~~ term is the integral over the boundary Λ

$$\frac{1}{\sqrt{1+\gamma^2}} \int_{\Lambda(s)} \left\{ -\frac{x-x_0}{|x-x_0|} \cdot \left(u_t \lambda (\nabla \cdot u) + \nu u_t \cdot (\nabla \underline{u} + (\nabla \underline{u})^T) \right) + \gamma \left(\frac{1}{2} |u_t|^2 + \frac{1}{2} \lambda (\nabla \cdot u)^2 + \frac{\nu}{4} |\nabla u + (\nabla u)^T|^2 \right) \right\} ds$$

So we have

$$-\int_{C_0} (\quad)(x,0)\, dx + \underbrace{\int_{C_s} (\quad)(x,s)\, ds}_{e(s)} + \underbrace{\text{that integral}}_{\uparrow} = 0$$

~~The two integrals over C_0 and C_s~~ can

Note, $|\underline{a} \cdot \underline{b}| \le |\underline{a}|\,|\underline{b}|$

we have

$$0 \ge e(s) - \frac{1}{\sqrt{1+\gamma^2}} \int_{\Lambda(s)} \left[|u_t| \lambda |\nabla \cdot u| + \nu |u_t \cdot (\nabla u + (\nabla u)^T)| \right] ds$$

$$+ \frac{1}{\sqrt{1+\gamma^2}} \int_{\Lambda(s)} \gamma (\quad)\, ds$$

Now set $\gamma = \sqrt{\lambda + 2\nu}$ and rewrite

$$0 \ge e(s) - \frac{\gamma}{2\sqrt{1+\gamma^2}} \int_{\Lambda(s)} \frac{2}{\sqrt{\lambda+2\nu}} \left[|u_t| \lambda |\nabla \cdot u| + \nu |u_t|\, |\nabla u + (\nabla u)^T| \right] ds$$

$$+ \frac{\gamma}{2\sqrt{1+\gamma^2}} \int_{\Lambda(s)} |u_t|^2 + \lambda (\nabla \cdot u)^2 + \frac{\nu}{2} |\nabla u + (\nabla u)^T|^2 \, ds$$

and Finally we obtain

$$0 = e(s) + \frac{\gamma}{2\sqrt{1+\gamma^2}} \int_{\Lambda(s)} \lambda\left[\frac{|u_t|}{\sqrt{\lambda+2\nu}} - |\nabla \cdot u|\right]^2 + 2\nu\left[\frac{|u_t|}{\sqrt{\lambda+2\nu}} - |\nabla u + (\nabla u)^T|\right]^2 ds$$

This analysis implies that we have to pick

$$\gamma > \sqrt{\lambda+2\nu} \quad \forall x$$

and this defines the cone.

$$\left.\begin{aligned} -\nabla\cdot(A\nabla u) &= f \qquad x\in U \\ u &= 0 \qquad x\in\partial U \end{aligned}\right\} \ *$$

where A is a symmetric matrix, $n\times n$,
with bounded components

$\theta_0|\xi|^2 \le \xi^T A \xi \le \theta_1|\xi|^2$, $\xi\in\mathbb{R}^n$ $\forall x\in\overline{U}$, $\theta_0>0$

Definition: u is a weak solution in $H_0^1(U)$
iff

$$\int_U \nabla v^T \cdot A\nabla u = \int_U f v \qquad \forall v\in H_0^1(U)$$

$\exists\,!$ weak solution of $*$

this was showed

Prove existance of unique solution to

$$(**)\quad -\nabla\cdot(A\nabla u) + c(x)\,u = f\ ,\quad x\in U$$

$c_1 > c(x) > c_0 > 0$

$u=0\ ,\ x\in\partial U,\ \forall x\in\overline{U}$

working with

Real Hilbert space, such that

$$(au, v) = a(u, v) = (u, av)$$

so that we don't have to deal with complex conjugates

$L^2(\mho) = \{ f \mid \int f^2 dx < \infty \}$ is Hilbert space

with $(f, g)_{L^2(\mho)} = \int_{\mho} fg \, dx$

Note,

$$H_0^1(\mho) : (u, v)_{H_0^1(\mho)} = \int_{\mho} \nabla u \nabla v \, dx$$

Definition, real bilinear mapping $B : H \times H \mapsto R$, where H is a real Hilbert space

$$B[au, v] = a B[u, v] \quad , a \in \mathbb{R}$$
$$B[u, av] = a B[u, v] \quad , a \in \mathbb{R}$$
$$B[u + w, v] = B[u, v] + B[w, v]$$
$$B[u, v + w] = B[u, v] + B(v, w)$$

Lax Milgram theorem

Given real Hilbert space H and bilinear mapping $H \times H \to R$ with

$$\left| B[u, v] \right| \leq \beta \|u\|_H \|v\|_H \quad , \quad \alpha \|u\|_H^2 \leq \left| B[u, u] \right|$$

for $\alpha, \beta > 0$, AND given real bounded linear functional $f : H \to R$ then $\exists !\ u \in H$ with $B[u, v] = \langle f, v \rangle \ \forall v \in H$

Can we apply LM theorem?

Is $\int_\Omega \nabla v^T \cdot A \nabla v$ a bilinear mapping on $H = H_0^1(\Omega)$? Fix one, is it linear in the other? YES

Is $\int_\Omega fv$ a bounded linear functional?

Yes, because this is an inner product in $L^2(\Omega)$ so it's bounded above,

$$\left| \int_\Omega fv \right| \le \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \le \hat{C} \|f\|_{L^2(\Omega)} \|v\|_{H_0^1(\Omega)}$$

$$\to \left| \int_\Omega fv \right| \le C \|v\|_{H_0^1(\Omega)}$$

More carefully,

if $v \in H_0^1(\Omega)$ then $\|v\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla v|^2 \, dx$

and $\|v\|^2_{L^2(\Omega)} = \int_\Omega v^2 \, dx$

and we know from Poincaré that

$$\int_\Omega v^2 = \|v\|_{L^2(\Omega)} \le \hat{C} \|v\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla v|^2$$

Ok so we have a bilinear mapping and we have a bounded linear functional, but what about inequalities

Multiply by test function, integrate over U

$$\int_U v\left(-\nabla\cdot(Au)\right) + \int_U cuv = \int_U fv$$

~~what if sign is positive?~~

$$\rightarrow \quad \int_U \left(\nabla v^T A \nabla u + cuv\right) dx = \int_U fv$$

this is positive definite (apperently)
but if put negative sign in front of cuv,
then this is problematic

<u>Definition</u>: $u \in H_0^1(U)$ is a weak solution of (**) iff

$$\cancel{\int_U v\left(-\nabla\cdot(A\nabla u)\right) + \int_U cuv = \int_U fv \quad \forall v \in H_0^1(U)}$$

$$\int_U \nabla v^T A \nabla u + cuv \, dx = \int_U fv \qquad \forall v \in H_0^1(U)$$

positive sign

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<u>Theorem</u> Given f, A, c (that satisfy our conditions), $\exists\, !\ u \in H_0^1$ which is a weak solution of $(**)$.

<u>Proof - apply LM theorem</u>

$$B[u,v] = \int_U \left[(\nabla v^T A \nabla u) + c u v \right] dx$$

$$|B[u,v]| \le \int_U \hat{c} |\nabla v| |\nabla u| + c_1 |u||v| \, dx$$

since $|(\nabla v)^T A \nabla u| \le |\nabla v| A |\nabla u|$

$\le \hat{c} |\nabla v| |\nabla u|$, $\hat{c} > 0$

$$\le \max[\hat{c}, c_1] \left[\underbrace{\left(\int_U |\nabla v|^2 \right)^{2}}_{\|v\|_{H_0^1(U)}} \underbrace{\left(\int_U |\nabla u|^2 \right)^{2}}_{\|u\|_{H_0^1(U)}} + \|u\|_{L^2} \|v\|_{L^2(U)} \right]$$

but $\|u\|_{L^2} \|v\|_{L^2(U)} \le D \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}$

with $D > 0$, from Poincaré

~~$\le \max\{\hat{c}, c, D\}$~~

$$\le \left\{ \max[c_1, \hat{c}] + D \right\} \|v\|_{H_0^1(U)} \|u\|_{H_0^1(U)}$$

so that we have

$$|B[u,v]| \leq \left(\max(\hat{c}, c_0) + D\right) \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}$$

Furthermore

$$B[u,u] = \int_U \left(\nabla u A \nabla u + c u^2 \right) dx$$

$$\geq \theta_0 \int_U |\nabla u|^2 + c_0 \int_U u^2$$

$$\geq \min(\theta_0, c_0) \int_U \left(|\nabla u|^2 + u^2 \right) dx$$

$$\geq \min(\theta_0, c_0) \left[1 + \frac{1}{\hat{c}} \right] \int_U |\nabla u|^2 = \|u\|_{H_0^1(U)}$$

Note, if change sign in front of $c(x)\, u$, with $c_1 > c(x) > c_0 > 0$,

to obtain

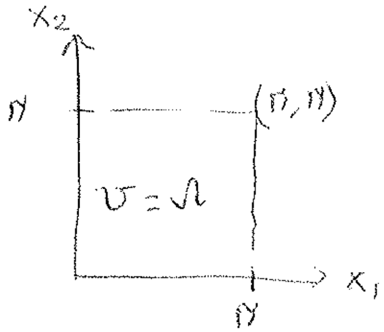
$$-\nabla \cdot (A \nabla u) - c(x)\, u = f, \quad x \in U$$

$$u = 0, \quad x \in \partial U, \quad \forall x \in \bar{U}$$

then this becomes very difficult.

As a simple example,

consider $U = \Omega = [0, \pi] \times [0, \pi]$



$$\boxed{\begin{aligned} \Delta u + (n^2 + m^2)\, u &= 0, \quad x \in \Omega \\ u &= 0, \quad x \in \partial\Omega \end{aligned}}$$

linear eqn in square domain, apply separation of variables ...

A nonzero solution to this problem is

$$u_1 = \sin(n x_1) \sin(m x_2)$$

$$u_2 = \sin(m x_1) \sin(n x_2)$$

and we also know $u_3 = 0$ is a solution

and there are others ...

IF v satisfies $\Delta v + (n^2 + m^2) v = f, \; x \in \Omega$

$v = 0, \; x \in \partial\Omega,$

Then $v + a u_1 + b u_2$ is also solution

Similarly, suppose $u = u(x, t)$

$\nabla \cdot (A \nabla u) = u_{tt}$

assume $u = v(x) e^{-i\omega t}$ then $\nabla \cdot (A \nabla v) + \omega^2 v = 0$

General outline

$$\int_U \nabla v A \nabla u + \int_U -c(x)\, uv = \int f v$$

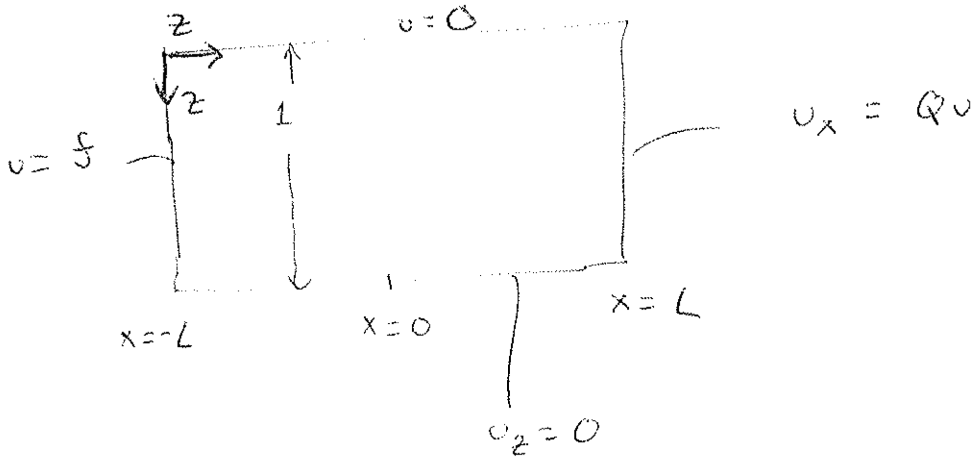
not workable from LM point of view,
so add constant to make $-c(x)uv > 0$

$$\underbrace{\int_U \nabla v A \nabla u + \int (\hat{c} - c(x))\, uv}_{B[u,v]} = \int f v + \int \hat{c}\, uv$$

$$B[u,v] = \langle F, v \rangle = K_u(v) + \langle f, v \rangle$$

Helmholtz Equation

$$\Delta u + n^2(x,z)\, u = 0$$



Complex Lax Milgram Theorem

Let H be a complex Hilbert space

- inner product, $(u,v)_H$, $u,v \in H$

$$(u,v) = \overline{(v,u)}$$
$$(au, v) = a(u,v)$$
$$(u, av) = \bar{a}(u,v)$$
$$(u+w, v) = (u,v) + (w,v)$$
$$(u, v+w) = (u,v) + (u,w)$$
$$\|u\|^2 = (u,u) = \overline{(u,u)}$$

Suppose that B satisfies
$\exists\; c, c_1 > 0$ with

$$|B[u,v]| \leq c_1 \|u\|_H \|v\|_H$$

$$c\|u\|^2 \leq \operatorname{Re} B[u,u]$$

Let $\hat{w} \in H$, then $\exists!\; u \in H$ satisfying

$$B[v,u] = (v, \hat{w}) \quad \forall\, v \in H$$

$$\Delta u + n^2(x,z)\, u = 0$$

use separation of variables :

$$u = \sum_{m=1}^{\infty} \psi_m(z)\left[a_m e^{i\sqrt{\lambda_m}(x-L)} + \cancel{b_m e^{-i\sqrt{\lambda_m}\, x}} \right]$$

where ψ_m satisfies $\quad \psi_{zz} = \nu \psi \,, \quad \nu = (m-1/2)^2 \pi^2$

$$\psi(0) = 0 \quad \psi_z(1)$$

$$\Rightarrow \quad \psi_m = \sqrt{2}\, \sin (m-1/2)\pi z$$

$$\lambda_m = n_0^2 - (m-1/2)^2 \pi^2 \,, \quad \int_0^1 \psi_m = 1 \,, \quad \int_0^1 \psi_m \psi_n = \delta_{mn}$$

$\sqrt{\lambda_m}$ positive imaginary - eliminate $e^{-i\sqrt{\lambda_m}\, x}$ in physical arguments

$\Rightarrow$

$$u(L,z) = \sum_{m=1}^{\infty} a_m \psi_m(z)$$

$$a_m = \big(u(L,\cdot), \psi_m(\cdot) \big)_{L^2(0,1)}$$

$$u_x(L,z) = \sum_{m=1}^{\infty} i\sqrt{\lambda_m}\, a_m \psi_m(z)$$

$$= \sum i\sqrt{\lambda_m} \big(u(L,\cdot), \psi_m(\cdot) \big)_{L^2(0,1)} \psi_m(z)$$

$$= Q u$$

Dirichlet to Neumann map

(DtN)

$$-\nabla\cdot(A\nabla u) - c(x)\,u = f \qquad x \in U$$
$$u = 0 \qquad x \in \partial U$$

$0 < c_0 \leq c(x) \leq c \quad \forall x \in \overline{U}$ and $f \in L^2(U)$

Let $c_2 = c_1 + n$, $n > 0$

$$-\nabla\cdot(A\nabla u) + (c_2 - c(x))\,u = c_2 u + f = F, \quad x \in U$$

where $F \in L^2(U)$, $u = 0$, $x \in \partial U$

---

define weak solution: Find $u \in H_0^1(U)$

$$B[u,v] = \int_U \nabla v\, A\, \nabla u + \int_U (c_2 - c)\, uv = (F, v)_{L^2(U)}$$

$\forall\, v \in H_0^1(U)$

Elements of proof

$$\left|(F,v)_{L^2(U)}\right| \leq \|F\|_{L^2(U)} \|u\|_{L^2(U)} \leq \|F\|_{L^2(U)} \|u\|_{H_0^1(U)}$$

$(F, v)_{L^2(U)} = (\hat{w}, v)_{H_0^1(U)}$ for some $\hat{w} \in H_0^1(U)$ and $\forall\, v \in H_0^1(U)$

$\hat{w} = \hat{\Gamma} F$, $\hat{\Gamma} : L^2(U) \to H_0^1(U)$

$\hat{w} = \Gamma(u)$, $\Gamma : H_0^1(U) \to H_0^1(U)$

Γ is 1-1 and $\underbrace{R(\Gamma)}_{\text{Range}} = H_0^1(\mho)$

$u = \Gamma^{-1}\hat{w}$, Γ^{-1} is 1-1 , $R(\Gamma^{-1}) = H_0^1(\mho)$

$u = \underbrace{\Gamma^{-1}\hat{\Gamma}}_{\hat{K}} F \longrightarrow \hat{K} : L^2(\mho) \longrightarrow H_0^1(\mho)$

our real problem is :

Find u so that $u = \hat{K}[c_2 u + F]$

$I u = c_2 \hat{K} u + \hat{K} F$

$\underbrace{[I - c_2\hat{K}]}_{\text{want}} u = \hat{K} F \longrightarrow u = [I - c_2\hat{K}]^{-1}\hat{K} F$

$(I - \hat{K}_{c_2})$ is 1-1 and $R[I - \hat{K}c_2] = H_0^1(\mho)$

We must change how we think of $\hat{K}$

$K(F) = u \leftarrow$ change size measure , change norm

$K : L^2(\mho) \to L^2(\mho)$

$\uparrow$

compact , ~~compact~~ completely continuous

IF $\{u_j\}_{j=1}^{\infty}$ is a bounded set, $\|u_j\|_{L^2(\mho)} \le C \ \forall j$

then $\exists \{u_{j_k}\}_{k=1}^{\infty}$ with $K u_{j_k}$ converges in $L^2(\mho)$

When is $(I - c_2 K)$ not 1-1 ?

iff there is a $\hat{u} \in L^2(\mathcal{V})$, $\hat{u} \neq 0$

with $[I - c_2 K]\hat{u} = 0$

$i : H_0^1(\mathcal{V}) \to L^2(\mathcal{V})$

i compact or completeley continuous

$i \circ \hat{K} : L^2(\mathcal{V}) \to \cancel{H_0^1(\mathcal{V})}\ L^2(\mathcal{V})$

Want $i \circ \hat{K}$ to be compact

we're ok as long as $\hat{K}$ maps bounded sets to bounded sets

Want to show if $\{F_j\}_{j=1}^{\infty}$ is a bounded set in $L^2(\mathcal{V})$ then $\{\hat{K}(F_j)\}_{j=1}^{\infty}$ is a bounded set in $H_0^1(\mathcal{V})$, $\hat{K} = \Gamma^{-1}\hat{\Gamma}$

so $B[u,v] = (\hat{w}, v) = (\Gamma(u), v)$, $\hat{w} = \Gamma(u)$

$$\|\hat{w}\|_{H_0^1(\mathcal{V})} = \|\Gamma(u)\|_{H_0^1(\mathcal{V})}$$

$$\hat{c}\|u\|^2 \le |B[u,u]| = |(\Gamma(u), u)| \le \|\Gamma(u)\|\,\|u\| \quad , \hat{c} > 0$$

$$\hat{c}\|u\| \le \|\underbrace{\Gamma(u)}_{\hat{w}}\| \quad \to \quad \hat{c}\|\Gamma^{-1}\hat{w}\| \le \|\hat{w}\|$$

$$\to \hat{c}\|\hat{K}F\| = \hat{c}\|\Gamma^{-1}\hat{\Gamma}F\| \le \|\hat{\Gamma}F\| \le \hat{\hat{C}}\|F\|$$

claim is that $\exists\ \hat{\hat{c}}$ so that

$$\|\hat{w}\| = \|\hat{\Gamma} F\| \leq \hat{\hat{c}} \|F\|$$

$$(F, v)_{L^2(\mho)} = (\hat{w}, v)_{H_0^1(\mho)} \quad \forall\ v \in H_0^1(\mho)$$

put $v = \hat{w}$, $\|\hat{w}\|^2_{H_0^1(\mho)} = (\hat{w}, \hat{w})_{H_0^1(\mho)} = (F, \hat{w})_{L^2(\mho)}$

$$= \int_{\mho} F \hat{w}\, dx$$

take absolute values, apply Poincaré inequality

$$\|\hat{w}\|^2_{H_0^1(\mho)} \leq \|F\|_{L^2(\mho)} \|\hat{w}\|_{L^2(\mho)} \leq \hat{\hat{c}} \|F\|_{L^2(\mho)} \|\hat{w}\|_{H_0^1(\mho)}$$

$$\rightarrow \quad \|\hat{w}\|_{H_0^1(\mho)} \leq \hat{\hat{c}} \|F\|_{L^2(\mho)}$$

---

the result is that $\hat{K}$ maps bounded sets to bounded sets. This implies that $i \circ \hat{K}$ maps bounded sets to sets with convergent subsequences, ~~se~~

$\Rightarrow i \circ \hat{K} = K$, $L^2(\mho) \rightarrow L^2(\mho)$ is compact

Now take a look at $[I - c_2 K]$

$[I - c_2 K] : L^2(\mathcal{V}) \to L^2(\mathcal{V})$

this operator keeps Function the same

this operator squeezes things together

Another way to write is

$[I - c_2 K] = c_2[\lambda I - K]$

where $\lambda = \frac{1}{c_2}$

Theorem:

Let $K : L^2(\mathcal{V}) \to L^2(\mathcal{V})$. we say $\lambda \in C$ (complex space) is an ~~eigenfunction~~ eigenvalue for K iff $\exists\ u \in L^2(\mathcal{V})$, $u \neq 0$, such that

$$(\lambda I - K)u = 0$$

Remark: when λ is an eigenvalue, $\lambda I - K$ is NOT 1-1!

Facts: ① If λ is not an eigenvalue then $[\lambda I - K]$ is 1-1 and $R[\lambda I - K] = L^2(\mathcal{V})$. Therefore we can find a solution to our problem, which is $u = [I - \frac{1}{\lambda}K]^{-1} K f$

② If λ is large then

$$\left[1 + \frac{1}{\lambda}K + \frac{1}{\lambda^2}K \circ K + \dots\right] K f$$

$$\Rightarrow \quad u \sim K f + \frac{1}{\lambda} K \circ K f + \dots$$

③ Suppose λ is an eigenvalue, then trying to solve $[I - \frac{1}{\lambda}K]u = Kf$ (★). If $\tilde{u}$ is a solution to this problem and $\hat{u}$ is an eigenfunction that satisfies $(I - \frac{1}{\lambda}\hat{K})\hat{u} = 0$, then this implies

$\Rightarrow \quad \tilde{u} + \beta\hat{u}$ is a solution of (★), so that we have nonuniqueness.

over ⟶

It is also true that in this case
$R(I - \frac{1}{\lambda}K)$ is not all of $L^2(\mathcal{V})$

$$\Delta u + \frac{\omega^2}{c^2(x,z)} u = 0$$

$$K = \frac{\omega}{c(x,z)} = \text{wave number}$$

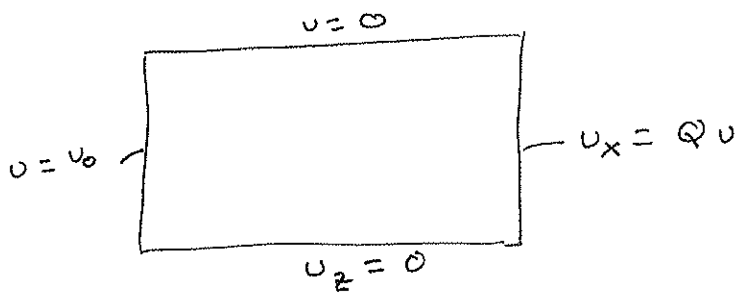
$u = 0$ at $z = 0$

$u_z = 0$ at $z = 1$

$u =$ outgoing at $x = \infty$

$u_n = Qu$

$u(0,z) = u_0$



$$Q = \underbrace{i\sqrt{\lambda_n}(\cdot, \varphi_n)\varphi_n}_{\text{From separation of variables}}$$

Use Lax Milgram Theorem over Complex Hilbert spaces

§

- H complex space
- $B : H \times H \to \mathbb{C}$

$$|B(u,v)| \leq \alpha \|u\|_H \|v\|_H$$

$$\beta \|u\|_H \leq \operatorname{Re} B(u,u)$$

- $f : H \to \mathbb{C}$

then $B[u,v] = (f, v) \quad \forall v \in H$

$\exists !\, u \in H$

$$\varphi'' + \frac{\omega^2}{c_\infty^2(z)} \varphi = \lambda \varphi, \quad \varphi'(0) = \varphi'(1) = 0$$

From separation of variables in z, obtain these eigenfunctions

obtain exponential in x

$$u_{xx} + u_{zz} + \frac{\omega^2}{c^2(x,z)} u = 0$$

$\cancel{u = 0,\ u_z = 0}$

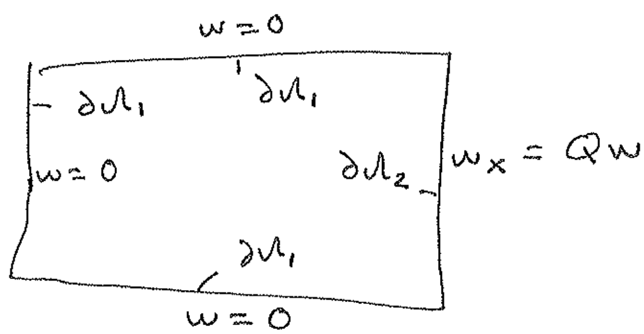
$$u = 0 \quad \text{at} \quad z = 0$$
$$u_z = 0 \quad \text{at} \quad z = 1$$
$$u(0,z) = u_0$$
$$u_x = Q u$$

Transform problem to nicer domain with better BCs

$$u = w + v, \qquad v = \begin{cases} u_0(z)\,\dfrac{(x-\varepsilon)^{2n}}{\varepsilon^{2n}} & 0 < x - \epsilon \\ 0 & x > \varepsilon \end{cases}$$

and reflect, $u(x,z) = u(x, 2-z)$ ← exploiting symmetry from Neumann BC at $z = 1$

$$\Rightarrow \begin{cases} w_{xx} + w_{zz} + \dfrac{\omega^2}{c^2(x,z)} w = -\Delta v - \dfrac{\omega^2}{c^2(x,z)} v \\ w = 0 \quad \text{at } z = 0 \\ w = 0 \quad \text{at } z = 2 \\ w = 0 \quad \text{at } x = 0 \\ w_x = Q w \end{cases}$$



equivalently, for the functional analysis point of view

$$\boxed{\begin{aligned} &\Delta w + \frac{\omega^2}{c^2(x,z)} w = f \\ &w = 0 \quad \text{on } \partial\Omega_1 \\ &w_x = Q w \quad \text{on } \partial\Omega_2 \end{aligned}} \qquad f = -\Delta v - \frac{\omega^2}{c^2(x,z)} v$$

3/4/05
2/4

~~$w_{xx} + w_{zz}$~~

Multiply PDE by test function

$v \in C_0^{\infty}(\Omega \cup \partial\Omega_2)$

then integrate by parts.

(Actually multiply by conjugate of v)

$$\underbrace{\int_\Omega \Delta w \,\bar{v}\, dx}_{\text{apply Green's Formula}} + \int_\Omega \frac{\omega^2}{c^2(x,z)} w \,\bar{v}\, dx = \int_\Omega f \bar{v}\, dx$$

$\bar{v}$ vanishes on $\partial\Omega_1$ and $\frac{\partial w}{\partial n} = Qw$ on $\partial\Omega_2$

$$-\int_\Omega \nabla w \cdot \nabla \bar{v}\, dx + \overbrace{\int_{\partial\Omega} \bar{v} \frac{\partial w}{\partial n}\, d\sigma} + \int_\Omega \frac{\omega^2}{c^2} w\bar{v} = \int_\Omega f\bar{v}\, dx$$

$$-\int_\Omega \nabla w \cdot \nabla \bar{v}\, dx + \int_0^2 \bar{v} \cdot Qw\, dz + \int_\Omega \frac{\omega^2}{c^2} w\bar{v}\, dx = \int_\Omega f\bar{v}\, dx$$

substitute $Qw = \sum i\sqrt{\lambda_n}(w,\varphi_n)\varphi_n$
such that second integral becomes
after some reduction

$$\int_0^2 \bar{v} \sum_{n=1}^{\infty} i\sqrt{\lambda_n}(w,\varphi_n)\varphi_n = \sum_{n=1}^{\infty} i\sqrt{\lambda_n}(w,\varphi_n)(\bar{v},\varphi_n)$$

both inner products over z

so that the weak formulation becomes

$$\underbrace{-\int_\Omega \nabla w \cdot \nabla \bar{v}\, dx + \sum_{n=1}^{\infty} i\sqrt{\lambda_n}(w,\varphi_n)(\bar{v},\varphi_n) + \int_\Omega \frac{\omega^2}{c^2(x,z)} w \bar{v}\, dx}_{-B[u,v]} = \int_\Omega f \bar{v}\, dx$$

(note, $B[u,v]$ is the negative of what's in the bracket)

we need to associate Hilbert space with Bilinear form.
Must define inner product to handle the summation term

define $\hat{H}_1(\Omega) = \overline{C_0^\infty(\Omega \cup \partial\Omega_2)}$ = new Hilbert space

completion

with inner product

$$\langle\langle u, w \rangle\rangle = \underbrace{\int_\Omega \nabla u \cdot \nabla \bar{w} + \int_\Omega u \cdot \bar{w}}_{\text{usual Hilbert space inner product}} + \underbrace{\sum_{n=N+1}^{\infty} \sqrt{|\lambda_n|}\,(u(L,\cdot),\varphi_n)(\overline{w}(L,\cdot),\varphi_n)}_{\text{boundary contribution}}$$

where $(\hat{H}_1(\Omega), \langle\langle\cdot,\cdot\rangle\rangle)$ pre Hilbert space.
Is it complete?
Consider $(\hat{\hat{H}}_1(\Omega), \langle\langle\cdot,\cdot\rangle\rangle)$ to be the completion of $(\hat{H}_1(\Omega), \langle\langle\cdot,\cdot\rangle\rangle)$ with $\|\ \|_{\langle\langle,\rangle\rangle}$ induced by $\langle\langle,\rangle\rangle$

$$\rightarrow \quad \|u\|^2_{\langle\langle,\rangle\rangle} = \int_\Omega |\nabla u|^2 + \int_\Omega u^2 + \sum_{n=N+1}^{\infty} \sqrt{|\lambda_n|}\,(u,\varphi_n)^2$$

L^2 norm of gradient ∇u

L^2 norm of u

the eigenvalues λ_n, $n \geq N+1$ are negative, which is why we have the absolute value

3/4/05
3/4

BVP ① $\begin{cases} Lw = f & \text{in } \Omega \\ w(x, 2-z) = w(x, z) \\ w = 0 & \text{on } \partial\Omega_1 \\ w_x = Qw & \text{on } \partial\Omega_2 \end{cases}$

<u>Def:</u> We say $w \in (\hat{\hat{H}}_1(\Omega), \langle\langle, \rangle\rangle)$ is a weak solution of BVP ① if $B[w, v] = (-f, v)_{L^2}$ for all $v \in \hat{\hat{H}}_1(\Omega)$

We want to show that B is bounded and that Re(B) is coercive (strongly positive) because the goal is to apply Lax Milgram theorem.

$$|B(u,v)| \leq \left| \int_\Omega \nabla u \cdot \nabla \bar{v} \, dx dz \right| + \left| \int_\Omega u \bar{v} \frac{\omega^2}{c^2(x,z)} \, dx dz \right|$$

$$+ \left| \sum_{n=1}^{\infty} i\sqrt{\lambda_n} \left(u(L,\cdot), \varphi_n \right) \left(\bar{v}(L,\cdot), \varphi_n \right) \right|$$

apply Cauchy-Schwarz to first two integrals and split summation into two parts corresponding to positive & negative eigenvalues

$$\Rightarrow |B(u,v)| \leq \|\nabla u\|_{L^2} + \|u\|_{L^2} \|v\|_{L^2} M + \left| \sum_{n=1}^{N} (\cdots) \right| + \left| \sum_{n=N+1}^{\infty} (\cdots) \right|$$

From definition of induced norm

$$\|v\|_{L^2} \|u\|_{L^2} \text{ and } \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} \leq \|u\|_{\langle\langle,\rangle\rangle} \|v\|_{\langle\langle,\rangle\rangle}$$

$$\to \quad \left| B[u,v] \right| \le \|u\|_{\langle\langle,\rangle\rangle} \|v\|_{\langle\langle,\rangle\rangle} + M \|u\|_{\langle\langle,\rangle\rangle} \|v\|_{\langle\langle,\rangle\rangle}$$

$$+ \sum_{n=1}^{N} \sqrt{\lambda_n} \left| u(t,\cdot)\varphi_n \right|_{L^2} \left| \bar{v}(t,\cdot)\varphi_n \right|_{L^2}$$

$$+ \sum_{n=N+1}^{\infty} \sqrt{\lambda_n} \left| u(t,\cdot)\varphi_n \right|_{L^2} \left| \bar{v}(t,\cdot)\varphi_n \right|_{L^2}$$

use inequalities

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \le \left(a_1^2 + \cdots + a_n^2 \right)^{1/2} \left(b_1^2 + \cdots + b_n^2 \right)^{1/2}$$

$$\to \quad \sum_{n=1}^{\infty} a_n b_n \le \left(\sum_{n=1}^{\infty} a_n^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} b_n^2 \right)^{1/2}$$

such that the first sum is bounded by

$$(+) \quad \left(\sum_{n=1}^{N} \sqrt{\lambda_n} \left| u(t,\cdot)\varphi_n \right|^2 \right)^{1/2} \left(\sum_{n=1}^{N} \sqrt{\lambda_n} \left| \bar{v}(t,\cdot)\varphi_n \right|^2 \right)^{1/2}$$

and the second sum is bonded by

$$(*) \quad \left(\sum_{n=N+1}^{\infty} \sqrt{\lambda_n} \left| u(t,\cdot)\varphi_n \right|^2 \right)^{1/2} \left(\sum_{n=N+1}^{\infty} \sqrt{\lambda_n} \left| \bar{v}(t,\cdot)\varphi_n \right|^2 \right)^{1/2} \le \|u\|_{\langle\langle,\rangle\rangle} + \|v\|_{\langle\langle,\rangle\rangle}$$

Use definition of norm to bound ~~these~~ the last ~~two~~ bounds (*). For the first bound (+), use trace theorems

see Evans pg 258

$$\|\gamma u\|_{L^2(\partial\Omega)} \le \tilde{c}\|u\| \quad , \text{where } \gamma \text{ is a trace operator}$$

so this allows us to bound (+)

such that we have

$$|B[u,v]| \le \|u\|_{\langle\langle , \rangle\rangle} \|v\|_{\langle\langle , \rangle\rangle} (L + M + \alpha)$$

and therefore B is bounded. $\boxed{\text{PWOH}}$

Now we want to show that B is coercive.
However, we can only show a weaker
coercivity condition, i.e.,

$\exists\, \beta > 0$, $\gamma > 0$ s.t.

$$\beta \|u\|^2_{\hat{H}_1(\Omega)} \le \operatorname{Re} B[u,v] + \gamma \|u\|^2_{L^2(\Omega)}$$

simple to prove using definitions.

we cannot apply Lax Milgram due to
this extra term $\gamma \|u\|^2_{L^2(\Omega)}$, however we
can apply a weake form of LM.

<u>**Thm**</u> $\boxed{Lw - \nu w = f \text{ in } \Omega}$ $*$

There exists a $\gamma \ge 0$ st for each $f \in L^2(\Omega)$
and each $\nu \ge \gamma$ then $*$ has a unique
weak solution.

Define
$$B_\nu[u,v] = B[u,v] + \nu (u,v)_{L^2}$$
then B_ν satisfies pure coercivity condition
so that can apply LM to $B_\nu[u,v]$
so we have proved existence and uniqueness
of $*$, not with original problem $Lw = f$

So we have showed existence and uniqueness of

$$\Delta w + \frac{\omega^2}{c^2(x,z)} w - \nu w = F$$
$$w = 0 \quad \text{on } \partial\Omega_1$$
$$w_x = Qw \quad \text{on } \partial\Omega_2$$

Note, $\frac{\omega^2}{c^2(x,z)} - \nu < 0$, so that we have to both sides to problems. Add something make coefficient of u positive (as we did in the last lecture).

<u>Fredholm Alternative</u>

only one of the following statements hold

ⓐ $\forall f \in L^2(\Omega)$ there exists a unique weak solution W,

$$Lw = f \quad \text{in } \Omega$$
$$w = 0 \quad \text{on } \partial\Omega_1$$
$$w_x = Qw \quad \text{on } \partial\Omega_2$$

ⓑ there exists weak solution $w \neq 0$ of the homogeneous problem

$$Lw = 0 \quad \text{in } \Omega$$
$$w = 0 \quad \text{on } \partial\Omega_1$$
$$w_x = Qw \quad \text{on } \partial\Omega_2$$

- rerun NO8 for much longer domain
- discuss w/ Schwendeman & Kapila
  results concerning similarity soln.
  sound speed, v, $v-c$, $v+c$
- write up of NO8
- edit steady state document

Chapter 10

derive Eikonal equation

$$\nabla \cdot (\nu \nabla w) = \rho w_{tt}$$

omega

Let $w = A(x,t,\omega) e^{i\omega t \phi(x,t)}$

$$= \left[A_0(x,t) + \frac{1}{i\omega} A_1(x,t) + \dots\right] e^{i\omega \phi(x,t)}$$

to leading order, obtain $\boxed{\phi_t = \sqrt{\frac{\nu}{\rho}} |\nabla \phi|}$ Eikonal equation

more generally, write $\boxed{v_t^\varepsilon + H(\nabla v^\varepsilon, x) \underbrace{- \varepsilon \Delta v^\varepsilon}_{\text{added on Laplacian}} = 0}$, $\varepsilon > 0$

"we regularize the solution by adding on higher order term."

What happens when $\varepsilon \to 0$

v^ε is known as the viscosity solution

Cannot define solutions as we did for linear eqns because Eikonal is nonlinear.

we'll need to define weak solutions

- what is the weak solution? how many are there?
- what is the weak solution

we'll show there is only one weak solution as per our definition of weak solution and we'll show this is a solution of a control theory problem

$$u_t^\varepsilon + H(\nabla u^\varepsilon, x) - \varepsilon \Delta u^\varepsilon = 0, \quad \varepsilon > 0, \quad x \in \mathbb{R}^n, \quad 0 < t < \infty$$

$$u^\varepsilon(x,0) = g(x) \quad \text{at } t = 0$$

---

this is a nonlinear parabolic equation

For each $\varepsilon > 0$, the solution is smooth. This can be proven. Well known for linear parabolic PDE. For now, the only requirement we impose on $u^\varepsilon(x,t)$ is that it is continuous ...

we want to show that $\{u^\varepsilon\}_{0 \le \varepsilon \le n}$ is an <u>equicontinuous</u> family

$$\text{equicontinuous} \begin{cases} \text{given } \xi > 0, \ \exists\, \delta > 0 \text{ with} \\ \left| u^\varepsilon(x,t) - u^\varepsilon(\hat{x},\hat{t}) \right| < \xi \\ \text{when } |x - \hat{x}| + |t - \hat{t}| < \delta \quad \forall \varepsilon \end{cases}$$

this will imply that there exists subsequence $\{u^{\varepsilon_j}\}_{j=1}^{\infty}$ such that $u^{\varepsilon_j} \to u$, uniformly, and this implies u is uniformly continuous.

This only tells us that u is continuous, but it tells us nothing about the solution itself.

We want to define the weak solution.

<u>Definition of weak solution</u> of

$$\bigstar \quad \begin{aligned} u_t + H(\nabla u, x) &= 0 \quad x \in \mathbb{R}^n,\ t > 0 \\ u(x,0) &= g(x) \quad x \in \mathbb{R}^n,\ t \geq 0 \end{aligned}$$

A bounded, uniformly continuous function u is called a viscosity solution of $\bigstar$ iff

i) $u(x,0) = g(x) \quad \forall x \in \mathbb{R}^n$

ii) For each $v \in C^\infty(\mathbb{R}^n)$, if $u - v$ has a local maximum at a point $(x_0, t_0) \in \mathbb{R}^n \times (0,\infty)$

then $v_t(x_0,t_0) + H(\nabla_x v(x_0,t_0), x_0) \leq 0$

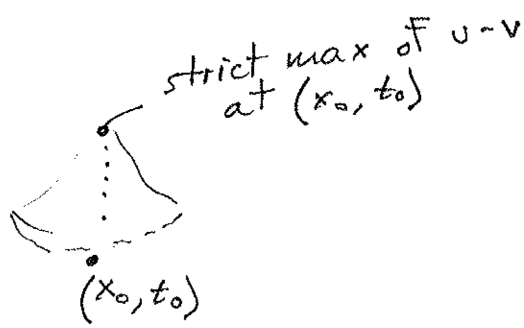
iii) For each $v \in C^\infty(\mathbb{R}^n)$, if $u - v$ has a local minimum at a point $(x_0,t_0) \in \mathbb{R}^n \times (0,\infty)$

then $v_t(x_0,t_0) + H(\nabla_x v(x_0,t_0), x_0) \geq 0$

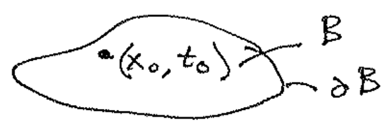
Suppose $u = \lim_{j \to \infty} u^{\varepsilon_j}(x,t)$

suppose $u - v$ has a strict maximum at (x_0, t_0)

show $v_t + H(\nabla_x v(x_0,t_0), x_0) \leq 0$



If strict max, one can take small ball around x_0, t_0



then $(u-v)(x_0,t_0) > (u-v)(x,t)$, $(x,t) \in \partial B$

we also know $\lim_{j\to\infty} u^{\varepsilon_j} = u$ (uniform convergence) consider $u^{\varepsilon_j} - v$

Apply the operator

$$(9) \quad u_t^{\varepsilon_j} - v_t^{\varepsilon_j} + H(\nabla u^{\varepsilon}(x,t), x) - H(\nabla v, x) - \varepsilon^j \Delta u^{\varepsilon_j}$$

we want to say something about the sign about this quantity

Claim, if ε_j is small enough,

$$(u^{\varepsilon_j} - v)(x_0, t_0) > (u^{\varepsilon_j} - v)(x,t) \quad , \quad (x,t) \in \partial B$$

$\Rightarrow \exists \ (x^{\varepsilon_j}, t^{\varepsilon_j}) \in B$ where $(u^{\varepsilon_j} - v)$ has a maximum (not saying strict maximum)

$$\Rightarrow \quad u_t^{\varepsilon_j} = v_t \quad \text{at} \quad (x^{\varepsilon_j}, t^{\varepsilon_j})$$

$$\text{and} \quad \nabla_x u^{\varepsilon_j} = \nabla_x v \quad \text{at} \quad (x^{\varepsilon_j}, t^{\varepsilon_j})$$

Therefore evaluate (9) at $x^{\varepsilon_j}, t^{\varepsilon_j}$, this boils down to

$$- \varepsilon^j \Delta u^{\varepsilon_j}$$

Chapter 10

$$u_t + H(\nabla u, x) = 0 \quad , \quad \mathbb{R}^n \times (0,\infty)$$ ★

$$u(x,0) = g(x)$$

In chapter 3, problem is studied as $H(p,x)$, where $p = \nabla u$ and we assume $\dfrac{|H(p,x)|}{|p|} \to \infty$ as $p \to \infty$

For our purposes, we are thinking about the Eikonal equation, ie $H(\phi, x) = \sqrt{\dfrac{\nu(x)}{\rho(x)}}\, |\nabla \phi|$

one approach is to regularize the solution

$$u_t^\varepsilon + H(\nabla u^\varepsilon, x) = \varepsilon \Delta u^\varepsilon \quad , \quad u^\varepsilon(x,0) = g(x)$$

parabolic problem, smooth solution

people can compute this

but what happens as $\varepsilon \to 0$?

one can show that $u^{\varepsilon_j} \to u$ (uniformly)

which implies u is uniformly ~~continuous~~

<u>Def</u> A bounded uniformly continuous function $\hat{u}$ is a viscosity solution of $\bigstar$ if $\hat{u}(x,0) = g(x)$, $x \in \mathbb{R}^n$ and for each $v \in C^\infty(\mathbb{R}^n \times (0,\infty))$ if $\hat{u} - v$ has a local max (min) at (x_0, t_0) then

$$\left[v_t + H(\nabla v, x)\right]\Big|_{x_0,t_0} \underset{(\geq)}{\leq} 0$$

~~Show~~ $\cancel{u = \lim_{j\to\infty} u^{\varepsilon_j}}$

Suppose $u - v$ has a strict local <u>max</u> at (x_0, t_0) then

$$(u-v)\Big|_{(x_0,t_0)} > (u-v)(x,t) \quad , \quad (x,t) \in \overline{B} - (x_0,t_0)$$

as long as we are on the boundary ∂B we cand add constant and still satisfy inequality

$$(u-v)\Big|_{(x_0,t_0)} > (u-v)(x,t) + c_0 \quad , \quad \begin{matrix} c_0 > 0 \\ (x,t) \in \partial B \end{matrix}$$

Select ε_j small enough so that

$$\left(u^{\varepsilon_j} - v\right)(x_0, t_0) > \left(u^{\varepsilon_j} - v\right)(x,t) + \frac{c_0}{2} \quad , \partial B$$

$\exists\, (x_{\varepsilon_j}, t_{\varepsilon_j})$ local max for $\left(u^{\varepsilon_j} - v\right)$ in B

$$\left(u^{\varepsilon_j} - v\right)(x_{\varepsilon_j}, t_{\varepsilon_j}) > \left(u^{\varepsilon_j} - v\right)(x,t) + \frac{c_0}{2} \quad , x,t \in \partial B$$

where we know u^{ε_j} satisfies viscous equation and converges uniformly to u as $\varepsilon_j \to 0$

Because $u^{\varepsilon_j} - v$ is a local max, we know

$$\nabla u^{\varepsilon_j} - \nabla v = 0 \quad \text{at } (x,t) = (x^{\varepsilon_j}, t^{\varepsilon_j})$$

$$u_t^{\varepsilon_j} - v_t = 0$$

$$\left[u_t^{\varepsilon_j} + H(\nabla u^{\varepsilon_j}, x)\right]_{(x^{\varepsilon_j}, t^{\varepsilon_j})} = \left[v_t + H(\nabla v, x)\right]_{(x^{\varepsilon_j}, t^{\varepsilon_j})} = \varepsilon_j \Delta u^{\varepsilon_j}$$

But we don't know about $\varepsilon_j \Delta u^{\varepsilon_j}$, ~~aside~~ but we do know (from basic calculus)

$$\Delta (u^{\varepsilon_j} - v)\Big|_{x^{\varepsilon_j}, t^{\varepsilon_j}} \leq 0$$

so that we have

$$\left[u_t^{\varepsilon_j} + H(\nabla u^{\varepsilon_j}, x)\right]_{(x^{\varepsilon_j}, t^{\varepsilon_j})} = \left[v_t + H(\nabla v, x)\right]_{(x^{\varepsilon_j}, t^{\varepsilon_j})} \leq \varepsilon_j \Delta v_{(x^{\varepsilon_j}, t^{\varepsilon_j})}$$

However, in the limit as $j \to \infty$

$$x^{\varepsilon_j}, t^{\varepsilon_j} \to x_0, t_0$$

and $\quad \Delta v\big|_{(x^{\varepsilon_j}, t^{\varepsilon_j})} \to \Delta v\big|_{(x_0, t_0)}$

such that $\quad \boxed{v_t + H(\nabla v, x) \leq 0 \quad \text{at } (x_0, t_0)}$

For the strict minimum, it is east to reverse these argument and to come up with the correspond inequality, but we want to remove the assumption of <u>strict</u> max or min and replace with <u>local</u> max or min

Suppose $u-v$ has a local max at (x_0, t_0)
then construct

$$\tilde{v} = v - \delta\left[(x-x_0)^2 + (t-t_0)^2 \right]$$

then $u - \tilde{v}$ has a strict local max at (x_0, t_0)

$\Rightarrow \quad \tilde{v}_t + H(\nabla \tilde{v}, x) \leq 0 \quad$ at (x_0, t_0)

(from previous results)

but $\tilde{v}_t \Big|_{(x_0,t_0)} = v_t - \delta 2(t-t_0)\Big|_{(x_0,t_0)} = v_t \Big|_{x_0,t_0}$

and $\nabla \tilde{v}\Big|_{x_0,t_0} = \nabla v \Big|$

I) We want to continue to check that our definition is consistent. Suppose we have a continuous ~~$\S$~~ solution of ★.

$\tilde{u}$ is a C^1 solution of ★

Is $\tilde{u}$ a viscosity solution ?

suppose $v \in C^\infty(\mathbb{R}^n \times (0, \infty))$
suppose $u-v$ has a local max at (x_0, t_0)
we want to show that v satisfies

$$v_t + H(\nabla v, x) \leq 0 \quad \text{at } (x_0, t_0)$$

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since $\tilde{u} \in C^1$ and $v \in C^\infty$ at x_0, t_0
then at local max (or min), we have

$$\nabla \tilde{u} = \nabla v \quad , \text{ at } (x_0, t_0)$$

$$\tilde{u}_t = v_t \quad , \text{ at } (x_0, t_0)$$

$$0 = \tilde{u}_t + H(\nabla \tilde{u}, x) \quad \text{(everywhere)}$$

$$= v_t + H(\nabla v, x) \quad \text{at } (x_0, t_0)$$

$$\Rightarrow \quad v_t + H(\nabla v, x) \leq 0 \quad \text{at } (x_0, t_0)$$

(II) suppose $\hat{u}$ is a viscosity solution to ★
and $\hat{u} \in C^1(\mathbb{R}^n \times (0,\infty))$ then we want to
show $\hat{u}_t + H(\nabla \hat{u}, t) = 0 \quad \forall\, x, t \in \mathbb{R}^n \times (0,\infty)$

proof starts with Lemma pg 544, Evans
and then Theorem 1 pg 545.

Semigroup Theory (Evans pg 412-424)

$u_t = \nabla \cdot (B \nabla u)$, $x \in U$, $0 < t < T$

B is a symmetric matrix

$B = (b_{ij})_{i,j=1}^{n} \in C(\bar{U})$

$u(x,0) = u$ - (notation used in Evans) , $x \in U$, $t = 0$

$u = 0$ on ∂U, $0 \le t \le T$

} parabolic problem

$u_{tt} = \nabla \cdot (B \nabla u)$, $x \in U$, $0 < t < T$

$u(x,0) = u$, $x \in U$, $t = 0$

$u_t(x,0) = v$, $x \in U$, $t = 0$

$u(x,t) = 0$, $x \in \partial U$, $0 \le t \le T$

} hyperbolic problem

transform to a system

let $v = u_t$, $v_t = \nabla \cdot (B \nabla u)$ } $x \in U$, $0 < t \le T$

$\begin{pmatrix} u \\ v \end{pmatrix}(x,0) = \begin{pmatrix} u \\ v \end{pmatrix}$ $x \in U$, $t = 0$

$\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $x \in \partial U$, $0 \le t \le T$

$\Rightarrow$ $u_t = A u$, $u(x,0)$ given

$A : X \supset D(A) \to X$

$D(A)$: domain of A, dense in X, for every $w \in X$ $\exists$ c.s. $\{w_j\}_{j=1}^{\infty} \subset D(A)$ and $\lim_{j \to \infty} \|w_j - w\|_X = 0$

X: Banach space (no inner product)

$\| \cdot \|_X$ norm

space is complete, all c.s. have limits

eg $\underline{X} = L^2(U)$

$$D(A) = H^2(U) \cap H_0^1(U)$$

$\dot{x} = a(t)\,x$, $x(0) = x_0$, $x(t) = x_0 e^{\int_0^t a(s)ds}$

Now consider a system

$\dot{\underline{x}} = A\underline{x}$, $\underline{x}$ - n-vector , A symmetric

$\underline{x}(0) = \underline{x}_0$, $\underline{x}(t) = \underline{x}_0 e^{\int_0^t A(s)ds}$

$$\underline{x}(t) = \underline{x}_0 \left[I + \int_0^t A(s)ds + \frac{1}{2!}\int_0^t A(s)ds \int_0^{t_1} A(s_1)ds_1 + \dots \right]$$

Now differentiate this:

$$\dot{\underline{x}} = \underline{x}_0 \left[A(t) + \frac{1}{2}A(t)\int_0^t A(s_1)ds + \cancel{\frac{1}{2}A(t)\int_0^t A(s)ds + \dots} \right.$$

$$\left. + \frac{1}{2}\int_0^t A(s)ds\, A(t) \right]$$

we need these ~~operators~~ matrices to commute

the point is : the parabolic and hyperbolic problems will be different since one is a system

Notation: $u(t) = \underbrace{S(t)}_{\text{operator}}\, \underbrace{u}_{\text{initial condition}}$

$S(\cdot)u : [0,\infty] \to \underline{X}$, real Banach space

$S(0)u = u \quad \to \quad S(0) = I$, identity

$S(t+s)u = S(t)S(s)u = S(s)S(t)u$

the map $t \to S(t)u$ is continuous

we have identity, commutativity, and others, so have some properties of Groups, why called semi-groups

Def: (1) A Family $\{S(t)\}_{t\geq 0}$ of bounded linear operators $S(t) : X \to X$ is called semigroup if ★ is satisfied

(2) we say $\{S(t)\}_{t\geq 0}$ is a contraction semi-group if (1) is satisfied and

operator norm → $\|S(t)\| \leq 1 \quad \forall\, t \geq 0$

Remark:

$$\|S(t)\| = \sup_{\substack{u \in X \\ u \neq 0}} \frac{\|S(t)u\|_X}{\|u\|_X} = \sup_{\substack{u \in X \\ \|u\|_X = 1}} \|S(t)u\|_X$$

contraction semigroup

$$\begin{aligned} \|S(t)u - S(t)u_1\|_X &= \|S(t)(u-u_1)\| \\ &\leq \|S(t)\| \, \|u-u_1\|_X \\ &\leq \|u-u_1\|_X \end{aligned}$$

operator norm

Elementary Properties

Def [define D(A)] ⟨Let $\{S(t)\}_{t\geq 0}$ be a contraction semigroup on X, a real Banach space.⟩

$$D(A) = \left\{ u \in X \;\middle|\; \lim_{t\to 0} \frac{S(t)u - S(0)u}{t} \text{ exists in } X \right\}$$

and $Au = \lim_{t \to 0} \frac{S(t)u - u}{t}$. We call $A : X \supset D(A) \to X$ the infinitesimal generator of the semigroup $\{S(t)\}_{t\geq 0}$

D(A) is the domain of A.

we want $S(t)u$ to be solution of

$$\frac{d}{dt}\left(S(t)u\right) = A\left(S(t)u\right) \quad , \quad S(0)u = u$$

Theorem (Differential properties of semigroups of operators)

Assume $\langle\rangle$. Assume $u \in D(A)$.

i) $S(t)u \in D(A)$

ii) $S(t)A = AS(t) \quad \forall t$

iii) $t \to S(t)u \; , \; [0,\infty] \to D(A)$ is differentiable (in t)

iv) $\frac{d}{dt}\left(S(t)u\right) = AS(t)u$

Demonstration

$$D(A) = \left\{ u \in X \;\middle|\; \lim_{t \to 0} \frac{S(t)u - u}{t} \text{ exists in } X \right\}$$

we need to show that

$$\lim_{s \to 0} \frac{S(s)\left[S(t)u\right] - S(t)u}{s} \quad \text{exists}$$

$$= \lim_{s \to 0} \frac{S(t)S(s)u - S(t)u}{s} = \lim_{s \to 0} S(t)\left[\frac{S(s)u - u}{s}\right]$$

bounded linear operator, so can take limit inside

$$= S(t) \underbrace{\lim_{s \to 0}\left[\frac{S(s)u - u}{s}\right]}_{\text{this limit exists}}$$

we have showed that $S(t)u \in D(A)$.

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- Note, we have defined A as

$$Au = \lim_{s \to 0} \frac{S(s)u - u}{s}$$

$$\lim_{s \to 0} \frac{S(s)\,S(t)\,u - S(t)\,u}{s} = A\,S(t)\,u = S(t)\,A u \implies \underline{\text{commutativity}}$$

- Now consider (iv):

show: $$\lim_{\substack{h \to 0 \\ h > 0}} \frac{S(t-h)u - S(t)u}{h} - A\,S(t)\,u \quad \text{exists and equals } 0$$

$$= \cancel{\lim_{h\to 0}}_{h>0} \left\| \frac{S(t-h)u - S(t-h)\,S(h)\,u}{h} - A\,S(t)\,u \right\|_X$$

$$= \cancel{\lim_{h\to 0}}_{h>0} \left\| S(t-h)\left[\frac{u - S(h)\,u}{h}\right] - A\,S(t)\,u \right\|_X$$

$$= \cancel{\lim_{h\to 0}}_{h>0} \left\| S(t-h)\left[\frac{u - S(h)u}{h} - Au\right] + S(t-h)Au - S(t)Au \right\|_X$$

$$\leq \underbrace{\|S(t-h)\|}_{\substack{\leq 1 \\ \text{contraction} \\ \text{semigroup}}} \underbrace{\left\| \frac{u - S(h)u}{h} - Au \right\|}_{\substack{\downarrow 0 \text{ as } h \to 0 \\ \text{since} \\ u \in D(A)}} + \underbrace{\left\| [S(t-h) - S(t)]\,Au \right\|}_{\substack{\to 0 \text{ as } h \to 0 \\ \text{since } S(\cdot) \text{ is} \\ \text{continuous}}}$$

$\implies$ (iii) and (iv)

derivative of $S(t)$ exists and

$$\frac{d}{dt}(S(t)\,u) = A\,S(t)\,u$$

---

starting with contraction semigroup, get a differential equation

A is the infinitesimal generator of S

<u>Theorem 2</u> : (properties of generators)

i) $D(A)$ is dense in $\mathbb{X}$

ii) A is a closed operator

<u>Def:</u> Suppose $A : \mathbb{X} \supset D(A) \to \mathbb{X}$ and $D(A)$ is dense in $\mathbb{X}$, then A is closed iff:

given C.S. $\{u_k\}_{k=1}^{\infty} \subset D(A)$ with

$$\lim_{k\to\infty} u_k = u \in \mathbb{X}$$

having also the property that $\{Au_k\}$ is a C.S. in $\mathbb{X}$ with

$$\lim_{k\to\infty} Au_k = w \in \mathbb{X}$$

then $u \in D(A)$, $Au = w$

This says $\left\{ (u, Au) \mid u \in D(A) \right\}$ is closed in $\mathbb{X} \times \mathbb{X}$.

---

these differential operators A are unbounded in $\mathbb{X}$ but they are closed

---

<u>Proof of Theorem 2</u>

First, show $u^t := \int_0^t S(s) u \, ds \in D(A)$

goal: $\lim_{s\to 0} \dfrac{S(s) u^t - u^t}{s}$ exists

but first look at $\lim_{t\to 0} \dfrac{u^t - u}{t}$ and show this limit is 0

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$$\left\| \frac{u_t - ut}{t} \right\|_X = \left\| \frac{\int_0^t S(s)\,u\,ds - ut}{t} \right\|$$

$$= \left\| \frac{\int_0^t S(s)\,u\,ds - tS(0)\,u}{t} \right\|$$

$$= \left\| \frac{\int_0^t [S(s) - S(0)]\,u\,ds}{t} \right\|$$

$$\leq \frac{\int_0^t \| (S(s) - S(0))\,u \|\,ds}{t} \leq \frac{\int_0^t \varepsilon\,ds}{t}$$ when $|t| < \delta$ according to continuity of S

$$= \varepsilon$$

Let $\varepsilon \to 0 \Rightarrow \delta \to 0$

$$\lim_{t\to 0} \frac{u^t - u}{t} = 0 \Rightarrow \boxed{\lim_{t\to 0} \frac{u^t}{t} = u}$$

Now $u^t \in D(A)$ – intuitive because integrating things that are in domain of A, but it has to be shown

show $\displaystyle\lim_{r\to 0^+} \frac{S(r)\,u^t - u^t}{r}$ exists

$$\frac{1}{r}\left(S(r)\,u^t - u^t \right) = \frac{1}{r}\left[S(r) \int_0^t S(s)\,u\,ds - \int_0^t S(s)\,u\,ds \right]$$

$$\int_0^t S(s) \left[\frac{S(r)\,u - u}{r} \right] ds$$

Now take limit

$$\lim_{r\to 0^+} \int_0^t S(s) \left[\frac{S(r)\,u - u}{r} \right] ds = \int_0^t S(s) \underbrace{\lim_{r\to 0^+} \left[\frac{S(r)\,u - S(r)}{r} \right]}_{\text{this limit exists}} ds$$

$$\Rightarrow u^t \in D(A)$$

~~Note, $\lim_{r\to 0} \frac{S(r)u-u}{r} = Au$~~

~~so that in the limit, the integral becomes~~

Note, $\lim_{r \to 0} \frac{S(r)u - u}{r} = Au$

so that in the limit, we have

$$Au^t = \lim_{r\to 0} \frac{1}{r}\left(S(r)u^t - u^t \right)$$

$$= \int_0^t S(s) \lim_{r\to 0} \left(\frac{S(r)u-u}{r}\right) ds = \int_0^t S(s) Au \, ds$$

$$= \int_0^t A S(s) u \, ds = \int_0^t \frac{d}{ds}\left(S(s)u\right) ds = S(t)u - u$$

$$\Rightarrow \qquad Au^t = S(t)u - u$$

Show A is closed:

suppose $\{u_k\}_{k=1}^{\infty} \subset D(A)$

$u_k \to u$, $Au_k \to v \in X$

$$\underbrace{S(t)u_k}_{S(t)u} - \underbrace{u_k}_{u} = \int_0^t S(s) \underbrace{Au_k}_{v} \, ds \Rightarrow \text{in the limit as } k \to \infty$$

$$S(t)u - u = \int_0^t S(s) Av \, ds$$

A is closed.

<u>Def:</u> We say a real number λ is in the <u>resolvent set</u> of A iff

$$(\lambda I - A) : D(A) \to X$$

is 1-1 and $\underset{\substack{\uparrow \\ \text{range}}}{R}(\lambda I - A) = X$.

Denote the resolvent set $\rho(A)$.

<u>Def:</u> If $\lambda \in \rho(A)$ then the <u>resolvent operator</u>

$$R_\lambda u = (I\lambda - A)^{-1} u$$

$$R_\lambda : X \to X$$

Claim: R_λ is closed.

Demonstration: $\{u_k\}_{k=1}^{\infty}$ is a Cauchy sequence in X

$\{R_\lambda u_k = w_k\}_{k=1}^{\infty}$ is a Cauchy sequence in X

$u_k \to u \in X$, $R_\lambda u_k = w_k \to w \in X$

Show: $R_\lambda u = w$

$(\lambda I - A)^{-1} u_k = R_\lambda u_k = w_k$

$u_k = (\lambda I - A) w_k = \lambda w_k - A w_k$

A is closed
$$\begin{array}{ccc} A w_k & = \lambda w_k & - u_k \\ \downarrow & \downarrow & \downarrow \\ w & w & u \end{array}$$

since A is closed, $Aw = \lambda w - u \Rightarrow (\lambda I - A) w = u$

$R_\lambda^{-1} w = u \to R_\lambda u = w \Rightarrow$ $\boxed{R_\lambda \text{ is closed}}$

closed graph theorem says

$$\boxed{R_\lambda : X \to X \text{ , closed } \Rightarrow R_\lambda \text{ is bounded}}$$

claim: $A R_\lambda = R_\lambda A$

where
$$\begin{aligned}
A R_\lambda &= A[\lambda I - A]^{-1} \\
&= \big((A - \lambda I) + \lambda I\big)(\lambda I - A)^{-1} \\
&= -I + \lambda(\lambda I - A)^{-1} \\
&= (\lambda I - A)^{-1}(A - \lambda I) + (\lambda I - A)^{-1}\lambda I \\
&= (\lambda I - A)^{-1}[A - \lambda I + \lambda I] \\
&= (\lambda I - A)^{-1} A \\
&= R_\lambda A
\end{aligned}$$

Suppose $S(t)$ is a contraction semigroup and A is the infinitesimal generator:

Theorem

i) suppose $\lambda, \nu \in \rho(A)$. Then
$$R_\lambda - R_\nu = R_\lambda R_\nu (\nu - \lambda) = R_\nu R_\lambda (\nu - \lambda)$$

ii) If $\lambda > 0$ and $\lambda \in \rho(A)$
$$R_\lambda u = \int_0^\infty e^{-\lambda t} S(t) u \, dt \quad \forall u \in X$$ (Laplace transform)

and $$\|R_\lambda\| = \sup_{\substack{u \in X \\ u \neq 0}} \frac{\|R_\lambda u\|}{\|u\|} \leq \frac{1}{\lambda}$$

$$R_\lambda - R_\nu = (\lambda I - A)^{-1} - (\nu I - A)^{-1}$$

$$(\lambda I - A)\left[(\lambda I - A)^{-1} - (\nu I - A)^{-1}\right] = I - (\lambda I - A)(\nu I - A)^{-1}$$

blah blah blah

$$\int_0^\infty e^{-\lambda t} S(t) u \, dt$$

$\uparrow$ operator → since continuous in t, not worried about integration

$$\left\| \int_0^\infty e^{-\lambda t} S(t) u \, dt \right\| \le \int_0^\infty \| e^{-\lambda t} S(t) u \| \, dt$$

contraction semigroup implies $\|S(0)\| \le 1$

$$\le \int_0^\infty e^{-\lambda t} \|S(0)\| \, \|u\| \, dt$$

$$\le \|u\| \int_0^\infty e^{-\lambda t} \, dt = \frac{1}{\lambda}\|u\|$$

IF $R_\lambda u = \int_0^\infty e^{-\lambda t} S(t) u \, dt$, which we haven't proved yet, then we have showed that

$$\|R_\lambda u\| \le \frac{1}{\lambda}\|u\|$$

$$\Rightarrow \sup_{\substack{u \in X \\ u \ne 0}} \frac{\|R_\lambda u\|}{\|u\|} \le \frac{1}{\lambda}$$

Define $\tilde{R}_\lambda u = \int_0^\infty e^{-\lambda t} S(t) u \, dt$

Look at quantity $\lim_{t \to 0} \dfrac{S(t)\tilde{R}_\lambda u - \tilde{R}_\lambda u}{t}$

if this limit exists then $\tilde{R}_\lambda u \in D(A)$ and this limit equals $A\tilde{R}_\lambda u$

the goal is to show:

$$\tilde{R}_\lambda(\lambda I - A)u = Iu \qquad \text{where } \tilde{R}_\lambda u = \int_0^\infty e^{-\lambda t} S(t)u\,dt$$

$$(\lambda I - A)\tilde{R}_\lambda u = u$$

$$\frac{1}{h}\left[S(h)\tilde{R}_\lambda u - S(0)\tilde{R}_\lambda u\right] = \frac{1}{h}\left[S(h)\int_0^\infty e^{-\lambda t} S(t)u\,dt - S(0)\int_0^\infty e^{-\lambda t} S(t)u\,dt\right]$$

$$= \frac{1}{h}\left[\int_0^\infty e^{-\lambda t} S(t+h)u\,dt - \int_0^\infty e^{-\lambda t} S(t)u\,dt\right]$$

$$= \frac{1}{h}\int_h^\infty e^{-\lambda(p-h)} S(p)u\,dp$$

$$= -\frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda p} S(p)u\,dp + \underbrace{\frac{e^{\lambda h}-1}{h}}_{\lambda,\ \text{as } h\to 0}\int_0^\infty e^{-\lambda t} S(t)\,dt$$

In the limit as $h \to 0$,

$$\lim_{h\to 0}\frac{1}{h}\left[S(h)\tilde{R}_\lambda u - S(0)\tilde{R}_\lambda u\right] = -S(0)u + \lambda\tilde{R}_\lambda$$

$$\downarrow$$

$$A\tilde{R}_\lambda u = -Iu + \lambda\tilde{R}_\lambda \quad \longrightarrow \quad u = (\lambda I - A)\underbrace{\tilde{R}_\lambda}_{\text{right inverse function}} u$$

Now show $\tilde{R}_\lambda(\lambda I - A)u = Iu$

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Show $\tilde{R}_\lambda(\lambda I - A)u = Iu$, $\tilde{R}_\lambda u = \int_0^\infty e^{-\lambda t} S(t) u \, dt$

$$A\tilde{R}_\lambda u = A\int_0^\infty e^{-\lambda t} S(t) u \, dt = \int_0^\infty e^{-\lambda t} A S(t) u \, dt$$

they commute

$$= \int_0^\infty e^{-\lambda t} S(t) A u \, dt$$

$$= \left(\int_0^\infty e^{-\lambda t} S(t) \, dt\right)[Au] = R_\lambda A u$$

~~$A\tilde{R}_\lambda u - R_\lambda A u = 0$~~

$$u = \left(\lambda I \tilde{R}_\lambda - A\tilde{R}_\lambda\right) u$$

$$= \left(\lambda I \tilde{R}_\lambda - \tilde{R}_\lambda A\right) u$$

$$= \tilde{R}_\lambda (\lambda I - A) u$$

$$\Rightarrow \boxed{\tilde{R}_\lambda = R_\lambda}$$

Thm: (Hille - Yoshida)

Let $A : X \supset D(A) \to X$, X a Banach space, $\overline{D(A)} = X$

Then A is the infinitesimal generator of a contraction semigroup $\{S(t)\}_{t \geq 0}$ iff

$$(0, \infty) \in \rho(A) \text{ and } \|R_\lambda\| \leq \frac{1}{\lambda}$$

Examples

$u_t = \Delta u$, $x \in \mho$, $0 < t < \infty$, $\mho$ is bounded, open, connected

$u = 0$ $x \in \partial\mho$

$u(x,0) = u$

Claim: let $X = L^2(\mho)$

$$D(A) = H^2(\mho) \cap H_0^1(\mho)$$

Show: 1) $A = \Delta$ is closed

2) $(0,\infty) \in \rho(A)$ and $\| R_\lambda \| \le \frac{1}{\lambda}$

Fact: ~~[illegible]~~

(section 6.2.2 of text)

if $u \in H^2(\mho) \cap H_0^1(\mho)$ then $\exists\ c > 0$ with

$$\|u\|_{H^2(\mho)} \le c\left[\|\Delta u\|_{L^2(\mho)} + \|u\|_{L^2(\mho)} \right]$$

1. Show A is closed. Suppose $u_k \to u \in X$, with $\{u_k\}_{k=1}^{\infty} \subset D(A)$ and $Au_k = w_k \to w \in X$, $\{w_k\}_{k=1}^{\infty} \subset X$. Show $u \in D(A)$ and $Au = w$.

$$\|u_k - u_\ell\|_{H^2(\mho)} \le c\left[\underbrace{\|\Delta u_k - \Delta u_\ell\|_{L^2(\mho)}}_{\|w_k - w_\ell\|_{L^2(\mho)}} + \|u_k - u_\ell\|_{L^2(\mho)} \right]$$

$\Rightarrow \{u_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $H^2(\mho)$

$\Rightarrow u \in H^2(\mho) \cap H_0^1(\mho) \Rightarrow u \in D(A)$

this also implies

$Du_k \to Du$, in $L^2(\mho)$

$\frac{\partial^2}{\partial x_i \partial x_j} u_k \to \frac{\partial^2}{\partial x_i \partial x_j} u$ in $L^2(\mho)$, $i,j = 1,\dots,n$

$\to \underset{\downarrow\atop w}{\Delta u_k} \to \Delta u = Au$

$\Rightarrow Au = w$

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Now ~~show~~ need to show $(0, \infty) \in \rho(A)$

- show $\lambda I - A$ is one-to-one for $\lambda > 0$

  suppose this is not the case. Then $\exists\, v \in D(A)$ $(v \neq 0)$ with $Av = \lambda v$, (where $A = \Delta$)

  $$\Delta v = \lambda v, \quad v \in H_0^1(\mho) \cap H^2(\mho)$$

  multiply by v, integrate.

  $$\int_\mho v \Delta v = -\int_\mho |\nabla v|^2 = \lambda \int_\mho v^2 \Rightarrow \text{contradiction}$$

  so that $\lambda I - A$ is one-to-one.

- Show $R(\lambda I - A) = L^2(\mho)$

  range

Demonstration of Hille-Yoshida proof (continued)

Suppose A is a closed operator

$$A : \mathbb{X} \supseteq D(A) \to \mathbb{X},$$

$\overline{D(A)} = \mathbb{X}$, where $\mathbb{X}$ is a Banach space.

Then A is the infinitesimal generator of a contraction semigroup $\{S(t)\}_{t \geq 0}$ iff

$$\underbrace{(0,\infty) \in \rho(A)}_{\text{resolvent set}} \quad \text{and} \quad \underbrace{\|R_\lambda\|}_{\text{resolvent}} \leq \frac{1}{\lambda} \qquad (\bigstar)$$

Demonstration (continued):

$\Rightarrow$ already shown

$\Leftarrow$ suppose $(\bigstar)$ holds

1) Define $A_\lambda = \lambda(-I + \lambda R_\lambda) = \lambda(-\overbrace{(\lambda I - A)R_\lambda}^{I} + \lambda R_\lambda)$

$$\boxed{A_\lambda = \lambda A R_\lambda}$$

Note: A_λ is bounded though blows up in the limit as $\lambda \to \infty$!!!

since $R_\lambda : \mathbb{X} \to D(A)$

then $A_\lambda : \mathbb{X} \to \mathbb{X}$

show, for each fixed $u \in D(A)$ $A_\lambda u \to Au$ as $\lambda \to \infty$ $\forall u \in D(A)$

First show $\lambda R_\lambda u \to u$ as $\lambda \to \infty$, $u \in D(A)$

$$\lambda R_\lambda u - u = (\lambda R_\lambda - I)u = A R_\lambda u = R_\lambda A u$$

$$\|\lambda R_\lambda u - u\| = \|R_\lambda A u\| \leq \|R_\lambda\| \, \|Au\|$$

but we assume that $\|R_\lambda\| \leq 1/\lambda$

$$\to \quad \|\lambda R_\lambda u - u\| \leq \frac{\|Au\|}{\lambda} \to 0 \text{ as } \lambda \to \infty$$

so that $\lambda R_\lambda u \to u$ as $\lambda \to \infty$

Now,

$$A_\lambda u = \lambda A R_\lambda u = A(\lambda R_\lambda u)$$

A is not a bounded operator, if it was, we "could pass the limit inside and we'd be done." we have to be more careful though because A is a closed operator.

~~let $\{u_k\}_{k=1}^{\infty} \in D(A)$, $u_k \to u$ as $k \to \infty$~~

~~$w_k = \lambda R_\lambda u_k$ bounded operator for each λ~~

$$A_\lambda u = A(\lambda R_\lambda - I)u + Au$$
$$= (\lambda R_\lambda - I)Au + Au \quad \leftarrow \text{by commutativity}$$

and so $\quad \forall\, Au \in D(A)$

$$(\lambda R_\lambda - I)Au \to 0 \quad \text{as } \lambda \to \infty$$

Extend this result to all of $\mathbb{X}$

$\{u_k\}_{k=1}^{\infty} \subset D(A)$, $u_k \to u$, $k \to \infty$

$$\underbrace{\lambda R_\lambda}_{\text{bounded operator}} u_\lambda \to \lambda R_\lambda u \quad \text{as } k \to \infty \text{ for fixed } \lambda.$$

$$\|\lambda R_\lambda u - u\| = \|\lambda R_\lambda u - \lambda R_\lambda u_k + \lambda R_\lambda u_k - u_k + u_k - u\|$$
$$\le \|\lambda R_\lambda(u-u_k) - (u-u_k)\| + \|\lambda R_\lambda u_k - u_k\|$$
$$\le 2\|u-u_k\| + \|\lambda R_\lambda u_k - u_k\|$$

given ~~select~~ $\varepsilon > 0$ ~~so that~~ $\exists N$ with $\|u-u_k\| < \frac{\varepsilon}{2}$ for $k > N$

$$\Longrightarrow \|\lambda R_\lambda u - u\| \le \varepsilon + \|\lambda R_\lambda u_k - u_k\|$$

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~~Let k→∞ then ∀λ~~

$\| \lambda R_\lambda u - u \| \leq 2\varepsilon$ for sufficiently large λ

because for $\lambda > \Lambda$ for fixed $k > N$, $\| \lambda R_\lambda u_k - u_k \| < \varepsilon$

Now let $\varepsilon \to 0 \Rightarrow \lambda R_\lambda u = u \quad \forall u \in X$

Define $S_\lambda(t) = e^{tA_\lambda} = I + tA_\lambda + \frac{t^2 A_\lambda^2}{2!} + \cdots$

$$= e^{t\lambda[-I + \lambda R_\lambda]} = e^{-\lambda t} e^{\lambda^2 t R_\lambda}$$

$$= e^{-\lambda t}\left[I + \lambda^2 t R_\lambda + \frac{(\lambda^2 t)^2 R_\lambda R_\lambda}{2!} + \cdots \right]$$

since R_λ is bounded, these operations are fine

$$= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda^2 t)^n R_\lambda^n}{n!}$$

apply triangle inequality

$$\| S_\lambda(t) \| \leq e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda^2 t)^n}{n!} \| R_\lambda \|^n$$

$$= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} \| \lambda R_\lambda \|^n \quad \text{but} \quad \| \lambda R_\lambda \| \leq 1$$

$$\leq e^{-\lambda t} e^{\lambda t} = 1$$

and therefore, for fixed λ and t, $S_\lambda(t)$ is a bounded linear mapping and it is a contraction

Likewise, $S(0) = I$

Now show

$$S_\lambda(t+s) = S_\lambda(t) S_\lambda(s) = S_\lambda(s) S_\lambda(t)$$

which is easy from exponential definition ... so that $S_\lambda(t)$ obeys properties of contraction semigroup

$\Rightarrow$ For fixed λ, $S_\lambda(t)$ is a contraction semigroup

Finally, show A_λ is the infinitesimal generator

claim: $\displaystyle \lim_{t \to 0} \frac{S_\lambda(t)u - S_\lambda(0)u}{t} = A_\lambda$

Follows from series expansion for $S_\lambda(t)$.

and this is defined for all $u \in X$

such that $D(A_\lambda) = X$

Now show $S_\lambda(t) \to S(t)$ as $\lambda \to \infty$

and show $S(t)$ is a contraction semigroup and A is its infinitesimal generator.

Hille-Yosida Theorem:

Let A be a closed linear operator
$$A : \mathbb{X} \supset D(A) \to \mathbb{X}$$
where $\mathbb{X}$ is a Banach space, $\overline{D(A)} = \mathbb{X}$.
Then A is the infinitesimal generator of a contraction semigroup iff
$$(0, \infty) \subset \rho(A) \quad \text{and} \quad \| R_\lambda \| \le \frac{1}{\lambda} \quad \text{for } \lambda > 0.$$

Given $u_t = Au$, $u(0) = u$
then the solution is given by $S(t)u$

$$u_t - \nabla\cdot(B\nabla u) = 0 \quad , (x,t) \in U_T = \{x,t \mid x \in U, 0 < t \le T\}$$
$$u(x,0) = g(x) \quad , x \in U, \ t = 0$$
$$u(x,t) = 0 \quad , x \in \partial U, \ 0 \le t \le T$$

with $B = (b_{ij})_{i,j=1}^n$ is symmetric, $b_{ij} \in C^1(\bar{U})$
and positive semi-definite, $0|\xi|^2 \le \xi^T B \xi$, $\xi \in \mathbb{R}^n$

Therefore the operator A is defined by
$$Au = \nabla\cdot(B\nabla u) \quad , \quad \mathbb{X} = L^2(U)$$
$$D(A) = H_0^1(U) \cap H^2(U)$$

Show that A is closed.

$\{\hat{u}_k\}_{k=1}^\infty \subset D(A)$ is a Cauchy sequence in $\mathbb{X}$
$\hat{u}_k \to \hat{u} \in \mathbb{X}$ and $A\hat{u}_k = \hat{w}_k \to \hat{w} \in \mathbb{X}$.
Then show that $\hat{u} \in D(A)$ and $A\hat{u} = \hat{w}$.

From Evans regularity: Note, if $u \in H_0^1(U) \cap H^2(U)$, you can show
$$\|u\|_{H^2(U)} \le C\left[\|\Delta u\|_{L^2(U)} + \|u\|_{L^2(U)}\right],$$
or replace Laplacian with our operator, i.e.,
$$\|\hat{u}\|_{H^2(U)} \le C\left[\|\nabla\cdot(B\nabla\hat{u})\|_{L^2(U)} + \|\hat{u}\|_{L^2(U)}\right]$$

and so we have

$$\|\hat{u}_k - \hat{u}_\ell\|_{H^2(U)} \leq C\Big[\underbrace{\|A\hat{u}_k - A\hat{u}_\ell\|_{L^2(U)}}_{\text{C.S. in } L^2} + \underbrace{\|\hat{u}_k - \hat{u}_\ell\|_{L^2(U)}}_{\text{C.S. in } L^2}\Big]$$

$\Rightarrow \{\hat{u}_k\}_{k=1}^{\infty}$ is a Cauchy sequence in H^2

and therefore $A\hat{u}_k \to A\hat{u}$ since $\hat{u} \in H^2$

what about the boundary ?

$H_0^1(U)$ is a closed subspace of $H^1(U)$ ■ PWOH

Now show $(\lambda I - A)$ is one-to-one $\forall$ $0 < \lambda < \infty$.

$$(\lambda I - A) \supseteq D(A) \to X$$

Suppose this is not true. Then $\exists\, w \neq 0$, $w \in D(A) = H_0^1 \cap H^2$

satisfying $\lambda w - \nabla\cdot(B\nabla w) = 0$

$$\lambda w^2 - w\,\nabla\cdot(B\nabla w) = 0$$

$$\lambda\int_U w^2\,dx + \int_U (\nabla w)^T B\,\nabla w = 0 \quad \Rightarrow \quad w = 0$$

contradiction ■

Show that range $\lambda I - A = X$

$(\lambda I - A)^{-1}$: Range $(\lambda I - A) \to D(A)$

solve $(\lambda I - A)w = f \in X$, $w \in H_0^1(U) \cap H^2(U)$

$$\lambda w - \nabla\cdot(B\nabla u) = f$$

(★) $\int_U \lambda w v + \int_U \nabla v\, B\,\nabla w = \int_U f v$, $v \in H_0^1(U) \cap H^2(U)$

we know for every $f \in L^2(U)$, $\exists !\ u \in H_0^1(U)$ that satisfies (★) $\forall\, v \in H_0^1(U)$; regularity theory says, in addition, $u \in H^2(U)$. And therefore the range is $X = L^2(U)$, $u = (\lambda I - A)^{-1} f$

consider:

$$\underbrace{\lambda \int_{\mho} u^2\, dx}_{\lambda \| u\|^2_{L^2(\mho)}} + \int_{\mho} (\nabla u)^T B \nabla u = \int_{\mho} f u$$

since B is strictly positive definite, left-hand side is positive, so add absolute value to the right hand side. Also, positive definiteness of B gives lower bound, $\theta \int_{\mho} |\nabla u|^2\, dx$, so that we have

$$\lambda \| u\|^2_{L^2(\mho)} + \theta \| u\|^2_{H_0^1(\mho)} \le \| f\|_{L^2} \| u\|_{L^2}$$

goal: $\| u\|_{L^2} = \| (\lambda I - A)^{-1} f \| \le \frac{1}{\lambda} \| f\|$

$$\sup_{\substack{f \in L^2 \\ f \neq 0}} \frac{\| (\lambda I - A)^{-1} f\|_{L^2}}{\| f\|_{L^2}} = \| (\lambda I - A)^{-1}\| \le \frac{1}{\lambda}$$ we must show this

apply Poincaré inequality

$$(\lambda + \theta c) \| u\|^2_{L^2} \le \| f\|_{L^2} \| u\|_{L^2}$$

$$\rightarrow \quad \| u\|_{L^2} \le \frac{1}{(\lambda + \theta c)} \| f\|_{L^2} \le \frac{1}{\lambda} \| f\|_{L^2}$$

HYPERBOLIC EXAMPLE

$$u_{tt} = \nabla\cdot(B\nabla u) \quad , \quad (x,t)\in \mho_T$$
$$u(x,t) = 0$$
$$u(x,0) = g$$
$$u_t(x,0) = h$$

Rewrite as system:

let $u_t = v$

$v_t = \nabla\cdot(B\nabla u)$

to obtain the system

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \underbrace{\begin{pmatrix} 0 & 1 \\ \nabla\cdot B\nabla & 0 \end{pmatrix}}_{\text{this is our } A}\begin{pmatrix} u \\ v \end{pmatrix}$$

what is the domain of A?

$A : H_0^1(\mho)\times X \supset D(A) \to H_0^1(\mho)\times X$

with $X \in L^2(\mho)$ and

$D(A) = H_0^1(\mho)\cap H^2(\mho) \times H_0^1(\mho)$

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showt that A is a closed operator:

suppose $\begin{pmatrix} u_k \\ v_k \end{pmatrix}_{k=1}^{\infty} \in D(A)$ and it's a Cauchy sequence in $H_0^1\times X$. Then $\begin{pmatrix} u_k \\ v_k \end{pmatrix} \to \begin{pmatrix} u \\ v \end{pmatrix}$, $u\in H_0^1(\mho)$, $v\in X$.

Suppose $A\begin{pmatrix} u_k \\ v_k \end{pmatrix} \to \begin{pmatrix} w \\ z \end{pmatrix}$ in $H_0^1(\mho)\times L^2(\mho)$. Then show $\begin{pmatrix} u \\ v \end{pmatrix} \in D(A)$.

we know $\nabla\cdot(B\nabla u_k)$ converges in $H_0^1(\mho)$

IF $v_k \to v$ in $L^2(\mho)$. Show $u\in H_0^1(\mho)\cap H^2(\mho)$

use previous argument to say $u\in H^2(\mho)$ and $\{u_k\}$ converges in $H^2(\mho)$

$$\Rightarrow \nabla\cdot(B\nabla u_k) \xrightarrow[\text{in } L^2]{} \nabla\cdot B\nabla u$$

and $v_k \to v$ in L^2

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how to do hmwk problem #2 contraction

show $\sup|u_1 - u_2| \leq \sup|g_1 - g_2| = \alpha$

assume $\sup|u_1 - u_2| \geq \alpha + \sigma \quad , \quad \sigma > 0$

$$\Phi = u_1(x,t) - u_2(y,s) - \lambda|t+s| + \cdots \qquad \text{eq 5}$$

(7) becomes

$$\Phi(x_0, y_0, t_0, s_0) \geq \sup \Phi(x, x, t, t) \geq \frac{\sigma}{2} + \alpha$$

we get (8), (9), (10), (11)

$$\frac{\alpha}{2} + \frac{\sigma}{4} \leq \omega_1(t_0) + \omega_2(t_0)$$

construct the same v

we get (12)