Drifting Games

Combining expert advice

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- Goal: make least mistakes under the assumption (no statistics!)

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- Naive implementation requires explicitly maintaining each meta-expert.

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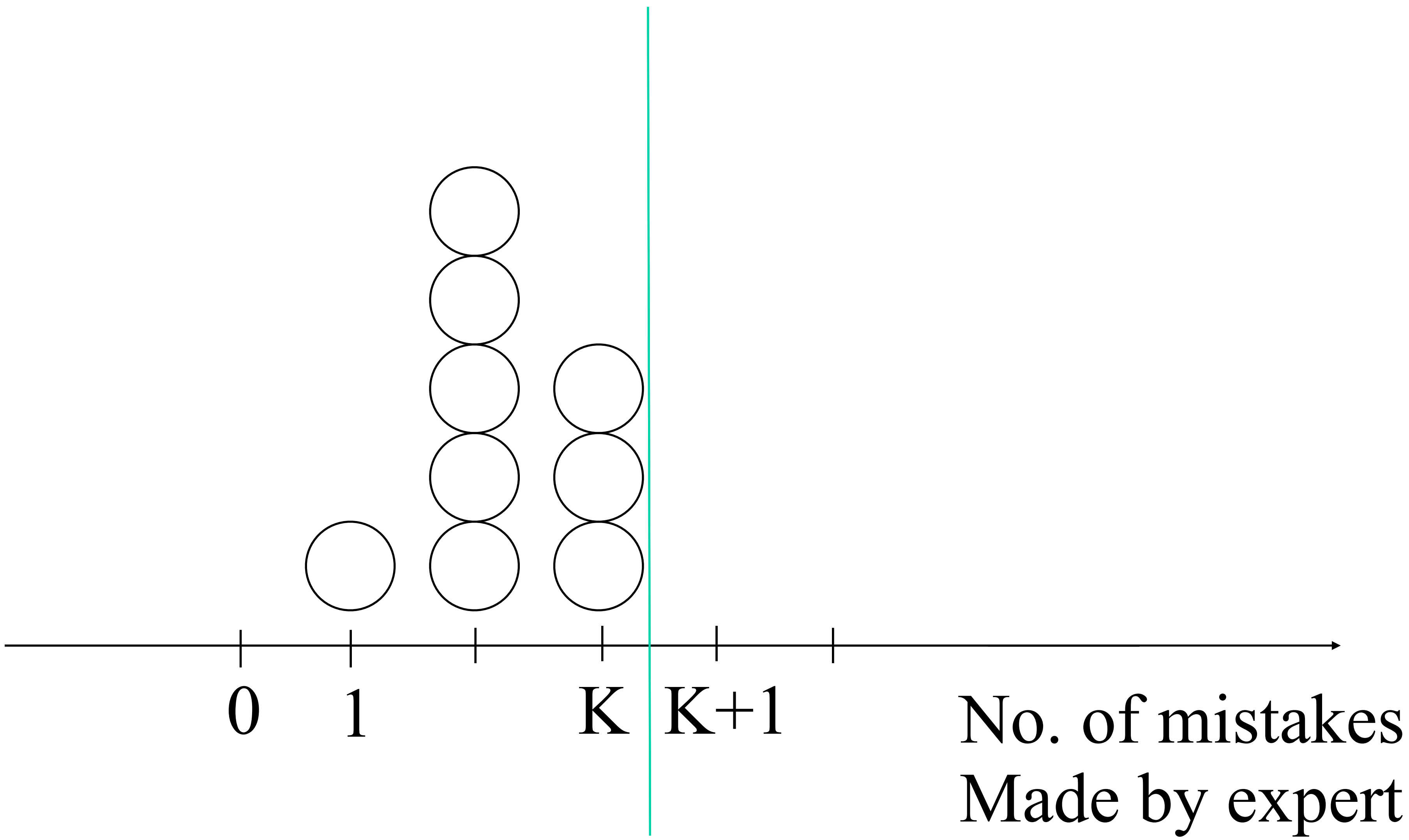
- Game step = algorithm's prediction is incorrect.

Chip game for expert advice



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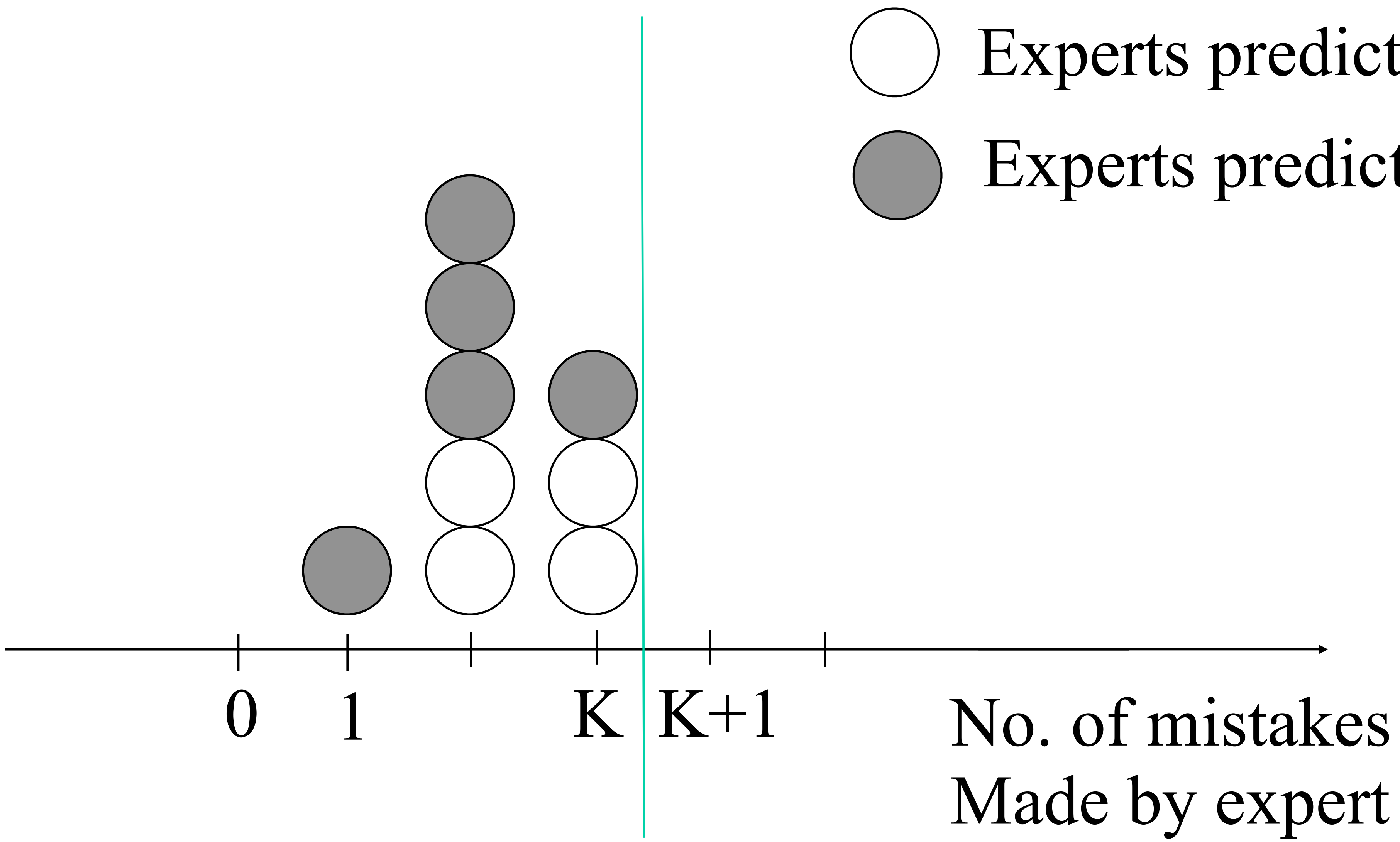

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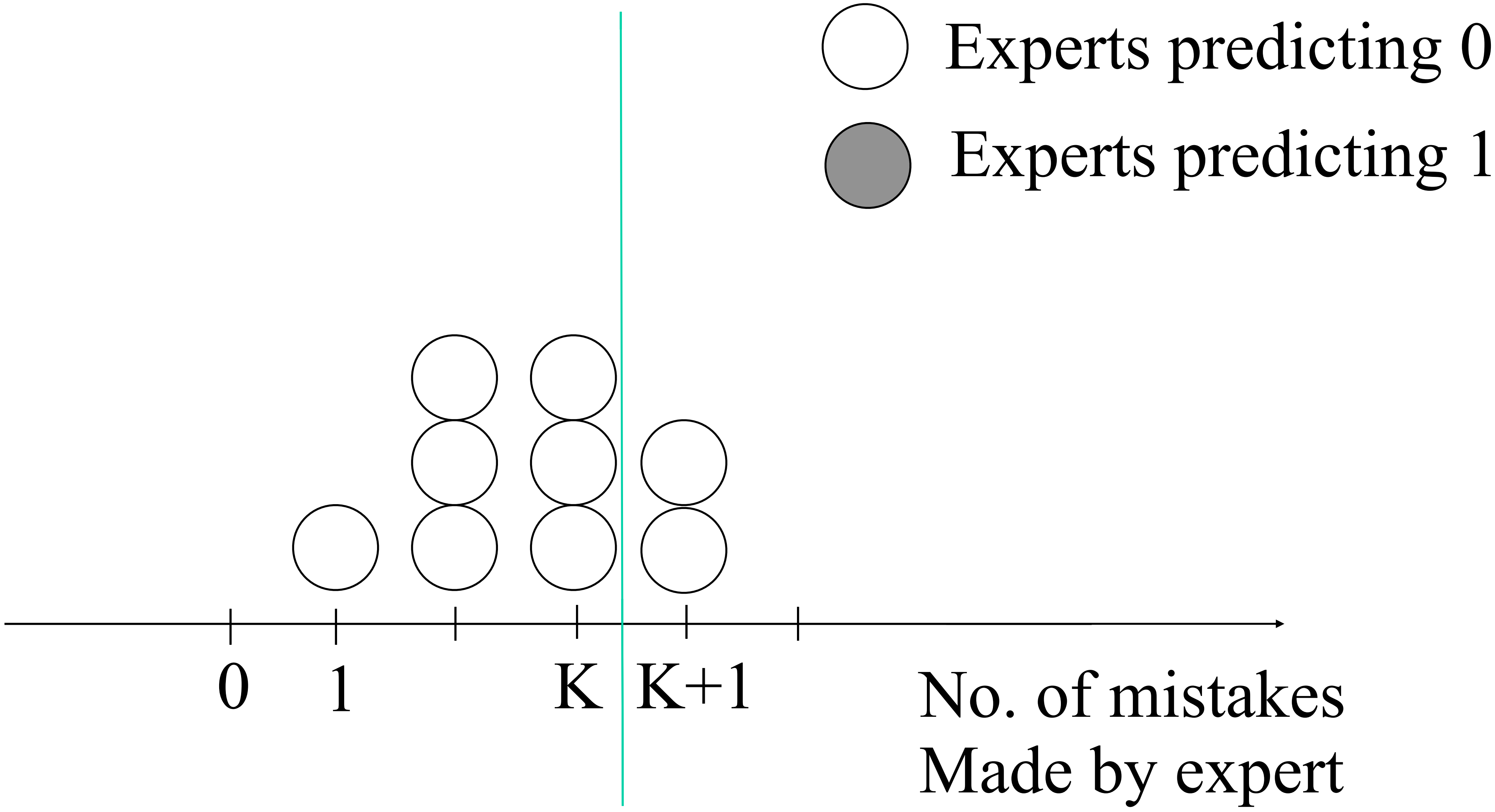

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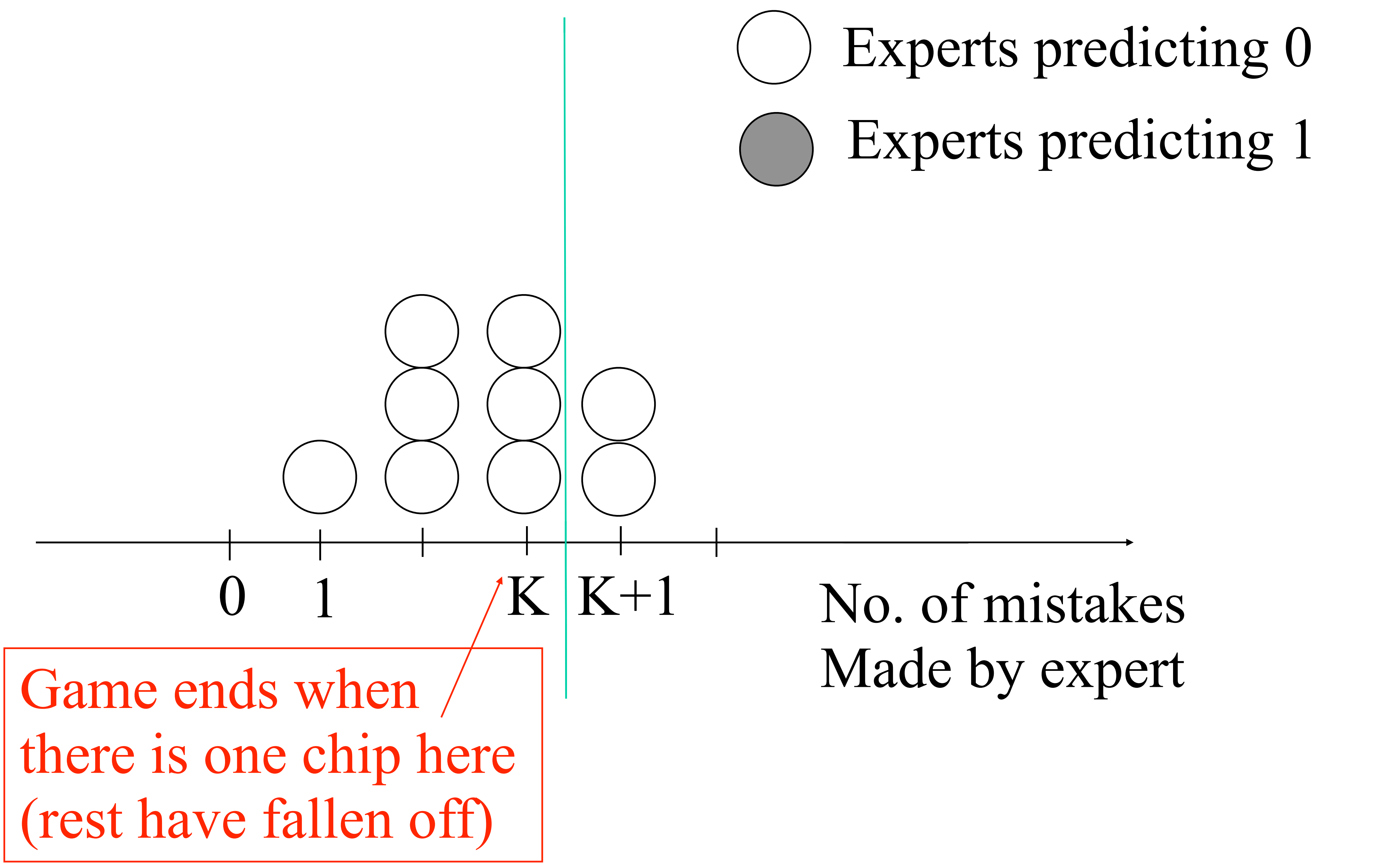

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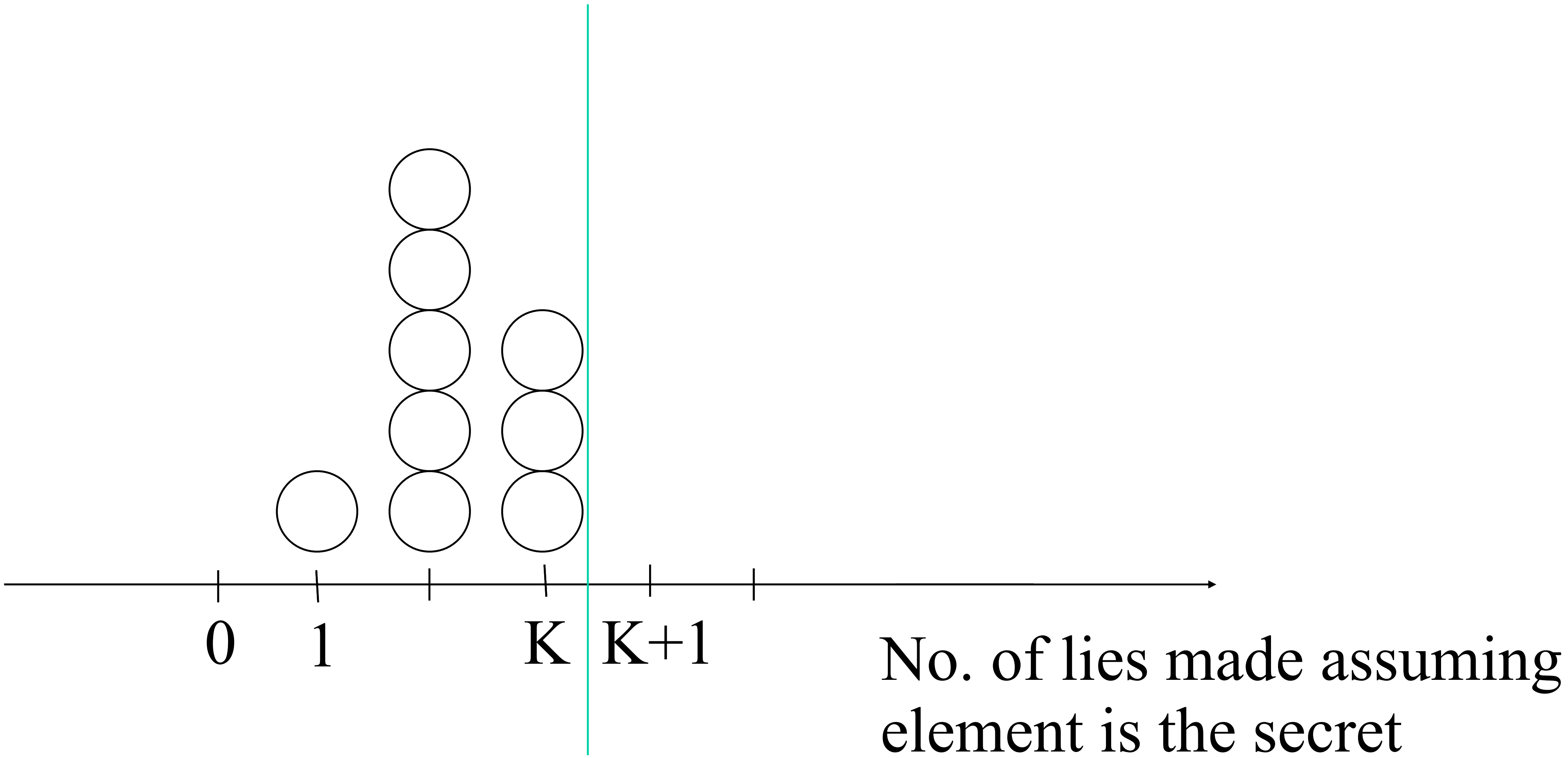

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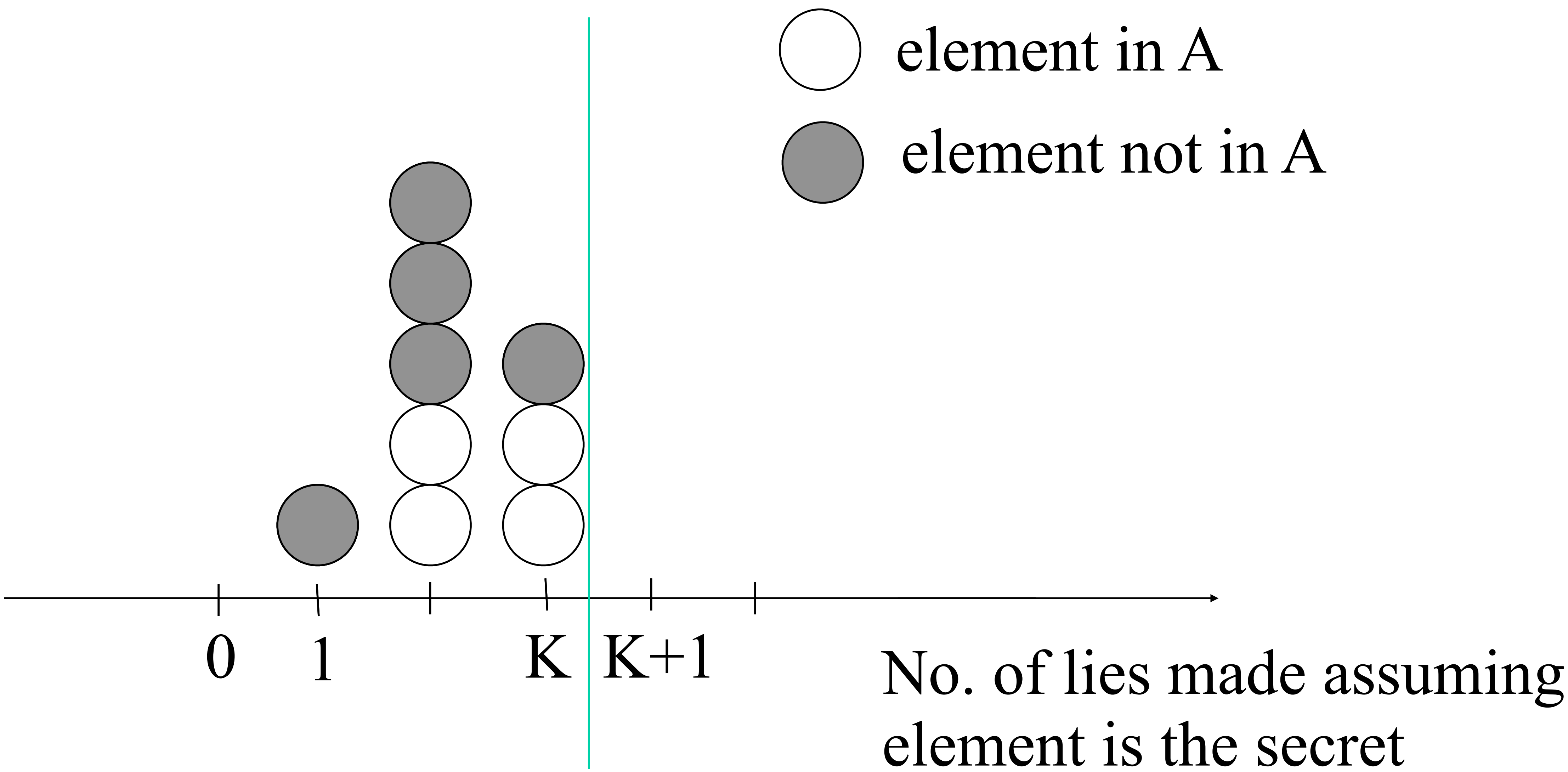

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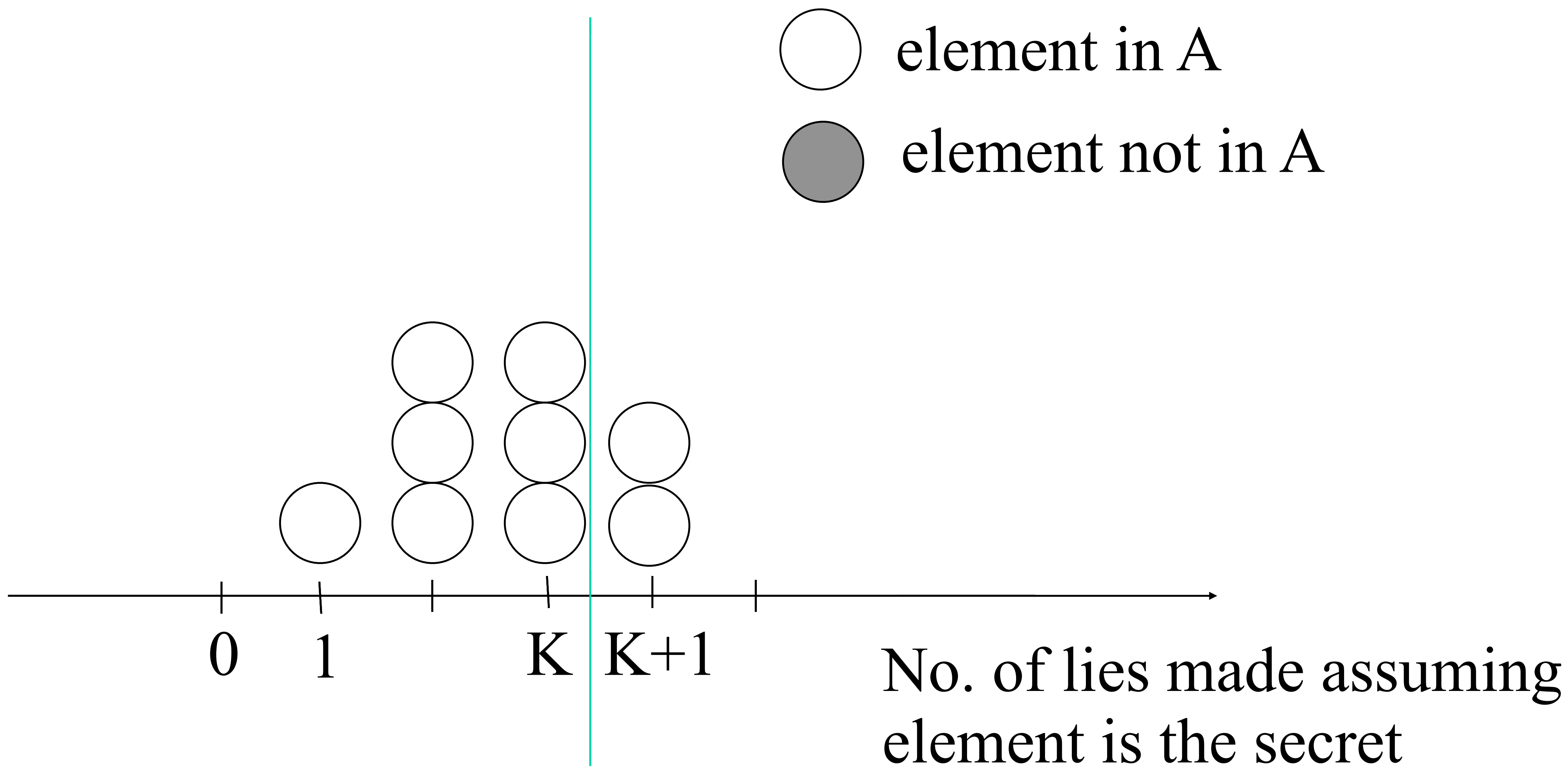

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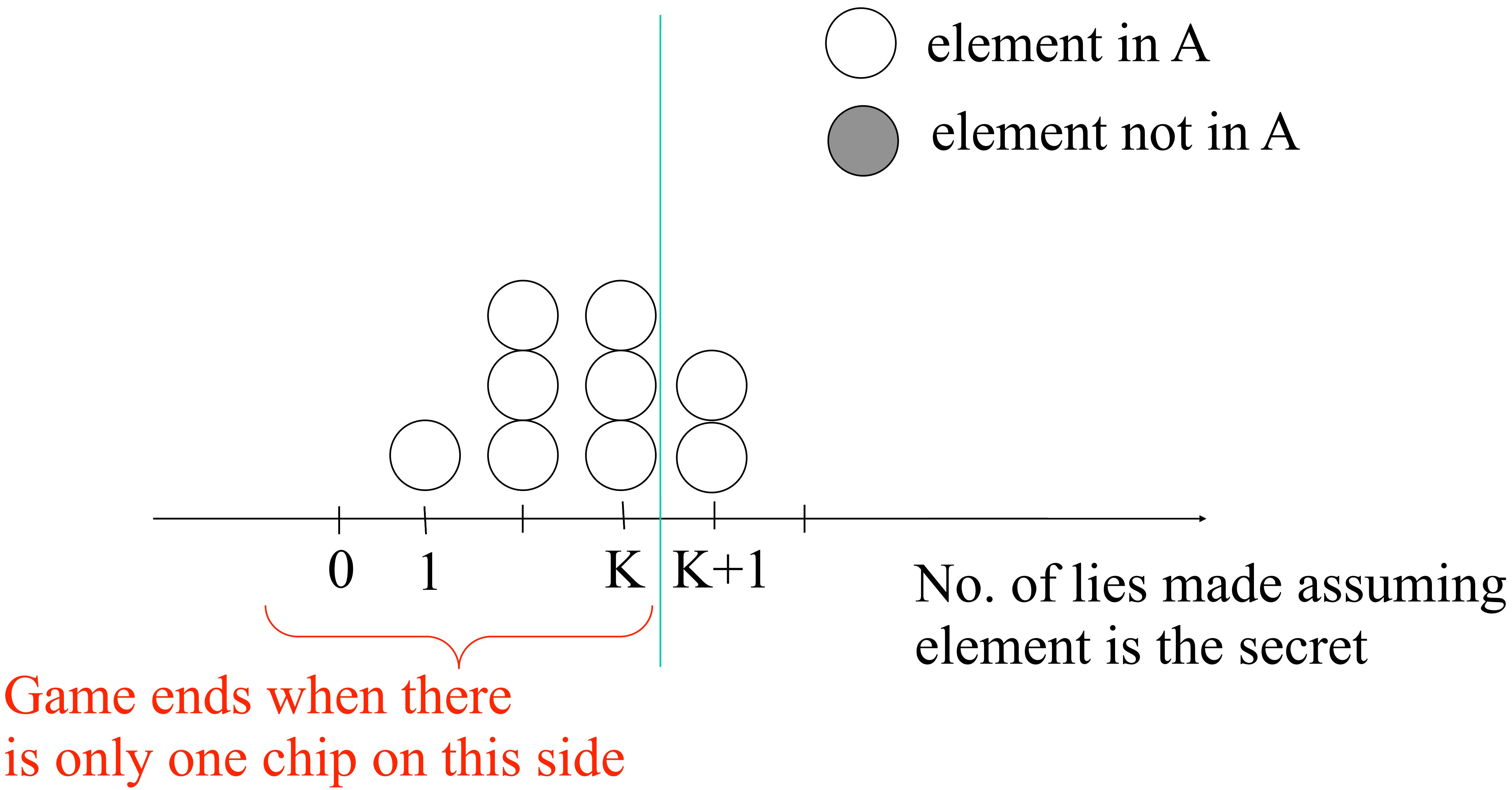

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- Potential: The number of chips that will remain in bins 0..k when m iterations remain and both sides play optimally:

$$\Psi(\vec{n},k,m) = \sum_{i=0}^{k} n_i \Phi(i,k,m)$$

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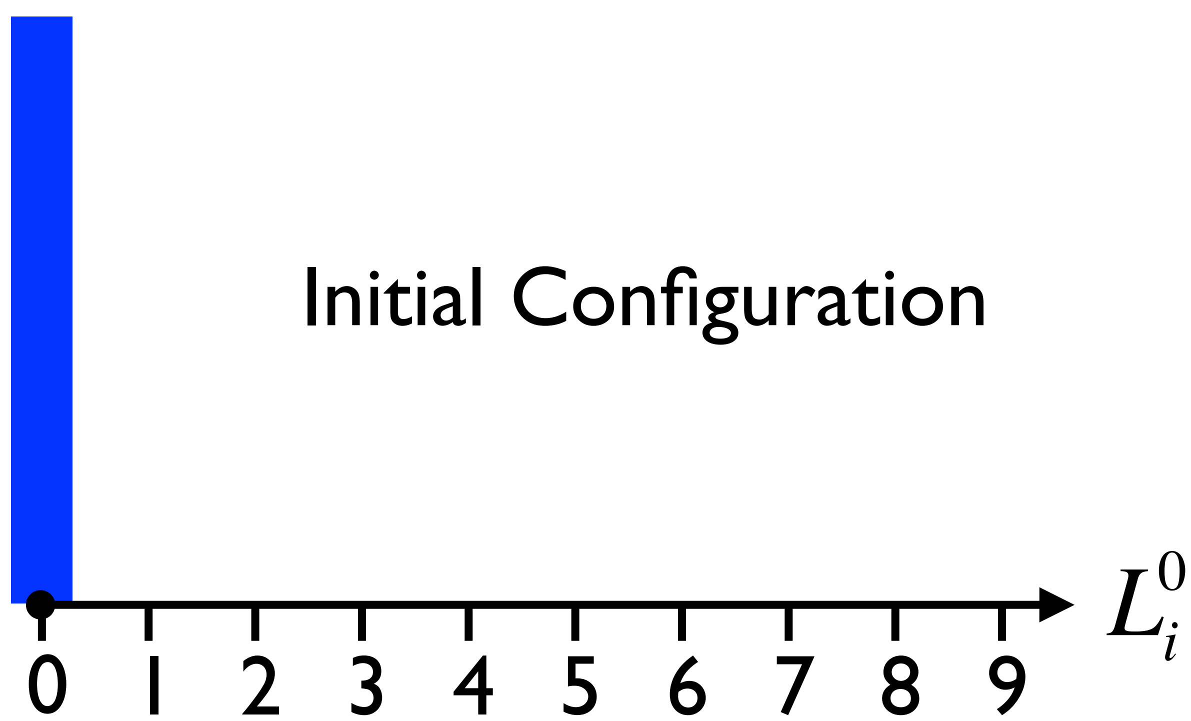
- k=0 corresponds to the halving algorithm.

Combining experts as a drifting game

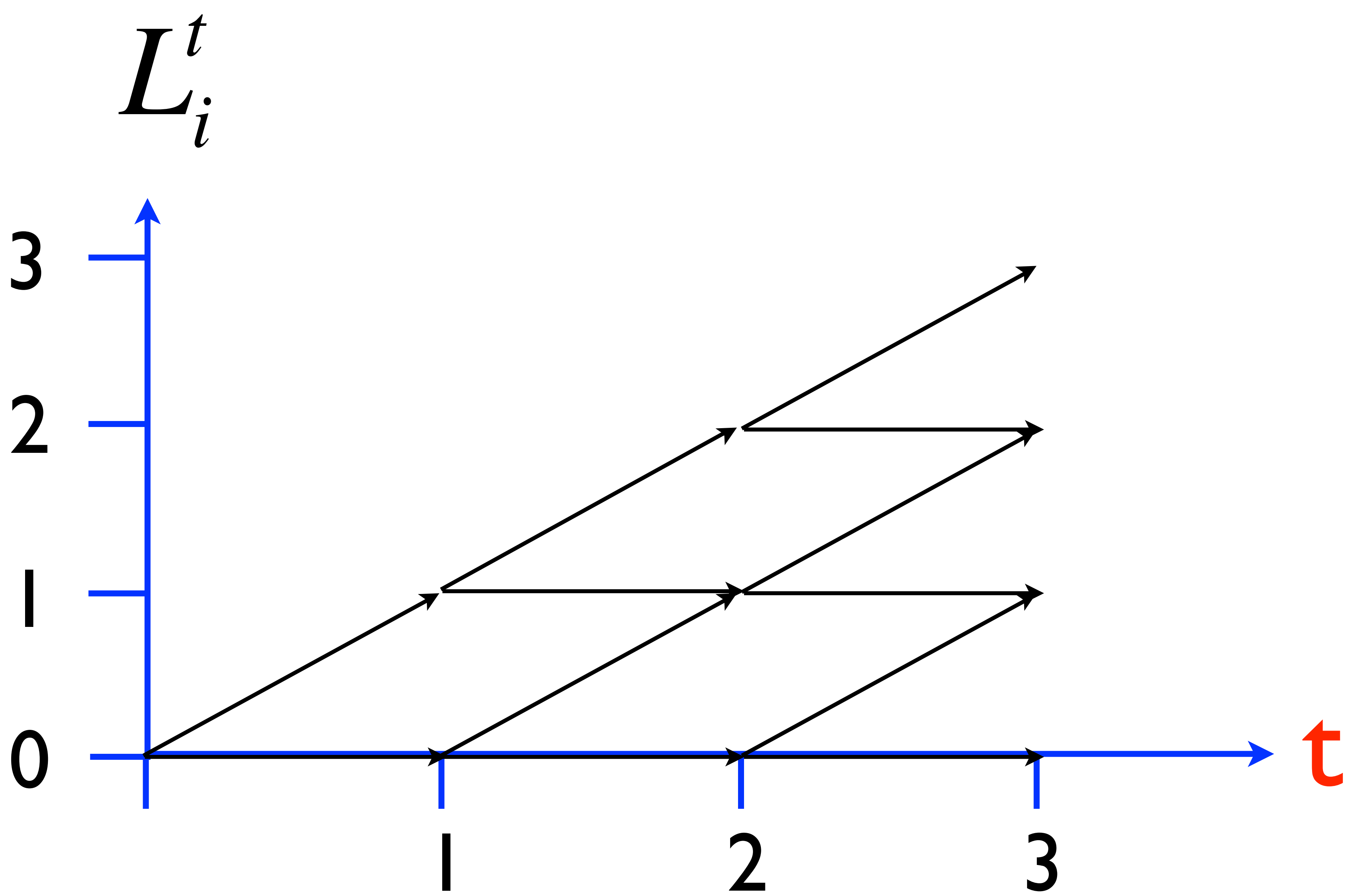
[Cesa-Bianchi, Freund, Helmbold, Warmuth 96]

Binary instantanous loss $l_i^t, l_A^t \in \{0,1\}$

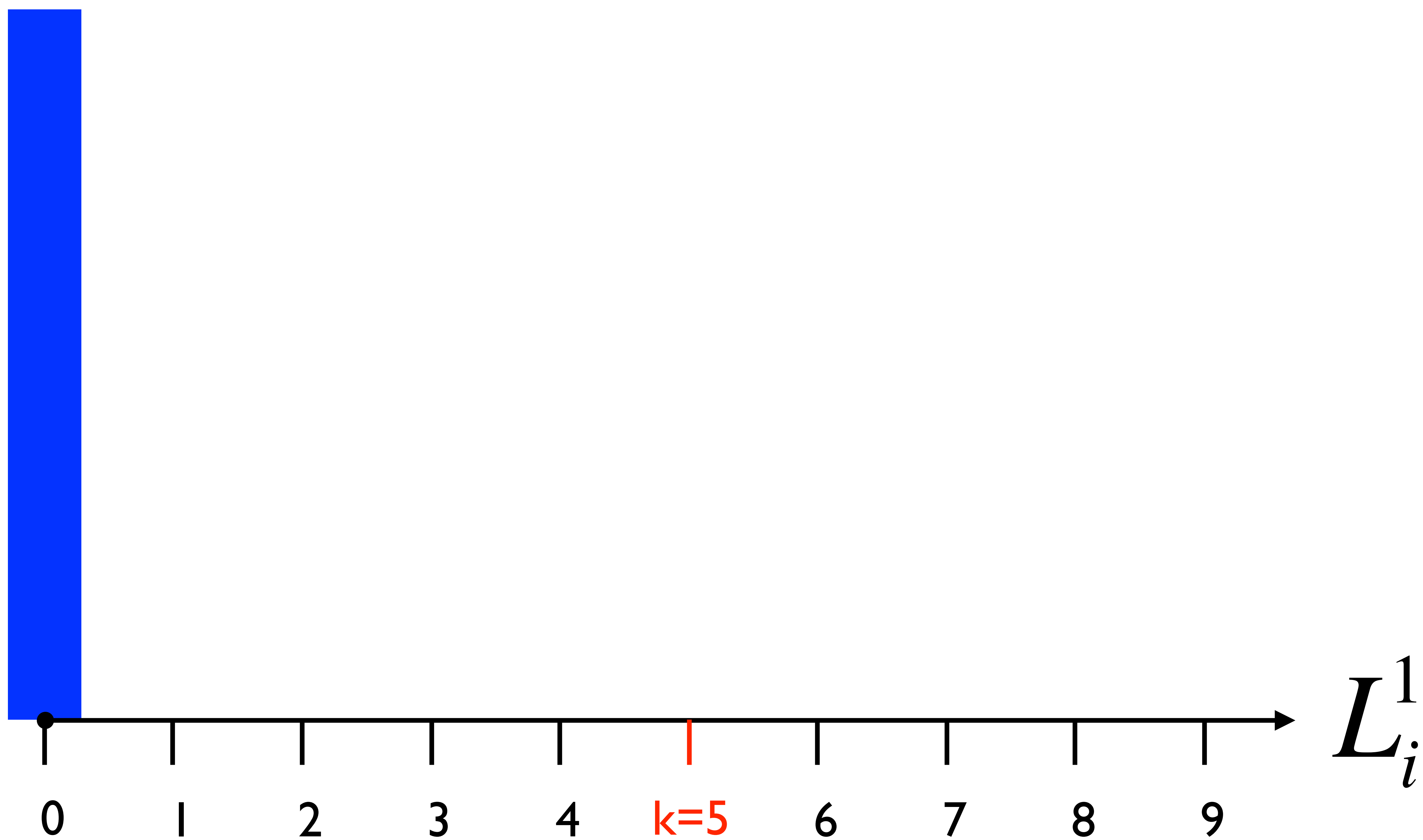
Bin s contains all experts for which $L_i^t = s$



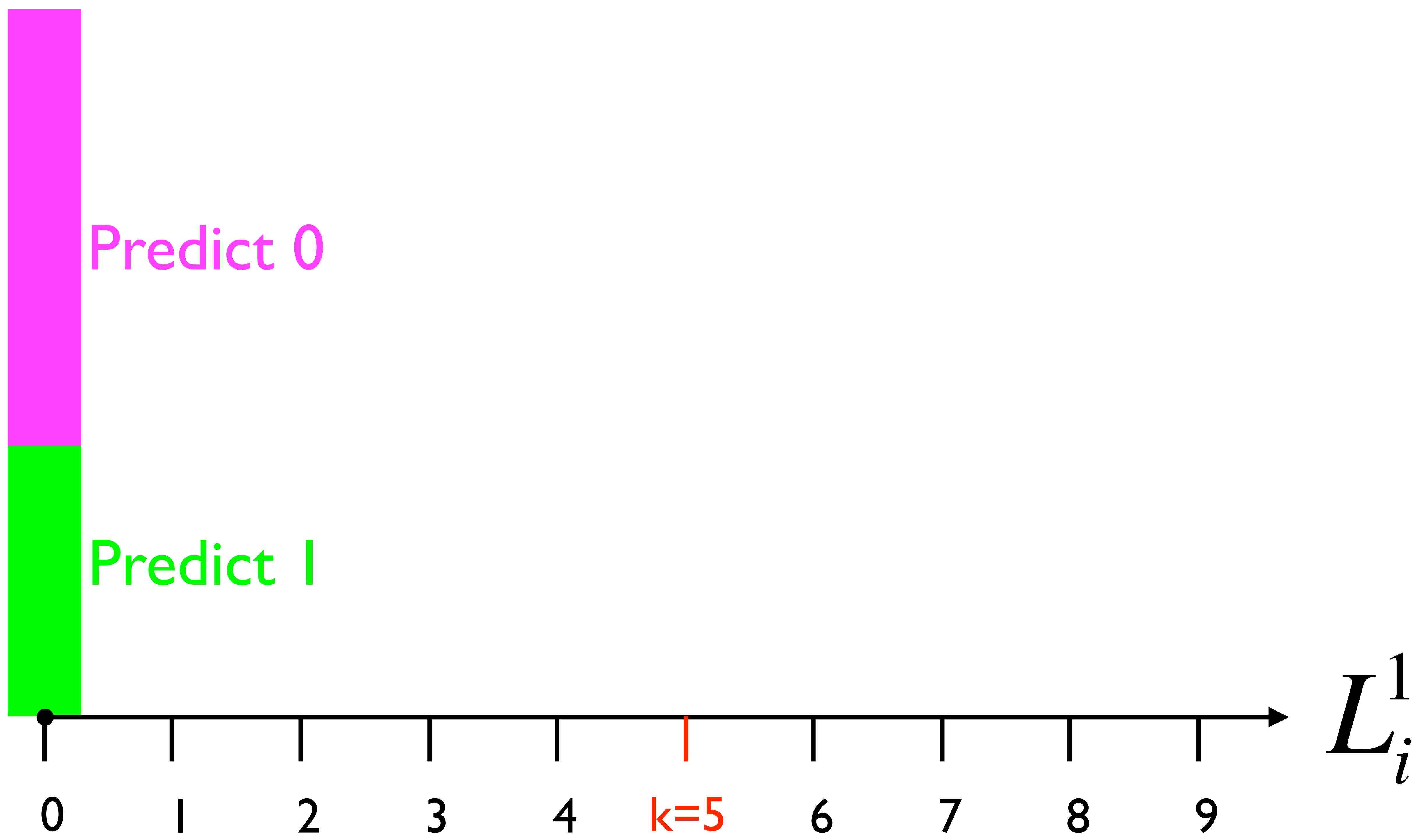
The game lattice



Initial configuration t=1



Experts predictions t=1



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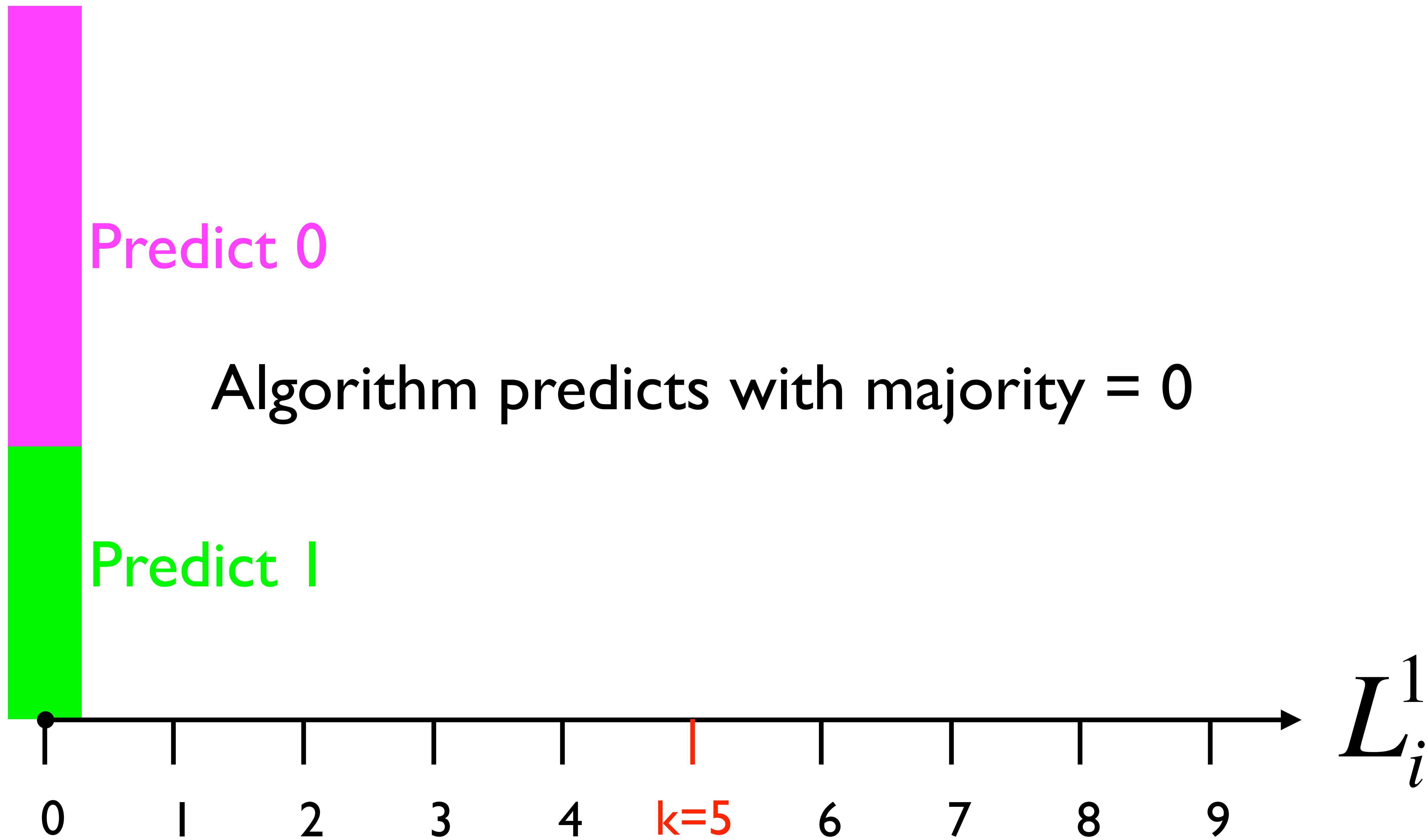
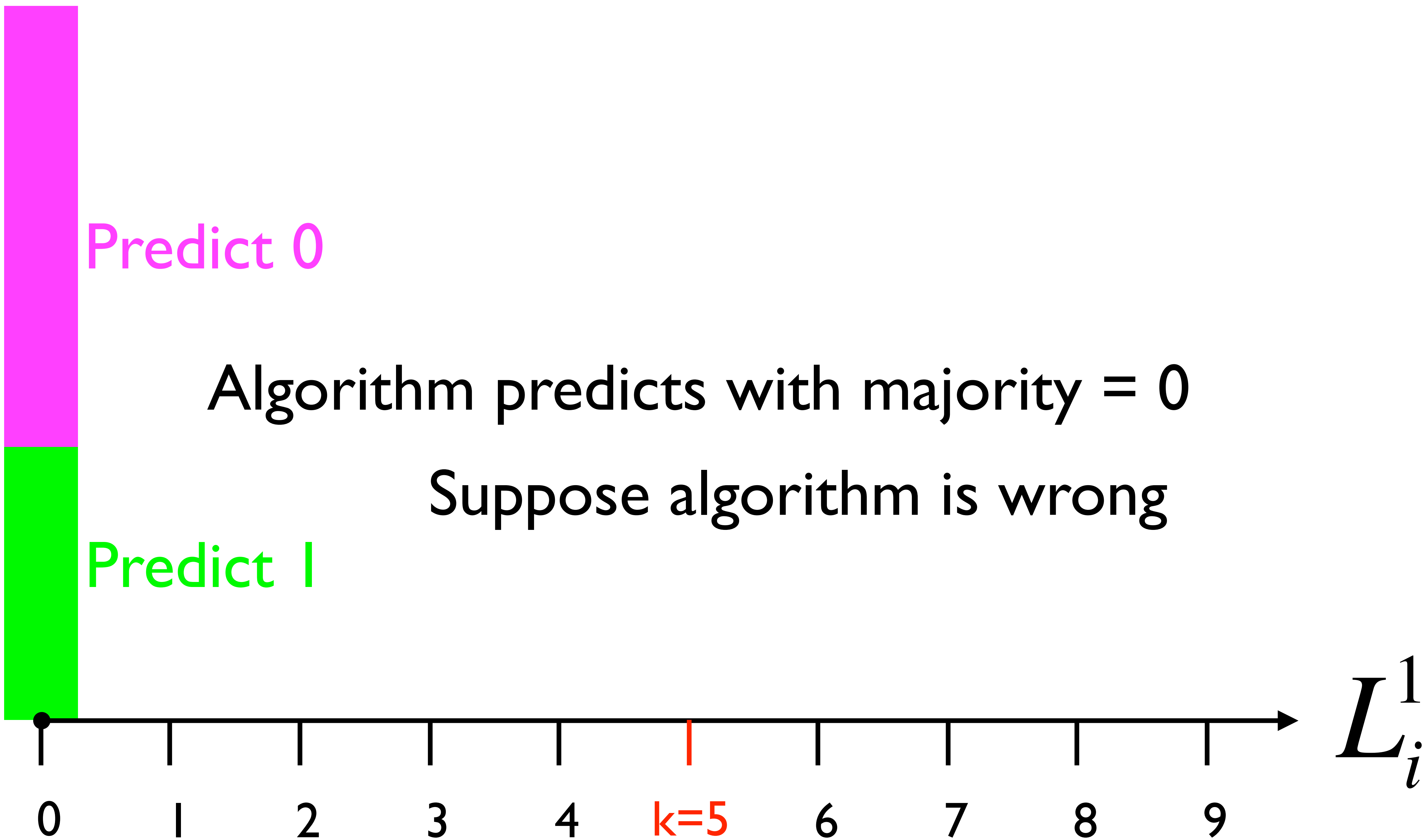
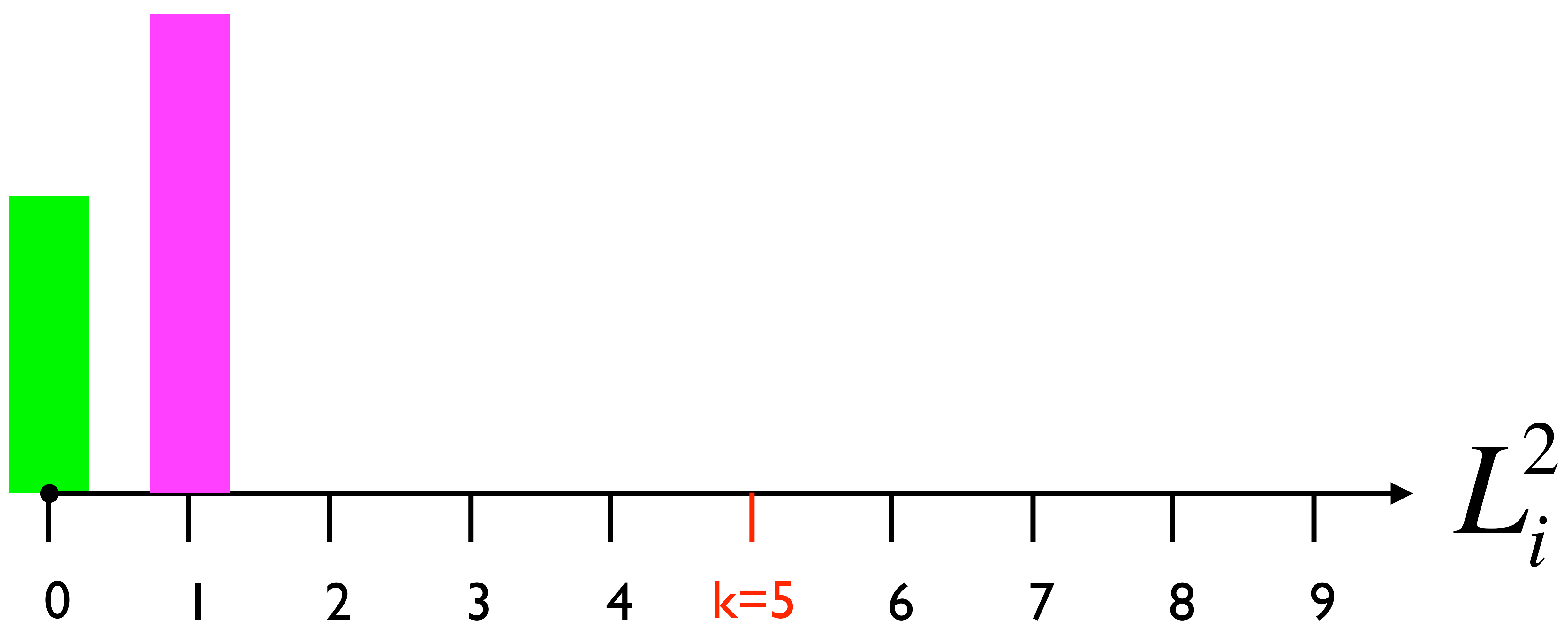


Experts predictions t=1

Algorithm predicts with majority = 0

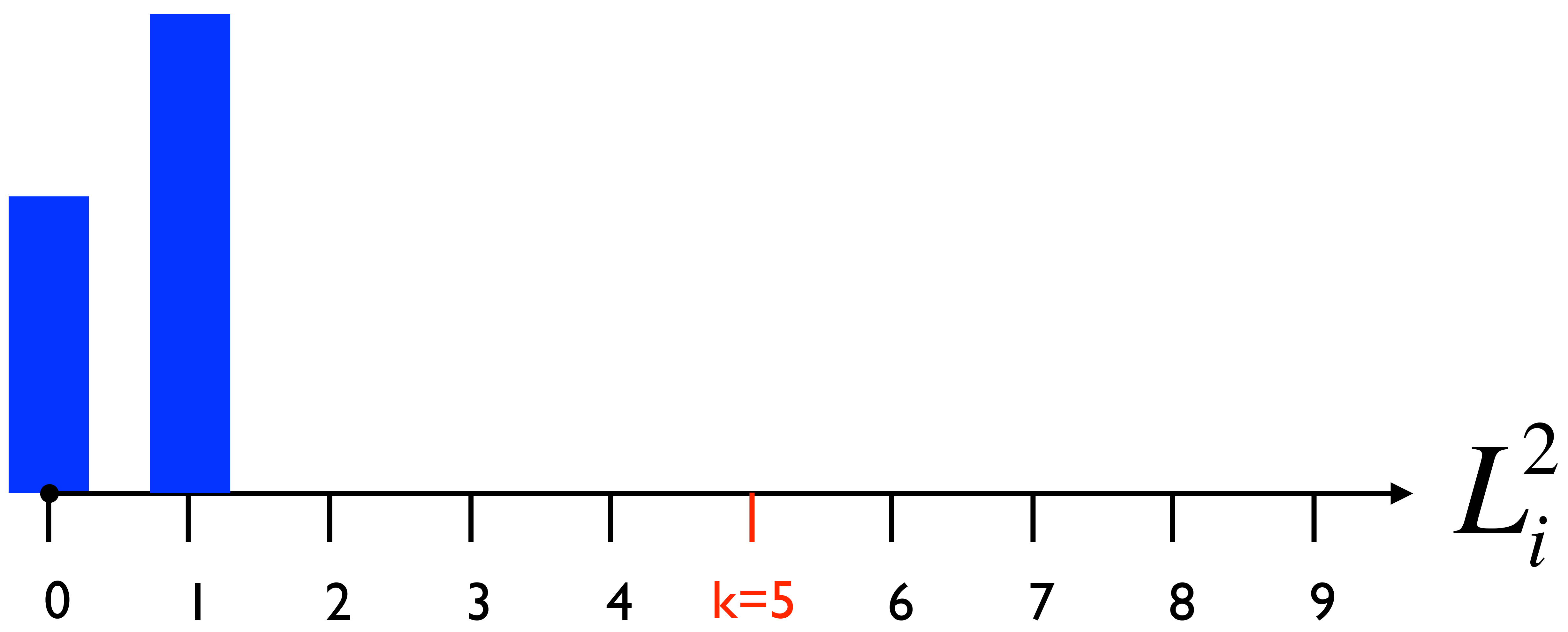
Suppose algorithm is wrong





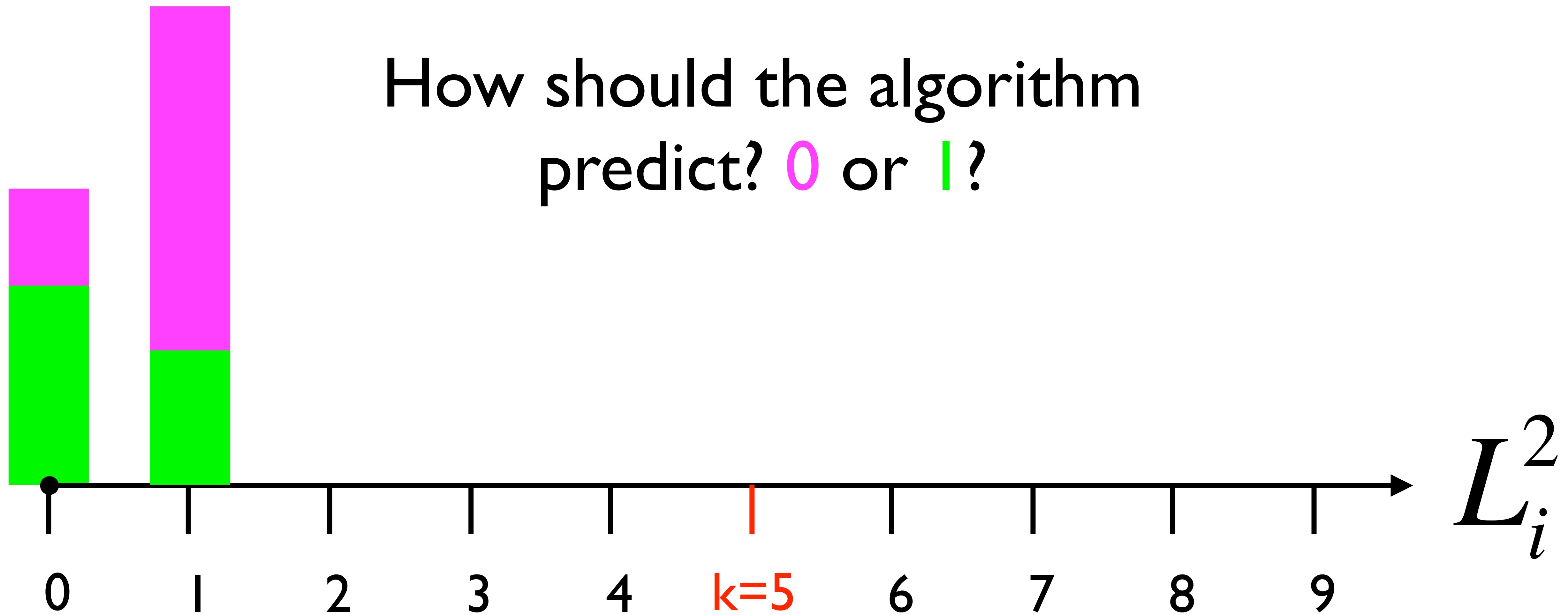
L_i^2
0
1
2
3
4
k=5
6
7
8
9

configuration at t=2



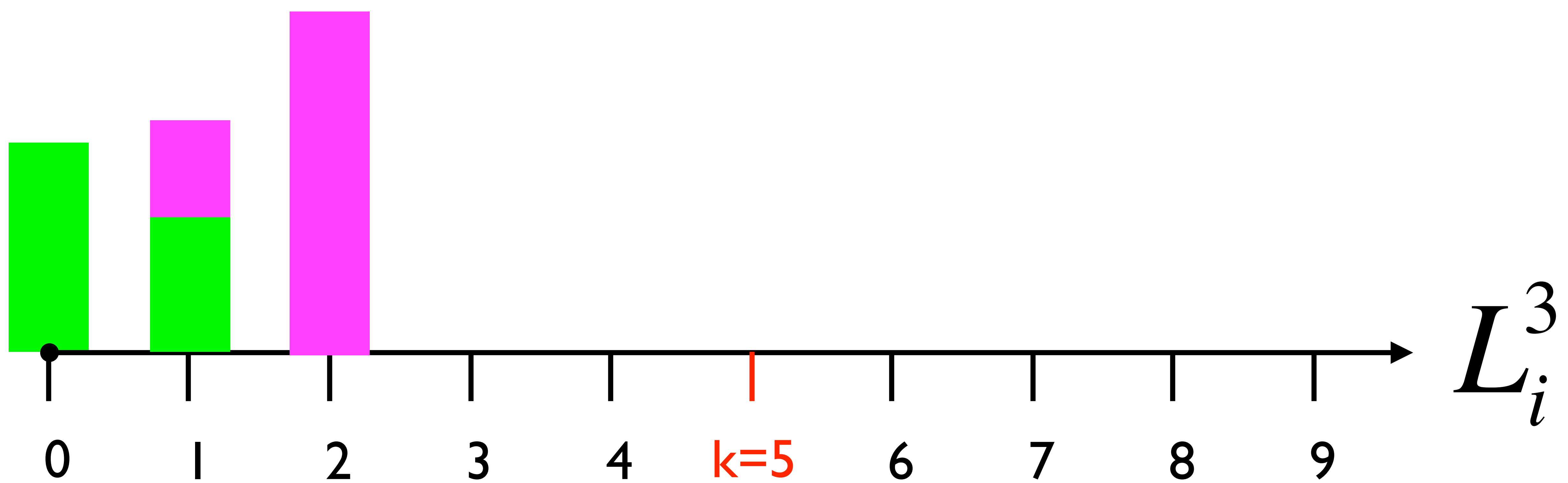
Experts predictions t=2

How should the algorithm predict? 0 or 1?



configuration t=3

Prediction is 0 and outcome is 1



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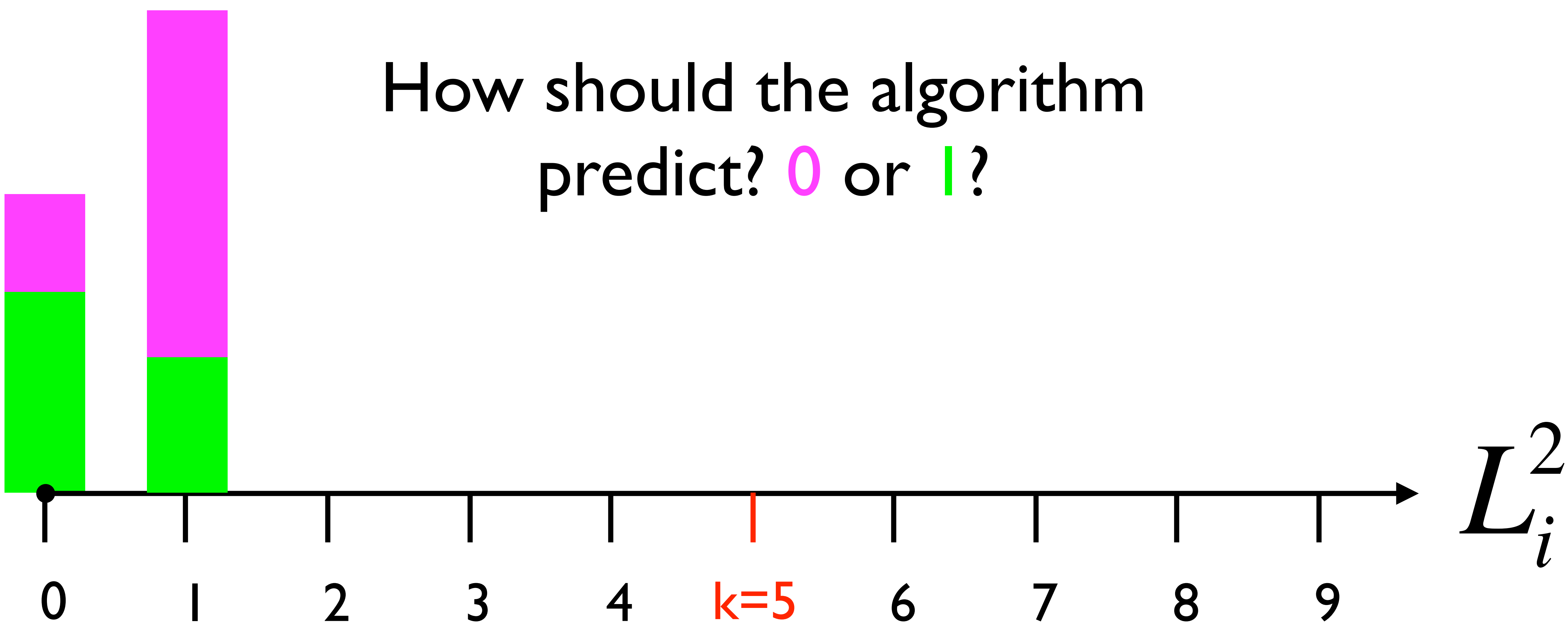
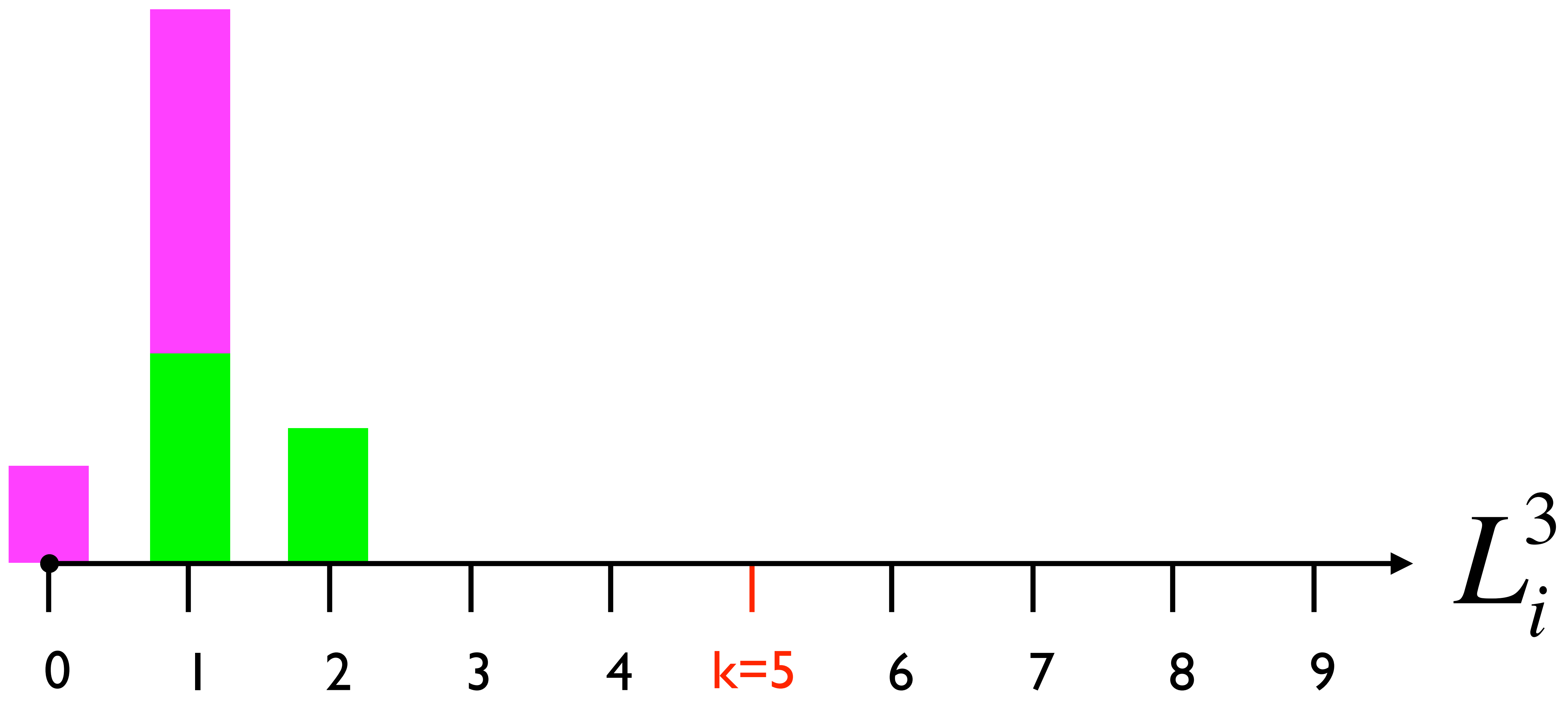


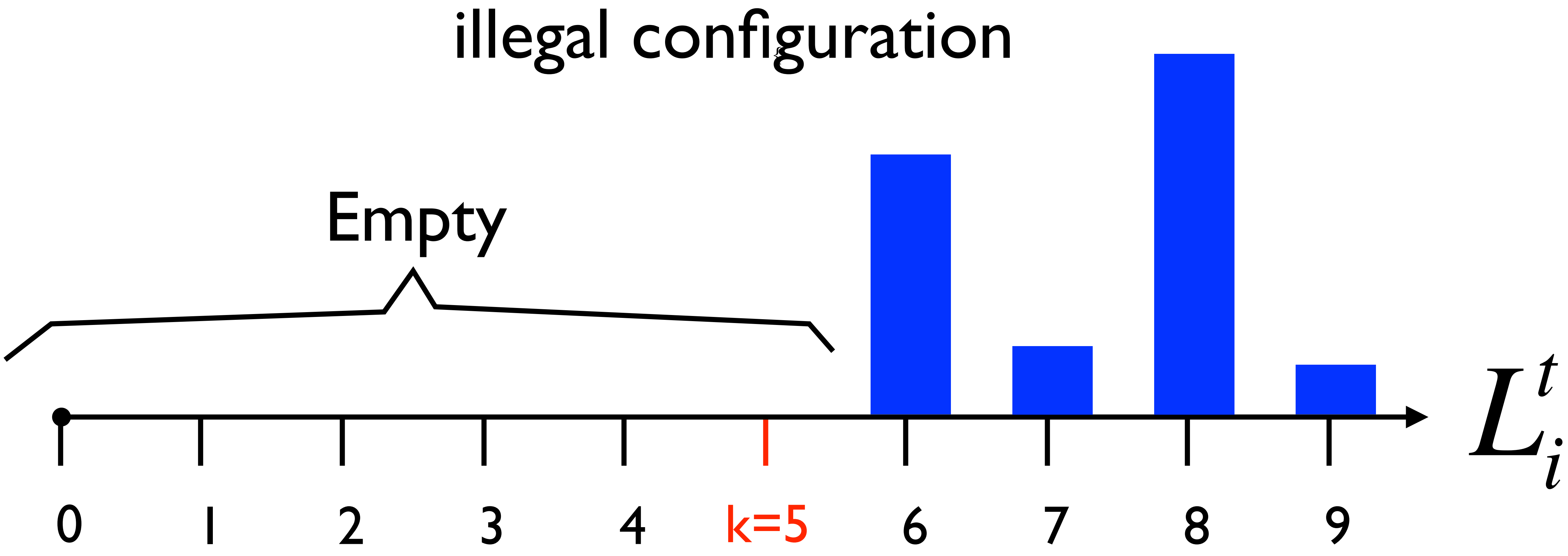
configuration t=3

Prediction is 1 and outcome is 0



If an error will lead to this configuration then an error is not possible

$\Rightarrow$ this is a safe prediction



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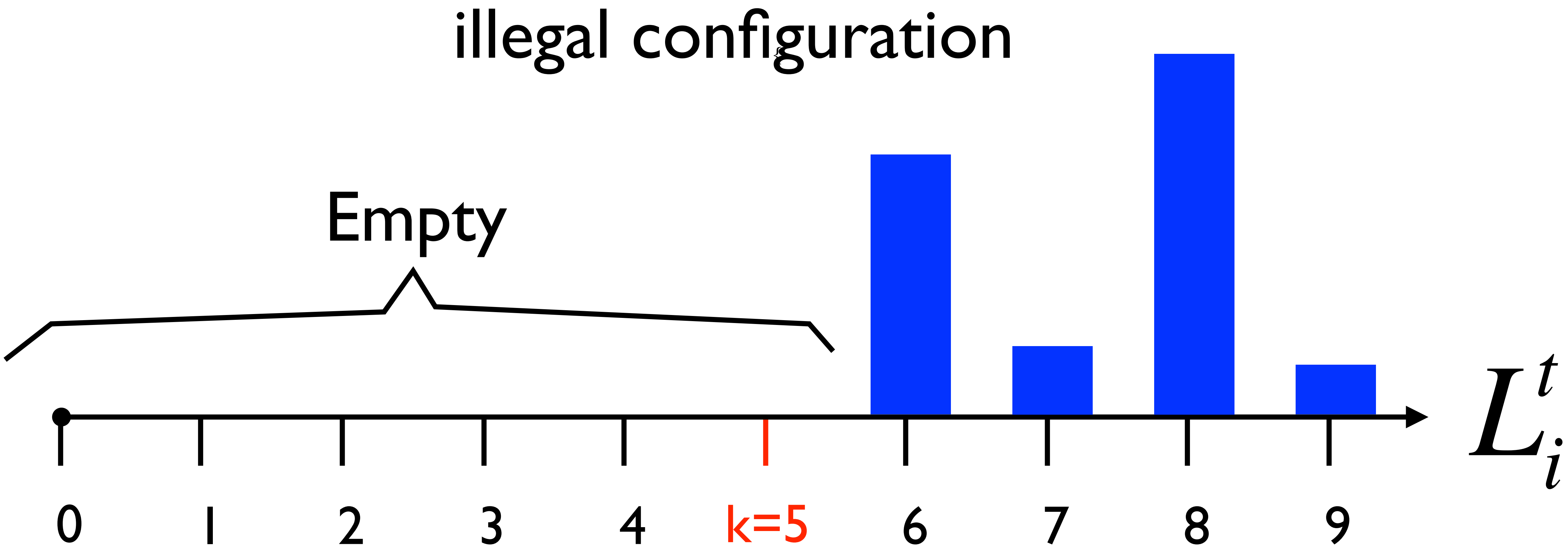

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- Find algorithm with the tightest uniform upper bound on the cumulative loss.

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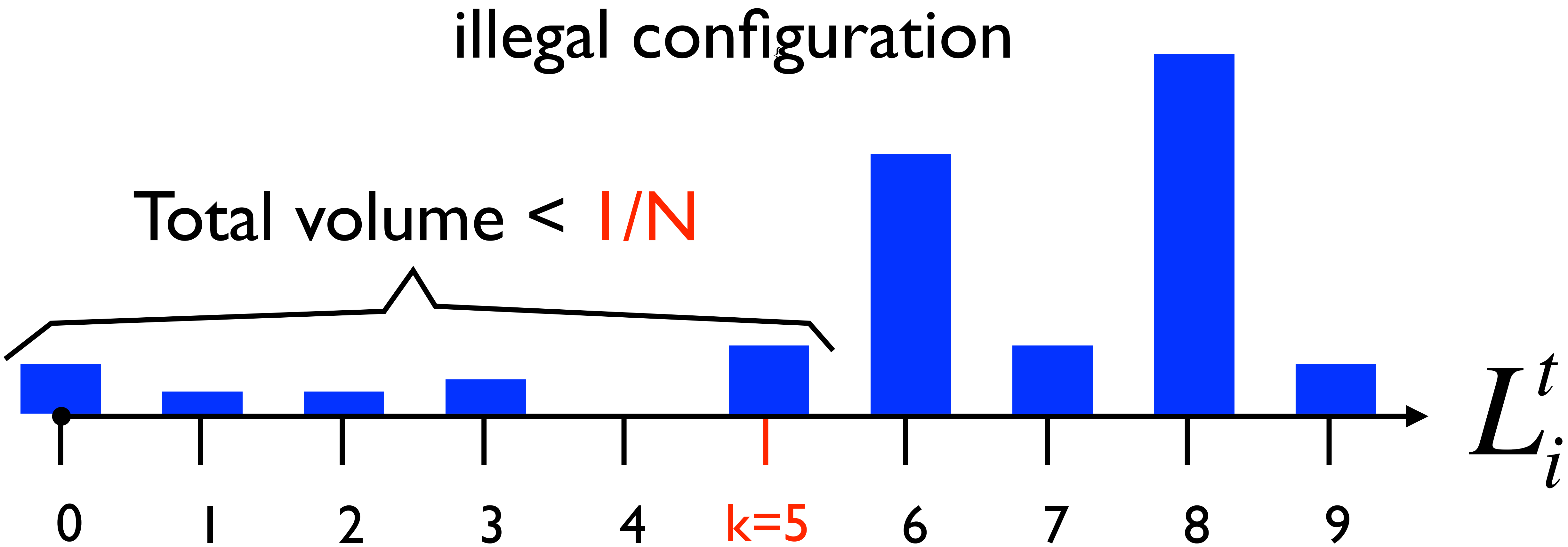

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- Same adversarial strategy was used to prove general lower bound on BLG

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- Relevant only when adversary plays sub-optimally, when adversary plays optimally the two next configurations are identical.

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$$\phi(i,k,m) \doteq \frac{1}{2^m}\binom{m}{\leq k-i} = \frac{1}{2^m}\sum_{j=0}^{k-i}\binom{m}{j}$$

- Configuration: $\vec{n} = < n_0, n_1, \ldots, n_k >$
- Potential: The number of chips that will remain in bins 0..k when m iterations remain and both sides play optimally:

$$\Psi(\vec{n},k,m) = \sum_{i=0}^{k} n_i \phi(i,k,m)$$

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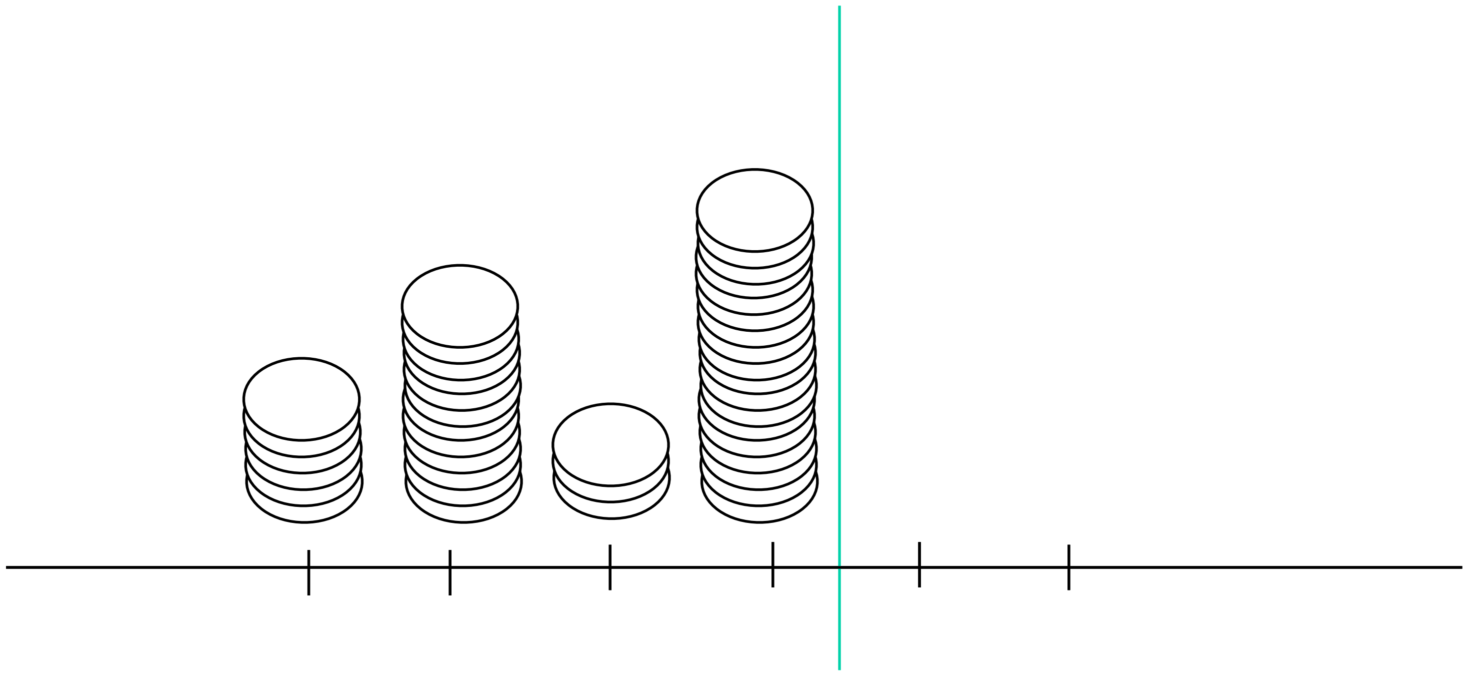

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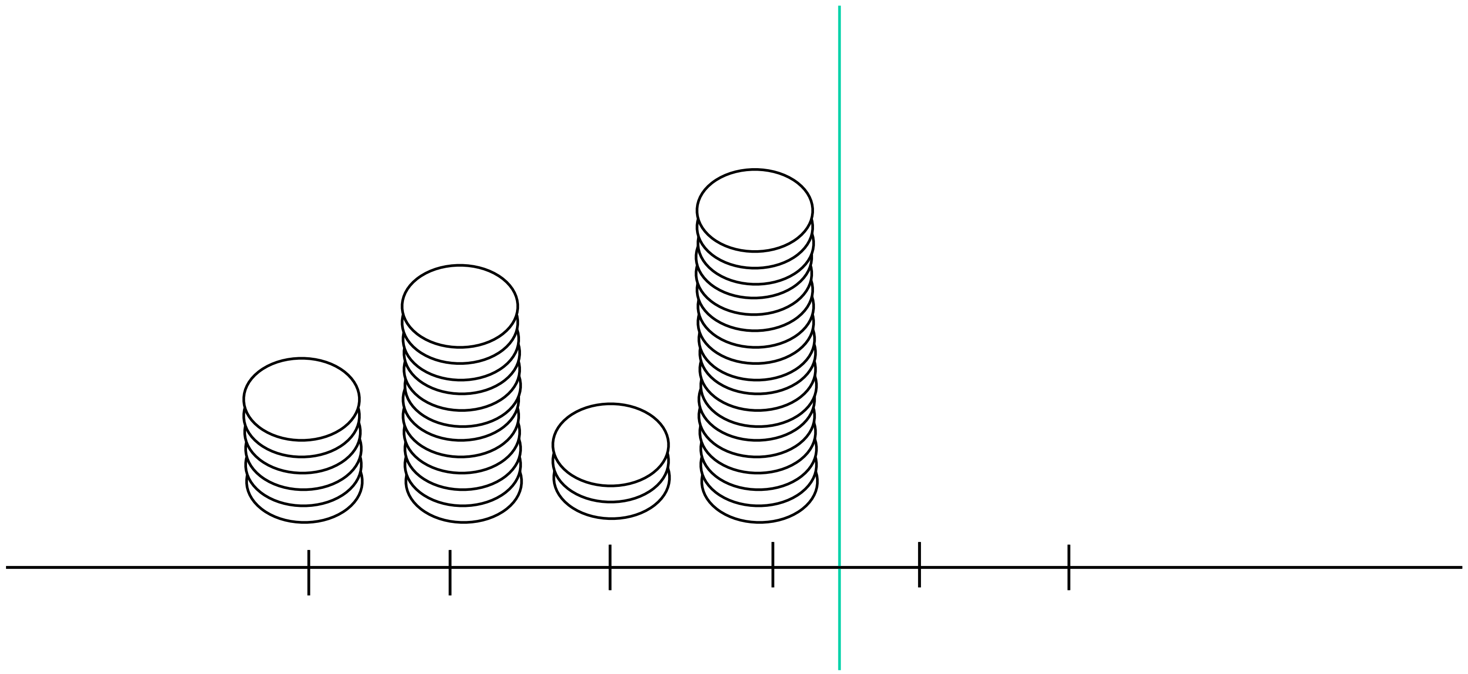

Optimal splitter strategy:
Split each bin into two equal parts

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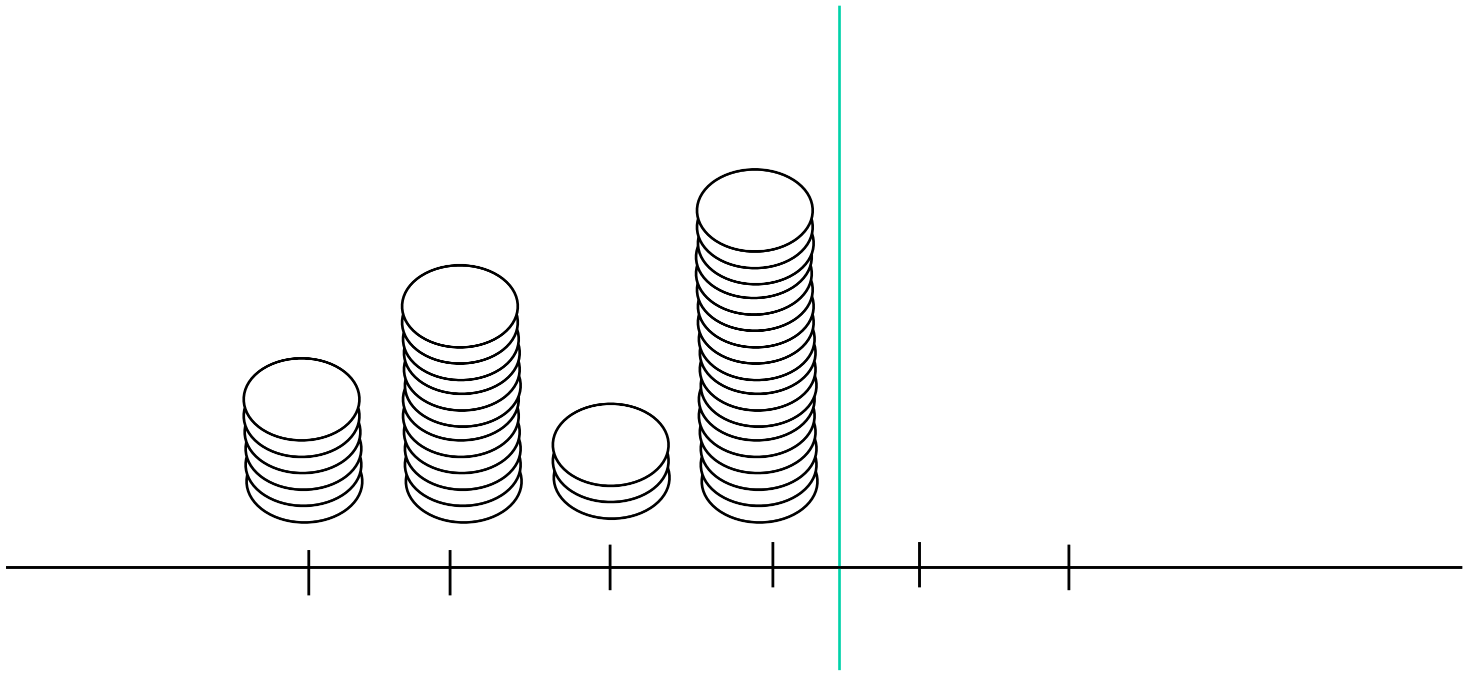

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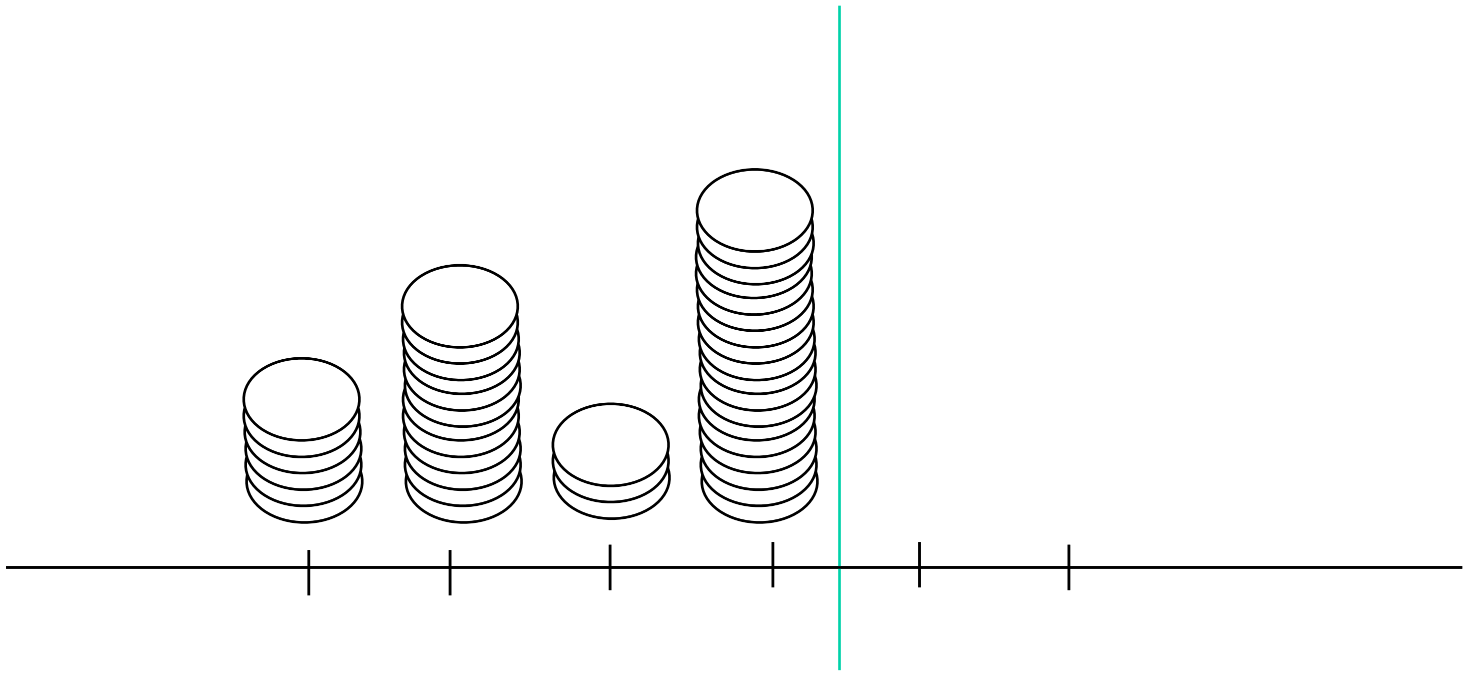

Optimal splitter strategy:
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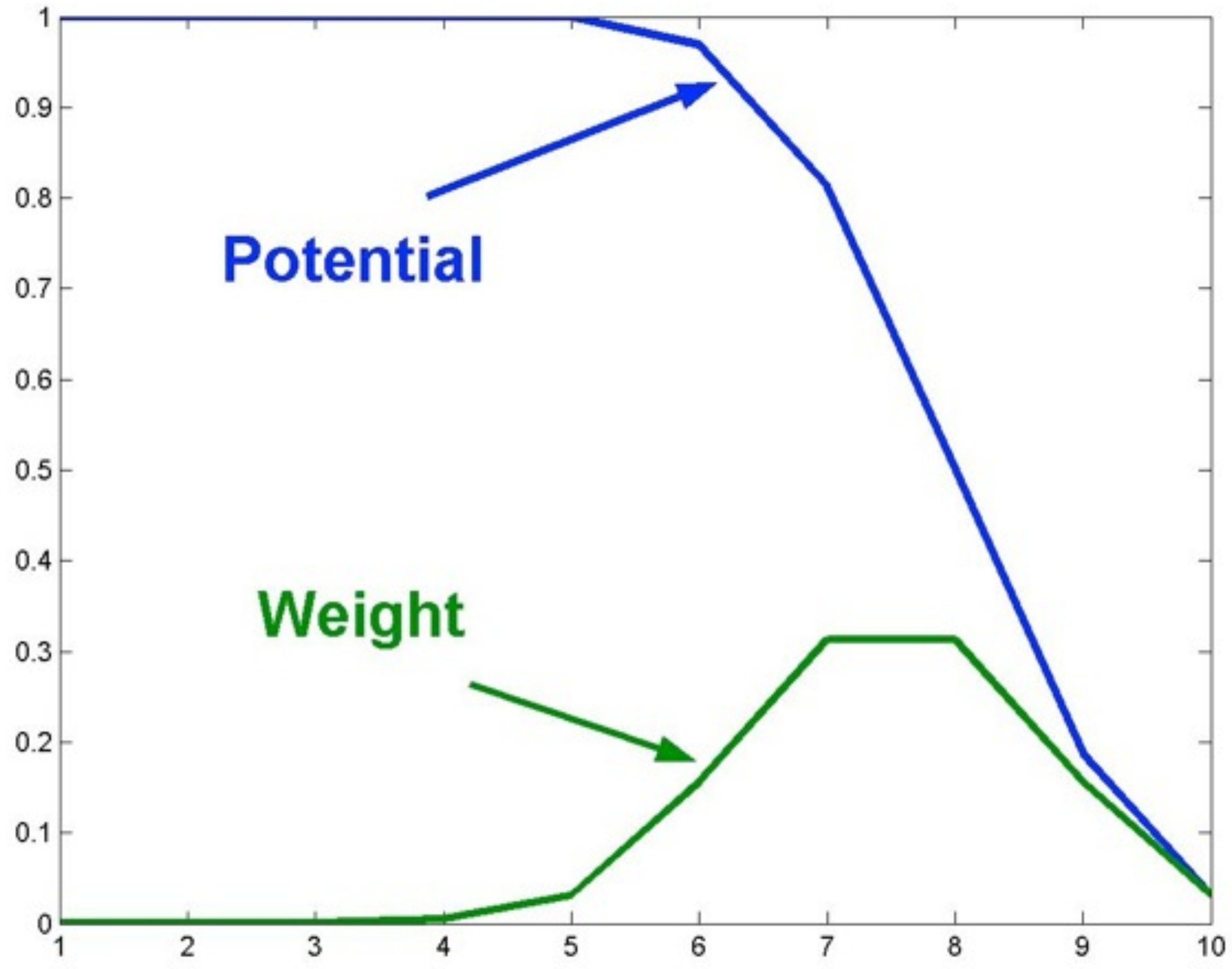
Example potential and weight

m=5; k=10



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- Basic idea: re-weight training examples to force weak learner into different parts of the space.
- Combine weak rules by a majority vote.

Boost by Majority (BBM)

[Freund 95]

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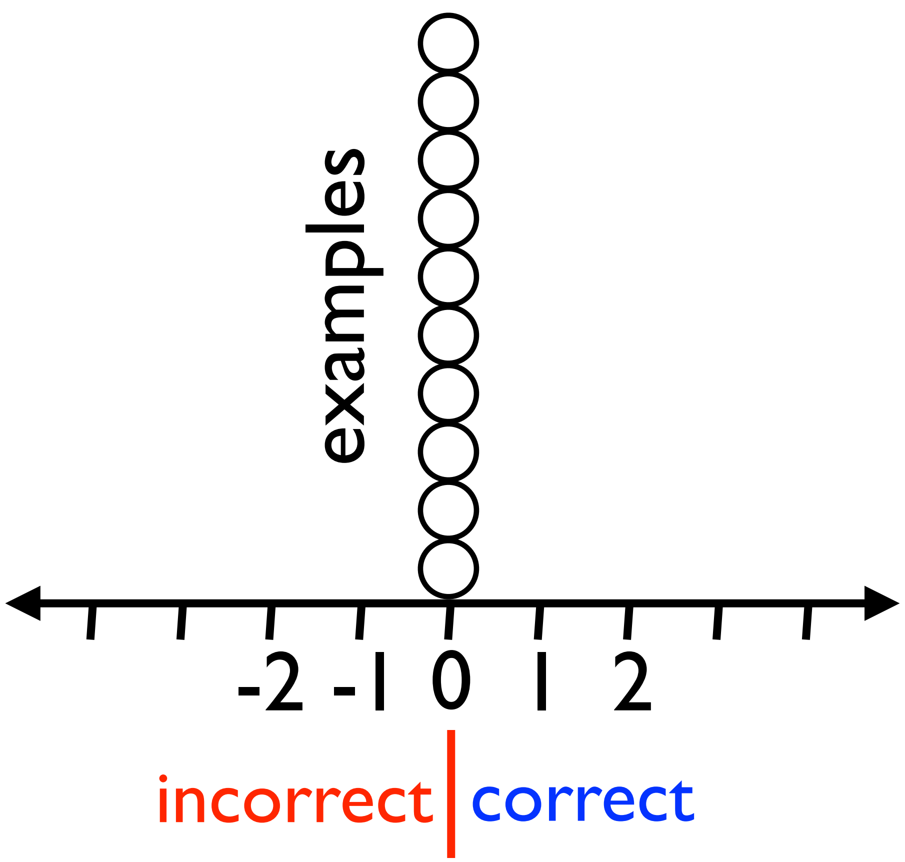
  - Rule is added to majority rule
- Goal of booster is to minimize number of errors of final majority rule.

BBM as a drifting game

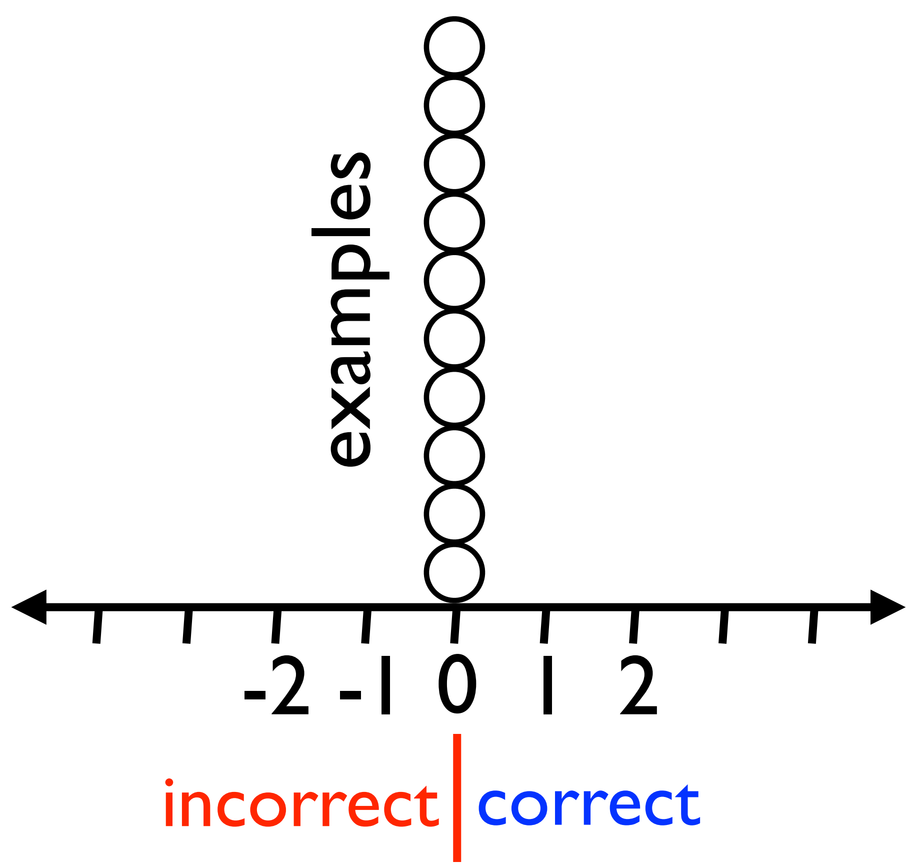


Initial Configuration

$$s \doteq y \sum_{t=1}^{T} h_t(x)$$

BBM as a drifting game

- Chips = examples



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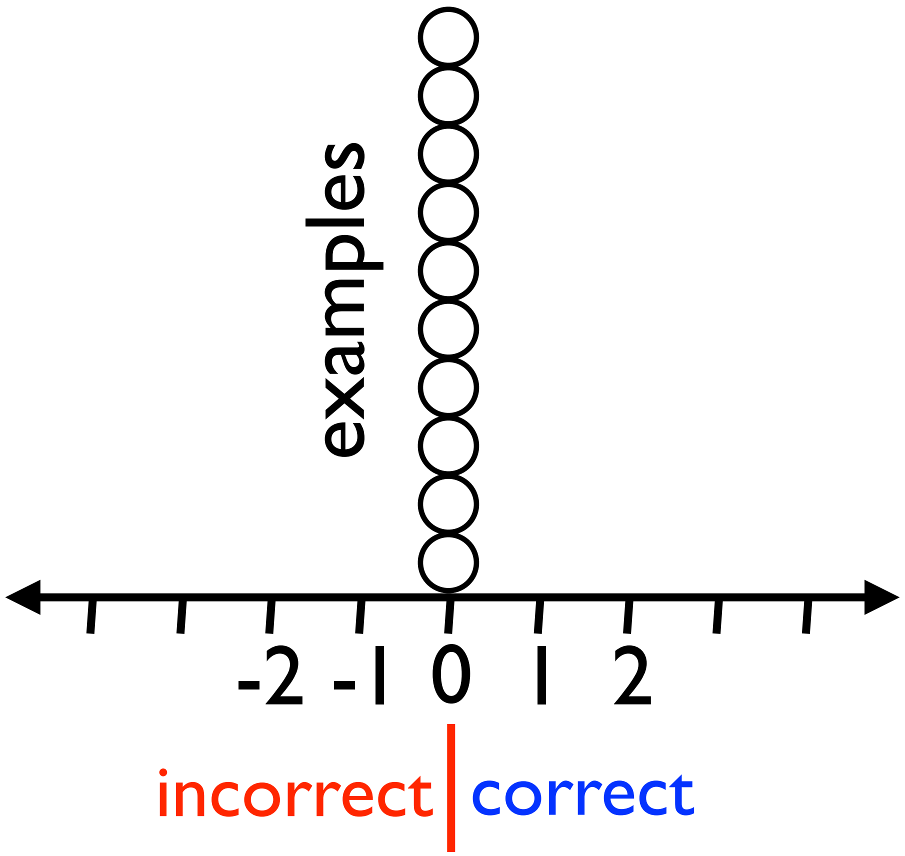
BBM as a drifting game

- Chips = examples
- bin s contains the examples for which the difference between the number of correct and incorrect base rules is s



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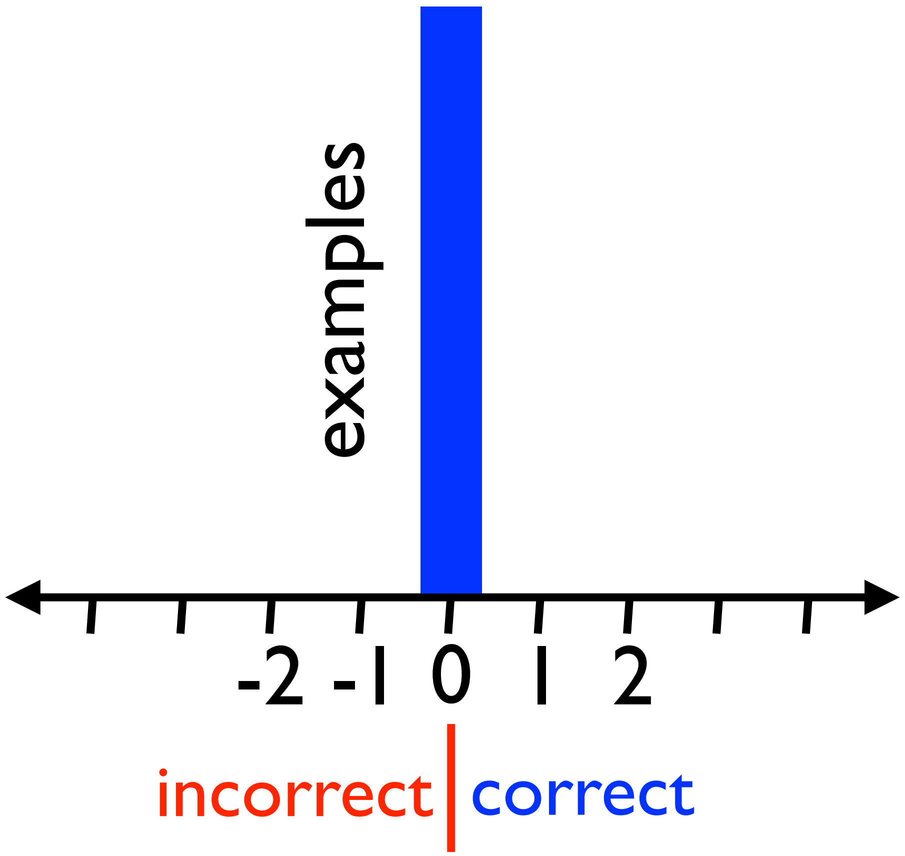
The continuous chip limit



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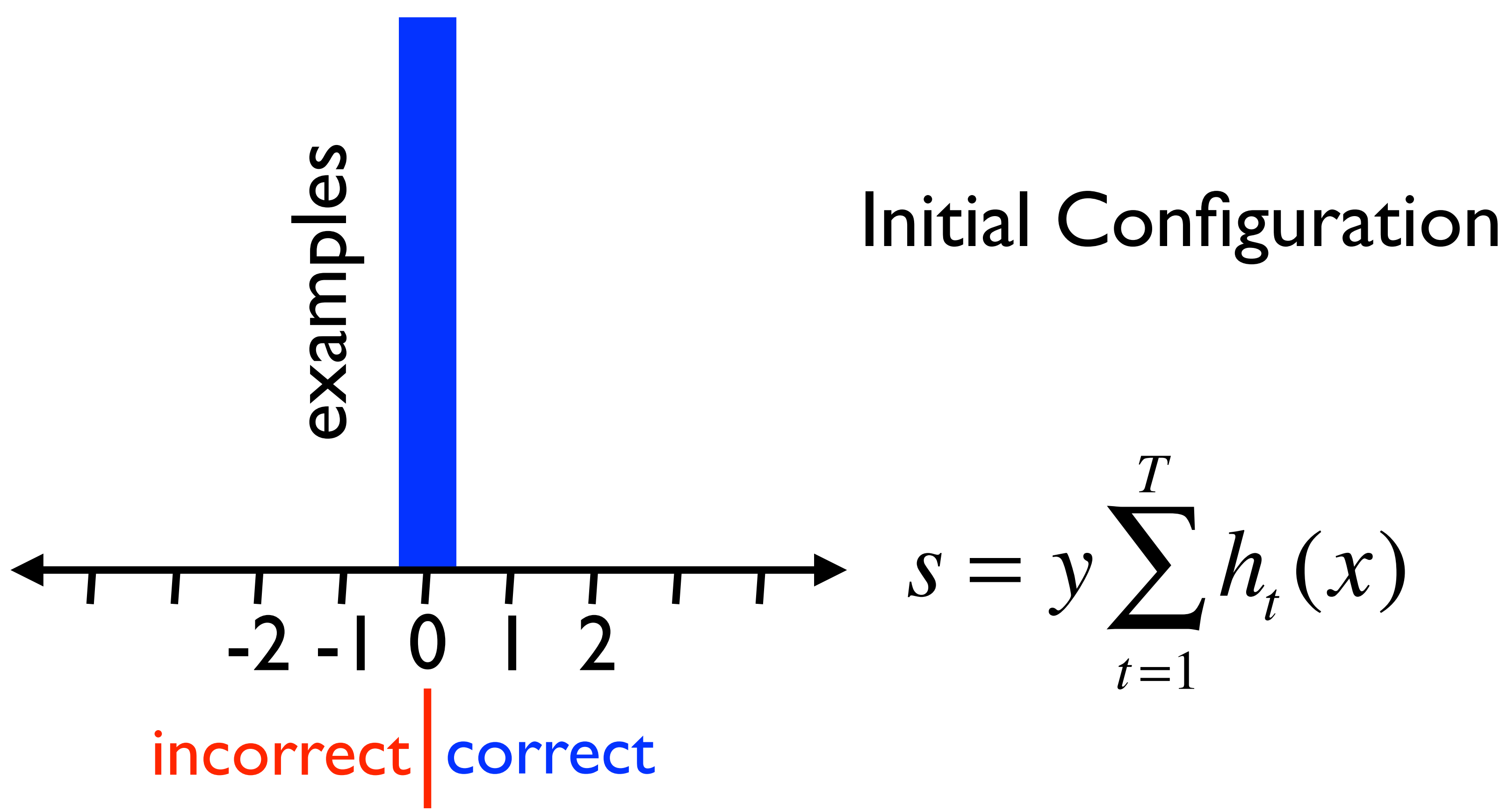
- Number of training examples increases to infinity.



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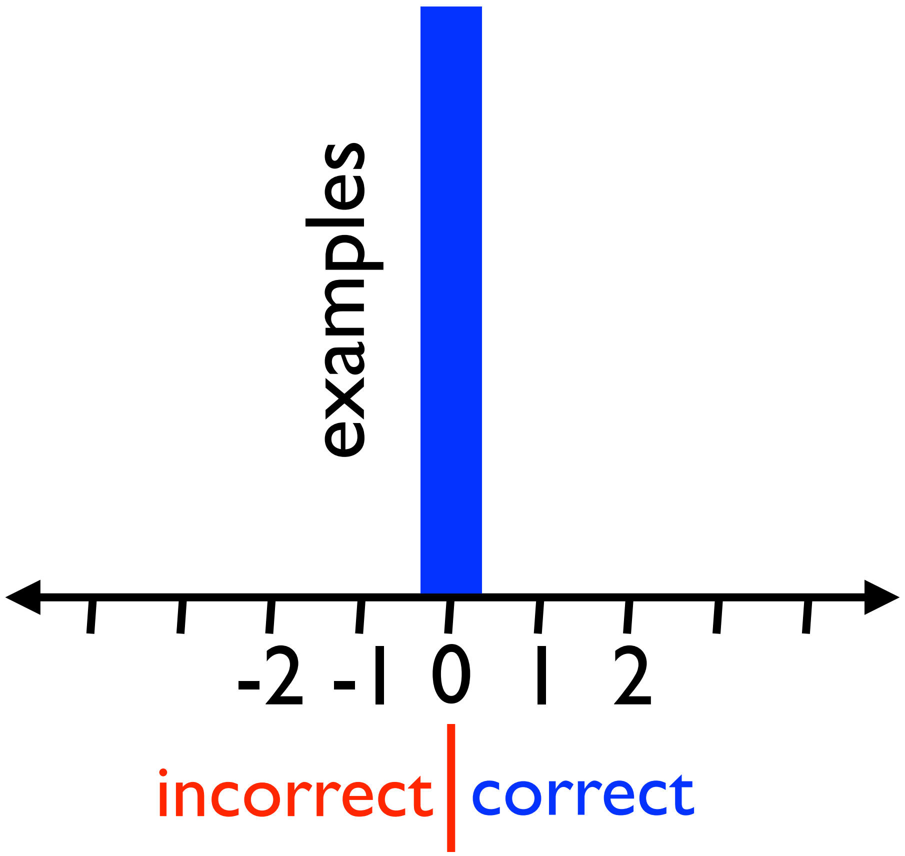
The continuous chip limit

- Number of training examples increases to infinity.
- Alternatively - think of examples as a probability mass with probability measure μ defined on it.



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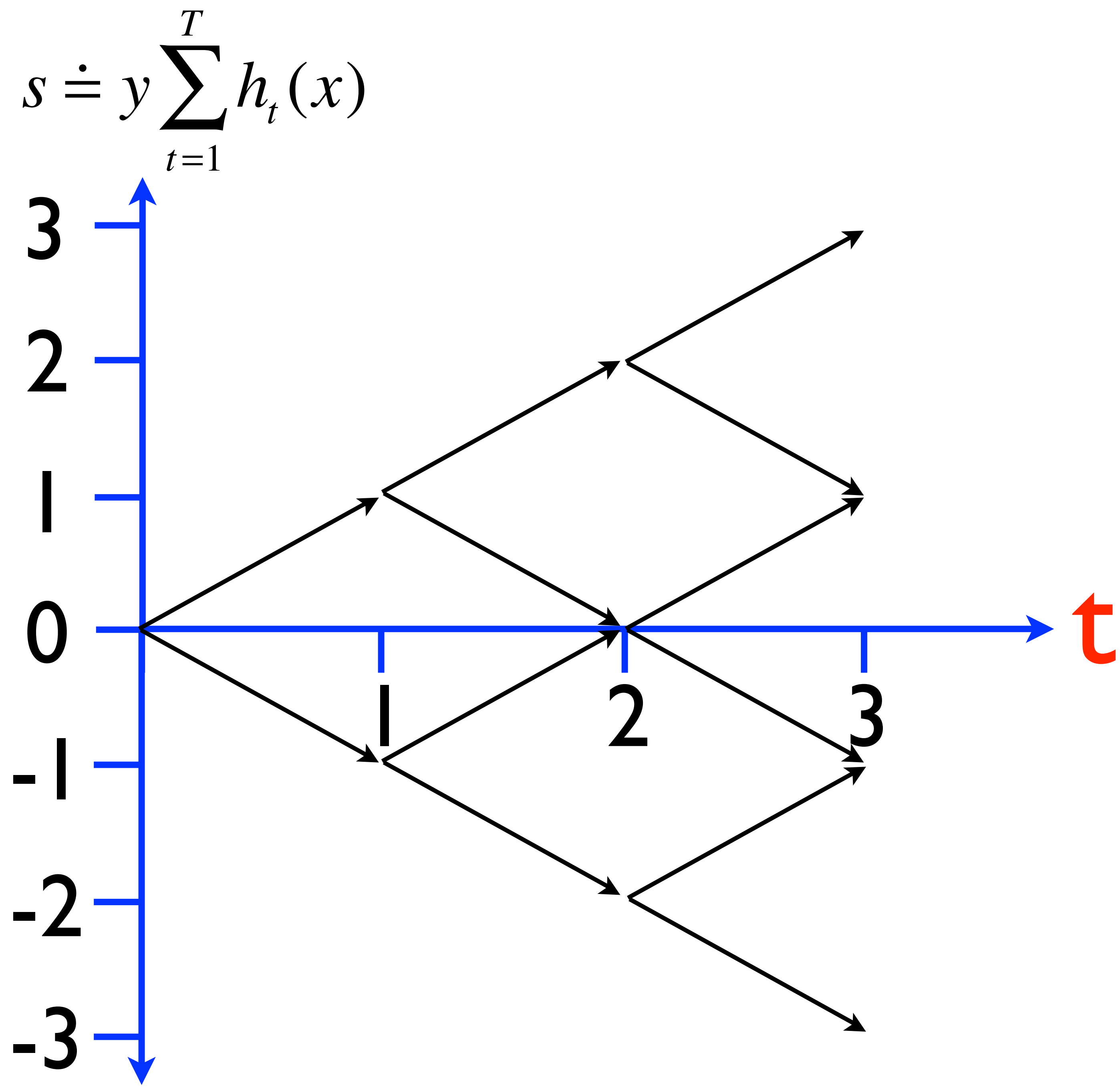

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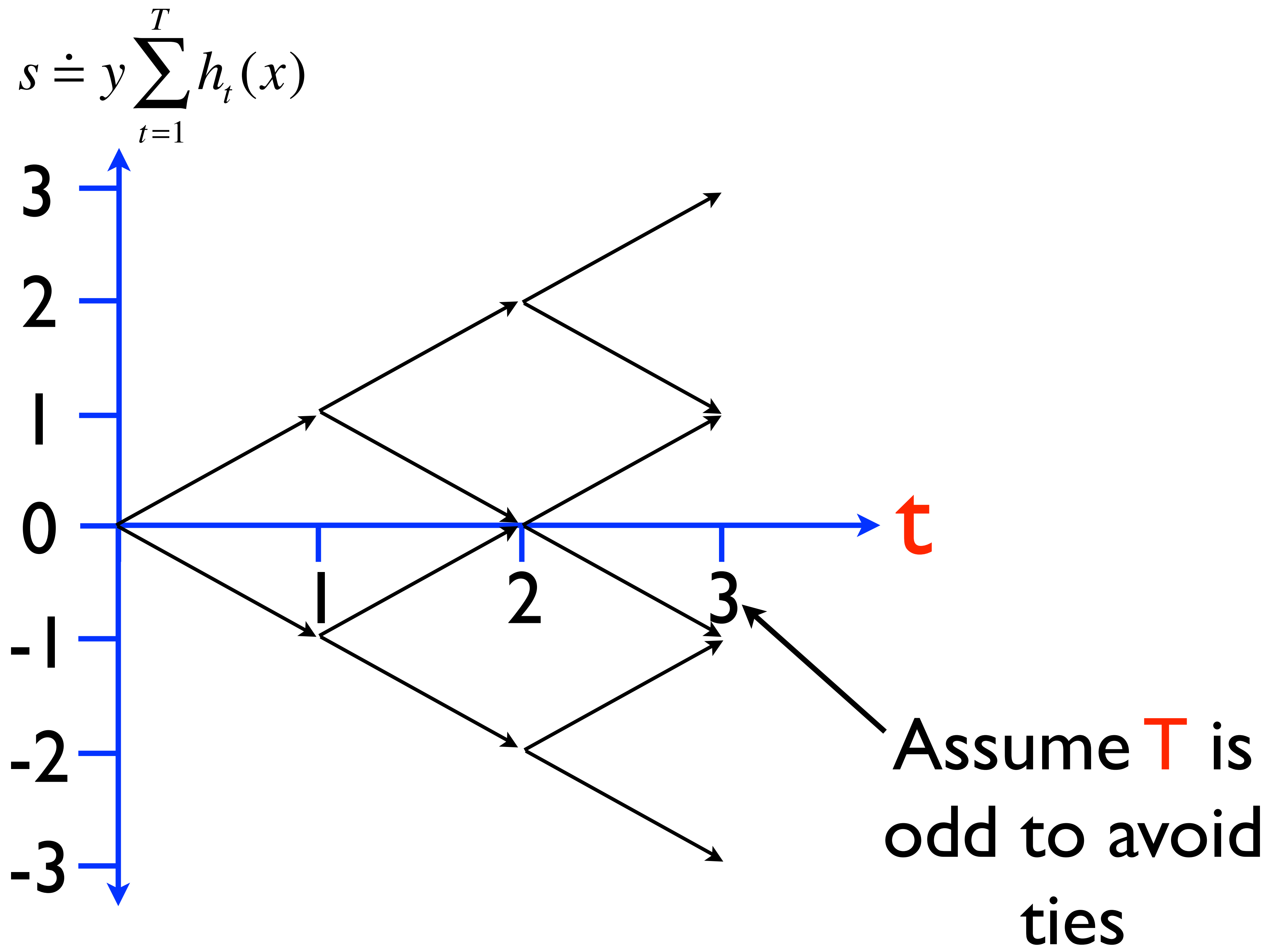
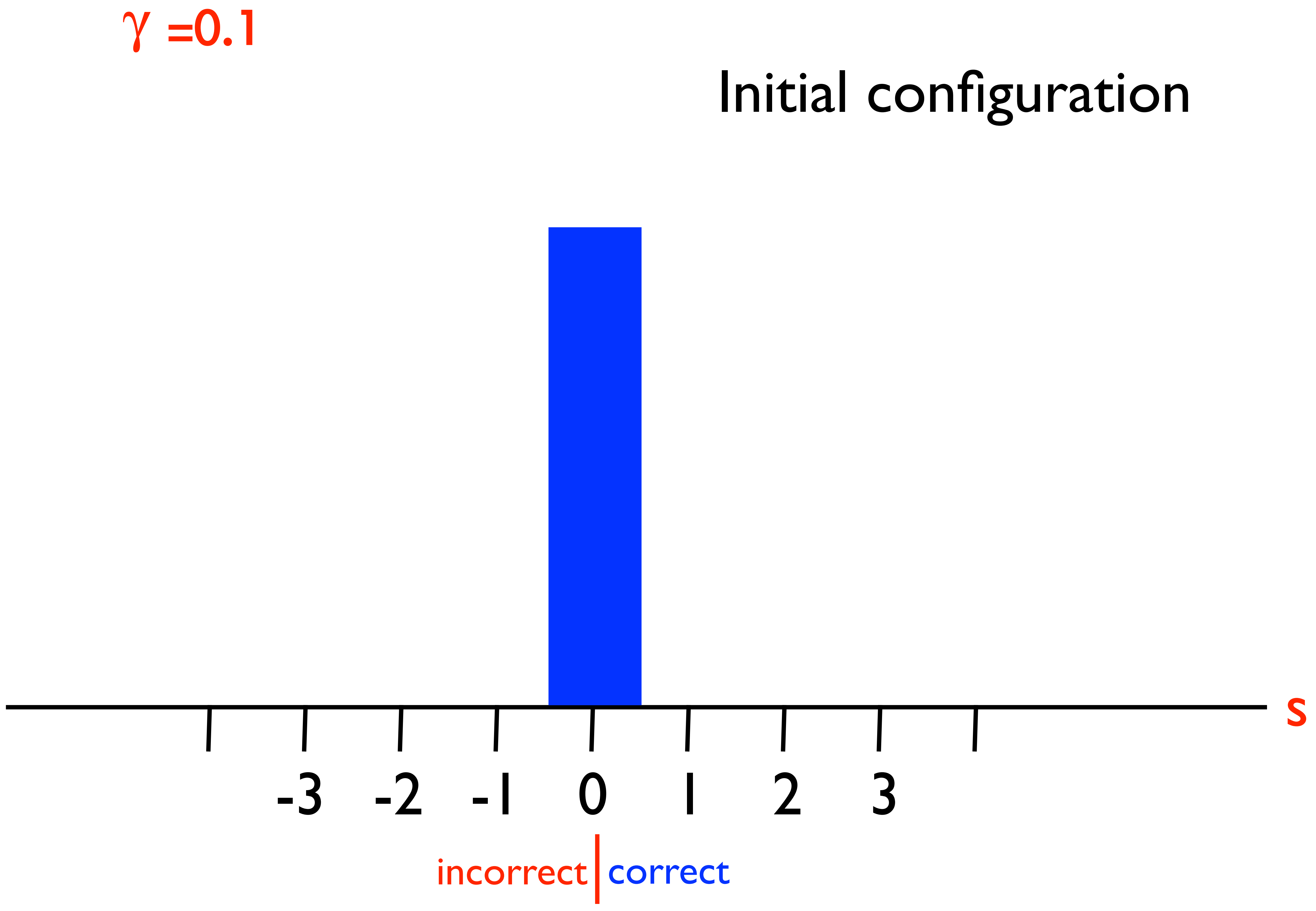
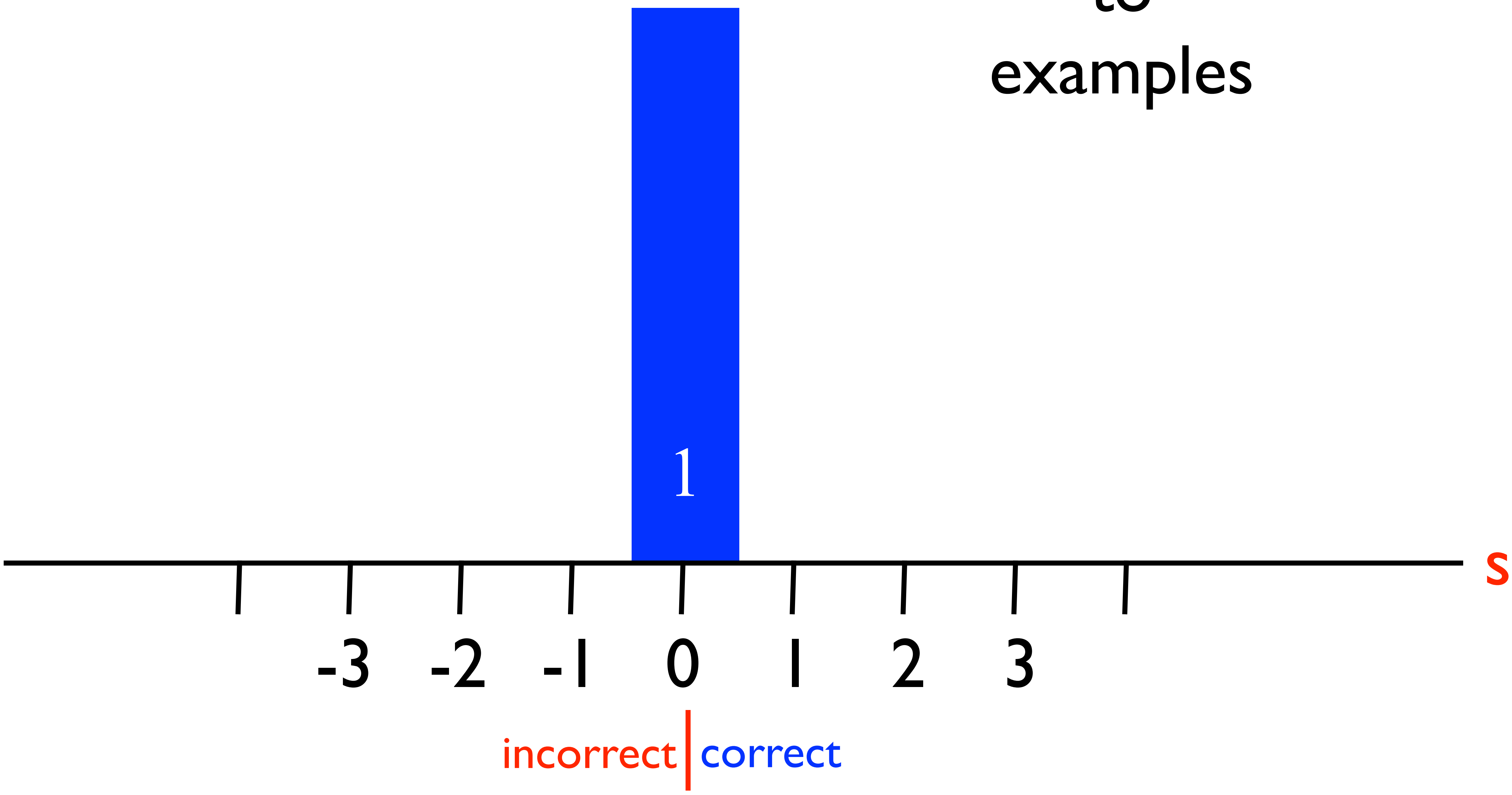




γ =0.1
Initial configuration
s
-3
-2
-1
0
1
2
3
incorrect
correct

$\gamma = 0.1$

Booster assigns weights to examples



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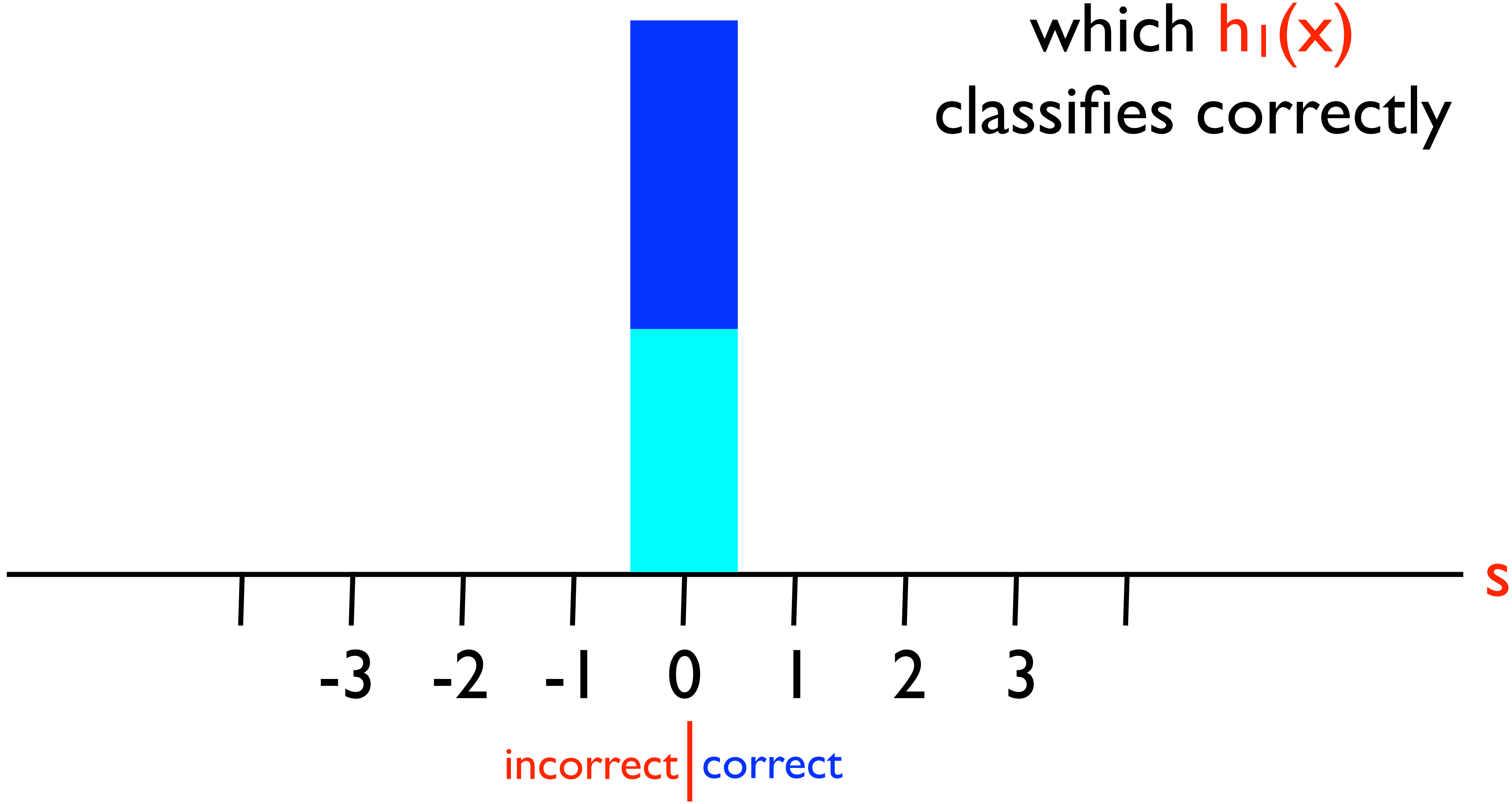
Weak learner chooses subset with weight $1/2+\gamma$ which $h_1(x)$ classifies correctly



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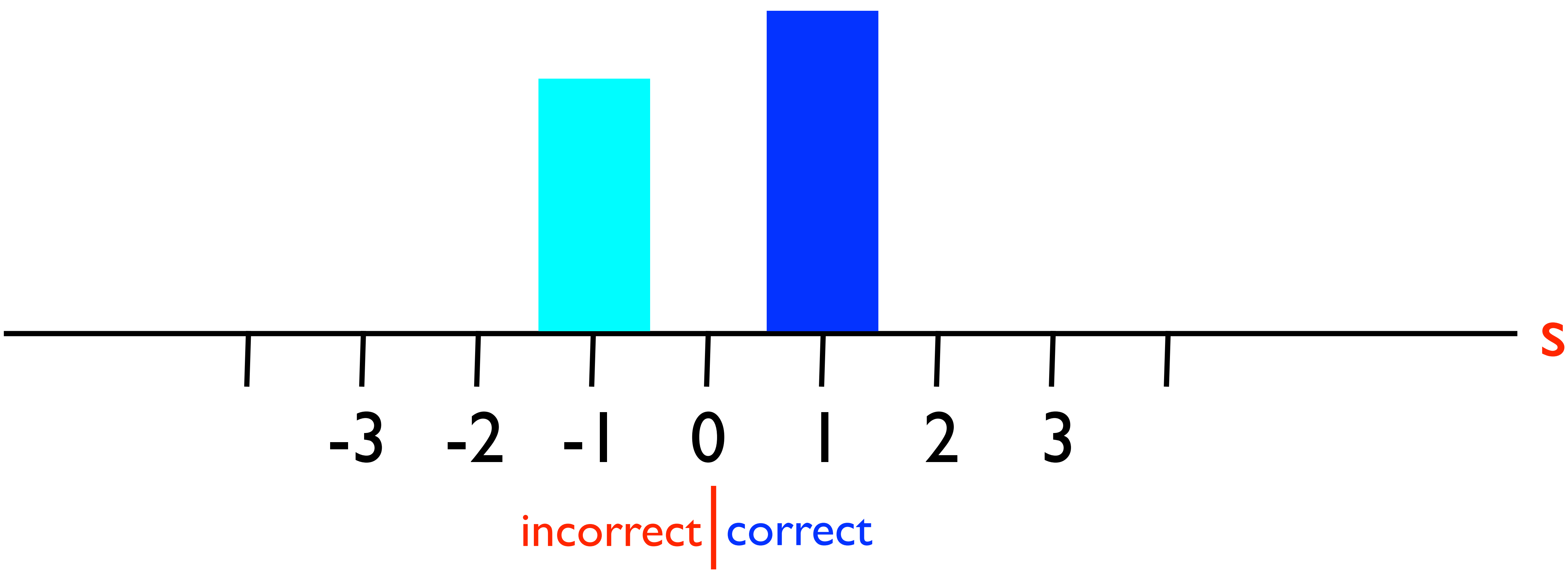

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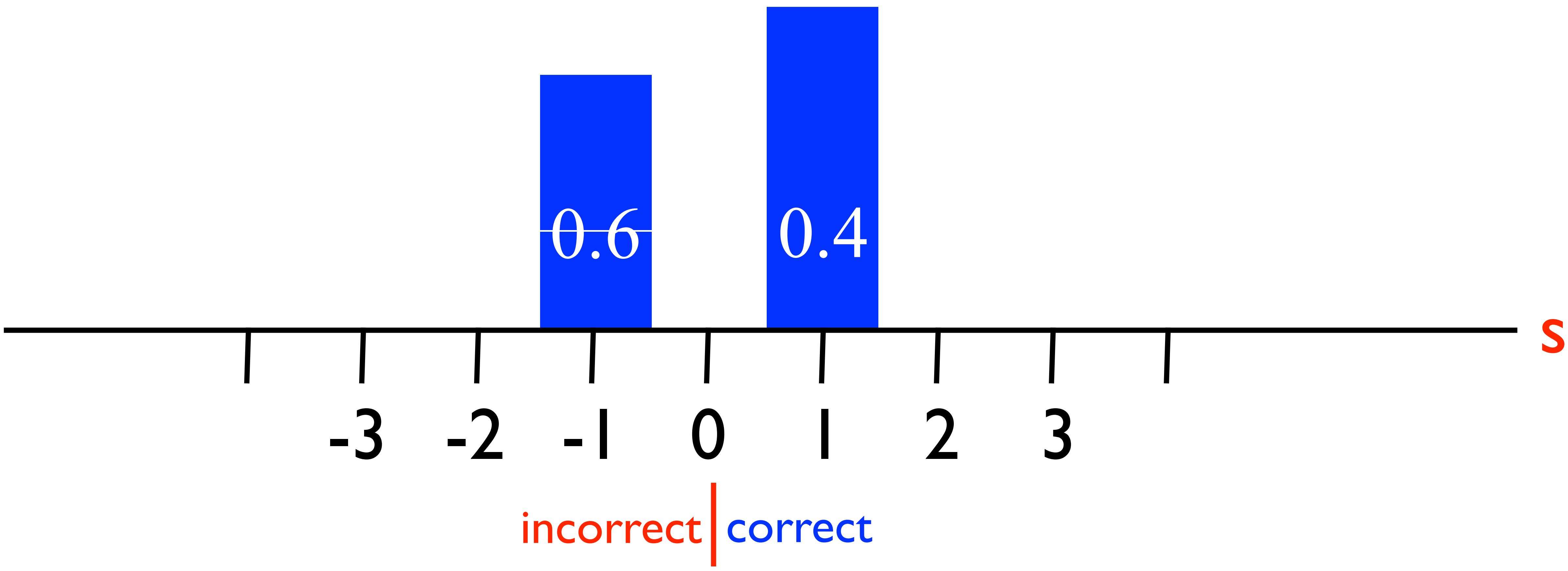

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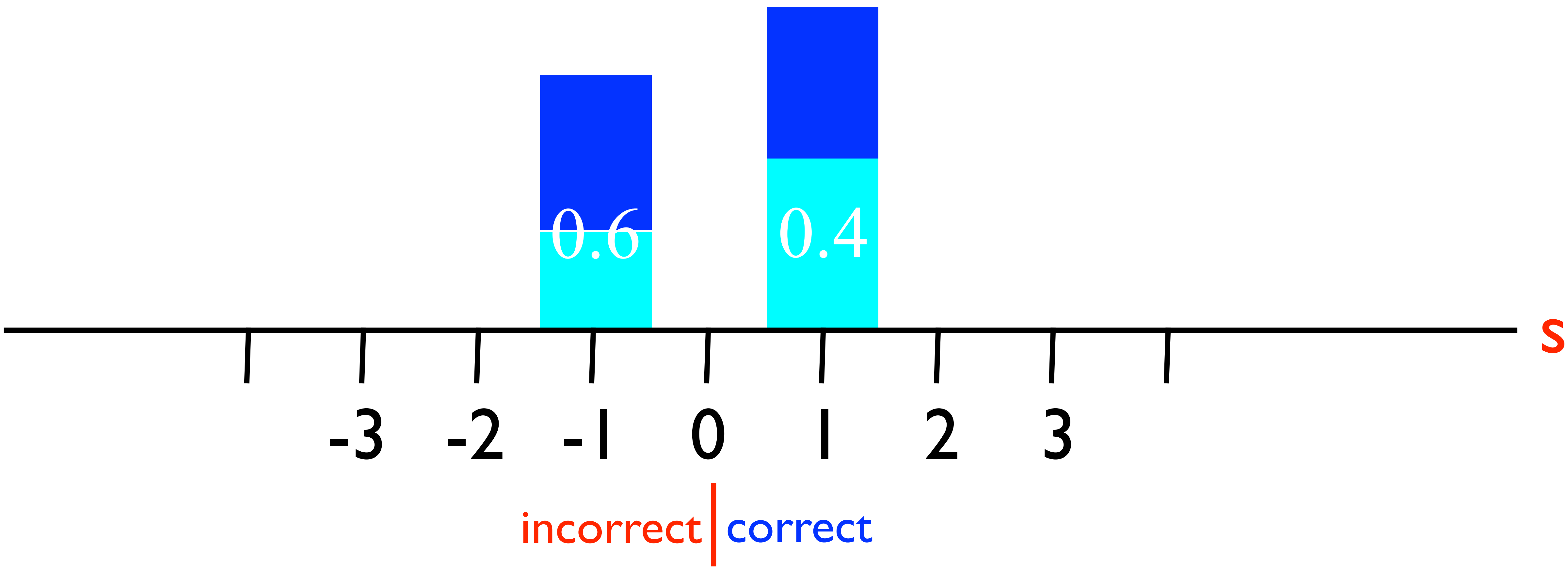

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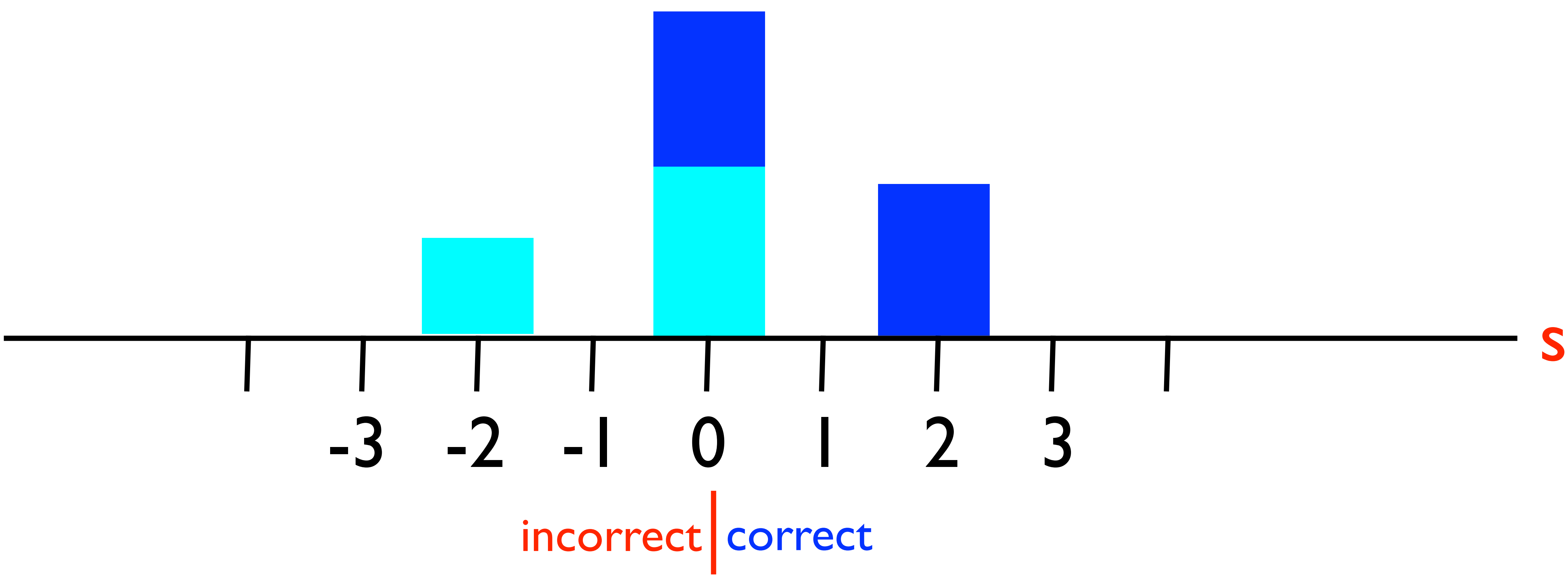

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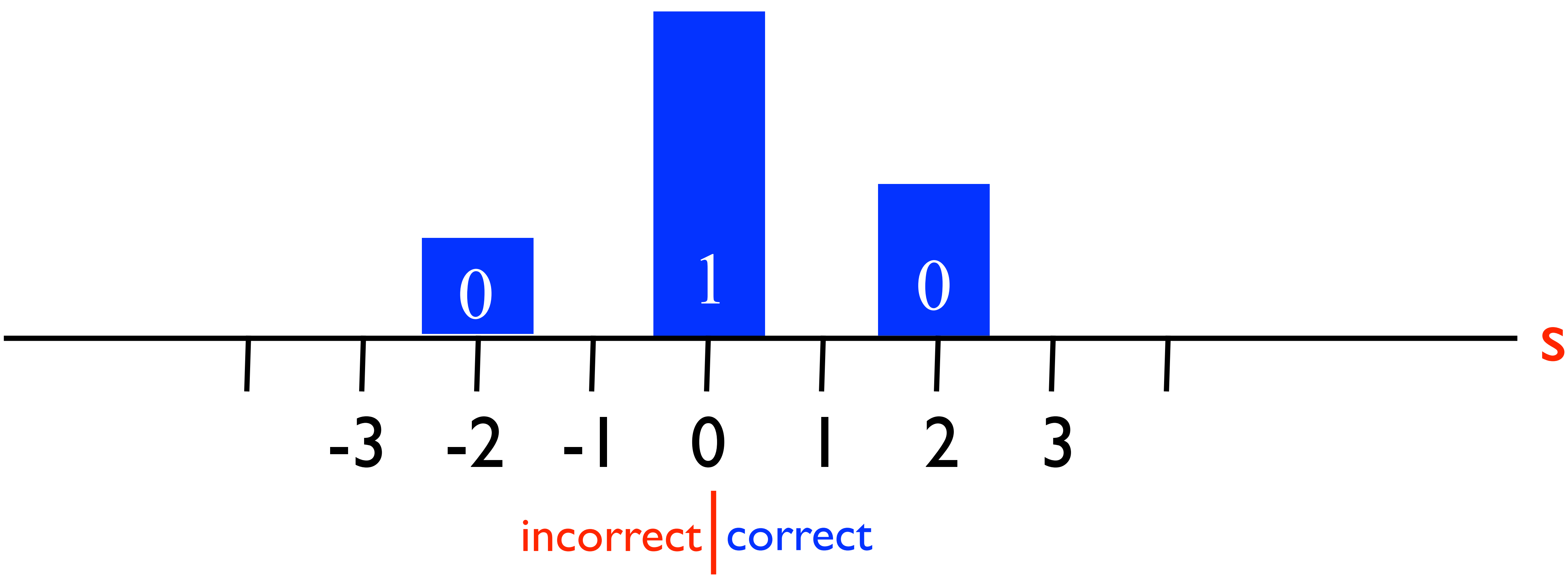

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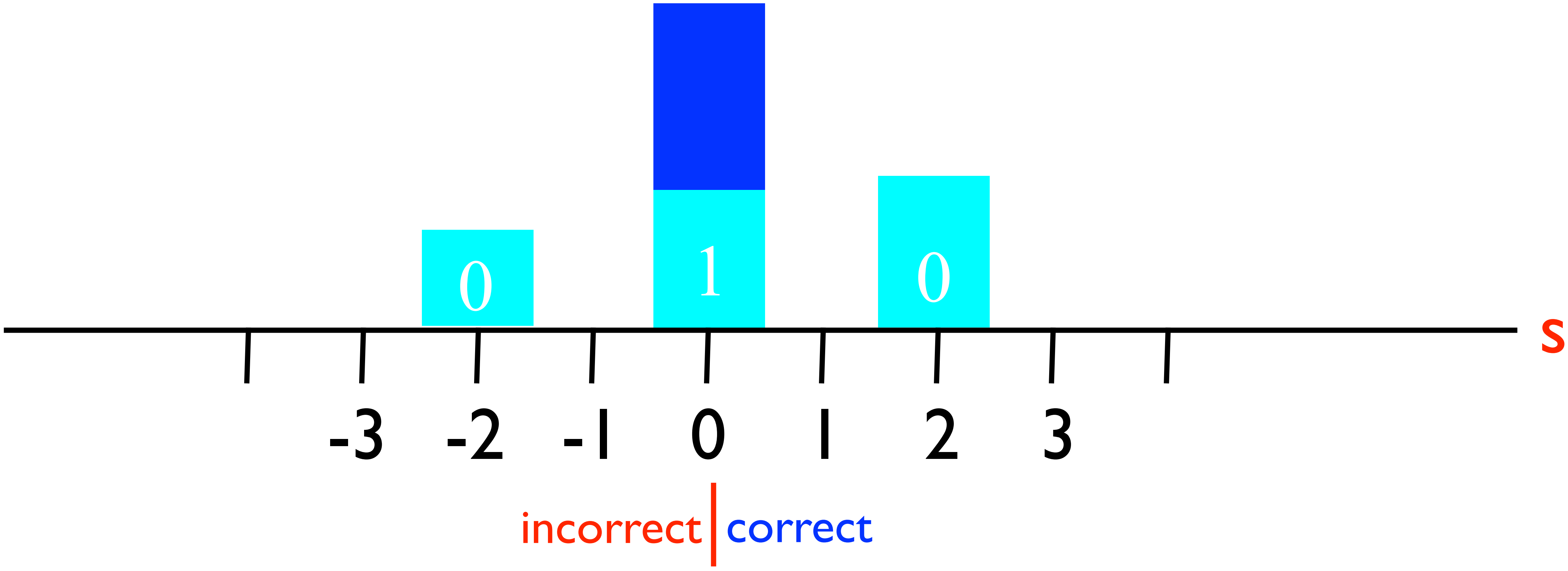

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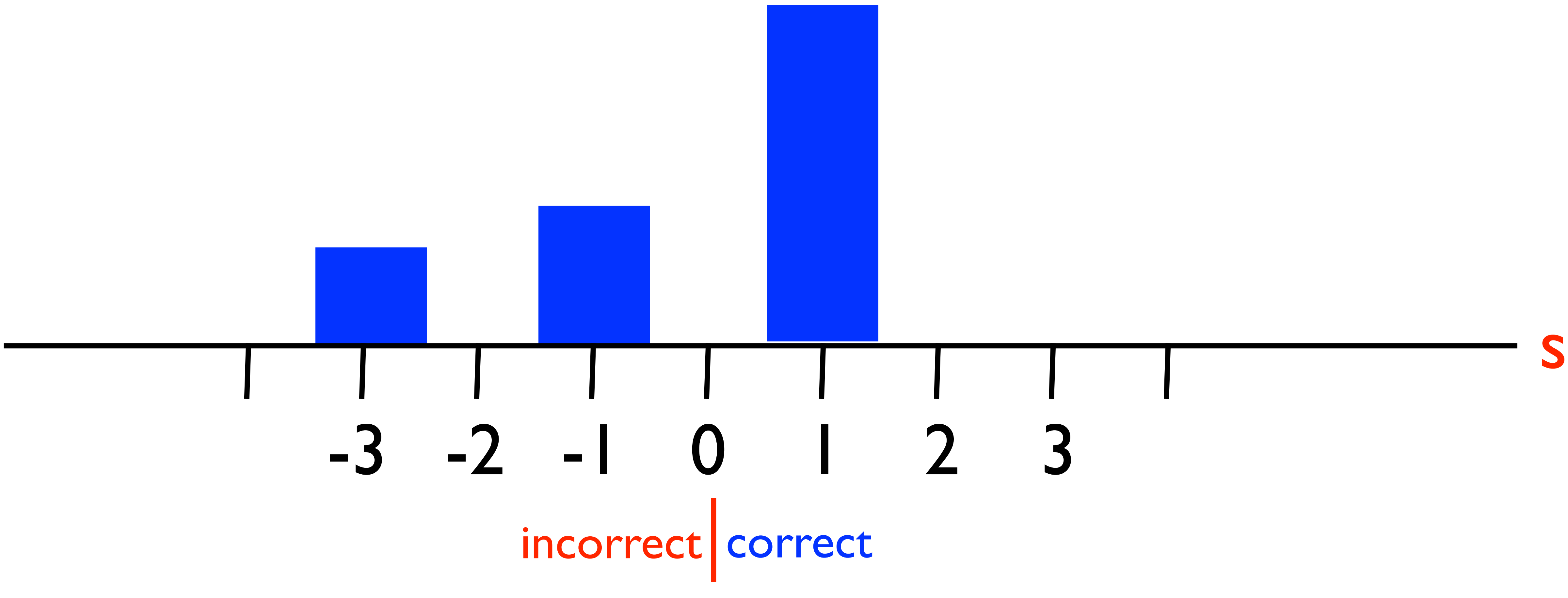

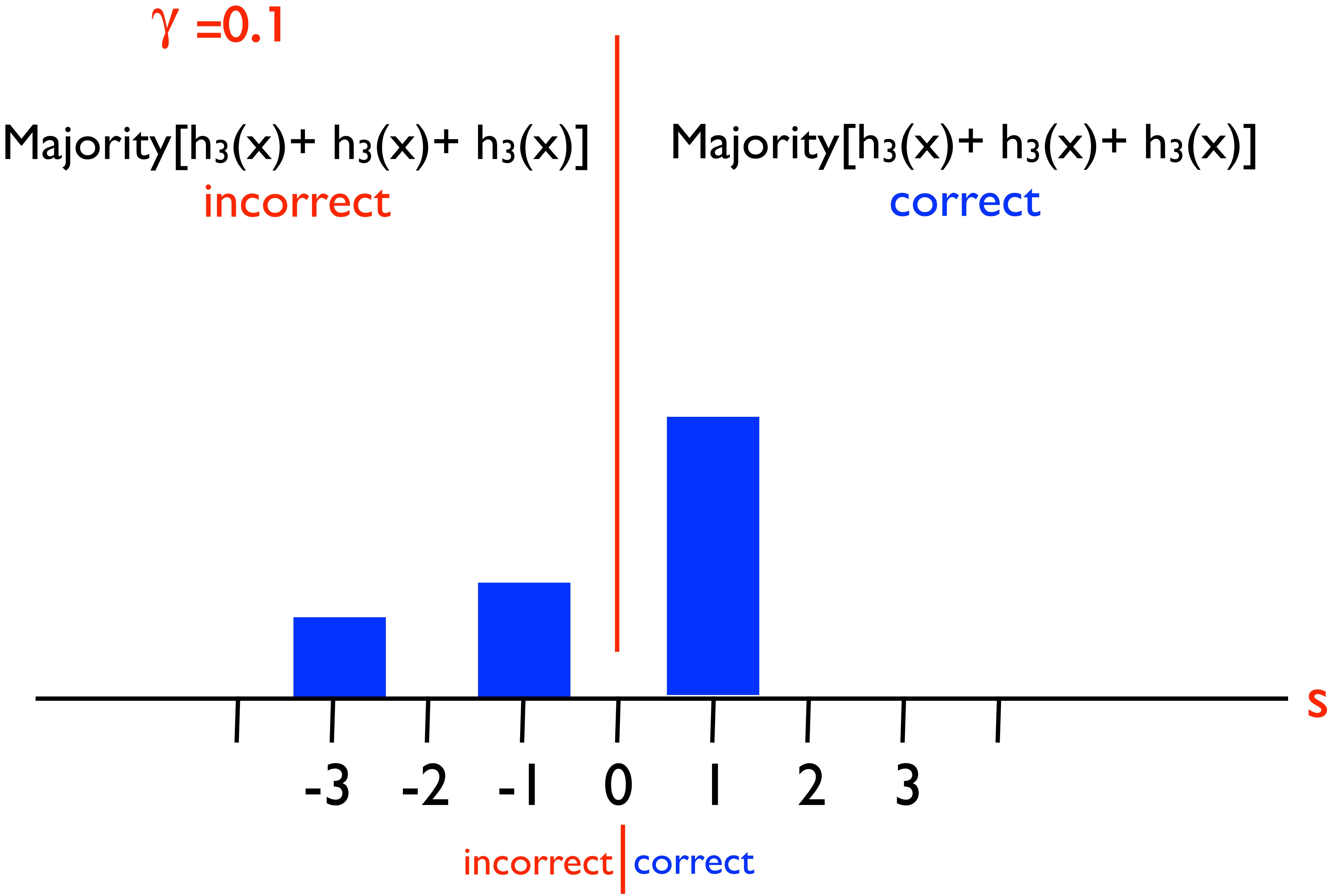

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incorrect
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correct
s
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-2
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- Equivalently: prediction of each base rule on each example is chosen independently at random

$$P(h_t(x)=y) = 1/2+\gamma$$

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Definitions

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Example Weights: $w(t,s) \doteq \phi(t+1, s-1) - \phi(t+1, s+1)$

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Theorem

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Theorem

If, for step t, $\sum_s d(t,s) w(t,s) \geq 0 \quad$ then $\Psi(t+1) \leq \Psi(t)$

Corollary

If $\forall t$ [weighted error of $h_t(x)$] $\leq 1/2-\gamma$

Then

Initial potential $\geq$ final training error.

Proof of Theorem

Proof of Theorem

$$f(t+1,s) = f(t,s-1)\left(\frac{1}{2}+\gamma+d(t,s-1)\right)+f(t,s+1)\left(\frac{1}{2}-\gamma-d(t,s+1)\right)$$

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$$\Psi(f,t+1) = \sum_{s}\left[f(t,s-1)\left(\frac{1}{2}+\gamma+d(t,s-1)\right)+f(t,s+1)\left(\frac{1}{2}-\gamma-d(t,s+1)\right)\right]\phi(t+1,s)$$

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If $\sum_{s}d(t,s)w(t,s)\geq 0$ then $\Psi(f,t+1)\leq\Psi(f,t)$

Theorem about BBM

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setting the boosting weights at iteration t to be

$$w(t,s) = \binom{T-t}{\left\lfloor \frac{T-t-s+1}{2} \right\rfloor} \left(\frac{1}{2}+\gamma\right)^{\left\lfloor \frac{T-t-s+1}{2} \right\rfloor} \left(\frac{1}{2}-\gamma\right)^{\left\lceil \frac{T-t+s-1}{2} \right\rceil}$$

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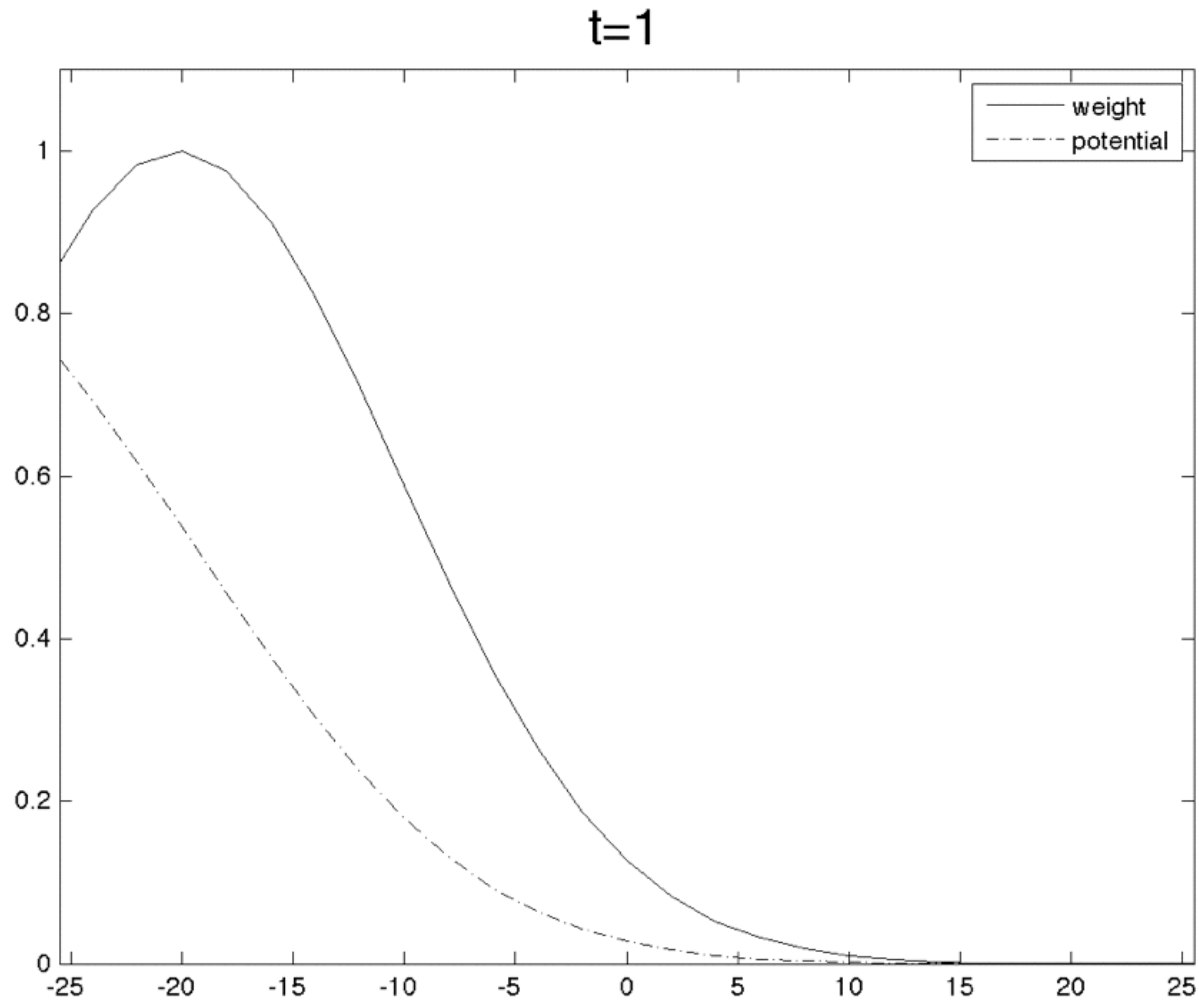
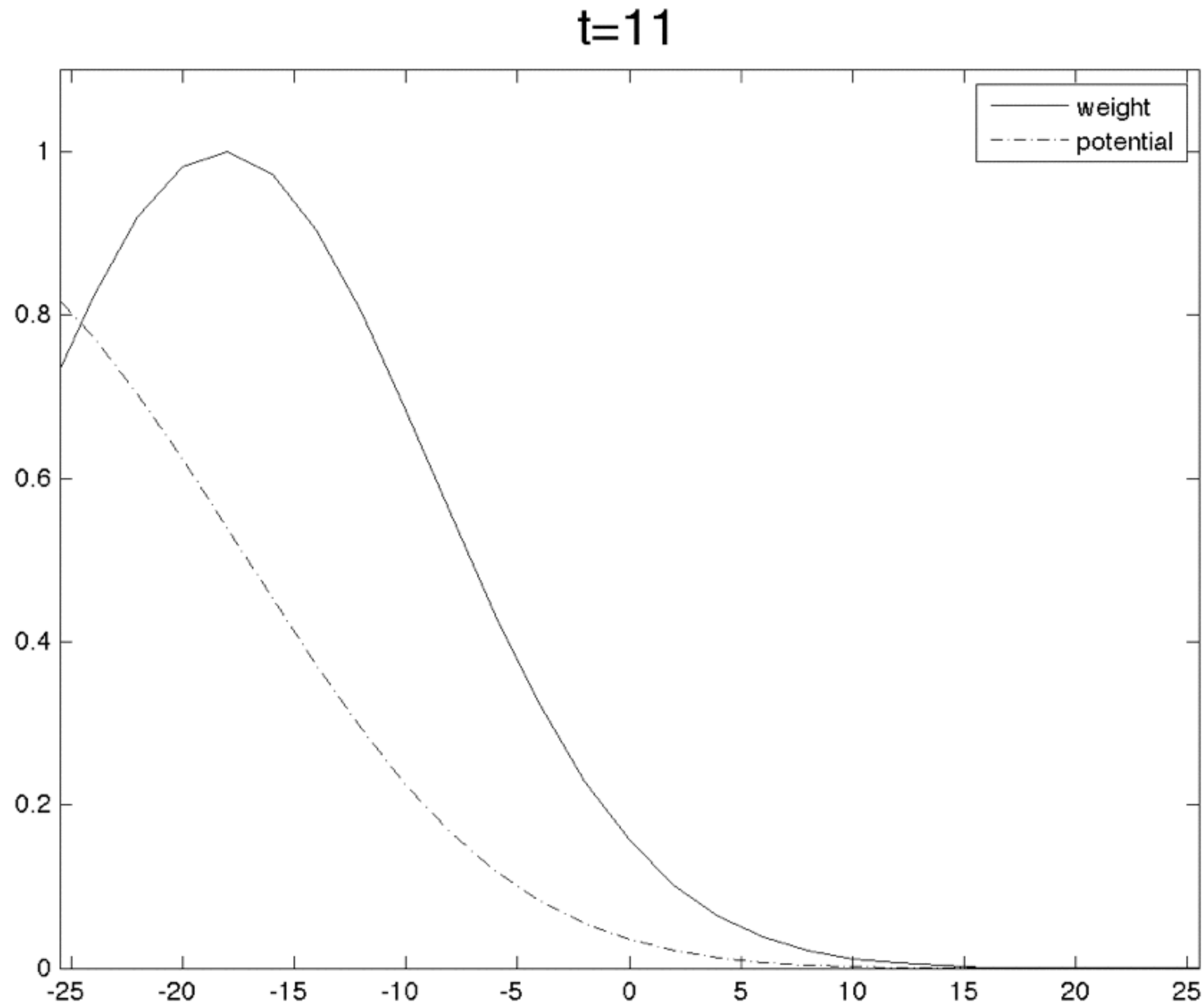
guarantees

$$\text{Initial potential} = \Psi(0) \geq \Psi(1) \geq \cdots \geq \Psi(T) = \text{ training error of sign}\left(\sum_{t=1}^{T} h_t(x)\right)$$

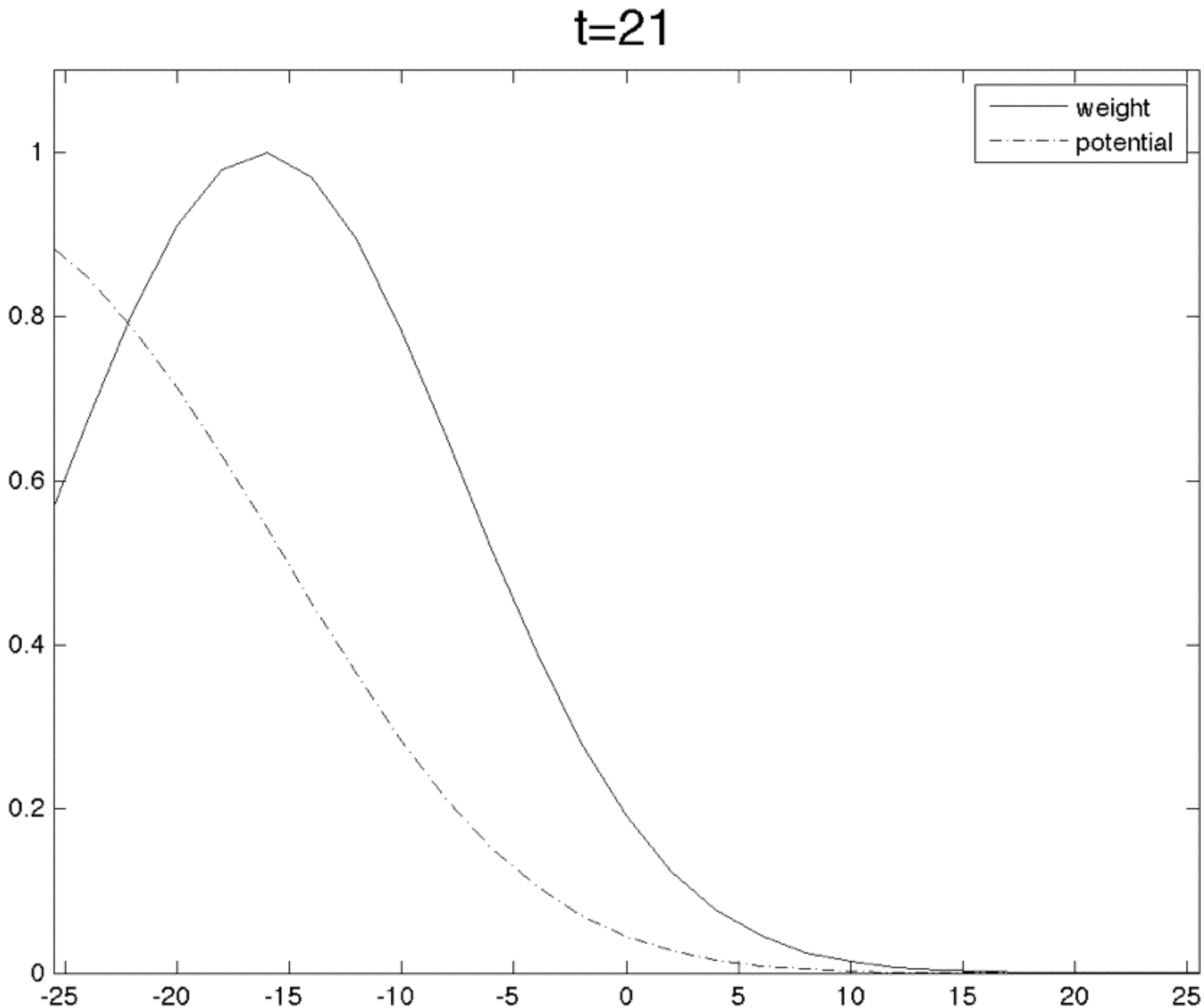
$$\varepsilon = \Psi(0) = \psi(0,0) = \text{Binom}\left(T, \frac{T}{2}, \frac{1}{2}+\gamma\right)$$

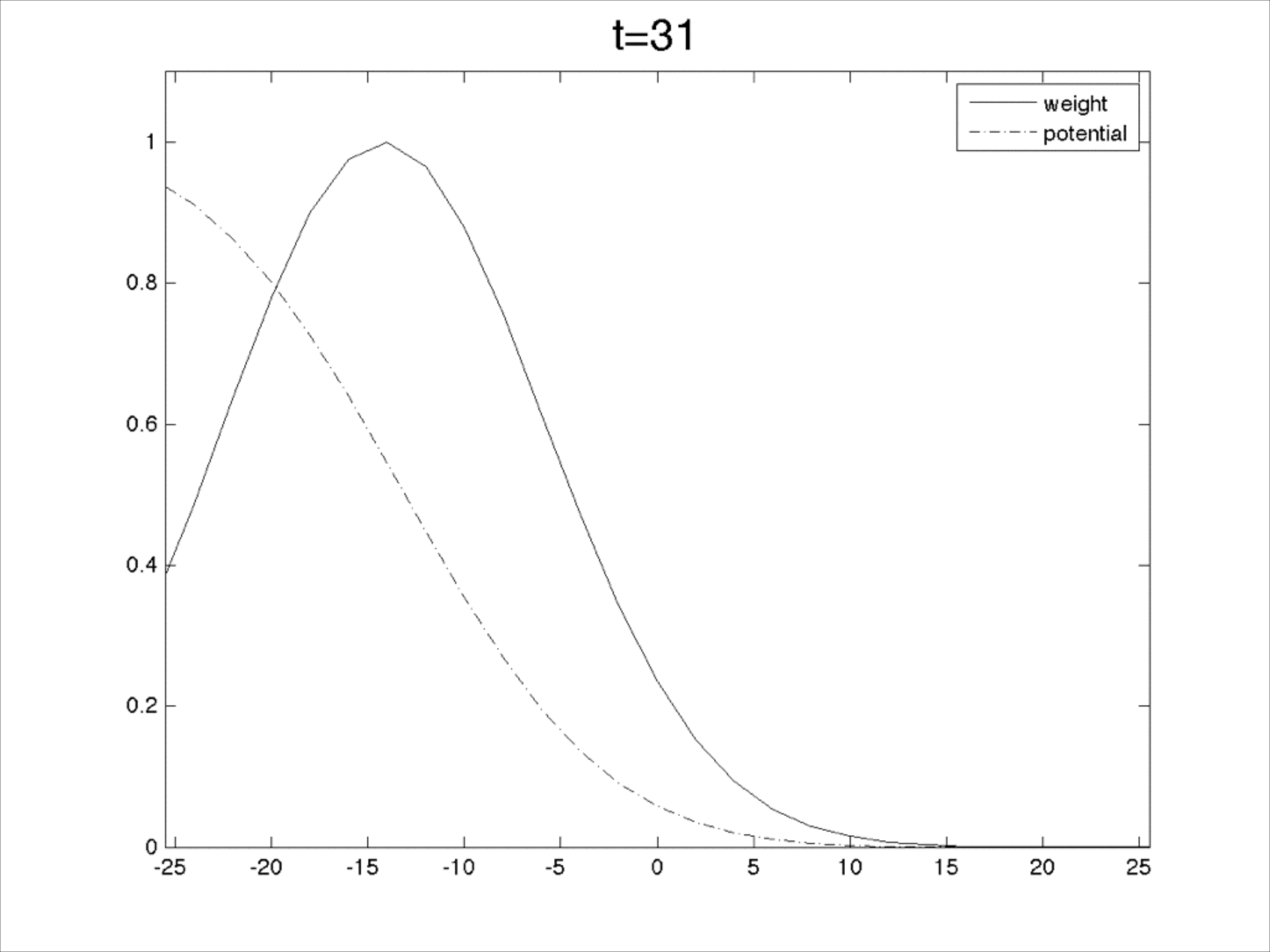


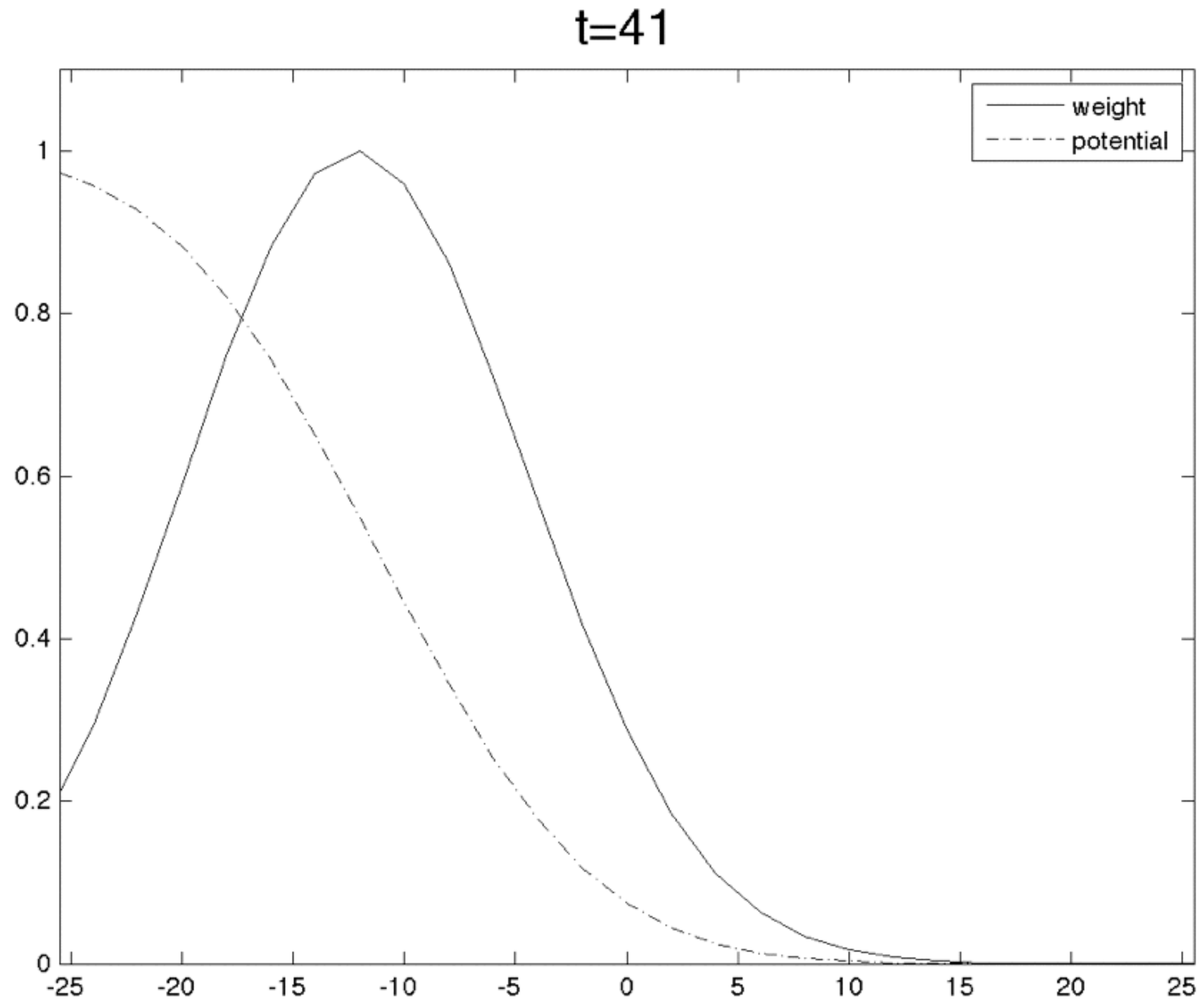
t=1
weight
potential
1
0.8
0.6
0.4
0.2
0
-25
-20
-15
-10
-5
0
5
10
15
20
25

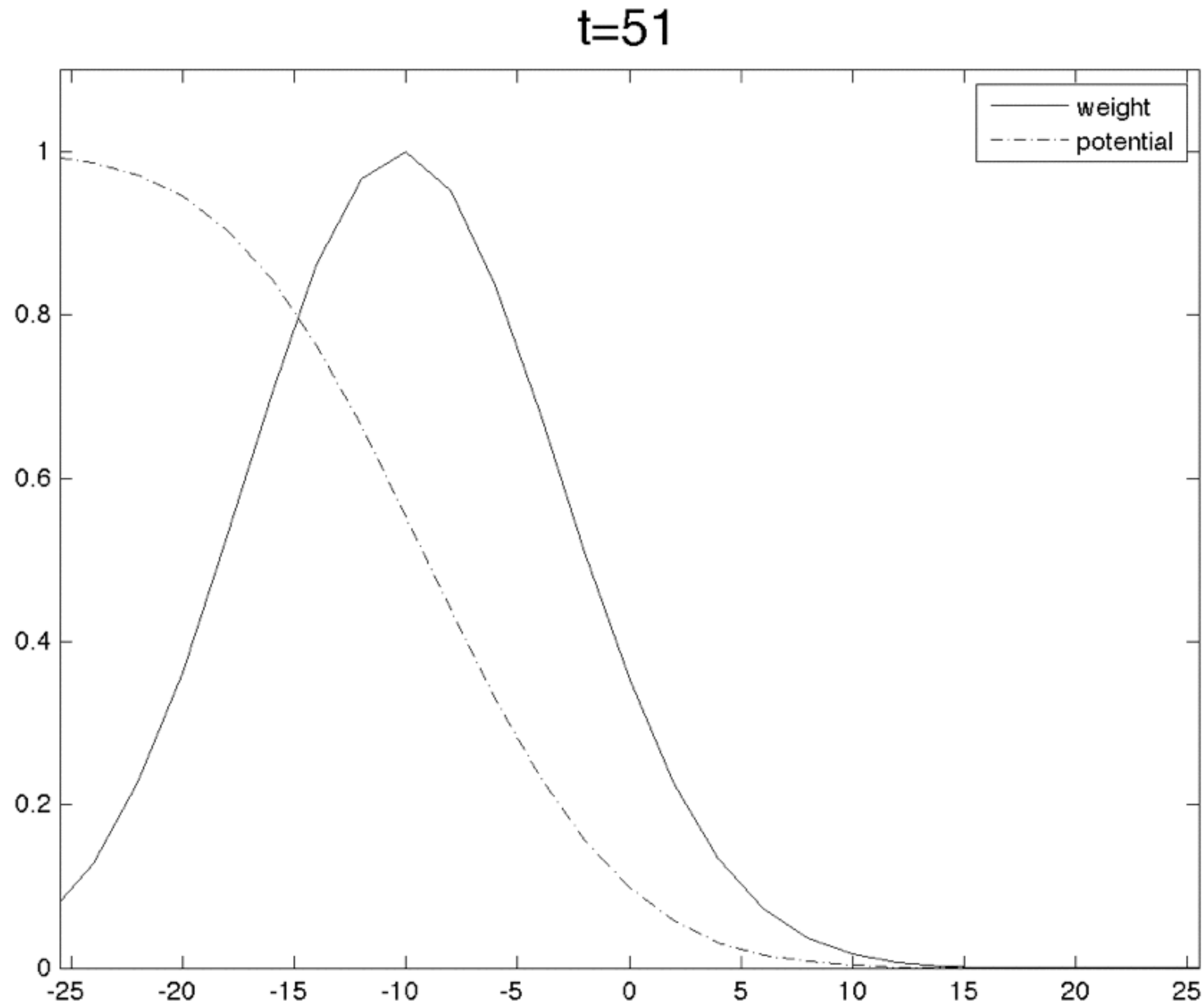


t=11
weight
potential
1
0.8
0.6
0.4
0.2
0
-25
-20
-15
-10
-5
0
5
10
15
20
25

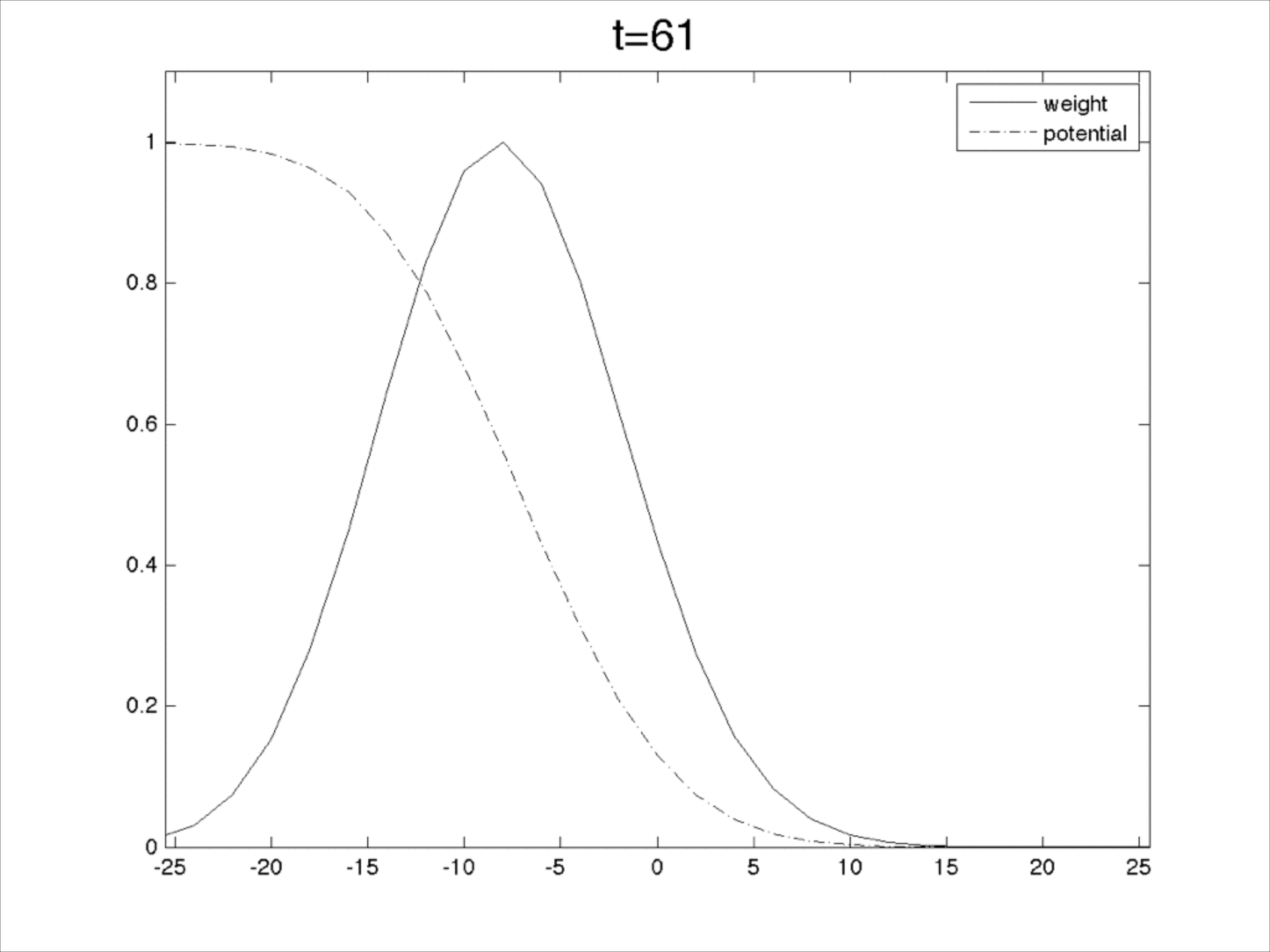

t=21
weight
potential
1
0.8
0.6
0.4
0.2
0
-25
-20
-15
-10
-5
0
5
10
15
20
25

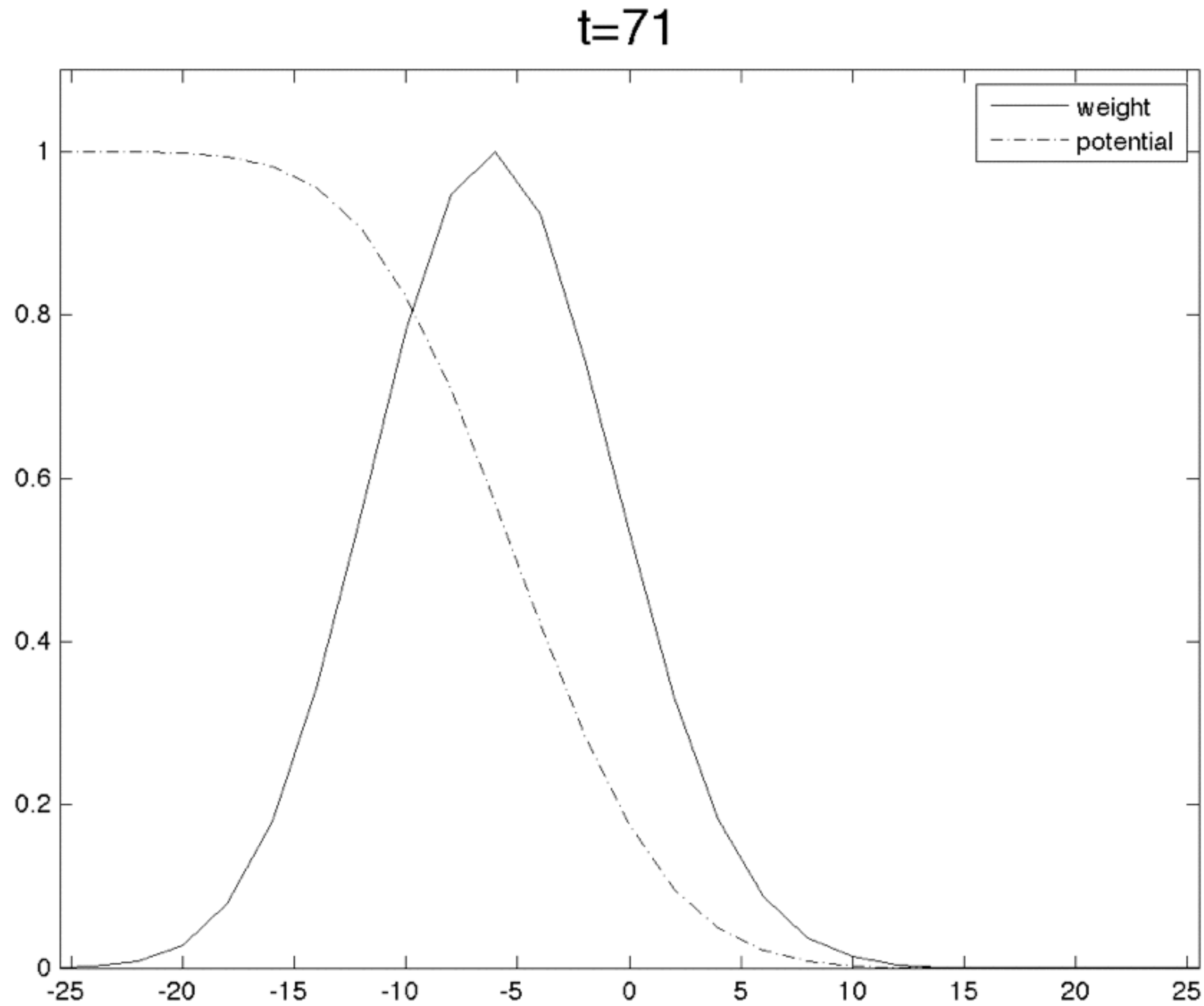

t=31
weight
potential
1
0.8
0.6
0.4
0.2
0
-25
-20
-15
-10
-5
0
5
10
15
20
25

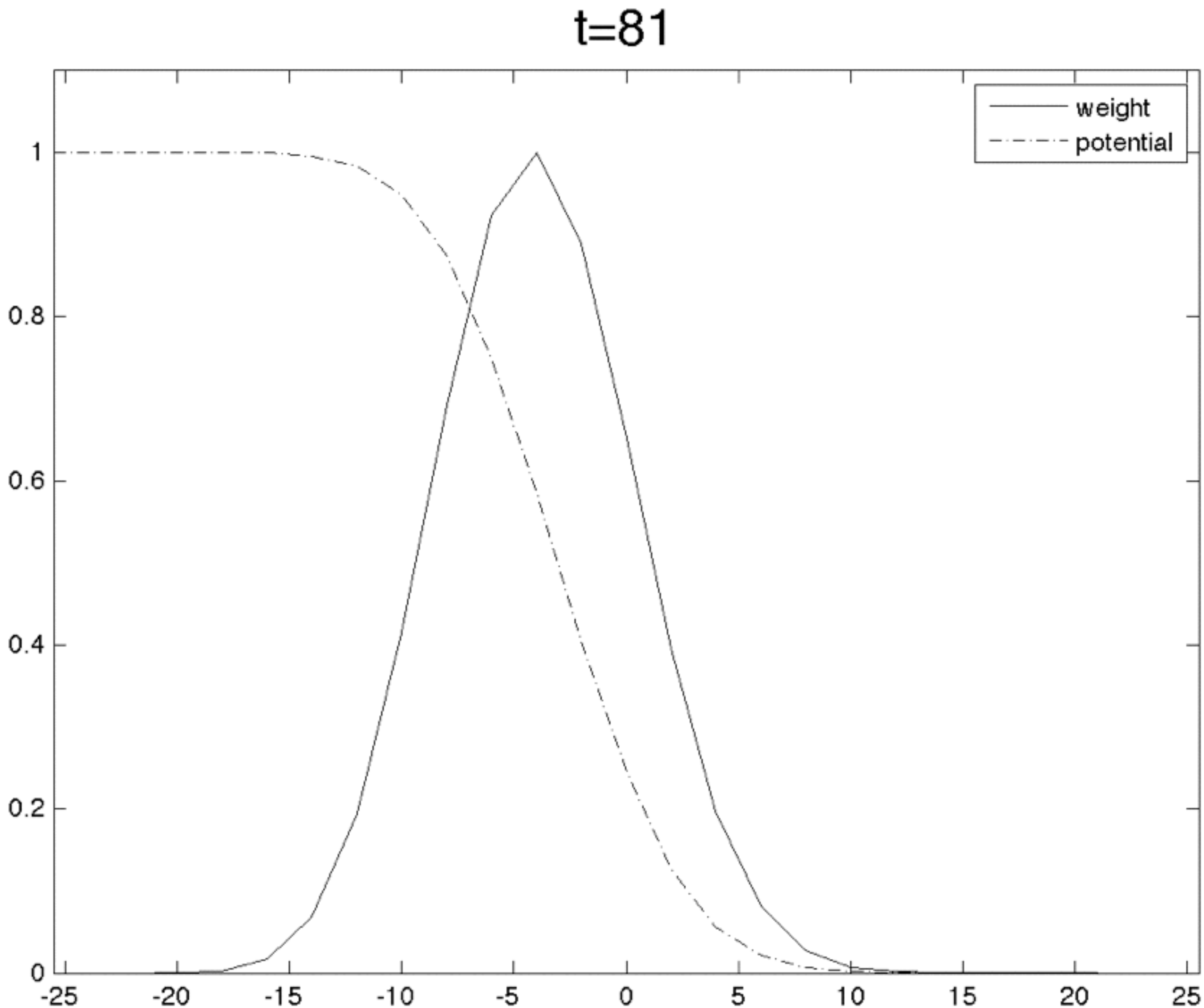

t=41
weight
potential
1
0.8
0.6
0.4
0.2
0
-25
-20
-15
-10
-5
0
5
10
15
20
25



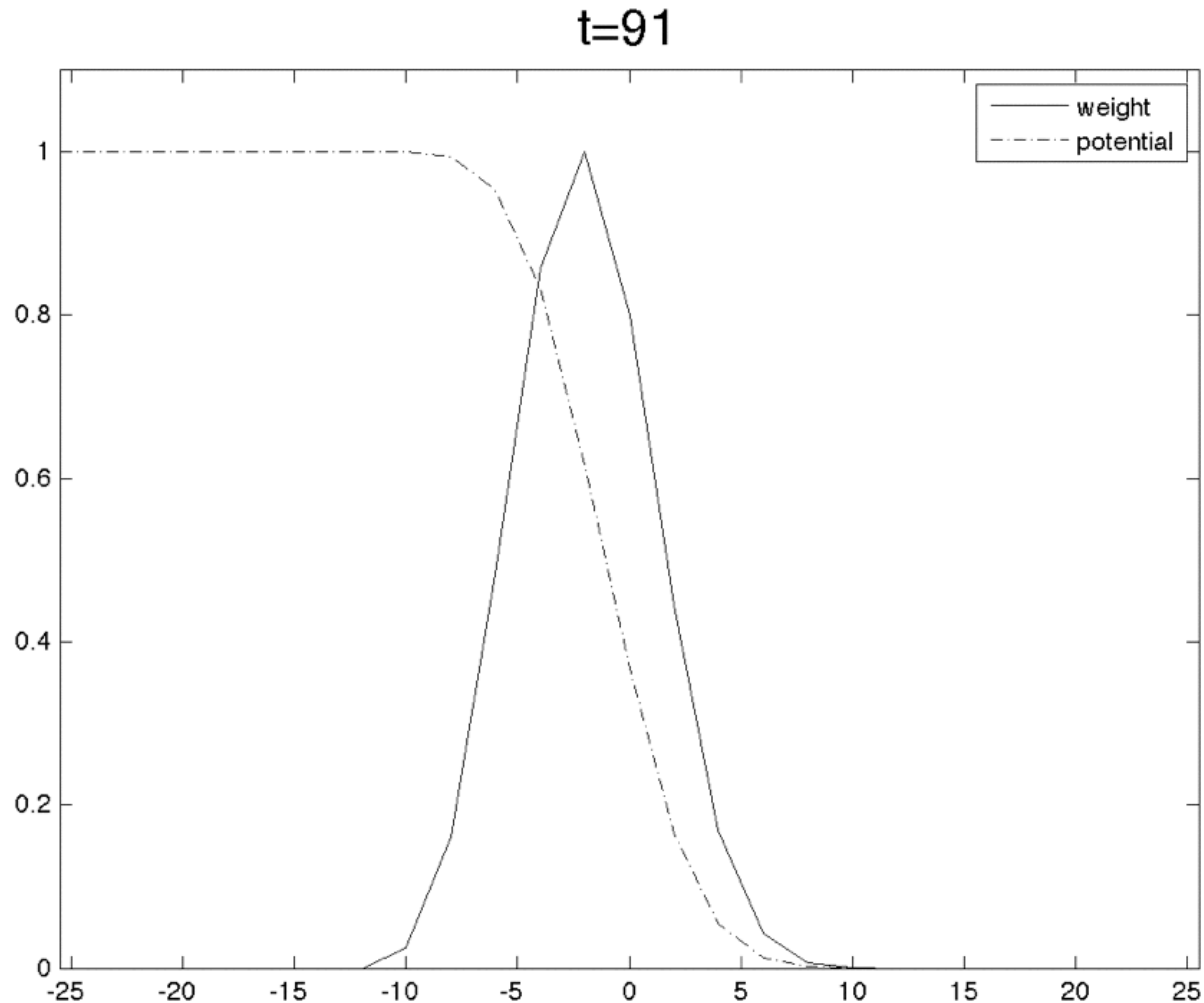
t=51
weight
potential
1
0.8
0.6
0.4
0.2
0
-25
-20
-15
-10
-5
0
5
10
15
20
25


t=61
weight
potential
0
0.2
0.4
0.6
0.8
1
-25
-20
-15
-10
-5
0
5
10
15
20
25

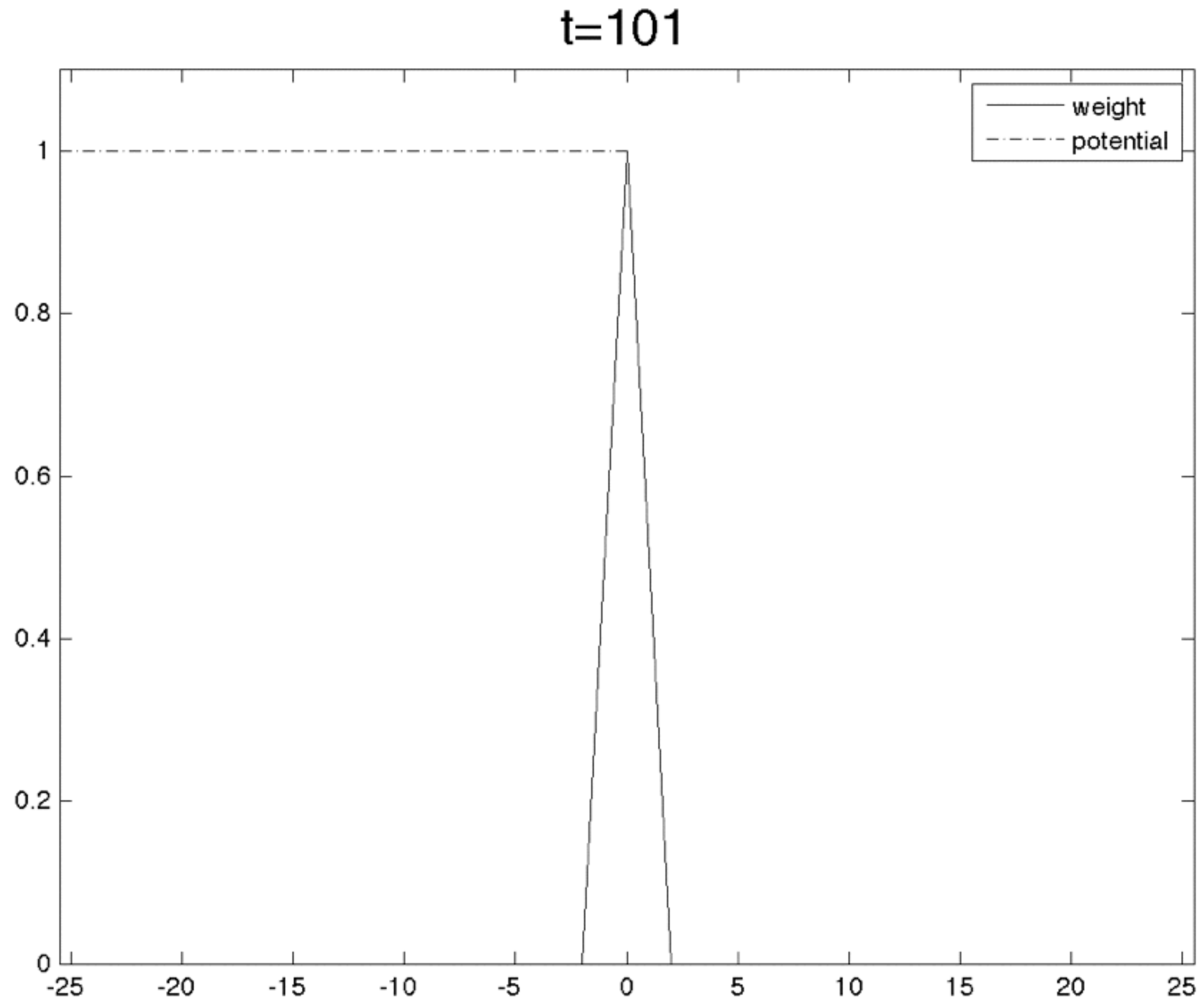

t=71
weight
potential
1
0.8
0.6
0.4
0.2
0
-25
-20
-15
-10
-5
0
5
10
15
20
25



t=81
weight
potential
1
0.8
0.6
0.4
0.2
0
-25
-20
-15
-10
-5
0
5
10
15
20
25

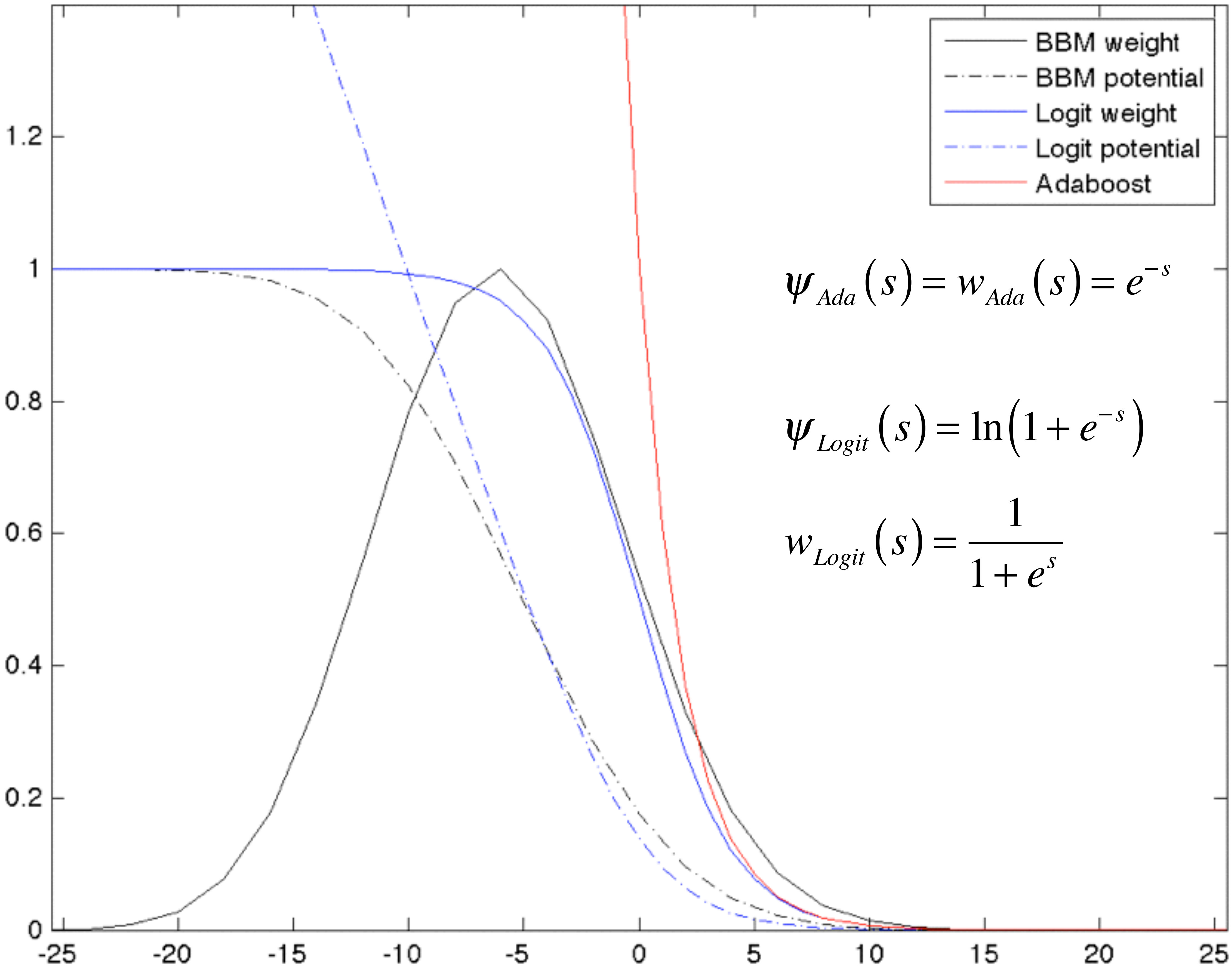


t=91
weight
potential
1
0.8
0.6
0.4
0.2
0
-25
-20
-15
-10
-5
0
5
10
15
20
25



t=101
weight
potential
1
0.8
0.6
0.4
0.2
0
-25
-20
-15
-10
-5
0
5
10
15
20
25

BBM/Logitboost/Adaboost



$$\psi_{Ada}(s) = w_{Ada}(s) = e^{-s}$$

$$\psi_{Logit}(s) = \ln\left(1 + e^{-s}\right)$$

$$w_{Logit}(s) = \frac{1}{1 + e^{s}}$$

High level summary

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- The worst case adversary splits each bin into:
  $1/2-\gamma$ incorrect / $1/2+\gamma$ correct

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- Alternative interpretation: Random walk with IID steps.

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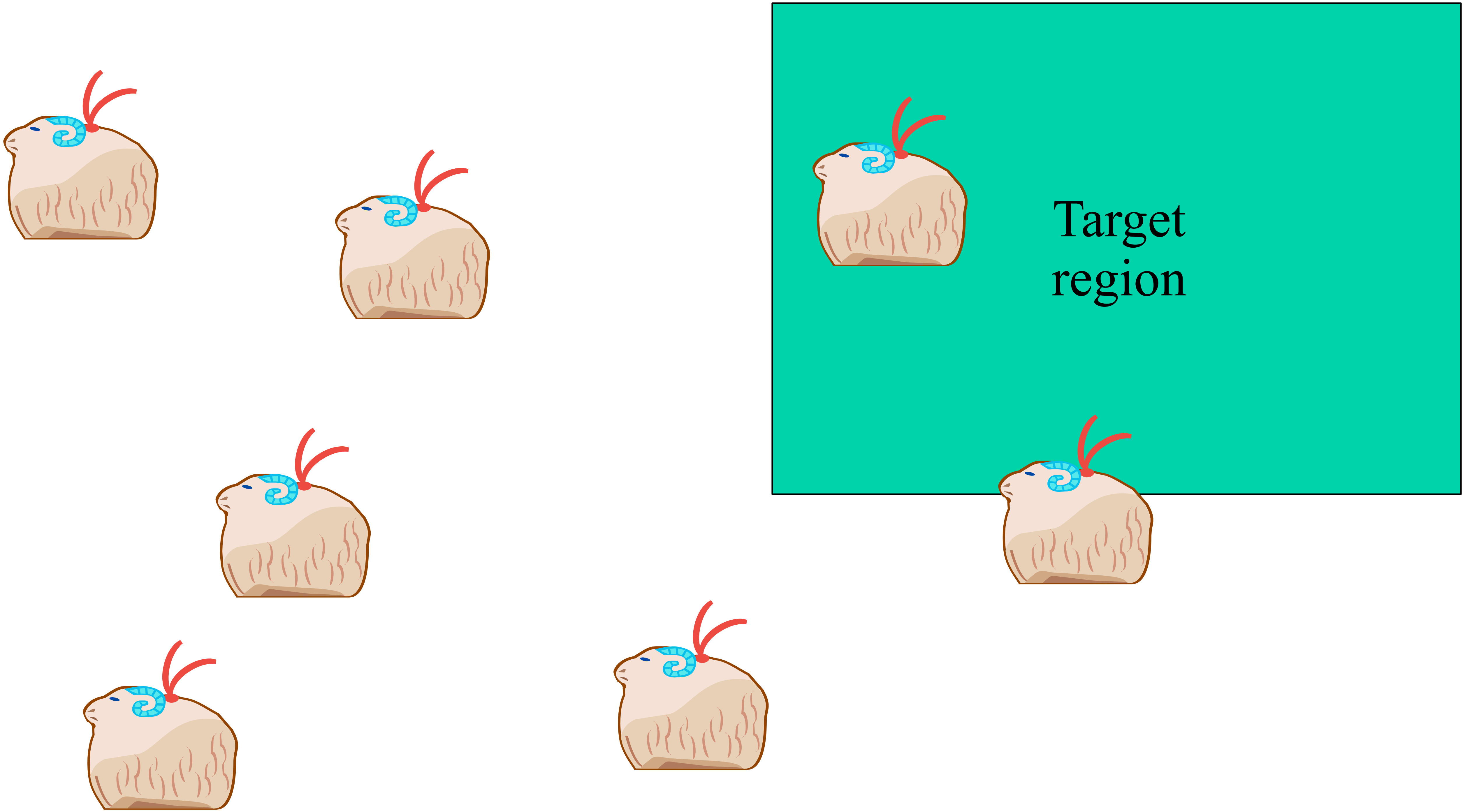
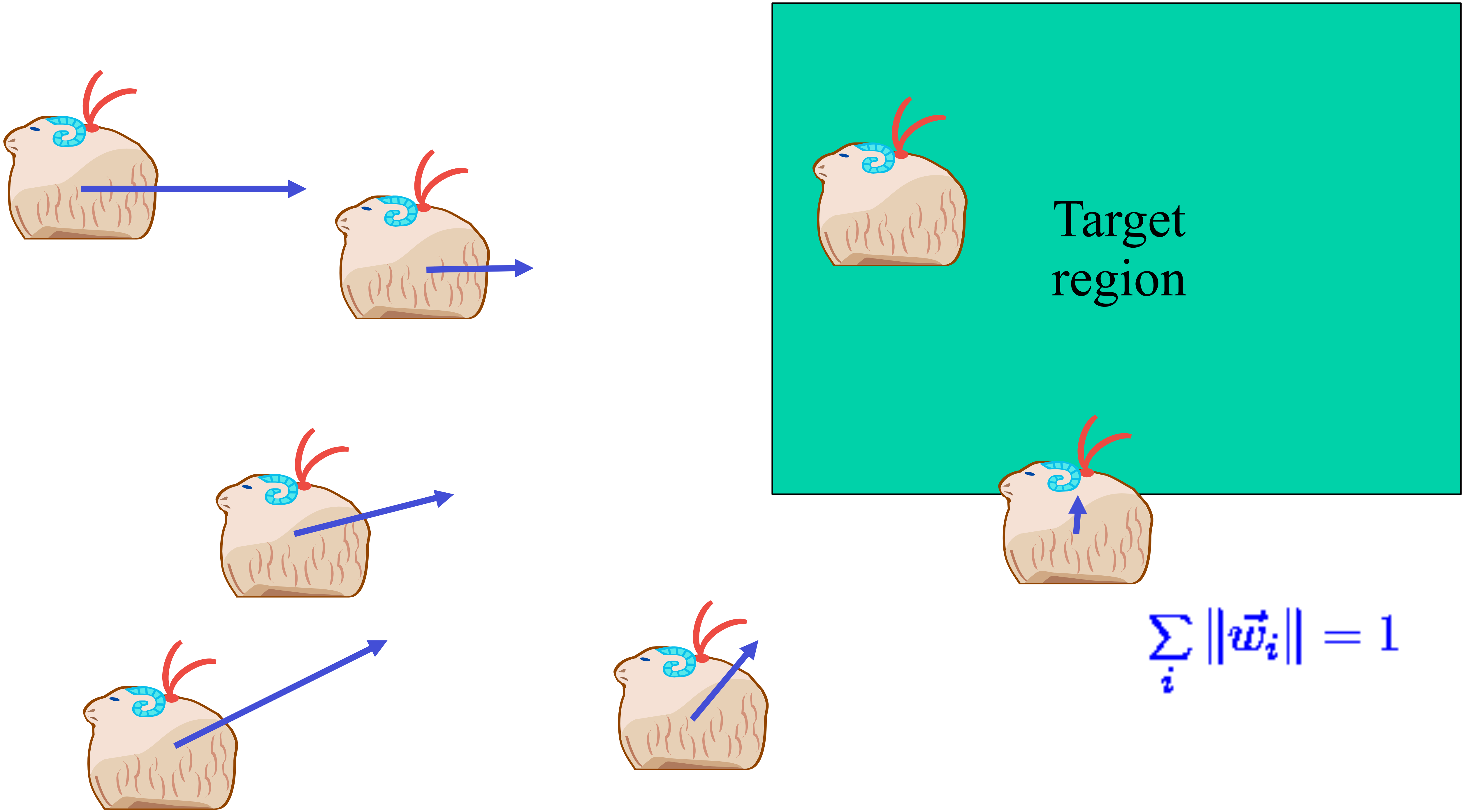
  $1/2-\gamma$ incorrect / $1/2+\gamma$ correct
- Alternative interpretation: Random walk with IID steps.
- Algorithm is derived as optimal response to this simple worst-case adversary.

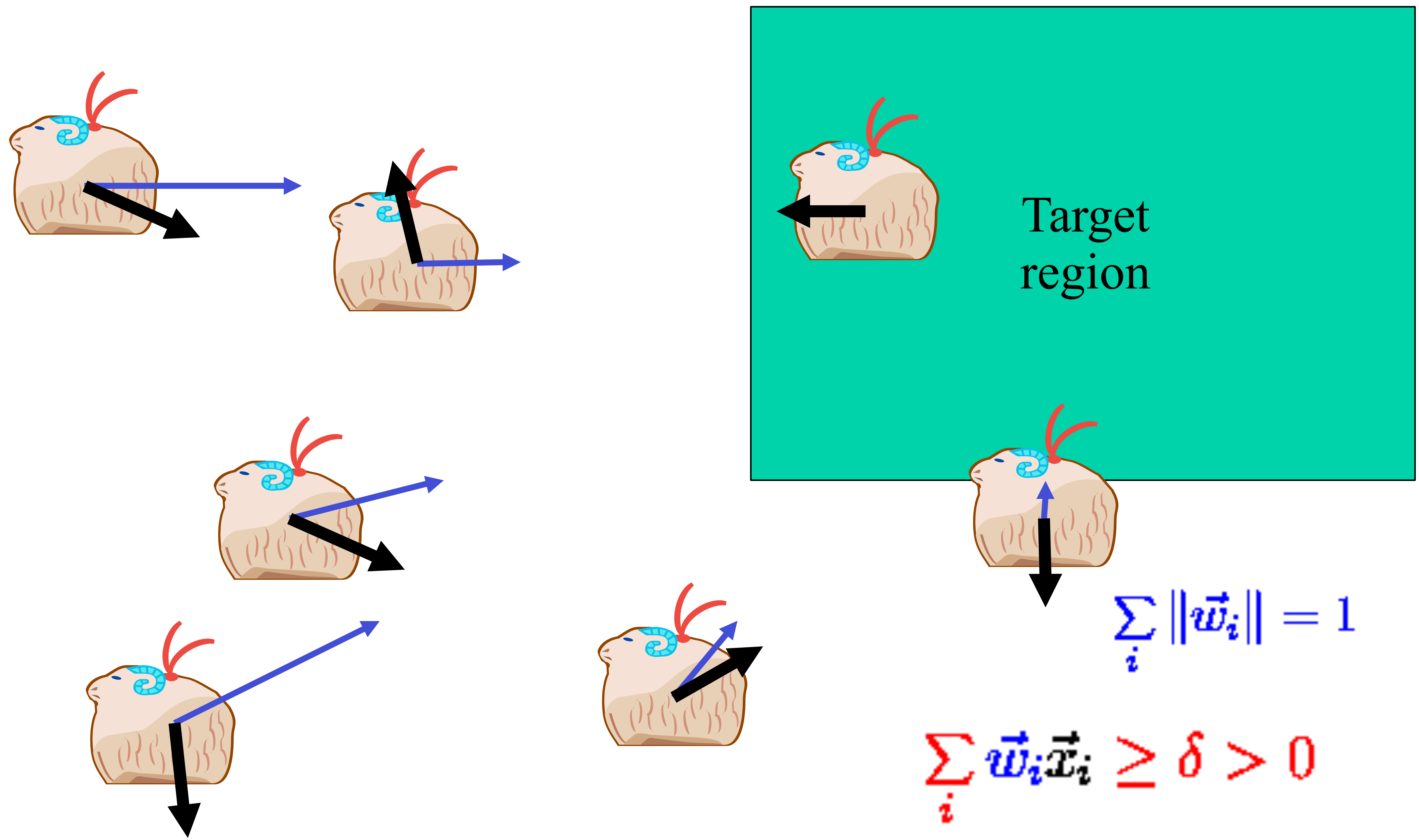
Drifting games (in 2d)



Drifting games (in 2d)



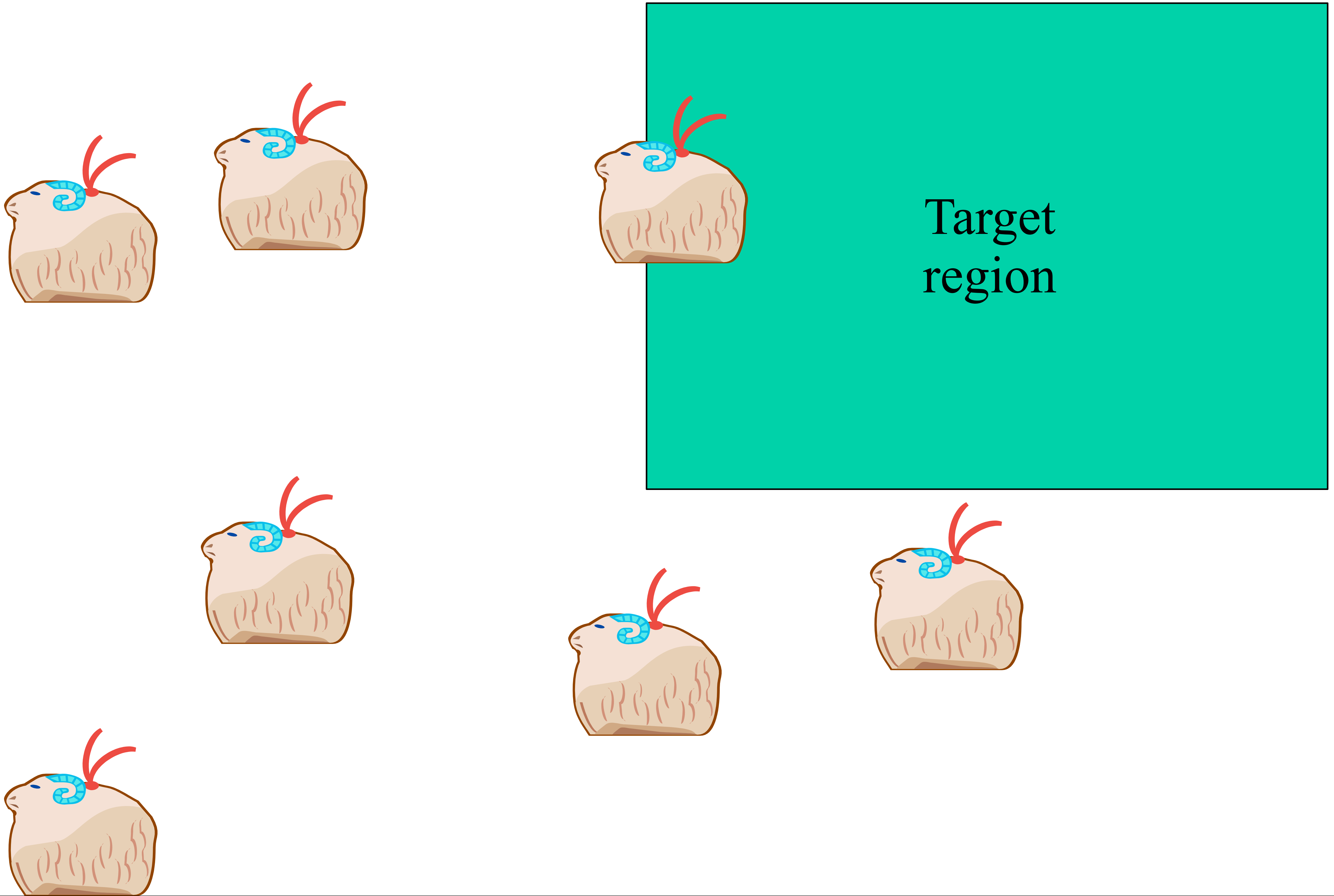
Drifting games (in 2d)



$$\sum_i \|\vec{w}_i\| = 1$$

$$\sum_i \vec{w}_i \vec{x}_i \geq \delta > 0$$

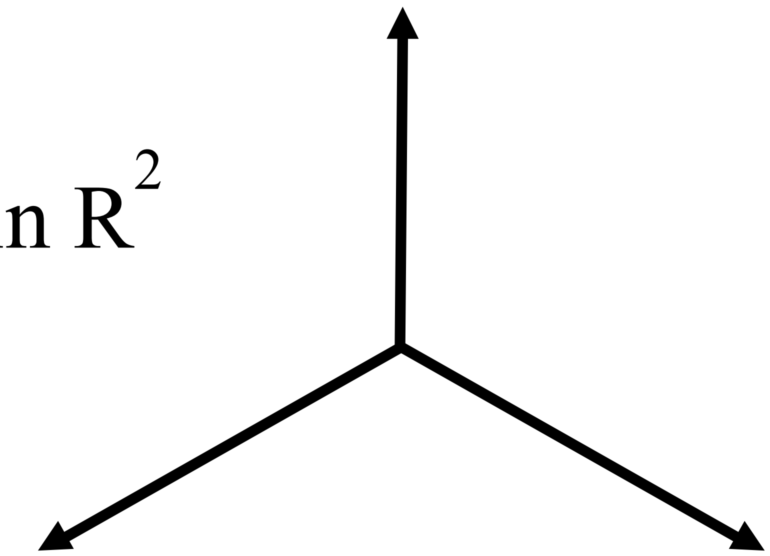
Drifting games (in 2d)



The allowable steps

B = the set of all allowable step
Normal B = minimal set that spans the space. (~basis)
Regular B = a symmetric regular set. (~orthonormal basis)

Regular step set in R^2



The min/max solution

[Schapire99]

- A potential defined by a min/max recursion

$$\phi_T(\mathbf{s}) = L(\mathbf{s})$$
$$\phi_{t-1}(\mathbf{s}) = \min_{\mathbf{w}\in\mathbb{R}^d} \sup_{\mathbf{z}\in B} (\phi_t(\mathbf{s}+\mathbf{z}) + \mathbf{w}\cdot\mathbf{z} - \delta\,\|\mathbf{w}\|)$$

$\phi_0(0) =$ the value of the game

- Shepherd's strategy

$$\mathbf{w}_i^t = \arg\min_{\mathbf{w}} \sup_{\mathbf{z}\in B} (\phi_t(\mathbf{s}+\mathbf{z}) + \mathbf{w}\cdot\mathbf{z} - \delta\,\|\mathbf{w}\|)$$

The solution simplifies when $\delta \to 0$

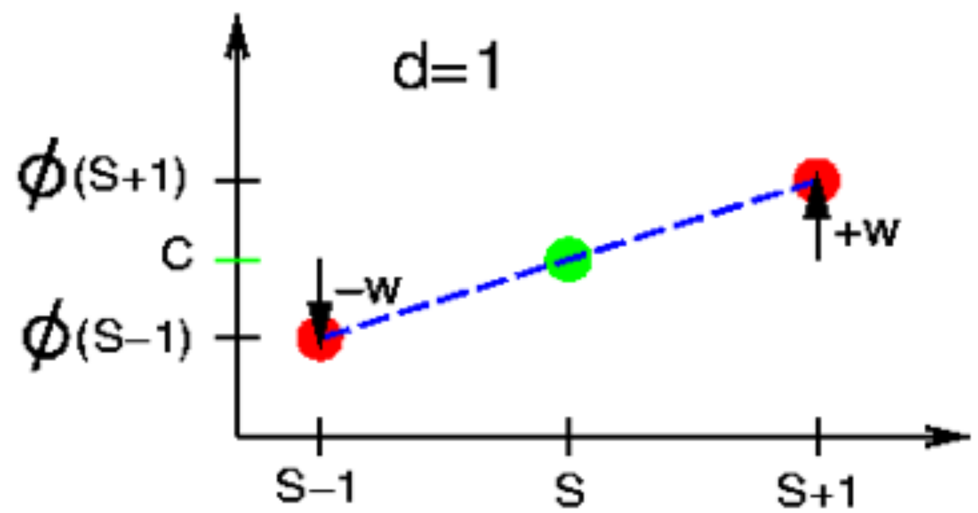
If B is normal, and δ is sufficiently small then $\exists \mathbf{w}^*$ such that

$$\phi_{t-1}(\mathbf{s}) = \phi_t(\mathbf{s} + \mathbf{z}) + \mathbf{w}^* \cdot \mathbf{z} - \delta \|\mathbf{w}^*\|$$

for all $\mathbf{z} \in B$ (and all $t = 1, 2 \ldots, \mathbf{s} \in \mathbb{R}^d$)

Implies that: $\mathbf{w}^*$ is the "local slope" at $\phi_t(\mathbf{s})$, i.e.

$$\phi_t(\mathbf{s} + \mathbf{z}_i) = C + \mathbf{w}^* \mathbf{z}_i \; ; \quad C \doteq \frac{\Sigma_{j=0}^d \phi_t(\mathbf{s} + \mathbf{z}_j)}{d+1}$$



and that

$$\phi_{t-1}(\mathbf{s}) = C - \delta \|\mathbf{w}^*\|$$

Increasing the number of steps

- Consider T steps in a unit time
- Drift δ should scale like $1/T$
- Step size $O(1/T)$ gives game to shepherd
- Step size $O\left(1/\sqrt{T}\right)$ keeps game balanced

The solution when $T \to \infty$

The local slope becomes the gradient

$$\mathbf{w}^* = \nabla \phi_\tau(\mathbf{s})$$

The recursion becomes a PDE

$$\begin{aligned}\frac{\partial \phi_\tau(\mathbf{s})}{\partial \tau} &= -\frac{1}{2}\sum_{k=1}^{d} \frac{\partial^2 \phi_\tau(\mathbf{s})}{\partial^2 s_k} + \delta \|\mathbf{w}^*\| \\ &= -\frac{1}{2}\Delta \phi_\tau(\mathbf{s}) + \delta \|\nabla \phi_\tau(\mathbf{s})\|\end{aligned}$$

Same PDE describes time development of
Brownian motion with drift

Plan of talk

- Label noise and convex loss functions.
- Boost by Majority and drifting games.
- Boosting in continuous time.
- RobustBoost
- Experimental results.

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  - generates a weighted majority rule.
  - Decide when to stop using cross-validation.
- How can we make BBM adaptive?

Letting time step decrease to zero.

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- Number of iterations required by BBM: $T = \frac{1}{\gamma^2} \ln \frac{1}{\varepsilon}$

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- The same weak rule is added many times, until it's advantage falls below γ .
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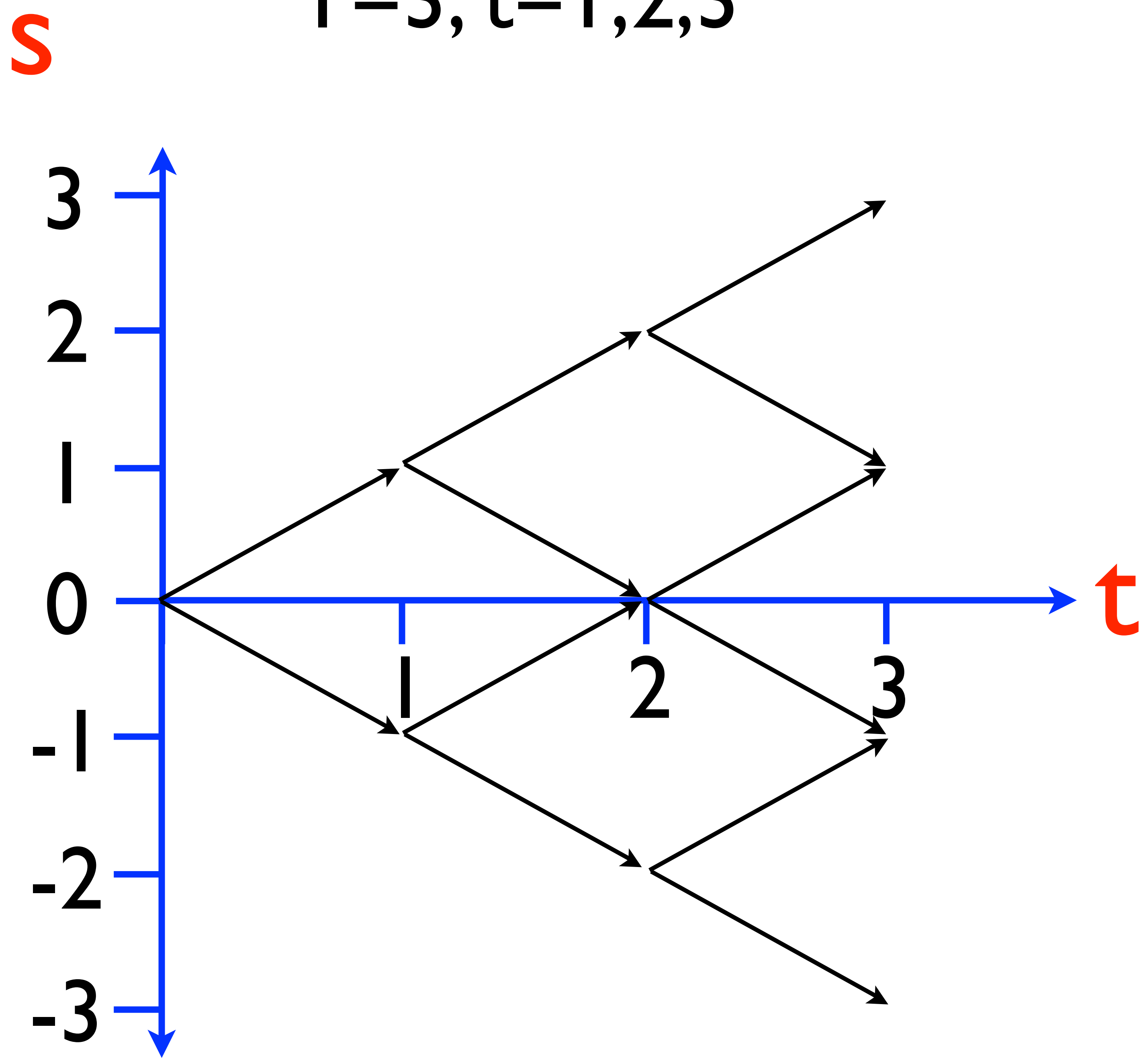
- In the limit, adversary uses random walk in continuous time = Brownian Motion.
- Instead of t=1,2,...,T use t=1/T,2/T,...,1

The game lattice

T=3, t=1,2,3



Using step $\Delta s = \pm \frac{1}{T}$

$$T = 1, \Delta t = 1, \Delta s = \pm 1$$

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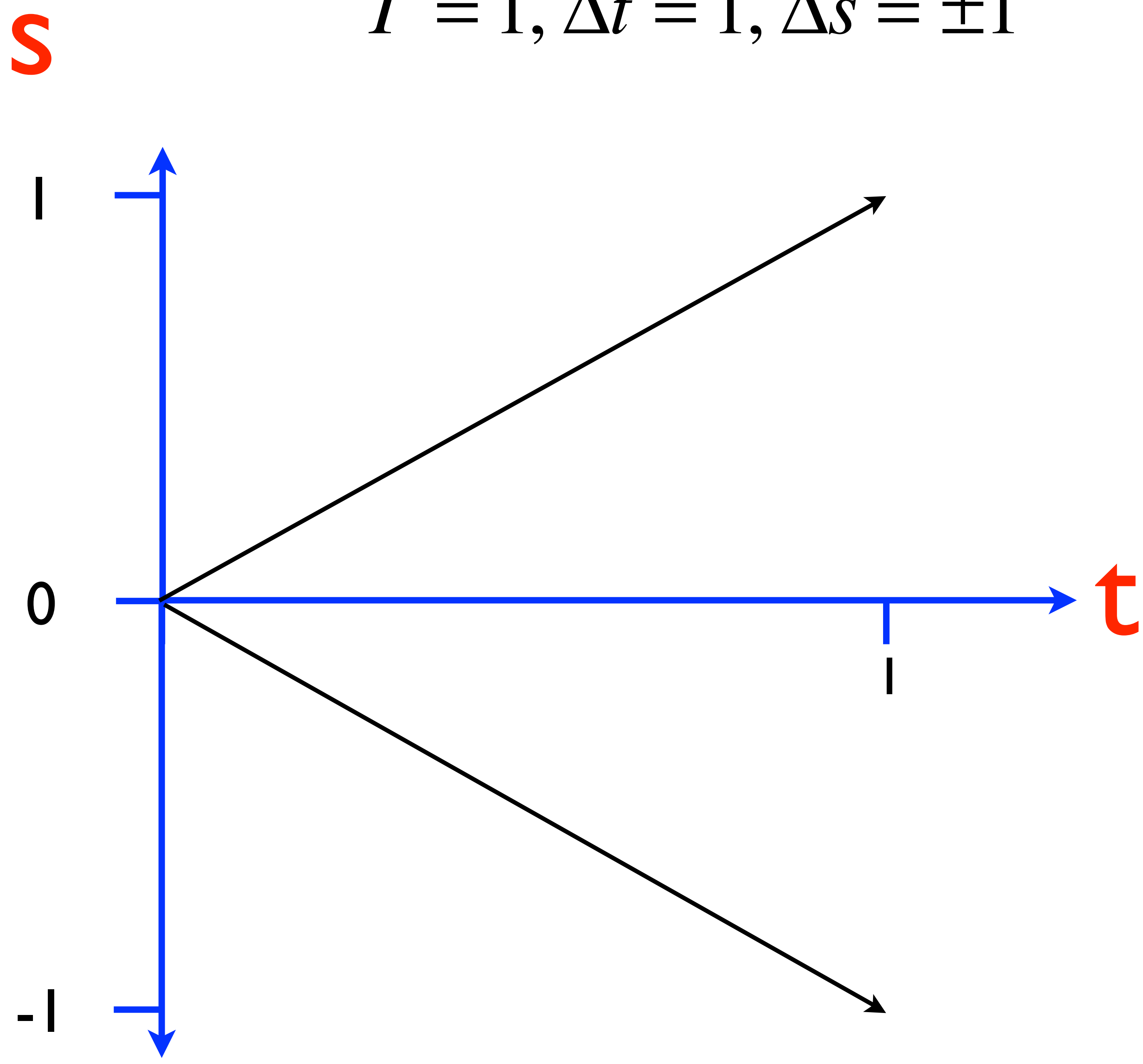

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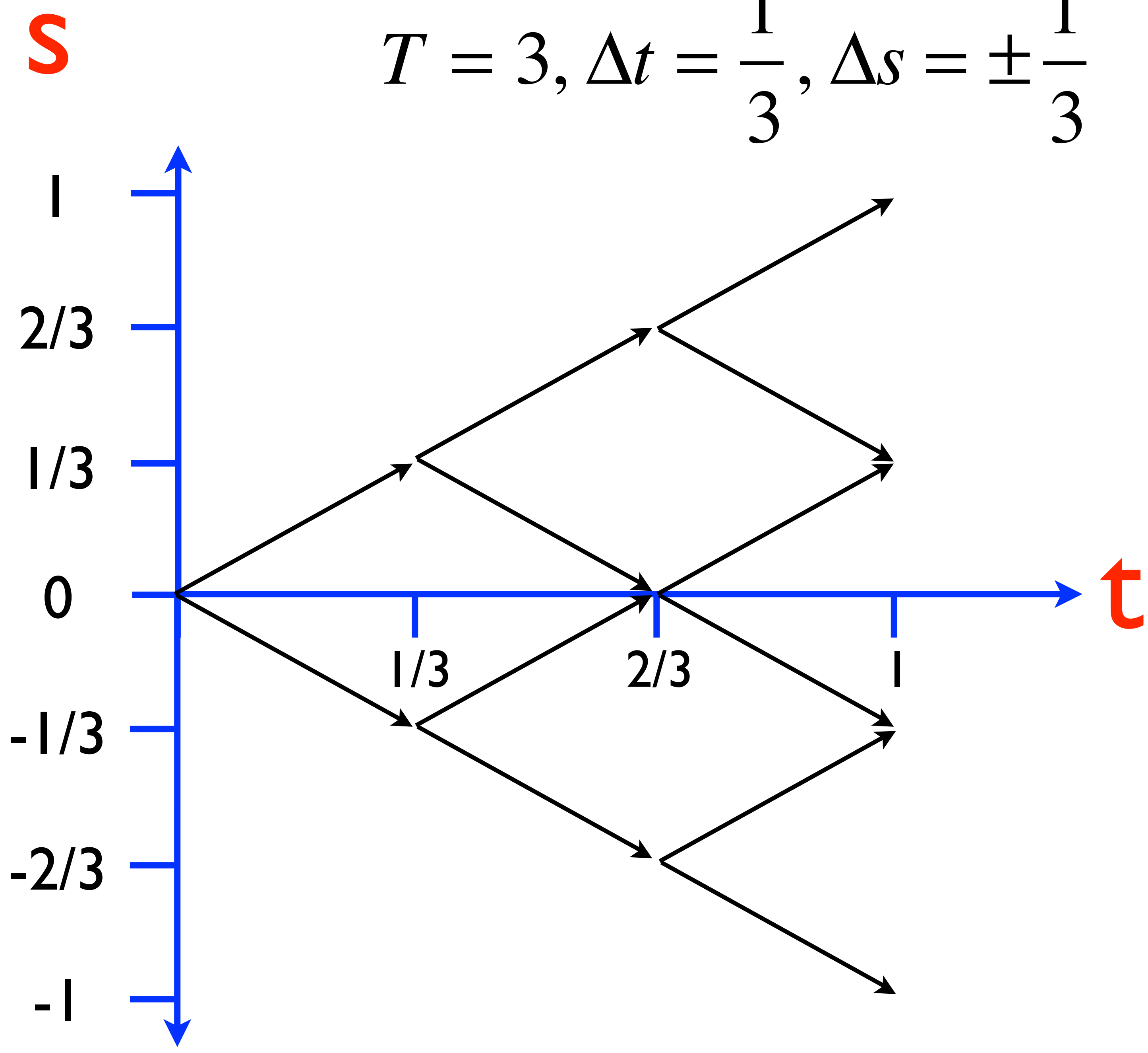

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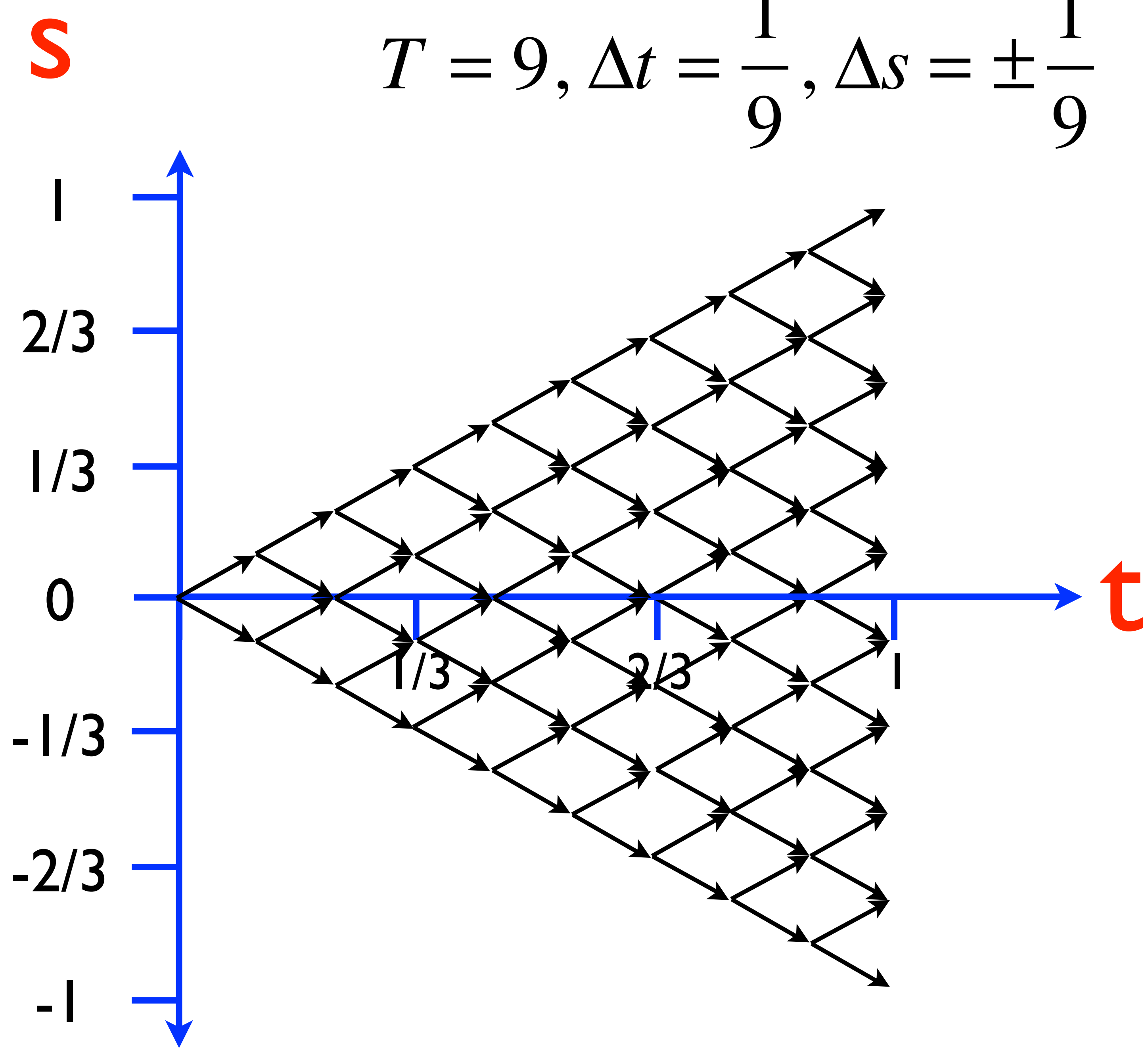

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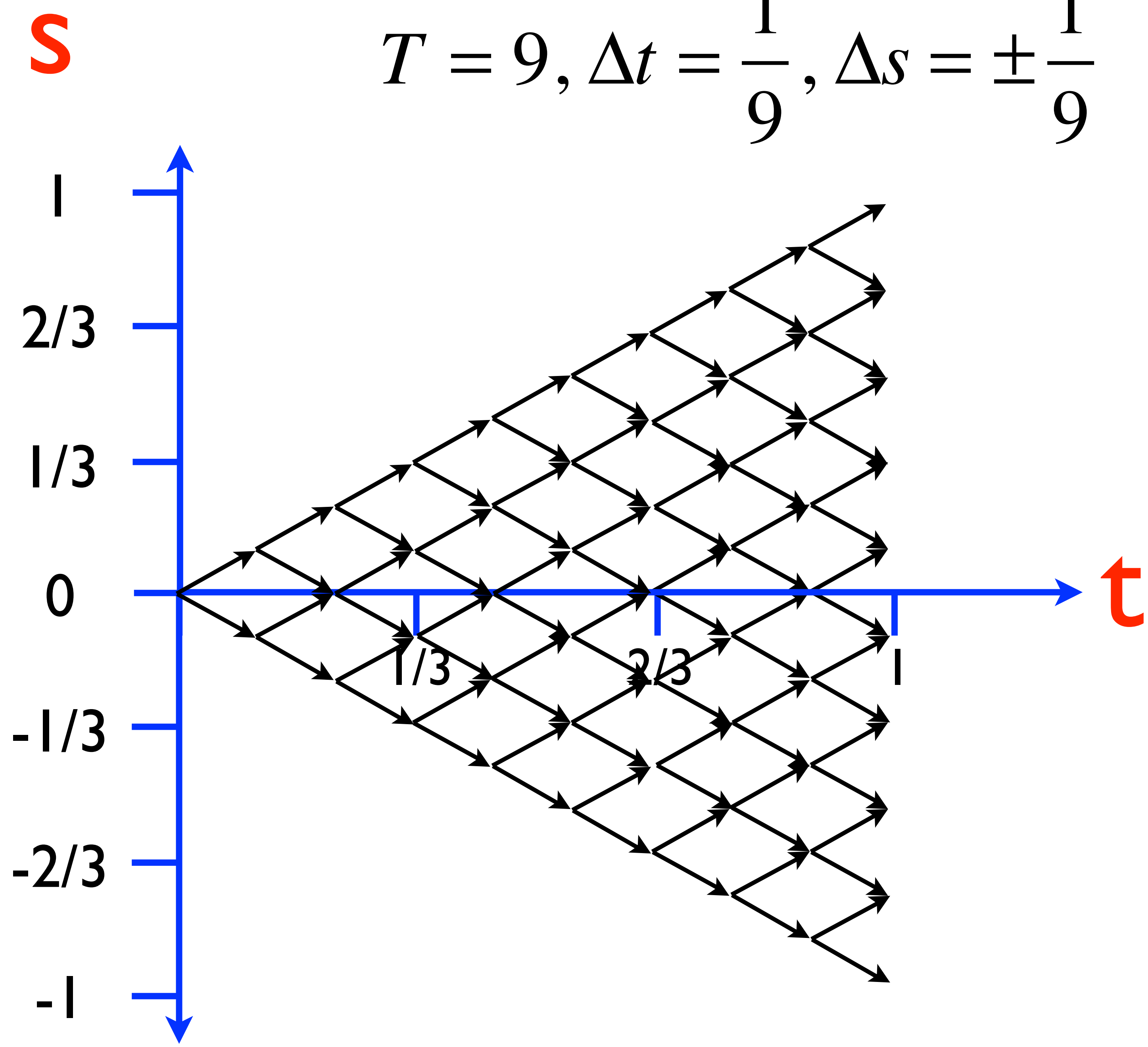

Looks fine but $\mathrm{var}(s) = T\frac{1}{T^2} = \frac{1}{T} \underset{T\to\infty}{\to} 0$

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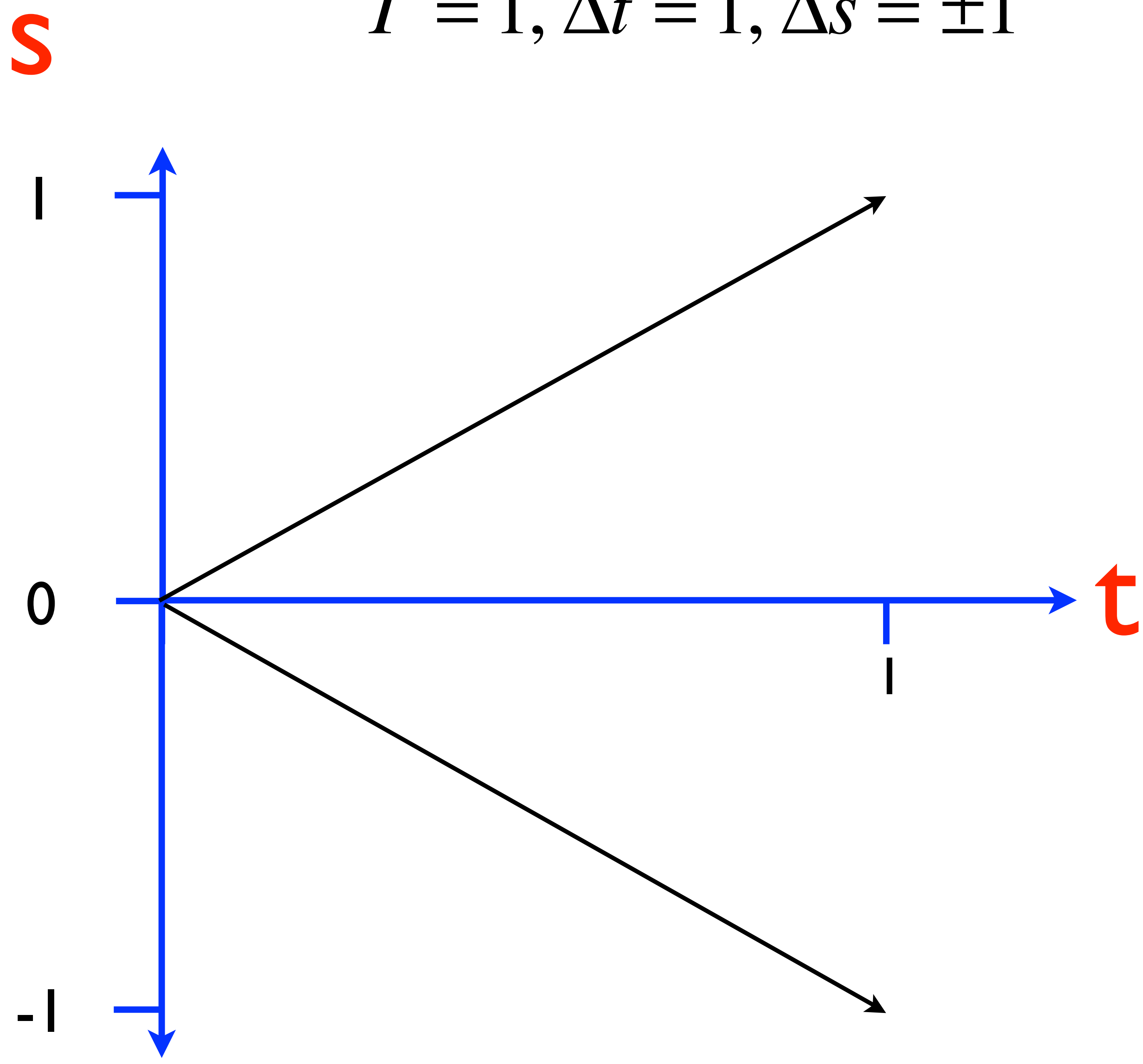

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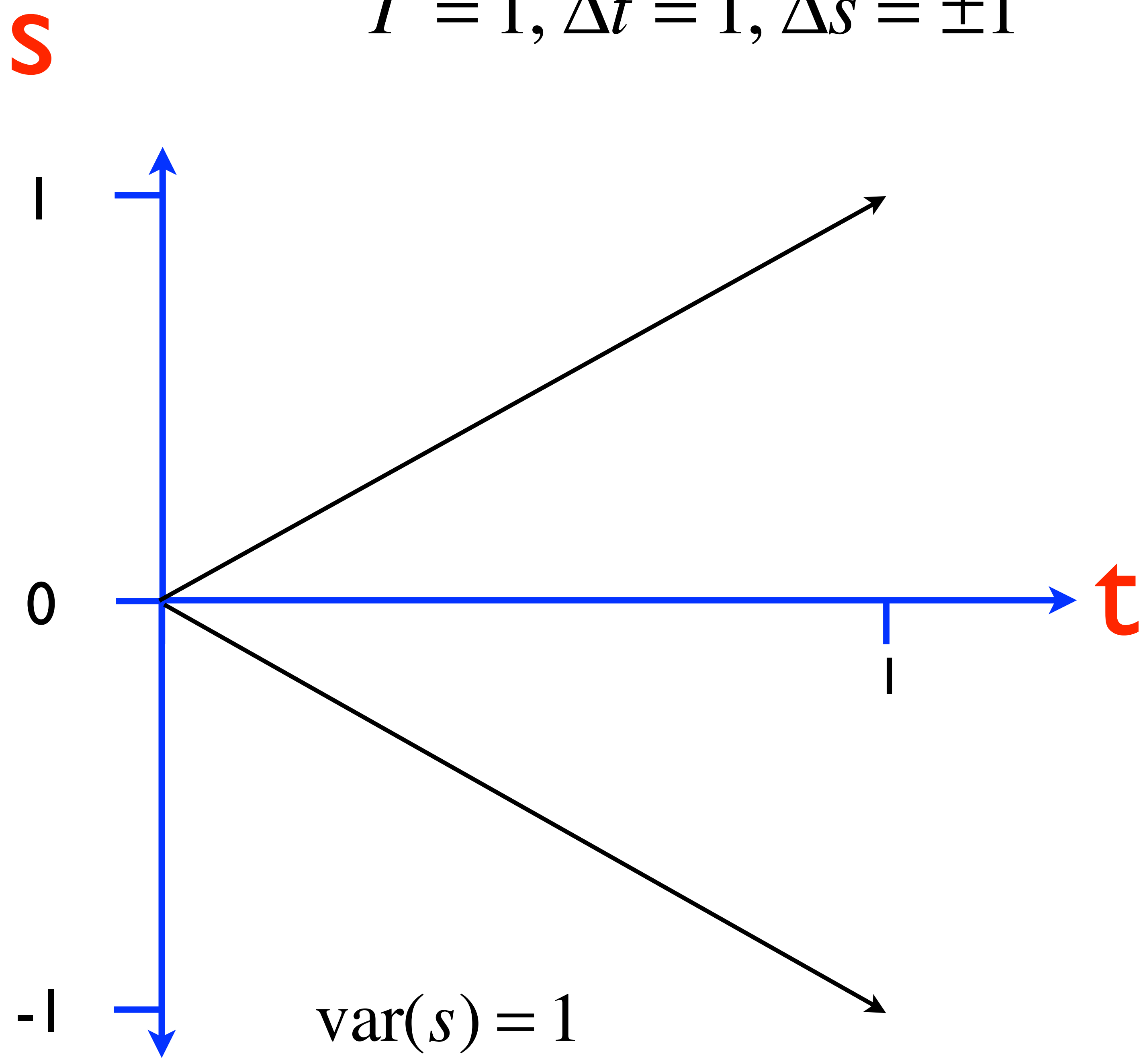

$$\mathrm{var}(s) = 1$$

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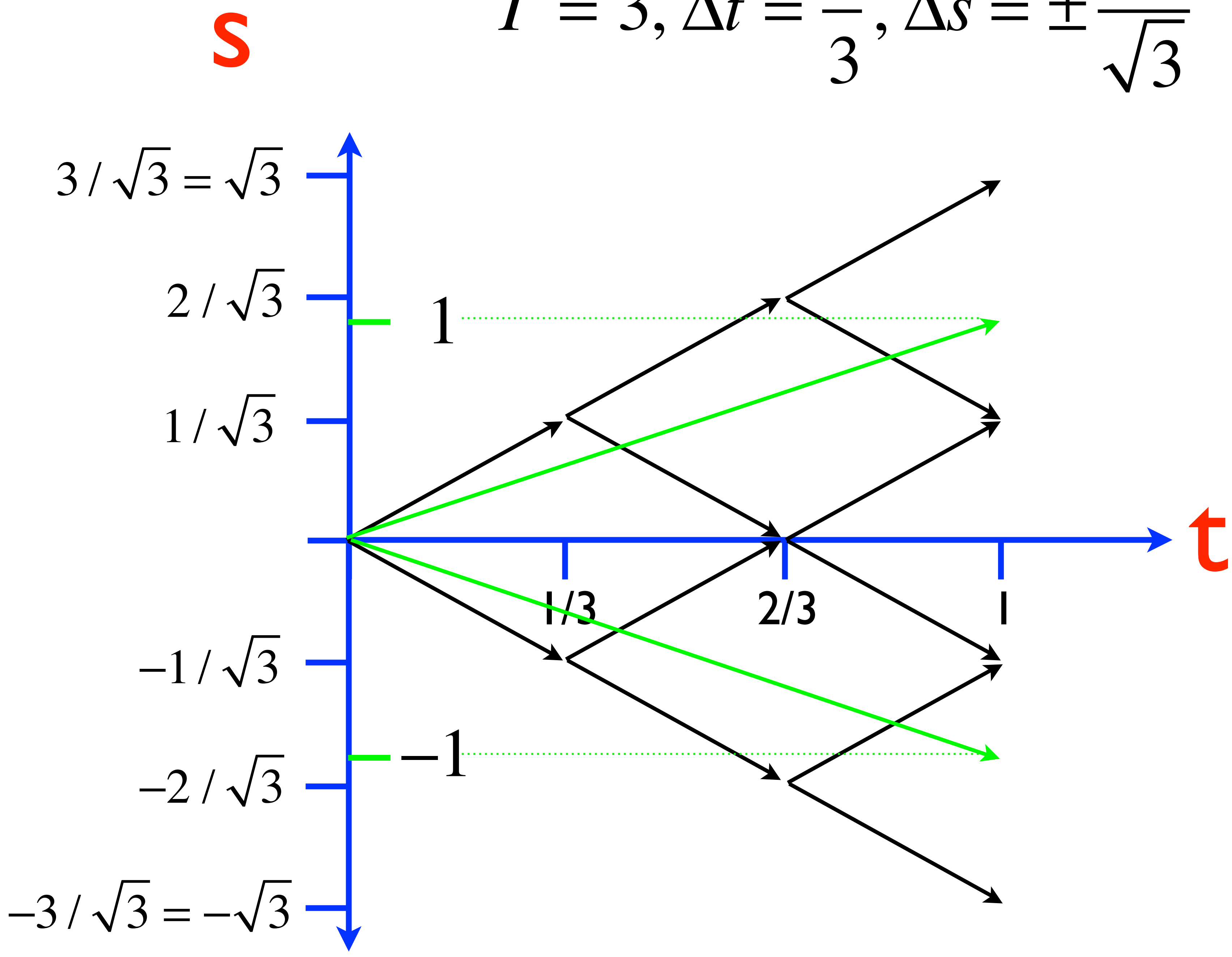

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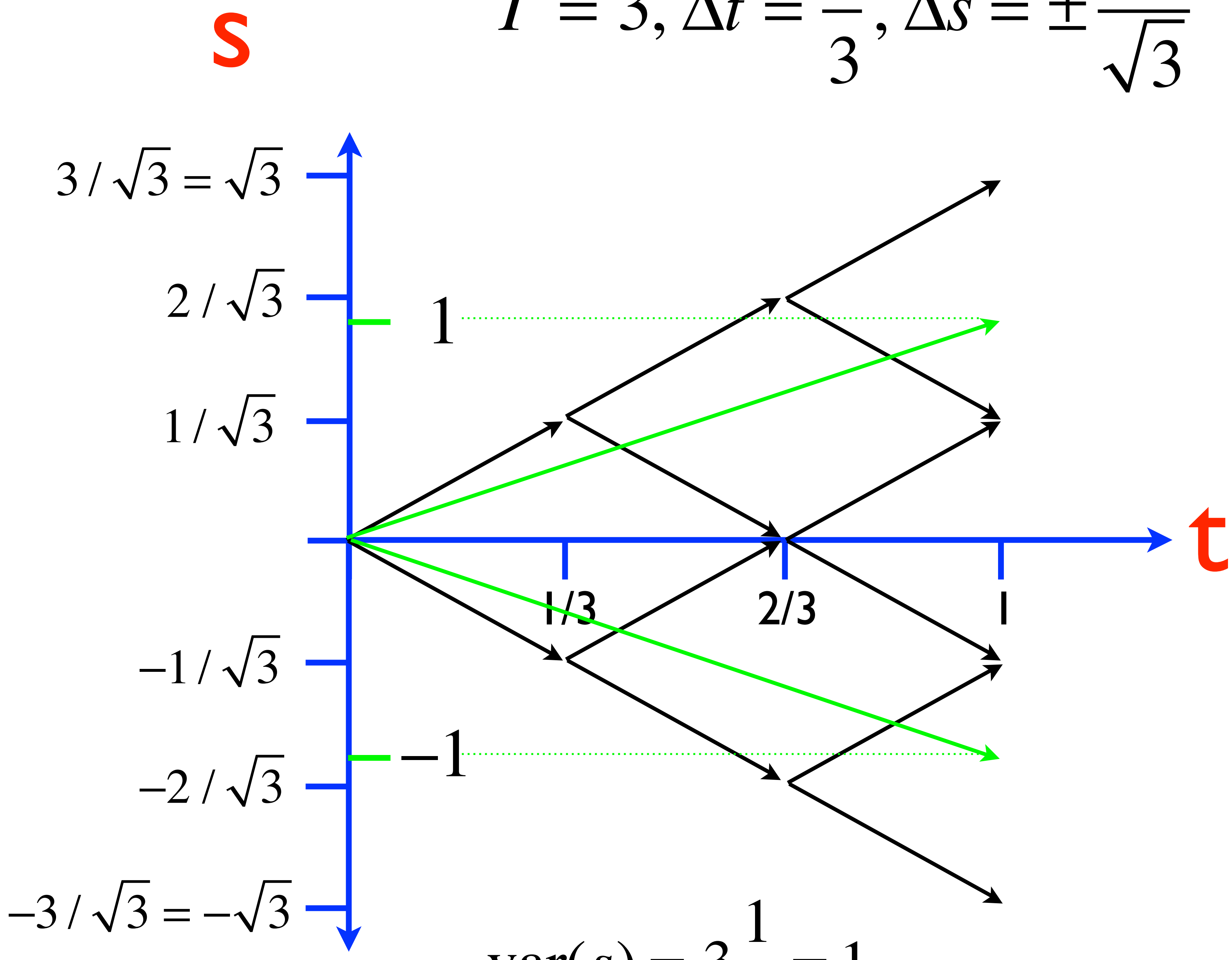

$$\mathrm{var}(s) = 3\frac{1}{3} = 1$$

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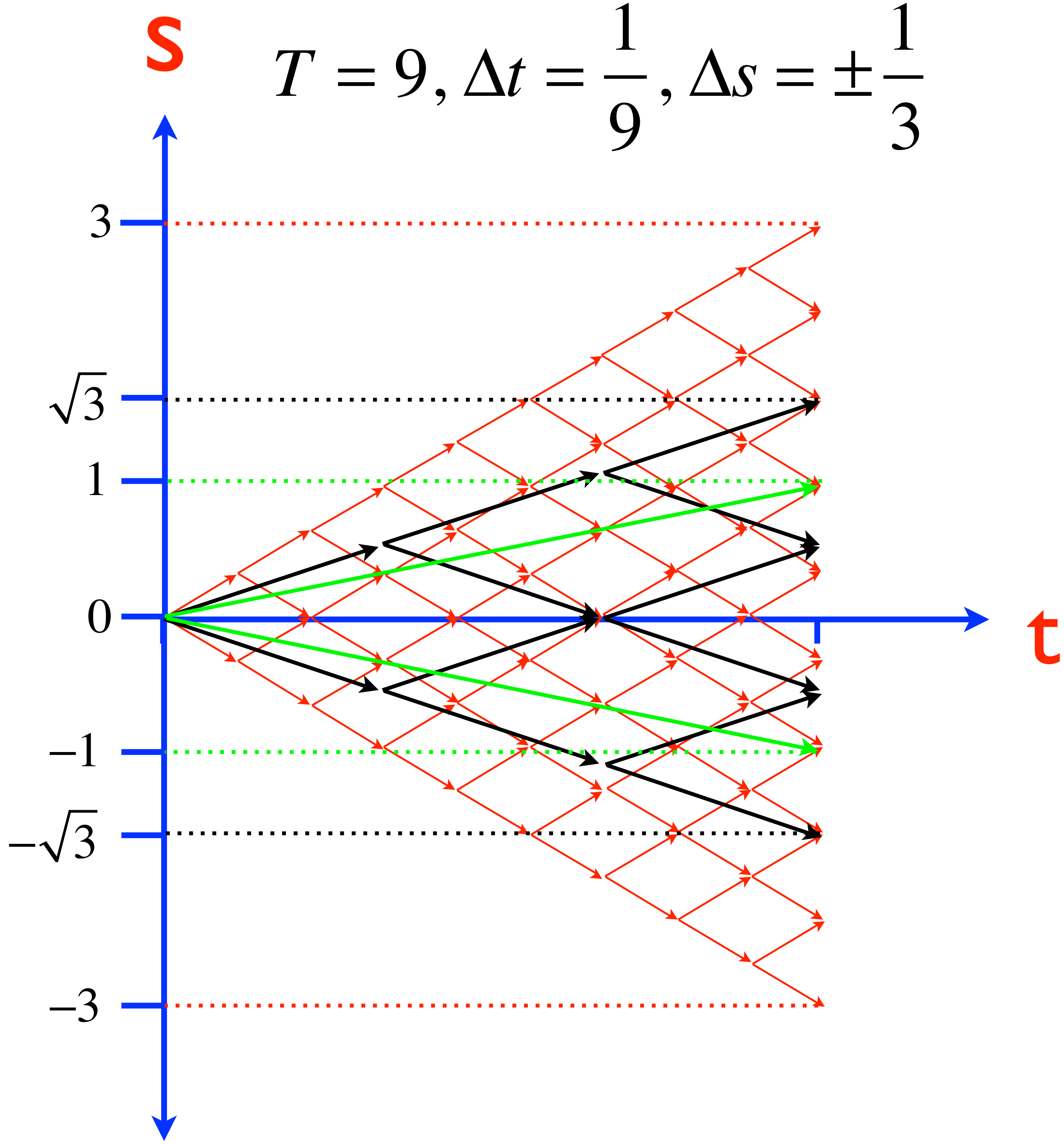

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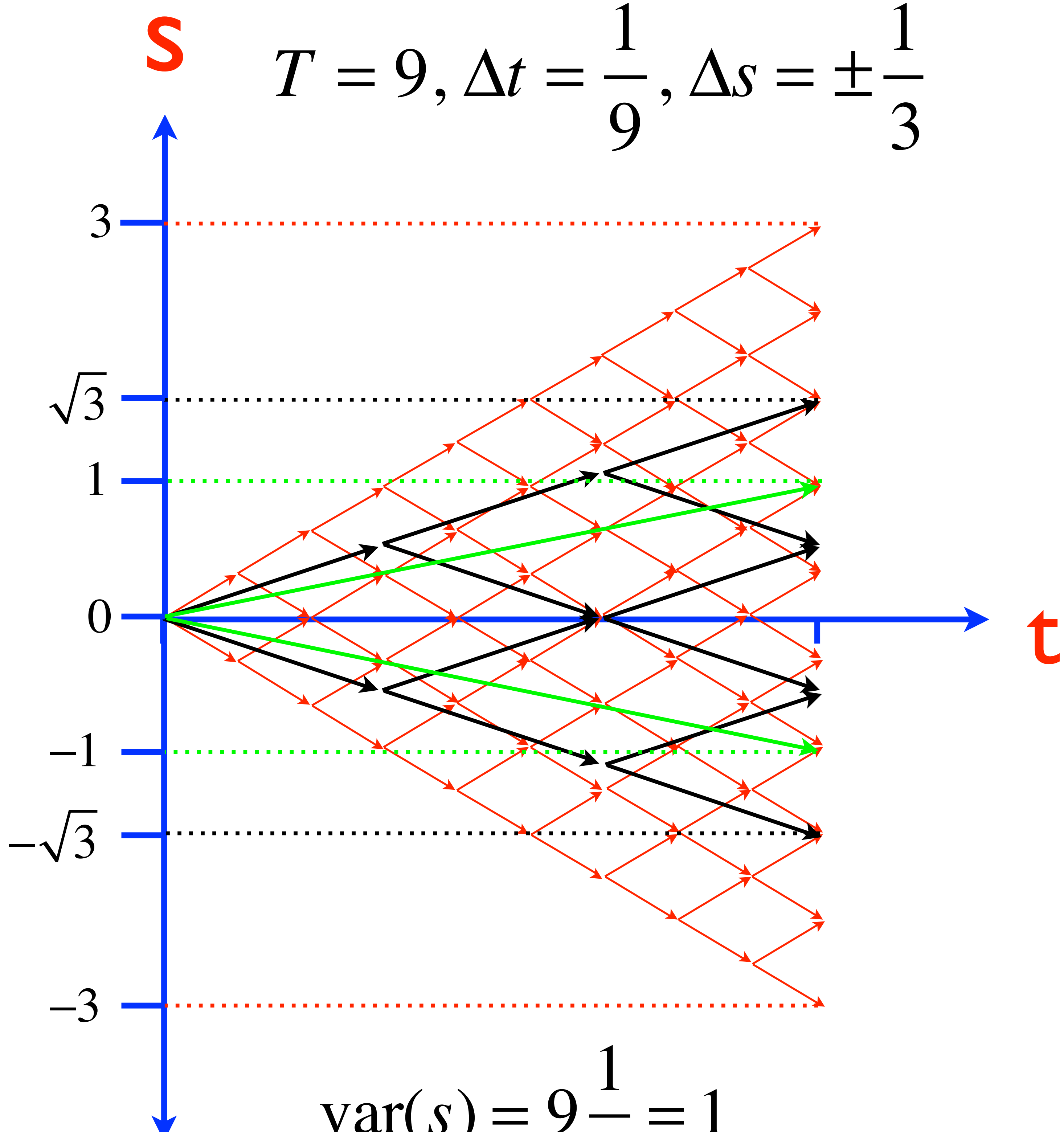

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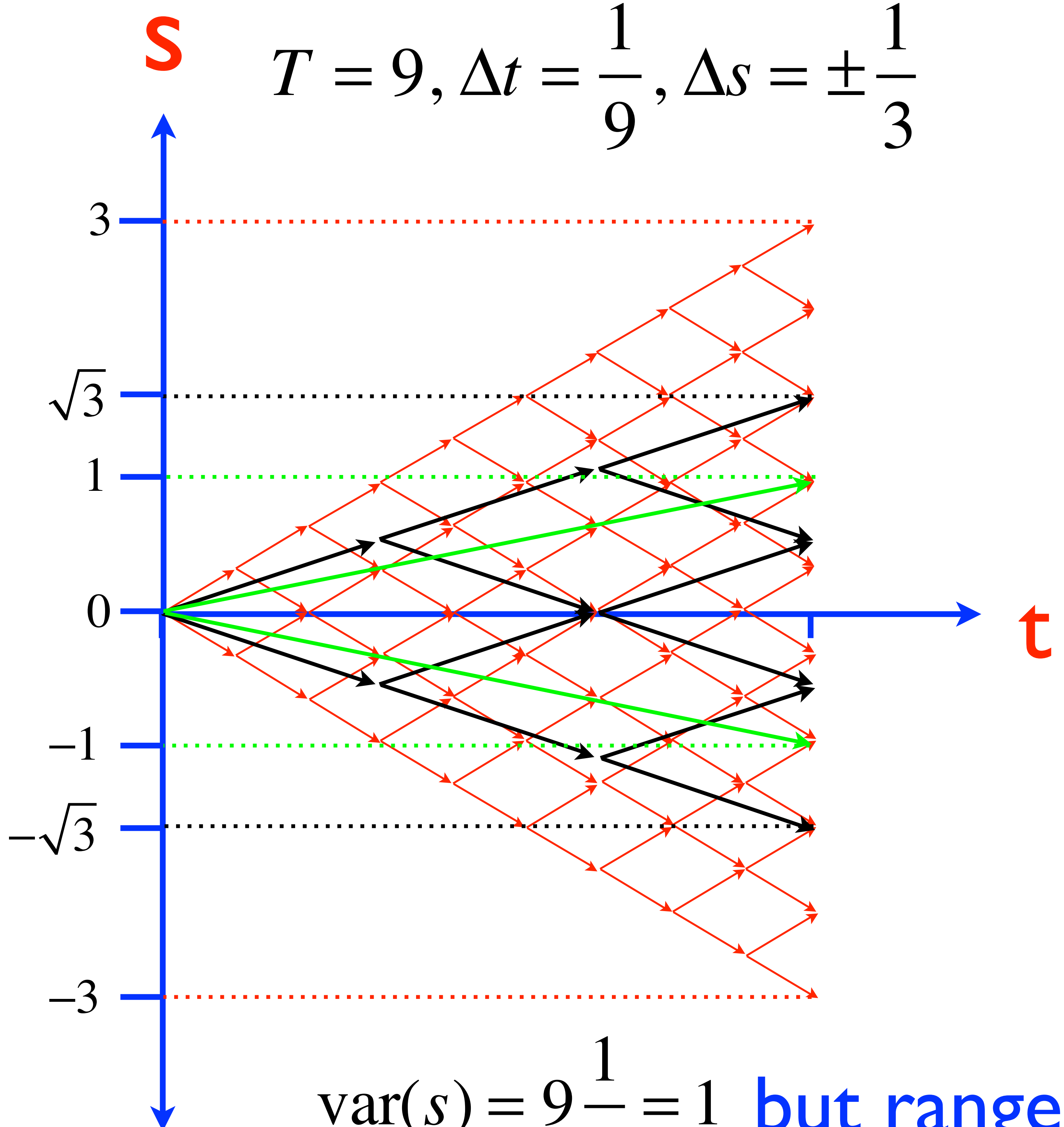

$\mathrm{var}(s) = 9\frac{1}{9} = 1$ but range(s) $\rightarrow \infty$

Potentials in continuous time

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- Continuous time:
  Differential Equations describing the density evolution for Brownian motion with drift (known as the Kolmogorov forward and backward equations).

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Illegal configuration:

$$\Psi(\text{configuration}) = \sum_{j=1}^{k} f(i)\psi(i,0) < \frac{1}{N}$$

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Beginning of game

Number of errors if adversary always plays optimally

$r-1$, where r is the smallest integer for which

$$\psi(0,r) < \frac{1}{N}$$

BW Prediction algorithm

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  - Removing assumption that expert set is divisible min/max optimality holds if $N > 2^{2^k}$
  - Based on relation to Ulam's game with k lies [Spencer 92]

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The optimal prediction is according to the the sign of this weighted sum.

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- Instantaneous loss is restricted to $\{0,1\}$, we want it to be any number in $[-1,+1]$.

Design of NormalHedge

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- Normal-Hedge: Potential function based on regret and variance of the positive regrets

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$$\text{Weight: } w(r,c) = \frac{\partial}{\partial r}\psi(r,c) = \begin{cases} \dfrac{r}{c}\exp\left(\dfrac{r^2}{2c}\right) & \text{if } r \geq 0 \\ 0 & \text{if } r \leq 0 \end{cases}$$

NormalHedge algorithm

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Update regrets: $R_i^{t+1} = R_i^t + l_A^t - l_i^t$

Illustrative Example

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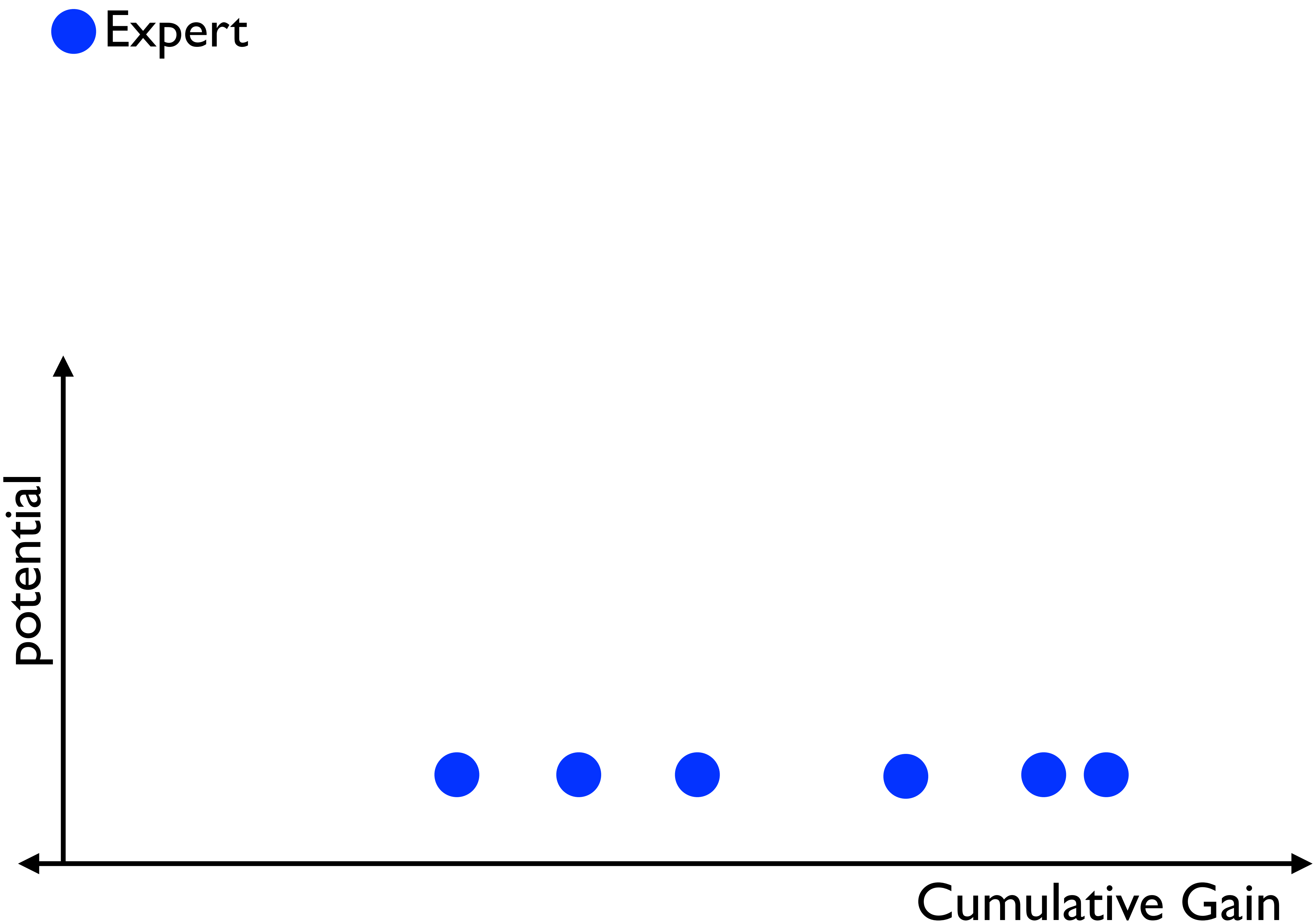

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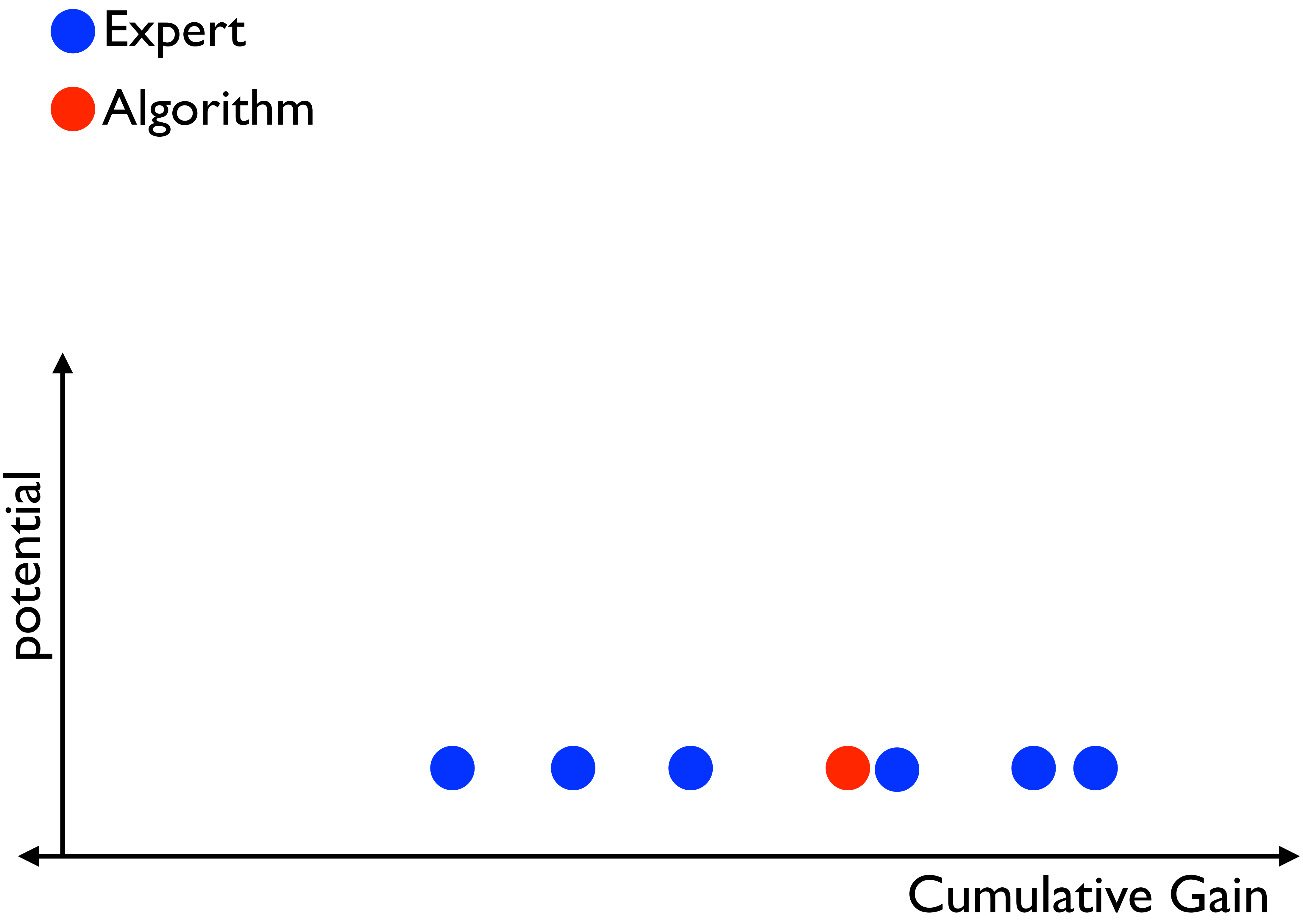
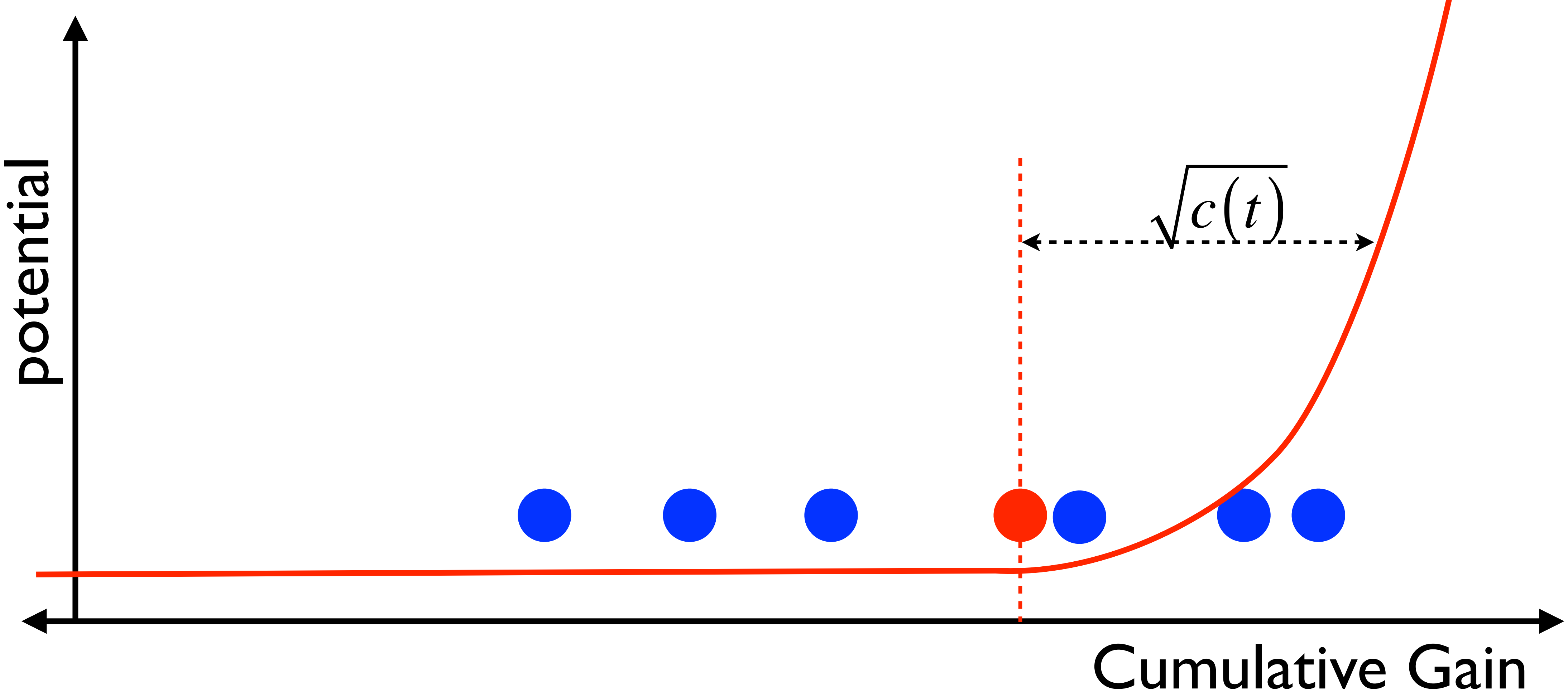


Illustrative Example

Expert

Algorithm

$$\begin{cases} \exp\left(\dfrac{R^2}{2c}\right) & \text{if R} \geq 0 \\ 1 & \text{if R} \leq 0 \end{cases}$$



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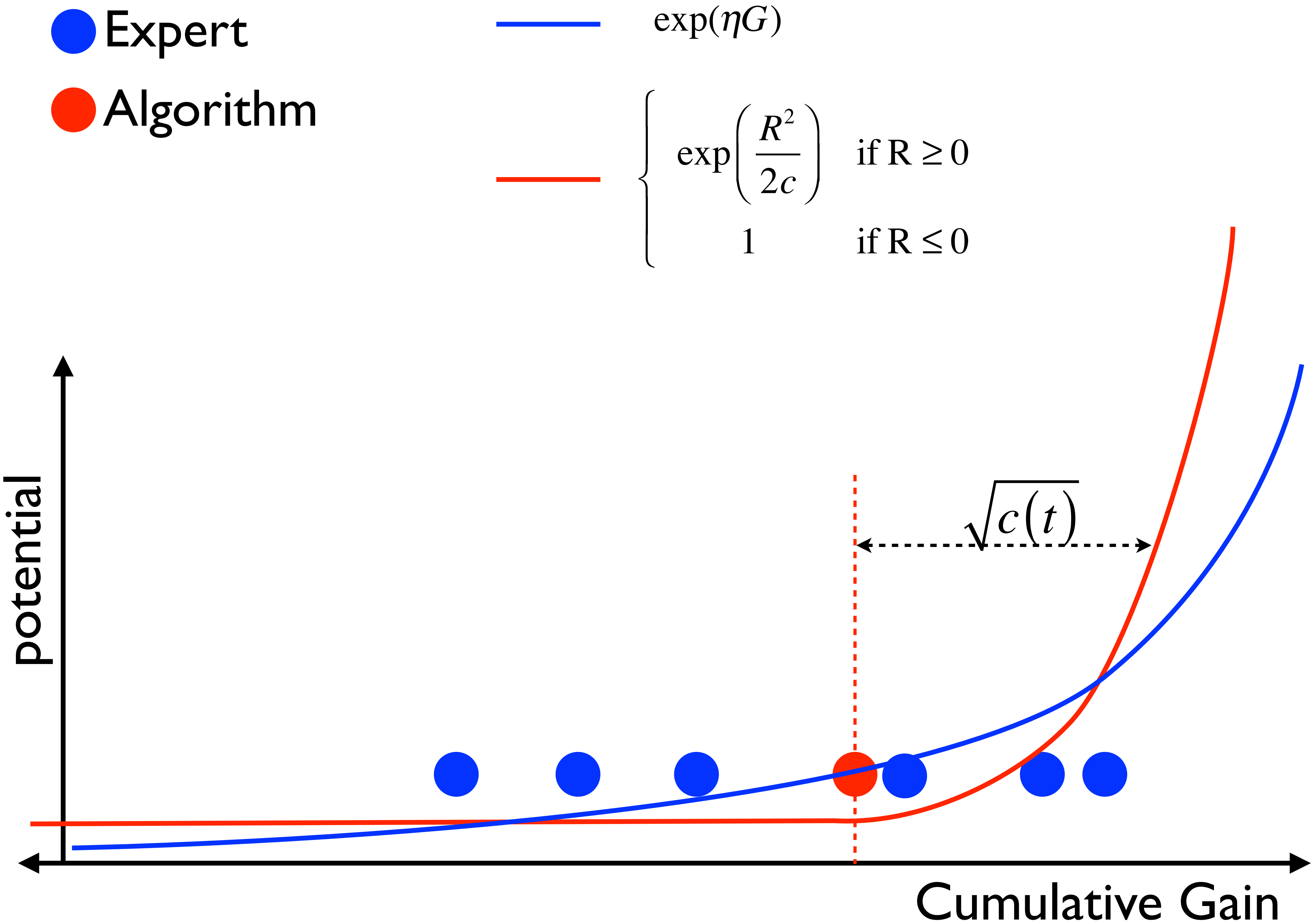

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$$\exp(\eta G)$$

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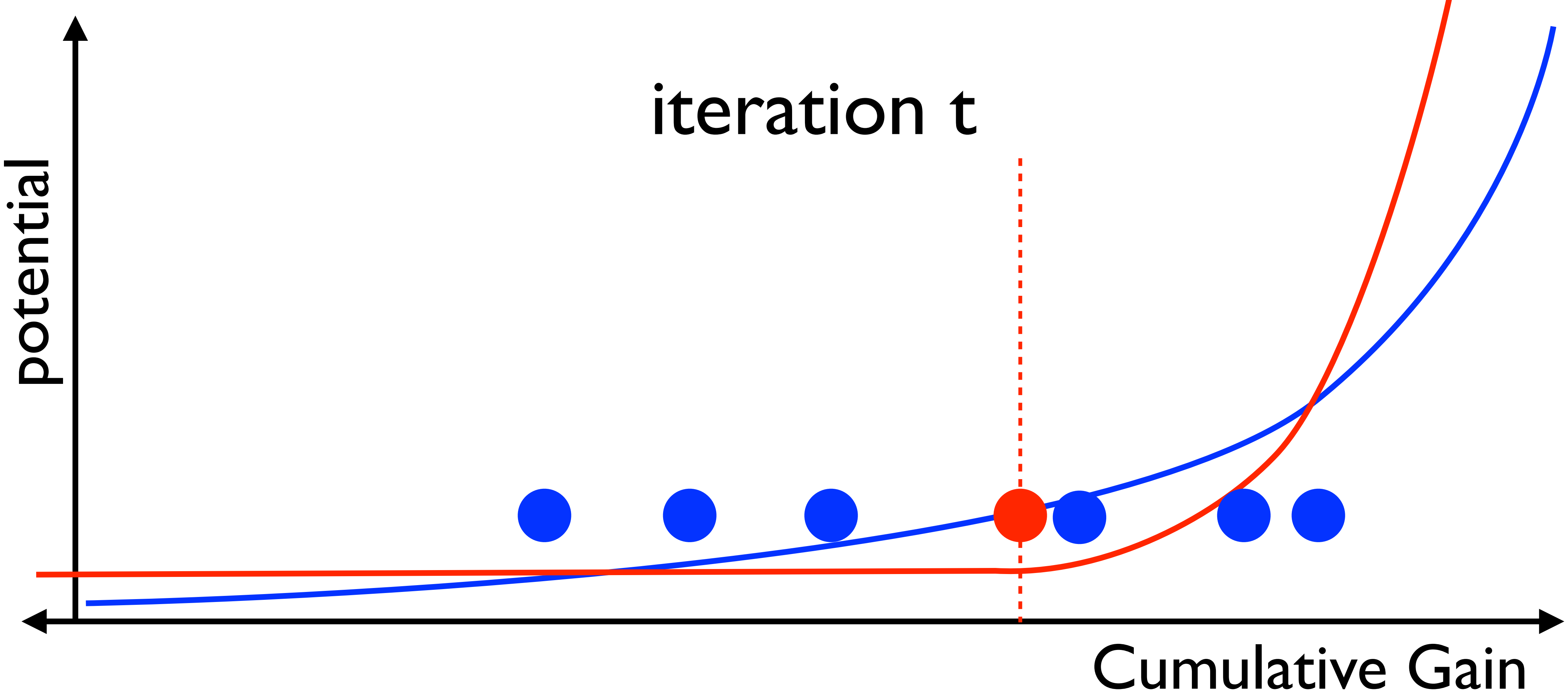

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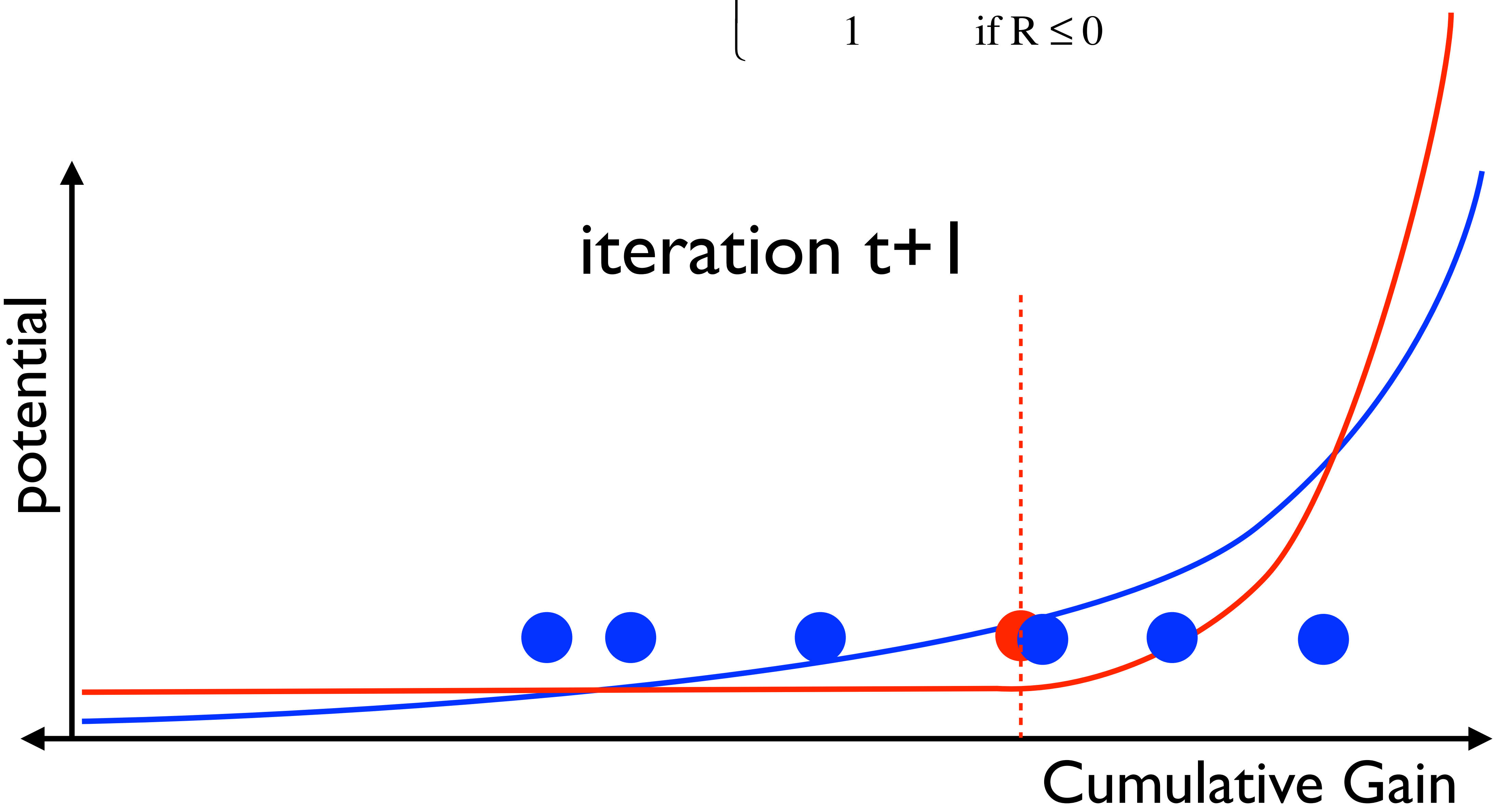

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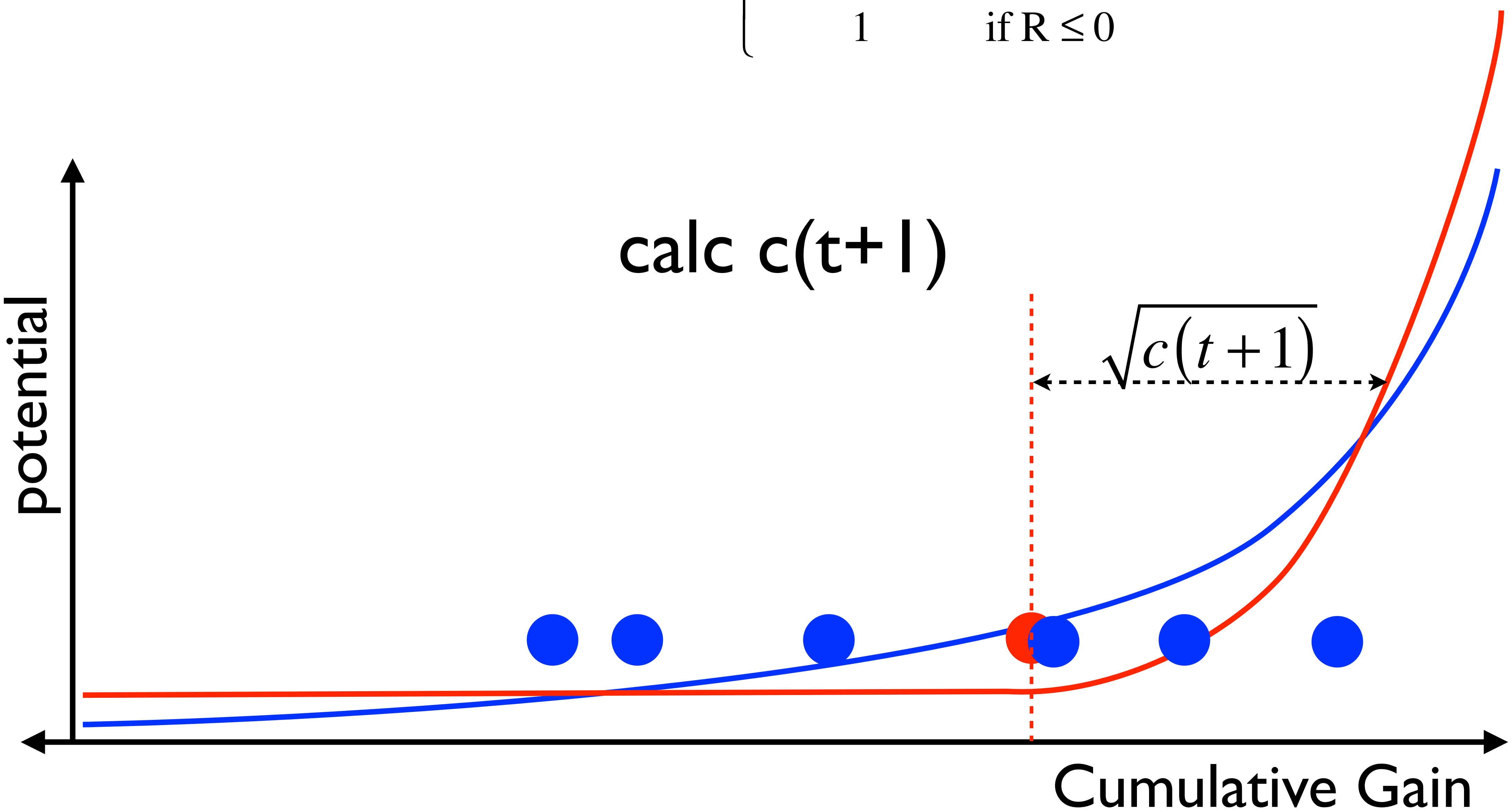


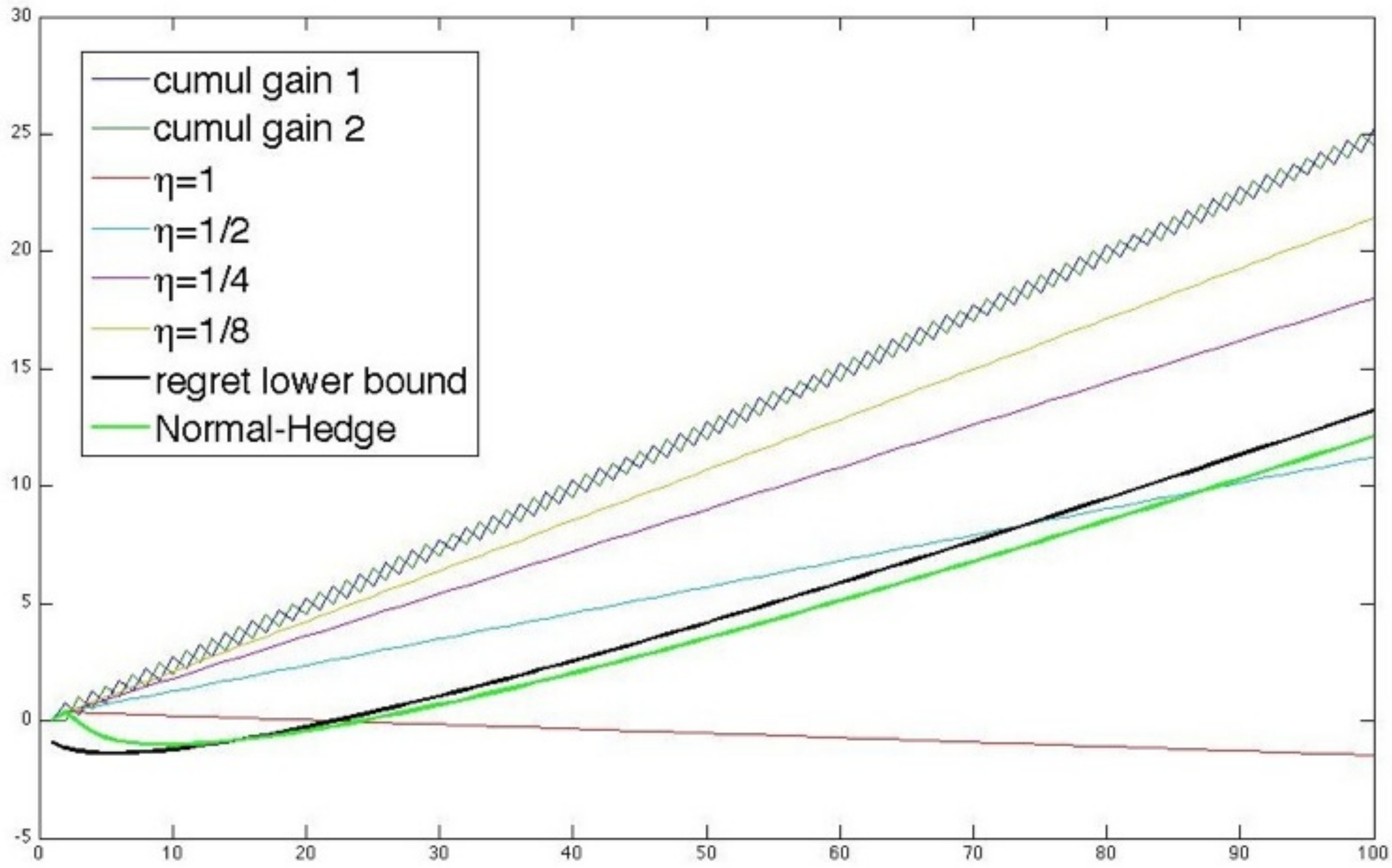
Normal-Hedge Performance bound

[Chaudhuri, Freund & Hsu 2009]

The regret of NormalHedge is upper bounded by

$$O\left(\sqrt{T \ln N + \ln^3 N}\right)$$

Performance on flip-flop



Summary

- Drifting games is a method for **deriving** new potential functions for online learning and boosting.
- Performed by working backwards in time, starting from final loss function and working backwards.
- Adversarial strategy: random walk / normal distribution.
- Yields Brown-boost
- Yields Normal-Hedge
- Both Brownboost and normal-hedge have difficult open problems.