MA 6792 8/26/03

Mathematical Fluid Mechanics

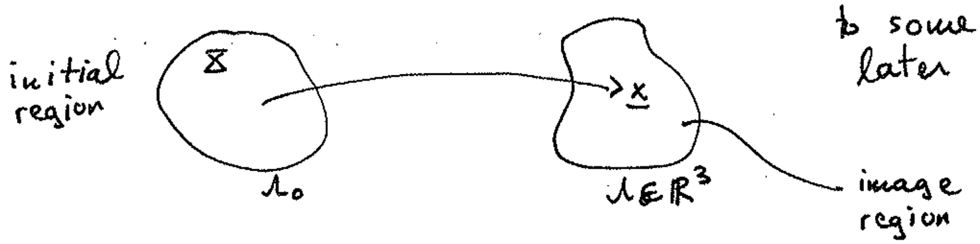
1) homeworks - from texts
2) midterm
3) Final exam
4) project

What is a Fluid?

consider a continuum:
"the idea is that you have points in a region"

$$\underline{\underline{X}} \in \Omega_0 \subset \mathbb{R}^3$$

"each pt in some starting region is mapped to some point in some later region"



$\underline{x} = f(\underline{X}, t)$ mapping from Ω_0 to $\Omega(t)$

a Fluid "forgets" its original configuration

Geometry, $\mathbb{R}^3$, Cartesian Tensors (not Generalized tensors)

consider two different bases, $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ and $(\underline{\bar{e}}_1, \underline{\bar{e}}_2, \underline{\bar{e}}_3)$

and $\underline{v} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3$

$$\underline{v} \cdot \underline{\bar{e}}_1 = \bar{v}_1 = v_1(\underline{e}_1, \underline{\bar{e}}_1) + v_2(\underline{e}_2, \underline{\bar{e}}_1) + v_3(\underline{e}_3, \underline{\bar{e}}_1)$$

$$\underline{v} \cdot \underline{\bar{e}}_j = \bar{v}_j = \sum_{i=1}^{3} v_i(\underline{e}_i, \underline{\bar{e}}_j)$$

$$\underline{v} \cdot \underline{e}_i = v_i = \sum_{j=1}^{3} \bar{v}_j(\underline{\bar{e}}_j, \underline{e}_i) = \sum_{j=1}^{3} \bar{v}_j(\underline{e}_i, \underline{\bar{e}}_j) = \text{change of coordinate "matrix}$$

and $\underline{e}_i \cdot \underline{\bar{e}}_j = |\underline{e}_i||\underline{\bar{e}}_j| \cos\beta_{ij} = \cos\beta_{ij}$

$\Rightarrow$ β_{ij} is the angle between the i and j axes

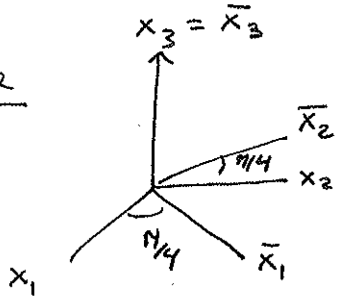
$\Rightarrow$ a vector is a rule for changing coordinates

$\bar{v}_j = \sum_{i=1}^{3} l_{ij} v_i$ For any two coordinate systems.

where $l_{ij} = \cos \beta_{ij} = \underline{e}_i \cdot \underline{\bar{e}}_j$

Rule: for any coordinate system, $v_1 = 1$, $v_2 = 0$, $v_3 = 0$

Example



compute l_{ij}

For $\underline{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

everything with subscript 3 is a zero

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\bar{v}_1 = 1 = v_1 l_{11} + v_2 l_{21} + v_3 l_{31} \neq 1$, Fails

$\bar{v}_2 = 0 = v_1 l_{12} + v_2 l_{22} + v_3 l_{32} = 1/\sqrt{2}$, Fails

$\bar{v}_3 = 0$ fails

Cartesian Tensor Shortcuts

$\boxed{1}$ $\underline{v} = \sum_{i=1}^{3} v_i e_i \longrightarrow \boxed{\underline{v} = v_i}$ (Notation, drop the sum, drop the unit vector)

$\boxed{2}$ $\bar{v}_j = \sum l_{ij} v_i \longrightarrow \boxed{\bar{v}_j = l_{ij} v_i}$ (Notation, drop the sum)
For repeated subscripts

Example

$v_i w_i = v_1 w_1 + v_2 w_2 + v_3 w_3$ – inner product

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we have $v_i = l_{ij}\bar{v}_j = l_{ik}\bar{v}_k$

$\bar{v}_j = l_{ij} v_i$

substitute first into second, take care with indeces

$$\rightarrow \quad \bar{v}_j = l_{ij}(l_{ik}\bar{v}_k) = (l_{ij}\, l_{ik})\,\bar{v}_k = \delta_{jk}\bar{v}_k$$

$$= \sum \delta_{j1}\bar{v}_1 + \delta_{j2}\bar{v}_2 + \delta_{j3}\bar{v}_3$$

$$\rightarrow \quad \boxed{l_{ij}\, l_{ik} = \delta_{jk}} \rightarrow \boxed{\sum \cos\beta_{ij}\cos\beta_{ik} = \begin{cases} 1, & j=k \\ 0, & j\neq k \end{cases}}$$

where $\delta_{jk} = \begin{cases} 1, & j=k \\ 0, & j\neq k \end{cases}$

notice $l_{ij}\, l_{kj} = \delta_{ik}$

<u>vector product</u>, $\underline{v}\times\underline{w} = (v_2w_3 - v_3w_2)\,\underline{e}_1 + (v_3w_1 - v_1w_3)\,\underline{e}_2 + (v_1w_2 - v_2w_1)\,\underline{e}_3$

$\underline{a} = \underline{v}\times\underline{w}$

$a_i = \epsilon_{ijk} v_j w_k$, ϵ_{ijk} is the alternating tensor, (Levi-civita)

$$\boxed{\epsilon_{ijk} = \begin{cases} 0 & \text{if any subscripts are repeated} \\ 1 & \text{if } ijk \text{ cyclic } (123, 231, 312) \\ -1 & \text{if } ijk \text{ anticyclic } (213, 321, 132) \end{cases}}$$

example, $\underline{u}\times(\underline{v}\times\underline{w}) = \epsilon_{imj}(\epsilon_{jkl} v_k w_l u_m)$

<u>ϵ-δ rule</u>: $\epsilon_{ij\underset{\uparrow}{k}}\,\epsilon_{lm\underset{\uparrow}{k}} = (\delta_{il}\,\delta_{jm} - \delta_{jl}\,\delta_{im})$

summed

"1st's" "2nd s" "inners" "outers"

"bac cab"

$$\underline{A} \times (\underline{B} \times \underline{C}) = \underline{B}(\underline{A}\cdot\underline{C}) - \underline{C}(\underline{A}\cdot\underline{B})$$

$$\epsilon_{ijk} A_j \epsilon_{klm} B_l C_m = \epsilon_{ijk} \epsilon_{klm} A_j B_l C_m$$ permute 2nd ϵ, cyclically

$$= \epsilon_{ijk} \epsilon_{lmk} A_j B_l C_M$$ NOW APPLY $\epsilon\delta$ rule

$$= (\delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}) A_j B_l C_m$$

$$= \underbrace{\delta_{il}\delta_{jm} A_j B_l C_m} - \underbrace{\delta_{jl}\delta_{im} A_j B_l C_m}$$

j, l, m are summed
i is free - the components

$$= \delta_{il} A_m B_l C_m - \delta$$

$$= (A_m C_m) B_l - (A_j B_j) C_i$$

Homework

Aris, pg 17 # 2.32.2, 2.32.3, 2.32.4
pg 22 # 2.42.1 (Prove directly)
pg 24 # 2.44.1
pg 29 # 2.61.2

Office Hours

Th 10-12, F 1-2

Class schedule for next week

Tu 9/2, 2-4
Fr 9/5, 8-10, 2-4

remember, $l_{ij} = \cos\beta_{ij} = \underline{e}_i \cdot \underline{\bar{e}}_j$

$$\bar{x}_j = l_{ij} x_i$$

$$x_i = l_{iK} \bar{x}_K$$

Linear transformation

$$\underline{v} = F(\underline{u})$$

$$F(c_1 \underline{u}_1 + c_2 \underline{u}_2) = c_1 F(\underline{u}_1) + c_2 F(\underline{u}_2)$$

if $\underline{u} = u_i \underline{e}_i$

then
$$
\begin{aligned}
F(\underline{u}) &= F(u_1 \underline{e}_1 + u_2 \underline{e}_2 + u_3 \underline{e}_3) \\
&= u_1 F(\underline{e}_1) + u_2 F(\underline{e}_2) + u_3 F(\underline{e}_3) \\
&= u_1 \underline{A}_1 + u_2 \underline{A}_2 + u_3 \underline{A}_3 \\
&= u_1 \begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \end{pmatrix} + u_2 \begin{pmatrix} A_{21} \\ A_{22} \\ A_{23} \end{pmatrix} + u_3 \begin{pmatrix} A_{31} \\ A_{32} \\ A_{33} \end{pmatrix} \\
&= \cancel{[u_1\ u_2\ u_3] \begin{bmatrix} A_{11} \\ A_{12} \\ A \end{bmatrix}} \\
&= \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}
\end{aligned}
$$

or essentially, a linear operator acting on a vector is equivalent to a matrix multiplication let $v_j = A_{ij} u_i$

$\bar{v}_k = \bar{A}_{\ell k} \bar{u}_k$

then $\ell_{mk} = \bar{A}_{\ell k} \ell_{n\ell} u_n \rightarrow \underset{\shortparallel}{\ell_{jk} \ell_{mk}} v_m = \ell_{jk} \underset{\shortparallel}{\bar{A}_{\ell k}} \ell_{n\ell} u_n$

$$\underset{\shortparallel}{\delta_{jm} v_m} = \ell_{jk} \ell_{n\ell} \bar{A}_{\ell k} u_n$$

$$v_j = A_{nj} u_n \quad \Rightarrow \boxed{A_{nj} = \ell_{jk} \ell_{n\ell} \bar{A}_{\ell k}}$$

$$\Updownarrow$$

$$\boxed{\bar{A}_{pq} = \ell_{rp} \ell_{sq} A_{rs}}$$

Def

a 2nd order tensor is a quantity A_{ij} which transforms according

$$\boxed{\bar{A}_{pq} = \ell_{ip} \ell_{jq} A_{ij}}$$

Example: if $\underline{w}$ is a vector and $\underline{z}$ is a vector then $w_i z_j$ is a second order tensor.

$$\overline{w_k z_\ell} = \bar{w}_k \bar{z}_\ell = \ell_{ik} w_i \ell_{j\ell} z_j = \ell_{ik} \ell_{j\ell} (w_i z_j)$$

Def

An nth order tensor is a quantity

$A_{i_1 i_2 \cdots i_n}$ (n indeces)

which transforms according to

$$\bar{A}_{j_1 \cdots j_n} = \ell_{i_1 j_1} \ell_{i_2 j_2} \ell_{i_3 j_3} \cdots \ell_{i_n j_n} A_{i_1 \cdots i_n}$$

notice, $A_{ij} v_k$ contains 27 numbers

if A_{ij} is a 2nd order tensor and v is a vector

then $A_{ij} v_k$ are 3rd order tensors

Quotient Rule

IF in an expression of the form

$$A_{i_1 \cdots i_n} = B_{i_1 \cdots i_n j_1 \cdots j_m} C_{j_1 \cdots j_m}$$

where $A_{i_1 \cdots i_n}$ is an $n^{\underline{th}}$ order tensor and $C_{j_1 \cdots j_m}$ is an $m^{\underline{th}}$ order tensor then $B_{i_1 \cdots i_n j_1 \cdots j_m}$ is an $n+m^{th}$ order tensor.

Contraction

suppose $A_{i_1 \cdots i_n}$ is an $n^{\underline{th}}$ order tensor then

$A_{i_1 \cdots i_{m-1} j i_{m+1} \cdots i_{p-1} j i_{p+1} \cdots i_n}$

(where we have replaced two sobscripts by j) is a tensor of order $n-2$. It is called a contraction.

example, $v_i v_i$ is a zeroth order tensor, ie a scalar

$A_{ij} v_j$ is a vector

Determinants

recall cross product of two vectors $(\underline{b} \times \underline{c})_i = \epsilon_{ijk} b_j c_k$

notationally conveniant way of calculating cross product

is $(\underline{b} \times \underline{c}) = \begin{vmatrix} \underline{e_1} & \underline{e_2} & \underline{e_3} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

But now for a general determinant,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \underline{a} \cdot (\underline{b} \times \underline{c}) = \epsilon_{ijk} a_i b_j c_k$$

Now consider

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \epsilon_{ijk} A_{1i} A_{2j} A_{3k} = \det \underline{\underline{A}}$$

Isotropic

$A_{i_1 \cdots i_n}$ is isotropic if $\bar{A}_{i_1 \cdots i_n} = A_{i_1 \cdots i_n}$ no matter what coordinate system is used.

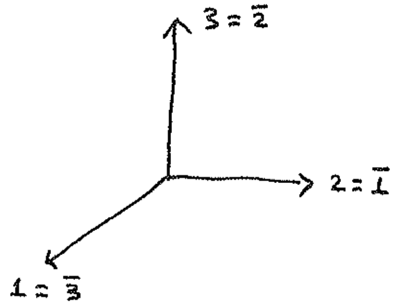
Example δ_{ij}

$$\bar{\delta}_{k\ell} = \ell_{ik} \ell_{j\ell} \delta_{ij} = \ell_{ik} \ell_{i\ell} = \delta_{k\ell}$$

therefore δ_{ij} is an isotropic 2nd order tensor

Question, is this the only 2nd order tensor?

$$\bar{A}_{pq} = A_{pq} = \ell_{ip} \ell_{jq} A_{ij}$$



$\ell_{11} = \underline{e}_1 \cdot \bar{\underline{e}}_1 = \underline{e}_1 \cdot \underline{e}_2 = 0$

$\ell_{12} = \underline{e}_1 \cdot \bar{\underline{e}}_2 = \underline{e}_1 \cdot \underline{e}_3 = 0$

$\ell_{13} = \underline{e}_1 \cdot \bar{\underline{e}}_3 = \underline{e}_1 \cdot \underline{e}_1 = 1$

$\ell_{21} = 1$

$\ell_{22} = 0$

$\ell_{23} = 0$

$\ell_{31} = 0$

$\ell_{32} = 1$

$\ell_{33} = 0$

so $\bar{A}_{11} = \ell_{i1} \ell_{j1} A_{ij} = 1 \cdot 1 \cdot A_{22} \rightarrow \cancel{\bar{A}_{11} = A_{22}}$ $A_{11} = A_{22}$

$\bar{A}_{12} = \ell_{i1} \ell_{j2} A_{ij} = A_{23} \rightarrow A_{12} = A_{23}$

$\bar{A}_{13} = \ell_{i1} \ell_{j3} A_{ij} = A_{21} \rightarrow A_{13} = A_{21}$

$\bar{A}_{21} = \ell_{i2} \ell_{j1} A_{ij} = A_{32} \rightarrow A_{21} = A_{32}$

$\bar{A}_{22} = \ell_{i2} \ell_{j2} A_{ij} = A_{33} \rightarrow A_{22} = A_{33}$

$\bar{A}_{23} = A_{31} = A_{23}$

$\bar{A}_{31} = A_{12} = A_{31}$

$\bar{A}_{32} = A_{13} = A_{31}$

$\bar{A}_{33} = A_{11} = A_{33}$

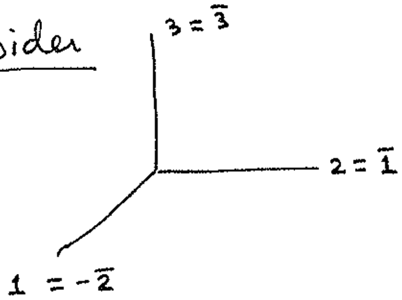
we have 2 seperate sets of equations

$$A_{11} = A_{22} = A_{33}$$

$$A_{12} = A_{31} = A_{32} = A_{21} = A_{23} = A_{13}$$

hence the matrix has the form $\begin{bmatrix} A & B & B \\ B & A & B \\ B & B & A \end{bmatrix}$

Now consider



then $l_{11} = 0$

$l_{12} = \underline{e}_1 \cdot \underline{\bar{e}}_2 = \underline{e}_1 \cdot (-\underline{e}_1) = -1$

$l_{13} = 0$

$l_{21} = 1$

$l_{22} = 0$

$l_{23} = 0$

$l_{31} = 0$

$l_{32} = 0$

$l_{33} = 1$

then $\bar{A}_{11} = l_{i1} l_{j1} A_{ij} = A_{22} = A_{11}$

$\bar{A}_{12} = l_{i1} l_{j2} A_{ij} = -A_{21} = A_{12}$

$\Rightarrow$ the off diagonal elements must equal zero

therefore we now have $\begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{bmatrix}$, so the only 2nd order isotrop tensor is a constant times the identity.

Now show $A \bar{\delta}_{kl} = A \delta_{kl}$

What is an isotropic vector? / Is there an isotropic vector?

- the zero vector is the only isotropic vector

3rd order

$A \epsilon_{ijk}$ is an isotropic 3rd order tensor <u>without reflections</u>

4th order , (A_{ijkl})

$\delta_{ij}\delta_{kl}$ is isotropic (remember, $A_{12} \neq A_{21}$)

$\delta_{ik}\delta_{jl}$

$\delta_{il}\delta_{kj}$

$A\delta_{ij}\delta_{kl} + B\delta_{ik}\delta_{jl} + C\delta_{il}\delta_{jk}$, where A, B, C scalars

$$\begin{gathered} A\delta_{ij}\delta_{kl} + B\delta_{ik}\delta_{jl} + C\delta_{il}\delta_{jk} \\ \shortparallel \\ \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \nu(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \end{gathered}$$

where $\lambda = A$, $\mu = \frac{B+C}{2}$, $\nu = \frac{B-C}{2}$

this is the most general 4th order isotropic tensor with or without reflections.

Tensor Calculus

vector , $\underline{v}(t)$

$$\underline{a}(t) = \frac{d\underline{v}}{dt} \quad , \quad a_i = \frac{dv_i}{dt}$$

$\underline{x}(t)$ - position

$\frac{d\underline{x}(t)}{dt} = \underline{v}(t)$, velocity

$\frac{d\underline{v}(t)}{dt} = \underline{a}(t)$, acceleration

$$\underline{v}(t) = \sum v_i(t)\, \underline{e}_i \quad , \quad \frac{d\underline{v}(t)}{dt} = \sum \frac{dv_i}{dt} \underline{e}_i$$

→ with this convention we assume the coordinate system does not move

$$\frac{d\bar{v}_j}{dt} = l_{ij} \frac{dv_i}{dt}$$

Now consider $F(\underline{x})$

$$\frac{\partial F}{\partial x_i} \quad , \quad \nabla F = \sum_{i=1}^{3} \frac{\partial F}{\partial x_i} \underline{e}_i$$

notation , $\frac{\partial F}{\partial x_i} \rightarrow F_{,i}$ "F comma i"

is ∇F a vector ?. yes
but to prove, we want to show that

$$\overline{\frac{\partial F}{\partial x_j}} = l_{ji} \frac{\partial F}{\partial x_i}$$

$F(\underline{x})$	$\overline{F(\underline{x})}$
$F(x_i)$	$\bar{F}(\bar{x}_j)$

show $\frac{\partial \bar{F}}{\partial \bar{x}_j} = l_{ij} \frac{\partial F}{\partial x_i}$

$$\begin{gathered} \bar{F}\big(\bar{x}_1(x_1,x_2,x_3), \bar{x}_2(x_1,x_2,x_3), \bar{x}_3(x_1,x_2,x_3)\big) \\ \| \\ F(x_1, x_2, x_3) \end{gathered}$$

$$F(x_1, x_2, x_3) = \bar{F}\left(\bar{x}_1(x_1,x_2,x_3), \bar{x}_2(x_1,x_2,x_3), \bar{x}_3(x_1,x_2,x_3)\right)$$

$$\frac{\partial F}{\partial x_i} = \frac{\partial \bar{F}}{\partial \bar{x}_1}\frac{\partial \bar{x}_1}{\partial x_i} + \frac{\partial \bar{F}}{\partial \bar{x}_2}\frac{\partial \bar{x}_2}{\partial x_i} + \frac{\partial \bar{F}}{\partial \bar{x}_3}\frac{\partial \bar{x}_3}{\partial x_i}$$

$$\frac{\partial F}{\partial x_i} = \frac{\partial \bar{F}}{\partial \bar{x}_j}\frac{\partial \bar{x}_j}{\partial x_i}$$

recall, $\bar{x}_j = l_{ij} x_i$

then $\dfrac{\partial \bar{x}_j}{\partial x_i} = l_{ij}$

$$\boxed{\frac{\partial F}{\partial x_i} = l_{ij}\frac{\partial \bar{F}}{\partial \bar{x}_j}}$$

, which shows that the gradiant of a scalar valued Function is a vector because it obeys the "transformation" rule

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Now consider $\nabla \underline{v}$ - a 2nd order tensor

$$\rightarrow \nabla \underline{v} = \left(\sum_{i=1}^{3} \underline{e}_i \frac{\partial}{\partial x_i}\right)\left(\sum_{j=1}^{3} \underline{e}_j v_j\right)$$

$$\nabla \underline{v} = \sum\sum \underline{e}_i\, \underline{e}_j \frac{\partial v_j}{\partial x_i}$$

gradient of a vector

||

2nd order tensor

$$\rightarrow (\nabla \underline{v})_{ij} = \frac{\partial v_j}{\partial x_i} = v_{j,i}$$

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gradient of a 2nd order tensor = 3rd order tensor

$T_{ij,k}$

$\nabla \underline{\underline{T}}$

$$T_{ij,j} = T_{1j,j}\,\underline{e}_1 + T_{2j,j}\,\underline{e}_2 + T_{3j,j}\,\underline{e}_3$$

$$= (T_{11,1} + T_{12,2} + T_{13,3})\,\underline{e}_1 + (T_{21,1} + T_{22,2} + T_{23,3})\,\underline{e}_2 + \ldots$$

$$= \left(\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3}\right)\underline{e}_1 + \left(\frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3}\right)\underline{e}_2 + \ldots$$

→ $T_{ij,j}$ = vector?

$$(\varphi_{,ii}) = (\varphi_{,i})_{,i}$$

$$= \varphi_{,11} + \varphi_{,22} + \varphi_{,33}$$

$$= \frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2} + \frac{\partial^2\varphi}{\partial x_3^2}$$

$$= \Delta\varphi$$

Div of vector

$$\nabla\cdot\underline{v} = v_{i,i}$$

Curl of vector

$$\nabla\times\underline{v} = \epsilon_{ijk}\, v_{k,j}$$

$\nabla\cdot(\nabla\times\underline{v}) = (\epsilon_{ijk}\, v_{k,j})_{,i} = \epsilon_{ijk}\, v_{k,ji}$, with the assumption that ϵ_{ijk} is independent of location, in the Cartesian coordinates

$= \epsilon_{ijk}\, v_{k,ij}$ ↖ if v has continuous partial derivatives to switch order of derivatives

$$\downarrow$$

$$\epsilon_{ijk}\, v_{k,ij} = \epsilon_{jik}\, v_{k,ji} = \underbrace{-\epsilon_{ijk}\, v_{k,ji} = \epsilon_{ijk}\, v_{k,ji}}$$

"subscript proof" that $\nabla\cdot(\nabla\times\underline{v}) = 0$

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curl - curl - v

$$\nabla\times(\nabla\times\underline{v}) = \epsilon_{mli}(\epsilon_{ijk}\, v_{k,j})_{,l}$$

$$= \epsilon_{mli}\,\epsilon_{ijk}\, v_{k,jl}$$

$$= \epsilon_{mli}\,\epsilon_{jki}\, v_{k,jl}$$ - cyclic permutation

$$= (\delta_{mj}\,\delta_{lk} - \delta_{lj}\,\delta_{mk})\, v_{k,jl}$$

$$= v_{k,mk} - v_{m,jj} = (v_{k,k})_{,m} - v_{m,jj} = \nabla(\nabla\cdot\underline{v}) - \nabla^2\underline{v}$$

$$\nabla \times (\nabla \varphi) = 0$$

Integrals - vector Theorems

$$\oiint_{\partial V} \underset{\substack{\uparrow \\ \text{unit normal} \\ \text{to surface}}}{\underline{n}} F \, ds = \iiint_V \nabla F \, dv ,$$ divergence theorem (Gauss's theorem)

in tensor notation,

$$\oiint n_i F \, ds = \iiint F_{,i} \, dv$$

For a vector,

$$\oiint n_i v_i \, ds = \iiint v_{i,i} \, dv$$

Stokes' Theorem

$$\oint_C \underset{\substack{\uparrow \\ \text{unit tangent} \\ \text{vector to C}}}{\underline{t}} \cdot \underline{v} \, dl = \iint_S \underset{\substack{\uparrow \\ \text{normal to S}}}{\underline{n}} \cdot \nabla \times \underline{v} \, dS$$

Green' Theorem

$$\oint \underline{v} \cdot \underline{t} \, dl = \iint (\nabla \times \underline{v})_3 \, dS$$

$$\oint (v_1 dx_1 + v_2 dx_2) = \iint \left(\frac{\partial v_2}{\partial x_2} - \frac{\partial v_1}{\partial x_2} \right) dS$$

$$\iiint F(\underline{x})\,dv = \iiint F(x_1,x_2,x_3)\,dx_1\,dx_2\,dx_3$$

remember, given $d\underline{x}^{(1)}, d\underline{x}^{(2)}, d\underline{x}^{(3)}$ then

$$dV = d\underline{x}^{(1)} \cdot \left(d\underline{x}^{(2)} \times d\underline{x}^{(3)}\right)$$

$$= \epsilon_{ijk}\, dx_i^{(1)}\, dx_j^{(2)}\, dx_k^{(3)}$$

consider $\bar{\underline{x}} = \bar{\underline{x}}(x_1, x_2, x_3)$

$$d\underline{x} = \bar{\underline{x}}(x_1+dx_1, x_2, x_3) - \bar{\underline{x}}(x_1, x_2, x_3)$$

$$\cong \bar{\underline{x}}(x_1, x_2, x_3) + dx_1 \frac{\partial \bar{\underline{x}}}{\partial x_1} - \bar{\underline{x}}(x_1, x_2, x_3)$$

$$\rightarrow d\bar{\underline{x}}^{(1)} = dx_1 \frac{\partial \bar{\underline{x}}}{\partial x_1}, \quad d\bar{\underline{x}}^{(2)} = dx_2 \frac{\partial \bar{\underline{x}}}{\partial x_2}, \quad d\bar{\underline{x}}^{(3)} = dx_3 \frac{\partial \bar{\underline{x}}}{\partial x_3}$$

$$\rightarrow d\bar{v} = \epsilon_{ijk}\left(dx_2 \frac{\partial \bar{x}_i}{\partial x_1}\right)\left(dx_2 \frac{\partial \bar{x}_j}{\partial x_2}\right)\left(dx_3 \frac{\partial \bar{x}_k}{\partial x_3}\right)$$

$$= dx_1\,dx_2\,dx_3\ \epsilon_{ijk}\left(\frac{\partial \bar{x}_i}{\partial x_1} \frac{\partial \bar{x}_j}{\partial x_2} \frac{\partial \bar{x}_k}{\partial x_3}\right)$$

$$= \left(\det \frac{\partial \bar{x}_j}{\partial x_i}\right) dx_1\,dx_2\,dx_3$$

$$= J(\bar{\underline{x}}, \underline{x})\,dx_1\,dx_2\,dx_3$$

if $\bar{x}_j = l_{ij} x_i$, $\dfrac{\partial \bar{x}_j}{\partial x_i} = l_{ij}$

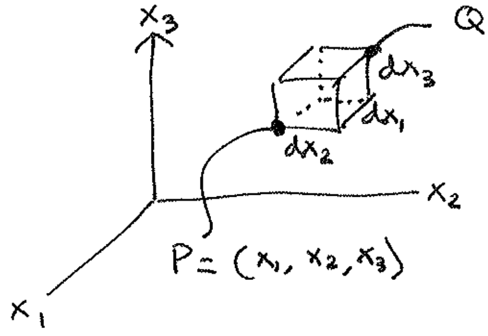
$\Rightarrow \det l_{ij} = \pm 1$, For Cartesian system

$$\Rightarrow \iiint \bar{F}(\bar{\underline{x}})\,d\bar{v} = \iiint \bar{F}(\underline{x})\,\cancel{J}^{1}\,d\bar{v}$$

$\Rightarrow$ integral of nth order tensor is an nth order tensor

$$\iint_{\partial V} \underline{n}\cdot\underline{v}\, dS = \iiint_{V} \nabla\cdot\underline{v}\, dV$$

$Q = (x_1+dx_1\,,\ x_2+dx_2\,,\ x_3+dx_3)$



"handwaving" proof of divergence theorem

$$\iint_{\partial V} = \iint_{S_1} + \iint_{S_2} + \cdots + \iint_{S_6}$$

$$\rightarrow \iint_{\partial V} = \underbrace{\int_{x_2}^{x_2+dx_2}\int_{x_3}^{x_3+dx_3} (-\underline{e}_1)\cdot\underline{v}(x_1, x_2', x_3')\, dx_2'\, dx_3'}_{\text{"backside"}}$$

$$+ \underbrace{\int_{x_2}^{x_2+dx_2}\int_{x_3}^{x_3+dx_3} (+\underline{e}_1)\cdot\underline{v}(x_1+dx_1, x_2', x_3')\, dx_2'\, dx_3'}_{\text{"front side"}} + \iint \text{4 more surface integrals}$$

$$= \iint v_1(x_1+dx_1, x_2', x_3') - v_1(x_1, x_2', x_3')\, dx_2'\, dx_3' + \iint \text{2 more similar integrals}$$

using Taylor's theorem

$$\cong \iint dx_1\, \frac{\partial v_1}{\partial x_1}(x_1, x_2', x_3')\, dx_2'\, dx_3' + \cdots$$

$$= \left(\iint \frac{\partial v_1}{\partial x_1}\, dx_2\, dx_3 \right) dx_1 + \cdots$$

$$= \iiint \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right) dx_1\, dx_2\, dx_3$$

Note, $\iint \underline{v}\cdot\underline{e}_1\, dx_2 dx_3$ can be interpetted as a Flux through the area $dx_2 dx_3$

Kinematics

"dealing with motions"

in Fluids, what you think can be measured is - the velocity field $\underline{v}(\underline{x},t)$

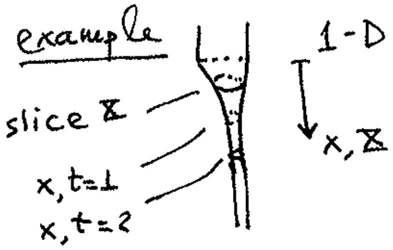
continuum mechanics idea, find out where any particle is at an instant of time

$$\underline{x} = \underline{\chi}(\underline{X}, t) \quad , \quad \underline{X} \in \Omega$$

↑ vector valued function

↓ location of a fluid "particle" at time t, starting at point $\underline{X}$

example



1-D nozzle or outlet

$$x = \chi(X,t)$$
$$x = X + tX^2$$

at $X = 0$, $x = 0$ for all t

$X = 1$, $x = 1+t$

$X = 2$, $x = 2+4t$

so each slice moves at different speeds

Given x, t, what is $\underline{X}$?

ie, if you know where the fluid is at time t, can you determine where it started from? ie, is the map invertible? Yes, by quadratic formula

$$X = \frac{-1}{2t} \pm \frac{\sqrt{1+4xt}}{2t}$$

two solutions? this is an issue. but there is a negative solution that is physically implausible, so we pick the positive position as the unique inverse, ie

$$X = \frac{-1}{2t} + \frac{\sqrt{1+4xt}}{2t}$$

Given $\underline{x} = \underline{\chi}(\underline{X}, t)$

what is the inverse? $\underline{X} = \underline{\Psi}(\underline{x}, t)$

by inverse, we mean

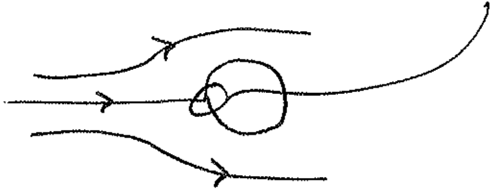
$$\left.\begin{aligned} \underline{x} &= \underline{\chi}(\underline{\Psi}(\underline{x},t), t) \\ \underline{X} &= \underline{\Psi}(\underline{\chi}(\underline{x},t), t) \end{aligned}\right\} \text{ for all } t > 0$$

this implies we can pick a point and follow it back to its "starting" position?

what is criterion for invertible map? non-zero Jacobian, ie

$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} \neq 0 \quad \text{, everywhere at all time}$$

But what about stagnation points?



we don't worry about them

velocity

$\underline{x} = \underline{\chi}(\underline{X}, t)$

$$\underline{V} = \text{velocity} = \frac{\partial \underline{\chi}}{\partial t}(\underline{X}, t) = \underline{V}(\underline{X}, t)$$

so velocity depends on starting point and time

→ Lagrangian coordinates, Material coordinates

so for our example,

$\underline{x} = \underline{X} + t\,\underline{X}^2$

$\underline{V}(\underline{X}, t) = \underline{X}^2$

But we want $\underline{v}(\underline{x},t) = \underline{V}(\underline{\Psi}(\underline{x},t),t)$

- Eulerian coordinates

~~$\underline{v}(\underline{x},t) = \partial V(\underline{X}$~~

For our example, (scalar)

$$v(x,t) = V(X,t)\Big|_{X = \Psi(x,t)}$$

$$v(x,t) = \left(\frac{-1}{2t} + \frac{\sqrt{1+4tx}}{2t}\right)^2$$

at $t=0$, there is a removable singularity

Acceleration

$$\underline{A}(\underline{X},t) = \frac{\partial \underline{V}}{\partial t}\Big|_{\underline{X}} = \frac{\partial^2 \underline{x}}{\partial t}(\underline{X},t) \quad , \text{ Lagrangian}$$

$$\underline{a}(\underline{x},t) = \underline{A}(\underline{\Psi}(\underline{x},t),t) \quad , \text{ Eulerian}$$

note, none of what we've said specifically applies to fluids - this is simply fluid mechanics. Where fluids people and solids people ~~disagree~~ differ starts about here. "Cause solids people work in Lagrangian coordinates, fluids people work in Eulerian coordinates."

in our example (scalar)

$$v(x,t) = \left(\frac{-1}{2t} + \frac{\sqrt{1+4xt}}{2t} \right)^2$$

at $t=1$, $x=1$, $v = \left(-\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^2$

at $t=2$, $x=1$, $v = \left(\frac{-1}{4} + \frac{3}{4}\right)^2$

at same place, different times, fluid accelerates, but if you ride on a chunk of fluid, you experience no acceleration

$$\underline{a}(\underline{x},t) = \underline{A}(\underline{\psi}(\underline{x},t),t)$$

- acceleration of the particle of fluid which is at $\underline{x}$ at time t

$$\underline{V}(\underline{X},t) = \underline{v}(\underline{\chi}(\underline{X},t),t)$$

$$\underline{A}(\underline{X},t) = \frac{\partial \underline{V}}{\partial t}(\underline{X},t)\bigg|_{\underline{X}} = \frac{\partial}{\partial t}\underline{v}(\underline{\chi}(\underline{X},t),t)\bigg|_{\underline{X}}$$

$$= \frac{\partial \underline{v}}{\partial t}\bigg|_{\underline{x}} + \frac{\partial \chi_1}{\partial t}\bigg|_{\underline{X}} \frac{\partial \underline{v}}{\partial x_1}\bigg|_{x_2,x_3 t} + \frac{\partial \chi_2}{\partial t}\bigg|_{\underline{X}} \frac{\partial \underline{v}}{\partial \chi_2}\bigg|_{x_1,x_3,t} + \frac{\partial \chi_3}{\partial t}\bigg|_{\underline{X}} \frac{\partial \underline{v}}{\partial \chi_3}\bigg|_{x_1,x_2,t}$$

$$\rightarrow \quad A_i(X_j,t) = \frac{\partial v_i}{\partial t} + \frac{\partial \chi_j}{\partial t} v_{i;j}$$

$$A_i(X_j,t) = \frac{\partial v_i}{\partial t} + V_j v_{i;j}$$

$$a_i(x_K,t) = \frac{\partial v_i}{\partial t}(x_K,t) + v_j(x_k,t)\, v_{i;j}(x_K,t)$$

$$\rightarrow \quad \boxed{\underline{a} = \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}}$$

$$\underline{a} = \frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}$$

local rate of change of velocity

convective rate of change

Density

$$R(\underline{X},t), \quad \rho(\underline{x},t)$$

$$\frac{\partial R}{\partial t}(\underline{X},t) = \frac{\partial \rho}{\partial t} + \underline{v}\cdot\nabla\rho$$

~~Temperature~~

Material derivative: $\left.\frac{\partial}{\partial t}\right|_{\underline{X}} = \left.\frac{\partial}{\partial t}\right|_{\underline{x}} + \underline{v}\cdot\nabla$

"most important application of chain rule ever"

Back to our example (scalar)

$$x = X + tX^2$$

compute $\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}$

where $v = \left(\frac{-1}{2t} + \frac{\sqrt{1+4xt}}{2t}\right)^2 = \frac{1}{4t^2}\left(-1+\sqrt{1+4xt}\right)^2$

$$\frac{\partial v}{\partial t} = \frac{-1}{2t^3}\left(-1+\sqrt{1+4xt}\right)^2 + \frac{2}{4t^2}\left(-1+\sqrt{1+4xt}\right)\left(\frac{1}{2}(1+4xt)^{-1/2}4x\right)$$

$$\frac{\partial v}{\partial t} = \frac{-1}{2t^3}\left(-1+\sqrt{1+4xt}\right)^2 + \frac{x}{t^2}\frac{\left(-1+\sqrt{1+4xt}\right)}{\sqrt{1+4xt}}$$

$$v = \frac{1}{4t^2}\left(-1+\sqrt{1+4xt}\right)^2$$

$$\frac{\partial v}{\partial x} = \frac{1}{4t^2}\cancel{2}\left(-1+\sqrt{1+4xt}\right)\left(\frac{1}{\cancel{2}}\left(\sqrt{1+4xt}\right)^{-1/2}\cdot 4t\right)$$

$$= \frac{1}{t}\,\frac{\left(-1+\sqrt{1+4xt}\right)}{\sqrt{1+4xt}}$$

$$\boxed{v\frac{\partial v}{\partial x} = \frac{1}{4t^3}\,\frac{\left(-1+\sqrt{1+4xt}\right)^3}{\sqrt{1+4xt}}}$$

$$\frac{\partial v}{\partial t} = \frac{-1}{2t^3}\left(-1+\sqrt{1+4xt}\right)^2 + \frac{x}{t^2}\,\frac{\left(-1+\sqrt{1+4xt}\right)}{\sqrt{1+4xt}}$$

$$= \frac{-\sqrt{1+4xt}\left(-1+\sqrt{1+4xt}\right)^2}{2t^3\sqrt{1+4xt}} + \frac{4x}{4t^3}\,\frac{\left(-1+\sqrt{1+4xt}\right)}{\sqrt{1+4xt}}$$

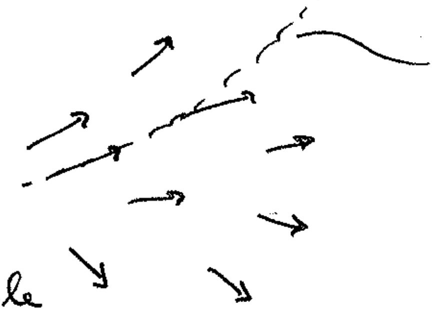
$$= \frac{-2\sqrt{1+4xt}\left(1-2\sqrt{1+4xt}+(1+4xt)\right)}{4t^3\sqrt{1+4xt}} + \frac{4x\left(-1+\sqrt{1+4xt}\right)}{4t^3\sqrt{1+4xt}}$$

$$= \frac{-2\sqrt{1+4xt}+4(1+4xt)-2(1+4xt)^{3/2}-4x+4x\sqrt{1+4xt}}{4t^3\sqrt{1+4xt}}$$

$\underline{x} = \underline{\chi}(\underline{X}, t) =$ particle paths

$\frac{d\underline{x}}{ds} =$ unit vector on curve $\underline{x} = \underline{f}(s)$ tangent to curve

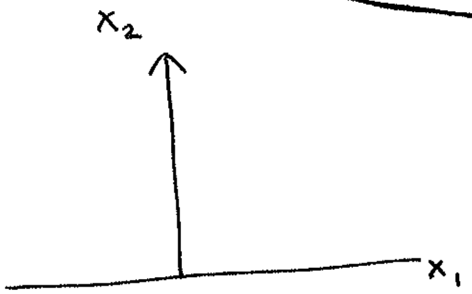
suppose you have a <u>velocity field</u>, at t fixed



<u>streak line</u> - tangent to velocity at time t

$$\frac{d\underline{x}}{ds} = \frac{\underline{v}(\underline{x},t)}{|\underline{v}(\underline{x},t)|}$$

<u>Example</u>



$v_1 = x_1$, $v_2 = -x_2$, $v_3 = 0$, $x_2 > 0$

$\underline{v} = x_1 \underline{e}_1 - x_2 \underline{e}_2$

"this is in Eulerian, spatial coordinates"

$$\underline{a} = \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} = 0 + v_j \frac{\partial v_i}{\partial x_j}$$

$$\rightarrow a_1 = v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} = x_1 \cdot 1 - x_2 \cdot 0 = x_1$$

$$a_2 = v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2} = 0 - x_2(-1) = x_2$$

so the acceleration points radially out and increases in magnitude the further out from origin

if the flow is steady, streaklines = streamlines
" particle paths

$$\frac{d\underline{x}}{dt} = \underline{v}(x)$$

$$\frac{dx_1}{dt} = x_1 \quad , \quad \frac{dx_2}{dt} = -x_2$$

$$x_1 = X_1 e^{t} \qquad x_2 = X_2 e^{-t}$$

$$x_1(0) = X_1 \qquad x_2(0) = X_2$$

$$\left.\begin{aligned} V_1 &= \frac{\partial x_1}{\partial t} = X_1 e^{t} \\ V_2 &= -X_2 e^{-t} \end{aligned}\right\}$$ Lagrangian velocities

"bit of Fluid moving, might well be a Function of time"

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Particle paths, $\left.\begin{aligned} \frac{d\underline{x}}{dt} &= \underline{v}(\underline{x},t) \\ \underline{x}(0) &= \underline{X} \end{aligned}\right\} \underline{x}(\underline{X},t)$

Streamlines, Field tangent to $v(\underline{x},t)$

$$\frac{d\underline{x}}{d\tau} = \underline{v}(\underline{x},t) \,, \quad t \text{ Fixed}$$

$$\underline{x}(0) = \underline{x}_0$$

streaklines: like particle paths, but for Finite time intervals

$$\frac{d\underline{x}}{d\rho} = \underline{v}(\underline{x},t) \quad , \quad \frac{d\rho}{dt} = 1 \quad , \quad \underline{x}(\rho=0) = \underline{x}_0$$

Preview of Coming Attractions

we'll be thinking about such terms as

$$\iiint_V F(\underline{x},t)\,dx_1dx_2dx_3 = \iiint_{V_0} F(\underline{x}(\underset{\substack{\uparrow\\ \text{initial}\\ \text{position}}}{\underline{X}},t),t)\,J\,\underset{\substack{|\\ dV_0 = dX_1 d\underline{X}_2 dX_3}}{dV_0}$$

$$J = \frac{\partial(x_1,x_2,x_3)}{\partial(X_1,X_2,X_3)}$$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_3}{\partial X_1} \\ \frac{\partial x_1}{\partial X_2} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_3}{\partial X_2} \\ \frac{\partial x_1}{\partial X_3} & \frac{\partial x_2}{\partial X_3} & \frac{\partial x_3}{\partial X_3} \end{vmatrix} = \epsilon_{ijk} \frac{\partial x_1}{\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{\partial x_3}{\partial X_3}$$

interested in time derivative of Jacobian

want
$$\left.\frac{\partial J}{\partial t}\right|_{\underline{X}} = \epsilon_{ijk}\left(\frac{\partial^2 x_1}{\partial t\,\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{\partial x_3}{\partial X_k} + \frac{\partial x_1}{\partial X_i}\frac{\partial^2 x_2}{\partial t\,\partial X_j}\frac{\partial x_3}{\partial X_k} + \cdots\right)$$

$$\frac{\partial V_i}{\partial X_j} = \frac{\partial x_k}{\partial X_j}\frac{\partial v_i}{\partial x_k}$$

$$\rightarrow \quad \left.\frac{\partial J}{\partial t}\right|_{\underline{X}} = \epsilon_{ijk}\left(\frac{\partial v_1}{\partial X_i}\frac{\partial x_2}{\partial X_j}\frac{\partial x_3}{\partial X_k} + \frac{\partial x_1}{\partial X_i}\frac{\partial V_2}{\partial X_j}\frac{\partial x_3}{\partial X_k} + \cdots\right)$$

$$= \epsilon_{ijk}\left(\frac{\partial x_\ell}{\partial X_i}\frac{\partial v_1}{\partial x_\ell}\frac{\partial x_2}{\partial X_j}\frac{\partial x_3}{\partial X_k} + \frac{\partial x_1}{\partial X_i}\frac{\partial x_\ell}{\partial X_j}\frac{\partial v_2}{\partial x_\ell}\frac{\partial x_3}{\partial X_k} + \cdots\right)$$

$$= \epsilon_{ijk}\left[\underbrace{\left(\frac{\partial v_1}{\partial x_1}\frac{\partial x_1}{\partial X_i} + \frac{\partial v_1}{\partial x_2}\frac{\partial x_2}{\partial X_i} + \frac{\partial v_1}{\partial x_3}\frac{\partial x_3}{\partial X_i}\right)}_{i^{th}\ \text{term}}\frac{\partial x_2}{\partial X_j}\frac{\partial x_3}{\partial X_k} + \cdots\right]$$

this is ~~the~~ a determinant

$$\therefore \left.\frac{\partial J}{\partial t}\right|_{\underline{\mathbf{X}}} = \frac{\partial v_1}{\partial x_1}\begin{vmatrix} \frac{\partial x_1}{\partial \mathbf{X}_1} & \frac{\partial x_1}{\partial \mathbf{X}_2} & \frac{\partial x_1}{\partial \mathbf{X}_3} \\ \frac{\partial x_2}{\partial \mathbf{X}_1} & \frac{\partial x_2}{\partial \mathbf{X}_2} & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} + \frac{\partial v_1}{\partial x_2}\cancelto{0}{\begin{vmatrix} \frac{\partial x_2}{\partial \mathbf{X}_1} & \frac{\partial x_2}{\partial \mathbf{X}_2} & \frac{\partial x_2}{\partial \mathbf{X}_3} \\ \frac{\partial x_2}{\partial \mathbf{X}_1} & \frac{\partial x_2}{\partial \mathbf{X}_2} & \frac{\partial x_2}{\partial \mathbf{X}_3} \\ \cdots & \cdots & \cdots \end{vmatrix}}$$

$$+ \frac{\partial v_1}{\partial x_3}\cancelto{0}{\begin{vmatrix} \; \end{vmatrix}} + \frac{\partial v_2}{\partial x_1}\cancelto{0}{\begin{vmatrix} \frac{\partial x_1}{\partial \mathbf{X}_1} & \frac{\partial x_1}{\partial \mathbf{X}_2} & \vdots \\ \frac{\partial x_1}{\partial \mathbf{X}_1} & \frac{\partial x_1}{\partial \mathbf{X}_2} & \vdots \\ \cdots & \cdots & \cdots \end{vmatrix}}$$

$$+ \frac{\partial v_2}{\partial x_2}\begin{vmatrix} \frac{\partial x_1}{\partial \mathbf{X}_1} & \frac{\partial x_1}{\partial \mathbf{X}_2} & \frac{\partial x_1}{\partial \mathbf{X}_3} \\ \frac{\partial x_2}{\partial \mathbf{X}_1} & \frac{\partial x_2}{\partial \mathbf{X}} & \vdots \\ \cdots & \cdots & \cdots \end{vmatrix} + \cdots$$

$$\longrightarrow \quad \left.\frac{\partial J}{\partial t}\right|_{\underline{\mathbf{X}}} = \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right) J = (\nabla \cdot v)\, J$$

→ Euler's dilatation Formula - the rate of change of the Jacobian is the divergence of the velocity times the Jacobian

note - dilate means "make bigger"

Example: $v_1 = x_1$, $v_2 = -x_2$, $v_3 = 0$

$$\nabla \cdot \underline{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 1-1+0 = 0$$

→ the fluid is incompressible

Reynolds Transport Theorem

$$\frac{d}{dt}\iiint_{V(t)} F(\underline{x},t)\,dv = \iiint\left[\frac{\partial F}{\partial t} + \nabla\cdot(F v)\right]dv$$

, F is a scalar valued function, eg temperature, density, ...

where V(t) moves with the fluid

~~Proof 1~~ Version 1

$$\frac{d}{dt}\iiint_{V(t)} F(\underline{x},t)\,dv = \frac{d}{dt}\iiint_{V(0)} F(\underline{x}(\underline{X},t),t)\,J\,dv_0$$

← can't differentiate wrt time yet but change variables to move time derivative inside

$$= \iiint_{V(0)} \frac{\partial}{\partial t}\left[F(\underline{x}(\underline{X},t),t)J\right]dV_0$$

$$= \iiint_{V(0)}\left(\frac{\partial F}{\partial t}\Big|_{\underline{X}} J + F\frac{\partial J}{\partial t}\Big|_{\underline{X}}\right)dV_0$$

← independent variables are $\underline{X}$ and t

$$= \iiint_{V(0)}\left[\left(\frac{\partial F}{\partial t} + \frac{\partial x_i}{\partial t}\frac{\partial F}{\partial x_i}\right)J + F\,\nabla\cdot\underline{v}\,J\right]dV_0$$

$$= \iiint_{V(0)}\left(\frac{\partial F}{\partial t} + v_i\frac{\partial F}{\partial x_i} + F\frac{\partial v_i}{\partial x_i}\right)J\,dV_0$$

$$= \iiint_{V(t)}\left(\frac{\partial F}{\partial t} + \frac{\partial}{\partial x_i}(F v_i)\right)dV$$

Ve~~rsion~~ 2
~~Proof 2~~

Version 2

$$\cancel{\frac{d}{dt}\iiint_{V(t)} F\,dV = \iiint_{V(t)} \frac{\partial F}{\partial t}\,dv + \iiint_{\partial V(t)} n \cdot F v\,ds}$$

$$\frac{d}{dt}\iiint_{V(t)} F\,dV = \iiint_{V(t)} \frac{\partial F}{\partial t}\,dv + \iint_{\partial(V(t))} \underline{n}\cdot(F\underline{v})\,ds$$

Version 3

IF $F(\underline{x},t) = \rho(\underline{x},t)$ = density of fluid

mass inside $V = \iiint_{V} \rho(\underline{x},t)\,dv$

conservation of mass:

$$\frac{d}{dt}\iiint_{V(t)} \rho(\underline{x},t)\,dV = 0$$

$$\Rightarrow \iiint_{V(t)} \left(\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{v})\right) dv = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho v) = 0$$

, provided $\frac{\partial \rho}{\partial t}$, $\nabla\rho$, $\nabla\cdot\underline{v}$ are continuous

What if $\mathcal{F}(x,t) = \rho(x,t)\,F(x,t)$??

$$\frac{d}{dt}\iiint_{V(t)} \rho(\underline{x},t)\,F(\underline{x},t)\,dv = \iiint \rho(x,t)\left(\frac{\partial F}{\partial t} + \underline{v}\cdot\nabla F\right) dV$$

Reynolds
Transport
Theorem

derivation:

$$\frac{d}{dt}\iiint_{V(t)} \rho F\,dv = \iiint_{V}\left[\frac{\partial}{\partial t}(\rho F) + \nabla\cdot(\rho F \underline{v})\right] dV$$

$$= \iiint_{V}\left[\rho\frac{\partial F}{\partial t} + \cancel{F\frac{\partial \rho}{\partial t} + (\nabla\cdot\rho\underline{v})F} + \rho\underline{v}\cdot\nabla F\right] dv$$

$$= \iiint_{V}\rho\left(\frac{\partial F}{\partial t} + \underline{v}\cdot\nabla F\right) dv$$

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Momentum

$$\underline{m}(t) = \iiint_{V(t)} \rho(\underline{x},t)\,\underline{v}(x,t)\,dV$$

$$\frac{d}{dt}\iiint_{V(t)} \rho(\underline{x},t)\,\underline{v}(x,t)\,dV = \iiint_{V(t)} \rho\,\underline{f}\,dV$$, $\underline{f}$ = force per unit mass, 'acceleration'

$$+ \iint_{\partial V(t)} \underline{t}\,ds$$, t = force per unit area on ds "traction vector"

apply Reynolds Transport Theorem

$$\frac{d}{dt}\iiint_{V(t)} \rho(\underline{x},t)\,\underline{v}(x,t)\,dV = \iiint_{V(t)} \rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right)dv$$

$$= \iiint_{V(t)} \rho\,\underline{f}\,dv + \iiint_{V(t)} ?\,dv$$

For now, assume $\underline{t}(\underline{n})$ is a linear function of $\underline{n}$

$\underline{t}(\underline{n}) = \underline{n}\cdot\underline{\underline{T}}$ or $t_i = n_j T_{ji}$, T_{ji} - 2nd order tensor "stress tensor"

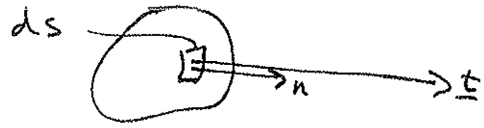
$$\rightarrow \frac{d}{dt}\iiint_{V(t)} \rho(\underline{x},t)\,\underline{v}(x,t)\,dv = \iiint_{V(t)} \rho\,\underline{f}\,dv + \iint_{\partial V(t)} \underline{n}\cdot\underline{\underline{T}}\,ds$$

$$\rightarrow \iiint_{V} \rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) - \nabla\cdot\underline{\underline{T}} - \rho\,\underline{f} = 0$$

$$\boxed{\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = \nabla\cdot\underline{\underline{T}} + \rho\,\underline{f}}$$ Cauchy's Equation of Motion

notice
- the divergence of the stress tensor term
- $\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}$ = <u>material</u> derivative of $\underline{v}$ = acceleration

IF we assume $T_{ij} = -p\,\delta_{ij}$



$\underline{t}(\underline{n}) = -p\underline{n}$, pressure

Then $$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -(p\,\delta_{ij})_{,j} + \rho\underline{f}$$
$$= -\nabla p + \rho\underline{f}$$

$$\rightarrow \boxed{\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla p + \rho\underline{f}}$$ Euler's Eqn

if $\underline{v} = 0$, $\underline{f} = -g\underline{e}_3$ $\rightarrow$ $\nabla p = -g\,\underline{e}_3$ and $\rho = \rho(x_1, x_2, x_3)$

$$\rightarrow \frac{\partial p}{\partial x_1} = 0, \quad \frac{\partial p}{\partial x_2} = 0, \quad \frac{\partial p}{\partial x_3} = -g\rho$$

$$\boxed{\begin{array}{l} p = -g\displaystyle\int_{a(x_1,x_2)}^{x_3} \rho(x_1, x_2, x_3')\,dx_3' + p_0(x_1, x_2) \\ \text{If } \rho = \text{const}, \text{ then } p = -\rho g x_3 + \text{const.} \end{array}}$$

Hydrostatic Equation

MA 6792 9/16/03

schedule

Friday 9/19, 8 am, not 2pm

Tues 9/23, no class

Fri 9/26, no class

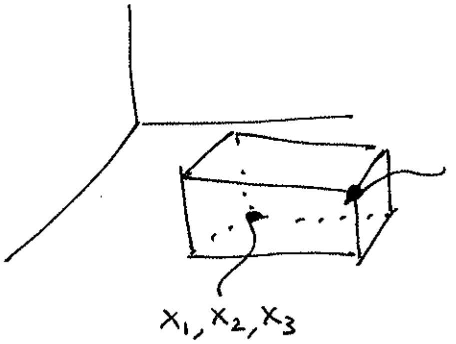
Tues 9/30, no class

Fri 10/03, no class

Tue 10/07, 2-4

Fri 10/10, 8-10, 2-4

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = \nabla\cdot\underline{\underline{T}} + \rho\underline{F}$$



$$v\cdot\nabla\underline{v}\ ,\quad v=(v_1,v_2,v_3)$$

$$\left(v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial x_2} + v_3\frac{\partial}{\partial x_3}\right)\left(v_1\underline{e}_1 + v_2\underline{e}_2 + v_3\underline{e}_3\right)$$

$$\frac{d}{dt}$$

$$\left(v_1\frac{\partial v_1}{\partial x_1} + v_2\frac{\partial v_1}{\partial x_2} + v_3\frac{\partial v_1}{\partial x_3}\right)\underline{e}_1$$

$$\left(v_1\frac{\partial v_2}{\partial x_1} + v_2\frac{\partial v_2}{\partial x_2} + v_3\frac{\partial v_2}{\partial x_3}\right)\underline{e}_2$$

$$\left(v_1\frac{\partial v_3}{\partial x_1} + v_2\frac{\partial v_3}{\partial x_2} + v_3\frac{\partial v_3}{\partial x_3}\right)\underline{e}_3$$

derive

$$\boxed{\frac{\partial \rho\underline{v}}{\partial t} + \frac{\partial}{\partial x_1}(\rho\underline{v}v_1) + \frac{\partial}{\partial x_2}(\rho\underline{v}v_2) + \frac{\partial}{\partial x_3}(\rho\underline{v}v_3) = \frac{\partial \underline{T}^{(1)}}{\partial x_1} + \frac{\partial \underline{T}^{(2)}}{\partial x_2} + \frac{\partial \underline{T}^{(3)}}{\partial x_3} + \rho\underline{F}}$$

1a

notice, $\frac{\partial}{\partial x_1}\left(\underline{T}^{(1)}\right) + \frac{\partial}{\partial x_2}\left(\underline{T}^{(2)}\right) + \frac{\partial}{\partial x_3}\left(\underline{T}^{(3)}\right)$

vectors

$$= \left(\frac{\partial}{\partial x_1}\underline{e}_1 + \frac{\partial}{\partial x_2}\underline{e}_2 + \frac{\partial}{\partial x_3}\underline{e}_3\right) \cdot \left(\underline{e}_1\underline{T}^{(1)} + e_2\,\underline{T}^{(2)} + e_3\,\underline{T}^{(3)}\right)$$

$$= \nabla \cdot \underline{\underline{T}}$$

where $T_{ij} = T_j^{(i)}$, ith vector, jth component

i refers to Face, j refers to stress

Notice

$$\frac{\partial(\rho\underline{v})}{\partial t} + \frac{\partial}{\partial x_1}(\rho\underline{v}v_1) + \frac{\partial}{\partial x_2}(\rho\underline{v}v_2) + \frac{\partial}{\partial x_3}(\rho\,\underline{v}v_3) =$$

$$= \underline{v}\frac{\partial\rho}{\partial t} + \rho\frac{\partial\underline{v}}{\partial t} + \frac{\partial}{\partial x_1}(\rho v_1)\underline{v} + \rho v_1\frac{\partial\underline{v}}{\partial x_1} + \frac{\partial}{\partial x_2}(\rho v_2)\underline{v} + \rho v_2\frac{\partial\underline{v}}{\partial x_2} + \frac{\partial}{\partial x_3}(\rho v_3)\underline{v} + \frac{\partial\underline{v}}{\partial x_3}\rho v_3$$

$$= \underline{v}\underbrace{\left(\frac{\partial\rho}{\partial t} + \frac{\partial\rho v_1}{\partial x_1} + \frac{\partial}{\partial x_2}\rho v_2 + \frac{\partial}{\partial x_3}\rho v_3\right)}_{0,\ \text{conservation of mass}} + \rho\left(\frac{\partial\underline{v}}{\partial t} + v_1\frac{\partial\underline{v}}{\partial x_1} + v_2\frac{\partial\underline{v}}{\partial x_2} + v_3\frac{\partial\underline{v}}{\partial x_3}\right)$$

$$= \rho\left(\frac{\partial\underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right)$$

Principle of Local Stress Equilibrium

consider moving control volume

$$\iiint_{V(t)} \rho\left(\frac{\partial\underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right)dV = \iint_{\partial V}\underline{t}_{(n)}\,dS + \iiint_{V}\rho\,\underline{f}\,dV$$

consider a sequence of shrinking volumes, measured by some value d, where $\iiint_{V} \propto d^3$ and $\iint_{\partial V} \propto d^2$

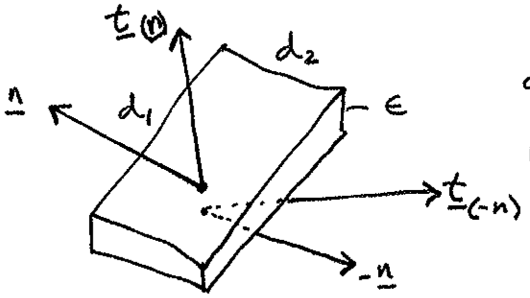
divide by d^2, take $\lim_{d\to 0}$

$$\frac{1}{d^2}\iiint_V \rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) dv = \frac{1}{d^2}\iint_{\partial V} \underline{t}_{(n)}\, ds + \frac{1}{d^2}\iiint_V \rho\underline{F}\, dv$$

since $\iiint_V \propto d^3$, $\iint_{\partial V} \propto d^2$,

$$\Rightarrow \quad \boxed{\frac{1}{d^2}\iint_{\partial V} \underline{t}_{(n)}\, ds = 0}$$

consider a flake, which looks like



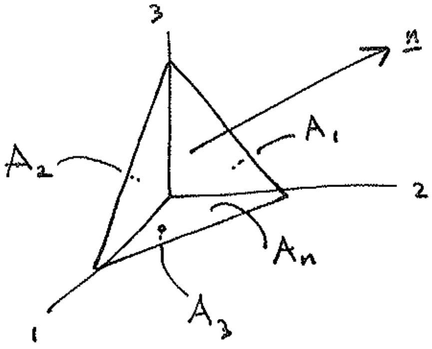
and which has x_1, x_2, x_3 inside flake
pick normal direction as shown

$$d_1 d_2\, \underline{t}_{(n)} + d_1 d_2\, \underline{t}_{(-n)} + O(\epsilon)(d_1, d_2) = 0$$

$$\longrightarrow \quad d_1 d_2\left(\underline{t}_{(n)} + \underline{t}_{(-n)}\right) = 0$$

$$\longrightarrow \quad \boxed{-\underline{t}_{(n)} = +\underline{t}_{(-n)}}$$

consider a tetrahedron



area of front face, A_n
area of bottom face, A_3
leftside area, A_2
right side area, A_1

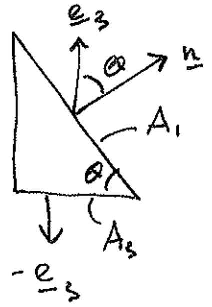
normal to right side is $-\underline{e}_1$

$$\rightarrow \quad \underline{t}_{(n)} A_n + \underline{t}_{(-\underline{e}_1)} A_1 + \underline{t}_{(-\underline{e}_2)} A_2 + \underline{t}_{(-\underline{e}_3)} A_3 = 0$$

but $\underline{t}_{(-\underline{e}_1)} = -\underline{t}_{(\underline{e}_1)}$

$$\underline{t}_n = \underline{t}_{(\underline{e}_1)} \frac{A_1}{A_n} + \underline{t}_{(e_2)} \frac{A_2}{A_n} + \underline{t}_{(e_3)} \frac{A_3}{A_n}$$

take a slice in plane of $\underline{n}$



$$\cos\theta = \underline{e}_3 \cdot \underline{n} = \frac{A_3}{A_n}$$

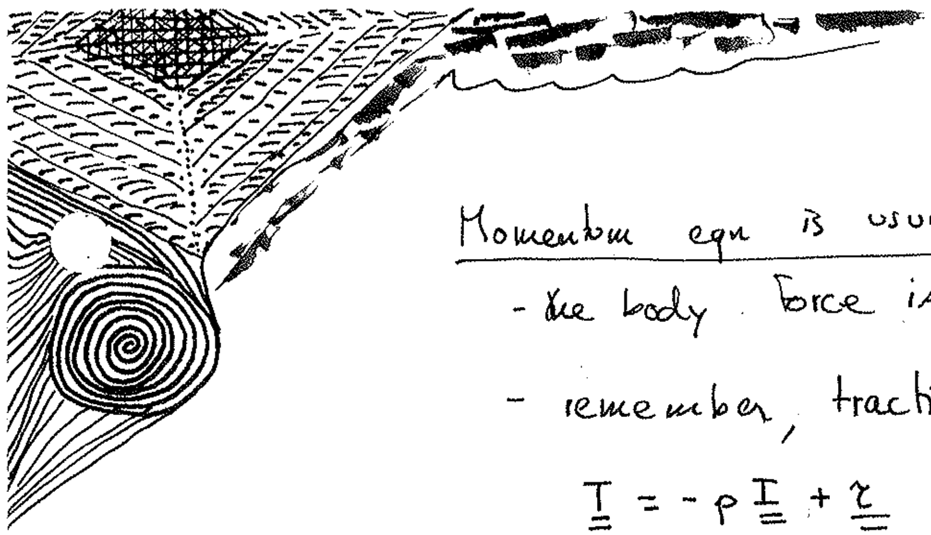
} not a complete argument

$$\rightarrow \quad \underline{t}_n = \underline{t}_{(e_1)}\, \underline{e}_1 \cdot \underline{n} + \underline{t}_{(e_2)}\, \underline{e}_2 \cdot \underline{n} + \underline{t}_{(\underline{e}_3)}\, \underline{e}_3 \cdot \underline{n}$$

← written backwards

$$\underline{t}_n = \underline{n} \cdot \Big(\underbrace{\underset{\underset{\underline{T}^{(1)}}{\downarrow}}{\underline{e}_1 \underline{t}_{e_1}} + \underset{\underset{\underline{T}^{(2)}}{\downarrow}}{\underline{e}_2 \underline{t}_{e_2}} + \underset{\underset{\underline{T}^{(3)}}{\downarrow}}{\underline{e}_3 \underline{t}_{e_3}}}_{\text{stress tensor}} \Big) = \underline{n} \cdot \underline{\underline{T}},$$

$$(t_{(n)})_i = n_j T_{ji}$$



Fluid Mechanics 10/07

Momentum eqn is usually called Cauchy's eqn

- the body force is assumed to be known
- remember, traction vector, $\underline{t} = \underline{n} \cdot \underline{\underline{T}} \leftarrow$ stress tensor

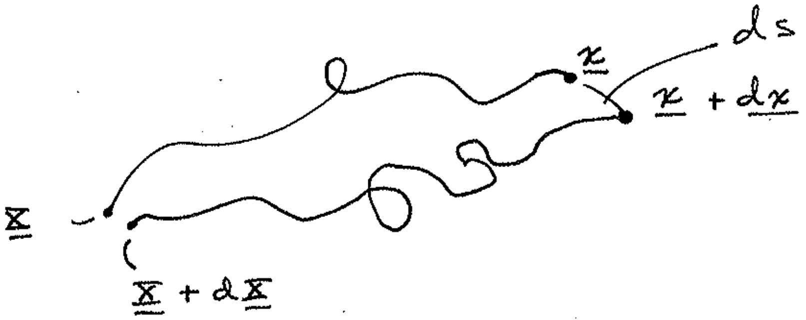
$$\underset{\substack{\uparrow \\ \text{}}}{}\underline{\underline{T}} = -\underset{\substack{\uparrow\\ \text{pressure}}}{p}\,\underset{\text{identity}}{\underline{\underline{I}}} + \underset{\text{shear stress}}{\underline{\underline{\tau}}}$$

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = \nabla\cdot\underline{\underline{T}} + \rho\underline{f}$$

"stress is a 'Functional' of the velocity", $\underline{\underline{T}} = \underline{\underline{T}}(\underline{v})$

Let's talk about the motion of bits of fluid think about two points, joined by some trajectory

$$\underline{x} = \underline{x}(\underline{X}, t)$$



$$dx_i \cong \frac{\partial x_i}{\partial X_j} dX_j$$

$$ds^2 = dx_i dx_i = dX_j dX_k \frac{\partial x_i}{\partial X_j}\frac{\partial x_i}{\partial X_k}$$

$$\frac{d}{dt}ds^2 = 2ds\frac{d(ds)}{dt} = \partial X_j \partial X_k\left(\frac{\partial^2 x_i}{\partial X_j \partial t}\frac{\partial x_i}{\partial X_k} + \frac{\partial x_i}{\partial X_j}\frac{\partial^2 x_i}{\partial X_k \partial t}\right)$$

$$2ds\frac{d}{dt}(ds) = \partial X_j \partial X_k\left(\frac{\partial v_i}{\partial X_j}\frac{\partial x_i}{\partial X_k} + \frac{\partial x_i}{\partial X_j}\frac{\partial v_i}{\partial X_k}\right)$$

$$2\,ds\,\frac{d}{dt}(ds) = dX_j\,dX_k\left(\frac{\partial v_i}{\partial X_j}\frac{\partial x_i}{\partial X_k} + \frac{\partial x_i}{\partial X_j}\frac{\partial v_i}{\partial X_k}\right)$$

$$= dX_j\,dX_k\left(\frac{\partial x_l}{\partial X_j}\frac{\partial v_i}{\partial x_l}\frac{\partial x_i}{\partial X_k} + \frac{\partial x_i}{\partial X_j}\frac{\partial x_m}{\partial X_k}\frac{\partial v_i}{\partial x_m}\right)$$

$$= \left(dX_j\,\frac{\partial x_l}{\partial X_j}\right)\left(dX_k\,\frac{\partial x_i}{\partial X_k}\right)\frac{\partial v_i}{\partial x_l} + \left(dX_j\,\frac{\partial x_i}{\partial X_j}\right)\left(dX_k\,\frac{\partial x_m}{\partial X_k}\right)\left(\frac{\partial v_i}{\partial x_m}\right)$$

$$= dx_l\,dx_i\,\frac{\partial v_i}{\partial x_l} + dx_i\,dx_m\,\frac{\partial v_i}{\partial x_m}$$

$$= 2\,dx_l\,dx_i\,\frac{\partial v_i}{\partial x_l}$$

$$\rightarrow \boxed{\frac{d}{dt}(ds) = \frac{1}{ds}\,dx_l\,dx_i\,\frac{dv_i}{\partial x_l}}$$

pick dX_j such that $dx_i = ds\,\underline{e}_1 + 0\,\underline{e}_2 + 0\,\underline{e}_3$

$$\rightarrow \frac{d}{dt}(ds) = \frac{1}{ds}\,ds \cdot ds\,\frac{\partial v_1}{\partial x_1} \quad \rightarrow \quad \frac{1}{ds}\frac{d}{dt}(ds) = \frac{\partial v_1}{\partial x_1}$$

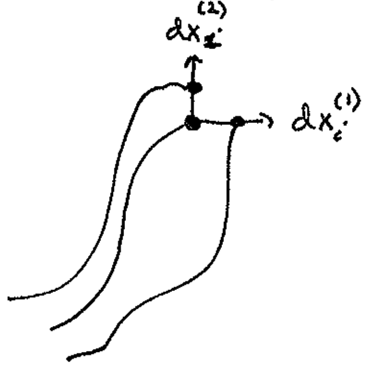
rate of change of length per unit length in x_1

similarly, we find

$\frac{\partial v_2}{\partial x_2}$, $\frac{\partial v_3}{\partial x_3}$

Fluid Mechanics 10/07

Now pick $d\mathbb{X}$ for three paths such that at some instant in time, the points are perpindicular to eachother



$$dx_i^{(1)} dx_i^{(2)} = ds^{(1)} ds^{(2)} \cos\Theta$$

we're seeking the rate of change of angle

$$\frac{d}{dt} dx_i^{(1)} dx_i^{(2)} = \frac{ds^{(1)}}{dt} s^{(2)} \cos\Theta + ds^{(1)} \frac{d}{dt}(ds^{(2)}) \cos\Theta$$

$$+ - ds^{(1)} ds^{(2)} \sin\Theta \frac{d\Theta}{dt}$$

the LHS of the equation is

$$\frac{d}{dt} dx_i^{(1)} dx_i^{(2)} = \frac{d}{dt}\left(d\mathbb{X}_j^{(1)} \frac{\partial x_i}{\partial \mathbb{X}_j} d\mathbb{X}_k^{(2)} \frac{\partial x_i}{\partial \mathbb{X}_k} \right)$$

$$= \cancel{d\mathbb{X}_j^{(1)} d\mathbb{X}_k^{(2)} \left(\frac{\partial x_i}{\partial \mathbb{X}_j} \frac{\partial v_i}{\partial x_l} \frac{\partial x_i}{\partial \mathbb{X}_k}\right.}$$

$$= d\underline{\mathbb{X}}_j^{(1)} d\mathbb{X}_k^{(2)} \left(\frac{\partial x_l}{\partial \mathbb{X}_j} \frac{\partial v_i}{\partial x_l} \frac{\partial x_i}{\partial \mathbb{X}_k} + \frac{\partial x_i}{\partial \mathbb{X}_j} \frac{\partial x_m}{\partial \mathbb{X}_n} \frac{\partial v_i}{\partial x_m} \right)$$

$$= dx_l^{(1)} dx_i^{(2)} \frac{\partial v_i}{\partial x_l} + dx_i^{(1)} dx_m^{(2)} \frac{\partial v_i}{\partial x_m}$$

$$= dx_l^{(1)} dx_i^{(2)} \left(\frac{\partial v_i}{\partial x_l} + \frac{\partial v_l}{\partial x_i} \right)$$

$$\rightarrow \quad \frac{d}{dt}\left(dx_i^{(1)}\, dx_i^{(2)}\right) = dx_\ell^{(1)}\, dx_i^{(2)} \left(\frac{\partial v_i}{\partial x_\ell} + \frac{\partial v_\ell}{\partial x_i}\right)$$

Pick $d\underline{X}_j^{(1)}$ and $d\underline{X}_K^{(2)}$ such that

$$d\underline{x}^{(1)} = ds^{(1)}\, \underline{e}_1 \ , \quad d\underline{x}^{(2)} = ds^{(2)}\, \underline{e}_2 \ , \quad \theta = \frac{\pi}{2}$$

then from

$$\frac{d}{dt}\left(dx_i^{(1)}\, dx_i^{(2)}\right) = \frac{d}{dt}(ds^{(1)})\, ds^{(2)} \cancelto{0}{\cos\theta} + ds^{(1)} \frac{d}{dt}(ds^{(2)}) \cancelto{0}{\cos\theta} - ds^{(1)} ds^{(2)} \cancelto{1}{\sin\theta} \frac{d\theta}{dt}$$

$$= -\, ds^{(1)} ds^{(2)} \frac{d\theta}{dt}$$

$$\rightarrow \quad \cancel{dx_\ell^{(1)}\, dx_i^{(2)} \left(\frac{\partial v_i}{\partial x_\ell} + \frac{\partial v_\ell}{\partial x_i}\right) = -\, ds^{(1)} ds^{(2)} \frac{d\theta}{dt}}$$

$$\frac{d}{dt}\left(dx_1^{(1)}\, dx_2^{(2)}\right) = -\, ds^{(1)} ds^{(2)} \frac{d\theta}{dt} = \underset{\substack{\downarrow \\ ds^{(1)}}}{dx_1^{(1)}}\, \underset{\substack{\downarrow \\ ds^{(2)}}}{dx_2^{(2)}} \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)$$

$$\rightarrow \quad \boxed{\frac{d\theta}{dt} = -\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right)}$$

and similarly,

$$\frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} \ , \quad \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2}$$

Fluid Mechanics 10/07

$$\frac{\partial v_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)}_{\text{symmetric}} + \underbrace{\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)}_{\text{antisymmetric}} \quad (A_{ij} = -A_{ji})$$

~~tells how stuff gets "squished around"~~

deformation

rotation

↘ the digonal must be zero

$$\begin{pmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{pmatrix}$$

any motion may be thought of as a translation + deformation + rotation

$$\underline{\underline{T}} = \underline{\underline{T}}(\underline{v}) = \underline{\underline{T}}\left(\frac{\partial v_i}{\partial x_j}\right) = \underline{\underline{T}}\left(\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right)$$

10/10/03
Fluid Mechanics

$\underline{\underline{T}} = \underline{\underline{T}}(\underline{\underline{D}})$

$\underline{\underline{T}} = \underline{\underline{T}}(\underline{v})$, functional

at a point, the only thing we can assume about the stress dependence on the fluid at that point is that it is only related to what happened before and not what hasn't happened

$$\underline{\underline{T}}(\underline{x},t) = \underline{\underline{T}}(\underline{v}(\underline{y},\tau), \underline{x}, t)$$ so $\underline{y}$ in body, $\tau < t$

$$\underline{\underline{T}} = \underline{\underline{T}}(\underline{\underline{D}}) \rightarrow \underline{\underline{T}}(\underline{x},t) = \underline{\underline{T}}(\underline{\underline{D}}(\underline{x},t))$$

$$\underline{\underline{D}} = \frac{1}{2}(\nabla\underline{v} + \nabla\underline{v}^T), \quad D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$$

$\underline{\underline{D}}(\underline{x},t) \approx$ "state" of fluid at $\underline{x}$

see Aris for Representation theorem

Cauchy Representation Theorem

$\underline{\underline{T}} = \underline{\underline{T}}(\underline{\underline{D}})$

function is same in any coordinate system

consider the idea for a vector:

$$\underline{v} = \underline{F}(\underline{u}) = \text{[illegible]} = \underline{g}(|\underline{u}|, \hat{\underline{u}})$$

"length of vector is invariant under coordinate transformation"

$$v_j = g_j(|\underline{u}|, \hat{u}_1, \hat{u}_2, \hat{u}_3), \quad \bar{v}_K = \bar{g}_K(|\bar{\underline{u}}|, \bar{\hat{u}}_1, \bar{\hat{u}}_2, \bar{\hat{u}}_3)$$

$\bar{v}_k$ should equal $g_k(|\bar{\underline{u}}|, \bar{\hat{u}}_1, \bar{\hat{u}}_2, \bar{\hat{u}}_3)$

now change coordinates $\bar{v}_k = l_{jk} v_j$, Substitute $M_0 = v_0/a_0$

$$v_1 \;=\; a_0 \left(\frac{1 + \frac{1}{2}(\gamma - 1)M_0^2}{\frac{1}{2}(\gamma + 1)M_0} \right)$$

Now make use of the transformation $u = U - v$ to solve for the jump in velocity across the shock $u_1 - u_0$

$$u_1 - u_0 \;=\; (U - v_1) - (U - v_0) \;=\; v_0 - v_1 \;=\; a_0 M_0 - a_0 \left(\frac{1 + \frac{1}{2}(\gamma - 1)M_0^2}{\frac{1}{2}(\gamma + 1)M_0} \right)$$

$$\rightarrow \qquad \frac{u_1 - u_0}{a_0} \;=\; \frac{\frac{1}{2}(\gamma + 1)M_0^2 - \left(1 + \frac{1}{2}(\gamma - 1)M_0^2\right)}{\frac{1}{2}(\gamma + 1)M_0} \qquad \rightarrow \qquad \frac{u_1 - u_0}{a_0} \;=\; \frac{M_0^2 - 1}{\frac{1}{2}(\gamma + 1)M_0}$$

We now derive an expression for the pressure jump by considering jump condition (b)

$$\left[\, \rho a^2 (1 + \gamma M^2) \,\right]$$

Substitute $p = \rho a^2/\gamma$ and expand the jump condition

$$p_1 \gamma (1 + \gamma M_1^2) \;=\; p_0 \gamma (1 + \gamma M_0^2) \qquad \rightarrow \qquad \frac{p_1}{p_0} \;=\; \frac{1 + \gamma M_0^2}{1 + \gamma M_1^2}$$

Substitute expression (e)

$$\frac{p_1}{p_0} \;=\; \frac{\gamma M_0^2 - \frac{1}{2}(\gamma - 1)}{\frac{1}{2}(\gamma + 1)}$$

We can now solve for the pressure jump $p_1 - p_0$ across the shock

$$p_1 - p_0 \;=\; p_0 \left(\frac{\gamma M_0^2 - \frac{1}{2}(\gamma - 1)}{\frac{1}{2}(\gamma + 1)} \right) - p_0 \qquad \rightarrow \qquad \frac{p_1 - p_0}{p_0} \;=\; \frac{\gamma (M_0^2 - 1)}{\frac{1}{2}(\gamma + 1)}$$

In summary, the jump conditions for the velocity, density and pressure across the shock are

$$\begin{aligned} \frac{u_1 - u_0}{a_0} &= \frac{2(M_0^2 - 1)}{(\gamma + 1)M_0} \\ \frac{\rho_1}{\rho_0} &= \frac{(\gamma + 1)M_0^2}{2 + (\gamma - 1)M_0^2} \\ \frac{p_1 - p_0}{p_0} &= \frac{2\gamma (M_0^2 - 1)}{\gamma + 1} \end{aligned}$$

10/10/03 2/4

$$\bar{v}_k = l_{jk} v_j \longrightarrow g_k(|\underline{v}|, \bar{v}_1, \bar{v}_2, \bar{v}_3) = l_{jk}\, g_j(|\underline{v}|, \hat{v}_1, \hat{v}_2, \hat{v}_3)$$

multiply by l_{ik} with $l_{ik} l_{jk} = \delta_{ij}$

~~$l_{ik} g_k$~~ $$l_{ik}\, g_k(|\underline{v}|, \bar{v}_1, \bar{v}_2, \bar{v}_3) = \delta_{ij}\, g_j(|\underline{v}|, \hat{v}_1, \hat{v}_2, \hat{v}_3)$$

$$l_{ik}\, g_k(|\underline{v}|, \bar{v}_1, \bar{v}_2, \bar{v}_3) = g_i(|\underline{v}|, \hat{v}_1, \hat{v}_2, \hat{v}_3)$$

$$\parallel$$

$$v_i \quad (\underline{e}_i \cdot \bar{\underline{e}}_k)\, g_k(|\underline{v}|, \bar{v}_1, \bar{v}_2, \bar{v}_3)$$

since this is true for any coordinate frame, pick one.

pick $\bar{\underline{e}}_1 = \hat{\underline{v}}$, $\bar{\underline{e}}_2 = \underline{t}$ where $\underline{t} \perp \hat{\underline{v}}$, $\bar{\underline{e}}_3 = \underline{b}$

$$= (\underline{e}_1 \cdot \hat{\underline{v}})\, g_1(|\underline{v}|, 1, 0, 0) + (\underline{e}_2 \cdot \underline{t})\, g_2(|\underline{v}|, 1, 0, 0) + (\underline{e}_3 \cdot \underline{b})\, g_3(|\underline{v}|, 1, 0, 0)$$

since $\underline{t}$, $\underline{b}$ are arbitrary (not necessarily perpindicular)

$$g_2(|\underline{v}|, 1, 0, 0) = 0, \quad g_3(|v|, 1, 0, 0) = 0$$

$$\rightarrow v_i = \underline{e}_i \cdot \hat{\underline{v}}\, g_1(|\underline{v}|, 1, 0, 0)$$

$$\rightarrow \underline{v} = v_i \underline{e}_i = \sum_{i=1}^{3} \underline{e}_i (\underline{e}_i \cdot \hat{\underline{v}})\, g_1(|\underline{v}|, 1, 0, 0)$$

$$= |\underline{v}|\, \hat{\underline{v}} \cdot \left(\sum_{i=1}^{3} \underline{e}_i \underline{e}_i \right) \frac{g_1(|\underline{v}|, 1, 0, 0)}{|\underline{v}|}$$

$$= \underline{v} \cdot \underbrace{\left(\sum_{i=1}^{3} \underline{e}_i \underline{e}_i \right)}_{\text{identity}} h(|\underline{v}|)$$

$$\boxed{\underline{v} = \underline{v}\, h(|\underline{v}|)} \quad \rightarrow \quad \boxed{\underline{v} = \underline{F}(\underline{v}) = g(|\underline{v}|)\, \underline{v}}$$

$$\underline{v} = \underline{F}(\underline{v}_1, \underline{v}_2)$$

$$= h_1(\underbrace{|\underline{v}_1|, |\underline{v}_2|, \underline{v}_1 \cdot \underline{v}_2}_{\text{lengths and inner products are invariants}})\, \underline{v}_1 + h_2(|\underline{v}_1|, |\underline{v}_2|, \underline{v}_1 \cdot \underline{v}_2)$$

We arrive at the following equation for M_1^2

$$M_1^2 = \frac{1 + \frac{1}{2}(\gamma - 1)M_0^2}{\gamma M_0^2 - \frac{1}{2}(\gamma - 1)} \qquad (d)$$

It will be useful for coming calculations to derive expressions for $1 + \gamma M_1^2$ and $1 + \frac{1}{2}(\gamma - 1)M_1^2$

$$\begin{aligned}
1 + \gamma M_1^2 &= \frac{\gamma M_0^2 - \frac{1}{2}(\gamma - 1) + \gamma\left(1 + \frac{1}{2}(\gamma - 1)M_0^2\right)}{\gamma M_0^2 - \frac{1}{2}(\gamma - 1)} \\
1 + \gamma M_1^2 &= \frac{\frac{1}{2}(\gamma + 1) + \frac{\gamma}{2}M_0^2(1 + \gamma)}{\gamma M_0^2 - \frac{1}{2}(\gamma - 1)} \\
1 + \gamma M_1^2 &= \frac{\frac{1}{2}(\gamma + 1)(1 + \gamma M_0^2)}{\gamma M_0^2 - \frac{1}{2}(\gamma - 1)} && (e) \\
1 + \frac{1}{2}(\gamma - 1)M_1^2 &= \frac{\gamma M_0^2 - \frac{1}{2}(\gamma - 1) + \frac{1}{2}(\gamma - 1)\left(1 + \frac{1}{2}(\gamma - 1)M_0^2\right)}{\gamma M_0^2 - \frac{1}{2}(\gamma - 1)} \\
1 + \frac{1}{2}(\gamma - 1)M_1^2 &= \frac{\gamma M_0^2 + \frac{1}{4}(\gamma - 1)^2 M_0^2}{\gamma M_0^2 - \frac{1}{2}(\gamma - 1)} \\
1 + \frac{1}{2}(\gamma - 1)M_1^2 &= \frac{\gamma M_0^2 + \frac{1}{4}(\gamma^2 - 2\gamma + 1)M_0^2}{\gamma M_0^2 - \frac{1}{2}(\gamma - 1)} \\
1 + \frac{1}{2}(\gamma - 1)M_1^2 &= \frac{\frac{1}{4}(\gamma + 1)^2 M_0^2}{\gamma M_0^2 - \frac{1}{2}(\gamma - 1)} && (f)
\end{aligned}$$

We can now turn our attention to the jump density by calculating (b)/(c)

$$\left[\frac{\rho a^2(1 + \gamma M^2)}{a^2\left(1 + \frac{1}{2}(\gamma - 1)M^2\right)}\right] = 0 \quad \rightarrow \quad \left[\frac{\rho(1 + \gamma M^2)}{1 + \frac{1}{2}(\gamma - 1)M^2}\right] = 0$$

Expand the jump condition to solve for ρ_1/ρ_0

$$\frac{\rho_1(1 + \gamma M_1^2)}{1 + \frac{1}{2}(\gamma - 1)M_1^2} = \frac{\rho_0(1 + \gamma M_0^2)}{1 + \frac{1}{2}(\gamma - 1)M_0^2} \quad \rightarrow \quad \frac{\rho_1}{\rho_0} = \left(\frac{1 + \gamma M_0^2}{1 + \frac{1}{2}(\gamma - 1)M_0^2}\right)\left(\frac{1 + \frac{1}{2}(\gamma - 1)M_1^2}{1 + \gamma M_1^2}\right)$$

Substitute expressions (e) and (f)

$$\frac{\rho_1}{\rho_0} = \left(\frac{1 + \gamma M_0^2}{1 + \frac{1}{2}(\gamma - 1)M_0^2}\right)\left(\frac{\frac{1}{4}(\gamma + 1)^2 M_0^2}{\frac{1}{2}(\gamma + 1)(1 + \gamma M_0^2)}\right) \quad \rightarrow \quad \frac{\rho_1}{\rho_0} = \frac{\frac{1}{2}(\gamma + 1)M_0^2}{1 + \frac{1}{2}(\gamma - 1)M_0^2}$$

Now we turn out attention to equation (c) in order to derive an expression for velocity. Expanding the jump condition (c), we have

$$a_1^2\left(1 + \frac{1}{2}(\gamma - 1)M_1^2\right) = a_0^2\left(1 + \frac{1}{2}(\gamma - 1)M_0^2\right)$$

Substitute $M = v/a$ and solve for v_1/v_0

$$\frac{v_1^2}{v_0^2} = \frac{M_1^2}{M_0^2}\left(\frac{1 + \frac{1}{2}(\gamma - 1)M_0^2}{1 + \frac{1}{2}(\gamma - 1)M_1^2}\right)$$

Now substitute the expressions (d), (e) and (f)

$$\begin{aligned}
\frac{v_1^2}{v_0^2} &= \frac{1}{M_0^2}\left(\frac{1 + \frac{1}{2}(\gamma - 1)M_0^2}{\gamma M_0^2 - \frac{1}{2}(\gamma - 1)}\right)\left(1 + \frac{1}{2}(\gamma - 1)M_0^2\right)\left(\frac{\gamma M_0^2 - \frac{1}{2}(\gamma - 1)}{\frac{1}{4}(\gamma + 1)^2 M_0^2}\right) \\
\frac{v_1^2}{v_0^2} &= \frac{1}{M_0^2}\left(\frac{1 + \frac{1}{2}(\gamma - 1)M_0^2}{\frac{1}{2}(\gamma + 1)M_0}\right)^2 \\
\frac{v_1}{v_0} &= \frac{1 + \frac{1}{2}(\gamma - 1)M_0^2}{\frac{1}{2}(\gamma + 1)M_0^2}
\end{aligned}$$

Back to tensors

Most General Invariant theory

$$\boxed{\begin{aligned}\underline{\underline{T}} = \underline{\underline{T}}(\underline{\underline{D}}) &= \lambda(\text{invariants})\,\underline{\underline{I}} + \mu(\text{invariants})\,\underline{\underline{D}} \\ &+ \nu(\text{invariants})\,\underbrace{\underline{\underline{D}}\cdot\underline{\underline{D}}}_{D_{ik}D_{jk}}\end{aligned}} + \cancel{\pi(\text{invariants})\,\underline{\underline{D}}\cdot\underline{\underline{D}}\cdot\underline{\underline{D}}}$$

λ, μ, ν functions of invariants

due to Cayley-Hamilton

invariants: eigenvalues, $\det(D_{ij} - \lambda\delta_{ij}) = 0$

$$III_D - II_D\lambda + I_D\lambda^2 - \lambda^3 = 0$$

third invariant, $III_D = \det D_{ij} = \lambda_1\lambda_2\lambda_3$

2nd invariant, $II_D = (D_{22}D_{33} - D_{23}D_{32}) - (D_{11}D_{33} - D_{13}D_{31}) + (D_{11}D_{22} - D_{12}D_{21})$

1st invariant, $I_D = \lambda_1 + \lambda_2 + \lambda_3$

Cayley - Hamilton theorem

$\underline{\underline{D}}$ satisfies $III_D\,\underline{\underline{I}} - II_D\,\underline{\underline{D}} + I_D\,\underline{\underline{D}}\cdot\underline{\underline{D}} - \underline{\underline{D}}\cdot\underline{\underline{D}}\cdot\underline{\underline{D}} = 0$

which explains why there is no $\underline{\underline{D}}\cdot\underline{\underline{D}}\cdot\underline{\underline{D}}\cdot\underline{\underline{D}}$

Linear: $\underline{\underline{T}} = \underline{\underline{T}}(c_1\underline{\underline{D}}_1 + c_2\underline{\underline{D}}_2) = c_1\underline{\underline{T}}(\underline{\underline{D}}_1) + c_2\underline{\underline{T}}(\underline{\underline{D}}_2)$

$$\boxed{\underline{\underline{T}} = \left(-p + \lambda_0(\nabla\cdot\underline{v})\right)\underline{\underline{I}} + 2\mu_0\underline{\underline{D}} \quad \text{Newtonian Fluid}}$$

because $\nu = 0$, $\mu = \text{constant} = 2\mu_0$, μ_0 = shear viscosity

λ = bulk viscosity

$$T_{ij} = -p\delta_{ij} + \lambda_0 v_{k,k}\delta_{ij} + \mu_0(v_{i,j} + v_{j,i})$$

---

the equations of motion for a Newtonian Fluid

$$\boxed{\begin{gathered}\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\underline{v} = 0 \\ \rho\left(\frac{\partial\underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla p + (\lambda_0 + \mu_0)\nabla(\nabla\cdot\underline{v}) + \mu_0\nabla^2\underline{v} + \rho\underline{F} \\ \text{Compressible Navier-Stokes equations}\end{gathered}}$$

because $T_{ji,j} = -p\delta_{ij} + \lambda_0 v_{k,k}\delta_{ij} + \mu_0(v_{i,j} + v_{j,i}) = -p_{,i} + \lambda_0 v_{k,ki} + \mu_0(v_{i,jj} + v_{j,ij})$

Derivation of Shock Conditions

September 29, 2003

Consider a shock wave moving with speed U. The state behind the shock is given by p_1, ρ_1, u_1, a_1, M_1. The state ahead of the shock is at rest and is given by p_0, ρ_0, $u_0 = 0$, a_0, M_0. The jump conditions across the shock are given by

$$\begin{aligned} -U\,[\,\rho\,] + [\,\rho u\,] &= 0 \\ -U\,[\,\rho u\,] + [\,\rho u^2 + p\,] &= 0 \\ -U\left[\,\frac{1}{2}\rho u^2 + e\,\right] + \left[\,\left(\frac{1}{2}\rho u^2 + \rho e\right) + pu\,\right] &= 0 \end{aligned}$$

If we make the substitution $v = U - u$ and $h = e + p/\rho$ where h is the enthalpy, then the jump conditions simplify to

$$\begin{aligned} [\,\rho v\,] &= 0 \\ [\,p + \rho v^2\,] &= 0 \\ \left[\,\frac{v^2}{2} + h\,\right] &= 0 \end{aligned}$$

Make the following substitutions

$$M = \frac{v}{a} \qquad p = \frac{\rho a^2}{\gamma} \qquad h = \frac{a^2}{\gamma - 1}$$

such that the jump conditions are expressed in terms of ρ, M and a. This will allow us to solve for M_1 in terms of M_0.

$$\begin{aligned} [\,\rho M a\,] &= 0 && (a) \\ [\,\rho a^2\left(1 + \gamma M^2\right)\,] &= 0 && (b) \\ \left[\,a^2\left(1 + \frac{1}{2}(\gamma - 1)M^2\right)\,\right] &= 0 && (c) \end{aligned}$$

Now calculate $(b)^2/(a)^2(c)$

$$\left[\,\frac{\rho^2 a^4 (1 + \gamma M^2)^2}{\rho^2 M^2 a^2 (1 + \frac{1}{2}(\gamma - 1)M^2)a^2}\,\right] = 0 \qquad \rightarrow \qquad \left[\,\frac{(1 + \gamma M^2)^2}{M^2(1 + \frac{1}{2}(\gamma - 1)M^2)}\,\right] = 0$$

Now expand the jump condition and solve for M_1

$$\begin{aligned} \frac{(1 + \gamma M_1^2)^2}{M_1^2\left(1 + \frac{1}{2}(\gamma - 1)M_1^2\right)} &= \frac{(1 + \gamma M_0^2)^2}{M_0^2\left(1 + \frac{1}{2}(\gamma - 1)M_0^2\right)} \\ (1 + 2\gamma M_1^2 + \gamma^2 M_1^4)\left(M_0^2 + \frac{1}{2}(\gamma - 1)M_0^4\right) &= (1 + 2\gamma M_0^2 + \gamma^2 M_0^4)\left(M_1^2 + \frac{1}{2}(\gamma - 1)M_1^4\right)\right) \end{aligned}$$

$$\begin{aligned} &M_0^2 + 2\gamma M_0^2 M_1^2 + \gamma^2 M_0^2 M_1^4 + \frac{1}{2}(\gamma - 1)M_0^4 + \gamma(\gamma - 1)M_0^4 M_1^2 + \frac{\gamma^2}{2}(\gamma - 1)M_0^4 M_1^4 \\ &\quad = \quad M_1^2 + 2\gamma M_0^2 M_1^2 + \gamma^2 M_0^4 M_1^2 + \frac{1}{2}(\gamma - 1)M_1^4 + \gamma(\gamma - 1)M_0^2 M_1^4 + \frac{\gamma^2}{2}(\gamma - 1)M_0^4 M_1^4 \\ &M_0^2 - M_1^2 + \gamma^2 M_0^2 M_1^2\left(M_1^2 - M_0^2\right) + \frac{1}{2}(\gamma - 1)\left(M_0^4 - M_1^4\right) + \gamma(\gamma - 1)M_0^2 M_1^2\left(M_0^2 - M_1^2\right) \;=\; 0 \end{aligned}$$

Divide by $M_0^2 - M_1^2$

$$\begin{aligned} 1 - \gamma^2 M_0^2 M_1^2 + \frac{1}{2}(\gamma - 1)\left(M_0^2 + M_1^2\right) + \gamma(\gamma - 1)M_0^2 M_1^2 &= 0 \\ 1 + \frac{1}{2}(\gamma - 1)\left(M_0^2 + M_1^2\right) - \gamma M_0^2 M_1^2 &= 0 \\ M_1^2\left(\frac{1}{2}(\gamma - 1) - \gamma M_0^2\right) &= -1 - \frac{1}{2}(\gamma - 1)M_0^2 \end{aligned}$$

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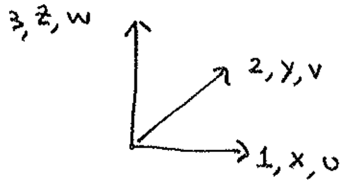
if the Fluid is incompressible, $\frac{\partial \rho}{\partial t} + v \cdot \nabla \rho = 0$

$$\Rightarrow \boxed{\nabla \cdot v = 0}$$

incompressible
Navier
Stokes

$$\Rightarrow \rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}\right) = -\nabla p + \nu_0 \nabla^2 \underline{v} + \rho \underline{f}$$

write out the components of the incompressible equations



$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \nu_0\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \rho f$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \nu_0\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \rho g$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \nu_0\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \rho h$$

4 equations, 5 unknowns

To solve these equations, consider the following generalization, with $f = f(x)$ and $g(\xi)$ an arbitrary function

$$\left(\frac{\partial}{\partial t} + \lambda\frac{\partial}{\partial x}\right)\eta = \frac{\partial f}{\partial x} \quad \rightarrow \quad \left(\frac{\partial}{\partial t} + \lambda\frac{\partial}{\partial x}\right)\eta = \left(\frac{\partial}{\partial t} + \lambda\frac{\partial}{\partial x}\right)\frac{f}{\lambda} \quad \rightarrow \quad \eta = \frac{f}{\lambda} + g(x - \lambda t)$$

Therefore integration of equations (4), (5), (6) results in

$$\begin{aligned}
\tilde{p} + a_1\rho_1\tilde{u} &= \frac{-a_1^2\rho_1 u_1}{(u_1 + a_1)A_0}\tilde{A} + F[x - (u_1 + a_1)t] \\
a_1^2\tilde{\rho} - \tilde{p} &= H(x - u_1 t) \\
\tilde{p} - a_1\rho_1\tilde{u} &= \frac{-a_1^2\rho_1 u_1}{(u_1 - a_1)A_0}\tilde{A} + G[x - (u_1 - a_1)t]
\end{aligned}$$

where F, G, and H are arbitrary functions.

10/17/03

Scaling of the Navier-Stokes Equations

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \nu\nabla^2\underline{u}$$

let $\underline{u} = U\,\underline{u}'$

$x = L\,x'$

$t = T\,t'$

$p = P\,p'$

$$\rightarrow \nabla = \frac{\partial}{\partial x_i} = \frac{\partial x_i'}{\partial x_i}\frac{\partial}{\partial x_i'} = \frac{1}{L}\frac{\partial}{\partial x_i'} = \frac{1}{L}\nabla'$$

$$\rightarrow \rho\frac{U}{T}\frac{\partial\underline{u}'}{\partial t'} + \rho\frac{U^2}{L}\,\underline{u}'\cdot\nabla'\underline{u}' = -\frac{P}{L}\nabla'p' + \text{[illegible]} + \frac{\nu U}{L^2}\nabla'^2\underline{u}'$$

$\uparrow T = \frac{L}{U}$ $\qquad$ $\uparrow P = \rho U^2$

$$\rho\frac{U^2}{L}\left(\frac{\partial\underline{u}'}{\partial t'} + \underline{u}'\cdot\nabla'\underline{u}'\right) = -\frac{\rho U^2}{L}\nabla'p' + \frac{\nu U}{L^2}\nabla'^2\underline{u}'$$

divide by $\rho\frac{U^2}{L}$

$$\frac{\nu U}{L^2}\cdot\frac{L}{\rho U^2} = \frac{1}{\frac{\rho U L}{\nu}} = \frac{1}{Re}$$

$$\rightarrow \frac{\partial\underline{u}'}{\partial t} + \underline{u}'\cdot\nabla'\underline{u}' = -\nabla'p' + \frac{1}{Re}\nabla'^2\underline{u}'$$

10/17/03

Derivation of Boundary Layer Eqns

Navier Stokes for an incompressible fluid

$$\nabla \cdot \underline{u} = 0$$

$$\frac{\partial u}{\partial t} + \underline{u} \cdot \nabla \underline{u} = \frac{-1}{\rho} \nabla p + \nu \nabla^2 \underline{u}$$

Consider a 2D flow: $\underline{u} = \left[u(x,y,t), v(x,y,t), 0 \right]$

$$\boxed{\begin{aligned}
&① \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\
&② \quad \rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\
&③ \quad \rho \frac{\partial v}{\partial t} + \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)
\end{aligned}}$$

Nondimensionalize with

$u = U u' \qquad x = L x' \qquad t = T t'$

$v = V v' \qquad y = \delta y' \qquad p = P p'$

where $P = \rho U^2$, $U = \frac{L}{T}$ and $\delta << L$

Equation 1

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{U}{L} \frac{\partial u'}{\partial x'} + \frac{V}{\delta} \frac{\partial v'}{\partial y'} = 0$$

$$\rightarrow \quad \frac{\partial u'}{\partial x'} + \frac{V}{U} \frac{L}{\delta} \frac{\partial v'}{\partial y'} = 0$$

define $\boxed{\frac{V}{U} \frac{L}{\delta} \equiv 1}$

Equation 2

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$\rightarrow \rho \frac{U}{T} \frac{\partial u'}{\partial t'} + \left(\rho \frac{U^2}{L} \right) u' \frac{\partial u'}{\partial x'} + \left(\rho \frac{U V}{\delta} \right) v' \frac{\partial u'}{\partial y'} = \frac{-P}{L} \frac{\partial p'}{\partial x'} + \nu \left(\frac{U}{L^2} \frac{\partial^2 u'}{\partial x'^2} + \frac{U}{\delta^2} \frac{\partial^2 u'}{\partial y'^2} \right)$$

$\uparrow$ $T = \frac{L}{U}$

$\frac{v}{u} \frac{L}{\delta} = 1$, from equation 1

$\uparrow$ $P = \rho U^2$

$$\rightarrow \frac{\rho U^2}{L} \left(\frac{\partial u'}{\partial t'} + u' \frac{\partial u'}{\partial x'} + \overbrace{\left(\frac{V}{U} \frac{L}{\delta} \right)} v' \frac{\partial u'}{\partial y'} \right) = - \frac{\rho U^2}{L} \frac{\partial p'}{\partial x'} + \frac{\nu U}{\delta^2} \left(\frac{\delta^2}{L^2} \frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2} \right)$$

divide through by $\frac{\rho U^2}{L}$ with

$$\rightarrow \cancel{\frac{\nu U}{\delta^2} \Big/ \frac{L}{\rho U^2}} \quad \frac{\rho U^2}{L} \cdot \frac{\delta^2}{\nu U} = \frac{\rho U L}{\nu} \frac{\delta^2}{L^2} = \boxed{Re \left(\frac{\delta}{L} \right)^2 \equiv 1} \rightarrow \frac{\delta^2}{L^2} = \frac{1}{Re}$$

$$\frac{\partial u'}{\partial t'} + u' \frac{\partial u'}{\partial x'} + v' \frac{\partial u'}{\partial y'} = - \frac{\partial p'}{\partial x'} + \frac{1}{Re} \frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2}$$

the order 1 equation is

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}$$

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Equation 3

$$\rho \frac{\partial v}{\partial t} + \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\rightarrow \underset{\uparrow\, T = \frac{L}{U}}{\frac{\rho V}{T}} \frac{\partial v'}{\partial t'} + \left(\rho \frac{UV}{L}\right) u' \frac{\partial v'}{\partial x'} + \left(\rho \frac{V^2}{\delta}\right) v' \frac{\partial v'}{\partial y'} = -\underset{\uparrow}{\frac{P}{\delta}} \frac{\partial p'}{\partial y'} + \nu\left(\frac{V}{L^2} \frac{\partial^2 v'}{\partial x'^2} + \frac{V}{\delta^2} \frac{\partial^2 v'}{\partial y'}\right)$$

$P = \rho U^2 \sim \frac{\nu L U}{\delta^2}$ (From Eq 2)

$$\frac{\rho V U}{L}\left(\frac{\partial v'}{\partial t'} + u' \frac{\partial v'}{\partial x'} + \overbrace{\frac{V}{U}\frac{L}{\delta}}^{1,\text{ from Eq } 1} v' \frac{\partial v'}{\partial y'}\right) = -\frac{\nu L U}{\delta^3} \frac{\partial p'}{\partial y'} + \frac{\nu V}{\delta^2}\left(\underset{\uparrow\, \frac{1}{Re}}{\frac{\delta^2}{L^2}} \frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right)$$

divide by $\frac{\nu U L}{\delta^3}$

$$\frac{\rho V \cancel{U}}{L} \frac{\delta^3}{\nu \cancel{U} L} = \frac{\rho}{\nu} \frac{V}{L^2} \delta^3 = \frac{\rho}{\nu} \frac{\delta^3}{L^2}\left(\frac{U\delta}{L}\right) = \frac{\rho U L}{\nu}\left(\frac{\delta^4}{L^4}\right) = Re \frac{1}{Re^2} = \frac{1}{Re}$$

$$\frac{\cancel{\nu} V}{\delta^2} \cdot \frac{\delta^3}{\cancel{\nu} U L} = \frac{V\delta}{UL} = \left(\frac{\delta}{L}\right)^2 = \frac{1}{Re}$$

$$\rightarrow \frac{1}{Re}\left(\frac{\partial v'}{\partial t'} + u' \frac{\partial v'}{\partial x'} + v' \frac{\partial v'}{\partial y'}\right) = \frac{\partial p'}{\partial y'} + \frac{1}{Re}\left(\frac{1}{Re} \frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right)$$

the order 1 equation is $\frac{\partial p}{\partial y} = 0$ → p is independent of y such that $p = p(x)$

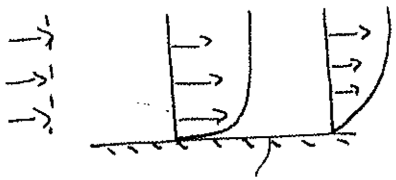
$$\Rightarrow \boxed{\begin{gathered} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial^2 u}{\partial y^2} \end{gathered}}$$

steady boundary layer, $\frac{dP}{dx} = 0$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2}$$

$\rightarrow P = P(x,\infty)$, $\frac{\partial P}{\partial x}(x, \infty = 0)$



$y=0, u=0, v=0$

stream function in 2D flow

satisfy $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ without any loss of generality by assuming there exists some function $\psi(x,y)$ such that $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$

if you substitute $u = \psi_y$ and $v = -\psi_x$ into $u_x + v_y = 0$, then you see the equation is satisfied in ψ, (if the velocities are continuous).

Now substitute into momentum:

$$\psi_y \, \psi_{xy} - \psi_x \, \psi_{yy} = \psi_{yyy}$$

Similarity

$u(x,y) = h\left(\frac{y}{g(x)}\right)$, $\eta = \frac{y}{g(x)}$

this satisfies BC at plate: $u(x,0) = h(0) = 0$

$\rightarrow \psi_y = h\left(\frac{y}{g(x)}\right) \rightarrow \psi = \int^y h\left(\frac{y}{g(x)}\right) dy + C(x)$

a note on stream function - parametrize a curve, $x = x(s)$, $y = y(s)$ and calculate derivative wrt s:

consider $\psi(x(s), y(s))$

$$\frac{d}{ds} \psi(x(s), y(s)) = \psi_x \, x_s + \psi_y \, y_s$$

IF these curves are particle paths ($s = t$) then

$$\frac{d\psi}{ds} = \psi_x \frac{dx}{dt} + \psi_y \frac{dy}{dt} = (-v)u + u(v) = 0$$

Therefore on a particle path, $\psi =$ constant

In the boundary layer, we can immediately identify a particle path on which $\psi =$ constant, and that is the line $y = 0 \Rightarrow \psi = 0$

This implies that the constant of integration in

$$\frac{\partial \psi}{\partial y} = h\left(\frac{y}{g(x)}\right)$$

$$\psi = \int_0^y h\left(\frac{y}{g(x)}\right) dy + C(x)$$

$$\psi(x,0) = 0 = C(x) \quad \rightarrow \quad \psi = \int_0^y h\left(\frac{y}{g(x)}\right) dy$$

$$\eta = \frac{y}{g(x)} \quad \rightarrow \quad y = \eta\, g(x) \quad , dy = d\eta\, g(x)$$

$$\rightarrow \quad \psi = g(x) \int_0^\eta h(\eta)\, d\eta = g(x)\, f(\eta) \quad \longrightarrow \quad \boxed{\psi = g(x)\, f(\eta)}$$

Substitute into

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = \psi_{yyy}$$

$$\psi_y = g(x)\, f'(\eta) \frac{1}{g(x)} = f'(\eta)$$

$$\psi_x = g'(x)\, f(\eta) + g(x)\, f'(\eta) \left(\frac{-y}{g(x)^2} g'(x)\right) = g'(x) f(\eta) - \frac{y}{g(x)} g'(x)\, f'(\eta)$$

$$\psi_{yy} = f''(\eta) \frac{1}{g(x)}$$

$$\psi_{yyy} = f'''(\eta) \frac{1}{g(x)^2}$$

~~$\psi_{xy} = g''(x)\, g'(x)\, f'(x) \frac{1}{g(x)} - \frac{g'(x)}{g(x)} f'(\eta) - \frac{y}{g(x)} g'(x)$~~

$$\psi_{xy} = \psi_{yx} = f''(\eta) \left(\frac{-y}{g(x)^2} g'(x)\right)$$

combining,

$$\cancel{f'(\eta)\, f''(\eta)\left(\frac{-y}{g(x)^2}\, g'(x)\right)} - \left(g'(x)\, f(\eta) - \cancel{\frac{y}{g(x)}\, g'(x)\, f'(\eta)}\right) f''(\eta)\,\frac{1}{g(x)} = f'''(\eta)\,\frac{1}{g(x)^2}$$

$\rightarrow$ $f'''(\eta) = -g(x)\, g'(x)\, f(\eta)\, f''(\eta)$ nonlinear ordinary differential equation

let $g(x)\, g'(x) = 1$ $\rightarrow$ $g\,dg = dx$

$\rightarrow$ $f'''(\eta) + f f''(\eta) = 0$

$$\frac{g^2}{2} = x + d$$

$$\downarrow$$

$$g = \sqrt{2x + 2d}$$

what is the value of the constant d?

$u = \frac{\partial \psi}{\partial y}$ $\rightarrow$ $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} f'(\eta) = \underbrace{f''(\eta)\,\frac{1}{g(x)}}_{\text{shear is infinite when } g(x) = 0?,\ d = 0}$ } shear

at leading edge of plate, flow at left is uniform and just to the right, on plate, so $u(x,0) = 0$, therefore singularity at $x = 0$,

" $\frac{\partial u}{\partial y} = \infty$ "

Pick the boundary conditions

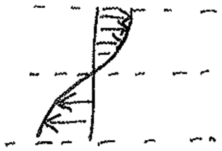
$f(0) = 0$ $\rightarrow$ makes $y = 0$ a streamline

$f'(0) = 0$

$f'(\infty) = 1$

Consider the following example

shear layer



vorticity equation, $\underline{\omega} = \nabla \times \underline{v}$

NS : $\frac{\partial v}{\partial t} + \underline{v} \cdot \nabla \underline{v} = - \nabla p + \frac{1}{Re} \nabla^2 \underline{v}$

take curle of NS :

term by term, • $\nabla \times \frac{\partial v}{\partial t} = \frac{\partial}{\partial t}(\nabla \times \underline{v}) = \frac{\partial \omega}{\partial t}$ assuming can interchange order of differentiations

• $\nabla \times \nabla p = 0$

• $\nabla \times \nabla^2 \underline{v} = \nabla^2 (\nabla \times \underline{v}) = \nabla^2 \underline{\omega}$

notice, $\underline{v} \times \underline{\omega} = \underline{v} \times (\nabla \times \underline{v})$

$$\rightarrow (\underline{v} \times \underline{\omega})_i = \epsilon_{ijk} v_j \, \epsilon_{klm} v_{m,l}$$

$$= \left(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}\right) v_j v_{m,l}$$

$$= v_j v_{j,i} - v_j v_{i,j}$$

$$= v_j \frac{\partial}{\partial x_i} v_j - v_j \frac{\partial v_i}{\partial x_j}$$

$$= \frac{1}{2} \frac{\partial v_j^2}{\partial x_i} - v_j \frac{\partial}{\partial x_j}(v_i)$$

$$= \frac{1}{2} \nabla |\underline{v}|^2 - v \cdot \nabla v$$

$$\rightarrow \underline{v} \cdot \nabla \underline{v} = \frac{1}{2} \nabla |\underline{v}|^2 - \underline{v} \times \underline{\omega}$$

$$\rightarrow \frac{\partial v}{\partial t} + \nabla \frac{1}{2} |\underline{v}|^2 - \underline{v} \times \underline{\omega} = - \nabla p + \frac{1}{Re} \nabla^2 \underline{v}$$

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$$\rightarrow \nabla\times(\underline{v}\cdot\nabla\underline{v}) = \nabla\times\left(\nabla\tfrac{1}{2}|\underline{v}|^2 - \underline{v}\times\underline{w}\right)$$

$$= -\nabla\times(\underline{v}\times\underline{w})$$, because curl of gradient vanishes so $\nabla\times\nabla|\underline{v}|^2 = 0$

$$\nabla\times(\underline{v}\times\underline{w}) = \epsilon_{ijk}\left(\epsilon_{klm}v_l w_m\right)_{,j}$$

$$= \epsilon_{ijk}\,\epsilon_{klm}\,(v_l w_m)_{,j}$$

$$= \left(\delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}\right)\left(v_{l,j}w_m + v_l w_{m,j}\right)$$

$$= v_{i,j}w_j + v_i \underbrace{\cancel{w_{j,j}}}_{0} - \cancel{v_{j,j}}w_i - v_j w_{i,j}$$

$$= w_j\frac{\partial}{\partial x_j}v_i - v_j\frac{\partial}{\partial x_j}w_i$$

0, because $v_{j,j} = \nabla\cdot\underline{v} = 0$, by conservation of mass

notice, $w_{j,j} = \nabla\cdot(\nabla\times\underline{v}) = \epsilon_{ijk}(v_{k,j})_{,i}$

$$= \epsilon_{ijk}v_{k,ji} = -\epsilon_{jik}v_{k,ji}$$

$= 0$ because equal to minus of itself

$$= \underline{w}\cdot\nabla\underline{v} - \underline{v}\cdot\nabla\underline{w}$$

$\Rightarrow$ **vorticity equation**

$$\boxed{\frac{\partial\underline{w}}{\partial t} + \underline{v}\cdot\nabla\underline{w} - \underline{w}\cdot\nabla\underline{v} = \frac{1}{R}\nabla^2\underline{w}}$$

- nonliner
- still has the diffusive term, therefore vorticity DIFFUSES

- notice, $\frac{\partial\underline{w}}{\partial t} + \underline{v}\cdot\nabla\underline{w} = \frac{D\underline{w}}{Dt}$, material derivative → vorticity is transported with fluid

the term $\underline{w}\cdot\nabla\underline{v}$ refers to vortex stretching

$$\frac{D\underline{w}}{Dt} = \underline{w}\cdot\nabla\underline{v} + \frac{1}{R}\nabla^2\underline{w}$$

vortex stretching

$$\underline{\omega} \cdot \nabla \underline{v} = \underline{\omega} \cdot \left[\tfrac{1}{2} \left(\nabla \underline{v} + \nabla v^T \right) + \tfrac{1}{2} \left(\nabla \underline{v} - \nabla \underline{v}^T \right) \right]$$

$$= \underline{\omega} \cdot \underline{\underline{D}} + \underline{\omega} \cdot \underline{\underline{W}}$$

where $\underline{\underline{W}} = \begin{bmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & \omega_1 \\ w_2 & -w_1 & 0 \end{bmatrix}$

$$\underline{\omega} \cdot \underline{\underline{W}} = [\omega_1, \omega_2, \omega_3] \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}$$

$$= [0, 0, 0]$$

consider the velocity vector $\underline{v} = (u, v, 0)$

ie a 2D motion, where $u = u(x,y)$, $v = v(x,y)$

then $\nabla \times v = \det \begin{vmatrix} i & j & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & 0 \end{vmatrix} = \underline{k} \left(v_x - u_y \right)$

$$\underline{\underline{D}} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}(\quad) & 0 \\ (\quad) & \frac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \underline{\omega} \cdot \underline{\underline{D}} = 0 \rightarrow \text{no stretching in 2-D}$$

~~If 2D,~~

If $R \rightarrow \infty$, $\dfrac{D\underline{\omega}}{Dt} = \underline{\omega} \cdot \underline{\underline{D}}$

If $\underline{\omega}(\underline{x}, 0) = 0$, $\underline{\omega}(\underline{x}, t) = 0$ ∀ time

If $\underline{\underline{D}}$ = constant, $\underline{\omega} = \underline{\omega}_0 e^{\int \underline{\underline{D}}\, dt} = w_0 e^{\underline{\underline{D}} t}$ for $\underline{\omega}_0$ = constant

an~~d $\underline{w}_0$~~

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Midterm Tuesday 10/28 2-4 pm

covers: tensors, vector calculus, eqns of motion (Reynolds transport) stress/rate of strain, simple solutions

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if $Re \gg 1$

$$\frac{\partial \underline{w}}{\partial t} + \underline{v} \cdot \nabla \underline{w} = \frac{D\underline{w}}{Dt} = \underline{w} \cdot \nabla \underline{v} = \underline{w} \cdot \underline{\underline{D}}$$

$$\underline{w}(\underline{x}, 0) = 0 \rightarrow \underline{w}(\underline{x}, t) = 0 \text{ for all } t > 0$$

recall

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} = -\nabla p + \frac{1}{Re} \nabla^2 \underline{v}$$

$$\frac{\partial \underline{v}}{\partial t} + \nabla \frac{1}{2} |\underline{v}|^2 - \underline{v} \times \underline{w} = -\nabla p + \frac{1}{Re} \nabla^2 \underline{v}$$

assume $Re \gg 1$ and steady flow
(nearly inviscid)

$$\rightarrow \nabla \frac{1}{2} |\underline{v}|^2 + \nabla p = \underline{v} \times \underline{w}$$

$$\nabla \left(\frac{1}{2} |\underline{v}|^2 + p \right) = \underline{v} \times \underline{w}$$

Now assume vorticity $\underline{w} = 0$

$$\rightarrow \nabla \left(\frac{1}{2} |\underline{v}|^2 + p \right) = 0$$

$$\rightarrow \frac{\partial}{\partial x} \left(\frac{1}{2} |\underline{v}|^2 + p \right) = 0, \quad \frac{\partial}{\partial y} \left(\frac{1}{2} |\underline{v}|^2 + p \right) = 0, \quad \frac{\partial}{\partial z} \left(\frac{1}{2} |\underline{v}|^2 + p \right) = 0$$

$\Rightarrow \frac{1}{2} |\underline{v}|^2 + p$ can't depend on x, y, z

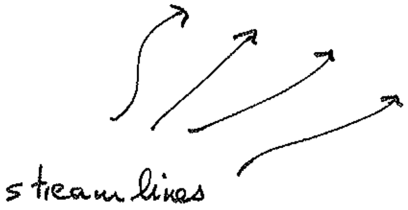
$$\Rightarrow \boxed{\frac{1}{2} |\underline{v}|^2 + p = \text{constant}, \text{ Bernoulli's Equation}}$$

Back to

$$\nabla\left(\tfrac{1}{2}|\underline{v}|^2+p\right)=\underline{v}\times\underline{w}$$

Consider the case where $\underline{w}\neq 0$ (nonzero vorticity)

(still inviscid steady flow)



we want to integrate this equation along streamlines

$$\int_{\underline{\xi}}^{\underline{x}}\nabla\left(p+\tfrac{1}{2}|\underline{v}|^2\right)\cdot d\underline{x}=\int_{\underline{\xi}}^{\underline{x}}(\underline{v}\times\underline{w})\cdot d\underline{x}$$

line integral in 3 D

since $d\underline{x}$ is ~~perpindicular~~ parallel to velocity

and velocity perpindicular to $\underline{v}\times\underline{w}$

then $d\underline{x}\perp\underline{v}\times\underline{w}$ → the right hand

side vanishes

$$\int_{\underline{\xi}}^{\underline{x}}\nabla\left(p+\tfrac{1}{2}|\underline{v}|^2\right)\cdot d\underline{x}=\int_{\underline{\xi}}^{\underline{x}}\frac{\partial}{\partial x}(\ )dx+\frac{\partial}{\partial y}(\ )dy+\frac{\partial}{\partial z}(\ )dy$$

$$=\int_{\underline{\xi}}^{\underline{x}}d(\ )\quad=\left[p+\tfrac{1}{2}|\underline{v}|^2\right]_{\underline{\xi}}^{\underline{x}}$$

$$\to\quad p+\tfrac{1}{2}|\underline{v}|^2\Big]_{\underline{x}}=p+\tfrac{1}{2}|\underline{v}|^2\Big]_{\underline{\xi}}$$

$$\Rightarrow\quad\boxed{p+\tfrac{1}{2}|\underline{v}|^2\ \text{is constant on streamlines}}$$

- Now consider vortex lines, which at each point are tangent to vorticity $\underline{\omega}$

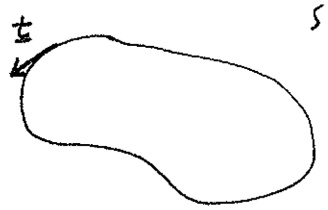
$$\int \nabla\left(p + \tfrac{1}{2}|\underline{v}|^2\right)\cdot dx = \int \underline{v}\times\underline{\omega}\cdot \underline{dx} = 0$$

so $\underline{dx} \parallel \underline{\omega}$ → $\underline{v}\times\underline{\omega} \perp \underline{dx}$ cause $\underline{\omega} \perp \underline{v}\times\underline{\omega}$

→ $\boxed{p + \tfrac{1}{2}|\underline{v}|^2 = \text{constant on vortex lines}}$

- If $\underline{\omega} = 0$ (But possibly unsteady)

$\underline{\omega} = \nabla\times\underline{v}$



arclength

circulation

stoke's Thm: $\underbrace{\oint \underline{v}\cdot\underline{t}\, dl}_{\text{circulation}} = \iint \underline{n}\cdot(\nabla\times\underline{v})\, dA$

if ~~velocity~~ ~~is~~ vorticity is zero

then $\oint_C \underline{v}\cdot\underline{t}\, dl = 0$ ~~everywhere~~

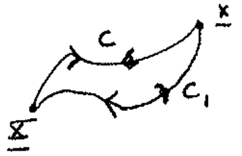
For all curves C (nice curves)

$\underline{v}\cdot\underline{t}$ is the component of the velocity along the direction of the curve

→ $\boxed{\text{circulation} = 0 \text{ for any curve for irrotational flows}}$

Now integrate from starting point $\underline{X}$ to some end point $\underline{x}$ (variable) on some curve C

$$\int_{\underline{X}}^{\underline{x}} \underline{t}\cdot\underline{v}\, dl$$



Now consider same points, different curve

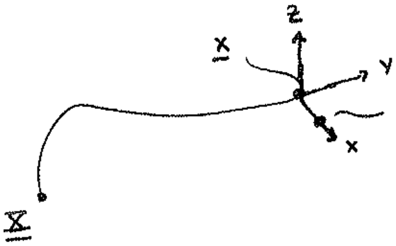
$$\oint_{C^*} \underline{v}\cdot\underline{t}\, dl = \int_{C} \underline{v}\cdot\underline{t}\, dl - \int_{C_1} \underline{v}\cdot\underline{t}\, dl = 0$$

→ $\boxed{\int_C \underline{v}\cdot\underline{t}\, dl = \int_{C_1} \underline{v}\cdot\underline{t}\, dl}$ the integral is path independent

Define

$$\phi(\underline{x}) = \int_{\underline{X}}^{\underline{x}} \underline{v}\cdot\underline{t}\; dl$$

the line integral doesn't depend on path (for irrotational flow) therefore only depends on endpoint $\underline{x}$



$\underline{x} + \underline{\Delta x}$, $\underline{\Delta x} = (\Delta x, 0, 0)$

$\underline{t} = \underline{e}_x$ because path in direction of x-axis

$$\phi(\underline{x}+\underline{\Delta x}) = \int_{\underline{X}}^{\underline{x}} \underline{v}\cdot\underline{t}\; dl + \int_{\underline{x}}^{\underline{x}+\underline{\Delta x}} \underline{v}\cdot\underline{t}\; dl$$

$$\phi(\underline{x}+\Delta\underline{x}) = \phi(\underline{x}) + \int_{\underline{x}}^{\underline{x}+\Delta x} \underline{v}\cdot\underline{e}_x\; dx$$

, let $\underline{v} = (u, v, w)$

$$\rightarrow \phi(\underline{x}+\Delta\underline{x}) - \phi(\underline{x}) = \int_{\underline{x}}^{\underline{x}+\Delta x} u\, dx$$

divide by Δx, take limit

$$\rightarrow \lim_{\Delta x\to 0} \frac{\phi(\underline{x}+\Delta\underline{x}) - \phi(\underline{x})}{\Delta x} = \lim_{\Delta x\to 0} \frac{1}{\Delta x}\int_{\underline{x}}^{\underline{x}+\Delta x} u\, dx$$

$$\longrightarrow \boxed{\frac{\partial\phi(\underline{x})}{\partial x} = u(\underline{x}) \quad \text{velocity potential}}$$

and similarly,

$$\boxed{\frac{\partial\phi}{\partial y} = v \quad , \quad \frac{\partial\phi(\underline{x})}{\partial z} = w(\underline{x})}$$

$\rightarrow$ velocity potential , $\underline{v} = \nabla\phi$

$$\underline{\omega} = \nabla\times\underline{v} = \nabla\times(\nabla\phi) = 0$$

example

$u = x, \quad v = -y, \quad w = 0$

vorticity

$$\underline{\omega} = \nabla \times v = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x & -y & 0 \end{vmatrix}$$

$= 0$, irrotational flow

velocity potential

$$\frac{\partial \phi}{\partial x} = u = x \quad , \quad \frac{\partial \phi}{\partial y} = v = -y \quad , \quad \frac{\partial \phi}{\partial z} = w = 0$$

$$\phi = \frac{x^2}{2} + f(y) \rightarrow \frac{\partial \phi}{\partial y} = f'(y) = -y \rightarrow f(y) = -\frac{y^2}{2} + c$$

$$\rightarrow \boxed{\phi = \frac{x^2}{2} - \frac{y^2}{2} + c}$$

stream function

$$\frac{\partial \psi}{\partial y} = u, \quad \frac{\partial \psi}{\partial x} = -v$$

$$\frac{\partial \psi}{\partial y} = x \rightarrow \quad \psi = xy + g(x)$$

$$\frac{\partial \psi}{\partial x} = y + g'(x) = y \rightarrow g'(x) = 0 \rightarrow g(x) = \hat{c}$$

$$\boxed{\psi = xy + \tilde{c}}$$

For $\nabla \cdot \underline{v} = 0$ and ~~ϕ~~ $\underline{v} = \nabla \phi$

then $\nabla \cdot \nabla \phi = \boxed{\nabla^2 \phi = 0}$

there are many solution methods

boundary $\underline{v} \cdot \underline{n} = 0 \Rightarrow \underline{n} \cdot \nabla \phi = \frac{\partial \phi}{\partial n}$, Neumann condition

suppose vorticity is zero, so there exists a velocity potential, then

$$\frac{\partial \underline{v}}{\partial t} + \nabla \frac{1}{2} |v|^2 - \overset{0}{\cancel{v \times \underline{\omega}}} = -\nabla p + \frac{1}{Re} \nabla^2 \underline{v}$$

$$\frac{\partial}{\partial t} \nabla \phi + \nabla \frac{1}{2} |\nabla \phi|^2 = -\nabla p + \frac{1}{Re} \nabla^2 \nabla \phi$$

$$\rightarrow \nabla \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + p - \overset{0}{\cancel{\frac{1}{Re} \nabla^2 \phi}} \right) = 0$$

$$\rightarrow \boxed{\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + p = F(t) \quad \text{Bernoulli Eqn}}$$

$\Rightarrow$ if flow is irrotational,

$\nabla^2 \phi = 0$ tells us velocity

and this last equation tells us pressure

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Complex Variables to Solve ~~For~~ 2D steady, irrotational inviscid flows

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \rightarrow \text{ stream function (2D, steady)}$$

$$p + \frac{1}{2}\left(u^2 + v^2\right) = \text{const.}$$

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y} \quad \rightarrow \text{ due to irrotational}$$

stream function,

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

$$\underline{\omega} = \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \hat{k}\left(-\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}\right) \quad \rightarrow \underline{\text{stream function harmon}}$$

Define complex potential

$$F(z) = \phi + i\psi$$

check Cauchy-Riemann conditions:

$$\frac{\partial F}{\partial z} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial y} = u - iv$$

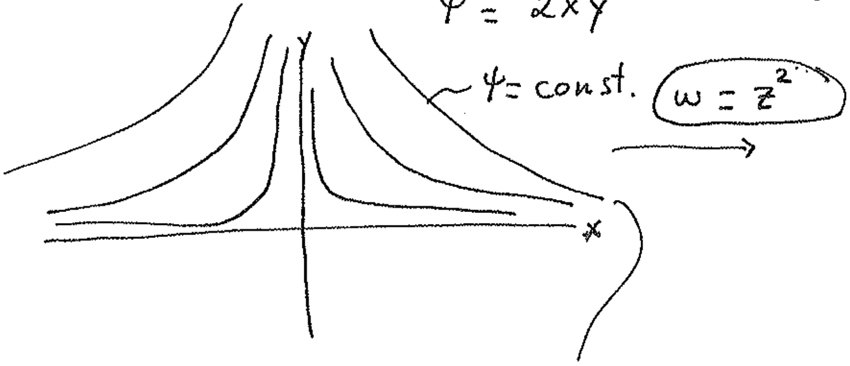
$$\frac{\partial \phi}{\partial x} = u, \quad \frac{\partial \psi}{\partial y} = u \quad \rightarrow \quad \phi_x = \psi_y$$

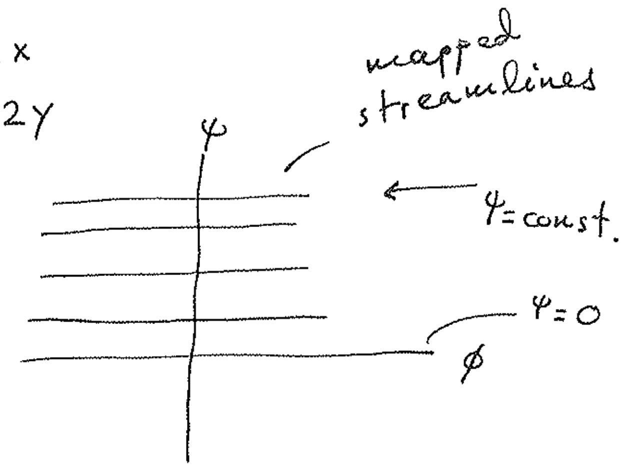
$$\frac{\partial \phi}{\partial y} = +v, \quad \frac{\partial \psi}{\partial x} = -v \quad \rightarrow \quad \phi_y = -\psi_x$$

example: $F(z) = z^2$

$\phi = x^2 - y^2$, $u = 2x$

$\psi = 2xy$ $\quad v = -2y$



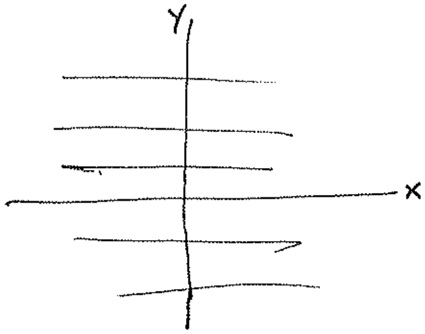


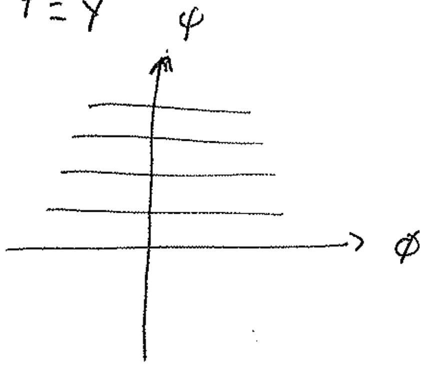
streamline, on this ψ = constant

$\psi = 2xy \rightarrow y = \frac{1}{2x}$

on streamline $\psi = 0 \rightarrow x = 0$ or $y = 0$

example $F(z) = z \quad \longrightarrow \quad \psi = y$





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Example $F(z) = Cz$, where C is a complex constant

$$C = U e^{i\alpha}$$

~~$\rightarrow F(z) = U e^{i\alpha} r e^{i\theta}$, $r = 1$~~

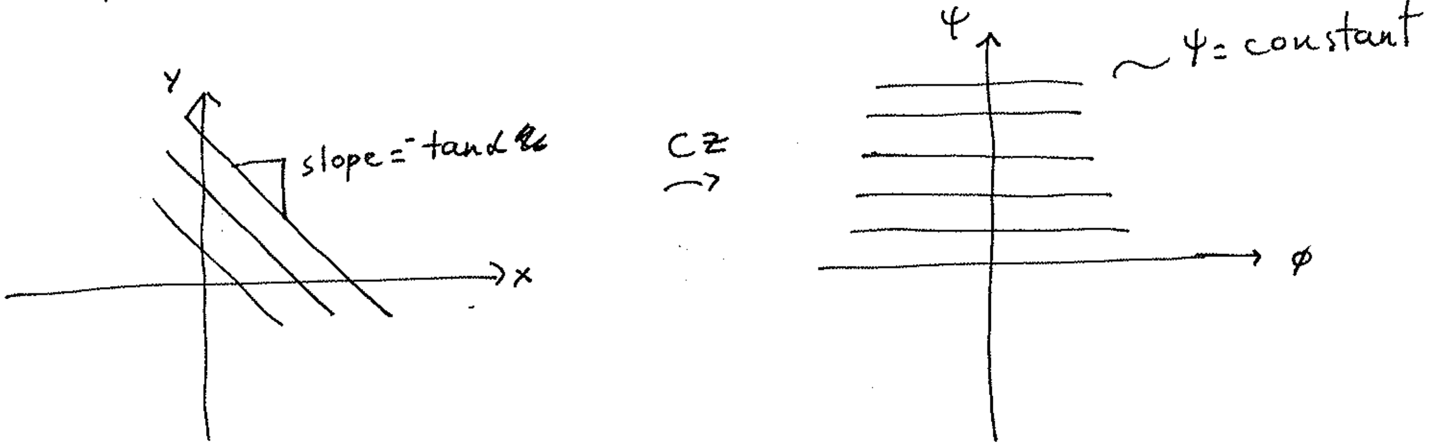
~~$= U e^{i(\theta+\alpha)}$~~

$$f(z) = U e^{i\alpha}(x+iy) = U(\cos\alpha + i\sin\alpha)(x+iy)$$

$\text{Re } F(z) = \phi$, $\text{Im } F(z) = \psi$

$\psi = U(y\cos\alpha + x\sin\alpha)$ ← this is linear

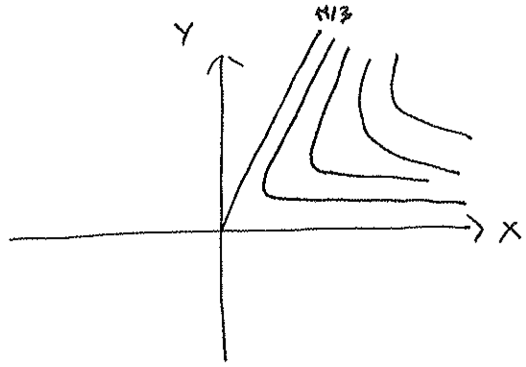
$\psi = \text{constant} \rightarrow y = \left(\frac{c}{U} - x\sin\alpha\right)\frac{1}{\cos\alpha}$

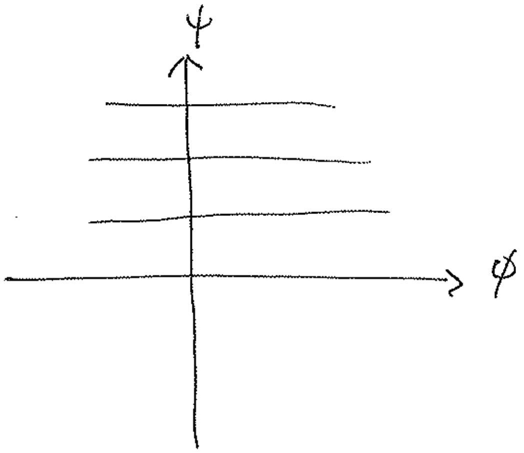


Example $f(z) = z^3$

58
62
66
70
74
78
82
86

90
94
98
02
2002
1958
44





let $f(z) = (re^{i\theta})^3 = r^3 e^{i3\theta} = r^3(\cos 3\theta + i \sin 3\theta)$

$\psi = \text{Im } f(z)$

when $\theta = 0$, $\psi = r^3 \sin 3\theta = 0$ → the line $y=0$ gets mapped to $\psi = 0$

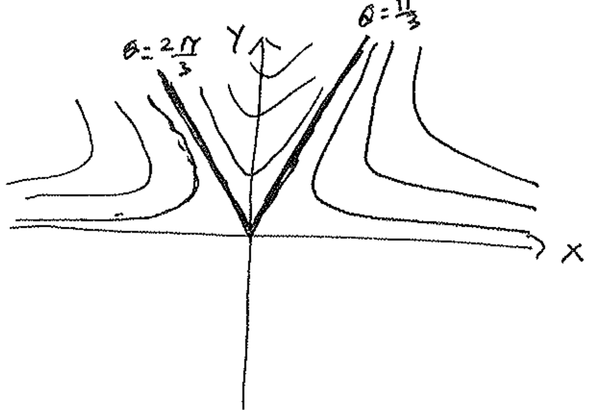
when $\theta = \frac{\pi}{3}$, $\psi = r^3 \sin 3 \cdot \frac{\pi}{3} = 0$ → the line $y = \frac{\sqrt{3}}{2}x$ gets mapped to $\psi = 0$

the stream function is:

$(x+iy)^3 = x^3 + 3ix^2y - 3xy^2 - iy^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$

→ $\psi = 3x^2y - y^3$

what does the rest look like?



$y(3x^2 - y^2) = c$

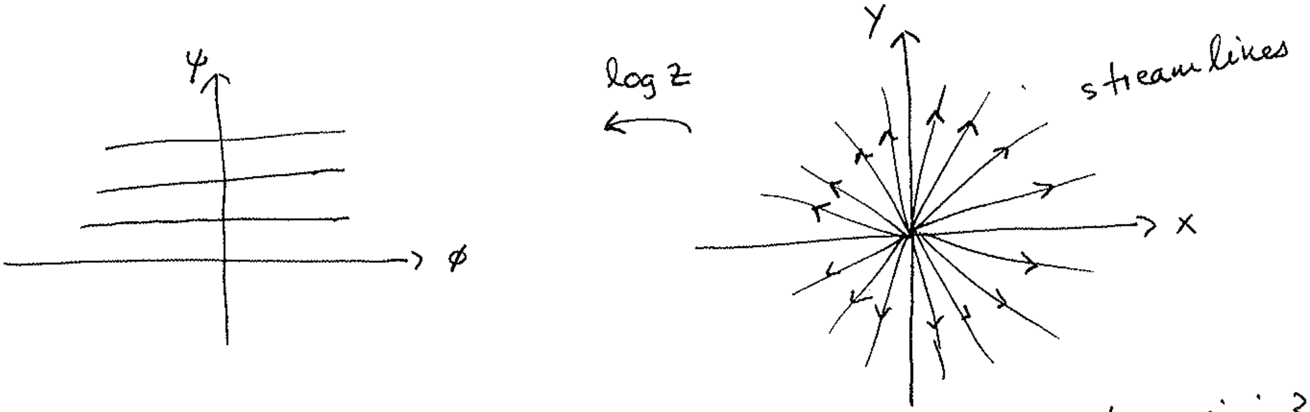
example

$$\boxed{f(z) = \log z}$$

$$z = re^{i\theta} \rightarrow f(z) = \log r + i\theta$$

$$f(z) = \ln\sqrt{x^2+y^2} + i\,\text{atan}\left(\frac{y}{x}\right) = \phi + i\psi$$

5 Br
2 Ver
2 Arg
1 It
1 En
1 Fr
12 sinc
58



what do the stream lines look like?

$$\psi = \arctan\left(\frac{y}{x}\right) = \text{const}$$

$$\boxed{y = x\tan(\text{const})}$$

straight lines coming through the origin

what about origin?

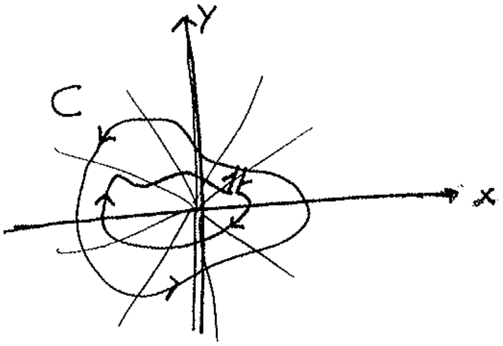
note, $f'(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$

$$u = \frac{x}{x^2+y^2}, \quad v = \frac{y}{x^2+y^2}$$

notice

$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$, except at $(x,y) = (0,0)$

what if you consider counterclockwise contour C?



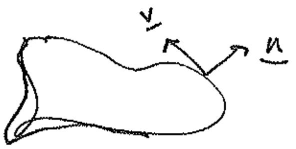
Let us calculate circulation

$$\oint_{|z|=1} \frac{x}{1}dx + \frac{y}{1}dy = \oint \frac{1}{2}d(x^2+y^2)$$

$$= \left.\frac{x^2+y^2}{2}\right]_{(x,y)}^{(x,y)} = 0$$

6/7

what is the flux of mass through C ?



$\int \underline{n} \cdot \underline{v} \, ds$

$$\int_{|z|=a} \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) \cdot \left(\frac{x}{a}, \frac{y}{a}\right) \overset{a\,d\theta}{/} = \int \left(\frac{x^2}{a^3} + \frac{y^2}{a^3}\right) \overset{a\,d\theta}{/} = \int \frac{1}{a} \cdot a\,d\theta = \int d\theta = 2\pi$$

Consider source + uniform Flow

$f_1(z) = A \log z$ — A is a measure of source strength

$f_2(z) = Cz$

$\rightarrow f(z) = A \log z + Cz$

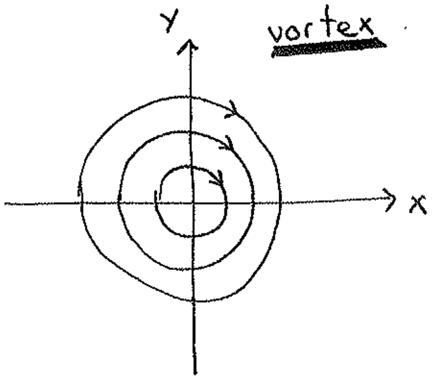
blah blah blah

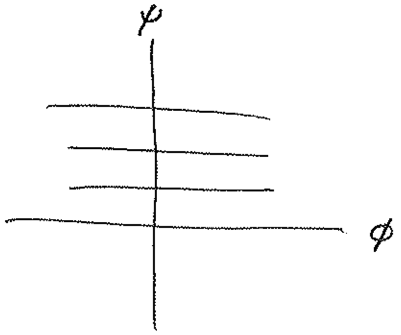
What about $f(z) = i \log z = i(\ln r + i\theta)$

$= \phi + i\psi$

Now instead of having $\theta = \text{const}$, we have $\psi = \ln r = \text{constant}$ ← which are concentric circles of radius ~~r=~~ ~~ln r~~ $= e^{\psi}$

vortex





$\psi = \ln r = \cancel{\ln} \sqrt{x^2+y^2} = \frac{1}{2} \ln(x^2+y^2)$

$u = \psi_y = \frac{1}{2} \frac{2y}{x^2+y^2} = \frac{y}{x^2+y^2}$, $v = \frac{-x}{x^2+y^2}$ } to determine direction of Flow at $(x,y) = (0,1)$ velocity is $(u,v) = (1,0)$ so clockwise

Is it still true that if I take a curve that doesn't encircle the origin, the circulation is zero.

$$\oint u\,dx + v\,dy = \oint \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$$

$$\phi = -\tan^{-1}\left(\frac{y}{x}\right) \qquad \phi_x = \frac{-\left(-y/x^2\right)}{1+\left(y/x\right)^2} = \frac{+y}{x^2+y^2}$$

$$\phi_y = \frac{-\left(y/x\right)}{1+\left(y/x\right)^2} = \frac{-x}{x^2+y^2}$$

$$\rightarrow \oint \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = \cancel{\oint \frac{+y\,dx \overset{\ominus}{+} x\,dy}{x^2+y^2}}$$

$$= \oint \frac{+y/x^2}{1+\left(y/x\right)^2} dx - \frac{y\,x}{1+\left(y/x\right)^2} dy = \begin{cases} 0 & \text{if origin not inside} \\ -2n\pi & \text{if origin encircled } n \text{ times} \end{cases}$$

$$\cancel{= \oint d(\arctan y/x)}$$

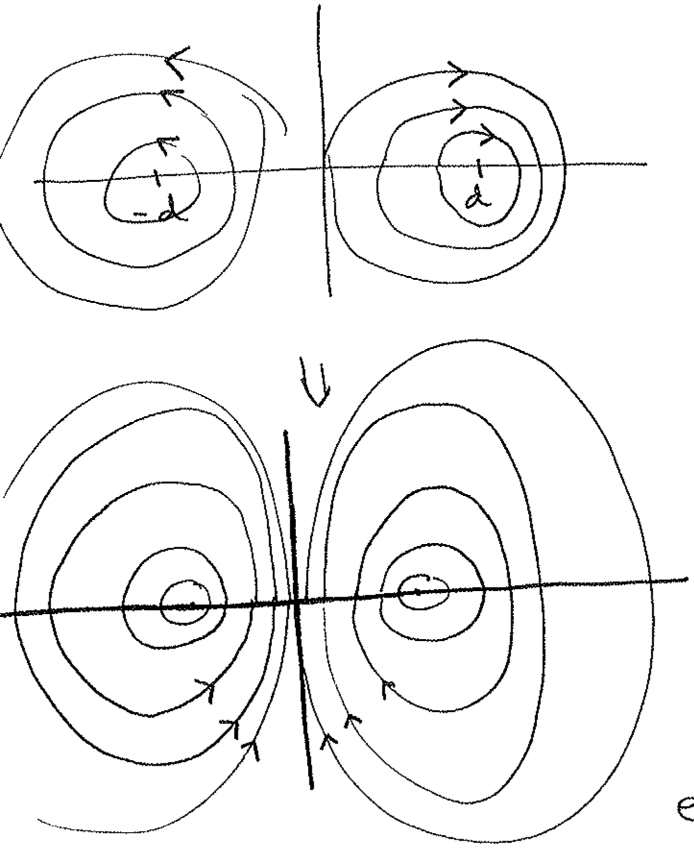
$$u = \frac{y}{x^2+y^2} \quad , \quad v = \frac{-x}{x^2+y^2}$$

if you get farther and farther away
from origin, the velocity decreases,
though circulation is still zero

Example

$i\log(z-d) - i\log(z+d)$, $d \in$ Real

two vertexes, translated \ $i\log\left(\frac{z-d}{z+d}\right)$



$= i\log\left|\frac{z-d}{z+a}\right| + i^2$(something)

$\psi = \ln\left|\frac{z-d}{z+d}\right|$

$e^{\psi} = \frac{|z-d|}{|z+d|} = \frac{\sqrt{(x-d)^2+y^2}}{\sqrt{(x+d)^2+y^2}}$

$e^{2\psi} = \frac{(x-d)^2+y^2}{(x+d)^2+y^2}$

~~$e^{2\psi}((x+d)^2+y^2$~~

$e^{2\psi}(x^2+2xd+d^2+y^2) = x^2-2xd+d^2+y^2$

$$\underbrace{x^2\underbrace{(1-e^{2\psi})}_{r} + y^2\underbrace{(1-e^{2\psi})}_{r} + (\;)x + (\;)}_{\text{this is a circle!!!!}} = 0$$

the streamlines are all circles

when $\psi=0$, obtain the straight vertical line

10/31/03

Class schedule

today 8-10, 2-4
tuesday 11/4, 2-4
friday 11/7 8-10, 2-4

$$\left.\begin{array}{l}\text{Steady}\\ \text{two-dimensional}\\ \text{inviscid}\\ \text{irrotational}\\ \text{incompressible}\end{array}\right\}\ \text{use complex potential}$$

velocity potential, $u = \frac{\partial \phi}{\partial x}, v = \frac{\partial \phi}{\partial y}$

stream function, $u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x}$

$z = x + iy$, $\quad w = f(z) = \phi + \psi$

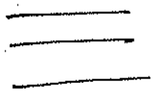
$$\frac{dw}{dz} = \frac{d\phi}{dx} + i\frac{\partial \psi}{\partial x} = u - iv = \frac{\partial \phi}{\partial (iy)} + i\frac{\partial \psi}{\partial (iy)}$$

For $\chi = \chi(x, y) = \text{const.}$

Remember

$w = f(z) = z$

the stream lines look like this



Now add $1/z$

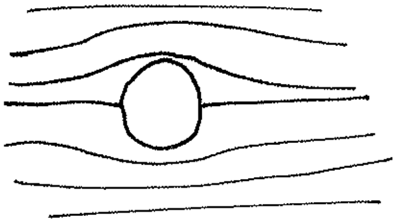
$\rightarrow w = f(z) = z + 1/z = x + iy + \frac{1}{x+iy} = x + iy + \frac{x - iy}{x^2 + y^2}$

$\Rightarrow \quad \phi = x + \frac{x}{x^2+y^2}, \quad \psi = y - \frac{y}{x^2+y^2}$

notice, $\psi = y\left(1 - \frac{1}{x^2+y^2}\right) \rightarrow x^2 + y^2 = 1$, $\psi = 0$ is a streamline

$y = 0$ also satisfies $\psi = 0$

exclude the interior of the circle - don't have to deal with singularity



streamlines ψ

- let $\psi = \text{const}, c$

$$\psi - y = \frac{-1}{x^2+y^2} \longrightarrow \quad x = \pm\sqrt{\frac{-1}{(\psi - y)} - y^2}$$

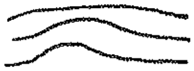
this implies the streamlines are left-right symmetric

- notice, as $x \to \pm\infty$, $\psi \to y$

- at $x = 0$, $\psi = y\left(1 - \frac{1}{y^2}\right) = y - \frac{1}{y}$

$$\longrightarrow y^2 - \psi y - 1 = 0 \longrightarrow \quad y = \frac{\psi}{2} \pm \frac{\sqrt{\psi^2+4}}{2}$$

– ψ is a little higher on y-axis than at $x = \pm\infty$ for the same streamline, ie explains bulge



Calculate velocity components

$$u = \frac{\partial\phi}{\partial x} = 1 + \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}$$

$$\boxed{u = 1 + \frac{y^2 - x^2}{(x^2+y^2)^2}}$$

$$\boxed{v = \frac{\partial\phi}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}}$$

notice, in 1st quadrant, v is negative – streamline going down, in the 2nd quadrant, v is positive, streamline going up.

what is the stress?

the stress tensor, $\underline{\underline{T}} = -p\,\underline{\underline{I}}$ due to inviscid nature

force per unit area, $= -p\,\underline{n}$

~~$dF = -p\underline{n}\, r\, d\theta \big|_{r=1}$~~ $\quad dF = -p\,\underline{n}\,dl$

element of boundary

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Force vector, $F = F_x + i F_y$

F_x: force in x-direction

F_y: Force in y-direction

$$\overline{F} = F_x - i F_y = \frac{1}{2} i \oint \left(\frac{dF}{dz}\right)^2 dz$$

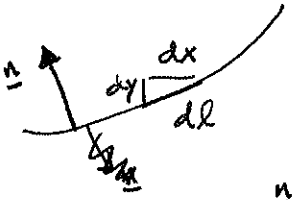
Bernouilli's eqn: $p + \frac{1}{2}|\underline{v}|^2 = p_\infty = \text{const}$

$$p = p_\infty - \frac{1}{2}|\underline{v}|^2 = p_\infty - \frac{1}{2}(u^2+v^2)$$

$$\left.\begin{array}{l} u = |\underline{v}| \cos\theta \\ v = |\underline{v}| \sin\theta \end{array}\right\} u + iv = |v|(\cos\theta + i\sin\theta) = |\underline{v}|\, e^{i\theta}$$

$$\longrightarrow \quad |\underline{v}| = (u+iv)\, e^{-i\theta}$$

$$\Rightarrow \quad p = p_\infty - \frac{1}{2}(u+iv)^2 e^{-2i\theta}$$



$d\ell^2 = dx^2 + dy^2$ and $dz = dx + idy$

$$\longrightarrow \quad dz = d\ell\, e^{i\theta}$$

$dx = d\ell \cos\theta$

$dy = d\ell \sin\theta$

$$\underline{n} = \frac{-dy + idx}{\sqrt{dx^2+dy^2}} = \frac{-d\ell(\sin\theta + i\cos\theta)}{d\ell}$$

$$\underline{n} = i(\cos\theta - i\sin\theta) = i e^{i\theta} = e^{i(\theta + \pi/2)}$$

↑ there is a mistake here ⤴

but the general idea is:

$$\text{Force} = \int p\,\underline{n}\, d\ell = \int p_\infty \underline{n}\, d\ell + \frac{1}{2}\int (u+iv)^2 e^{-2i\theta} e^{i(\theta+\frac{\pi}{2})}$$

blah blah blah

$$f(z) = z + \frac{1}{z}$$

$$f'(z) = 1 - \frac{1}{z^2} \qquad \left[f'(z)\right]^2 = 1 - \frac{2}{z^2} + \frac{1}{z^4}$$

$$\overline{F} = \frac{i}{2} \oint_{|z|=1} \left(1 - \frac{2}{z^2} + \frac{1}{z^4}\right) dz = 0$$

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

$$r = 1 \rightarrow z = e^{i\theta}$$

$$\rightarrow dz = ie^{i\theta}\, d\theta$$

$$\frac{1}{z^2} = \frac{1}{e^{2i\theta}} = e^{-2i\theta}, \quad \frac{1}{z^4} = e^{-4i\theta}$$

$$\overline{F} = \frac{i}{2} \int_0^{2\pi} \left(1 - 2e^{-2i\theta} + e^{-4i\theta}\right) ie^{i\theta}\, d\theta$$

$$= -\frac{1}{2} \int_0^{2\pi} \left(e^{i\theta} - 2e^{-i\theta} + e^{-3i\theta}\right) d\theta$$

$$= -\frac{1}{2} \left[\frac{e^{i\theta}}{i} + \frac{2e^{-i\theta}}{i} - \frac{e^{-3i\theta}}{3i} \right]_0^{2\pi} = 0 \quad \boxed{!!!}$$

10/31/03

3/

$$f(z) = z + \frac{1}{z} + \frac{i\Gamma}{2\pi}\log z$$

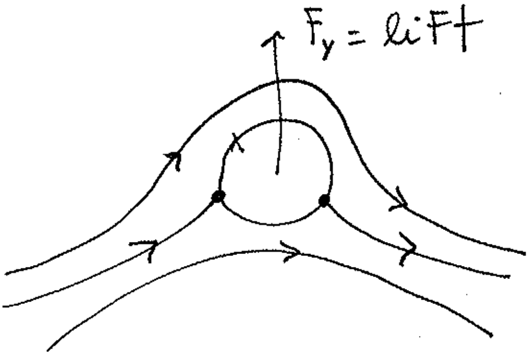
$$f'(z) = 1 - \frac{1}{z^2} + \frac{i\Gamma}{2\pi}\frac{1}{z}$$

~~$[f'(z)]^2 = 1 - \frac{1}{z^4} - \left(\frac{\Gamma}{2\pi}\right)^2\frac{1}{z^2} - \frac{2}{z^2} + 2\frac{i\Gamma}{2\pi}\frac{1}{z} - 2\frac{i\Gamma}{2\pi}\frac{1}{z^3}$~~

$$F_x = 0\,, \quad F_y = \Gamma$$

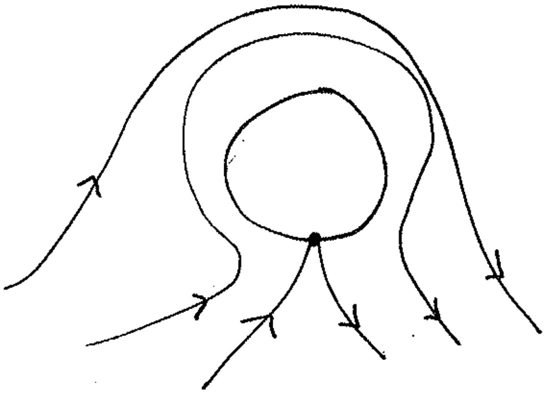
$$f(z) = x + iy + \frac{x-iy}{x^2+y^2} + \frac{i\Gamma}{2\pi}\left(\ln\sqrt{x^2+y^2} + \tan^{-1}\left(\frac{y}{x}\right)\right)$$

$$\rightarrow \boxed{\psi = y - \frac{y}{x^2+y^2} + \frac{\Gamma}{2\pi}\left(\ln\sqrt{x^2+y^2} + \tan^{-1}\left(\frac{y}{x}\right)\right)}$$

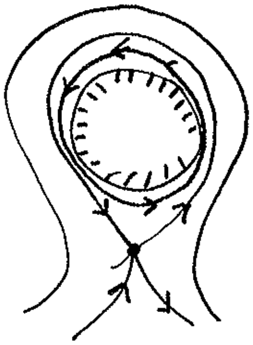


the circulation ~~[illegible]~~ results in lift...

$$\frac{|\Gamma|}{2\pi} < 2$$



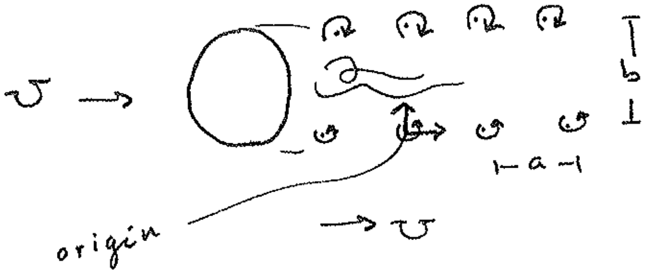
$$\frac{|\Gamma|}{2\pi} = 2$$



$$\frac{|\Gamma|}{2\pi} > 2$$

$\frac{i\Gamma}{2\pi}\log z$ - line vortex

von Karman vortex street



$$\underset{\text{bottom}}{f(z)} = \frac{i\Gamma}{2\pi}\left(\log z + \log(z-a) + \log(z+a) + \log(z \pm an)\right)$$

$$= \cancel{\frac{i\Gamma}{2\pi}\log} \quad \cancel{\sin \pi z}$$

$$= \frac{i\Gamma}{2\pi}\log\left(\sin\frac{\pi z}{a}\right) = \frac{i\Gamma}{2\pi}\log\left[\sin\left(\frac{\pi}{a}(z-na) + n\pi\right)\right]$$

$$\text{notice} \quad \sin(y+n\pi) = \begin{cases} \sin y, & n \text{ even} \\ -\sin y, & n \text{ odd} \end{cases}$$

$$u - iv = \frac{dF}{dz} = \frac{i\Gamma}{2\pi}\,\frac{\pi}{a}\,\frac{\cos\frac{\pi z}{a}}{\sin\frac{\pi z}{a}}\Bigg|_{z=(n+\frac{1}{2})a+ib}$$

$$= \frac{\pi}{a}\frac{i\Gamma}{2\pi}\,\frac{\cos\frac{\pi}{a}(an+\frac{1}{2}a+ib)}{\sin\frac{\pi}{a}(an+\frac{1}{2}a+ib)} = \frac{i\Gamma}{2a}\,\frac{\cos\left(\left(n\pi+\frac{\pi}{2}\right)+\frac{i\pi b}{a}\right)}{\sin\left(\left(n\pi+\frac{\pi}{2}\right)+i\frac{\pi b}{a}\right)}$$

$$= \frac{i\Gamma}{2a}\,\frac{\sin\frac{i\pi b}{a}}{\cos\frac{i\pi b}{a}} = \frac{\cancel{i}\Gamma}{2a}\,\frac{\left[e^{i\frac{i\pi b}{a}} - e^{\frac{i^2\pi b}{a}}\right]}{\left[e^{\frac{i^2\pi b}{a}} + e^{\frac{-i^2\pi b}{a}}\right]}\,\frac{1}{i} = \frac{\Gamma}{2a}\,\frac{e^{\frac{\pi b}{a}} - e^{-\frac{\pi b}{a}}}{e^{\frac{\pi b}{a}} + e^{-\frac{\pi b}{a}}} = \underline{\underline{\text{Real}}}$$

$$= \frac{\Gamma}{2a}\tanh\left(\frac{\pi b}{a}\right) \Rightarrow \underline{\text{drift of the vortex street}}$$

Fluid Mechanics 11/04

1/1

<u>class schedule</u>

Friday 11/7/03, 8-10, 2-4

thereafter, T 2-4, F 2-4

<u>class presentations</u>, Dec 2, 5

Very Viscous Flow

if neglect inertia terms, scale the equation, obtain

$$\nabla^2 \underline{v} = -\nabla p$$

$$\nabla \cdot \underline{v} = 0$$

take curl, $\nabla \times \underline{\omega} = \nabla \times (\nabla \times \underline{v}) = -\nabla^2 \underline{v}$

$$\rightarrow \quad -\nabla \times \underline{\omega} = -\nabla p$$

$$\rightarrow \quad -\nabla \times (\nabla \times \underline{\omega}) = -\nabla \times \nabla p = 0$$

$$\Rightarrow \quad \nabla^2 \underline{\omega} = 0$$

<u>2 dim flow</u>

$u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$, $\underline{v} = \nabla \times (\psi \underline{k}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & \psi \end{vmatrix}$

$(\frac{\partial}{\partial z} = 0)$

$$= \underline{i}\frac{\partial \psi}{\partial y} - \underline{j}\frac{\partial \psi}{\partial x} + \underline{k}\,0$$

$$\rightarrow \underline{\omega} = \nabla \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ \frac{\partial \psi}{\partial y} & -\frac{\partial \psi}{\partial x} & 0 \end{vmatrix} = \underline{k}\left(-\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}\right) = -\underline{k}\,\nabla^2 \psi$$

$$\rightarrow \nabla \times \underline{\omega} = \underline{i}\frac{\partial}{\partial y}(\nabla^2 \psi) + \underline{j}\frac{\partial}{\partial x}(\nabla^2 \psi)$$

$$\rightarrow \nabla \times (\nabla \times \underline{\omega}) = \underline{k}\,\nabla^2 \nabla^2 \psi = 0$$

two dimension Stokes flow governed by $\nabla^2 \nabla^2 \psi = 0$
biharmonic equation

Stokes Flow - the flow is reversible

consider ~~of~~ flagellae , 

by considering a surface given by $y = a\cos(kx - \omega t)$

assume $\underset{\frac{\partial \psi}{\partial y}}{\underset{\|}{u}} = 0$ on $y = a\cos(kx - \omega t)$ and $\underset{-\frac{\partial \psi}{\partial x}}{\underset{\|}{v = \frac{dy}{dt}}} = a\omega \sin(kx - \omega t)$,

nondimensionalize, $\underset{\frac{\partial}{\partial x} = k\frac{\partial}{\partial x'}}{\underset{\downarrow}{x' = kx - \omega t}}$, $\underset{\frac{\partial}{\partial y} = k\frac{\partial}{\partial y'}}{\underset{\downarrow}{y' = ky}}$

$\rightarrow \nabla'^2 \nabla'^2 \psi = 0$, $u = k\frac{\partial \psi}{\partial y'} = 0$, $v = (\omega a)\sin \cancel{x}' = -k\frac{\partial \psi}{\partial x'}$

$\psi = \frac{\omega a}{k}\psi'$

$\rightarrow \nabla'^2 \nabla'^2 \psi' = 0$, $u' = \frac{\partial \psi}{\partial y'}$ at $y' = (ka)\cos(x')$

$v' = -\frac{\partial \psi}{\partial x'} = \sin x'$, at $y' = \underset{\text{parameter}}{\underset{\vee}{(ka)}}\cos(x')$

replace parameter ka by ϵ, ϵ small

drop primes

$\psi(x, y, \epsilon) = \psi_0(x,y) + \epsilon \psi_1(x,y) + \epsilon^2 \psi_2(x,y) + \dots$

$\nabla^2\nabla^2\psi = 0 \rightarrow \nabla^2\nabla^2\psi_0 + \epsilon\nabla^2\nabla^2\psi_1 + \epsilon^2\nabla\nabla^2\psi_2 + \dots = 0$

$\rightarrow \nabla^2\nabla^2\psi_0 = 0$, $\nabla^2\nabla^2\psi_1 = 0$, $\nabla^2\nabla^2\psi_2 = 0$