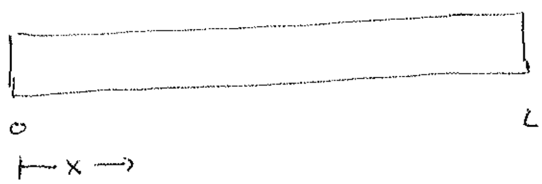
Eigenvalues and Eigenvectors of continuous systems
- Dynamic Models

<u>example</u>: Heat Conduction (uniform temperature)



heat eqn : $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial t^2}$

separation of variables : look for solution $u(x,t) = y(x)\, e^{-\sigma t}$
(negative exponent to account for decreasing temperature)

$u = y e^{-\sigma t}$

$u_t = -\sigma y e^{-\sigma t}$

$u_{xx} = y'' e^{-\sigma t}$ substitute into PDE $\Rightarrow$ $k y'' e^{-\sigma t} = -\sigma y e^{-\sigma t}$

$\rightarrow \; y'' + \frac{\sigma}{k} y = 0$

let $\lambda = \frac{\sigma}{k}$ then $\boxed{y'' + \lambda y = 0}$

<u>assume BC</u> : $\cancel{u(0,t) = 0,\ u(L,t) = 0}$

$\rightarrow \; \boxed{y(0) = y(L) = 0 \;,\; y'' + \lambda y = 0}$

OR <u>assume IC</u> : initial condition in x, not t

$y(0) = y'(0) = 1$

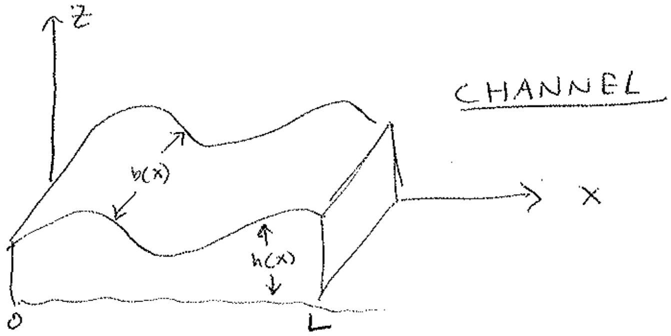
ICs are often preferred because they often lead to a unique solution. BCs often behave non-uniquely.

Reference : Cochran Chpt 2

Eigenvalue problems for differential equations

Application : Shallow Water Waves

Imagine fluid flow with free surface



$b(x)$ - breadth

$h(x)$ - heighth

$\eta(x,t)$ is the displacement of the surface of the fluid in the channel.

By a careful analysis, Cochran shows that for small displacements (linearized equations of motion)

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{g}{b}\frac{\partial}{\partial x}\left(bh\frac{\partial \eta}{\partial x}\right)$$

If b and h are constants

$$\frac{\partial^2 \eta}{\partial t^2} = gh\frac{\partial^2 \eta}{\partial x^2} \qquad \text{(wave eqn)}$$

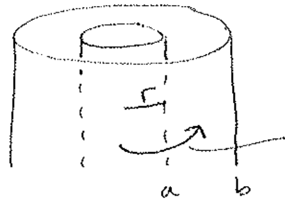
Note $g \sim \frac{L}{T^2}$, $h \sim L$ $\longrightarrow$ $gh \sim \frac{L^2}{T^2} = \left(\frac{L}{T}\right)^2$

$$\Rightarrow \boxed{\eta_{tt} = c^2 \eta_{xx}}$$

Rotating Flow

Rayleigh's Criterion

~~Rayleigh's Flow~~ (inviscid)



$\bar{v}(r)$ velocity , $\rho(r)$ density

A necessary and sufficient condition for stability is

$$\frac{d}{dr}\left(\bar{\rho}\, r^2 \bar{v}^2\right) > 0$$

Synge (1933) proved Rayleigh's criterion by Sturm-Liouville theory.

$$\boxed{\begin{array}{l} \left(p(x)y'\right)' + \left(\lambda r(x) - q(x)\right) y = 0 \quad , \quad a < x < b \\ A\,y(a) + B\,y'(a) + C\,y(b) + D\,y'(b) = 0 \\ E\,y(a) + F\,y'(a) + G\,y(b) + H\,y'(b) = 0 \end{array}}$$

Consider channel case again: $\dfrac{\partial^2 \eta}{\partial t^2} = \dfrac{g}{b}\dfrac{\partial}{\partial x}\left(bh\dfrac{\partial \eta}{\partial x}\right)$

Try $\eta(x,t) = y(x)\sin(\omega t + \theta)$ $\quad$ ω - frequency, θ - phase angle

$$\eta_{tt} = -\omega^2 y \sin(\omega t + \theta)$$

$$\eta_x = y'(x)\sin(\omega t + \theta)$$

$$\frac{\partial}{\partial x}\left(bh\frac{\partial \eta}{\partial x}\right) = (bhy')'\sin(\omega t + \theta)$$

$$\Rightarrow \eta_{tt} = \frac{g}{b}\frac{\partial}{\partial x}\left(bh\frac{\partial \eta}{\partial x}\right) \Rightarrow -\omega^2 y \sin(\omega t + \theta) = \frac{g}{b}(bhy')'\sin(\omega t + \theta)$$

$$\Rightarrow \boxed{(bhy')' + \frac{b\omega^2}{g} y = 0}$$ ODE , ω^2 is the eigenvalue

possible Boundary conditions

$$y(0) = y(L) = 0$$

bh plays the role of $p(x)$
ω^2 plays the role of λ
b/g plays the role of $r(x)$
$q(x)$ vanishes

In Cochran, page 51 #2

- Consider the general $2^{\underline{nd}}$ order linear ODE

$$\alpha y'' + \beta y' + \gamma y = 0 \quad , \quad \alpha \neq 0$$

let $F \neq 0$ be a multiplier

$$\alpha F y'' + \beta F y' + \gamma F y = 0$$

let $\alpha F = p$ and $\beta F = p'$

$$\rightarrow \frac{p'}{p} = \frac{\beta}{\alpha} \quad \rightarrow \quad \frac{d}{dx} \log p = \frac{\beta}{\alpha} \quad \rightarrow \quad \log p = \int \frac{\beta}{\alpha} dx$$

Exponentiate $\rightarrow$ $p(x) = e^{\int \beta/\alpha \, dx}$

$$\boxed{F = \frac{1}{\alpha} e^{\int \beta/\alpha \, dx}}$$

$$\Rightarrow \frac{d}{dx}(py') + \frac{\gamma}{\alpha}\left(e^{\int \beta/\alpha \, dx}\right) y = 0$$

<u>Differential Operator</u>

$$W(u,v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - u'v$$

$$Ly \equiv (py')' - qy \quad , \quad a < x < b$$

$p(x) > 0$ for $a \leq x \leq b$

p' and q' are continuous on $[a, b]$

$$\int_a^b (uLv - vLu)\, dx = \Big[p(x) W(u,v)\Big]_a^b \quad , \quad W \text{ is the Wronskian of } u \text{ and } v$$

The right side depends only on the boundary values

The matrix of coefficients $\begin{bmatrix} A & B & C & D \\ E & F & G & H \end{bmatrix}$ has rank 2.

$\Rightarrow$ the BCs are distinct.

There are the <u>General mixed BC</u>

If $C = D = 0$ and $E = F = 0$ the BCs are called <u>separated</u> or <u>unmixed</u>.

<u>Def</u>: BCs of general mixed homogeneous type are called <u>regular</u> if for all u, v that satisfy them $\left[p(x) W(u,v) \right]_a^b = 0$

<u>Property 1</u>: Unmixed (separated) BCs are regular.

<u>Property 2</u>: Periodic BCs $y(b) = y(a)$, $y'(b) = y'(a)$ are also regular provided $p(b) = p(a)$.

The operator L with regular BCs is called <u>self-adjoint</u>.

<u>Example 1</u>: $Ly = y''$, $0 < x < \pi$, $y(0) = y(\pi) = 0$

Eigenvalue problem $y'' + \lambda y = 0$

Eigenfunctions $y_n = c \sin(\sqrt{\lambda_n}\, x)$, $\lambda_n = n^2$, $n = 1, 2, 3, \ldots$

Note that $\lambda = 0$ is not an eigenvalue

Normalize the eigenfunctions with the inner product $\langle f, g \rangle = \int_0^\pi f(x) g(x)\, dx$

$$\rightarrow \quad y_n = \sqrt{\frac{2}{\pi}} \sin(nx)$$

note $\langle y_n, y_n \rangle = \int_0^\pi y_n(x)\, y_n(x)\, dx = \delta_{mn}$

$$\langle Ly, y \rangle = \int_0^\pi y'' y\, dx = y'y\Big]_0^\pi - \int_0^\pi y'y'\, dx = -\int_0^\pi y'^2 dx \le 0$$

Example 2 : $\quad y'' + \lambda y = 0 \quad , \; 0 < x < 1 \quad , \; y'(0) = 0$
$$y(1) + y'(1) = 0$$

Assume that $y = e^{mx}$

$\Rightarrow \quad m^2 + \lambda = 0 \quad \Rightarrow \quad m = \pm i\sqrt{\lambda} \quad$ if $\lambda \geq 0$

- If $\lambda = 0$: $\quad y'' = 0 \quad \rightarrow \quad y = c_1 + c_2 x$

$$\boxed{y' = c_2}$$

$y'(0) = c_2 = 0 \quad , \; \boxed{y = c_1}$ $\qquad$ contradiction

and $y(1) + y'(1) = 0 \quad \rightarrow \quad \boxed{y'(1) = -c_1}$

$$\Rightarrow \boxed{\lambda \neq 0}$$

- If $\lambda < 0$ take $\lambda = -k^2 \quad , \; y = c_1 e^{kx} + c_2 e^{-kx}$

or take $\quad y = c_1 \cosh(kx) + c_2 \sinh(kx)$

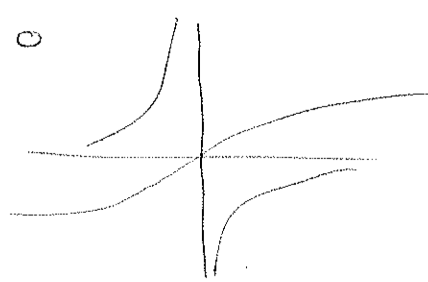
$$y' = k\left(c_1 \sinh(kx) + c_2 \cosh(kx)\right)$$

$$y'(0) = k c_2 = 0 \quad \rightarrow \quad \boxed{c_2 = 0}$$

$$y(1) + y'(1) = 0 = c_1 \cosh(k) + k c_1 \sinh k = 0$$

$\rightarrow \quad \cosh(k\cancel{x}) + k\sinh k = 0$

$\rightarrow \quad \tanh k = -\frac{1}{k}$



no intersection

therefore $\boxed{\lambda \not< 0}$

- Now examine in more detail $\lambda > 0$

$m = \pm i\sqrt{\lambda}$

$y = c_1 \cos(mx) + c_2 \sin(mx)$

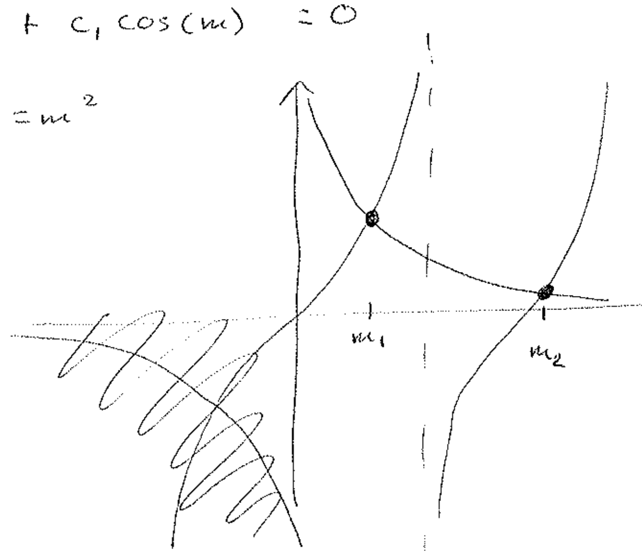
$y' = -mc_1 \sin(mx) + mc_2 \cos(mx)$

$y'(0) = mc_2 = 0$

$\Rightarrow \; y = c_1 \cos(mx)$

$y'(1) + y(1) = -mc_1 \sin(m) + c_1 \cos(m) = 0$

$\longrightarrow \boxed{\tan(m) = \frac{1}{m}} \qquad \lambda = m^2$



$m_1 = 0.86033$

$\lambda_1 = 0.7402$

Notice, as $n \to \infty$

$m_n \longrightarrow n\pi$

$\lambda_n \longrightarrow n^2\pi^2$

Next Time: Cochran 2.3, pg 52 # 10

Eigenvalue Probems For ODEs

{ Cochran, Chpt 2
{ My notes on Eigenvalue Probems } available on line

Regular Sturm-Liouville Problems

Separated Boundary conditions

$$Ly + \lambda r y = (py')' + (\lambda r - q)y = 0, \quad a<x<b$$

$$A\,y(a) - B\,y'(a) = 0$$

$$G\,y(b) + H\,y'(b) = 0$$

$$\Rightarrow \quad Ly = (py')' - qy$$

$$uLv - vLu = u\left[(pv')' - \cancel{qv}\right] - v\left[(pu')' - \cancel{qu}\right]$$

$$= u(pv')' - v(pu')' = \frac{d}{dx}\left(pW(u,v)\right)$$

$$\boxed{\begin{gathered} \text{Wronskian: } W(u,v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - u'v \\ pW(u,v) = p(uv' - u'v) \\ \frac{d}{dx}\left(pW(u,v)\right) = (pv')'u + \cancel{pv'u'} - (pu')'v - \cancel{pu'v'} \\ = u(pv')' - v(pu')' = uLv - vLu \end{gathered}}$$

$$\int_a^b (uLv - vLu)\,dx = \int_a^b \frac{d}{dx}\left(pW(u,v)\right) = \Big[pW(u,v)\Big]_a^b = 0 \quad \left(\begin{array}{l}\text{in the} \\ \text{regular case}\end{array}\right)$$

Herron
- see notes

IF $Lu + \lambda r u = 0$

$Lv + \nu r v = 0$ then u and v are eigenfunctions.

multiply 1st by v, 2nd by u and subtract.

$$\Rightarrow (\lambda - \nu) r v u = \frac{d}{dx}\left[pW(u,v)\right]$$

$$\Rightarrow (\lambda - \nu)\int_a^b ruv\,dx = 0$$

$\Rightarrow$ Property 1 IF $\lambda = \nu$, then $W(u,v) = 0$ so u and v are linearly dependent.

$\Rightarrow$ Property 2 IF $\lambda \neq \nu$ then $\int_a^b ruv\,dx = 0$ and u and v are linearly independent. That is u and v (both nonzero) are orthogonal with weight function r.

$\Rightarrow$ Property 3 Every eigenvalue is real. Each corresponding eigenfunction can be chosen to be real.

- Suppose L is symmetric $\left(\int_a^b (uLv - vLu)\,dx = 0 \right)$

$$Ly + \lambda r y = 0$$

- Suppose λ is complex, then $y(x) = y_1(x) + i y_2(x)$

Take the complex conjugate: $\bar{y}(x) = y_1(x) - i y_2(x)$

$$L\bar{y} + \bar{\lambda} r \bar{y} = 0$$

Use the previous argument where $u = y$, $v = \bar{y}$

$$(\lambda - \bar{\lambda}) \int_a^b r y \bar{y}\,dx = 0$$

$$(\lambda - \bar{\lambda}) \int_a^b r \|y\|^2\,dx = 0 \rightarrow \boxed{\lambda \text{ is real}}$$

L is symmetric if $\langle Lu, v \rangle = \langle u, Lv \rangle$

Example: $y'' + \lambda y = 0 \quad , \quad 0 < x < 1$

$y'(0) = 0 \quad , \quad y(1) + y'(1) = 0$

we fond last time that $y_n(x) = c_n \cos(k_n x) \quad , n = 1, \ldots$

where $\tan k_n = \frac{1}{k_n} \quad , n = 1, \ldots$

orthogonality:

$$\int_0^1 \cos(k_1 x)\cos(k_2 x)\,dx = \frac{1}{2}\int_0^1 \cos((k_1 + k_2)x)\,dx + \frac{1}{2}\int_0^1 \cos((k_1 - k_2)x)\,dx$$

$$= \left[\frac{1}{2} \frac{\sin((k_1 + k_2)x)}{k_1 + k_2} + \frac{1}{2} \frac{\sin((k_1 - k_2)x)}{k_1 - k_2} \right]_0^1 \qquad \boxed{\text{Note, } k_1 \neq k_2}$$

$$= \frac{1}{2} \left(\frac{\sin(k_1 + k_2)}{k_1 + k_2} + \frac{\sin(k_1 - k_2)}{k_1 - k_2} \right)$$

$$= \frac{1}{2} \left(\frac{\sin k_1 \cos k_2 + \cos k_1 \sin k_2}{k_1 + k_2} + \frac{\sin k_1 \cos k_2 - \cos k_1 \sin k_2}{k_1 - k_2} \right)$$

we know $\frac{\sin k_1}{\cos k_1} = \frac{1}{k_1}$

$$\Rightarrow \quad = \frac{1}{2}\left[\frac{k_2 \sin k_1 \sin k_2}{k_1 + k_2} + \frac{k_1 \sin k_1 \sin k_2}{k_1 + k_2} + \frac{(k_2 - k_1) \sin k_1 \sin k_2}{k_1 \cancel{+} k_2}\right.$$

$$= \frac{1}{2}\left[\frac{(k_2 + k_1) \sin k_1 \sin k_2}{k_1 + k_2} + \frac{(k_2 - k_1) \sin k_1 \sin k_2}{k_1 - k_2} \right]$$

$$= \frac{1}{2} \sin k_1 \sin k_2 \left(\frac{(k_2 + k_1)(k_1 - k_2) + (k_2 + k_1)(k_2 - k_1)}{(k_1 - k_2)(k_1 + k_2)} \right)$$

$$= 0$$

Normalize the eigenfunctions: $\int_0^1 Y_i(x)\, Y_j(x)\, dx = \delta_{ij}$

choose c_1 so that

$$\int_0^1 Y_1^2\, dx = 1$$

$Y_1 = c_1 \cos(k_1 x)$

$$c_1^2 \int_0^1 \cos^2(k_1 x)\, dx = \frac{1}{2} c_1^2 \int_0^1 (\cos 2k_1 x + 1)\, dx = \left[\frac{c_1^2 \sin(2k_1 x)}{4k_1} + \frac{x c_1^2}{2} \right]_0^1$$

$$= \frac{c_1^2}{4k_1} \sin 2k_1 + \frac{c_1^2}{2} = 1$$

$$\Rightarrow \quad c_1^2 \left(\frac{\sin 2k_1}{2(2k_1)} + \frac{1}{2} \right) = 1 \quad \Rightarrow \quad c_1^2 \left(\frac{2 \sin k_1 \cos k_1}{4 k_1} + \frac{1}{2} \right) = 1$$

But $\frac{\cos k_1}{k_1} = \sin k_1 \Rightarrow c_1^2 \left(\frac{\sin^2 k_1}{2} + \frac{1}{2} \right) = 1$

$$\Rightarrow \boxed{c_1 = \sqrt{\frac{2}{\sin^2 k_1 + 1}}}$$

the normalized constant depends on the eigenvalue.

Example: $y'' + \lambda y = 0, \quad 0 < x < \pi$

$$y(0) = y(\pi) = 0$$

$$\lambda_n = n^2, \quad n = 1,$$

Cochran pg 52 #10 a

$$Ly + \lambda r y = (py')' + (\lambda r - q) y = 0$$

reality holds

$$\int_a^b yLy\, dx + \lambda \int_a^b r y^2\, dx = 0$$

$$\int_a^b yLy\, dx = \int_a^b y(py')'\, dx - \int_a^b q y^2\, dx$$

$$= \left[p y y' \right]_a^b - \int_a^b \left(p y'^2 + q y^2 \right) dx$$

$$\rightarrow \quad \lambda \int_a^b r y^2 dx = \int_a^b \left(p y'^2 + q y^2 \right) dx + p(a) y'(a) y(a) - p(b) y'(b) y(b)$$

By assumption, $y(a) = \frac{B}{A} y'(a)$

$$y(b) = \frac{-H}{G} y'(b)$$

$$= \int_a^b \left(p y'^2 + q y^2 \right) dx + p(a) \left(y'(a) \right)^2 \frac{B}{A} + p(b) \left(y'(b) \right)^2 \frac{H}{G}$$

If $q(x) \geq 0$, $AB \geq 0$, $GH \geq 0$

Hence $\lambda > 0$

Cochran pg 52 # 10 b

Show that a similar conclusion is valid if $p(a) = p(b)$,

$y(a) = y(b)$, $y'(a) = y'(b)$

Cochran pg 52 # 10 c

Are there mixed homogeneous BCs other than periodic ones for which the eigenvalues of the associated Sturm-Liouville problem are non-negative?

Yes need real eigenvalues

Ex: $y'' + \lambda y = 0$, $0 \le x \le +1$

$y'(1) = y'(0)$

$y'(1) - y(1) + y(0) = 0$

The previous analyses do not apply since the BCs are neither separated or periodic.

Let $\lambda = k^2$

Then $\cancel{y = c_1 \cos k_1 x + c_2 \sin k_2 x}$

$\cancel{y' = -c_1 k \sin kx}$

$y = c_1 \cos kx + c_2 \sin kx$

$y' = -c_1 k \sin kx + c_2 k \cos kx$

$y'(1) = y'(0) \Rightarrow -c_1 k \sin k + c_2 k \cos k = c_2 k$

$y'(1) - y(1) + y(0) = 0 \Rightarrow -c_1 k \sin k + c_2 k \cos k - c_1 \cos k - c_2 \sin k + c_1 = 0$

$$\begin{cases} -c_1 k \sin k + c_2 (k \cos k - k) = 0 \\ c_1 (-k \sin k - \cos k + 1) + c_2 (k \cos k - \sin k) = 0 \end{cases}$$

Is $k = 0$ an eigenvalue? IF it is, $y'' = 0 \Rightarrow y = ax + b$

$y'(0) = y'(1)$ is satisfied

$y'(1) - y(1) + y(0) = a - (a+b) + b = 0$, so satisfied

Hence, corresponding to $\lambda = 0$, $y = 1$ and $y = x$ are both eigenfunctions.

cancel out the k:

$$-c_1 \sin k + c_2(\cos k - 1) = 0$$

$$c_1(-k\sin k - \cos k + 1) + c_2(k\cos k - \sin k) = 0$$

Require that determinant of coefficients vanish:

$$\begin{vmatrix} -\sin k & \cos k - 1 \\ -k\sin k - \cos k + 1 & k\cos k - \sin k \end{vmatrix} = 0$$

$$\Rightarrow \cancel{-k\sin k\cos k} + \cancel{\sin^2 k} + \cancel{k\sin k\cos k} - k\sin k + \cancel{\cos^2 k} - \cos k - \cos k + 1 = 0$$

$$2 - k\sin k - 2\cos k = 0$$

$$\cancel{1 - 2\cos^2\frac{k}{2} = \sin k} \qquad 1 - 2\sin^2\frac{k}{2} = \cos(k)$$

$$2\sin\frac{k}{2}\cos\frac{k}{2} = \sin k$$

$$\cancel{2} - 2k\sin\frac{k}{2}\cos\frac{k}{2} - \cancel{2} + 4\sin^2\frac{k}{2} = 0$$

$$2\sin\frac{k}{2}\left(2\sin\frac{k}{2} - k\cos\frac{k}{2}\right) = 0$$

$$\Rightarrow 2\sin\frac{k}{2} = 0, \qquad 2\sin\frac{k}{2} - k\cos\frac{k}{2} = 0$$

$$\Rightarrow \frac{k}{2} = n\pi, \; n = 1, 2, \ldots, \qquad \tan\frac{k}{2} = \frac{k}{2}$$

or use the type of argument developed so far:

$$Lu + \lambda u = 0$$
$$Lv + \nu v = 0$$

show that $\int_0^1 (\lambda - \nu)\, uv\, dx = 0$ and that if $v = \bar{u}$, $\lambda = \bar{\lambda}$ then λ is real. Accept the reality of the eigenvalue.

$$\int_0^1 (y'' + \lambda y)\, y\, dx = 0$$

integrate by parts: $\left[y'y\right]_0^1 - \int_0^1 y'^2 dx + \lambda \int_0^1 y^2 dx = 0$

Examine: $\quad [y'y]_0^1 = y'(1)y(1) - y'(0)y(0)$

The BCs were $\quad y'(1) = y'(0) \quad$ and $\quad y'(1) - y(1) + y(0) = 0$

$$\Rightarrow [y'y]_0^1 = y'(1)y(1) + y'(1)\big(y'(1) - y(1)\big)$$
$$= \big(y'(1)\big)^2$$

$$\Rightarrow \lambda \int_0^1 y^2\,dx = \int_0^1 (y')^2\,dx - \big(y'(1)\big)^2$$

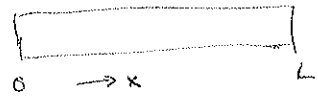
If $y = 1,\ y' = 0 \quad \rightarrow \quad \lambda = 0$

If $y = x,\ y' = 1 \quad \rightarrow \quad \lambda = 0$

Now if you take $\lambda = -\nu^2$, you find that there are no negative eigenvalues.

Motivation: Heat conduction in a non-uniform medium

one-dimensional, $u(x,t)$, temperature



$$\rho c \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}\right)$$

where ρ = density, c = thermal capacity, $K_0(x)$ conductivity
ρ, c, K_0 time-independent

Let $u(x,t) = \varphi(x) e^{-\lambda t}$, separation of variables

$$\rho c(-\lambda)\varphi e^{-\lambda t} = \frac{d}{dx}\left(K_0(x)\varphi'\right)e^{-\lambda t}, \quad e^{-\lambda t} \neq 0$$

$$\to (K_0(x)\varphi')' + \lambda \rho c \varphi = 0$$

typical boundary conditions, temp given at the ends

Regular Sturm-Liouville

Model Example, $L = 1$

$$K_0(x) = \frac{1}{(1+x)^2}, \quad \rho c = \frac{1}{(1+x)^2}$$

we arrive at

$$\left(\frac{1}{(1+x)^2}\varphi'\right)' + \frac{\lambda}{(1+x)^2}\varphi = 0$$

Boundary condition: $\varphi(0) = \varphi(1) = 0$

Since this is ~~a~~ regular Sturm-Liouville problem and because of the boundary conditions all eigenvalues are non-negative.

$$\text{Set } \lambda = k^2 \to \sqrt{\lambda} = k$$

$$\left(\frac{\varphi'}{(1+x)^2}\right)' + \frac{\lambda}{(1+x)^2}\varphi = 0$$

$$\frac{\varphi''}{(1+x)^2} - \frac{2\varphi'}{(1+x)^3} + \frac{\lambda}{(1+x)^2}\varphi = 0$$

General solution:

$$\varphi(x) = c_1\big(k(1+x)\cos(kx) - \sin kx\big) + c_2\big(k(1+x)\sin(kx) + \cos(kx)\big)$$

Apply BCs

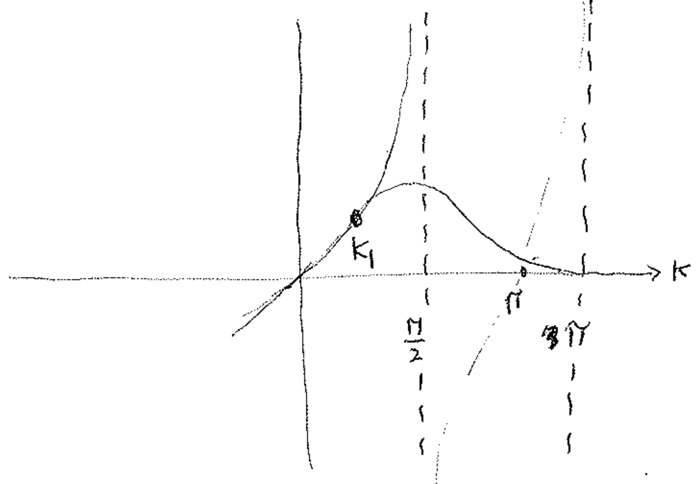
$$\varphi(0) = c_1 k + c_2(1) = 0 \quad \Rightarrow \quad \boxed{c_2 = -kc_1}$$

$$\varphi(1) = c_1(2k\cos k - \sin k) + c_2(2k\sin k + \cos k) = 0$$

$$\Rightarrow \quad c_1(2k\cos k - \sin k) - kc_1(2k\sin k + \cos k) = 0$$

For non-trivial solution, $c_1 \neq 0$

$$k\cos k - (1+2k^2)\sin k = 0 \quad \longrightarrow \quad \tan k = \frac{k}{1+k^2}$$



$k_1 \approx 3.286 \rightarrow \lambda_1 = 10.80$

notice, as $n \to \infty$, the eigenvalues look mor and more like $\tan(nk) = 0$
$\Rightarrow k_n \sim n\pi$ as $n \to \infty$

Is $\lambda = 0$ an eigenvalue?

$$\left(\frac{\varphi'}{(1+x)^2}\right)' = 0 \quad \longrightarrow \quad \varphi = c_2 + \frac{c_1}{3}(1+x)^3$$

apply BCs:

$$\left.\begin{aligned}\varphi(0) &= \frac{c_1}{3} + c_2 = 0\\ \varphi(1) &= \frac{8}{3}c_1 + c_2 = 0\end{aligned}\right\} \quad c_1 = c_2 = 0$$

$$\boxed{\lambda \neq 0}$$

MOAM, 9/13/02

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Last period we solved Cochran, pg 52 # 10(a)

$$(py')' + (\lambda r - q) y = 0$$

$$A\, y(a) + B y'(a) = 0$$

$$G\, y(b) - H y'(b) = 0$$

IF $q(x)$ as well as AB and GH are nonnegative, show that all eigenvalues are nonnegative.

Cochran Section 2.5 Eigenfunction Expansions

Flow in a channel. Sought $\eta(x,t)$ - wave height.

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{g}{b} \frac{\partial}{\partial x}\left(bh \frac{\partial \eta}{\partial x}\right) \quad , \quad t > 0 \quad , \quad a < x < c$$

Initial value problem:

$$\eta(x,0) = \eta_0(x) \quad , \quad \eta_t(x,0) = 0 \quad , \quad \eta(a,t) = \eta(c,t) = 0$$

Look for solutions $\eta(x,t) = F(x)\, G(t)$, obtain two equations.

- $(bh F')' + \frac{\lambda b}{g} F = 0$ - Sturm-Liouville equation

  $F(a) = F(c)$

- $\frac{G''}{G} = -\lambda_n \rightarrow G'' + \lambda_n G = 0$ - time-dependent problem, $G = G(t)$

Suppose $\eta(x,t) = \sum_{n=1}^{\infty} F_n(x)\, G_n(t)$

$$\eta(a,t) = \eta(c,t) = 0 \implies F_n(a) = F_n(c) = 0 \; , \; \forall n$$

$$\eta(x,0) = \sum_{n=1}^{\infty} F_n(x)\, G_n(0) = \eta_0(x)$$

$$\eta_t(x,0) = \sum_{n=1}^{\infty} F_n(x)\, G_n'(0) = 0$$

Thus $G_n(t) = \cos(\omega_n t)$, where $\omega_n = \sqrt{\lambda_n}$

$$\rightarrow \quad u_0(x) = \sum_{n=1}^{\infty} F_n(x)$$

Take $F_n(x) = c_n y_n(x)$ where ~~the~~ y_n are the eigenfunctions of $(bhy')' + \frac{w^2}{g} b\, y = 0$

Questions:

① What are the constants c_n?

② What about the convergence of the series?

Return to Example ①: $Ly = y''$, $0 < x < \pi$

$y(0) = y(\pi) = 0$

- solved $y'' + \lambda y = 0$
- Find that ~~[illegible]~~ $y = c \sin(\sqrt{\lambda}\, x)$, $\lambda = n^2$

$\rightarrow \quad y_n = \sqrt{\frac{2}{\pi}} \sin(nx)$, $n = 1, 2, \ldots$ — normalized coefficient from $\int_0^{\pi} y_n^2\, dx = 1$

- $\int_0^{\pi} y_n(x)\, y_m(x)\, dx = \delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$

To make use of this in describing a function $(0, \pi)$ we make use of theory of Fourier series,

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

Multiply by $\sin(mx)$, integrate from 0 to π

$$\int_0^{\pi} f(x) \sin(mx)\, dx = \int_0^{\pi} \sum_{n=1}^{\infty} a_n \sin(nx) \sin(mx)\, dx = \underbrace{\int_0^{\pi} a_m \sin^2(mx)\, dx}_{\text{only 1 term survives}}$$

$$= \int_0^{\pi} \left(\frac{a_m}{2} - \frac{a_m}{2} \cos 2x \right) dx$$

$$= \left. \frac{a_m x}{2} - \frac{a_m}{4} \sin 2x \right]_0^{\pi}$$

$$= \frac{\pi a_m}{2}$$

$$\Rightarrow \quad \boxed{a_m = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(mx)\, dx}$$

$\cos^2 x - \sin^2 x = \cos 2a$

$1 - 2\sin^2 x = \cos 2x$

$\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$

The equality $F(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$

is to be understood in the sense that

$$\lim_{N\to\infty} \int_0^{\pi} \left[F(x) - \sum_{n=1}^{N} a_n \sin(nx) \right]^2 dx = 0$$

the sine functions are complete in the sense that any square integrable F can be written in such a series and the series converges to F in the above sense.

Cochran 2.5

Property 2 : The eigenfunction of the Sturm-Liouville problem form a complet set.

Property 3 : If $F(x)$ is continuous on $[a,b]$ and $\{y_i\}$ is the orthonormal sequence of eigenfunctions associated with the eigenvalue problem (2.3-1), then

Generalized Fourier Series $\longrightarrow$ $F = \sum_{n=1}^{\infty} \langle F, y_n \rangle y_n$

where the equality is meant in the sense that

$$\lim_{N\to\infty} \left\| F - \sum_{n=1}^{\infty} \langle F, y_n \rangle y_n \right\| = 0$$

Parseval's Equality is valid :

$$\sum_{n=1}^{\infty} \langle F, y_n \rangle^2 = \| F \|^2$$

Fourier sine series , $y_n = \sqrt{\frac{2}{\pi}} \sin(nx)$

$$\langle f, y_n \rangle = \sqrt{\frac{2}{\pi}} \int_0^{\pi} f(x) \sin(nx)\, dx$$

$$f = \sum_{n=1}^{\infty} \langle f, y_n \rangle y_n = \sum_{n=1}^{\infty} \sqrt{\frac{2}{\pi}} \int_0^{\pi} f(\xi) \sin(n\xi)\, d\xi \sqrt{\frac{2}{\pi}} \sin(nx)$$

Example: $x^2 y'' + xy' + \lambda y = 0$, $1 < x < e$

a) Given that the eigenvalues and eigenfunctions are $\lambda_n = n^2\pi^2$, $y_n = \sin(n\pi \ln x)$, $n = 1, 2, \ldots$

b) If $f(x)$ is piecewise differentiable in the interval $(1, e)$ and if

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \sin(n\pi \ln x)$$

then show that

$$\alpha_n = \frac{\int_1^e f(x) \sin(n\pi \ln x) \frac{1}{x}\, dx}{\int_1^e \sin^2(n\pi \ln x) \frac{1}{x}\, dx}$$

Divide diff equation by x: $xy'' + y' + \frac{\lambda}{x} y = 0$

$\rightarrow (xy')' + \frac{\lambda}{x} y = 0$, so $p(x) = x$, $q(x) = 0$, $r(x) = \frac{1}{x}$

~~Normalize the eigenfunctions:~~

Begin with assumed form for $f(x)$, multiply both sides by $y_n(x)$ and integrate.

$$\therefore \int_1^e F(x)\frac{1}{x}\sin(n\pi \ln x) = \int_1^e \sum_{n=1}^{\infty} \alpha_n \sin(n\pi \ln x)\frac{1}{x}\sin(\underbrace{m\pi \ln x}_{m})\,dx$$

we know $\displaystyle\int_1^e \sin(n\pi \ln x)\frac{1}{x}\sin(m\pi \ln x)\,dx = 0, \quad m \neq n$

then $\displaystyle\int_1^e F(x)\frac{1}{x}\sin(m\pi \ln x) = \alpha_n \int_1^e \frac{\sin^2(m\pi \ln x)}{x}\,dx$

$$\Rightarrow \quad \alpha_n = \frac{\int_1^e F(x)\frac{1}{x}\sin(m\pi \ln x)\,dx}{\int_1^e \frac{\sin^2(m\pi \ln x)}{x}\,dx}$$

- By introducing an appropriate change of variables, reduce this result to form

$$\alpha_n = 2\int_0^1 F(e^t)\sin(n\pi t)\,dt$$

Let $x = e^t, \quad t = \ln x$

$dx = e^t\,dt$

$[1, e] \to [0, 1]$

$$\alpha_m = \frac{\int_0^1 F(e^t)\cancel{e^{-t}}\sin(m\pi t)\cancel{e^t}\,dt}{\underbrace{\int_0^1 \sin^2(m\pi t)\,dt}_{\frac{1}{2}}} = 2\int_0^1 F(e^t)\sin(m\pi t)\,dt$$

- Evaluate α_n when $F(x) = 1$ and when $F(x) = x$

$$\alpha_n = 2\int_0^1 F(e^t)\sin(n\pi t)\,dt$$

$$F = 1 \Rightarrow \quad \alpha_n = 2\int_0^1 \sin(n\pi t)\,dt = \left[-\frac{2}{n\pi}\cos(n\pi t)\right]_0^1$$

$$\boxed{\alpha_n = \frac{2}{n\pi}\left(1 - (-1)^n\right)}$$

when $F(x) = x$ then $\alpha_n = 2\int_0^1 e^t \sin(n\pi t)\, dt$

$$\alpha_n = \frac{e}{1+n^2\pi^2}\Big[\sin(n\pi t) - n\pi\cos(n\pi t)\Big]_0^1$$

$$\boxed{\alpha_n = \frac{n\pi}{1+n^2\pi^2}\left(1 - e(-1)^n\right)}$$

<u>Next time:</u> More discussion of convergence

Special Functions (Cochran, Chpt 3)

Last time - <u>Heat Conduction</u> $\rho c u_x = (K_0(x) u_x)_x$

where ρ = density, c = specific heat

Sturm-Liouville problem obtained by setting $u(x,t) = e^{-\lambda t}\varphi(x)$

$\Rightarrow (K_0(x)\varphi')' + \rho c \lambda \varphi = 0$, with boundary conditions.

In general, if the ODE is written

$$(P(x) y')' + (\lambda R(x) - Q(x)) y = 0 \quad , \quad a < x < b$$

may be <u>transformed "Liouville Normal Form"</u>

$$u''(t) + (\lambda - q(t)) u(t) = 0 \,,$$

which is advantageous because reduced problem with three coefficients (functions) to one function coeff ($q(t)$).

Change independent and dependent variables:

Let $y(x) = \dfrac{u(t)}{\sqrt[4]{P(x) R(x)}}$, $t = \dfrac{\int_a^x \sqrt{R(x)/P(x)}\,dx}{\int_a^b \sqrt{R(x)/P(x)}\,dx}$, $0 \le t \le 1$

$\boxed{\text{See details in Courant \& Hilbert, vol 1}}$

~~we chan~~

Example: $\left(\dfrac{\varphi'}{(1+x)^2}\right)' + \dfrac{\lambda}{(1+x)^2}\varphi = 0$, $0 < x < 1$

$\Rightarrow P(x) = \dfrac{1}{(1+x)^2}$, $R(x) = \dfrac{1}{(1+x)^2}$, $Q(x) = 0$

$\Rightarrow t = \dfrac{\int_0^x \sqrt{\dfrac{(1+x)^2}{(1+x)^2}}}{\int_0^1 dx} = \int_0^x dx = x \Rightarrow \boxed{t = x}$

$\varphi(x) = \dfrac{u(x)}{\sqrt[4]{\dfrac{1}{(1+x)^4}}} = u(x)(1+x) \Rightarrow \boxed{u(x) = \dfrac{\varphi(x)}{1+x}}$

$\varphi'(x) = (1+x)u' + u$

$\dfrac{\varphi'(x)}{(1+x)^2} = \dfrac{u'}{1+x} + \dfrac{u}{(1+x)^2}$

$\left(\dfrac{\varphi'(x)}{(1+x)^2}\right)' = \dfrac{u''}{1+x} \longrightarrow$ see next

The ODE becomes: $\boxed{u'' + \left(\lambda - \dfrac{2}{(1+x)^2}\right)u = 0 \,, \quad 0 < x < 1}$

SEE NEXT PAGE

Exercise 8.3, page 86

Verify that the curvature of the integral curves and the Hugoniot locus are the same by computing $m'(\hat\rho)$ for the Hugoniot locus from

$$\tilde m = \tilde\rho\hat m/\hat\rho \pm a(\tilde\rho - \hat\rho)\sqrt{\tilde\rho/\hat\rho}$$

and for the integral curves from

$$m(\rho) = \rho\frac{m_l}{\rho_l} - a\rho\log\frac{\rho}{\rho_1} \qquad or \qquad m(\rho) = \rho\frac{m_l}{\rho_l} + a\rho\log\frac{\rho}{\rho_1}$$

Also verify that the third derivatives are not equal.

From the Hugoniot locus we have:

$$\begin{aligned}
\tilde m(\tilde\rho) &= \tilde\rho\hat m/\hat\rho \pm a(\tilde\rho - \hat\rho)\sqrt{\tilde\rho/\hat\rho}\\
\tilde m'(\tilde\rho) &= \hat m/\hat\rho \pm \left(a\sqrt{\tilde\rho/\hat\rho} + \frac{a(\tilde\rho-\hat\rho)}{2\sqrt{\tilde\rho\hat\rho}}\right)\\
\tilde m''(\tilde\rho) &= \pm\left(\frac{a}{2\sqrt{\tilde\rho\hat\rho}} + \frac{a}{2\sqrt{\tilde\rho\hat\rho}} - \frac{a(\tilde\rho-\hat\rho)}{4\tilde\rho\sqrt{\tilde\rho\hat\rho}}\right)\\
\tilde m''(\hat\rho) &= \pm\frac{a}{\hat\rho}
\end{aligned}$$

From example 8.1, the 1-rarefactions are given by:

$$\begin{aligned}
m(\rho) &= \rho\frac{m_l}{\rho_l} - a\rho\log\frac{\rho}{\rho_1}\\
m'(\rho) &= \frac{m_l}{\rho_l} - a - a\log\frac{\rho}{\rho_1}\\
m''(\rho) &= \frac{-a}{\rho}\\
m''(\hat\rho) &= \frac{-a}{\hat\rho}
\end{aligned}$$

From exercise 8.1, the 2-rarefactions are given by:

$$\begin{aligned}
m(\rho) &= \rho\frac{m_l}{\rho_l} + a\rho\log\frac{\rho}{\rho_1}\\
m'(\rho) &= \frac{m_l}{\rho_l} + a + a\log\frac{\rho}{\rho_1}\\
m''(\rho) &= \frac{a}{\rho}\\
m''(\hat\rho) &= \frac{a}{\hat\rho}
\end{aligned}$$

The third derivative is, from the Hugoniot locus:

$$\begin{aligned}
\tilde m'''(\tilde\rho) &= \pm\left(\frac{-a}{6\tilde\rho\sqrt{\tilde\rho\hat\rho}} + \frac{a(\tilde\rho-\hat\rho)}{6\tilde\rho^2\sqrt{\tilde\rho\hat\rho}}\right)\\
\tilde m'''(\hat\rho) &= \pm\frac{a}{2\hat\rho^2}
\end{aligned}$$

The third derivatives are, from the 1 and 2-rarefactions:

$$\begin{aligned}
m'''(\rho) &= \pm\frac{a}{\rho^2}\\
m'''(\hat\rho) &= \pm\frac{a}{\hat\rho^2}
\end{aligned}$$

$$\left(\frac{\varphi'(x)}{(1+x)^2}\right)' = \frac{u''}{1+x} - \frac{u'}{(1+x)^2} + \frac{u'}{(1+x)^2} - \frac{2u}{(1+x)^3}$$

$$\frac{\lambda\varphi}{(1+x)^2} = \frac{\lambda(1+x)u}{(1+x)^2}$$

$$\Rightarrow \left(\frac{\varphi'(x)}{(1+x)^2}\right)' + \frac{\lambda\varphi}{(1+x)^2} = \frac{u''}{1+x} - \frac{2u}{(1+x)^3} + \frac{\lambda(1+x)u}{(1+x)^2} = 0$$

$$\Rightarrow \frac{u''}{1+x} - \frac{2u}{(1+x)^3} + \frac{\lambda u}{1+x} = 0$$

$$\Rightarrow u'' + \left(\lambda - \frac{2u}{(1+x)^2}\right)u = 0$$

Bari's Thm: (1960s) Let $\{\varphi_K\}$ be any complete sequence of orthonormal functions in an inner product space E and let $\{\psi_K\}$ be any sequence of orthonormal vectors in E which satisfy the inequality $\sum_{K=1}^{\infty} \|\psi_K - \varphi_K\|^2 < \infty$

Then $\{\psi_K\}$ is complete in E.

In the notes, looking at $(py')' + (\lambda R - Q)y = 0$

with BCs $\alpha_0 y(a) + \alpha_1 y'(a) = 0$, $\beta_0 y(b) + \beta_1 y'(b) = 0$,

specific example: $u'' + (\lambda - q(t))u = 0$

- IF $\alpha_1\beta_1 \neq 0 \rightarrow u_n(t) \sim \sqrt{2}\cos(n\pi t) + O(\frac{1}{n})$ as $n \to \infty$
  - ↳ eigenfunctions corresponding to the higher eigenvalu
- IF $(\alpha_1)(\beta_1) = 0 \rightarrow u_n(t) \sim \sqrt{2}\sin(n\pi t) + O(\frac{1}{n})$ as $n \to \infty$

Note - ~~geometric~~ trigonometric sequences are complete

so $\varphi_n = \sqrt{2}\sin(n\pi t)$, $\|u_n - \varphi_n\|^2 = O(\frac{1}{n^2})$

Exercise 8.2, page 85

Verify that $m'(\rho_l) = \lambda_p(u_l)$ and explain why this should be so.

From example 8.1, the 1-rarefactions are given by:

$$\begin{aligned} m(\rho) &= \rho\frac{m_l}{\rho_l} - a\rho\log\frac{\rho}{\rho_1} \\ m'(\rho) &= \frac{m_l}{\rho_l} - a - a\log\frac{\rho}{\rho_1} \\ m'(\rho_l) &= \frac{m_l}{\rho_l} - a \\ m'(\rho_l) &= \lambda_1(u_l) \end{aligned}$$

From exercise 8.1, the 2-rarefactions are given by:

$$\begin{aligned} m(\rho) &= \rho\frac{m_l}{\rho_l} + a\rho\log\frac{\rho}{\rho_1} \\ m'(\rho) &= \frac{m_l}{\rho_l} + a + a\log\frac{\rho}{\rho_1} \\ m'(\rho_l) &= \frac{m_l}{\rho_l} + a \\ m'(\rho_l) &= \lambda_2(u_l) \end{aligned}$$

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Example from last time

$$\left(\frac{\varphi'}{(1+x)^2}\right)' + \frac{\lambda}{(1+x)^2}\varphi = 0, \quad 0<x<1, \quad \varphi(0)=\varphi(1)=0$$

General solution: $\varphi(x) = c_1\left(\cos(kx) + k(1+x)\sin(kx)\right)$

$$+ c_2\left(\sin(kx) - k(1+x)\cos(kx)\right)$$

$$\Rightarrow \quad u = c_1\left(\frac{\cos kx}{1+x} + k\sin(kx)\right) + c_2\left(\frac{\sin kx}{1+x} - k\cos(kx)\right)$$

$u(0) = u(1) = 0$

$\Rightarrow \quad c_1 - kc_2 = 0$

$$u = c_2\left(\frac{k\cos kx}{1+x} + \underset{\uparrow}{k^2\sin kx} + \frac{\sin kx}{1+x} - k\cos kx\right)$$

dominant term for large k

Related problem: $y'' + \lambda y, \quad 0<x<2\pi$

$y(0) = y(2\pi)$

$y'(0) = y'(2\pi)$, periodic BCs

We can show that all eigenvalues are real and nonnegative.

$y = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x, \quad \sqrt{\lambda} = k$

$y(0) = c_1, \quad y(2\pi) = c_1\cos(2\pi k) + c_2\sin(2\pi k)$

$y'(0) = c_2 k, \quad y'(2\pi) = -kc_1\sin(2\pi k) + kc_2\cos(2\pi k)$

$$\Rightarrow \quad c_1(\cos 2\pi k - 1) + c_2\sin(2\pi k) = 0$$

$$-c_1\sin 2\pi k + c_2(\cos 2\pi k - 1) = 0$$

For non-trivial solution

$$\Rightarrow \begin{vmatrix} \cos 2\pi k - 1 & \sin 2\pi k \\ -\sin 2\pi k & \cos 2\pi k - 1 \end{vmatrix} = 0$$

$\Rightarrow (\cos 2\pi k - 1)^2 + \sin^2(2\pi k) = 0 \Rightarrow \cos^2(2\pi k) - 2\cos(2\pi k) + 1 + \sin^2(2\pi k) = 0$

$\Rightarrow \cos 2\pi k = 1 \Rightarrow \boxed{\lambda_n = \cancel{(\ldots)}\, n^2} \Rightarrow y_n = c_1\cos(nx) + c_2\sin(nx)$

$\Rightarrow k = 1, 2, 3, \ldots$

$y'' = 0, \quad y = c_1 + c_2 x$

$y(0) = y(2\pi) \Rightarrow c_1, \quad y'(0) = y'(2\pi)$

Exercise 8.1, page 85

Derive the expressions for 2-rarefactons for the isothermal equations.

The eigenvalues and eigenvectors for the isothermal equations are

$$\lambda_1 = \frac{m}{\rho} - a \qquad r_1 = \begin{pmatrix} 1 \\ \frac{m}{\rho} - a \end{pmatrix} \qquad \lambda_2 = \frac{m}{\rho} + a \qquad r_2 = \begin{pmatrix} 1 \\ \frac{m}{\rho} + a \end{pmatrix}$$

We have

$$\begin{aligned} \nabla\lambda_1(u)\cdot r_1(u) &= \frac{-a}{\rho} \\ \nabla\lambda_1(u)\cdot r_1(u) &= \frac{-a}{\rho} \end{aligned}$$

Therefore substituting into the system of ordinary differential equations for $w(\xi)$ with boundary conditions $w(\xi_1) = u_l$

$$\xi'(\xi) = \frac{r_p(w(\xi))}{\nabla\lambda_p(w(\xi))\cdot r_p(x(\xi))}$$

For the 2-rarefactions, we use $p = 2$ and so the system of equations becomes

$$\begin{aligned} \rho'(\xi) &= \frac{\rho(\xi)}{a} && , \rho(\xi_1) = \rho_l \\ m'(\xi) &= \frac{m(\xi)/\rho(\xi) + a}{a/\rho(\xi)} && , m(\xi_1) = m_l \end{aligned}$$

Solving the first equation we obtain:

$$\rho(\xi) = \rho_l e^{(\xi-\xi_1)/a}$$

The second equation becomes:

$$m'(\xi) = \frac{m(\xi)}{a} + \rho_l e^{(\xi-\xi_1)/a} \qquad , m(\xi_1) = m_l$$

We solve this to find:

$$m(\xi) = \rho_l(\xi - 1)e^{(\xi-\xi_1)/a}$$

Now solve for ξ in terms of ρ:

$$\xi = \xi_1 + a\log\frac{\rho}{\rho_l}$$

Substitute for ξ in the expression for $m(\xi)$ to solve for $m(\rho)$

$$m(\rho) = \rho(\xi_1 - a) + a\rho\log\frac{\rho}{\rho_1}$$

Substitute $\xi_1 = \lambda_2(u_l) = m_l/\rho_l - a$ to find

$$m(\rho) = \rho\frac{m_l}{\rho_l} + a\rho\log\frac{\rho}{\rho_1}$$

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The relevant expansion is the Full range Fourier Series, $f(x)$ square integrable on $[0, 2\pi]$, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$

• **Boyce & DiPrima:**

Suppose f and f' are piecewise continuous on $-L < x < L$. Suppose f is defined outside to be periodic with period $2L$.

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right)$$

where
$$a_m = \frac{1}{\pi} \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) f(x)\, dx$$

$$b_m = \frac{1}{\pi} \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) f(x)\, dx$$

The Fourier series converges to $f(x)$ at all points where f is continuous, and to $\frac{f(x^+) + f(x^-)}{2}$ at all points where $f(x)$ is discontinuous.

~~For~~ ~~Watch~~ see <u>Cochran Text</u> Appendix 3 - Series solutions of ODEs

see also Chpt 3: introduces solving Maxwell's eqns in a cylindrical geometry.

Many of the important special functions are defined as solutions to ODEs.

For example, solution by power series:

$$y'' + f(x)\, y' + g(x)\, y = 0$$

seek solutions of the form: $y = \sum_{k=0}^{\infty} c_k (x - x_0)^k$, $|x - x_0| < R$

<u>Def:</u> A point x_0 is an <u>ordinary point</u> of the diff eqn if there is an interval $|x - x_0| < R$ in which

$f(x) = \sum_{k=0}^{\infty} \alpha_k (x - x_0)^k$, $g(x) = \sum_{k=0}^{\infty} \beta_k (x - x_0)^k$ both converge.

Therefore we have

$$\begin{aligned}\tilde{v} &= \hat{v} \pm (\tilde{\varphi} - \hat{\varphi})\sqrt{\frac{2}{\tilde{\varphi} + \hat{\varphi}}} \\ s &= \hat{v} \pm \tilde{\varphi}\sqrt{\frac{2}{\tilde{\varphi} + \hat{\varphi}}}\end{aligned}$$

We pick the parameterization $\tilde{\varphi} = \hat{\varphi} + \xi$ and obtain

$$\begin{aligned}\tilde{v} &= \hat{v} \pm \xi\sqrt{\frac{2}{\xi + 2\hat{\varphi}}} \\ s &= \hat{v} \pm (\hat{\varphi} + \xi)\sqrt{\frac{2}{\xi + 2\hat{\varphi}}}\end{aligned}$$

Therefore we have

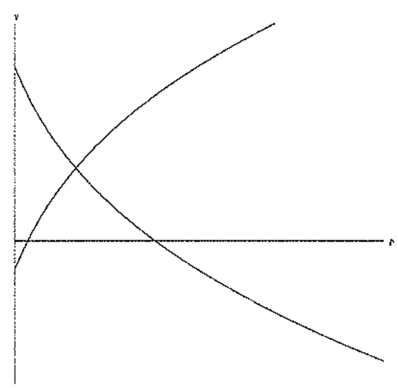
$$\tilde{u}_1(\xi;\hat{u}) = \begin{pmatrix} \hat{v} - \xi\sqrt{\frac{2}{\xi+2\hat{\varphi}}} \\ \hat{\varphi} + \xi \end{pmatrix}, \qquad s_1(\xi;\hat{u}) = \hat{v} - (\hat{\varphi} + \xi)\sqrt{\frac{2}{\xi + 2\hat{\varphi}}}$$

$$\tilde{u}_2(\xi;\hat{u}) = \begin{pmatrix} \hat{v} + \xi\sqrt{\frac{2}{\xi+2\hat{\varphi}}} \\ \hat{\varphi} + \xi \end{pmatrix}, \qquad s_2(\xi;\hat{u}) = \hat{v} + (\hat{\varphi} + \xi)\sqrt{\frac{2}{\xi + 2\hat{\varphi}}}$$

Notice that

$$\begin{aligned}\frac{\partial}{\partial \xi}\tilde{u}_p(0;\hat{u}) &= \hat{\varphi} r_p(\hat{u}) \\ s_p(0;\hat{u}) &= \lambda_p(\hat{u}) \qquad \text{for } p = 1, 2\end{aligned}$$

The hugoniot locus is plotted below for $\hat{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



ν = constant

For example, $y'' + \nu^2 y = 0$ – Every point is an ordinary point. Every solution can be written as a convergent power series about any point. The radius of convergence $R = \infty$ for all these solutions.

For example: $y'' + xy = 0$, Airy's equation

Chebyshev's Eqn: $(1-x^2)y'' - xy' + p^2 y = 0, \quad -1 < x < 1$

where p = constant

it is known that $x=0$ is an ordinary point.

$\Rightarrow$ assume $y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$

substitute into ODE

$$y'(x) = c_1 + 2c_2 x + \cdots + n c_n x^{n-1} = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y''(x) = 2c_2 + \cdots + n(n-1) c_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$0 = (1-x^2)y'' - xy' + p^2 y$$

$$= (1-x^2) \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - x \sum_{n=1}^{\infty} n c_n x^{n-1} + p^2 \sum_{n=0}^{\infty} c_n x^n$$

$$= \underbrace{\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}}_{\text{change}} - \underbrace{\sum_{n=2}^{\infty} n(n-1) c_n x^n}_{\text{no change}} - \sum_{n=1}^{\infty} n c_n x^n + p^2 \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=0}^{\infty} n(n-1) c_n x^n - \sum_{n=0}^{\infty} n c_n x^n + p^2 \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1) c_{n+2} + (-n^2 + n - n + p^2) c_n\right] x^n$$

each power must vanish separately for ODE to be satisfied:

$$(n+2)(n+1) c_{n+2} + (p^2 - n^2) c_n = 0, \quad n = 0, 1, 2, \ldots$$

$$\Rightarrow \boxed{c_{n+2} = \frac{(n^2 - p^2) c_n}{(n+2)(n+1)}} \Big\} \text{ two-term recurrence formula}$$

Example 7.1, page 71

The isothermal equations of gas dynamics with momentum $m = \rho v$ are

$$\begin{aligned} \rho_t + m_x &= 0 \\ m_t + (m^2/\rho + a^2\rho)_x &= 0 \end{aligned}$$

or $u_t + f(u)_x = 0$ where $u = u(\rho, m)$. The Jacobian matrix is

$$f'(u) = \begin{bmatrix} 0 & 1 \\ a^2 - m^2/\rho^2 & 2m/\rho \end{bmatrix}$$

Calculate eigenvalues:

$$\begin{vmatrix} 0-\lambda & 1 \\ a^2 - m^2/\rho^2 & 2m/\rho - \lambda \end{vmatrix} = 0 \quad \rightarrow \quad -\lambda\left(\frac{2m}{\rho} - \lambda\right) - a^2 + \frac{m^2}{\rho} = 0$$

$$\rightarrow \quad \lambda^2 - \frac{2m}{\rho}\lambda - a^2 + \frac{m^2}{\rho} = 0 \quad \rightarrow \quad \lambda = \frac{2m}{2\rho} \pm \sqrt{\frac{4m^2}{\rho} + 4a^2 - \frac{4m^2}{\rho}} \quad \rightarrow \quad \lambda = \frac{m}{\rho} \pm a$$

Calculate corresponding eigenvectors:

$$\lambda = \frac{m}{\rho} + a: \quad \begin{bmatrix} -\frac{m}{\rho} - a & 1 \\ a^2 - \frac{m^2}{\rho^2} & \frac{m}{\rho} - a \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \quad \rightarrow \quad \begin{matrix} -(\frac{m}{\rho} + a)v_1 + v_2 = 0 \\ (\frac{m}{\rho} + a)v_1 - v_2 = 0 \end{matrix} \quad \rightarrow \quad v = \begin{pmatrix} 1 \\ \frac{m}{\rho} + a \end{pmatrix}$$

$$\lambda = \frac{m}{\rho} - a: \quad \begin{bmatrix} -\frac{m}{\rho} + a & 1 \\ a^2 - \frac{m^2}{\rho^2} & \frac{m}{\rho} + a \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \quad \rightarrow \quad \begin{matrix} (-\frac{m}{\rho} + a)v_1 + v_2 = 0 \\ (-\frac{m}{\rho} + a)v_1 + v_2 = 0 \end{matrix} \quad \rightarrow \quad v = \begin{pmatrix} 1 \\ \frac{m}{\rho} - a \end{pmatrix}$$

Fix a state $\hat{u}$ and determine the set of states $\tilde{u}$ that can be connected by a discontinuity. The Rankine-Hugoniot condition for the system is

$$f(\tilde{u}) - f(\hat{u}) = s(\tilde{u} - \hat{u}) \quad \rightarrow \quad \begin{aligned} \tilde{m} - \hat{m} &= s(\tilde{\rho} - \hat{\rho}) \\ (\tilde{m}^2/\tilde{\rho} + a^2\tilde{\rho}) - (\hat{m}^2/\hat{\rho} + a^2\hat{\rho}) &= s(\tilde{m} - \hat{m}) \end{aligned}$$

This gives two equations in three unknowns $\tilde{\rho}$, $\tilde{m}$ and s. Solve for $\tilde{m}$ in terms of $\tilde{\rho}$:

$$\frac{\tilde{m}^2}{\tilde{\rho}} + a^2\tilde{\rho} - \frac{\hat{m}^2}{\hat{\rho}} - a^2\hat{\rho} = \frac{(\tilde{m} - \hat{m})^2}{\tilde{\rho} - \hat{\rho}}$$

$$\frac{\tilde{m}^2}{\tilde{\rho}} - \frac{\tilde{m}^2}{\tilde{\rho} - \hat{\rho}} + \frac{2\tilde{m}\hat{m}}{\tilde{\rho} - \hat{\rho}} - \frac{\hat{m}^2}{\tilde{\rho} - \hat{\rho}} + a^2(\tilde{\rho} - \hat{\rho}) - \frac{\hat{m}^2}{\hat{\rho}} = 0$$

$$-\tilde{m}^2\frac{\hat{\rho}}{\tilde{\rho}} + 2\tilde{m}\hat{m} - \hat{m}\frac{\tilde{\rho}}{\hat{\rho}} + a^2(\tilde{\rho} - \hat{\rho}) = 0 \quad \rightarrow \quad \tilde{m}^2\frac{\hat{\rho}}{\tilde{\rho}} - 2\tilde{m}\hat{m} + \left(\hat{m}\frac{\tilde{\rho}}{\hat{\rho}} - a^2(\tilde{\rho} - \hat{\rho})\right) = 0$$

$$\tilde{m} = \frac{2\hat{m}\tilde{\rho}}{2\hat{\rho}} \pm \frac{\tilde{\rho}}{2\hat{\rho}}\sqrt{4\hat{m}^2 - 4\frac{\hat{\rho}}{\tilde{\rho}}\hat{m}^2\frac{\tilde{\rho}}{\hat{\rho}} + 4\frac{\hat{\rho}}{\tilde{\rho}}a^2(\tilde{\rho} - \hat{\rho})^2} \quad \rightarrow \quad \tilde{m} = \frac{\hat{m}\tilde{\rho}}{\hat{\rho}} \pm \frac{\tilde{\rho}}{\hat{\rho}}a(\tilde{\rho} - \hat{\rho})\sqrt{\frac{\hat{\rho}}{\tilde{\rho}}}$$

$$\tilde{m} = \frac{\hat{m}\tilde{\rho}}{\hat{\rho}} \pm a(\tilde{\rho} - \hat{\rho})\sqrt{\frac{\tilde{\rho}}{\hat{\rho}}}$$

Solve for s in terms of $\tilde{\rho}$:

$$\frac{(\hat{m} + s(\tilde{\rho} - \hat{\rho}))^2}{\tilde{\rho}} + a^2\tilde{\rho} - \frac{\hat{m}^2}{\hat{\rho}} - a^2\hat{\rho} = s^2(\tilde{\rho} - \hat{\rho})$$

$$\frac{\hat{m}^2}{\tilde{\rho}} + \frac{2\hat{m}s(\tilde{\rho} - \hat{\rho})}{\tilde{\rho}} + \frac{s^2(\tilde{\rho} - \hat{\rho})^2}{\tilde{\rho}} + a^2(\tilde{\rho} - \hat{\rho}) - \frac{\hat{m}^2}{\hat{\rho}} = s^2(\tilde{\rho} - \hat{\rho})$$

$$\frac{s^2(\tilde{\rho} - \hat{\rho})}{\tilde{\rho}} - s^2 + \frac{2\hat{m}}{\tilde{\rho}}s + a^2 - \frac{\hat{m}^2}{\tilde{\rho}\hat{\rho}} = 0 \quad \rightarrow \quad \frac{s^2\hat{\rho}}{\tilde{\rho}} - \frac{2\hat{m}}{\tilde{\rho}}s + \left(\frac{\hat{m}^2}{\tilde{\rho}\hat{\rho}} - a^2\right) = 0$$

$$s = \frac{2\hat{m}}{\tilde{\rho}}\frac{\tilde{\rho}}{2\hat{\rho}} \pm \frac{\tilde{\rho}}{2\hat{\rho}}\sqrt{\frac{4\hat{m}^2}{\tilde{\rho}^2} - \frac{4\hat{\rho}}{\tilde{\rho}}\left(\frac{\hat{m}^2}{\tilde{\rho}\hat{\rho}} - a^2\right)} \quad \rightarrow \quad s = \frac{\hat{m}}{\hat{\rho}} \pm \frac{a\tilde{\rho}}{\hat{\rho}}\sqrt{\frac{\hat{\rho}}{\tilde{\rho}}} \quad \rightarrow \quad s = \frac{\hat{m}}{\hat{\rho}} \pm a\sqrt{\frac{\tilde{\rho}}{\hat{\rho}}}$$

Normally take c_0 and c_1 as the arbitrary constants.

If p is an integer, then one series terminates at c_p.

$\Rightarrow$ these are called the Chebyshev polynomials.

Herron's theorem: American Mathematical Monthy 1989 824-827

> If the recurrence formula is of the form
>
> $$c_{n+s} = F(n)\, c_n \quad , n = 0, 1, 2, \ldots \quad , s > 0 \ (s = \text{integer})$$
>
> then the radius of convergence is
>
> $$R = \left[\lim_{n \to \infty} |F(sn)| \right]^{-1/s}$$

In our case: $F(n) = \frac{n^2 - p^2}{(n+2)(n+1)}$ for any p

and $R = 1$.

---

this mirrors the general theory that

$y'' + f(x)\, y' + g(x)\, y = 0$

has two solutions which converge on the intersection of the intervals of $f(x)$ and $g(x)$

Exercise 7.1, page 73

Calculate the Hugoniot locus for the shallow water equations.

$$\begin{pmatrix} v \\ \varphi \end{pmatrix}_t + \begin{pmatrix} v^2/2+\varphi \\ v\varphi \end{pmatrix}_x = 0 \quad \rightarrow \quad \begin{pmatrix} v \\ \varphi \end{pmatrix}_t + \begin{bmatrix} v & 1 \\ \varphi & v \end{bmatrix} \begin{pmatrix} v \\ \varphi \end{pmatrix}_x$$

The eigenvalues and corresponding eigenvectors are

$$\lambda_1 = v - \sqrt{\varphi} \qquad r_1 = \begin{pmatrix} -\sqrt{\varphi} \\ 1 \end{pmatrix}$$

$$\lambda_2 = v + \sqrt{\varphi} \qquad r_2 = \begin{pmatrix} \sqrt{\varphi} \\ 1 \end{pmatrix}$$

Fix a state $\hat{u} = \begin{pmatrix} \hat{v} \\ \hat{\varphi} \end{pmatrix}$ and determine states $\tilde{u} = \begin{pmatrix} \tilde{v} \\ \tilde{\varphi} \end{pmatrix}$ that can be connected by a discontinuity.

The Rankine-Hugoniot condition $f(\tilde{u}) - f(\hat{u}) = s(\tilde{u} - \hat{u})$ becomes

$$\begin{aligned} \left(\frac{\tilde{v}^2}{2} + \tilde{\varphi}\right) - \left(\frac{\hat{v}^2}{2} + \hat{\varphi}\right) &= s(\tilde{v} - \hat{v}) \\ \tilde{v}\tilde{\varphi} - \hat{v}\hat{\varphi} &= s(\tilde{\varphi} - \hat{\varphi}) \end{aligned}$$

Solve for s in terms of $\tilde{\varphi}$

$$\tilde{v} = \frac{s(\tilde{\varphi} - \hat{\varphi}) + \hat{v}\hat{\varphi}}{\tilde{\varphi}} \qquad \rightarrow$$

$$\frac{s^2(\tilde{\varphi} - \hat{\varphi})^2}{2\tilde{\varphi}^2} + \frac{2s(\tilde{\varphi} - \hat{\varphi})\hat{v}\hat{\varphi}}{2\varphi^2} + \frac{\hat{v}^2\hat{\varphi}^2}{2\tilde{\varphi}^2} + (\tilde{\varphi} - \hat{\varphi}) - \frac{\hat{v}^2}{2} = \frac{s^2(\tilde{\varphi} - \hat{\varphi})}{\tilde{\varphi}} + \frac{s\hat{v}\hat{\varphi}}{\tilde{\varphi}} - s\hat{v}$$

$$\frac{s^2(\tilde{\varphi} - \hat{\varphi})}{\tilde{\varphi}^2}\left(\frac{\tilde{\varphi} - \hat{\varphi}}{2} - \tilde{\varphi}\right) + \frac{s\hat{v}(\tilde{\varphi} - \hat{\varphi})}{\varphi}\left(\frac{\hat{\varphi}}{\tilde{\varphi}} + 1\right) + (\tilde{\varphi} - \hat{\varphi}) + \frac{\hat{v}^2}{2}\left(\frac{\hat{\varphi}^2 - \tilde{\varphi}^2}{\tilde{\varphi}^2}\right) = 0$$

$$\frac{s^2(\tilde{\varphi} - \hat{\varphi})(\tilde{\varphi} + \hat{\varphi})}{2\tilde{\varphi}^2} - \frac{s\hat{v}(\tilde{\varphi} - \hat{\varphi})(\tilde{\varphi} + \hat{\varphi})}{\varphi^2} - (\tilde{\varphi} - \hat{\varphi}) - \frac{\hat{v}^2(\hat{\varphi} - \tilde{\varphi})(\tilde{\varphi} + \hat{\varphi})}{2\tilde{\varphi}^2} = 0$$

$$s^2 - 2\hat{v}s - \frac{2\tilde{\varphi}^2}{\tilde{\varphi} + \hat{\varphi}} + \hat{v}^2 = 0 \qquad \rightarrow \qquad s = \hat{v} \pm \frac{1}{2}\sqrt{4\hat{v}^2 - 4\hat{v}^2 + 4\left(\frac{2\tilde{\varphi}^2}{\tilde{\varphi} + \hat{\varphi}}\right)}$$

$$\rightarrow \qquad s = \hat{v} \pm \tilde{\varphi}\sqrt{\frac{2}{\tilde{\varphi} + \hat{\varphi}}}$$

Solve for $\tilde{v}$ in terms of $\tilde{\varphi}$

$$s = \frac{\tilde{v}\tilde{\varphi} - \hat{v}\hat{\varphi}}{\tilde{\varphi} - \hat{\varphi}} \qquad \rightarrow$$

$$\frac{\tilde{v}^2}{2} - \frac{\hat{v}^2}{2} + \tilde{\varphi} - \hat{\varphi} = \frac{(\tilde{v} - \hat{v})(\tilde{v}\tilde{\varphi} - \hat{v}\hat{\varphi})}{\tilde{\varphi} - \hat{\varphi}}$$

$$\frac{\tilde{v}^2}{2}(\tilde{\varphi} - \hat{\varphi}) - \frac{\hat{v}^2}{2}(\tilde{\varphi} - \hat{\varphi}) + (\tilde{\varphi} - \hat{\varphi})^2 = \tilde{v}^2\tilde{\varphi} - \tilde{v}\hat{v}\tilde{\varphi} - \tilde{v}\hat{v}\hat{\varphi} + \hat{v}^2\hat{\varphi}$$

$$-\frac{\tilde{v}^2}{2}(\tilde{\varphi} + \hat{\varphi}) + \tilde{v}\hat{v}(\tilde{\varphi} + \hat{\varphi}) - \frac{\hat{v}^2}{2}(\tilde{\varphi} + \hat{\varphi}) + (\tilde{\varphi} - \hat{\varphi})^2 = 0$$

$$\tilde{v}^2 - 2\tilde{v}\hat{v} + \hat{v}^2 - \frac{(\tilde{\varphi} - \hat{\varphi})^2}{\tilde{\varphi} + \hat{\varphi}} = 0 \qquad \rightarrow \qquad \tilde{v} = \hat{v} \pm \frac{1}{2}\sqrt{4\hat{v}^2 - 4\left(\hat{v}^2 - \frac{(\tilde{\varphi} - \hat{\varphi})^2}{\tilde{\varphi} + \hat{\varphi}}\right)}$$

$$\rightarrow \qquad \tilde{v} = \hat{v} \pm (\tilde{\varphi} - \hat{\varphi})\sqrt{\frac{2}{\tilde{\varphi} + \hat{\varphi}}}$$

Topics

Eigenvectors & Eigenvectors - finite dimensional
Eigenvalue Problems for ODEs
Special Functions — defined by diFF eqns

Keener { Green's Functions
Calculus of Variations } Hildebrand (bookstore)

special function: Gamma Function (App 5 in Cochran)
- defined by an integral

$y'' + f(x)\, y' + g(x)\, y = 0$ (standard form - determines if pt is ordinary or singular)

power series approach
- When f and g are analytic at a point, this is an <u>ordinary point</u>.

Last time $(1-x^2)\, y'' - xy' + p^2 y = 0$, p - constant

$$y'' - \frac{x}{1-x^2} y' + \frac{p^2}{1-x^2} y = 0$$

hence f and g are analytic on $-1 < x < 1$

IF either f or g fails to be analytic at $x = x_0$ then x_0 is a <u>singular point</u>.

- In the example, $x = \pm 1$ are singular points of the Chebyshev equation.
- $y'' + \lambda y$ has no singular points on $-\infty < x < \infty$

<u>Def</u>: A point x_0 is termed a <u>regular singular point</u> of the diFF eqn iF there exists an interval $|x - x_0| < R$ in which both $(x - x_0) f(x)$ and $(x - x_0)^2 g(x)$ have converging power series representations.

~~In~~ ~~some~~

Example: $y'' + \frac{x}{x^2-1} y' - \frac{p^2}{x^2-1} y = 0$

$$\rightarrow \quad y'' + \frac{x}{(x-1)(x+1)} y' + \frac{p^2}{(x-1)(x+1)} y = 0$$

$$(x-1)F(x) = \frac{(x-1)\,x}{(x-1)(x+1)} = \frac{x}{x+1} = P(x)$$

$$(x-1)^2 q(x) = (x-1)^2\left[\frac{-p^2}{(x-1)(x+1)}\right] = \frac{-p^2(x-1)}{x+1} = Q(x)$$

Both power series have convergent power series about $x=1$.

$$P(x) = P(1) + P'(1)(x-1) + \ldots$$
$$Q(x) = Q(1) + Q'(1)(x-1) + \ldots$$

Theorem 2: Let x_0 be a regular singular point. In this case, $\exists$ at least one solution which has the representation

$$y(x) = \sum_{k=0}^{\infty} c_k (x-x_0)^{k+\alpha_0}$$

$$= (x-x_0)^{\alpha_0} \sum_{k=0}^{\infty} c_k (x-x_0)^k$$

converges for $0 < |x-x_0| < R$

The constant α is a root of the initial equation

$$\alpha^2 + \alpha(a_0 - 1) + b_0 = 0,$$

$$a(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k, \quad b(x) = \sum_{k=0}^{\infty} b_k (x-x_0)^k$$

Simple prototype - Euler- Cauchy Equation

$$x^2 y'' + a_0 x y' + b_0 y = 0 \quad , \quad x > 0$$

Try $y = x^m$, $y' = m x^{m-1}$, $y'' = m(m-1) x^{m-2}$

$$x^2\left(m(m-1)x^{m-2}\right) + a_0 x \left(m x^{m-1}\right) + b_0 x^m = 0$$

$$x^m\left(m(m-1) + a_0 m + b_0\right) = 0$$

$$\rightarrow m^2 + (a_0 - 1) m + b_0 = 0$$

In general, $y = \sum_{k=0}^{\infty} c_k (x - x_0)^{k+\alpha}$ the coefficients are expressed in terms of the a_k and b_k.

If the roots are complex, or if both are real and do not differ by an integer, then there are two solutions of the assumed form.

These two solutions are independent.

The exceptional case is a repeated root $\alpha = \alpha_1, \alpha_1$ or $\alpha_2 - \alpha_1$. The second solution may be shown to be of the form:

$$y_2(x) = B y_1(x) \ln(x - x_0) + \sum_{k=0}^{\infty} c_k (x - x_0)^{k+\alpha_2}$$

Many of the special functions solve ODEs with regular single points.

Example:

Cochran in chpt 3 solves the hypergeometric eqn.

$$x(1-x)y'' + (c-(a+b+1)x)y' - aby = 0$$

a, b, c constants.

First general solution is called ${}_2F_1(a,b,c,x)$

$${}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \frac{x^k}{k!}, \quad |x|<1$$

about $x=0$

- $\boxed{\text{Assume } y = \sum_{k=0}^{\infty} A_k x^{k+\alpha}}$, singular point at $x=0$

$$y' = \sum_{k=0}^{\infty} (k+\alpha) A_k x^{k+\alpha-1}$$

$$y'' = \sum_{k=0}^{\infty} (k+\alpha)(k+\alpha-1) A_k^{k+\alpha-2}$$

$$0 = x(1-x)y'' + (c-(a+b+1)x)y' - aby$$

$$= x(1-x)\sum_{k=0}^{\infty}(k+\alpha)(k+\alpha-1)A_k^{k+\alpha-2} + [c-(a+b+1)x]\sum_{k=0}^{\infty}(k+\alpha)A_k x^{k+\alpha-1} - ab\sum_{k=0}^{\infty} A_k x^{k+\alpha}$$

$$= \sum_{k=0}^{\infty}(k+\alpha)(k+\alpha-1)A_k\left(x^{k+\alpha-1} - x^{k+\alpha}\right) + \sum_{k=0}^{\infty}(k+\alpha)A_k\left(cx^{k+\alpha-1} - (a+b+1)x^{k+\alpha}\right) - ab\sum_{k=0}^{\infty}A_k x^{k+\alpha}$$

- leading term

$k=0$: $x^{\alpha-1}$: $\boxed{\alpha(\alpha-1)A_0 + \alpha A_0 c = 0 \quad \text{Initial Eqn, } A_0 \neq 0}$

$$\alpha^2 - \alpha + \alpha c = 0 \Rightarrow \alpha = 0,\ \alpha = 1-c$$

- nex term, $k=1 \rightarrow x^{\alpha}$

Let $\alpha = 0$

$$0 = \sum_{k=0}^{\infty} k(k-1) A_k \left(x^{k-1} - x^k\right) + \sum_{k=0}^{\infty} k A_k \left(c x^{k-1} - (a+b+1) x^k\right) - ab \sum_{k=0}^{\infty} A_k x^k$$

$$= \sum_{k=0}^{\infty} \left(k(k-1) + kc\right) A_k x^{k-1} - \sum_{k=0}^{\infty} \left[k(k-1) + k(a+b+1) + ab\right] A_k x^k$$

$$= \sum_{k=0}^{\infty} \left[(k+1)(k+c) A_{k+1} - \left(k(k-1) + k(a+b+1) + ab\right) A_k\right] x^k$$

Require the coefficients of x_k to vanish for each k:

$$(k+1)(k+c) A_{k+1} - (k-a)(k+b) A_k = 0 \qquad \text{(recursive formula)}$$

$$\left|\frac{A_k}{A_{k+1}}\right| = \left|\frac{(a+k)(b+k)}{(c+k)(k+1)}\right| \to 1 \quad \text{as} \quad k \to \infty$$

This series converges for $|x| < 1$.

What happens at $x = \pm 1$?

Weirestrauss Convergence criteria

Suppose $\sum c_k z^k$ has a unit radius of convergence.

Let $\dfrac{c_{k+1}}{c_k} = 1 - \dfrac{\beta}{k} + \dfrac{B(k)}{k^\lambda}$

where β is a complex constant, $\lambda > 1$, $B(k)$ is bounded as $k \to \infty$. Then for points on the circle of convergence $|z| = 1$, the series will: ① converge absolutely if $Re(\beta) > 1$; ② diverge if $Re(\beta) \le 0$; ③ converge but not absolutely if $0 < Re(\beta) \le 1$ except at $z = 1$ where it will diverge.

Simple Example: $\sum \frac{z^k}{k}$

Converges everywhere on the unit circle except at $z = 1$

$$\frac{c_{k+1}}{c_k} = \frac{k+1}{k} = 1 + \frac{1}{k}$$

Back to hyper geometric series

$$\frac{A_{k+1}}{A_k} = \frac{k^2 + (a+b)k + ab}{k^2 + (c+1)k + c}$$

$$= 1 + \frac{(a+b-c-1)k + ab - c}{k^2 + (c+1)k + c}$$

$$= 1 + \frac{a+b-c-1}{k} + \frac{B(k)}{k^2} \qquad \boxed{\beta = c + 1 - a - b}$$

(1) Series converges on $|z| = 1$ if $Re(c - a - b) > 0$

(2) Series diverges if $Re(c - a - b) \leq -1$

(3) converges conditionally if $0 < Re(c - a - b + 1) \leq 1$

$$A_{k+1} = \frac{(a+k)(b+k)}{(c+k)(k+1)} A_k \quad , k \geq 0 \quad \text{- RECURRENCE RELATION}$$

- Gamma function is a generalization of the factorial function.

It can be defined by $\Gamma(a+1) = a\,\Gamma(a)$ for $Re(a) > 0$

$$\to \quad A_k = \frac{(a+k-1)(b+k-1)}{(c+k-1)k} A_{k-1}$$

substitute

$$\Rightarrow \quad A_k = A_0 \frac{a(a+1)\cdots(a+k-1)\,b(b-1)\ldots(b+k-1)}{c(c+1)\ldots c(c+k-1)\cdot 1\cdot 2\ldots k}$$

$$A_k = A_0 \frac{\Gamma(a+k)\,\Gamma(b+k)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b)\,\Gamma(c+k)\,!}$$

$$\Rightarrow \quad {}_2F_1(a;b;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \frac{x^k}{k!}$$

Euler's formula

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_0^1 (1-xt)^{-b} t^{a-1} (1-t)^{c-a-1}\, dt$$

since $\displaystyle (1-xt)^{-b} = \sum_{k=0}^{\infty} \frac{(-xt)^k}{k!} \frac{\Gamma(-b+1)}{\Gamma(-b-k+1)}$

and $\displaystyle \int_0^1 t^{z-1}(1-t)^{w-1}\,dt = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$

Confluent hypergeometric eqn

$$xy'' + (c-x)y' - ay = 0$$

sometimes called Kummer's eqn

$$M(a;c;x) = {}_1F_1(a,c;x) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(c+k)} \frac{x^k}{k!}$$

The second solution to the hypergeometric eqn:

write $y = x^{1-c} w$

substitute into hypergeometric eqn

$$\Rightarrow \quad x(1-x)w'' + \left[(2-c) - (a+b-2c+3)x\right] w' - (a-c+1)(b-c+1)w = 0$$

$$w = {}_2F_1(a-c+1,\, b-c+1;\, 2-c,\, x)$$

$$y = {}_2F_1(a;b;c;x) \quad , \quad c \neq 1$$

- Cochran treats the **Hypergeometric Diff Eqn**

$$x(1-x)y'' + \left[c-(a+b+1)x\right]y' - aby = 0$$

- Many important eqns can be transformed to or from the hypergeometric equation.

- Chebyshev's Eqn

$$(1-x^2)y'' - xy' + p^2 y = 0 \quad , \quad -1 < x < 1 \quad , \quad p = \text{constant}$$

we found the solution in series about $x=0$
It may also be transformed to the hypergeometric DE.
Let the independent variable be $z = \frac{1-x}{2}$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = -\frac{1}{2}\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \left(-\frac{1}{2}\right)^2 \frac{d^2y}{dz^2} = \frac{1}{4}\frac{d^2y}{dz^2}$$

$$\Rightarrow \left(1-(1-2z)^2\right)\frac{1}{4}\frac{d^2y}{dz^2} - (1-2z)\left(-\frac{1}{2}\frac{dy}{dz}\right) + p^2 y(z) = 0$$

$$\frac{2z(2-2z)}{4} \cancel{4(x)}\, y''(z) + \left(\frac{1}{2}-z\right)y'(z) + p^2 y(z) = 0$$

$$z(1-z)y''(z) + \left(\frac{1}{2}-z\right)y'(z) + p^2 y(z) = 0$$

hypergeometric eqn with: $c = \frac{1}{2}$, $a+b=0$, $-ab = p^2$, $x = z$

take $a = -p$, $b = p$

$$\Rightarrow y(z) = {}_2F_1\left(-p; p; \tfrac{1}{2}; z\right) = T_p(z)$$

$$\boxed{{}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\,\Gamma(b+k)}{\Gamma(c+k)} \frac{x^k}{k!}}$$

$${}_2F_1(-p;p;\tfrac{1}{2};z) = T_p(z) = \frac{\Gamma(\frac{1}{2})}{\Gamma(-p)\,\Gamma(p)} \sum_{k=0}^{\infty} \frac{\Gamma(-p+k)\,\Gamma(p+k)}{\Gamma(\frac{1}{2}+k)} \frac{z^k}{k!}$$

- IF p is an integer, the series terminates...

$$\Rightarrow \cancel{T_p(x) = \frac{\Gamma(\frac{1}{2})}{\Gamma(-p)\,\Gamma(p)} \sum_{k=0}^{\infty}}$$

$${}_2F_1\left(-p,p;\tfrac{1}{2};\tfrac{1-x}{2}\right) = T_p(x) = \frac{\Gamma(\frac{1}{2})}{\Gamma(-p)\,\Gamma(p)} \sum_{k=0}^{\infty} \frac{\Gamma(-p+k)\,\Gamma(p+k)}{\Gamma(\frac{1}{2}+k)} \left(\frac{1-x}{2}\right)^k \cdot \frac{1}{k!}$$

at $x = 1$, this becomes finite

what happens at $x = -1$? Does the series converge, or converge absolutely?

Notice $A_k = \dfrac{\Gamma(-p+k)\,\Gamma(p+k)}{\Gamma(\frac{1}{2}+k)} \cdot \dfrac{1}{k!}$

$$\frac{A_{k+1}}{A_k} = \cancel{\frac{\Gamma(\frac{1}{2}+k+1)\,(k+1)!}{\Gamma(-p+k+1)\,\Gamma}} \; \frac{\Gamma(-p+k+1)\,\Gamma(p+k+1)}{\Gamma(\frac{1}{2}+k+1)\,(k+1)!} \; \frac{\Gamma(\frac{1}{2}+k)\;k!}{\Gamma(-p+k)\,\Gamma(p+k)}$$

From definition of gamma function, $\Gamma(a+1) = a\,\Gamma(a)$

$$\Rightarrow \frac{A_{k+1}}{A_k} = \frac{(-p+k)\,\cancel{\Gamma(-p+k)}\,(p+k)\,\cancel{\Gamma(p+k)}}{(\frac{1}{2}+k)\,\cancel{\Gamma(\frac{1}{2}+k)}\;(k+1)} \; \frac{\cancel{\Gamma(\frac{1}{2}+k)}}{\cancel{\Gamma(-p+k)}\,\cancel{\Gamma(p+k)}}$$

$$= \frac{k^2-p^2}{(k+1)(\frac{1}{2}+k)} \longrightarrow 1 \text{ as } k \to \infty \quad \left(\begin{array}{l}\text{ratio test inconclusive}\\ \text{if } \lim = 1\end{array}\right)$$

Use the Weierstrauss criterion since that is inconclusive

$$\frac{A_{k+1}}{A_k} = \frac{k^2-p^2}{k^2+\frac{3}{2}k+\frac{1}{2}} = 1 - \frac{3/2}{k} + \frac{B(k)}{k^2}$$

$B(k)$ is bounded as $k \to \infty$

$\boxed{\text{Long Division}}$

$\beta = 3/2 > 1 \Rightarrow \boxed{\text{Converges Absolutely}}$

Direct Treatment of Chebyshev's Eqn

$$(1-x^2)\,y'' - xy' + p^2 y = 0, \qquad -1 < x < 1$$

expand about $x=1$, set $t = x-1$ (transformation - a shift)
($x=1$ is a singular point)

$$\Downarrow$$
$$y'(t) = y'(x)$$

$$\Rightarrow \left(1-(1+t)^2\right) y''(t) - (t+1)\,y'(t) + p^2 y(t) = 0$$
$$-t(t+2)\,y''(t) - (t+1)\,y'(t) + p^2 y(t) = 0$$
$$t(t+2)\,y''(t) + (t+1)\,y'(t) - p^2 y(t) = 0$$

- **Frobenius**:

assume $y = \sum_{n=0}^{\infty} c_n t^{n+\alpha}$ where α is to be determined

$$y'(t) = \sum_{n=0}^{\infty} c_n (n+\alpha)\, t^{n+\alpha-1}, \qquad y''(t) = \sum_{n=0}^{\infty} c_n (n+\alpha)(n+\alpha-1)\, t^{n+\alpha-2}$$

- substitute into Diff Eqn

$$0 = \sum_{n=0}^{\infty} (t^2+2t)\, t^{n+\alpha-2} (n+\alpha)(n+\alpha-1)\, c_n + \sum_{n=0}^{\infty} (t+1)\, t^{n+\alpha-1} (n+\alpha)\, c_n - \sum_{n=0}^{\infty} p^2 t^{n+\alpha} c_n$$

$n=0$: $t^{\alpha-1}$: $2\alpha(\alpha-1)c_0 + \alpha c_0 = 0 \Rightarrow c_0(2\alpha^2-\alpha) = 0 \Rightarrow c_0\alpha(2\alpha-1) = 0$

we have $\alpha = 0, \ 1/2$

$$\Rightarrow 0 = \sum_{n=0}^{\infty} c_n t^{n+\alpha}\left((n+\alpha)^2 - p^2\right) + \sum_{n=0}^{\infty} (n+\alpha)(2n+2\alpha-1)\, c_n t^{n+\alpha-1} = 0$$

$\alpha = 0$: $\quad c_{n-1}\left[(n-1)^2 - p^2\right] + n(2n-1)\, c_n = 0$

$$\Rightarrow \quad c_{n+1} = \frac{-(n+p)(n-p)}{(n+1)(2n+1)}\, c_n$$

$$\left|\frac{c_{n+1}}{c_n}\right| \to \frac{1}{2} \text{ as } n \to \infty \quad \Rightarrow \text{ Radius of convergence } = 2$$

- Bessel Functions : arise when there is a circular or cylindrical geometry

motivation: vibrating circular drum

displacement: $u(r, \theta, t)$

wave eqn: $\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right)$

Assume $u(r, \theta, t) = \varphi(r, \theta) h(t) \Rightarrow$ $\frac{d^2 h}{dt^2} = -\lambda c^2 h$

$\nabla^2 \varphi + \lambda \varphi = 0$

$\varphi(r, \theta) = f(r) g(\theta)$

Reduces to $\frac{d^2 g}{d\theta^2} = -\nu g$

$$\frac{d^2 F}{dr^2} + \frac{1}{r} \frac{dF}{dr} + \left(\lambda - \frac{\nu}{r^2} \right) F = 0$$

$$r F''(r) + F'(r) + \left(\lambda r - \frac{\nu}{r} \right) F = 0$$

$$r \frac{d}{dr} \left(r \frac{dF}{dr} \right) + \left(\lambda r^2 - m^2 \right) F = 0 \quad , \quad m^2 = \nu$$

general solution of $r^2 F''(r) + r F'(r) + (\lambda r^2 - m^2) F = 0$

is $F(r) = c_1 J_m(\sqrt{\lambda}, r) + c_2 Y_m(\sqrt{\lambda}, r)$

- Let z be the independent variable, $z = \sqrt{\lambda} r$

$\Rightarrow \frac{dF}{dr} = \frac{dF}{dz} \frac{dz}{dr} = \sqrt{\lambda} F'(z)$

$\frac{d^2 F}{dr^2} = \lambda F''(z)$

$\Rightarrow$ ~~$r^2 F''(z) + r F'(z)$~~ $\frac{z^2}{\lambda} \lambda F''(z) + \frac{z}{\sqrt{\lambda}} \sqrt{\lambda} F'(z) + (z^2 - m^2) F(z) = 0$

$\Rightarrow$ $\boxed{z^2 F''(z) + z F'(z) + (z^2 - m^2) F(z) = 0}$ standard Form of Bessel's Eqn

- Note, $z=0$ is a regular singular point

$$F''(z) + \frac{1}{z}F'(z) + \left(1-\frac{m^2}{z^2}\right)F(z) = 0$$

- Assume $F(z) = \sum_{n=0}^{\infty} c_n z^{n+\alpha}$, determine α (Frobenius Method)

Previously, $x^2 y''(x) + x a(x) y'(x) + b(x) y(x) = 0$

if $x_0 = 0$ is a regular singular point

$$a(x) = \sum_{n=0}^{\infty} a_n x^n, \quad b(x) = \sum_{n=0}^{\infty} b_n x^n$$

- Initial Eqn: $\alpha^2 + \alpha(a_0 - 1) + b_0 = 0$

- For Bessel's equation

$$a_0 = -1, \quad b_0 = -m^2 \Rightarrow \alpha^2 = m^2 \Rightarrow \boxed{\alpha = \pm m}$$

- Good reference: Bessel Functions by G. N. Watson

For $\alpha = m$: $$J_m(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+m}}{k!\,(k+m)!}$$ (assumes m = integer)

~~If m is not an integer~~

For $\alpha = -m$: $$J_{-m}(z) = \sum_{k=m}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k-m}}{k!\,(k-m)!}$$

~~Suppose~~ Since m is an integer: let $n = k - m \Rightarrow k = n + m$

$$J_{-m}(z) = \sum_{n=0}^{\infty} \frac{(-1)^{n+m} (z/2)^{2n+m}}{n!\,(n+m)!} = (-1)^m J_m(z)$$

Therefore linearly dependent (assuming m is an integer).

$$J_m^{(m)}(0) = \frac{1}{2^m} \qquad (m^{th} \text{ derivative})$$

$$J_0(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{2^2 \cdot 4^2} - \frac{z^6}{2^2 4^2 6^2} + \ldots$$

$$J_0'(z) = -J_1(z)$$

There exist many such relationships between the integer order Bessel functions of consecutive order.

The particular choice of the constants is actually dictated by integral representations of the Bessel functions.

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x \sin\theta - n\theta)\, d\theta \qquad \text{where } n \text{ is an integer}$$

Special Functions – mainly defined by differential equations. Many are also defined by integrals.

The Gamma Function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt \quad , \; x > 0$$

Complex form is sometimes useful

$$\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1$$

$$\Gamma(2) = \int_0^\infty t e^{-t}\,dt = \left[t e^{-t}\right]_0^\infty + \int_0^\infty e^{-t}\,dt = 1$$

$$\Gamma(3) = 2$$

$$\Gamma(n) = (n-1)! \quad , \; n = 1, 2, 3, \ldots$$

Use $\Gamma(x+1) = x\,\Gamma(x)$ as a way of extending the definition

Bessel Function

$$J_m(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k+m}}{k!\,(m+k)!} \quad , \; m \text{ is an integer}$$

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(z/2\right)^{2k+\nu}}{k!\,\Gamma(\nu+k-1)!} \quad , \; \nu \text{ is a real number}$$

we say that

$$J_m(z) = (-1)^m J_{-m}(z)$$

Then a 2^{nd} linearly independent solution is Y_m

Define $Y_\nu(z) = \dfrac{J_\nu(z)\cos(\pi\nu) - J_{-\nu}(z)}{\sin(\pi\nu)}$

As $\nu \to m$ an integer,

$$Y_m(z) \to \frac{J_m(z)\cos(m\pi) - J_{-m}(z)}{\sin(\pi m)} = \frac{J_m(z)(-1)^m - J_{-m}(z)}{0} = \frac{0}{0}$$

use l'Hopital's rule $\longleftarrow$ Indeterminate form $\longleftarrow$

$$Y_m(z) = \lim_{\nu \to m} \frac{\frac{d}{d\nu}\left(J_\nu(z)\cos(\pi z) - J_{-\nu}(z)\right)}{\pi\cos(\pi m)}$$

The derivative of $\Gamma(z)$ is needed to evaluate this expression

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt, \quad x>0$$

$$= \int_0^\infty e^{(x-1)\ln t} e^{-t}\,dt.$$

$$\Rightarrow \Gamma'(x) = \int_0^\infty \ln t\, e^{(x-1)\ln t} e^{-t}\,dt = \int_0^\infty t^{x-1}\ln t\, e^{-t}\,dt$$

Therefore the expression involves natural logarithms; it is given in Cochran Chpt 3.

- $x^2 R(x) y'' + x P(x) y' + Q(x) y = 0$ → $x = 0$ is a regular singular point

First solution $y_1(x)$

second solution $y_2(x)$

$$Y_2(x) = y_1(x) \int \frac{e^{-\int \frac{P(x)}{x(R(x))} dx}}{(y_1(x))^2} dx$$

This is advantageous if only a few terms of a series expansion of the solution are needed.

- Possibilities: Bessel's equation may be solved on a membrane

$0 < r < a$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right)$$

Bounded solutions are expected physically at $r = 0$. The solutions involving $Y_m(\sqrt{\lambda} r)$ are not needed. They are unbonded as $r \to 0$.

The introduction of terms involving $\ln r$ are acceptable and necessary to compute the solution.

Back to the separated form of Bessel's eqn:

$$(r F'(r))' + \left(\lambda r - \frac{m^2}{r}\right) F = 0 \quad \star$$

Boundary condition for clamped edges: $u(a, \theta, t) = 0$

$\Rightarrow F(a) = 0$ (is a possibility)

Due to homogeneity of $\star$, $F = 0$ is a possible solution, and so $F \neq 0$ has eigenfunctions, $F_n(r)$, $n = 1, 2, \ldots$

$F(r) = J_m(\sqrt{\lambda}\, a)$ a (we exclude $Y_m(\sqrt{\lambda} r)$ - unbounded at $r=0$)
in general we would need the 2- linearly independent solutions, but not in this case

the eigenvalues are given by $J_m(\sqrt{\lambda_n}\, r) = 0$

the zeros of $J_m(z)$ are given by ~~the~~ $z_n = j_{m,n}$

$$\Rightarrow \sqrt{\lambda_n}\, a = j_{m,n} \quad \Rightarrow \quad \boxed{\lambda_n = \frac{j_{m,n}^2}{a^2} \quad \text{eigenvalues}}$$

IF the solution of a problem is desired in which θ is absent, axisymmetric, then $m = 0$

ASIDE

IF θ is absent : $\dfrac{\partial^2 u}{\partial t^2} = c^2\left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r}\right)$

let $u(r,t) = F(r)\, g(t)$

$$F g'' = c^2\left(F'' g + \frac{1}{r} F' g\right)$$

$$\frac{g''}{c^2 g} = \frac{F'' + \frac{1}{r}F'}{F} = -\lambda \quad \Rightarrow \quad \boxed{F'' + \frac{1}{r}F' + \lambda F = 0 \quad \text{hence } m \text{ is absent}}$$

$$\left(r F'(r)\right)' + \left(\lambda r - \frac{m^2}{r}\right) F = 0$$

Let λ_1 and λ_2 be two different eigenvalues,
then solutions will have ~~the~~ $F_1 = J_m(\sqrt{\lambda_1}\, r)$ and $F_2 = J_m(\sqrt{\lambda_2}\, r)$ as corresponding eigenvalues

$$\Rightarrow \begin{array}{ll} \left(r F_1'\right)' + \left(\lambda_1 r - \frac{m^2}{r}\right) F_1 = 0 & (1) \\ \left(r F_2'\right)' + \left(\lambda_2 r - \frac{m^2}{r}\right) F_2 = 0 & (2) \end{array}$$

The eigenvalues are real. Suppose they are different.

Choose inner product $\langle F, g \rangle = \displaystyle\int_0^a r\, F(r)\, g(r)\, dr$

where the weight function r is based on coeff of λ_1, λ_2

Multiply ① by F_2, ② by F_1 and subtract, and integrate

$$(rF_1')'F_2 + \left(\lambda_1 r - \frac{m^2}{r}\right)F_1F_2 = 0$$

$$(rF_2')'F_1 + \left(\lambda_2 r - \frac{m^2}{r}\right)F_2F_1 = 0$$

$$\Rightarrow \int_{0^+}^{a}\left[(rF_1')'F_2 - (rF_2')'F_1\right]dr + (\lambda_1-\lambda_2)\int_0^a rF_1F_2\,dr = 0$$

$$\left[rF_1'F_2\right]_{0^+}^{a} - \left[rF_2'F_1\right]_{0^+}^{a} + \int_0^a rF_1'F_2' - rF_2'F_1'\,dr + (\lambda_1-\lambda_2)\int_0^a rF_1F_2\,dr = 0$$

Need $\lim_{r\to 0^+}\left[r\left(F_1'(r)F_2(r) - F_2'(r)F_1(r)\right)\right] = 0$

so that $(\lambda_1-\lambda_2)\int_0^a rF_1F_2\,dr = 0$

$\lambda_1 \neq \lambda_2$ by assumption then $\int_0^a rF_1F_2\,dr = 0 = \langle F_1, F_2\rangle$

This is an example of a singular Sturm-Liouville problem $(p(x)y')' + (\lambda R(x) - q(x))y = 0$, $a<x<b$

where $p(x)$ has a zero at the end point.

Expand a function in a series of Bessel functions

$$F(r) = \sum_{n=1}^{\infty} c_n J_m(\sqrt{\lambda_m}\, r), \quad 0<r<a$$

$$J_m(\sqrt{\lambda_n}\, a) = 0$$

To find c_n, $\int_0^a F(r)\, r J_n(\sqrt{\lambda_p}\, r)\,dr = \int_0^a \sum_{n=1}^{\infty} c_n J_m(\sqrt{\lambda_m}\, r)\, r J_n(\sqrt{\lambda_p}\, r)\,dr$

$$= c_p\int_0^a J_m(\sqrt{\lambda_p}\, r)\, J_m(\sqrt{\lambda_p}\, r)\, r\,dr$$

$$= c_p\int_0^a r J_m^2(\sqrt{\lambda_p}\, r)\,dr$$

Find $\int_0^a r J_m^2(\sqrt{\lambda_p} r)\, dr$ from the DE

~~$r^2 J_m'' + r J_m' + (\lambda r^2 - m^2) J_m = 0$~~

~~for $J_m(\sqrt{\lambda} r)$~~

and ~~$r^2 J_m'' + r J_m' + (\nu r^2 - m^2) J_m = 0$~~

~~for $J_m(\sqrt{\nu} r)$~~

$$r^2 J_m'' + r J_m' + (\nu^2 r^2 - m^2) J_m = 0 \quad \text{For } J_m(\nu r)$$

and $r^2 J_m'' + r J_m' + (\upsilon^2 r^2 - m^2) J_m = 0$ For $J_m(\upsilon r)$

For $\int_0^a J_m^2(\sqrt{\lambda} r)\, dr$, set $x = \frac{r}{a}$, $\rightarrow a \int_0^1 J_m^2(\sqrt{\lambda} a x)\, x\, dx$

so we're considering $\int_0^1 x J_m^2(\nu x)\, dx$ where $\nu = \sqrt{\lambda} a$

$$\lim_{\upsilon \to \nu} \frac{\nu J_m(\upsilon) J_m'(\nu) - \upsilon J_m(\nu) J_m'(\upsilon)}{\upsilon - \nu}$$

l'Hopital's rule necessary

$$= 2\nu \int_0^1 x J_m^2(\nu x)\, dx$$

$$2 \int_0^1 x J_m^2$$

Normalization of Eigenfunctions

$(p\varphi')' + (\lambda r - q)\varphi = 0$ eigenfunction φ, eigenvalue λ

$(p\psi')' + (\nu r - q)\psi = 0$ eigenfunction ψ, eigenvalue ν

both for $a < x < b$

$$\int_a^b \left((p\varphi')'\psi - (p\psi')'\varphi\right) dx + (\lambda - \nu)\int_a^b \psi\varphi r\, dx = 0$$

integrate by parts, boundary terms remain

$$\frac{\left[p(\varphi'\psi - \varphi\psi')\right]_a^b}{\nu - \lambda} = \int_a^b r\varphi\psi\, dx$$

Let $\nu \to \lambda$, have to apply l'Hopitals rule since φ, ψ solutions to differential equation, then assume differential

For $\varphi(r,\lambda)$, $\psi(r,\lambda)$ assume $\psi \to \varphi$

Invoking continuity and differentiability of the solutions with the parameter

$$\lim_{\nu \to \lambda} \frac{\partial}{\partial \nu}\Big[p(b)\big(\varphi'(b,\lambda)\,\psi(b,\nu) - \varphi(b,\lambda)\psi'(b,\nu)\big) - p(a)\big(\varphi'(a,\lambda)\psi(a,\nu) - \varphi(a,\lambda)\psi'(a,\nu)\big)\Big] = \lim_{\nu\to\lambda}\int_a^b r(x)\varphi(x,\lambda)\psi(x,\nu)\, dx$$

$$\Rightarrow \; p(b)\varphi'(b,\lambda)\frac{\partial \varphi}{\partial \lambda}(b,\lambda) - p(b)\frac{\partial \varphi'}{\partial \lambda}(b,\lambda)\varphi'(b,\lambda) + p(a)\varphi'(a,\lambda)\frac{\partial \varphi}{\partial \lambda}(a,\lambda) + p(a)\varphi(a,\lambda)\frac{\partial \varphi'}{\partial \lambda}(a,\lambda) = \int_a^b r(x)\left[\varphi(x,\lambda)\right]^2 dx$$

- Apply appropriate BC in x :

For example: consider BC $\varphi(a) = \varphi(b) = 0$

$$\Rightarrow \int_a^b r(x)(\varphi(x,\lambda))^2\,dx = p(b)\,\varphi'(b,\lambda)\frac{\partial\varphi}{\partial\lambda}(b,\lambda) - p(a)\,\varphi'(a,\lambda)\frac{\partial\varphi}{\partial\lambda}(a,\lambda)$$

Further evaluation depends on your actual knowledge of the behavior of the solutions.

Bessel Functions:

Bessel's equation: $xy'' + xy' + (x^2 - \nu^2)y = 0$

Liouville transformation: let $y(x) = u(x)\,x^{-1/2}$

then find: $u'' + \left(1 - \dfrac{\nu^2 - 1/4}{x^2}\right)u = 0$

Notice that if $\nu^2 = 1/4$ then $u'' + u = 0$

$\Rightarrow u = c_1 \cos x + c_2 \sin x$

$\Rightarrow y(x) = \dfrac{c_1 \cos x}{\sqrt{x}} + \dfrac{c_2 \sin x}{\sqrt{x}}$, general solution

This is a member of the family of spherical Bessel functions

$\nu = n + \frac{1}{2}$, $n = 0, \pm 1, \pm 2, \ldots$

See Handbook of Mathematical Functions by Abramowitz & Stegan

The Liouville transformation is valuable in approximating

$J_\nu(x)$

Modified Bessel Functions

They do not oscillate
Start with Bessel's equation. Set $z = ix$. $\rightarrow x^2 = -z^2$
or $x = -iz$

call $w(z) = y(ix)$

$$\frac{dw}{dz} = \frac{dy}{dx}\frac{dx}{dz} = -i\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = i\frac{dw}{dz}$$

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$$

$$-z^2\left(i^2 \frac{d^2w}{dz^2}\right) + (-iz)\, i\frac{dw}{dz} - (z^2 + \nu^2)w = 0$$

$$\Rightarrow \quad z^2 w''(z) + z w'(z) - (z^2 + \nu^2) w = 0$$

This problem occurs in heat conduction in a cylinder.
It arises naturally through separation of variables.

Basic pair: $I_\nu(z)$, $K_\nu(z)$ where

$$I_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}z^2\right)^k}{k!\,\Gamma(\nu+k+1)}$$

$$K_\nu(z) = \frac{1}{2}\pi\left[\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\pi\nu)}\right]$$

If ν is an integer, K_ν has natural logarithms

Green's Functions

Keener Section 4.2
Hildebrand Chpt 3

<u>Motivation</u>: used in solving non-homogeneous problems

$$(p(x)u'(x))' + (\lambda r(x) - q(x))u = F(x)$$, with boundary problems

<u>Example</u>: $u''(x) = F(x)$, $0<x<L$, $u(0) = u(L) = 0$
solve by integration $(p(x)=1,\ r(x)=q(x)=0)$

$$\Rightarrow u'(x) = \int F(x)\,dx + c_1$$

$$u'(x) = \int_0^x F(t)\,dt + c_1$$

To Find the answer and to apply BCs, integrate again

$$\Rightarrow u(x) = \int_0^x \int_0^y F(t)\,dt\,dy + c_1 x + c_2$$

$$u(0) = 0 \quad \Rightarrow \quad \int_0^0 (\ )\,dy + c_1(0) + c_2 = 0 \quad \Rightarrow c_2 = 0$$

$$\Rightarrow u(x) = \int_0^x \int_0^y F(t)\,dt\,dy + c_1 x$$

$$u(L) = \int_0^L \int_0^y F(t)\,dt\,dy + c_1 L = 0 \quad \Rightarrow \quad c_1 = \frac{-1}{L}\int_0^L \int_0^y F(t)\,dt\,dy$$

$$\Rightarrow \boxed{u(x) = \int_0^x \int_0^y F(t)\,dt\,dy - \frac{x}{L}\int_0^L \int_0^y F(t)\,dt\,dy}$$

The solution clearly satisfies the boundary conditions

why is this example beneficial to explain Green's Functions

Rewrite the solution using integration by parts

Name: $F(y) = \int_0^y f(t)\, dt \longrightarrow F'(y) = f(y)$

Then $u(x) = \int_0^x F(y)\, dy - \frac{x}{L}\int_0^L F(y)\, dy$

$$= \Big[y F(y)\Big]_0^x - \int_0^x y F'(y)\, dy - \frac{x}{L}\left[\Big[yF(y)\Big]_0^L - \int_0^L yF'(y)\, dy\right]$$

$$= x F(x) - \int_0^x y f(y)\, dy - \frac{x}{L}\left[L F(L) - \int_0^L y f(y)\, dy\right]$$

$$= x\int_0^x f(y)\, dy - \int_0^x y f(y)\, dy - \frac{x}{L}\left[L\int_0^L f(y)\, dy - \int_0^L y f(y)\, dy\right]$$

$$= \int_0^x (x-y) f(y)\, dy - x\int_0^L f(y)\, dy + \frac{x}{L}\int_0^L y f(y)\, dy$$

$$= \int_0^x (x-y) f(y)\, dy + \int_0^L \left(\frac{xy - xL}{L}\right) f(y)\, dy$$

$$= \int_0^L g(x,y) f(y)\, dy$$

$$= u(x)$$

where $g(x,y) = \begin{cases} x - y + \dfrac{xy - xL}{L}, & 0 < y < x \\ \dfrac{xy - xL}{L}, & x < y < L \end{cases}$

$$= \begin{cases} \frac{y}{L}(x - L), & 0 < y < x \\ \frac{x}{L}(y - L), & x < y < L \end{cases}$$

Keener in section 4.2 outlines "every thing"

$$Lu = a_2(x)\, x'' + a_1(x)\, u' + a_0(x)\, u = f(x), \text{ with BC's}$$

$$u(x) = \int_a^b g(x,y)\, f(y)\, dy$$

so $g(x,y)$ behaves as an "inverse"

symbolically, $Lu = L\int_a^b g(x,y) f(y)\, dy = \int_a^b L_x\, g(x,y)\, f(y)\, dy$

where L_x denotes differentiation wrt x.

$$L_x g = a_2(x) \frac{\partial^2 g}{\partial x^2} + a_1(x) \frac{\partial g}{\partial x} + a_0(x)\, g$$

$$\Rightarrow \int_a^b L_x\, g(x,y)\, f(y)\, dy = f(x)$$

then $L_x g(x,y)$ behaves like a delta function

$$L_x\, g(x,y) = \delta(x-y)$$

so that $\int_a^b \delta(x-y)\, f(y)\, dy = f(x)$

Back to the example,

$$g(x,y) = \begin{cases} \frac{y}{L}(x-L), & 0<y<x \\ \frac{x}{L}(y-L), & x<y<L \end{cases}$$

$$g_x(x,y) = \begin{cases} y/L, & 0<y<x \\ y/L - L, & x<y<L \end{cases}$$

Notice $\lim\limits_{x\uparrow y} g(x,y) = \frac{y^2-yL}{L}$, $\lim\limits_{x\downarrow y} g(x,y) = \frac{y^2-yL}{L}$

therefore $g(x,y)$ is ~~both~~ continuous in both arguments

However, g_x has discontinuity at at $x=y$

$\lim\limits_{x\uparrow y} g_x(x,y) = \frac{y}{L} - L$, $\lim\limits_{x\downarrow y} g_x(x,y) = \frac{y}{L}$

- Therefore, in this example, ~~every~~ $g(x,y)$ has a jump discontinuity in its first derivative.

$$\left[g_x(x,y) \right]_{x=y^-}^{x=y^+} = \frac{y}{L} - \left(\frac{y}{L} - 1 \right) = +1$$

- Notice additionally, in this example, $g(x,y)$ is symmetric:

$$g(x,y) = g(y,x)$$

- The text has a picture of this Green's function for $0<x<1$ $(L=1)$, on page 146, but there is a small typo: It should be written as the plot of $g(x,y)$ for $Lu = u''$. (a question of negatives.)

Variation of Parameters to derive Green's Functions

$$Lu = a_2(x)\,u'' + a_1(x)\,u' + a_0(x)\,u = F(x)$$

Assume u_1 and u_2 are known solutions of the homogeneous equation. Now look for another solution of the form: $u_p(x) = c_1(x)\,u_1(x) + c_2(x)\,u_2(x)$

Find that
$$c_1'(x)\,u_1(x) + c_2'(x)\,u_2(x) = 0$$
$$c_1'(x)\,u_1'(x) + c_2'(x)\,u_2'(x) = \frac{F(x)}{a_2(x)}$$

$$c_1'(x) = \frac{\begin{vmatrix} 0 & u_2(x) \\ \frac{F(x)}{a_2(x)} & u_2'(x) \end{vmatrix}}{\begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix}} = \frac{-\frac{F(x)}{a_2(x)}\,u_2(x)}{u_1(x)\,u_2'(x) - u_1'(x)\,u_2(x)}$$

Similarly, $$c_2'(x) = \frac{\frac{F(x)}{a_2(x)}\,u_1(x)}{u_1(x)\,u_2'(x) - u_1'(x)\,u_2(x)}$$

$\Rightarrow$ particular solution $\Rightarrow$

$$u(x) = -u_1(x)\int \frac{F(x)\,u_2(x)}{a_2(x)\,W(x)}\,dx + u_2(x)\int \frac{F(x)\,u_1(x)}{a_2(x)\,W(x)}\,dx$$

where $W(x) = u_1(x)\,u_2'(x) - u_1'(x)\,u_2(x)$

Let us assume separated or regular BCs :

$$B_1 u(a) = \alpha_0 u(a) + \alpha_1 u'(a) = 0$$

$$B_2 u(b) = \beta_0 u(b) + \beta_1 u'(b) = 0$$

If $\lambda = 0$ is not an eigenvalue, there is no eigenfunction that is a solution to $Lu = 0$.

$$B_1 u(a),\ B_2 u(b) = 0$$

choose $B_1 u_1(a) = 0$, $B_2 u_2(b) = 0$

for linearly independent u_1 and u_2

$$u_1 L u_2 - u_2 L u_1 = \frac{d}{dx} a_2(x)\, W(u_1, u_2) = 0$$

$\Rightarrow$ $a_2(x)\, W(u_1, u_2) = \text{constant}$

$$\Rightarrow u(x) = -u_1(x)\int_b^x \frac{F(y)\,u_2(y)}{a_2(y)\,W(y)}\,dy + u_2(x)\int_a^x \frac{F(y)\,u_1(y)}{a_2(y)\,W(y)}\,dy$$

$$= \int_a^b g(x,y)\,F(y)\,dy$$

$$\Rightarrow g(x,y) = \begin{cases} \dfrac{u_1(x)\,u_2(y)}{a_2(y)\,W(y)}, & x < y < b \\[2ex] \dfrac{u_1(y)\,u_2(x)}{a_2(y)\,W(y)}, & a < y < x \end{cases}$$

$$\boxed{\text{symmetric}: \quad g(x,y) = g(y,x)}$$

Additional reference book now on reserve:

Principles of Techniques of Applied Mathematics
by Bernard Friedman
(available in Dover paperback).

Green's Functions

$$(p(x)u')' + (\lambda r(x) - q(x))u = f(x) \quad , \quad a < x < b$$

λ is not an eigenvalue

boundary conditions: $B_1 u(a) = 0$, $B_2 u(b) = 0$ (separated BCs)

developed in detail in Sec 4.2 Keener

we found that

$$u(x) = \int_a^b g(x,y) f(y)\, dy$$

with
$$g(x,y) = \begin{cases} \dfrac{u_1(x)u_2(y)}{pW} & , x < y \\ \dfrac{u_1(y)u_2(x)}{pW} & , x > y \end{cases}$$

where $W(u_1,u_2) = u_1 u_2' - u_1' u_2$, $Lu_1 = 0$, $Lu_2 = 0$

notice the symmetry: $g(x,y) = g(y,x)$

if $Lu = f$ then we can think that

$$u(x) = \int_a^b g(x,y) f(y)\, dy = L^{-1} f(x)$$

Example: Keener pg 156

$$Lu = u'' + k^2 u, \quad k \neq 0, \quad u(0) = u(1) = 0$$

Show that
$$g(x,y) = \begin{cases} \dfrac{\sin kx \sin(k(y-1))}{k \sin k}, & 0 \leq x < y < 1 \\[2ex] \dfrac{\sin ky \sin(k(x-1))}{k \sin k}, & 0 \leq y < x < 1 \end{cases}$$

General solution to $Lu = 0$, $0 < x < 1$ is $u = c_1 \cos kx + c_2 \sin kx$

$u_1(0) = 0 \Rightarrow B_1 u_1(0) = 0$, $u_1 = \sin(kx)$

$B_2 u_2(1) = u_2(1) = 0 \Rightarrow u_2 = \sin(k(x-1))$

We assume that k is not an eigenvalue so u_1 and u_2 are independent.

$$W = \begin{vmatrix} \sin kx & \sin(k(x-1)) \\ k\cos kx & k\cos(k(x-1)) \end{vmatrix} = k \sin kx \cos(k(x-1)) - k \cos kx \sin(k(x-1))$$
$$= k \sin(kx - k(x-1)) = k \sin k$$

Notice:
$$g_x = \begin{cases} \dfrac{\overset{\sin}{\cos} kx \sin(k(y-1))}{\sin k} & x < y \\[2ex] \dfrac{\sin ky \overset{\sin}{\cos}(k(x-1))}{\sin k}, & y < x \end{cases}$$

As expected,
$$g_x \Big]_{x=y^-}^{x=y^+} = \frac{\sin ky \cos(k(y-1)) - \cos ky \sin(k(y-1))}{\sin k} = \frac{\sin(ky - k(y-1))}{\sin k}$$
$$= \frac{\sin k}{\sin k} = 1$$

Alternatively, by trig identity

$$g(x,y) = \frac{\cos[k(1-|x-y|)] - \cos[k(1+x+y)]}{2k \sin k}$$

Example: Non-separated BCs

$$Lu = u'' = F(x), \quad 0 < x < 1$$

$$2u(0) + u(1) = 0$$
$$2u'(0) + u'(1) = 0$$

From Assignment #2, we know that $\lambda = 0$ is NOT eigenvalue of $u'' + \lambda u = 0$, with these boundary conditions.

Since $\lambda = 0$ is NOT an eigenvalue, we expect L^{-1} to exist.

Theorem 4.2 (Keener)

For L order n,

1. $L_x g = 0$, for $x \neq y$
2. $g(x,y)$ is $n-2$ continuous in x at $x = y$
3. $$\left.\frac{\partial^{n-1} g(x,y)}{\partial x^{n-1}}\right]_{x=y^-}^{x=y^+} = \frac{1}{a_n(y)}$$
4. g must satisfy the BCs in x

In the example, $g_{xx} = 0$ for $x \neq y$

$$\Rightarrow \quad g(x) = \begin{cases} a_1 + a_2 x, & x < y \\ b_1 + b_2 x, & x > y \end{cases}$$

where $g(0,y) = a_1$, $g(1,y) = b_1 + b_2$
$g_x(0,y) = a_2$, $g_x(1,y) = b_2$ } Four conditions to find the four unknowns

note

$g(x,y)$ is continuous at $x = y$

$$a_1 + a_2 y = b_1 + b_2 y$$

jump: $\left. g_x(x,y)\right|_{x=y^-}^{x=y^+} = b_2 - a_2 = 1$

BC: $2a_1 + b_1 + b_2 = 0$

$2a_2 + b_2 = 0$

subtract $\Rightarrow$ $3a_2 = -1$

$a_2 = -1/3$

$b_2 = \frac{2}{3}$, $a_1 = -\frac{2}{9} + \frac{y}{3}$, $b_1 = a_1 - y = -\frac{2}{9} - \frac{2}{3}y$

$\Downarrow \quad \Downarrow \quad \Downarrow$

$$g(x,y) = \begin{cases} -\frac{2}{9} + \frac{y}{3} - \frac{x}{3}, & x \le y \\ -\frac{2}{9} - \frac{2}{3}y + \frac{2}{3}x, & x > y \end{cases}$$

or equivalently

$$g(x,y) = -\frac{2}{9} + (y-x)H(y-x) - \frac{2}{3}(y-x)$$

Use Keener pg 173 # 11

$$(xu')' - \frac{n^2}{x}u' + \lambda u = F(x), \quad 0 < x < 1$$

$u(0) =$ finite, $u(1) = 0$

- For general λ this could be solved in terms of Bessel functions, assuming λ is not an eigenvalue
- Convert this to an integral equation of the form

$$y = \lambda \int_0^1 k(x,y)\,u(y)\,dy + F(y)$$

solve $Lu = 0 \Rightarrow (xu')' + \frac{n^2}{x}u = 0 \rightarrow$

$$x'' + u''\left(\frac{-n^2}{x}\right)u = 0, \quad n > 0 \rightarrow \boxed{u_1 = x^n, \quad u_2 = x^n - x^{-n}}$$

satisfies left BC — satisfies BC at right

Same recipe for separated BCs

$$g(x,y) = \begin{cases} \frac{u_1(x)\,u_2(y)}{pW}, & x < y \\ \frac{u_1(y)\,u_2(x)}{pW}, & y < x \end{cases}$$

~~where~~ $\cancel{W = \begin{vmatrix} x^n & x^{n} \\ nx^{n-1} & (n-n)x^{n-1} \end{vmatrix} = x^{n+1}x^n x^{-n-1}(n-n) -}$

$$\text{when } W = \begin{vmatrix} x^n & x^n - x^{-n} \\ nx^{n-1} & nx^{n-1} + nx^{-n-1} \end{vmatrix} = nx^{2n-1} + nx^{-1} - nx^{2n-1} + nx^{-1} = \frac{2n}{x}$$

$$\Rightarrow \boxed{g(x,y) = \frac{1}{2n}x^n(y^n - y^{-n}), \quad 0 < x < y \le 1, \quad g(x,y) = g(y,x)}$$

1/4
Personal notes

Green's Functions - Hildebrand

$$Ly + \Phi(x) = 0 \quad , \quad \Phi(x) = \Phi(x, y(x))$$

where $L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$

homogeneous BCs $\alpha y(x) + \beta y'(x)$ for $x \in [a,b]$.

Determine a Green's function G given by

$$G(x) = \begin{cases} G_1(x) & , x < \xi \\ G_2(x) & , x > \xi \end{cases}$$

with the following four properties:

① $LG_1 = 0$ for $x < \xi$
$LG_2 = 0$ for $x > \xi$

② G_1 satisfies BC at $x = a$
G_2 satisfies BC at $x = b$

③ $G_1(\xi) = G_2(\xi)$ - continuity at $x = \xi$

④ $G_2'(\xi) - G_1'(\xi) = \frac{-1}{p(\xi)}$ (jump discontinuity in derivative)
where $p(x)$ is the coefficient of the 2nd order derivative of L

Then the solution is given by:

$$y(x) = \int_a^b G(x,\xi)\, \Phi(\xi)\, d\xi$$

Let $y = u(x)$ be a nontrivial solution to $Lu = 0$ with $\alpha u(a) + \beta u'(a) = 0$ and let $y = v(x)$ be a non-trivial solution to $Lv = 0$, $\alpha v(b) + \beta v'(b) = 0$.

then $G_1 = c_1 u(x)$ and $G_2 = c_2 v(x)$

$$\Rightarrow \quad G = \begin{cases} c_1 u(x), & x < \xi \\ c_2 v(x), & x > \xi \end{cases}$$

From continuity at $x = \xi$; $\quad c_2 v(\xi) - c_1 u(\xi) = 0 \quad$ (1)

From jump in derivative at ξ : $\quad c_2 v'(\xi) - c_1 u'(\xi) = \dfrac{-1}{p(\xi)} \quad$ (2)

This system of solution possesses a unique solution if $W[u(\xi), v(\xi)] \neq 0 \Rightarrow \begin{vmatrix} u(\xi) & v(\xi) \\ u'(\xi) & v'(\xi) \end{vmatrix} = u(\xi)v'(\xi) - u'(\xi)v(\xi) \neq 0$

Notice

$$Lu = 0 \quad \Rightarrow \quad (pu')' + qu = 0$$

$$Lv = 0 \quad \Rightarrow \quad (pv')' + qv = 0$$

$$\begin{aligned} vLu - uLv &= v(pu')' + \cancel{vqu} - u(pv')' - \cancel{uqv} \\ &= vp'u' + vpu'' - up'v' - upv'' + (pu'v' - pv'u') \\ &= \left[p(uv' - u'v)\right]' = 0 \end{aligned}$$

$$\Rightarrow \quad p(uv' - u'v) = A, \quad A \text{ is constant}$$

$$\boxed{uv' - u'v = \frac{A}{p}} \quad (3)$$

In ① solve for c_2 : $c_2 = \frac{c_1 u(\xi)}{v(\xi)}$

substitute into ② :

$$\frac{c_1 u(\xi)}{v(\xi)} v'(\xi) - c_1 u'(\xi) = \frac{-1}{p(\xi)}$$

$$\Rightarrow \quad u(\xi) v'(\xi) - u'(\xi) v(\xi) = \frac{-v(\xi)}{c_1 p(\xi)}$$

From this and ③ we have

$$c_1 = -\frac{v(\xi)}{A} \quad \text{and so} \quad c_2 = -\frac{u(\xi)}{A}$$

$$\Rightarrow \quad G = \begin{cases} -\frac{1}{A} u(x) v(\xi), & x < \xi \\ -\frac{1}{A} u(\xi) v(x), & x > \xi \end{cases}$$

Note if $A = 0$ then u and v are linearly independent.

Keener, Section 4.2 #2

$u''(x) = f(x)$, $u(0) = 0$, $u'(1) = u(1)$

$G(x,y) = \begin{cases} ax + b, & x < y \\ cx + d, & x > y \end{cases}$ • BCs: $u(0) = 0 \rightarrow \boxed{b = 0}$, $u'(1) = u(1) \rightarrow c = c + d \rightarrow \boxed{d = 0}$

• continuity at $x = y$: $ay = cy \rightarrow a = c$

$\Rightarrow G(x,y) = \begin{cases} ax, & x < y \\ ax, & x > y \end{cases}$ • jump in derivative: $\left.\frac{dG}{dx}\right|_{y^-}^{y^+} = a - a = 0 \neq 1$

$\boxed{\text{Green's Function does not exist!!}}$

Note: Wronskian: $\begin{vmatrix} ax + b & cx + d \\ a & c \end{vmatrix} = ac\cancel{x} + cd - a\cancel{cx} - ad = d(c - a)$

From BC, $d = 0$ therefore $|W(g_1(x), g_2(x))| = 0$ and so $g_1(x), g_2(x)$ linearly dependent

Keener, Example pg 145

$L u \equiv u''(x) = f(x) \quad , \quad u(0) = u(1) = 0$

$G(x,y) = \begin{cases} g_1(x) \,, & x<y \\ g_2(x) \,, & x>y \end{cases}$ where $L g_1 = 0 \,, \; L g_2 = 0$

$\Rightarrow G(x,y) = \begin{cases} ax+b \,, & x<y \\ cx+d \,, & x>y \end{cases}$

BCs: $g_1(0) = g_2(1) = 0 \Rightarrow b=0 \,, \; c=-d$

$\Rightarrow G(x,y) = \begin{cases} ax \,, & x<y \\ cx-c \,, & x>y \end{cases}$

jump in derivative: $c-a=1 \rightarrow a = c-1$

$G(x,y) = \begin{cases} (c-1)x \,, & x<y \\ (x-1)c \,, & x>y \end{cases}$

continuity: $(c-1)y = (y-1)c \rightarrow \cancel{cy} - y = \cancel{cy} - c \rightarrow c = y$

$\Rightarrow \boxed{G(x,y) = \begin{cases} x(y-1) \,, & x<y<1 \\ y(x-1) \,, & 0<y<x \end{cases}}$

BCs: $g_1(0) = g_2(1) = 0$ ✓

continuity: $y(y-1) = y(y-1)$ ✓

derivative: $y-(y-1) = 1$ ✓

$H(x) = \begin{cases} 1 \,, & x>0 \\ 0 \,, & x<0 \end{cases} \quad , \quad H(y-x) = \begin{cases} 1 \,, & y>x \\ 0 \,, & y<x \end{cases} = \begin{cases} 1 \,, & x<y<1 \\ 0 \,, & 0<y<x \end{cases}$

$G(x,y) = (y-x)H(y-x) + y(x-1) = \begin{cases} \cancel{y} - x + xy - \cancel{y} \,, & x<y<1 \\ 0 + y(x-1) \,, & 0<y<x \end{cases}$

$= \begin{cases} x(y-1) \,, & x<y<1 \\ y(x-1) \,, & 0<y<x \end{cases}$

$\Rightarrow \boxed{G(x,y) = (y-x)H(y-x) + y(x-1)}$

personal notes

Keener Section 4.2 #1

$u'' = f(x)$ $u(0) = 0$ $u'(1) = 0$

Method ①

$u'(x) = \int_0^x f(y)\,dy + c_1$, $u'(1) = 0 \Rightarrow$ $c_1 = -\int_0^1 f(y)\,dy$

$u'(x) = \int_0^x f(y)\,dy - \int_0^1 f(y)\,dy$

$u(x) = \int_0^x \int_0^y f(s)\,ds - x\int_0^1 f(y)\,dy + c_1$, $u(0) = 0 \Rightarrow c_1 = 0$

Let $\int_0^y f(s)\,ds = R(y) \rightarrow R'(y) = f(y)$

$\rightarrow u(x) = \int_0^x R(y)\,dy - x\int_0^1 f(y)\,dy = y\,R(y)\Big]_0^x - \int_0^x R'(y)\,y\,dy - x\int_0^1 f(y)\,dy$

$u(x) = x\,R(x) - \int_0^x y\,f(y)\,dy - x\int_0^1 f(y)\,dy = \int_0^x x\,f(y)\,dy - \int_0^x y\,f(y)\,dy - x\int_0^1 f(y)\,dy$

$u(x) = -x\int_x^0 f(y)\,dy - x\int_0^1 f(y)\,dy - \int_0^x y\,f(y)\,dy$

$\boxed{u(x) = -x\int_x^1 f(y)\,dy - \int_0^x y\,f(y)\,dy} = \int_0^1 g(x,y)\,f(y)\,dy$

$$\boxed{g(x,y) = \begin{cases} -y, & 0<y<x \\ -x, & x<y<1 \end{cases}}$$

Method ②

$u''(x) = 0$, $u(0) = 0 \rightarrow u'(x) = a_1$, $u(x) = a_1 x + a_2$

$v''(x) = 0$, $v'(1) = 0 \rightarrow v'(x) = b_1$, $v(x) = b_1 x + b_2$

$G = \begin{cases} c_1 u(x), & x<\xi \\ c_2 v(x), & x>\xi \end{cases}$

- From continuity $c_2 v(\xi) = c_1 u(\xi)$
- From jump in derivative: $c_2 v'(\xi) - c_1 u'(\xi)$

- from BCs: $u(0) = 0 \Rightarrow \boxed{u(x) = a_1 x}$, $v'(1) = 0 \Rightarrow \boxed{v(x) = b_2}$

$W = \begin{vmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{vmatrix} = \begin{vmatrix} a_1 x & b_2 \\ a_1 & 0 \end{vmatrix} = -a_1 b_2$ $G = \begin{cases} \frac{-1}{-a_1 b_2} a_1 x\, b_2, & x<\xi \\ \frac{-1}{-a_1 b_2} a_1 \xi\, b_2, & x>\xi \end{cases} \Rightarrow \boxed{G = \begin{cases} -x, & 0<x<\xi \\ -\xi, & \xi<x<1 \end{cases}}$

Keener Section 4.2 #2

Non-Separated Boundary Conditions

(problem done in class)

$Lu \equiv u'' = F(x)$, $0<x<1$

$2u(0) + u(1) = 0$

$2u'(0) + u'(1) = 0$ (It is known that $\lambda \neq 0$, therefore a Green's function $\exists$.)

Let $G(x,y) = \begin{cases} g_1(x), & x<y \\ g_2(x), & x>y \end{cases}$

where $Lg_1(x) = 0$, $Lg_2(x) = 0$ $\Rightarrow g_1(x) = ax+b$, $g_2(x) = cx+d$

$\Rightarrow G(x,y) = \begin{cases} ax+b, & x<y \\ cx+d, & x>y \end{cases}$

- continuity at $x=y$: $ay+b = cy+d$ (1)
- jump in derivative at $x=y$: $c-a=1$ (2)
- BCs: $2b+c+d=0$ (3)
  $2a+c=0$ (4)

(4) $c+2a=0$
(2) $c-a=1$ $\longrightarrow 3c=2 \longrightarrow \boxed{c=\frac{2}{3},\ a=-\frac{1}{3}}$

(3) $2b+d = -2/3$
(1) $b-d = y\left(\frac{2}{3}+\frac{1}{3}\right)$ $\longrightarrow 3b = y-\frac{2}{3} \longrightarrow \boxed{b=\frac{y}{3}-\frac{2}{9},\ d=-\frac{2}{3}y-\frac{2}{9}}$

$$\Rightarrow \boxed{G = \begin{cases} -\frac{x}{3}+\frac{y}{3}-\frac{2}{9}, & x<y<1 \\ \frac{2x}{3}-\frac{2y}{3}-\frac{2}{9}, & 0<x<y \end{cases}}$$

BCs: $\cancel{\frac{2y}{3}}-\frac{2}{9}+\frac{2}{3}-\cancel{\frac{2y}{3}}-\frac{2}{9} = \frac{2}{3}-\frac{6}{9} = 0$ ✓

$-\frac{2}{3}+\frac{2}{3}=0$

diff: $\frac{2}{3}-\left(-\frac{1}{3}\right) = 1$ ✓

cont: $-2/9 = -2/9$ ✓

$$\boxed{G(x,y) = (y-x)H(y-x) + \frac{2}{3}(x-y) - \frac{2}{9}}$$

$$= \begin{cases} y-x+\frac{2}{3}x-\frac{2}{3}y-\frac{2}{9} \\ 0+\frac{2}{3}x-\frac{2}{3}y-\frac{2}{9} \end{cases} = \begin{cases} \frac{y}{3}-\frac{x}{3}-\frac{2}{9}, & x<y<1 \\ \frac{2}{3}x-\frac{2}{3}y-\frac{2}{9}, & 0<y<x \end{cases}$$

"Generalized" Modified ~~Bessel~~ Green Functions, Keener Example page 159

4/4 personal notes

$Lu = u''$, $u'(0) = 0$
$\qquad\qquad u'(1) = 0$

Null space spanned by $\phi_0 = 1$.

$$L\, g(x,y) = -w(y) \sum_{i=1}^{K} \psi_i(x)\,\psi_i(y), \quad x \neq y$$

$\Rightarrow L g(x,y) = -1$

$\Rightarrow g'' = -1, \quad g'(0) = 0, \quad g'(1) = 0$

$$g(x,y) = \begin{cases} a_1 + a_2 x - \frac{1}{2}x^2, & x < y \\ b_1 + b_2 x - \frac{1}{2}x^2, & x > y \end{cases}$$

- continuity at $x = y$: $\quad a_1 + a_2 y = b_1 + b_2 y$
- jump-discontinuity in derivative: $\left.\frac{dg}{dx}\right]_{x=y^-}^{x=y^+} = b_2 - y - (a_2 - y) = b_2 - a_2 = 1$
- BC
  $g'(0) = 0 \Rightarrow a_2 = 0$
  $g'(1) = 0 \Rightarrow b_2 = 1$
- From continuity: $a_1 = b_1 + y$

$$\Rightarrow \quad g(x,y) = \begin{cases} b_1 + y - \frac{1}{2}x^2, & x < y \\ b_1 + x - \frac{1}{2}x^2, & x > y \end{cases}$$

- solve for b_1 by requirement that $\int_0^1 g(x,y)\,dx = 0$

$$\int_0^y b_1 + y - \tfrac{1}{2}x^2\,dx + \int_y^1 b_1 + x - \tfrac{1}{2}x^2\,dx = \frac{y^2}{2} + b_1 + \frac{1}{3} = 0 \quad \Rightarrow \quad b_1 = -\frac{y^2}{2} - \frac{1}{3}$$

$$g(x,y) = -\frac{1}{3} - \left(\frac{y^2}{2} + \frac{x^2}{2}\right) + \begin{cases} y, & 0 \leq x < y \\ x, & y < x \leq 1 \end{cases}$$

Chpt 14 Green's Function

14.1 Boundary Value Problems for ODEs

- linear, homogeneous self-adjoint differential operator

$$Lu = pu'' + p'u' - qu \quad , \quad x_0 \le x \le x_1 \quad , \quad p > 0$$

- associated non-homogeneous diff eq

$$Lu = -\phi(x)$$

- we desire a solution of the form

$$u(x) = \int_{x_0}^{x_1} K(x, \xi)\, \phi(\xi)\, d\xi$$

where $K(x,\xi)$ is called the <u>influence function</u> of <u>Green's Function</u>.

$K(x,\xi)$ satisfies the BCs at x_0 and x_1 from which it follows that $u(x)$ will satisfy these BCs,

$K(x,\xi)$ satisfies $LK(x,\xi) = 0 \quad , \quad x \ne \xi$

$\vdots$

$K(x,\xi)$ must satisfy the following requirements

1) $K(x,\xi)$ is continuous
2) $K(x,\xi)$ satisfies BC
3) $\left.\dfrac{dK(x,\xi)}{dx}\right|_{x=\xi^-}^{x=\xi^+} = \dfrac{-1}{p(\xi)}$
4) $LK = 0$, except at $x = \xi$

Green's Function of a self-adjoint differential operator is a symmetric function of the parameter ξ and the argument x, that is $K(x,\xi) = K(\xi, x)$

" If the force A, applied at the point ξ, produces the result $K(x,\xi)$ at the point x, then the force A acting at x produces the same result at ξ."

Construction of Green's Functions :

$Lu = 0$

$u = c_0 u_0(x)$ satisfies BC at $x = x_0$

$u = c_1 u_1(x)$ satisfies BC at $x = x_1$

Two cases: u_0 and u_1 are linearly independent or
u_0 and u_1 are linearly dependent

(1) u_0 and u_1 are linearly independent

$$u_0 u_0' - u_1' u_1 \neq 0$$

(2) u_0 and u_1 differ only by a constant factor

$u_0(x)$ satisfies BC at x_0 and x_1

$\lambda = 0$, where $Lu + \lambda u = 0$

MOAM 10/8/02

$$u(x) = \int_0^1 g(x,y) F(y)\, dy - \lambda \int_0^1 g(x,y)\, u(y)\, dy$$

- Integral equs are usually easier to solve than differential equations

Section 4.3 Differential Operators

- Domain of an operator
- Linear operator
- $Lu = F$
- Important Questions: inverse problem well posed
  the adjoint operator exists
  the Fredholm theorem applies

Domain (1) The domain of L is given by all $u \in L^2[a,b]$ for which $Lu \in L^2[a,b]$, with real inner product

$$\langle u, v\rangle = \int_a^b u(x)\, v(x)\, w(x)\, dx$$

(2) The domain of L is given by all u which satisfy the homogeneous boundary conditions

$\Rightarrow$ if $u \in$ Domain of L, $F \in$ Range of L.

This definition of domain is highlighted by the use of the Green's function, so that for example, Dirichlet problem

$$u'' = F, \quad 0 < x < 1, \quad u(0) = u(1) = 0$$

$$u = \int_0^1 g(x,y) F(y)\, dy, \quad g(x,y) = \begin{cases} x(1-y), & x < y \\ y(1-x), & y < x \end{cases}$$

Adjoint of an Operator

<u>Def:</u> The adjoint of a differential operator L is defined to be the operator L^* for which $\langle Lu, v\rangle = \langle u, L^* v\rangle$ for all u in the domain of L and v in the domain of L^*.

This definition of the adjoint arises by integration by parts:

$$Lu = (pu')' + qu, \qquad a < x < b, \quad + \text{ BCs}$$

$$\langle u, v\rangle = \int_a^b u(x)\, v(x)\, dx$$

$$\langle Lu, v\rangle = \int_a^b \left((pu')' + qu\right) v\, dx = pu'v\Big]_a^b - \int_a^b pu'v'\, dx + \int_a^b quv\, dx$$

$$= \Big[pu'v - puv'\Big]_a^b + \int_a^b u(pv')'\, dx + \int_a^b quv\, dx$$

$$= J(u,v)\Big]_a^b + \int_a^b \left((pv')' + qv\right) u\, dx$$

$$= J(u,v)\Big]_a^b + \int_a^b u L^* v\, dx$$

we require $J(u,v)\Big]_a^b = 0$

For instance, Dirichlet BCs $u(0) = u(1) = 0$, $Lu = u''$, $x \in [0,1]$

$$p(b)u'(b)v(b) - p\underbrace{\cancel{u(b)}}_{0}v'(b) - p(a)u'(a)v(a) + p(a)\underbrace{\cancel{u(a)}}_{0}v'(a) = 0$$

$$p(b)u'(b)v(b) - p(a)u'(a)v(a) = 0$$

we require $v(b) = v(a) = 0$

<u>Example</u> : $Lu = u''$, $0<x<1$, $u(0)=u(1)=0$

$p = 1$

$$\langle Lu, v\rangle = \int_0^1 u''(x)v(x)\,dx = \Big[u'v - uv'\Big]_0^1 + \int_0^1 u v''(x)\,dx$$

$$= u'(1)v(1) - u'(0)v(0) + \int_0^1 u L^* v\,dx$$

Force $v(1) = v(0) = 0$

$\Rightarrow \langle Lu, v\rangle = \langle u, Lv\rangle$ because <u>L is self-adjoint</u>

<u>Example</u> : $Lu = u''$, $0<x<1$, $2u(0)+u(1)=0$, $2u'(0)+u'(1)=0$

$$\langle Lu, v\rangle = \Big[u'v - uv'\Big]_0^1 + \int_0^1 uv''\,dx$$

Force boundary terms to zero

$$u'(1)v(1) - u(1)v'(1) - u'(0)v(0) + u(0)v'(0) = 0$$

$$-2u'(0)v(1) + 2u(0)v'(1) - u'(0)v(0) + u(0)v'(0) = 0$$

$$\Rightarrow u'(0)\big(-2v(1) - v(0)\big) + u(0)\big(2v'(1) + v'(0)\big) = 0$$

$\Rightarrow$ $L^* v = v''$, $0<x<1$

$2v(1) + v(0) = 0$

$2v'(1) + v'(0) = 0$

Study these eigenvalue problems and find that they must be complex conjugate (The complex inner product shows this).

Formally self-adjoint, the operators of L and L^* are the same, though the boundary conditions may be different.

Calculus of Variations

Ref: • F. B. Hildebrand, Methods of Applied Mathematics (2nd edition), chapter 2

• Keener

• Cochran

• Courant-Hilbert, Vol 1

Motivation

• Ordinary Calculus - the problem of finding the max or min of a function $f(x)$.

• Multivariable Calculus - max or min of $f(\underline{x})$ subject to constraints, $\underline{x}$ lies on $g(\underline{x}) = c$

• Variational Calculus

~~solve $Ax=b$~~

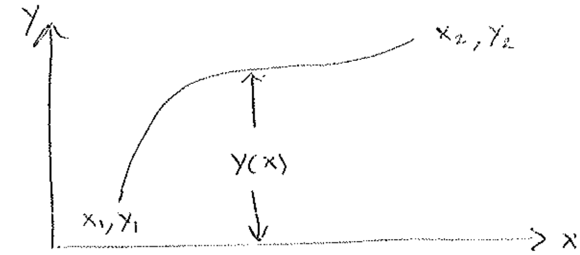
• Differentiable operator

$Lu = F$, minimize $\|Lu - F\|$
minimize $\|u\|$, u a function

• solve $A\underline{x} = \underline{b}$, A non-invertible, minimize $\|A\underline{x} - \underline{b}\|$,
minimize $\|\underline{x}\|$, $\underline{x}$ a vector

The resolution of these last two minimization problems is by linear equations. In general the calculus of variations may lead to non-linear problems.

Example



Rotate $y(x)$ about the x-axis

- Determine the function which minimizes the surface area,

$$S(y) = 2\pi \int_{x_1}^{x_2} y \sqrt{1 + [y'(x)]^2}\, dx$$

with $y(x_1) = y_1$, $y(x_2) = y_2$, $y_1 \geq 0$, $y_2 \geq 0$

If there is a minimum the smooth curve which generates it is a catenary. This is one of the classical problems of the calculus of variations.

Consider

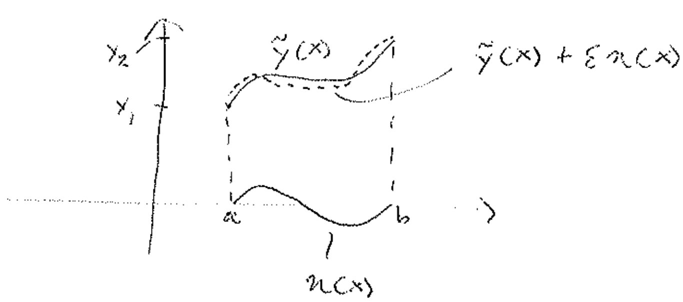
$$I(y) = \int_a^b F(x, y, y')\, dx \quad , y(a) = y_1 \quad , y(b) = y_2$$

where F is a known function of three variables.

Assume $F \in C^2$ with respect to its arguments.

We will relax the condition (of smoothness) later.

To fix ideas, suppose $I(y)$ is to be minimized (or maximized) by C^1 functions. Let $\tilde{y}(x)$ be the minimizing function. Choose any C^1 function $\eta(x)$ which vanishes at the endpoints, such that $\eta(a) = 0$, $\eta(b) = 0$.



Then for any ε, $\tilde{y}(x)+\varepsilon\eta(x)$ is a C^1 function satisfying the boundary conditions

$$y(a)=Y_1 \quad \text{and} \quad y(b)=Y_2$$

The integral $I(\tilde{y}+\varepsilon\eta)$ is a function of ε, when $\tilde{y}$ and η are fixed. It takes its minimum where $\varepsilon=0$. This is possible only if $\frac{d}{d\varepsilon}I(\tilde{y}+\varepsilon\eta)=0$ when $\varepsilon=0$.

Call $\tilde{F}=F(x,\tilde{y}+\varepsilon\eta,\tilde{y}'+\varepsilon\eta')$

then $$\frac{d\tilde{F}}{d\varepsilon}=\frac{\partial\tilde{F}}{\partial y}\eta+\frac{\partial\tilde{F}}{\partial y'}\eta'$$

Note $\tilde{F}\to F$ as $\varepsilon\to 0$, $\frac{\partial\tilde{F}}{\partial y}\to\frac{\partial F}{\partial y}$, $\frac{\partial\tilde{F}}{\partial y'}\to\frac{\partial F}{\partial y'}$

The necessary condition for $\varepsilon=0$ to be a stationary point is $\left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0}=0$

$$\left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0}=\int_a^b\left(\frac{\partial F}{\partial y}\eta+\frac{\partial F}{\partial y'}\eta'\right)dx=0 \quad \Bigg\} \text{ The first variation of } Y\text{, or } \delta I(y)$$

To expand the integral employ integration by parts

Consider $$\int_a^b\frac{\partial F}{\partial y'}\eta'\,dx=\underbrace{\cancel{\left.\frac{\partial F}{\partial y'}\eta\right]_{x=a}^{x=b}}}_{\to 0\text{ because }\eta(a)=\eta(b)=0}-\int_a^b\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta\,dx$$

$$\left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0}=\int_a^b\left[\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta\,dx-\cancel{\left[\frac{\partial F'}{\partial y'}\eta\right]_a^b}^{\,0}$$

$$\frac{dI}{d\varepsilon}\Big|_{\varepsilon=0} = \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y}\right)\right]\eta\,dx + \cancelto{0}{\left[\frac{\partial F}{\partial y'}\eta\right]_a^b} = 0$$

~~Since the brac~~

Because $\eta(a) = \eta(b) = 0$, the bracketed term vanishes.

Since $\eta(x)$ is arbitrary we conclude that the integrand must vanish identically otherwise we could choose a C^1 η which vanishes at the end points, which is a contradiction

The space of admissible variations:

class $N(a,b) = \left\{ \eta \mid \eta(a) = \eta(b) = 0,\ \eta \in C^1[a,b] \right\}$

Fondamental Lemma

Suppose $g(x)$ is continuous on (a,b) and

$$\int_a^b g(x)\eta(x)\,dx = 0,$$

for each $\eta \in N(a,b)$. Then $g(x) \equiv 0$.

Proof: If $g(x) \neq 0$, $g(x)$ must be positive or negative, say positive at $x = x_0 \in (a,b)$. Since $g(x)$ is continuous $\exists$ an interval (x_1, x_2), $x_1 < x_0 < x_2$ where $g(x)$ is positive.

Let $\eta = \begin{cases} 0, & x \notin (x_1, x_2) \\ (x-x_1)^2(x-x_2)^2, & x \in (x_1, x_2) \end{cases}$

For this choice of $\eta \in N(a,b)$ we get $\int_a^b g(x)\eta(x)\,dx > 0$. This is a contradiction so $g(x) \equiv 0$. ∎

Therefore a necessary condition is:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad , \text{ Euler's equation}$$

<u>Example:</u> Cochran, pg 122, # 2a

(a) Determine the 1^{st} variation of the functional

$$I(y) = \int_0^1 \left(y^2 + 2xy + y'^2\right) dx$$

$$\rightarrow \delta I(y) = \int_0^1 \left[(2y + 2x)\eta + 2y' \frac{d\eta}{dx}\right] dx \quad \Big\} \; \underline{1^{st} \text{ variation}}$$

(b) What is the relevant Euler-Lagrange equation?

Integrate by parts:

$$\delta I(y) = \int_0^1 (2y + 2x)\eta + \left[2y'\eta\right]_0^1 - \int_0^1 2y''\eta \, dx$$

$$\delta I(y) = \left[2y'\eta\right]_0^1 + \int_0^1 \left(2y + 2x - 2y''\right)\eta \, dx$$

$$\Rightarrow \quad 2y + 2x - 2y'' = 0$$

- Suppose $y(0) = 0$, $y(1) = 0$

then we have: $\left.\begin{array}{l} y'' = y + x \\ y(0) = 0 \;,\; y(1) = 0 \end{array}\right\}$ has a unique solution

Some simplifications

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

- Suppose $F(x,y,y') = F(x,y')$
  (F does not depend on y)
  then the Euler-Lagrange equation reduces to $\frac{-d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$
  this can be integrated
  $\frac{\partial F}{\partial y'} = c_1$ — 1^{st} order ODE

- Suppose $F(x,y,y') = F(y,y')$
  leads to an autonomous equation, *
  First integral: $F - y'\frac{\partial F}{\partial y'} = c_1$, First order ODE.

Notice that the Euler-Lagrange equation is 2^{nd} order even if the functional only involves y'. In general, we have

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad , F = F(x,y,y')$$

let $\frac{\partial F}{\partial y} = g(x,y,y')$, $\frac{\partial F}{\partial y'} = h(x,y,y')$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{dh(x,y,y')}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y}\frac{dy}{dx} + \frac{\partial h}{\partial y'}\frac{dy'}{dx}$$

$$\Rightarrow \quad g - h_x - h_y y_x - h_{y'} y'' = 0$$

In general this is a second order ODE.
This is linear only if F is quadratic in y and y'.

Integrating the Euler-Lagrange equation when x is absent:

$F_x \equiv 0 \Rightarrow$

Let $y' = v$, $y'' = v' = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy}$

then $g - h_x - h_y y' - h_{y'} y'' = 0$ in terms of v, y becomes

$$g(y,v) - h_y v - h_v v \frac{dv}{dy} = 0$$

separate variables and show that the resulting ODE is exact:

$$\frac{\partial}{\partial v}(g - h_y v) = g_v - h_y - h_{yv} v$$

$$\frac{\partial}{\partial y}(-v h_v) = -v h_{yv} - v_y h_v$$

Recall $g_v = \frac{\partial^2 F}{\partial y \partial v}$ and $h_v = \frac{\partial^2 F}{\partial v \partial y}$

$\rightarrow$ Total derivative: $d(F - vF_v) = 0$

$$\Rightarrow F - vF_v = c_1 \Rightarrow \boxed{F - y'\frac{\partial F}{\partial y'} = c_1}$$

Cochran, pg 122 # 4

$$S(y) = 2\pi \int_a^b y\sqrt{1+y'^2}\, dx = 2\pi\, I(y) \quad , y(a) = A, \; y(b) = B$$

(a) Solve for $y(x)$, stationary function,

(b) specialize to the case where $a = -b$, $A = B$

Show that depending on the relative size of b, B there may be none one or two candidates curves that satisfy the end point candidates

$$F = y\sqrt{1+y'^2} \quad \text{independent of } x$$

First integral

$$F - y'\frac{\partial F}{\partial y'} = y(1+y'^2)^{1/2} - \frac{(y')^2 y}{(1+y'^2)^{1/2}} = c_1$$

Find

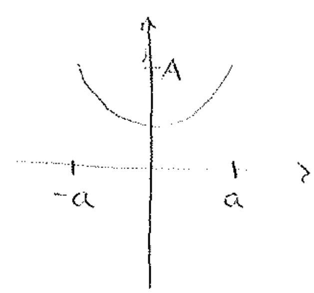
$$\frac{y(1+y'^2 - y'^2)}{(1+(y')^2)^{1/2}} = c_1 \quad \rightarrow \quad \frac{y}{(1+y'^2)^{1/2}} = c_1$$

solve for y'

$$\frac{dy}{dx} = \left(\frac{y^2}{c_1^2} - 1\right)^{1/2} \quad \Rightarrow \quad \frac{dy}{\sqrt{\frac{y^2}{c_1^2} - 1}} = dx$$

integrate both sides

$$x = c_1 \cosh^{-1}\left(\frac{y}{c_1}\right) + c_2 \quad \longrightarrow \quad \frac{x - c_2}{c_1} = \cosh^{-1}\left(\frac{y}{c_1}\right)$$



$$\rightarrow \quad y = c_1 \cosh\left(\frac{x - c_2}{c_1}\right)$$

$$\left.\begin{aligned} y(a) = A &= c_1 \cosh\left(\frac{a - c_2}{c_1}\right) \\ y(-a) = A &= c_1 \cosh\left(\frac{-a - c_2}{c_1}\right) = c_1 \cosh\left(\frac{a + c_2}{c_1}\right) \end{aligned}\right\} \quad c_2 = 0$$

$$\Rightarrow \quad y(x) = c_1 \cosh\left(\frac{x}{c_1}\right)$$

$y(a) = A = c_1 \cosh\left(\frac{a}{c_1}\right)$

Thus c_1 is to be determined

set $k = \frac{c_1}{a}$, $h = \frac{A}{a}$

$\Longrightarrow \frac{h}{k} = \cosh\left(\frac{1}{k}\right) \rightarrow h = k\cosh\left(\frac{1}{k}\right)$

the value of h is known and the value of k is then determined,

Consider the function $w(k) = k\cosh\left(\frac{1}{k}\right)$, $k > 0$

Notice $\lim_{k\to 0^+} k\cosh\left(\frac{1}{k}\right) = \lim_{v\to +\infty} \frac{\cosh v}{v} = +\infty \quad \left(\frac{1}{k} = v\right)$

$$w'(k) = \cosh\left(\frac{1}{k}\right) - \frac{1}{k}\sinh\left(\frac{1}{k}\right) \overset{?}{=} 0$$

Yes, there is one point where $w'(k) = 0$: $\tanh(k) = k$

where $k = k_0 = 0{,}833556$

concavity? $w''(k) = \frac{-1}{k^2}\sinh k + \frac{1}{k^2}\sinh k + \frac{1}{k^3}\cosh\left(\frac{1}{k}\right) > 0$

Thus $w(k)$ is concave up $\forall k$, and

$w''(k_0) > 0$, so $w(k_0)$ is a minimum.

$\Longrightarrow w_0(k_0) = k_0 \cosh\frac{1}{k_0} \approx 1{,}508$

Thus if $h < w_0$ there is no value of k such that $h = k\cosh\frac{1}{k}$. IF $h = w_0$, $\exists$ is exactly one value of k.
IF $h > w_0$, there are two values of k.

Calculus of Variations

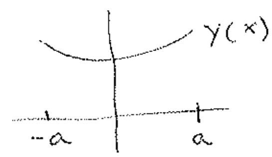
$$I(y) = \int_a^b F(x, y, y')\, dx \quad \text{- Functionals}$$

$\delta I = 0 \Rightarrow$ Euler equation (nonlinear 2^{nd} order ODE)

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \qquad (y(a) = y_1,\ y(b) = y_2)$$

Example: Minimal surface of Revolution

$$S(y) = 2\pi \int_{-a}^{a} y\sqrt{1 + (y')^2}\, dx$$

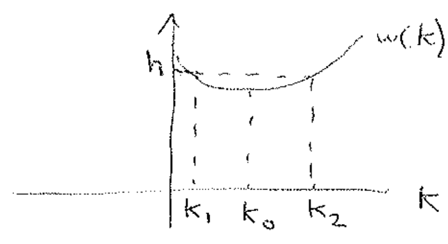


we found $y(x) = ak\cosh\left(\frac{x}{ak}\right)$

$k = \frac{c_1}{a}$, $h = \frac{A}{a}$ where $y(-a) = y(a) = A$

with this we found $w_0 = k_0 \cosh\frac{1}{k_0} \approx 1.5088796$

k_0 is the point where $w'(k_0) = 0$ and $w(k) = k\cosh\left(\frac{1}{k}\right)$



If h is too small, no solution.

If $w(k_0) = h$, exactly one solution

If $w(k_0) < h$, two solutions.

It turns out that $h = k_2$ provides the smallest surface area.

$$S(\tilde{y}) = 2\pi \int_{-a}^{a} \tilde{y}\sqrt{1 + (\tilde{y}')^2}\, dx$$

$$= 2\pi \int_{-a}^{a} ak\cosh\left(\frac{x}{ak}\right)\left[1 + \sinh^2\left(\frac{x}{ak}\right)\right]^{1/2} dx$$

$$= 2\pi \int_{-a}^{a} ak\left(\cosh\left(\frac{x}{ak}\right)\right)^2 dx = ak\pi \int_{-a}^{a} \left(\cosh\frac{2x}{ak} + 1\right) dx$$

$$S(\tilde{y}) = \pi a k \left[\frac{ak}{2} \sinh\left(\frac{2x}{ak}\right) + x \right]_{-a}^{a}$$

$$S(\tilde{y}) = \pi a^2 k \left[k \sinh \frac{2}{k} + 2 \right]$$ not monotonic in k

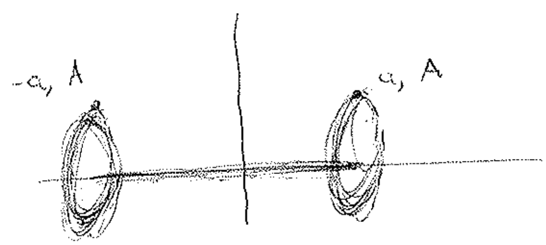
$$S(\tilde{y}) = \pi a^2 k \left[2k \cosh \frac{1}{k} \sinh \frac{1}{k} + 2 \right]$$

$$S(\tilde{y}) = \pi a^2 k \left(2k \sinh \frac{1}{k} + 2 \right)$$ This solution is monotonic decreasing in k for fixed h. Hence $k = k_2$ provides the smaller value of S.

What happens if $h = w_0$?

There is no catenary passing through $(-a, A)$ and (a, A).

The minimal surface consists of two wheels.



— broken extremal

$$S = 2\pi A^2 = 2\pi a^2 h^2$$

This area is smaller than that generated by any curve lying about the axis and passing through $(-a, A)$ and (a, A)

Hildebrand, Section 2.3 Natural Boundary Conditions

$I(y) = \int_a^b F(x, y, y')\,dx$ with either $y(a)$ and/or $y(b)$ not given

As previously, suppose $\tilde{y}(x)$ is a minimizing function

consider variations $\eta(x)$

~~$I(\tilde{y}+\varepsilon\eta)$~~ $I(\tilde{y}+\varepsilon\eta)$

evaluate $\frac{d}{d\varepsilon} I(\tilde{y}+\varepsilon\eta)\Big|_{\varepsilon=0} = 0$

$$\int_a^b \left[\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right] dx = 0$$

integrate by parts

$$\left[\frac{\partial F}{\partial y'}\eta\right]_a^b + \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta\,dx = 0$$

This is true for the extremal function $\tilde{y}$ for all $y \in C^1[a,b]$ and in particular for $\eta \in N[a,b]$,

since $N[a,b] \subseteq C^1[a,b]$.

Thus $\left[\frac{\partial F}{\partial y'}\eta\right]_a^b = 0$ and $\int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta\,dx = 0$, separately

Fundamental lemma gives $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ Euler Eqn

By a simple argument, consider C^1 variations that vanish at one endpoint say $x=a$ but not at $x=b$.

Then at $x=b$, $\frac{\partial F}{\partial y'}\Big|_{x=b} = 0$

Likewise, consider C^1 variations which vanish at $x=b$ but not at $x=a$, then at $x=a$, $\frac{\partial F}{\partial y'}\Big|_{x=a} = 0$

normal boundary conditions : if no boundary conditions are given at the end points then $\frac{\partial F}{\partial y'} = 0$ at the end points

Example : Cochran pg 123 # 10

Determine the function $y(x)$ that minimizes $I(y) = \int_0^{\pi} \left(2y \sin x + (y')^2\right) dx$ subject to the single condition $y(0) = 0$. Do first using the natural boundary condition and compare with the solution procedure that neglects to apply that auxiliary condition.

part a $\quad F = (y')^2 + 2y \sin x$

Euler eqn : $\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$

$\Rightarrow \quad 2\sin x - \frac{d}{dx}(2y') = 0 \quad \Rightarrow \quad y''(x) = \sin x$

solve by integration : $y'(x) = c_1 - \cos x \;, \quad y(x) = c_2 + c_1 x - \sin x$

$y(0) = 0 \Rightarrow c_2 = 0 \quad \Rightarrow \quad y(x) = c_1 x - \sin x$

To define c_1 use natural boundary condition at π :

$$\left.\frac{\partial F}{\partial y'}\right|_{x=\pi} = 0 \quad \Rightarrow \quad 2y'(\pi) = 0$$

$y'(x) = c_1 - \cos x \qquad 2y'(\pi) = 2c_1 - 2\cos\pi = 2 + 2c_1 = 0$

$\Rightarrow \quad c_1 = -1 \quad \Rightarrow \quad \boxed{y(x) = -x - \sin x}$

part b

↓

part b Evaluate $I(c_1 x - \sin x)$

$$I(c_1 x - \sin x) = \int_0^{\pi} \left[2\sin x(c_1 x - \sin x) + (c_1 - \cos x)^2\right] dx$$

$$= \int_0^{\pi}$$

$\vdots$

$$I(c_1) = \pi\left(c_1^2 + 2c_1 - \tfrac{1}{2}\right)$$

$$\frac{dI}{dc_1} = \pi(2c_1 + 2) = 0 \quad \Rightarrow \quad c_1 = -1 \text{ , as before}$$

How does the minimum in (a) compare with the minimum for $I(y)$ that results when the additional restriction $y(\pi) = 0$ is included?

$y = c_1 x - \sin x$ from part a

$y(\pi) = c_1 \pi - \sin\pi = 0 \Rightarrow c_1 = 0 \Rightarrow \boxed{y = -\sin x}$

$$I(-\sin x) = \int_0^{\pi} \left[-2\sin^2 x + \cos^2 x\right] dx = \frac{-\pi}{2}$$

from part (b) $I(-x - \sin x) = \pi\left(1 - 2 - \tfrac{1}{2}\right) = \frac{-3\pi}{2}$

The natural boundary condition provides the "smaller" minimum.

---

see Hildebrand examples.

Transition Conditions (see Hildebrand text)

In some cases the integrand F or its derivatives are discontinuous inside (a, b) but on sub intervals $F \in C^2$.

Still consider $I(y) = \int_a^b F(x, y, y')\, dx$

Consider a single discontinuity at $x = c$, $a < c < b$

$$I(y) = \int_a^{c^-} F(x, y, y')\, dx + \int_{c^+}^b F(x, y, y')\, dx$$

All of the steps involving $\tilde{y} + \varepsilon \eta$ may be repeated except that each has 2 parts

$$\int_a^{c^-} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) \right] \eta\, dx + \int_{c^+}^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) \right] \eta\, dx$$

$$+ \left[\frac{\partial F}{\partial y'} \eta(x) \right]_a^{c^-} + \left[\frac{\partial F}{\partial y'} \eta(x) \right]_{c^+}^b = 0$$

We require that $y(x)$ be continuous at $x = c$,

$\eta(x) \in C[a, b]$, then $\eta(c^+) = \eta(c^-) = \eta(c)$

Then $\delta I = 0$ becomes

$$\int_a^{c^-} \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right] \eta\, dx + \int_{c^+}^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) \right] + \left. \frac{\partial F}{\partial y'} \right|_b \eta(b) - \left. \frac{\partial F}{\partial y} \right|_a \eta(a)$$

$$- \left[\left. \frac{\partial F}{\partial y'} \right|_{c^+} - \left. \frac{\partial F}{\partial y'} \right|_{c^+} \right] \eta(c) = 0$$

The Fundamental lemma can be modified to continuous variations. Then the Euler equation is satisfied on each subinterval (a, c), (c, b), $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) = 0$ and $\frac{\partial F}{\partial y'}$ must vanish at an end-point where η is not given.

$$\boxed{\text{Hence } \left. \frac{\partial F}{\partial y'} \right]_{c^-}^{c^+} = 0 \implies \frac{\partial F}{\partial y'} \text{ is continuous} \implies \text{Transition Condition}}$$

Example

The input-output relationship of a continuous system is defined by

$$y''(t) + c(t)\, y'(t) + 6\, y(t) = 30 e^{-4t}$$

where $c(t) = \begin{cases} 5, & t < 1 \\ 7, & t > 1 \end{cases}$

$y\,\cancel{(0) = y(1)} \quad y\,\cancel{(0) =}\;\cancel{y} \qquad y(0) = 1, \quad y'(0) = -1.$

Find $y(t)$.

Define $p(t) = e^{\int c(t)\,dt} = \begin{cases} e^{5t}, & t < 1 \\ e^{7t}, & t > 1 \end{cases}$

Multiply by $p(t)$ ~~and integrate from t=0 to t=∞~~

$$\left(e^{\int c(t)\,dt}\, y' \right)' + 6 e^{\int c(t)\,dt}\, y = 30\, e^{\int c(t)\,dt}\, e^{-4t}$$

Multiply by $y(t)$ and then integrate

$$\int_0^\infty \left[(p(t)\, y')' y + 6 p(t) y^2 \right] dt = \int_0^\infty 30 e^{-4t} p(t)\, dt$$

Integrate by parts

$$\int_0^\infty \left[-p(t) y'^2 + 6 p(t) y^2 \right] dt = 30 \int_0^\infty e^{-4t} p(t)\, dt$$

$$\Rightarrow \quad \delta \int_0^\infty \left[-p(t)(y')^2 + 6 p(t) y^2 - 30 e^{-4t} p(t) \right] dt = 0$$

Assume $y, y' \to 0$ as $t \to \infty$

transition conditions

$$y(1^-) = y(1^+)$$

$$\left. \frac{\partial F}{\partial y'} \right|_{t=1^-}^{t=1^+} = \left[-2 y'(t)\, p(t) \right]_{1^-}^{1^+} = 0$$

$$\lim_{t \to 1^+} p(t)\, y'(t) = \lim_{t \to 1^-} p(t)\, y'(t)$$

$$e^5 y'(1^-) = e^7 y'(1^+)$$

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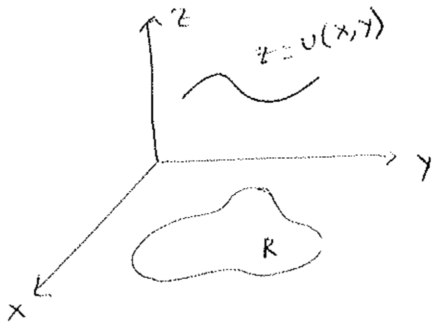
Functions of more than one independent variable

<u>Minimal surface</u>



$$I(u) = \iint_R \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}\, dx\, dy$$

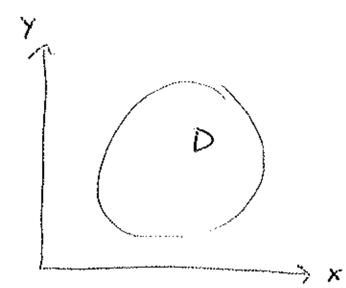
Analagous to our earlier definitions consider functionals $F(x, y, u, v, u_x, u_y, v_x, v_y)$ for functions $u(x,y)$, $v(x,y)$. Now we can vary v and u holding both x and y fixed. Variations $u + \varepsilon\xi$, $v + \varepsilon\eta$ where $\delta u = \varepsilon\xi(x,y)$ and $\delta v = \varepsilon\eta(x,y)$

$$\Delta F = \frac{\partial F}{\partial u}\varepsilon\xi + \frac{\partial F}{\partial v}\varepsilon\eta + \frac{\partial F}{\partial u_x}\varepsilon\xi_x + \frac{\partial F}{\partial v_x}\varepsilon\eta_x + \frac{\partial F}{\partial u_y}\varepsilon\xi_y + \frac{\partial F}{\partial v_y}\varepsilon\eta_y + \text{higher order terms} - F$$

$$\Delta F = \delta F + \text{higher order terms} - F$$

<u>Example:</u>

$$I(u) = \iint_D F(x, y, u_x, u_y, v_x, v_y, u, v)\, dx\, dy$$



D is a region for which Green's Thm in the plane is true

Blah Blah Blah

Example: Find the minimum of the functional

$$I(u) = \iint_G \left[\|\nabla u\|^2 + \frac{4u}{\sqrt{x^2+y^2}} \right] dx\, dy$$

among functions belonging to the class $C^1(G)$

where $G = \{ \underline{x} = (x, y) \mid 1 < \|\underline{x}\| < 3 \}$ and $u = 0$ on ∂G.

$$f = \|\nabla u\|^2 + \frac{4u}{\sqrt{x^2+y^2}} = u_x^2 + u_y^2 + \frac{4u}{\sqrt{x^2+y^2}}$$

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0 \qquad \underline{\text{Euler Lagrange}}$$

$$\frac{4}{\sqrt{x^2+y^2}} - \frac{d}{dx}(2u_x) - \frac{\partial}{\partial y}(2u_y) = 0$$

$$\Rightarrow \quad u_{xx} + u_{yy} = \frac{2}{\sqrt{x^2+y^2}}$$

polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$

$w(r, \theta) = u(x, y)$

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = \frac{2}{r}$$

A solution is independent of $\theta \rightarrow \dfrac{\partial w}{\partial \theta} = 0$

ODE; $\quad w_{rr} + \dfrac{1}{r} w_r = \dfrac{2}{r}$

$$\Rightarrow \quad r w''(r) + w'(r) = 2 \longrightarrow (r w')' = 2$$

$$r w' = 2r + c_1 \rightarrow w' = 2 + \frac{c_1}{r} \rightarrow \boxed{w(r) = 2r + c_1 \ln r + c_2}$$

$w(1) = 2 + c_2 = 0 \Longrightarrow c_2 = -2$

$w(3) = 6 + c_1 \ln 3 = 2 \Longrightarrow c_1 = \dfrac{-4}{\ln 3}$

$$\Longrightarrow \boxed{w(r) = 2r - \frac{4 \ln r}{\ln c} - 2}$$

Generalized Green's Functions

Cochran - Chpt on Green's Functions

① $L\, g(x,y) = -w(y) \sum_{i=1}^{k} \psi_i(x)\, \psi_i(y)$, $x \neq y$

where $\{\psi_i(x)\}_{i=1}^{k}$ is an orthonormal basis for the null space of L^* with homogeneous boundary conditions $B^* \psi_i = 0$.

$u'' = f(x)$, $u(0) = 0$, $u(1) = u'(1)$

$L = u'' = L^*$ - self adjoint , $B = B^*$, $i = 1$ in this case
$\psi_i = \phi_i$

② g is continuous at $x = y$

③ $\left.\frac{\partial g}{\partial x}\right|_{x=y^-}^{x=y^+}$ ~~ccu~~ $= \frac{1}{a_n(y)}$ jump condition

④ $\int_a^b g(x,y)\, \phi_i(x)\, w(x)\, dx = 0$

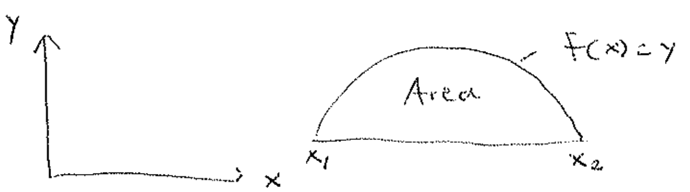
$i = 1$
$w = 1$
$\phi_1(x)$ is the basis for the null space of L
with $B \phi_1 = 0$

Assignment 4, problem 2
reference: "New directions in Linear Differential Equs"
SIAM Review, vol 21, 1979, pg 57-70

Exam 2 - November 15
- Special Functions
- Green's Functions
- Calculus of variations (without constraints)

Calculus of Variations with Constraints

Classical problem: "Isoperimetric Problem"



For all curves with a fixed perimeter L, maximize area A

$$\text{maximize} \quad A = \int_{x_1}^{x_2} y\,dx \quad \text{given} \quad L = \int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx + x_2 - x_1 \quad \text{is fixed}$$

$$y(x_1)=0,\ y(x_2)=0$$

this problem is done in Hildebrand

General situation

$$I = \int_{x_1}^{x_2} F(x,y,y')\,dx = \text{max or min} \quad, y(x_1)=y_1 \ , \ y(x_2)=y_2$$

$$\text{given} \quad J = \int_{x_1}^{x_2} G(x,y,y')\,dx = K \quad \text{is fixed}$$

Since there are two Functionals, introduce two variations

$$\delta y = \epsilon_1 \eta_1(x) + \epsilon_2 \eta_2(x)$$

$$\cancel{I(\epsilon_1,\epsilon_2) = \int_{x_1}^{x_2} F(x, y+\epsilon_1\eta_1(x), y}$$

$$I(\epsilon_1,\epsilon_2) = \int_{x_1}^{x_2} F\left(x, y+\epsilon_1\eta_1(x)+\epsilon_2\eta_2(x), y'+\epsilon_1\eta_1'(x)+\epsilon_2\eta_2'(x)\right) dx$$

$$J(\epsilon_1,\epsilon_2) = \int_{x_1}^{x_2} G\left(x, y+\epsilon_1\eta_1+\epsilon_2\eta_2, y'+\epsilon_1\eta_1'+\epsilon_2\eta_2'\right) dx = K$$

Introduce a Lagrange multiplier λ

$$\frac{\partial}{\partial \epsilon_1}\left[I(\epsilon_1,\epsilon_2) + \lambda J(\epsilon_1,\epsilon_2)\right] = 0$$

parallel

$$\frac{\partial}{\partial \epsilon_2}\left[I(\epsilon_1,\epsilon_2) + \lambda J(\epsilon_1,\epsilon_2)\right] = 0 \quad \Rightarrow \quad \nabla_\epsilon I \parallel \nabla_\epsilon J$$

Variational notation: $\delta(I+\lambda J) = 0$

$\Rightarrow$ Euler-Lagrange equation

$$\begin{cases} \dfrac{\partial}{\partial y}(F+\lambda G) - \dfrac{d}{dx}\dfrac{\partial}{\partial y'}(F+\lambda G) = 0 \\ \text{BCs: } y(x_1) = y_1, \quad y(x_2) = y_2 \\ J = \displaystyle\int_{x_1}^{x_2} G(x,y,y')\,dx = K \end{cases}$$

Example (Hildebrand) pg 142-144

$x_1 = 0$, $x_2 = 1$

The solutions satisfy $(x-c_1)^2 + (y-c_2)^2 = \lambda^2$

Which can be found by requiring the circle to pass through the end-points. If $L > \pi/2$, the solution will not determine y as a single-valued Function of x.

Blah Blah Blah Blah

- see Hiledrand

Sturm-Liouville Theory - Variational problems pg 145

Blah

Blah

$Ax = t$

Blah

Back ⇒

One-dimensional Poincare inequality

- consider Functions $f(x) \in L^2[0, L]$, $f(0) = f(L) = 0$

with $f'(x) \in L^2[0, L]$

then $\boxed{\frac{\pi^2}{L^2} \int_0^L [f(x)]^2 dx \leq \int_0^L [f'(x)]^2 dx}$

this relates to the Sturm-Liouville problem

$$y'' + \lambda y = 0, \qquad 0 < x < L \qquad , y(0) = y(L) = 0$$

let $\phi_n(x) = \sin\left(\frac{n\pi}{L} x\right)$, $\lambda_n = \frac{n\pi}{L}$

minimize $\dfrac{\int_0^L f'^2 dx}{\int_0^L f^2 dx} \geqq \lambda_1 = \frac{\pi^2}{L^2}$

Hence the Poincare inequality follows.

Wirtingen Inequality

if $f'(0) = f'(L) = 0$, $f' \in L^2[0, L]$ and $\int_0^L f dx = 0$ then

$$\frac{\pi^2}{L^2} \int_0^L f^2 dx \leqq \int_0^L f'^2 dx$$

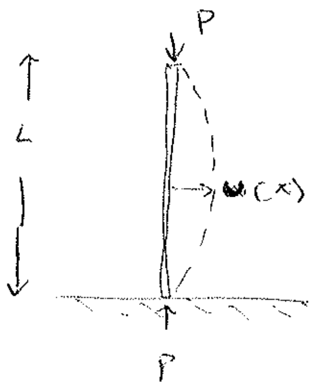
The associated Sturm-Liouville problem: $y'' + \lambda y = 0$, $0 < x < L$

$y'(0) = y'(L) = 0$, $\phi_n(x) = \cos\left(\frac{n\pi x}{L}\right)$, $\lambda_n = \left(\frac{n\pi}{L}\right)^2$

$\phi_0(x) = 1$, $\lambda_0 = 0$ here $\langle f, \phi_0 \rangle = 0$, f satisfies the B.C. s

Courant - Hilbert VOL 1, pg 272-274

- Vibrational problem of buckling



A rod is compressed by a loading force P.
(~~Pcrea~~ (P is constant at both ends)

$u(0) = u(L) = 0$

potential energy

$$V = \frac{1}{2}\int_0^L EI\, u'' dx - \frac{P}{2}\int_0^L u'^2 dx$$

For sufficiently small values of P
the minimum of V is zero
on the other hand, for sufficiently large P, V can be negative
For any admissible u we need only choose

$$P_0 > \frac{\int_0^L EI\, u'' dx}{\int_0^L (u')^2 dx} \quad \text{for buckling to occur.}$$

The buckling force P_0 is the largest value of P for which
min V = 0.

$$\frac{\int_0^L EI\, u'' dx}{\int_0^L (u')^2 dx} = \lambda \qquad \text{let } \delta\lambda = 0 \longrightarrow \frac{I_1}{I_2}$$

$$\boxed{\begin{gathered}\delta\left(\frac{I_1}{I_2}\right) = \frac{(\delta I_1) I_2 - I_1\, \delta I_2}{(I_2)^2} = 0 \\ \delta(I_1 - \lambda I_2) = 0\end{gathered}}$$

$$\Rightarrow \text{Euler-Lagrange equations: } \left.\begin{aligned} &u^{IV} + \frac{\lambda}{EI} u'' = 0 \\ &\text{natural boundary conditions: } u''(0) = u''(L) = 0 \end{aligned}\right\} \text{non-standard eigenvalue problem}$$

we already have: $u(0) = u(L) = 0$

$$v^{IV} + k^2 v'' = 0 \qquad , \quad k = \frac{\lambda}{EI}$$

$$\downarrow$$

$$v = c_1 + c_2 x + c_3 \cos kx + c_4 \sin kx$$

$$\left.\begin{aligned} v(0) &= c_1 + c_3 = 0 \\ v''(0) &= -k^2 c_3 = 0 \end{aligned}\right\} \quad c_1 = c_3 = 0$$

$$v(L) = c_2 L + c_4 \sin kL = 0$$

$$v''(L) = -k^2 c_4 \sin kL = 0$$

$$\Rightarrow \quad k_n L = n\pi \quad \rightarrow \quad k_n = \frac{n\pi}{L} \qquad , \quad c_2 = 0$$

$$\Rightarrow \boxed{v(x) = \sin \frac{n\pi}{L} x}$$

Green's Functions for Initial Value Problems

$Lg = \delta$ (L - second order)

$g(0) = 0$, $g'(0) = 0$

$$g = \begin{cases} c_1 u_1 + c_2 u_2 , & x < y \\ k_1 u_1 + k_2 u_2 , & x > y \end{cases} , \quad Lu_1 = 0 , \quad Lu_2 = 0$$

$c_1 u_1(0) + c_2 u_2(0) = 0$

$c_1 u_1'(0) + c_2 u_2'(0) = 0$

$\Rightarrow \quad c_1 = c_2 = 0$ unless $\begin{vmatrix} u_1(0) & u_2(0) \\ u_1'(0) & u_2'(0) \end{vmatrix} = 0 = |W(u_1, u_2)|$

but $|W(u_1, u_2)| \neq 0$ because u_1 and u_2 are linearly independent

$$\Rightarrow \quad g = \begin{cases} 0, & x < y \\ k_1 u_1 + k_2 u_2 , & x > y \end{cases}$$

apply jump condition , continuity

$$\left. \frac{dg(x,y)}{dx} \right|_{x=y^-}^{x=y^+} = \frac{1}{a_2(y)} \qquad\qquad g(x,y) \Big]_{x=y^-}^{x=y^+} = 0$$

$$\boxed{\begin{array}{l} \Rightarrow k_1 u_1'(y) + k_2 u_2'(y) = \dfrac{1}{a_2(y)} \qquad \Rightarrow k_1 u_1(y) + k_2 u_2(y) = 0 \\ \text{Now solve for } k_1 \text{ and } k_2 \end{array}}$$

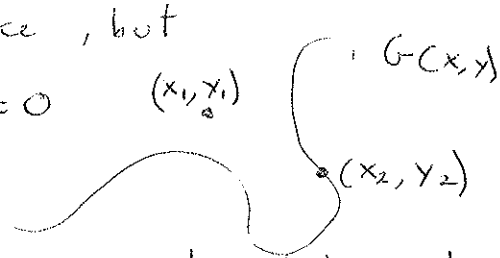
Calculus of Variations

- variable endpoints : Transversality condition

- the upper limit of the integral

$$I = \int_{x_1}^{x_2} F(x, y, y')\, dx \quad , \quad y(x_1) = y_1$$

and $y(x_2)$ is not given in advance , but (x_2, y_2) lies on a curve $G(x, y) = 0$



In Cochran , Hildebrand this problem is treated and also in Courant - Hilbert , Vol 1 .

Introduce a parameter t, $t_1 \le t \le t_2$.
The extremizing curve is $(x(t), y(t))$. Let

$$\mathcal{F}(x, y, \dot{x}, \dot{y}) = \dot{x}\, F\left(x, y, \frac{\dot{y}}{\dot{x}}\right)$$

where $\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$

The boundary conditions are $x(t_1)$, $y(t_1)$ Fixed

and $G(x(t_2), y(t_2)) = 0$

The variability of the upper limit is eliminated .
Also this will handle the case where it is inconvenient to solve G explicitly for y.

$$I = \int_{t_1}^{t_2} F(x, y, \dot{x}, \dot{y})\, dt \quad , \quad x(t_1) = x_1 \quad , \quad y(t_1) = y_1$$

constraint $G(x(t_2), y(t_2)) = 0$

Introduce two variations $\xi(t)$, $\eta(t)$ and two parameters ϵ_1, ϵ_2 such that $\xi(t_1) = 0$, $\eta(t_1) = 0$

$$\Psi(\epsilon_1, \epsilon_2) = G\left[x(t_2) + \epsilon\xi(t_2), y(t_2) + \epsilon_2\eta(t_2)\right] = 0$$

$$\Phi(\epsilon_1, \epsilon_2) = \int_{t_1}^{t_2} F(x + \epsilon_1\xi, y + \epsilon_2\eta, \dot{x} + \epsilon_1\dot{\xi}_1, \dot{y} + \epsilon_2\dot{\eta})\, dt -$$

is stationary at $\epsilon_1 = \epsilon_2 = 0$.

Introduce a Lagrange multiplier λ

$$\frac{\partial}{\partial\epsilon_1}\left(\Phi + \lambda\Psi\right)\Big|_{\epsilon_1 = \epsilon_2 = 0} = 0 \quad , \quad \frac{\partial}{\partial\epsilon_2}\left(\Phi + \lambda\Psi\right)\Big|_{\epsilon_1 = \epsilon_2 = 0} = 0$$

Since $(x(t), y(t))$ must satisfy the Euler-Lagrange equations

$$\int_{t_1}^{t_2} \left[F_x\xi + F_{\dot{x}}\dot{\xi}\right] dt + \lambda G_x(x(t_2), y(t_2))\,\xi(t_2) = 0$$

$$\int_{t_1}^{t_2} \left[F_y\eta + F_{\dot{y}}\dot{\eta}\right] dt + \lambda G_y(x(t_2), y(t_2))\,\eta(t_2) = 0$$

Integrate by parts:

$$\left[\left(F_{\dot{x}} + \lambda G_x\right)\xi\right]_{t=t_2} + \int_{t_1}^{t_2} \left[-\frac{d}{dt}F_{\dot{x}} + F_x\right]\xi\, dt = 0$$

$$\left[\left(F_{\dot{y}} + \lambda G_y\right)\eta\right]_{t=t_2} + \int_{t_1}^{t_2} \left[-\frac{d}{dt}F_{\dot{y}} + F_y\right]\eta\, dt = 0$$

Transversality ~~...~~ conditions are obtained by eliminating λ.

$$\left(F_{\dot{x}} + \lambda G_x\right)_{t=t_2} = 0$$

$$\left(F_{\dot{y}} + \lambda G_y\right)_{t=t_2} = 0$$

$$\Rightarrow \quad \lambda = \frac{-F_{\dot{x}}}{G_x} = \frac{-F_{\dot{y}}}{G_y}$$

Assume G_y and G_x do not both vanish at $t = t_2$

$$F_{\dot{x}} G_y - F_{\dot{y}} G_x = 0 \qquad \boxed{\#}$$

Go back to the original variables.

$$F_{\dot{x}} = f\left(x, y, \frac{\dot{y}}{\dot{x}}\right) + \dot{x} \frac{\partial F}{\partial y'}\left(x, y, \frac{\dot{y}}{\dot{x}}\right)\left(\frac{-\dot{y}}{\dot{x}^2}\right)$$

$$F_{\dot{x}} = F\left(x, y, \frac{\dot{y}}{\dot{x}}\right) - y' \frac{\partial F}{\partial y'}\left(x, y, \frac{\dot{y}}{\dot{x}}\right)$$

$$F_{\dot{y}} = \dot{x} \frac{\partial F}{\partial y'}\left(x, y, \frac{\dot{y}}{\dot{x}}\right)\left(\frac{1}{\dot{x}}\right) = \frac{\partial F}{\partial y'}\left(x, y, \frac{\dot{y}}{\dot{x}}\right)$$

Substitute into $\boxed{\#}$

$$\left(F - y' \frac{\partial F}{\partial y'}\right) G_y - \frac{\partial F}{\partial y'} G_x = 0 \qquad \Rightarrow \boxed{F G_y + \frac{\partial F}{\partial y'}\left(-y' G_y - G_x\right) = 0}$$

on the extremizing curve $y(x) = \tilde{y}$.

IF ~~...~~ $G(x,y) = y - g(x)$, $G_x = -g'(x)$, $G_y = 1$

$$\Rightarrow \boxed{\cancel{F + F_y(-y' + g') = 0}} \qquad \boxed{F + F_y\left(-y' + g'\right) = 0}$$

Classical Example: Brachistochrone problem

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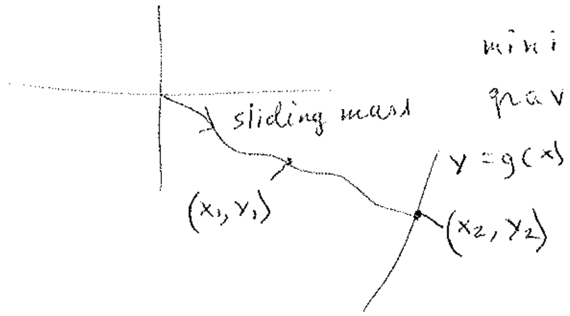


minimize the time for fall under gravity

If the endpoint lies on $y = g(x)$ then transversality is needed.

Example:

Minimize $\int_0^b (y')^2\, dx$ with $y(0) = 0$, where $(b, y(b))$ lies on the hyperbola $y^2 - x^2 = 1$, $y > 0$

so $G(x,y) = y^2 - x^2 - 1 = 0$

transversality condition

$$\left(F - y' F_{y'}\right) G_y - F_{y'} G_x = 0$$

$$\Rightarrow \left(y'^2 - y'(2y')\right) 2y - 2y'(-2x) = 0 \quad \Rightarrow \quad \boxed{yy' - 2x = 0 \quad , \quad x = b,\ y = y(b)}$$

Euler-Lagrange Eqn

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \Rightarrow \quad -2y'' = 0 \quad \Rightarrow \quad y(x) = Ax + B$$

$$y(0) = 0 \quad \Rightarrow \quad \boxed{y(x) = Ax}$$

$$(Ab)A - 2b = 0 \quad \Rightarrow \quad A^2 = 2 \quad \Rightarrow \quad A = \pm\sqrt{2}$$

$$\boxed{y(x) = \pm\sqrt{2}\, x}$$

Finite Constraints (Holonomic Constraints)

Render $I = \int_{t_1}^{t_2} F(t, x, y, z, \dot{x}, \dot{y}, \dot{z})\, dt$ stationary, given $G(x, y, z) = 0$. The stationary curve $(x(t), y(t), z(t))$ lies entirely on the surface. (Example, geodesic problem.)

Introduce Lagrange Multiplier $\lambda(t)$

$\delta I = 0$, $\delta G = 0$ $G_x \delta x + G_y \delta y + G_z \delta z = 0$

multiply by $\lambda(t)$ and integrate.

$$\int_{t_1}^{t_2} \left[\left(\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) + \lambda(t)\, G_x \right) \delta x + \left(\frac{\partial F}{\partial y} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) + \lambda(t)\, G_y \right) \delta y + \left(\frac{\partial F}{\partial z} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{z}}\right) + \lambda(t)\, G_z \right) \delta z \right] dt = 0$$

$\Rightarrow$ 3 Euler-Lagrange equations + constraint

Example: $G = x^2 + y^2 + z^2 - a^2 = 0$ sphere

$$L = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt, \quad G_x = 2x, \quad G_y = 2y, \quad G_z = 2z, \quad F = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}$$

Euler-Lagrange Eqns

$$\lambda G_x - \frac{d}{dt}\left(\frac{\dot{x}}{F}\right) = 0, \quad \lambda G_y - \frac{d}{dt}\left(\frac{\dot{y}}{F}\right) = 0, \quad \lambda G_z - \frac{d}{dt}\left(\frac{\dot{z}}{F}\right) = 0$$

Eliminate λ

$$\frac{\frac{d}{dt}\left(\frac{\dot{x}}{F}\right)}{G_x} = \frac{\frac{d}{dt}\left(\frac{\dot{y}}{F}\right)}{G_y} = \frac{\frac{d}{dt}\left(\frac{\dot{z}}{F}\right)}{G_z} \Rightarrow \frac{F\ddot{x} - \dot{x}\dot{F}}{2xF^2} = \frac{F\ddot{y} - \dot{y}\dot{F}}{2yF^2} = \frac{F\ddot{z} - \dot{z}\dot{F}}{2zF^2}$$

$$\frac{F\ddot{z} - \dot{z}\dot{F}}{z} = \frac{F\ddot{y} - \dot{y}\dot{F}}{y}$$

$$\Rightarrow \frac{\ddot{z} - \frac{\dot{z}\dot{F}}{F}}{z} = \frac{\ddot{y} - \frac{\dot{y}\dot{F}}{F}}{y} \Rightarrow y\left(\ddot{z} - \frac{\dot{z}F'}{F}\right) = z\left(\ddot{y} - \dot{y}\frac{F'}{F}\right)$$

$$\Rightarrow y\ddot{z} - y\dot{z}\frac{F'}{F} = z\ddot{y} - \dot{y}z\frac{F'}{F}$$

$$\Rightarrow \frac{F'}{F} = \frac{\ddot{y}z - y\ddot{z}}{\dot{y}z - y\dot{z}} \qquad \text{exact? } \frac{d}{dt}(z\dot{y} - y\dot{z}) = \ddot{y}z - y\ddot{z}$$

$$\Rightarrow \ln(y\dot{x} - x\dot{y}) = \log(z\dot{y} - y\dot{z}) + \log c_1$$

$$\Rightarrow y\dot{x} - x\dot{y} = c_1(z\dot{y} - y\dot{z})$$

$$\frac{\dot{x} + c_1\dot{z}}{x + c_1 z} = \frac{\dot{y}}{y}$$

integrate again $\ln(x + c_1 z) = \ln y + \log c_2$

$\Rightarrow X - c_2 y + c_1 z = 0$ plane through origin

intersection with sphere $x^2 + y^2 + z^2 = a^2$ gives a great circle.

Extension

Differential equations as constraints: Non-holonomic

Example: $\delta\int_{t_1}^{t_2} F(t, x, y, z, \dot{x}, \dot{y}, \dot{z})\, dt = 0$ given $G(x, y, z, \dot{x}, \dot{y}, \dot{z}) = 0$

Introduce $\lambda(t)$

$$\delta G = G_x \delta_x + G_y \delta_y + G_z \delta_z + G_{x'}\delta_{x'} + G_{y'}\delta_{y'} + G_{z'}\delta_{z'} = 0$$

Find $\int_{t_1}^{t_2} \delta F + \lambda D G \, dt = 0$

Transversality:

$$I = \int_{x_1}^{x_2} F(x, y, y')\,dx \quad , \quad y(x_1) = y_1 \ , \quad x_2 \text{ lies on } G(x,y) = 0$$

Necessary condition

$$(F - y'F_{y'})\,G_y = F_{y'}\,G_x \quad \text{for } x = x_2$$

We saw the special case: $y = g(x)$

Now consider $x = b$, b constant

Free boundary $G = x - b$, $G_x = 1$, $G_y = 0$

natural boundary condition: $F_{y'} = 0$ as before.

Differential Equations as constraints

Optimal control - Cochran Chpt 4

$$I(\underline{x}, \underline{u}) = \int_{t_1}^{t_2} F(t, \underline{x}, \underline{u})\,dt \quad , \quad \underline{x}(t_1) = \underline{x}^{(0)} \text{ known}$$

where $\dot{x}(t) = \underline{G}(t, \underline{x}, \underline{u})$, system of ODEs

$$\underline{x} = \underbrace{\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}}_{n} \text{ state vector}$$

$$\underline{u} = \underbrace{\begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{pmatrix}}_{m} \text{ control vector}$$

Ideally we could eliminate $\underline{u}$ otherwise we solve keeping $\underline{u}$ in the system.

Consider the case $n=1$, $m=1$ (one state vector, one control vector).

$$I(\underline{\dot{x}},\underline{u}) = \int_{t_1}^{t_2} F(t,\underline{x},\underline{u})\,dt$$

$$\Rightarrow I(x,u) = \int_{t_1}^{t_2} F(t,x,u)\,dt, \quad x(t_1) = x^{(0)}$$

& $\dot{x} = g(t,x,u)$ - this is like a nonholonomic constraint

Introduce a Lagrange multiplier $\lambda(t)$.

$$\dot{x} - g(t,x,u) = 0 \quad \text{(non-holonomic constraint)}$$

$$\delta(\dot{x} - g(t,x,u)) = 0$$

$$\delta\dot{x} - g_x\delta x - g_u\delta u = 0$$

Multiply my $\lambda(t)$ and integrate:

$$\int_{t_1}^{t_2} F_x\,\delta_x + F_u\,\delta_u + \lambda[\underbrace{\delta\dot{x}}_{\text{integrate by parts}} - g_x\delta x - g_u\delta u]\,dt = 0$$

$$\int_{t_1}^{t_2}\left[(F_x - \lambda g_x)\delta x + (F_u - \lambda g_u)\delta u\right]dt + \left[\lambda\delta x\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\dot{\lambda}\,\delta x\,dt = 0$$

require $\lambda(t_2)\,\delta(t_2) - \lambda(t_1)\cancelto{0}{\delta(t_1)} = 0$

hence $\lambda(t_2) = 0$

Euler equations

$$\begin{cases} F_x - \lambda g_x - \dot{\lambda} = 0 \\ F_u - \lambda g_u = 0 \\ \dot{x} = g(t,x,u) \end{cases}$$

Note it is possible to eliminate λ, but retaining λ may be the best strategy at times.

to eliminate λ (Cochran)

$$\dot{\lambda} + \lambda g_x = f_x$$

integrating factor, $\nu = e^{\int g_x \, dt}$ $\Rightarrow$ $\frac{d}{dt}\left(\lambda e^{\int g_x \, dt}\right) = f_x \, e^{\int g_x \, dt}$

$$\Rightarrow \quad \lambda e^{\int g_x \, dt} = \int f_x \, e^{\int g_x \, dt} \, dt + c$$

$$\cancel{\lambda(t) = e^{-\int_{t_1}^{t_2} g_x \, dt} \int_t^{t_2} F_x \, e^{\int g_x \, dt} \, dt + c = 0}$$

(~~BC~~ satisfies $\lambda(t_2) = 0$

we need $\lambda(t_2) = 0$, so for bounds of integration

$$\lambda(t) = e^{-\int_{t_0}^{t} g_x \, dt} \int_{t_0}^{t_2} F_x \, e^{\int g_x \, dt} + c$$

$$\lambda(t_2) = e^{-\int_{t_0}^{t_2} g_x \, dt} \int_{t_0}^{t_2} F_x \, e^{\int g_x \, dt} \, dt + c$$

$$\Rightarrow \quad c = -\left[\int_{t_0}^{t_2} F_x \, e^{\int g_x \, dt} \, dt\right] e^{-\int_{t_0}^{t_2} g_x \, dt}$$

$$\Rightarrow \lambda(t) = e^{-\int_{t_0}^{t} g_x \, dt}\left[\int_{t_0}^{t} F_x \, e^{\int g_x \, dt} \, dt - \int_{t_0}^{t_2} F_x \, e^{\int g_x \, dt} \, dt\right]$$

Take $t_0 = t_2$

$$\lambda(t) = e^{-\int_{t_2}^{t} g_x \, dt} \int_{t_2}^{t} F_x \, e^{\int g_x \, dt} \, dt$$

This may be substituted back into the system to find $x(t)$, $u(t)$.

For more general case :

$$I(\underline{x},\underline{u}) = \int_{t_1}^{t_2} F(t,\underline{x},\underline{u})\,dt \quad , \quad \dot{\underline{x}} = \underline{G}(t,\underline{x},\underline{u})$$

$$, \quad \underline{x}(t_1) = \underline{x}^{(0)}$$

Introduce $\underline{\lambda}(t) = \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \vdots \\ \lambda_n(t) \end{pmatrix}$

$$\dot{\underline{x}}_i = G_i(t,\underline{u},\underline{x}) \quad , i = 1,\dots,n$$

$$\delta \dot{\underline{x}}_i = \delta G_i = \sum_{j=1}^{n} \frac{\partial G_i}{\partial x_j}\delta x_j + \sum_{k=1}^{m} \frac{\partial G_i}{\partial u_k}\delta u_k$$

$$\sum_{i=1}^{n} \lambda_i(t)\,\delta \dot{x}_i(t) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} \lambda_i \frac{\partial G_i}{\partial x_j}\delta x_j + \sum_{k=1}^{m} \lambda_i \frac{\partial G_i}{\partial u_k}\delta u_k \right)$$

$$\delta I + (\quad) = 0$$

Einstein Summation notation (repeated indices

$$\int_{t_1}^{t_2} \left(\frac{\partial F}{\partial x_j}\delta x_j + \frac{\partial F}{\partial u_k}\delta u_k + \underbrace{\lambda_i \delta \dot{x}_i}_{\text{integrate by parts}} - \lambda_i \frac{\partial G_i}{\partial x_j}\delta x_j - \lambda_i \frac{\partial G_i}{\partial u_k}\delta u_k \right) dt = 0$$

$$\left[\lambda_i \delta x_i\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\left[\frac{\partial F}{\partial x_j} - \lambda_i \frac{\partial G_i}{\partial x_j} - \dot{\lambda}_j \right] \delta x_j + \left(\frac{\partial F}{\partial u_k} - \lambda_i \frac{\partial G_i}{\partial u_k} \right) \delta u_k \right) dt = 0$$

$\Rightarrow \delta x_i(t_1) = 0 \quad , \quad \lambda_i(t_2) = 0$, natural boundary conditions

System

$$\begin{cases} \dfrac{\partial F}{\partial x_j} - \displaystyle\sum_{i=1}^{n} \lambda_i \frac{\partial G_i}{\partial x_j} - \dot{\lambda}_j = 0 & , j = 0,1,2,\dots,n \\ \dfrac{\partial F}{\partial u_k} - \displaystyle\sum_{i=1}^{n} \lambda_i \frac{\partial G_i}{\partial u_k} = 0 & , k = 1,2,\dots,m \\ \underline{x}(t_1) = \underline{x}^{(0)} \\ \lambda(t_2) = 0 \end{cases}$$

Rayleigh-Ritz Method

Solve $Ly = F$, $a < x < b$, $Ly = (py')' - qy$

The eigenfunctions are ϕ_n

Approximate $y_n = c_1\phi_1 + \cdots + c_n\phi_n = \sum_{k=1}^{n} c_k \phi_k$

Suppose $y_n \to y$

variational formulation:

$$\int_a^b (Ly - F)\,\delta y \, dx = 0$$

$$\Rightarrow \int (Ly - F)(\phi_1 \delta c_1 + \phi_2 \delta c_2 + \cdots + \phi_n \delta c_n)\, dx = 0$$

$$\Rightarrow \int \cancel{\sum (c_k L\phi_k - F)}$$

$$\Rightarrow \int \left(\sum c_k L\phi_k - F\right)\left(\phi_1 \delta c_1 + \cdots + \phi_n \delta c_n\right) dx = 0$$

for each term

$$\int \left(\sum c_k L\phi_k - F\right)\phi_j \, dx = 0, \quad j = 1, 2, \ldots, n$$

what has been minimized is $\int (Ly - F)\, dx$

Galerkin method

impose from the beginning: $y_n = \sum_{k=1}^{n} c_k \phi_k$

choose c_k such that $\underbrace{\int (Ly_n - F)\,\phi_k \, dx = 0}_{\text{"weak formulation" of the solution}}$, $k = 1, \ldots, n$

ϕ_k forms an orthogonal set

If the trial functions $\{\phi_n\}$ are complete as $n \to \infty$ the Galerkin approximation converges in the weak sense to the solution of $Lu = F$.

Keener, Section 5.3: Approximate Methods, Review

Example pg 196

solve $u' = u$ on $[0,1]$ with $u(0) = a$ by

minimizing $F(u,u') = \int_0^1 (u'-u)^2\,dx$

assume $u = B_0 + B_1 x + B_2 x^2$

$u(0) = a \Rightarrow B_0 = a \Rightarrow$ $u = a + B_1 x + B_2 x^2$
$u' = B_1 + 2B_2 x$

$$F(u,u') = \int_0^1 (B_1 + 2B_2 x - a - B_1 x - B_2 x^2)^2\,dx$$

$$= \int_0^1 \left((B_1 - a) + (2B_2 - B_1)x - B_2 x^2 \right)^2 dx$$

$$F(u,u') = \int_0^1 \left\{ \begin{array}{l} (B_1-a)^2 + (2B_2-B_1)^2 x^2 + B_2^2 x^4 + 2(B_1-a)(2B_2-B_1)x \\ + 2(B_1-a)(-B_2x^2) + 2(2B_2-B_1)x(-B_2x^2) \end{array} \right\} dx$$

$$F(u,u') = \int_0^1 \left\{ \begin{array}{l} (B_1-a)^2 + (2B_2-B_1)^2 x^2 + B_2^2 x^4 + 2(2B_1B_2 - B_1^2 - 2aB_2 + aB_1)x \\ -2(B_1-a)B_2x^2 - 2(2B_2-B_1)B_2x^3 \end{array} \right\} dx$$

$$\boxed{F(u,u') = (B_1-a)^2 + \tfrac{1}{3}(2B_2-B_1)^2 + \frac{B_2^2}{5} + (2B_1B_2 - B_1^2 - 2aB_2 + aB_1) - \tfrac{2}{3}(B_1-a)B_2 - \tfrac{1}{2}(2B_2-B_1)B_2}$$

$$\frac{\partial F}{\partial B_1} = 2(B_1-a) - \tfrac{2}{3}(B_2-B_1) + 2B_2 - 2B_1 + a - \tfrac{2}{3}B_2 + \frac{B_2}{2} = \tfrac{2}{3}B_1 - \tfrac{4}{3}B_2 + 2B_2 - \tfrac{2}{3}B_2 + \frac{B_2}{2} - a$$

$$\frac{\partial F}{\partial B_1} = 0 \Rightarrow \boxed{\tfrac{2}{3}B_1 + \frac{B_2}{2} = a}$$

$$\frac{\partial F}{\partial B_2} = \tfrac{2}{3}(2B_2-B_1)(2) + \tfrac{2}{5}B_2 + 2B_1 - 2a - \tfrac{2}{3}(B_1-a) - B_2 - \tfrac{1}{2}(2B_2-B_1)$$

$$= B_1\left(-\cancel{\tfrac{4}{3}} + \cancel{2} - \cancel{\tfrac{2}{3}} + \tfrac{1}{2}\right) + B_2\left(\tfrac{8}{3} + \tfrac{2}{5} - 1 - 1\right) + a\left(-2 + \tfrac{2}{3}\right) = \frac{B_1}{2} + \frac{16}{15}B_2 - \frac{4}{3}a$$

$$\frac{\partial F}{\partial B_2} = 0 \Rightarrow \boxed{\frac{B_1}{2} + \frac{16}{15}B_2 = \frac{4}{3}a}$$

$$\frac{\partial F}{\partial B_1} = 0 \implies \boxed{\begin{aligned} \frac{2}{3} B_1 + \frac{B_2}{2} &= a \\ \frac{B_1}{2} + \frac{16}{15} B_2 &= \frac{4}{3} a \end{aligned}}$$

$$\frac{\partial F}{\partial B_2} = 0 \implies$$

$$\begin{aligned} 4B_1 + 3B_2 &= 6a \\ 15B_1 + 32B_2 &= 40a \end{aligned} \implies \begin{aligned} 60B_1 + 45B_2 &= 90a \\ 60B_1 + 128B_2 &= 160a \end{aligned} \implies 83B_2 = 70a \implies B_2 = \frac{70}{83} a$$

$$B_1 = \frac{3}{2} a - \frac{3}{4} B_2 = \left(\frac{3}{2} - \frac{3}{4}\left(\frac{70}{83}\right)\right) a = \left(\frac{249}{2 \cdot 83} - \frac{105}{2 \cdot 83}\right) a = \frac{144}{2 \cdot 83} a = \frac{72}{83} a$$

$$\implies \boxed{B_1 = \frac{72}{83} a, \quad B_2 = \frac{70}{83} a}$$

$$u(x) = a + B_1 x + B_2 x^2$$

$$u(x) = a + \frac{72}{83} a x + \frac{70}{83} a x^2$$

$$\boxed{u(x) = \frac{a}{83}\left(83 + 72x + 70x^2\right), \quad x \in [0,1]}$$

Review

Least squares approximation of exact solution $u(x) = ae^x$ on $[0,1]$ to $u' = u$, $u(0) = a$

~~If (e_1, e_2, e_3) is an or~~ Orthonormalize the basis $(1, x, x^2)$ using Gram Schmidt method:

$$\boxed{e_1 = 1}$$

$$\tilde{e}_2 = x - \langle x, 1\rangle = x - \frac{1}{2} \quad , \quad \langle x, 1\rangle = \int_0^1 x\,dx = \frac{x^2}{2}\Big]_0^1 = \frac{1}{2}$$

$$\|\tilde{e}_2\| = \sqrt{\langle \tilde{e}_2, \tilde{e}_2\rangle} = \sqrt{\int_0^1 \left(x-\frac{1}{2}\right)^2 dx} = \sqrt{\int_0^1 \left(x^2 - x + \frac{1}{4}\right) dx} = \sqrt{\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right]_0^1}$$

$$= \sqrt{\frac{1}{3} - \frac{1}{2} + \frac{1}{4}} = \sqrt{\frac{1}{12}} = \frac{1}{2\sqrt{3}}$$

$$\Rightarrow \boxed{e_2 = \frac{\tilde{e}_2}{\|\tilde{e}_2\|} = 2\sqrt{3}\left(x - \frac{1}{2}\right)}$$

$$\tilde{e}_3 = x^2 - \langle x^2, 1\rangle - \langle x^2, 2\sqrt{3}(x - 1/2)\rangle (x - 1/2) 2\sqrt{3}$$

$$\langle x^2, 1\rangle = \int_0^1 x^2\,dx = \frac{1}{3}$$

$$\langle x^2, 2\sqrt{3}(x-1/2)\rangle = 2\sqrt{3}\int_0^1 x^2\left(x - \frac{1}{2}\right)dx = 2\sqrt{3}\int_0^1 \left(x^3 - \frac{x^2}{2}\right)dx = 2\sqrt{3}\left[\frac{x^4}{4} - \frac{x^3}{6}\right]_0^1 = \frac{\sqrt{3}}{6}$$

$$\Rightarrow \tilde{e}_3 = x^2 - \frac{1}{3} - \left(x - \frac{1}{2}\right)(2\sqrt{3})\frac{\sqrt{3}}{6} = x^2 - x + \frac{1}{6}$$

$$\|\tilde{e}_3\| = \sqrt{\langle \tilde{e}_3, \tilde{e}_3\rangle} = \sqrt{\int_0^1 \left(x^2 - x + \frac{1}{6}\right) dx} = \sqrt{\int_0^1 \left(x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{x}{3} + \frac{1}{36}\right) dx}$$

$$= \sqrt{\frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36}} = \sqrt{\frac{1}{5} + \frac{-18 + 16 - 6 + 1}{36}} = \sqrt{\frac{1}{5} - \frac{7}{36}} = \sqrt{\frac{1}{180}} = \frac{1}{6\sqrt{5}}$$

$$\Rightarrow \boxed{e_3 = \frac{\tilde{e}_3}{\|\tilde{e}_3\|} = 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right)}$$

If $(e_1, \ldots, e_n)$ is an orthonormal basis of a vector space V then for all $w \in V$, $w = \langle w, e_1\rangle e_1 + \cdots + \langle w, e_n\rangle e_n$.

If $w = ae^x$

$\Rightarrow w = \langle ae^x, 1\rangle + \langle ae^x, 2\sqrt{3}(x-\tfrac{1}{2})\rangle 2\sqrt{3}(x-\tfrac{1}{2}) + \langle ae^x, 6\sqrt{5}(x^2-x+\tfrac{1}{6})\rangle 6\sqrt{5}(x^2-x+\tfrac{1}{6})$

$w = a\langle e^x, 1\rangle + a\langle e^x, x-\tfrac{1}{2}\rangle(12x-6) + a\langle e^x, x^2-x+\tfrac{1}{6}\rangle 180(x^2-x+\tfrac{1}{6})$

Note: $\int_0^1 xe^x = xe^x\Big]_0^1 - \int_0^1 e^x\,dx = e-(e-1) = 1$

$\int_0^1 x^2e^x\,dx = x^2e^x\Big]_0^1 - \int_0^1 2e^x x\,dx = e - 2(1) = e-2$

$\langle e^x, 1\rangle = \int_0^1 e^x\,dx = e-1$

$\langle e^x, x-\tfrac{1}{2}\rangle = \int_0^1 (xe^x - \tfrac{e^x}{2})\,dx = 1 - (\tfrac{e}{2}-\tfrac{1}{2}) = \tfrac{1}{2}(3-e)$

$\langle e^x, x^2-x+\tfrac{1}{6}\rangle = \int_0^1 (x^2e^x - xe^x + \tfrac{e^x}{6})\,dx = (e-2) - 1 + \tfrac{1}{6}(e-1) = \tfrac{7e}{6} - 3 - \tfrac{1}{6} = \tfrac{1}{6}(7e-19)$

$\Rightarrow w = a(e-1) + \tfrac{a}{2}(3-e)(12x-6) + \tfrac{a}{6}(7e-19)(180)(x^2-x+\tfrac{1}{6})$

$w = a[e-1 + (3-e)6x - (3-e)3 + 30(7e-19)x^2 - 30(7e-19)x + 5(7e-19)]$

$w = a[e-1-9+3e + (35e-95) + 588x - 216ex + 30(7e-19)x^2]$

$$\boxed{w = a\left(13(3e-35) + 12(49-18e)x + 30(7e-19)x^2\right), \quad x \in [0,1]}$$

Approximate Methods (Keener, Cochran, Hildebrand)

Problem (Keener pg 196)

solve $\left.\begin{array}{l} u' = u \\ u(0) = a \end{array}\right\}$ $u = ae^x$

approximation: minimize $F(u, u') = \int_0^1 (u' - u)^2 \, dx$, $u(0) = a$

use quadratic polynomials

possibilities - Least Squares, Taylor Polynomial

assume: $u(x) = \beta_0 + \beta_1 x + \beta_2 x^2$, $u(0) = a$

minimize: $F(\beta_0, \beta_1, \beta_2)$, take $\beta_0 = a$

$\Rightarrow u(x) = a + \beta_1 x + \beta_2 x^2$, $u'(x) = \beta_1 + 2\beta_2 x$

$$F(u, u') = \int_0^1 \left[(a - \beta_1) + (\beta_1 - 2\beta_2) x + \beta_2 x^2 \right]^2 dx$$

let $\beta_1 = \alpha_1 a$, $\beta_2 = \alpha_2 a$

$$= a \int_0^1 \left[(1 - \alpha_1) + (\alpha_1 - 2\alpha_2) x + \alpha_2 x^2 \right]^2 dx$$

MAPLE

set $\frac{\partial F}{\partial \alpha_1} = 0 \Rightarrow \frac{1}{2}\alpha_2 + \frac{2}{3}\alpha_1 - 1 = 0$

$\frac{\partial F}{\partial \alpha_2} = 0 \Rightarrow \frac{16}{15}\alpha_2 + \frac{1}{2}\alpha_1 - \frac{4}{3} = 0$

$\Rightarrow \alpha_1 = \frac{72}{83}$, $\alpha_2 = \frac{70}{83}$

$$\Rightarrow u = \frac{a}{83}\left(83 + 72x + 70x^2\right)$$

text: plot of exact solution versus several polynomial approximations

$$\Rightarrow \left[2y''\delta y'\right]_0^\infty - \int_0^\infty 2y'''\delta y'\,dx + \left[4y'\delta y\right]_0^\infty - \int_0^\infty 4y''\delta y\,dx + \int_0^\infty 2y\delta y\,dx - 3y(0)\,\delta y(0)$$

$$\Rightarrow -2y''(0)\,\delta y'(0) - \left[2y'''\delta y\right]_0^\infty + \int_0^\infty 2y''''\delta y\,dx + 4y'(0)\delta y(0) - \int_0^\infty 4y''\delta y\,dx + \int_0^\infty 2y\,\delta y\,dx - 3y(0)\delta y(0)$$

$$\Rightarrow -2y''(0)\,\delta y'(0) + 2y'''(0)\delta y(0) + \int_0^\infty (2y'''' - 4y'' + 2y)\,\delta y\,dx - 3y(0)\,\delta y(0) + 4y'(0)\,\delta y(0) = 0$$

Natural BCs:

$-2y'''(0)\delta y(0) + 4y'(0)\,\delta y(0) - 3y(0)\,\delta y(0) = 0$, $y''(0) = 0$

$\Rightarrow \quad -2y'''(0) - 4y'(0) - 3y(0) = 0$

$\Rightarrow$ Euler-Lagrange: $y'''' - 2y'' + y = 0$

Try $y = e^{mx}$; $m^4 - 2m^2 + 1 = 0 \rightarrow (m^2-1)^2 = 0$

$m = \pm 1, \pm 1$

$\Rightarrow \quad y(x) = (c_1 x + c_2)e^{x} + (c_3 x + c_4)e^{-x}$

$y, y' \to 0$ as $x \to \infty \Rightarrow c_1 = 0,\ c_2 = 0$

the other two conditions fix $c_3 = \frac{c_4}{2} \equiv c$

An alternative way of solving for $\tilde{y}$ is to let $y(0) = k$

minimizing $\int_0^\infty y''^2 + 2y'^2 + y^2\,dx$, $\tilde{y} = ce^{-x}(x+2)$

$I(\tilde{y}) = \int_0^\infty (\tilde{y}''^2 + 2\tilde{y}'^2 + \tilde{y}^2)\,dx = 6c^2$

IF $y = \tilde{y} + z$, $z(0) = 0$, $z''(0) = 0$, $z \neq 0$

$I(y) = I(\tilde{y}) + \int_0^\infty (z''^2 + 2z'^2 + z^2)\,dx > I(\tilde{y})$

$Lg = 0$ except at $x = y$

For a 2^{nd} order case

two boundary conditions must be satisfied

continuity must be satisfied

jump in 1^{st} derivative

$$g(x,y) = \begin{cases} \dfrac{u_1(x)\, u_2(y)}{pW}, & x < y \\ \dfrac{u_1(y)\, u_2(x)}{pW}, & x > y \end{cases}$$

For separated BCs

u_1 satisfies left BC

u_2 satisfies right BC

$$\int_0^1 2y'\,\delta y'\,dx + 2y(1)\,\delta y(1) = 0$$

$$= \left[2y'\,\delta y\right]_0^1 - \int_0^1 2y''\,\delta y\,dx + 2y(1)\,\delta y(1) = 0$$

$$= 2y'(1)\,\delta y(1) - 2y'(0)\,\delta y(0) - \int_0^{1} 2y''\,dx\,\delta y + 2y(1)\,\delta y(1) = 0$$

$2y'(1) + 2y(1) = 0$

$y'(1) = y(1)$

y

$y = 1 - \frac{1}{2}x$

$y'' = 0$

$y = ax + b$

$y(1) = a + b = 0$

$y(0) = b = 1$

$a = -1$

$y(x) = 1 - x$

- Systems of ODEs

  Boyce & Diprima

  Chpt 7 -- linear

  Chpt 9 - non linear

- Autonomous Systems

$$\frac{d\bar{x}}{dt} = \underline{F}(\underline{x}) \quad \text{or} \quad \frac{dx}{dt} = f(x,y), \quad \frac{dy}{dt} = g(x,y)$$

- Linear and Autonomous

  $\Rightarrow$ constant coefficient

$$\underbrace{\frac{dx}{dt} = ax + by, \quad \frac{dy}{dt} = cx + dy}, \quad a,b,c,d \text{ - constant}$$

$$\frac{d\vec{x}}{dt} = A\vec{x}$$ so it's all about eigenvalues and eigenvectors

critical point occurs where $\underline{F}(\underline{x}) = \underline{0}$

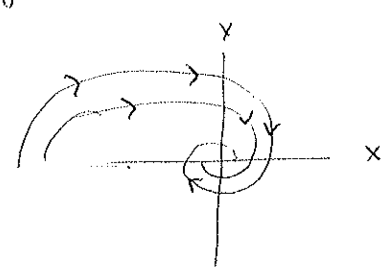
linear case implies $\underline{x} = \underline{0}$

Two - dimensions; $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

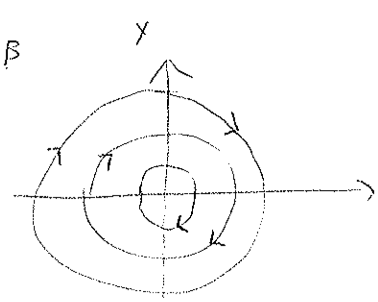
$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - (a+d)\lambda + ad - bc = 0$$

$\lambda = 0$ is NOT an eigenvalue of A

(a) spiral point: $\lambda = \alpha \pm i\beta$



(b) center: $\lambda = \pm i\beta$



General Results on Critical points

(Plane Hamiltonian systems)

suppose (ξ, η) is the critical point.

The gradient of the Hamiltonian must be 0 there,

$\nabla H = 0$

Write
$$H(x,y) = H(\xi,\eta) + H_x(\xi,\eta)(x-\xi) + H_y(\xi,\eta)(y-\eta)$$
$$+ \frac{a_{11}}{2}(x-\xi)^2 + a_{12}(x-\xi)(y-\eta) + \frac{a_{22}}{2}(y-\eta)^2 + \cdots$$

where
$$a_{11} = H_{xx}(\xi,\eta)$$
$$a_{12} = H_{xy}(\xi,\eta) = H_{yx}(\xi,\eta)$$
$$a_{22} = H_{yy}(\xi,\eta)$$

Thus
$$\frac{\partial H}{\partial y} = a_{12}(x-\xi) + a_{22}(y-\eta) + \cdots$$
$$-\frac{\partial H}{\partial x} = -\left[a_{11}(x-\xi) + a_{12}(y-\eta)\right] + \cdots$$

the linearized Hamiltonian system is:

$$\frac{d}{dt}\begin{pmatrix} x-\xi \\ y-\eta \end{pmatrix} = \begin{pmatrix} a_{12} & a_{22} \\ -a_{11} & -a_{12} \end{pmatrix}\begin{pmatrix} x-\xi \\ y-\eta \end{pmatrix}$$

The eigenvalues are $\begin{vmatrix} a_{12}-\lambda & a_{22} \\ -a_{11} & -a_{12}-\lambda \end{vmatrix} = 0$

$$\Rightarrow \lambda^2 - a_{12}^2 + a_{11}a_{22} = 0 \quad \Rightarrow \quad \lambda^2 = -\det\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$$

Thm: A nondegenerate critical point of a smooth plane Hamiltonian system is a center if $a_{11}a_{22} - a_{12}^2 > 0$ and is a saddlepoint if $a_{11}a_{22} - a_{12}^2 < 0$. When $a_{11}a_{22} = a_{12}^2$, the local behavior is indeterminate.

Hopf Bifurcation

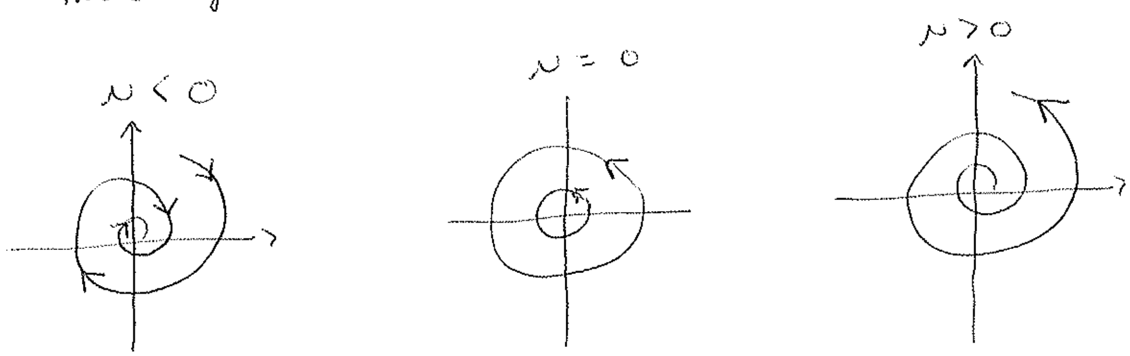
Example

system: $\dot{x} = -y - x(x^2+y^2-\nu)$
$\dot{y} = x - y(x^2+y^2-\nu)$, ν = constant

linearized system about $(0,0)$ $\dot{x} = -y + x\nu$
$\dot{y} = x + y\nu$

$\Rightarrow A = \begin{pmatrix} \nu & -1 \\ 1 & \nu \end{pmatrix} \Rightarrow (\nu-\lambda)^2 = -1 \Rightarrow$ ~~##~~ $\lambda = \nu \pm i$

If $\nu < 0$ then $Re(\lambda) < 0$ then stable spiral
if $\nu = 0$ then $Re(\lambda) = 0$ then center
if $\nu > 0$ then $Re(\lambda) > 0$ then unstable spiral
"increasing ν leads to instability"



This is a useful example because it can be integrated for an exact solution. Transform to polar coordinates.

$x = r\cos\theta$, $y = r\sin\theta$, $r = r(t)$, $\theta = \theta(t)$

Desire a system for $r(t)$, $\theta(t)$.
Multiply 1st eqn by x and 2nd by y.

$x\dot{x} = -xy - x^2(x^2+y^2-\nu)$
$y\dot{y} = xy - y^2(x^2+y^2-\nu)$

Add

$x\dot{x} + y\dot{y} = -(x^2+y^2)(x^2+y^2-\nu)$

$$x\dot{x} + y\dot{y} = -(x^2+y^2)(x^2+y^2-\nu)$$

Notice, $\frac{d}{dt}\left(\frac{x^2+y^2}{2}\right) = x\dot{x}+y\dot{y}$

$$x^2+y^2=r^2$$

$$\Rightarrow \frac{d}{dt}\left(\frac{r^2}{2}\right) = -r^2(r^2-\nu)$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \Rightarrow \dot{\theta} = \frac{x\dot{y}-y\dot{x}}{r^2}$$

Substitute $\dot{y} = x - y(x^2+y^2-\nu)$ and $\dot{x} = -y-x(x^2+y^2-\nu)$ into $\frac{d\theta}{dt}$

$$\Rightarrow \frac{d\theta}{dt} = \frac{x^2 - \cancel{xy(x^2+y^2-\nu)} + y^2 + \cancel{xy(x^2+y^2-\nu)}}{r^2} = \frac{r^2}{r^2} = 1$$

$$\Rightarrow \frac{d\theta}{dt} = 1 \Rightarrow \boxed{\theta = t + t_0}$$

$$\frac{d}{dt}\left(\frac{r^2}{2}\right) = -r^2(r^2-\nu) \quad \cancel{\text{Let } s(t) = (r(t))^2} \Rightarrow \boxed{\cancel{\frac{ds}{dt} = 2s(s-\nu)}}$$

$$\Rightarrow \cancel{\frac{1}{s}\frac{ds}{dt} = -2s + 2\nu} \quad \Rightarrow r\frac{dr}{dt} = -r^2(r^2-\nu)$$

$$\frac{dr}{dt} = -r(r^2-\nu) = -r^3+\nu r \Rightarrow$$

$$\frac{dr}{dt} = -r^3+\nu r \qquad \text{Let } w = r^{1-n} = r^{1-3} = r^{-2}$$

$$\Rightarrow w = r^{-2}, \quad \dot{w} = -2r^{-3}\dot{r}$$

$$\Downarrow$$

$$-2r^{-3}\frac{dr}{dt} = 2 - 2\nu r^{-2} \qquad \cancel{\Rightarrow \frac{dw}{dt} = 2 - 2\nu w}$$

$$\Downarrow$$

$$\frac{d}{dt}(r^{-2}) + 2\nu r^{-2} = 2 \qquad \text{integrating factor: } e^{2\nu t}$$

$$\Rightarrow \frac{d}{dt}\left(e^{2\nu t}r^{-2}\right) = 2e^{2\nu t} \Rightarrow e^{2\nu t}r^{-2} = \frac{e^{2\nu t}}{\nu} + r_0$$

$$\Rightarrow r^{-2} = \frac{1}{\nu} + r_0 e^{-2\nu t} \Rightarrow \boxed{r^2 = \frac{\nu}{1+\nu r_0 e^{-2\nu t}}}$$

$\theta = t + t_0$, $r^2 = \frac{\nu}{1+\nu r_0 e^{-2\nu t}}$

For $\nu > 0$

Let $r_0 \nu e^{-2\nu t_0} = 1$ $\Rightarrow$ $r = \frac{\sqrt{\nu}}{\sqrt{1+e^{-2\nu(t-t_0)}}}$ for $r_0 > N$

notice, $r \to \sqrt{\nu}$ as $t \to \infty$: Limit Cycle

The limit cycle exists for $\nu = 0$ as well.

The ODE can be integrated when $\nu = 0$.

You will find that $r(t) = \frac{1}{\sqrt{2t+c}}$, $\theta(t) = t - t_0$

What about $\nu < 0$?

Let $k^2 = -\nu > 0$

$\frac{ds}{dt} = -2s(k^2+s)$ where $s(t) = r^2(t)$

separation of variables

$\frac{ds}{s(k^2+s)} = -2dt$ $\Rightarrow$ partial fractions: $\frac{1}{s(k^2+s)} = \frac{1/k^2}{s} - \frac{1/k^2}{k^2+s}$

integrating: $\frac{1}{k^2} \ln\left|\frac{s}{k^2+s}\right| = -2(t-t_0)$

exponentiate: $\frac{s}{k^2+s} = \alpha e^{-2k^2(t-t_0)}$, where $\alpha = \pm 1$ due to absolute value

solve for s: $s = \frac{k^2 \alpha e^{-2k^2(t-t_0)}}{1-\alpha e^{-2k^2(t-t_0)}} = r^2$, $\theta = t - t_0$

this is a spiral, approaching origin as $t \to \infty$

As ν passes through the bifurcation point $\nu = 0$, the origin changes from a stable spiral point to an unstable spiral point and there appears a time-periodic solution (limit cycle).

Hopf Bifurcation

Keener, Theorem 11.3

Suppose that ~~the~~ $n \times n$ matrix $A(\nu)$ has eigenvalues $\lambda_j = \lambda_j(\nu)$ and that for $\nu = \nu_0$, $\lambda_1(\nu_0) = i\beta$, $\lambda_2(\nu_0) = -i\beta$ and that $Re(\lambda_j) \neq 0$ for $j > 2$. Suppose furthermore that $Re \dfrac{d\lambda_1(\nu_0)}{d\nu} \neq 0$

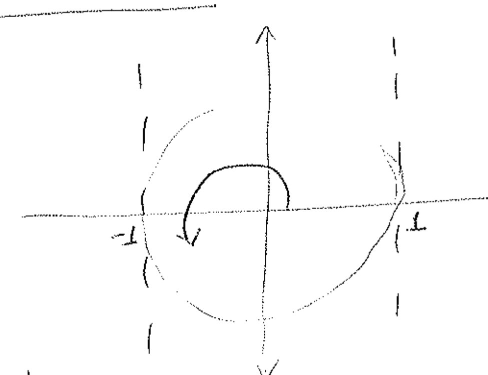
then the system of differential equations $\dfrac{d\vec{x}}{dt} = A(\nu)\vec{x} + F(\vec{x})$ where $F(\vec{0}) = \vec{0}$, $F(\vec{x})$ is a smooth function of $\vec{x}$, then $\vec{x}$ has a branch (continuum) of period solutions emanating from $\vec{x} = \vec{0}$, $\nu = \nu_0$.

In the previous example:

$$A = \begin{pmatrix} \nu & -1 \\ 1 & \nu \end{pmatrix}, \quad \lambda = \nu \pm i$$

$$F(\vec{x}) = \begin{pmatrix} -x(x^2+y^2) \\ -y(x^2+y^2) \end{pmatrix}, \quad Re \frac{d\lambda}{d\nu} = 1 \neq 0$$

Van der Pol eqn (Nonlinear Circuit Eqn)



$$\ddot{x} + \varepsilon \dot{x}(x^2-1) + x = 0$$

This eqn has a limit cycle

let $\dot{x} = y$

$\dot{y} = -x - \varepsilon y(x^2-1)$

As before: $\dfrac{d}{dt}\left(\dfrac{x^2+y^2}{2}\right) = -\varepsilon y^2(x^2-1)$

$|x| = 1$ is a delimiter if (x_0, y_0) is inside the strip $|x| = 1$. There is a limit cycle inside the strip.

Verhulst pg 200 (Bifurcations)

By a change of variables (dependent variables) obtain

$\ddot{u} + \dot{u}(u^2 - \mu) + u = 0$

Advantage: μ occurs only in the linear part, which is what Keener's Theorem "talks" about.

Write this ODE as a system:

$$\dot{u} = v$$
$$\dot{v} = -v(u^2 - \mu) - u$$

$$\Rightarrow \quad \frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix}}_{\text{linear}} + \underbrace{\begin{pmatrix} 0 \\ -vu^2 \end{pmatrix}}_{\text{non-linear}}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix} \longrightarrow |A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -1 & \mu - \lambda \end{vmatrix} \rightarrow \lambda^2 - \lambda\mu + 1 = 0$$

$$\rightarrow \lambda = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$$

when $\mu = 0$, $\lambda = \pm i$

$$\frac{d\lambda}{d\mu} = \frac{1}{2} \pm \frac{1}{2}\frac{\mu}{\sqrt{\mu^2 - 4}} \qquad Re\left(\frac{d\lambda}{dt}\right) = \frac{1}{2} \neq 0$$

Final Exam: Amos Eaton 215
Thursday Dec 12th
11:30 - 2:30 pm
open book, open notes

Past Final Problems

① Let $A^{(n\times n)}$ be a real antisymmetric matrix ($A^T = -A$).

i) Prove or disprove the assertion that every eigenvalue of A has zero real part.

ii) Show that $I - A$ is nonsingular and that $B = (I-A)^{-1}(I+A)$ is orthogonal.

i) example of real antisymmetric matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -A$$

If λ is an eigenvalue and x an eigenvector then

$Ax = \lambda x$

Take inner product: $\langle Ax, x\rangle = \langle \lambda x, x\rangle = \lambda\langle x, x\rangle$

$$\langle Ax, x\rangle = \langle x, A^* x\rangle = -\langle x, Ax\rangle = -\langle x, \lambda x\rangle = -\bar{\lambda}\langle x, x\rangle$$

$\Rightarrow \quad \lambda\langle x, x\rangle = -\bar{\lambda}\langle x, x\rangle \quad \Rightarrow \quad \lambda = -\bar{\lambda}$

$\Rightarrow \quad \lambda + \bar{\lambda} = 0 \quad \Rightarrow \quad 2\,\mathrm{Re}(\lambda) = 0 \quad \Rightarrow \boxed{\mathrm{Re}(\lambda) = 0}$

ii) Show that $I-A$ is nonsingular

$Re(\lambda) = 0 \Rightarrow Ax = x \Rightarrow x = 0$

$\Rightarrow (I-A)x = 0$ only if $x = 0$

therefore $I-A$ is nonsingular

iii) B is orthogonal if $B^{-1} = B^T$

$BB^{-1} = BB^T = I$

$B = (I-A)^{-1}(I+A)$

$(I+A)^{-1}$ exists by the same reasoning in part ii

$B^T = \left[(I-A)^{-1}(I+A)\right]^T$

$= (I+A)^T((I-A)^{-1})^T$ inverse and transpose commute

$= (I+A^T)(I-A^T)^{-1}$

$= (I-A)(I+A)^{-1}$

$BB^T = (I-A)^{-1}(I+A)(I-A)(I+A)^{-1}$ $(I+A)(I-A) = (I-A)(I+A)$

$= (I-A)^{-1}(I-A)(I+A)(I+A)^{-1}$

$= I$ ∎

Problem

Describe an orthogonal transformation that will diagonalize

$A = \begin{bmatrix} 11 & 4 \\ 4 & 5 \end{bmatrix}$

Problem Identify the eigenfunctions and eigenvalues of the boundary value problem

$$\left(\frac{1}{x} y'(x)\right)' + \frac{\lambda}{x} y(x) = 0 , \quad 0 < x < 1$$

$$\lim_{x \to 0^+} \frac{y}{x} \text{ Finite} , \quad 2y(1) - y'(1) = 0$$

From a past assignment, $y_n = cx J_1(\sqrt{\lambda} x)$, $\lambda \neq 0$

when $\cancel{X}(x) = 0$: $\left(\frac{1}{x} y'(x)\right)' = 0$

$\Rightarrow \frac{1}{x} y'(x) = a \quad \Rightarrow \quad y(x) = a\frac{x^2}{2} + b$

$\lim_{x \to 0^+} \frac{X(x)}{x}$ Finite $\Rightarrow$ $b = 0$

$2y(1) - y'(1) = 2\left(\frac{a}{2}\right) - a = 0 \Rightarrow$ let $a = 2$

then when $\lambda = 0$, $\cancel{y_0(x) = \frac{x^2}{2}}$ $y_0(x) = x^2$

Hence applying boundary condition at $x = 1$

$\sqrt{\lambda}\, J_0(\sqrt{\lambda}) - 2 J_1(\sqrt{\lambda}) = 0$ defines the other eigenvalues.

Consider the boundary/eigenvalue problem

$y'' + 2y' + \lambda x y = 0$, $0 < x < 1$, $y(0) = y'(1) = 0$

(a) convert it to a Storm-Liouville problem

(b) Determine the weight function $r(x)$ such that

$$\int_0^1 r(x)\, y_n(x)\, y_m(x) = 0$$

(c) Prove without explicitly finding them, that all eigenvalues are real and positive.

Sturm-Liouville standard form

$$\left(p(x)\,y'(x)\right)' + \left(\lambda r(x) - q(x)\right)y(x) = 0$$

$$p(x)y''(x) + p'(x)y'(x) + \left(\lambda r(x) - q(x)\right)y(x) = 0$$

$$\Rightarrow y'' + \frac{p'(x)}{p(x)}y' + \left(\lambda\frac{r(x)}{p(x)} - \frac{q(x)}{p(x)}\right)y(x) = 0$$

$$y'' + 2y' + \lambda x y = 0$$

$$\Rightarrow \frac{p'(x)}{p(x)} = 2 \quad \Rightarrow \quad p' = 2p \quad \Rightarrow \quad p = e^{2x}$$

Multiply through by e^{2x}

$$e^{2x}y'' + 2e^{2x}y' + \lambda x y = 0$$

$$\cancel{\left(e^{2x}y(x)\right)' + \lambda x y = 0} \quad \Rightarrow \quad \left(e^{2x}y'(x)\right)' + \left(\lambda x e^{2x}\right)y(x) = 0$$

hence $\boxed{r(x) = xe^{2x}}$

use complex inner product

$$\int_0^1 \left[\left(e^{2x}y'\right)'\bar{y} + \lambda x e^{2x}y\bar{y}\right]dx = 0$$

$$\left[e^{2x}y'\bar{y}\right]_0^1 - \int_0^1 e^{2x}y'\bar{y}'\,dx + \lambda\int_0^1 xe^{2x}|y|^2dx = 0$$

since $y(0) = 0 \Rightarrow \bar{y}(0) = 0$
hence BC terms drop out

$$\Rightarrow -\int_0^1 e^{2x}|y'|^2dx + \lambda\int_0^1 xe^{2x}|y|^2dx = 0$$

$\Rightarrow$ $\boxed{\lambda \text{ must be real}}$

IF $\lambda = 0$ is an eigenvalue then $|y'|^2 = 0$

$\Rightarrow y(x) = \text{constant}$ so $y(0) = 0 \Rightarrow y(x) = 0$

so $\lambda = 0$ is NOT an eigenvalue

so we have shown that $\lambda \neq 0$

$$\Rightarrow \lambda = \frac{\int_0^1 e^{2x}|y'|^2 dx}{\int_0^1 x e^{2x}|y|^2 dx} > 0$$

The entropy of a certain gas might be defined as

$$H = -\int_{-\infty}^{\infty} F \log F \, du$$

where ~~F(u)~~ $F = F(u) \geq 0$ is the velocity distribution ~~which maximizes~~ function. Writing $F = y^2$ for convenience determine $F(u)$ which maximizes H & for prescribed values of "density" and "temperature"

$$\int_{-\infty}^{\infty} F(u) = 1 \quad , \quad \int_{-\infty}^{\infty} F u^2 du = 1$$

Let $F = y^2$:
$$H = -\int_{-\infty}^{\infty} y^2 \log y^2 \, du = -2\int_{-\infty}^{\infty} y^2 \log y \, du$$

$$I_1 = \int_{-\infty}^{\infty} y^2 du = 1 \quad , \quad I_2 = \int_{-\infty}^{\infty} y^2 u^2 du = 1$$

Need two Lagrange multipliers λ_1, λ_2

$$\delta F = \delta\left(H + \lambda_1 I_1 + \lambda_2 I_2\right) = 0$$

Euler-Lagrange equations (notice: no dependence on y')

$$\frac{d}{dy}\left(-2y^2 \log y + \lambda_1 y^2 + \lambda_2 y^2 u^2\right) = 0$$

$$\Rightarrow -4y \log y - \frac{2y^2}{y} + 2\lambda_1 y + 2\lambda_2 y u^2 = 0$$

$$y \neq 0 \, , \quad -4 \log y - 2 + 2\lambda_1 + 2\lambda_2 u^2 = 0$$

$$\Rightarrow \log y = \frac{-1}{2} + \frac{\lambda_1}{2} + \frac{\lambda_2}{2} u^2$$

$$\Rightarrow \boxed{y = e^{+\left(-\frac{1}{2} + \frac{\lambda_1}{2} + \frac{\lambda_2}{2} u^2\right)}}$$

The optimal function is an exponential function

$$y = e^{\left(-\frac{1}{2} + \frac{\lambda_1}{2} + \frac{\lambda_2}{2} v^2\right)}$$

We require $\int_{-\infty}^{\infty} y^2 dv = 1$ and $\int_{-\infty}^{\infty} y^2 v^2 dv = 1$

$$y^2 = e^{-1+\lambda_1+\lambda_2 v^2} \Rightarrow \int_{-\infty}^{\infty} e^{\lambda_1 - 1 + \lambda_2 v^2} dv = 1$$

requires $\lambda_2 = -\nu_2^2 < 0$ for convergence

$$\int_{-\infty}^{\infty} y^2 v^2 dv = \int v^2 e^{\lambda_1 - 1 - \nu_2^2 v^2} dv = 1$$

Calculate the smallest value of k^2 st

$$\int_0^1 (y')^2 dx + (y(1))^2 \geq k^2 \int_0^1 y^2 dx \quad \forall \; y \in C^1[0,1], \; y(0) = 0$$

Minimize $\dfrac{\int_0^1 (y')^2 dx + (y(1))^2}{\int_0^1 y^2 dx}$ where $y(0) = 0$

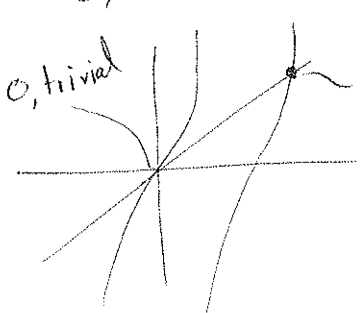
Euler-Lagrange

$$\delta\left[\int_0^1 (y')^2 dx + (y(1))^2 - \lambda \int_0^1 y^2 dx\right] = 0$$

$\Rightarrow \; y'' + \lambda y = 0, \quad y'(1) + y(1) = 0, \quad y(0) = 0$

$\Rightarrow \; y = c \sin(\sqrt{\lambda} x) = c \sin kx$

$\Rightarrow \; k \cos k + \sin k = 0 \Rightarrow k + \tan k = 0$



$k_{min} \simeq 2{,}029 \Rightarrow k^2_{min} = 4{,}116$