Introduction to Online Learning Algorithms

Yoav Freund

January 9, 2018

Outline

Example trace for Halving Algorithm

Example trace for Halving Algorithm

expert1
expert2
expert3
expert4
expert5
expert6
expert7
expert8

alg.

Example trace for Halving Algorithm

expert1
expert2
expert3
expert4
expert5
expert6
expert7
expert8

alg.
outcome

Example trace for Halving Algorithm

	$t = 1$
expert1	1
expert2	1
expert3	0
expert4	1
expert5	1
expert6	0
expert7	1
expert8	1
alg.	
outcome	

Example trace for Halving Algorithm

	$t = 1$
expert1	1
expert2	1
expert3	0
expert4	1
expert5	1
expert6	0
expert7	1
expert8	1
alg.	1
outcome	

Example trace for Halving Algorithm

	$t = 1$
expert1	1
expert2	1
expert3	0
expert4	1
expert5	1
expert6	0
expert7	1
expert8	1
alg.	1
outcome	1

Example trace for Halving Algorithm

	$t = 1$	$t = 2$
expert1	1	1
expert2	1	0
expert3	0	-
expert4	1	0
expert5	1	0
expert6	0	-
expert7	1	1
expert8	1	1
alg.	1	
outcome	1	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$
expert1	1	1
expert2	1	0
expert3	0	-
expert4	1	0
expert5	1	0
expert6	0	-
expert7	1	1
expert8	1	1
alg.	1	0
outcome	1	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$
expert1	1	1
expert2	1	0
expert3	0	-
expert4	1	0
expert5	1	0
expert6	0	-
expert7	1	1
expert8	1	1
alg.	1	0
outcome	1	1

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$
expert1	1	1	1
expert2	1	0	-
expert3	0	-	-
expert4	1	0	-
expert5	1	0	-
expert6	0	-	-
expert7	1	1	1
expert8	1	1	1
alg.	1	0	
outcome	1	1	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$
expert1	1	1	1
expert2	1	0	-
expert3	0	-	-
expert4	1	0	-
expert5	1	0	-
expert6	0	-	-
expert7	1	1	1
expert8	1	1	1
alg.	1	0	1
outcome	1	1	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$
expert1	1	1	1
expert2	1	0	-
expert3	0	-	-
expert4	1	0	-
expert5	1	0	-
expert6	0	-	-
expert7	1	1	1
expert8	1	1	1
alg.	1	0	1
outcome	1	1	1

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$
expert1	1	1	1	1
expert2	1	0	-	-
expert3	0	-	-	-
expert4	1	0	-	-
expert5	1	0	-	-
expert6	0	-	-	-
expert7	1	1	1	1
expert8	1	1	1	0
alg.	1	0	1	
outcome	1	1	1	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$
expert1	1	1	1	1
expert2	1	0	-	-
expert3	0	-	-	-
expert4	1	0	-	-
expert5	1	0	-	-
expert6	0	-	-	-
expert7	1	1	1	1
expert8	1	1	1	0
alg.	1	0	1	1
outcome	1	1	1	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$
expert1	1	1	1	1
expert2	1	0	-	-
expert3	0	-	-	-
expert4	1	0	-	-
expert5	1	0	-	-
expert6	0	-	-	-
expert7	1	1	1	1
expert8	1	1	1	0
alg.	1	0	1	1
outcome	1	1	1	0

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
expert1	1	1	1	1	-
expert2	1	0	-	-	-
expert3	0	-	-	-	-
expert4	1	0	-	-	-
expert5	1	0	-	-	-
expert6	0	-	-	-	-
expert7	1	1	1	1	0
expert8	1	1	1	0	-
alg.	1	0	1	1	
outcome	1	1	1	0	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
expert1	1	1	1	1	-
expert2	1	0	-	-	-
expert3	0	-	-	-	-
expert4	1	0	-	-	-
expert5	1	0	-	-	-
expert6	0	-	-	-	-
expert7	1	1	1	1	0
expert8	1	1	1	0	-
alg.	1	0	1	1	0
outcome	1	1	1	0	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
expert1	1	1	1	1	-
expert2	1	0	-	-	-
expert3	0	-	-	-	-
expert4	1	0	-	-	-
expert5	1	0	-	-	-
expert6	0	-	-	-	-
expert7	1	1	1	1	0
expert8	1	1	1	0	-
alg.	1	0	1	1	0
outcome	1	1	1	0	0

Mistake bound for Halving algorithm

- Each time algorithm makes a mistakes, the pool of perfect experts is halved (at least).

Mistake bound for Halving algorithm

- Each time algorithm makes a mistakes, the pool of perfect experts is halved (at least).
- We assume that at least one expert is perfect.

Mistake bound for Halving algorithm

- Each time algorithm makes a mistakes, the pool of perfect experts is halved (at least).
- We assume that at least one expert is perfect.
- Number of mistakes is at most $\log_2 N$.

Mistake bound for Halving algorithm

- Each time algorithm makes a mistakes, the pool of perfect experts is halved (at least).
- We assume that at least one expert is perfect.
- Number of mistakes is at most $\log_2 N$.
- No stochastic assumptions whatsoever.

Mistake bound for Halving algorithm

- Each time algorithm makes a mistakes, the pool of perfect experts is halved (at least).
- We assume that at least one expert is perfect.
- Number of mistakes is at most $\log_2 N$.
- No stochastic assumptions whatsoever.
- Proof is based on combining a lower and upper bounds on the number of perfect experts.

The hedging problem

- N possible actions

The hedging problem

- N possible actions
- At each time step t:

The hedging problem

- N possible actions
- At each time step t:
  - Algorithm chooses a distribution $\vec{P}^t$ over actions.

The hedging problem

- N possible actions
- At each time step t:
  - Algorithm chooses a distribution $\vec{P}^t$ over actions.
  - Losses $0 \leq \ell_i^t \leq 1$ of all actions $i = 1, \ldots, N$ are revealed.

The hedging problem

- N possible actions
- At each time step t:
  - Algorithm chooses a distribution $\vec{P}^t$ over actions.
  - Losses $0 \leq \ell_i^t \leq 1$ of all actions $i = 1, \ldots, N$ are revealed.
  - Algorithm suffers **expected** loss.

The hedging problem

- N possible actions
- At each time step t:
  - Algorithm chooses a distribution $\vec{P}^t$ over actions.
  - Losses $0 \leq \ell_i^t \leq 1$ of all actions $i = 1, \ldots, N$ are revealed.
  - Algorithm suffers **expected** loss.
- **Goal:** minimize total expected loss

The hedging problem

- N possible actions
- At each time step t:
  - Algorithm chooses a distribution $\vec{P}^t$ over actions.
  - Losses $0 \leq \ell_i^t \leq 1$ of all actions $i = 1, \ldots, N$ are revealed.
  - Algorithm suffers **expected** loss.
- **Goal:** minimize total expected loss
- Here we have stochasticity - but only in **algorithm**, not in **outcome**

The hedging problem

- N possible actions
- At each time step t:
  - Algorithm chooses a distribution $\vec{P}^t$ over actions.
  - Losses $0 \leq \ell_i^t \leq 1$ of all actions $i = 1, \ldots, N$ are revealed.
  - Algorithm suffers **expected** loss.
- **Goal:** minimize total expected loss
- Here we have stochasticity - but only in **algorithm**, not in **outcome**
- Fits nicely in game theory

Hedging vs. Halving

- Like halving - we want to zoom into best action (expert).

Hedging vs. Halving

- Like halving - we want to zoom into best action (expert).
- Unlike halving - no action is perfect.

Hedging vs. Halving

- Like halving - we want to zoom into best action (expert).
- Unlike halving - no action is perfect.
- Basic idea - reduce probability of lossy actions, but not all the way to zero.

Hedging vs. Halving

- Like halving - we want to zoom into best action (expert).
- Unlike halving - no action is perfect.
- Basic idea - reduce probability of lossy actions,
  but not all the way to zero.
- **Modified Goal:** minimize difference between
  expected total loss
  and
  minimal total loss of repeating one action.

The Hedge Algorithm

Consider action i at time t

- Total loss:

$$L_i^t = \sum_{s=1}^{t-1} \ell_i^s$$

The Hedge Algorithm

Consider action i at time t

- Total loss:

$$L_i^t = \sum_{s=1}^{t-1} \ell_i^s$$

- Weight:

$$W_i^t = e^{-\eta L_i^t}$$

The Hedge Algorithm

Consider action i at time t

- Total loss:

$$L_i^t = \sum_{s=1}^{t-1} \ell_i^s$$

- Weight:

$$W_i^t = e^{-\eta L_i^t}$$

- $\eta > 0$ is the learning rate parameter. Halving: $\eta = \infty$

The Hedge Algorithm

Consider action i at time t

- Total loss:

$$L_i^t = \sum_{s=1}^{t-1} \ell_i^s$$

- Weight:

$$W_i^t = e^{-\eta L_i^t}$$

- $\eta > 0$ is the learning rate parameter. Halving: $\eta = \infty$
- Probability:

$$P_i^t = \frac{W_i^t}{\sum_{j=1}^N W_j^t}$$

Example trace for Hedge Algorithm

$\eta = 1$

Example trace for Hedge Algorithm

$$\eta = 1$$

expert1
expert2
expert3
expert4
expert5
expert6
expert7
expert8

alg.

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$
expert1	1
expert2	1
expert3	1
expert4	1
expert5	1
expert6	1
expert7	1
expert8	1

alg.

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	$\vec{\ell}^1$
expert1	1	.1
expert2	1	.8
expert3	1	.3
expert4	1	.1
expert5	1	.9
expert6	1	0
expert7	1	1
expert8	1	.8

alg.

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	$\vec{\ell}^1$
expert1	1	.1
expert2	1	.8
expert3	1	.3
expert4	1	.1
expert5	1	.9
expert6	1	0
expert7	1	1
expert8	1	.8
alg.		.5

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	$\vec{\ell}^1$	$\vec{W}^2$
expert1	1	.1	.90
expert2	1	.8	.45
expert3	1	.3	.74
expert4	1	.1	.90
expert5	1	.9	.41
expert6	1	0	1
expert7	1	1	.37
expert8	1	.8	.45
alg.		.5	

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	$\vec{\ell}^1$	$\vec{W}^2$	$\vec{\ell}^2$
expert1	1	.1	.90	.1
expert2	1	.8	.45	.5
expert3	1	.3	.74	.2
expert4	1	.1	.90	.7
expert5	1	.9	.41	1
expert6	1	0	1	.1
expert7	1	1	.37	.5
expert8	1	.8	.45	.2
alg.		.5		

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	$\vec{\ell}^1$	$\vec{W}^2$	$\vec{\ell}^2$
expert1	1	.1	.90	.1
expert2	1	.8	.45	.5
expert3	1	.3	.74	.2
expert4	1	.1	.90	.7
expert5	1	.9	.41	1
expert6	1	0	1	.1
expert7	1	1	.37	.5
expert8	1	.8	.45	.2
alg.		.5		.36

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	$\vec{\ell}^1$	$\vec{W}^2$	$\vec{\ell}^2$	$\vec{W}^3$
expert1	1	.1	.90	.1	0.82
expert2	1	.8	.45	.5	0.27
expert3	1	.3	.74	.2	0.61
expert4	1	.1	.90	.7	0.45
expert5	1	.9	.41	1	0.15
expert6	1	0	1	.1	0.91
expert7	1	1	.37	.5	0.22
expert8	1	.8	.45	.2	0.37
alg.		.5		.36	

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	$\vec{\ell}^1$	$\vec{W}^2$	$\vec{\ell}^2$	$\vec{W}^3$	$\vec{\ell}^3$
expert1	1	.1	.90	.1	0.82	0
expert2	1	.8	.45	.5	0.27	.2
expert3	1	.3	.74	.2	0.61	.2
expert4	1	.1	.90	.7	0.45	.8
expert5	1	.9	.41	1	0.15	.8
expert6	1	0	1	.1	0.91	.2
expert7	1	1	.37	.5	0.22	.4
expert8	1	.8	.45	.2	0.37	.6
alg.		.5		.36		

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	$\vec{\ell}^1$	$\vec{W}^2$	$\vec{\ell}^2$	$\vec{W}^3$	$\vec{\ell}^3$
expert1	1	.1	.90	.1	0.82	0
expert2	1	.8	.45	.5	0.27	.2
expert3	1	.3	.74	.2	0.61	.2
expert4	1	.1	.90	.7	0.45	.8
expert5	1	.9	.41	1	0.15	.8
expert6	1	0	1	.1	0.91	.2
expert7	1	1	.37	.5	0.22	.4
expert8	1	.8	.45	.2	0.37	.6
alg.		.5		.36		.30

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	$\vec{\ell}^1$	$\vec{W}^2$	$\vec{\ell}^2$	$\vec{W}^3$	$\vec{\ell}^3$	total
expert1	1	.1	.90	.1	0.82	0	.2
expert2	1	.8	.45	.5	0.27	.2	1.5
expert3	1	.3	.74	.2	0.61	.2	.7
expert4	1	.1	.90	.7	0.45	.8	1.6
expert5	1	.9	.41	1	0.15	.8	2.7
expert6	1	0	1	.1	0.91	.2	.3
expert7	1	1	.37	.5	0.22	.4	1.9
expert8	1	.8	.45	.2	0.37	.6	1.6
alg.		.5		.36		.30	

Example trace for Hedge Algorithm

$\eta = 1$

	$\vec{W}^1$	$\vec{\ell}^1$	$\vec{W}^2$	$\vec{\ell}^2$	$\vec{W}^3$	$\vec{\ell}^3$	total
expert1	1	.1	.90	.1	0.82	0	.2
expert2	1	.8	.45	.5	0.27	.2	1.5
expert3	1	.3	.74	.2	0.61	.2	.7
expert4	1	.1	.90	.7	0.45	.8	1.6
expert5	1	.9	.41	1	0.15	.8	2.7
expert6	1	0	1	.1	0.91	.2	.3
expert7	1	1	.37	.5	0.22	.4	1.9
expert8	1	.8	.45	.2	0.37	.6	1.6
alg.		.5		.36		.30	1.16

Bound for Hedge Algorithm

- L^t_{Hedge}: Expected total loss of Hedge algorithm for time $1, 2, \ldots, t$

Bound for Hedge Algorithm

- L^t_{Hedge}: Expected total loss of Hedge algorithm for time $1, 2, \ldots, t$
- $$\forall t, i, \quad L_{\text{Hedge}} \leq \frac{\ln N + \eta L^t_i}{1 - e^{-\eta}}$$

Bound for Hedge Algorithm

- L^t_{Hedge}: Expected total loss of Hedge algorithm for time $1, 2, \ldots, t$
- $$\forall t, i, \quad L_{\text{Hedge}} \leq \frac{\ln N + \eta L_i^t}{1 - e^{-\eta}}$$
- Which implies

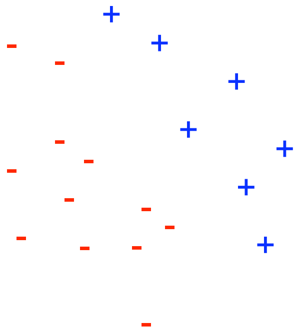
$$\forall t, \quad L_{\text{Hedge}} \leq \min_i \left(\frac{\ln N + \eta L_i^t}{1 - e^{-\eta}} \right)$$

Bound for Hedge Algorithm

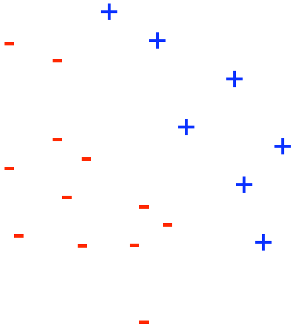
- L^t_{Hedge}: Expected total loss of Hedge algorithm for time $1, 2, \ldots, t$
- $$\forall t, i, \quad L_{\text{Hedge}} \leq \frac{\ln N + \eta L^t_i}{1 - e^{-\eta}}$$
- Which implies
$$\forall t, \quad L_{\text{Hedge}} \leq \min_i \left(\frac{\ln N + \eta L^t_i}{1 - e^{-\eta}} \right)$$
- Proof and choice of η: next class.

The Perceptron Problem

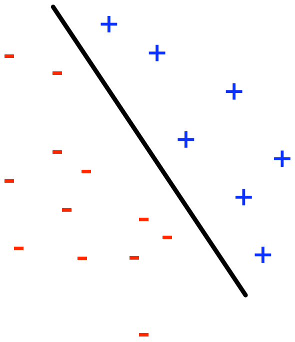
The Perceptron Problem



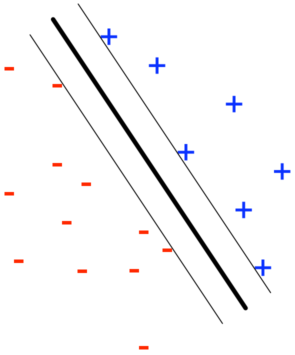
The Perceptron Problem



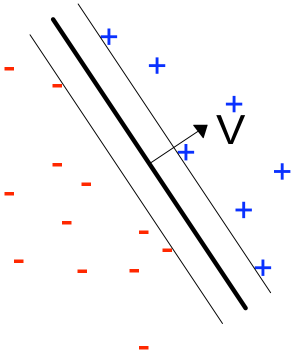
The Perceptron Problem



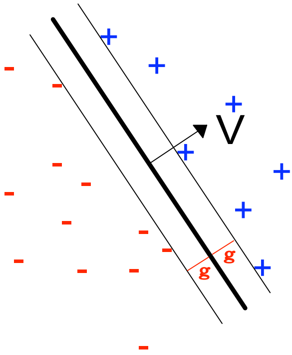
The Perceptron Problem



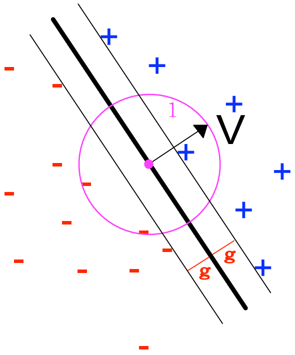
The Perceptron Problem



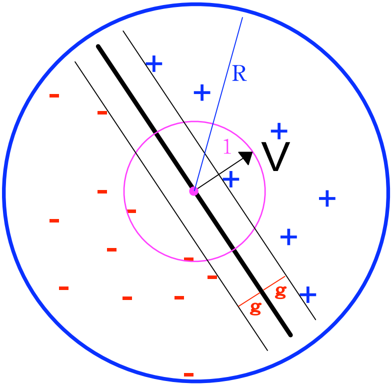
The Perceptron Problem



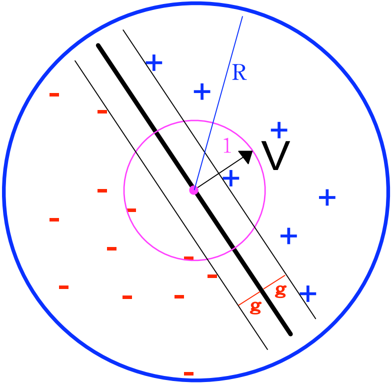
The Perceptron Problem



The Perceptron Problem

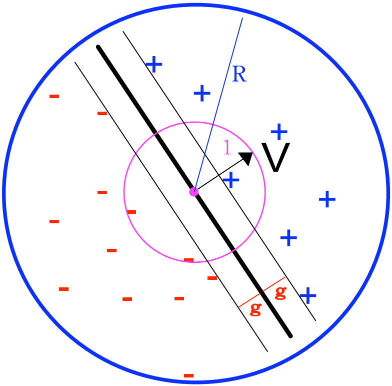


The Perceptron Problem



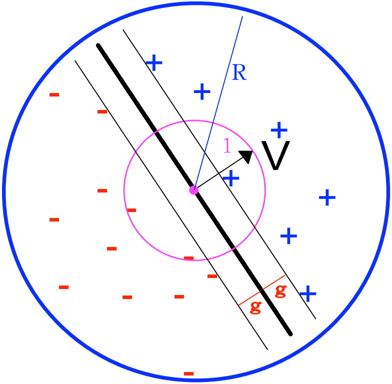
- $\|\vec{V}\| = 1$

The Perceptron Problem



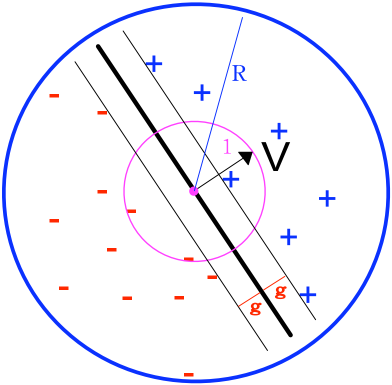
- $\|\vec{V}\| = 1$
- Example = $(\vec{X}, y)$, $y \in \{-1, +1\}$.

The Perceptron Problem



- $\|\vec{V}\| = 1$
- Example = $(\vec{X}, y)$, $y \in \{-1, +1\}$.
- $\forall \vec{X}, \quad \|\vec{X}\| \leq R$.

The Perceptron Problem



- $\|\vec{V}\| = 1$
- Example = $(\vec{X}, y)$, $y \in \{-1, +1\}$.
- $\forall \vec{X}, \quad \|\vec{X}\| \leq R$.
- $\forall (\vec{X}, y)$, $y(\vec{X} \cdot \vec{V}) \geq g$

The Perceptron learning algorithm

- An online algorithm. Examples presented one by one.

The Perceptron learning algorithm

- An online algorithm. Examples presented one by one.
- start with $\vec{W}_0 = \vec{0}$.

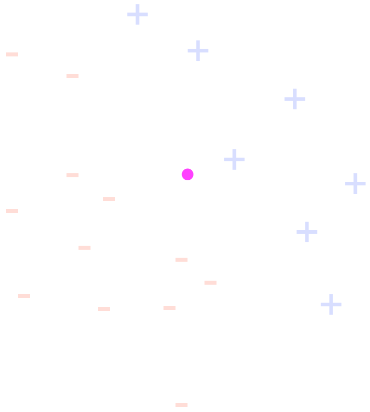
The Perceptron learning algorithm

- An online algorithm. Examples presented one by one.
- start with $\vec{W}_0 = \vec{0}$.
- If mistake: $(\vec{W}_i \cdot \vec{X}_i) y_i \leq 0$

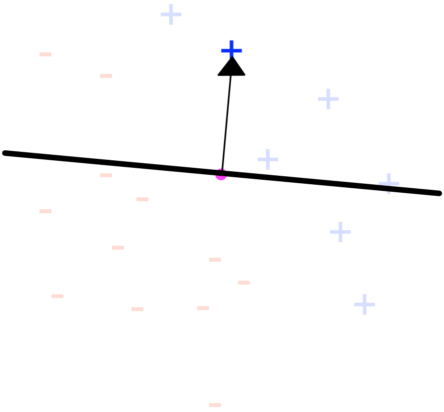
The Perceptron learning algorithm

- An online algorithm. Examples presented one by one.
- start with $\vec{W}_0 = \vec{0}$.
- If mistake: $(\vec{W}_i \cdot \vec{X}_i)y_i \leq 0$
    - Update $\vec{W}_{i+1} = \vec{W}_i + y_i X_i$.

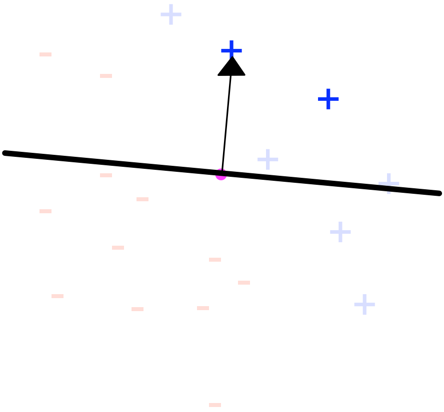
Example trace for the perceptron algorithm



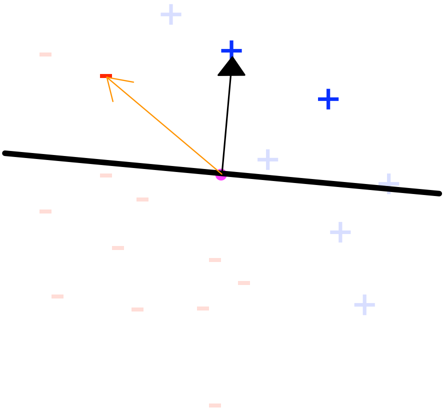
Example trace for the perceptron algorithm



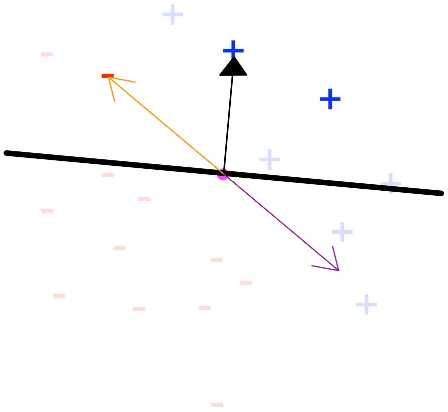
Example trace for the perceptron algorithm



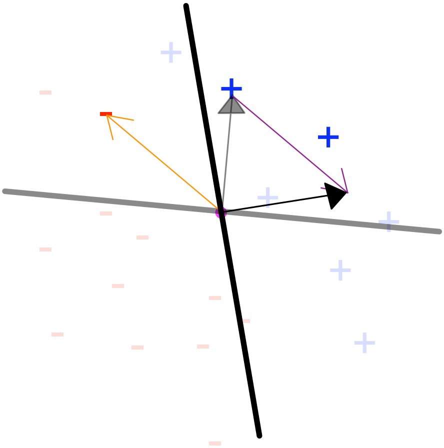
Example trace for the perceptron algorithm



Example trace for the perceptron algorithm



Example trace for the perceptron algorithm



Bound on number of mistakes

- The number of mistakes that the perceptron algorithm can make is at most $\left(\frac{R}{g}\right)^2$.

Bound on number of mistakes

- The number of mistakes that the perceptron algorithm can make is at most $\left(\frac{R}{g}\right)^2$.
- Proof by combining upper and lower bounds on $\|\vec{W}\|$.

Pythagorian Lemma

If $(\vec{W}_i \cdot X_i)y < 0$ then

Pythagorian Lemma

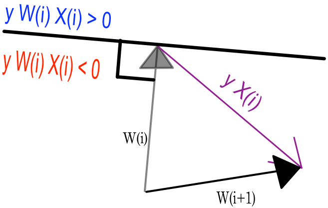
If $(\vec{W}_i \cdot X_i)y < 0$ then

$$\|\vec{W}_{i+1}\|^2 = \|\vec{W}_i + y_i\vec{X}_i\|^2 \leq \|\vec{W}_i\|^2 + \|\vec{X}_i\|^2$$

Pythagorian Lemma

If $(\vec{W}_i \cdot X_i) y < 0$ then

$$\|\vec{W}_{i+1}\|^2 = \|\vec{W}_i + y_i \vec{X}_i\|^2 \leq \|\vec{W}_i\|^2 + \|\vec{X}_i\|^2$$



Upper bound on $\|\vec{W}_i\|$

Upper bound on $\|\vec{W}_i\|$

Proof by induction

- Claim: $\|\vec{W}_i\|^2 \leq iR^2$

Upper bound on $\|\vec{W}_i\|$

Proof by induction

- Claim: $\|\vec{W}_i\|^2 \leq iR^2$
- Base: $i = 0$, $\|\vec{W}_0\|^2 = 0$

Upper bound on $\|\vec{W}_i\|$

Proof by induction

- Claim: $\|\vec{W}_i\|^2 \leq iR^2$
- Base: $i = 0$, $\|\vec{W}_0\|^2 = 0$
- Induction step (assume for i and prove for $i + 1$):
  $\|\vec{W}_{i+1}\|^2 \leq \|\vec{W}_i\|^2 + \|\vec{X}_i\|^2$
  $\leq \|\vec{W}_i\|^2 + R^2 \leq (i + 1)R^2$

Lower bound on $\|\vec{W}_i\|$

Lower bound on $\|\vec{W}_i\|$

$\|\vec{W}_i\| \geq \vec{W}_i \cdot \vec{V}$ because $\|\vec{V}\| = 1$.

Lower bound on $\|\vec{W}_i\|$

$\|\vec{W}_i\| \geq \vec{W}_i \cdot \vec{V}$ because $\|\vec{V}\| = 1$.
We prove a lower bound on $\vec{W}_i \cdot \vec{V}$ using induction over i

- Claim: $\vec{W}_i \cdot \vec{V} \geq ig$

Lower bound on $\|\vec{W}_i\|$

$\|\vec{W}_i\| \geq \vec{W}_i \cdot \vec{V}$ because $\|\vec{V}\| = 1$.
We prove a lower bound on $\vec{W}_i \cdot \vec{V}$ using induction over i

- Claim: $\vec{W}_i \cdot \vec{V} \geq ig$
- Base: $i = 0$, $\vec{W}_0 \cdot \vec{V} = 0$

Lower bound on $\|\vec{W}_i\|$

$\|\vec{W}_i\| \geq \vec{W}_i \cdot \vec{V}$ because $\|\vec{V}\| = 1$.
We prove a lower bound on $\vec{W}_i \cdot \vec{V}$ using induction over i

- Claim: $\vec{W}_i \cdot \vec{V} \geq ig$
- Base: $i = 0$, $\vec{W}_0 \cdot \vec{V} = 0$
- Induction step (assume for i and prove for $i + 1$):
  $\vec{W}_{i+1} \cdot \vec{V} = \left(\vec{W}_i + \vec{X}_i y_i\right) \vec{V}$

Lower bound on $\|\vec{W}_i\|$

$\|\vec{W}_i\| \geq \vec{W}_i \cdot \vec{V}$ because $\|\vec{V}\| = 1$.
We prove a lower bound on $\vec{W}_i \cdot \vec{V}$ using induction over i

- Claim: $\vec{W}_i \cdot \vec{V} \geq ig$
- Base: $i = 0$, $\vec{W}_0 \cdot \vec{V} = 0$
- Induction step (assume for i and prove for $i + 1$):
  $\vec{W}_{i+1} \cdot \vec{V} = \left(\vec{W}_i + \vec{X}_i y_i\right) \vec{V}$

Lower bound on $\|\vec{W}_i\|$

$\|\vec{W}_i\| \geq \vec{W}_i \cdot \vec{V}$ because $\|\vec{V}\| = 1$.

We prove a lower bound on $\vec{W}_i \cdot \vec{V}$ using induction over i

- Claim: $\vec{W}_i \cdot \vec{V} \geq ig$
- Base: $i = 0$, $\vec{W}_0 \cdot \vec{V} = 0$
- Induction step (assume for i and prove for $i+1$):
  $\vec{W}_{i+1} \cdot \vec{V} = \left(\vec{W}_i + \vec{X}_i y_i\right) \vec{V} = \vec{W}_i \cdot \vec{V} + y_i \vec{X}_i \cdot \vec{V}$
  $\geq ig + g = (i+1)g$

Combining the upper and lower bounds

Combining the upper and lower bounds

$$(ig)^2 \leq \|\vec{W}_i\|^2 \leq iR^2$$

Combining the upper and lower bounds

$$(ig)^2 \leq \|\vec{W}_i\|^2 \leq iR^2$$

Thus:

$$i \leq \left(\frac{R}{g}\right)^2$$

Estimating the bias of a coin

- We observe n coin flips:
  H,T,T,H,H,T,H,T,T

Estimating the bias of a coin

- We observe n coin flips:
  H,T,T,H,H,T,H,T,T
- We want to estimate the **probability** that the next flip will be **H**ead.

Estimating the bias of a coin

- We observe n coin flips: **H,T,T,H,H,T,H,T,T**
- We want to estimate the **probability** that the next flip will be **H**ead.
- Natural Answer:

$$\frac{\#\mathbf{H}}{n} = \frac{4}{9}$$

What if the estimation has to be done online?

- We observe a bf sequence of coin flips, and have to predict the probability of **H**ead at each step:
  $p_0,$

What if the estimation has to be done online?

- We observe a bf sequence of coin flips, and have to predict the probability of **H**ead at each step:

  p_0,

What if the estimation has to be done online?

- We observe a bf sequence of coin flips, and have to predict the probability of **H**ead at each step:
  p_0,**H**,

What if the estimation has to be done online?

- We observe a bf sequence of coin flips, and have to predict the probability of **H**ead at each step:
  p_0,**H**,p_1,

What if the estimation has to be done online?

- We observe a bf sequence of coin flips, and have to predict the probability of **H**ead at each step:
  p_0,**H**,p_1,**T**,

What if the estimation has to be done online?

- We observe a bf sequence of coin flips, and have to predict the probability of **H**ead at each step:
  p_0,**H**,p_1,**T**,p_2,

What if the estimation has to be done online?

- We observe a bf sequence of coin flips, and have to predict the probability of **H**ead at each step: p_0,**H**,p_1,**T**,p_2,...
- What would be a good value for p_0?

What if the estimation has to be done online?

- We observe a bf sequence of coin flips, and have to predict the probability of **H**ead at each step:
  p_0,**H**,p_1,**T**,p_2,...
- What would be a good value for p_0?
- For p_1?

What if the estimation has to be done online?

- We observe a bf sequence of coin flips, and have to predict the probability of **H**ead at each step:
  p_0,**H**,p_1,**T**,p_2,...
- What would be a good value for p_0?
- For p_1?
- Laplace Law of succession

$$\frac{\#\mathbf{H}+1}{n+2}$$

What if the estimation has to be done online?

- We observe a bf sequence of coin flips, and have to predict the probability of **H**ead at each step: p_0,**H**,p_1,**T**,p_2,...
- What would be a good value for p_0?
- For p_1?
- Laplace Law of succession
$$\frac{\#\mathbf{H}+1}{n+2}$$
- Turns out that a better rule is
$$\frac{\#\mathbf{H}+1/2}{n+1}$$
Krichevsky and Trofimov, 1981

What if the estimation has to be done online?

- We observe a bf sequence of coin flips, and have to predict the probability of **H**ead at each step:
  p_0,**H**,p_1,**T**,p_2,...
- What would be a good value for p_0?
- For p_1?
- Laplace Law of succession

$$\frac{\#\mathbf{H}+1}{n+2}$$

- Turns out that a better rule is

$$\frac{\#\mathbf{H}+1/2}{n+1}$$

  Krichevsky and Trofimov, 1981
- Why?

What if the estimation has to be done online?

- We observe a bf sequence of coin flips, and have to predict the probability of **H**ead at each step:
  p_0,**H**,p_1,**T**,p_2,. . .
- What would be a good value for p_0?
- For p_1?
- Laplace Law of succession
$$\frac{\#\mathbf{H}+1}{n+2}$$
- Turns out that a better rule is
$$\frac{\#\mathbf{H}+1/2}{n+1}$$
  Krichevsky and Trofimov, 1981
- Why?
- What does “better” mean?

To be continued...

Please

- Register on twiki (follow directions on my home page)

To be continued...

Please

- Register on twiki (follow directions on my home page)
- Follow link from main twiki page to “Online learning course”

To be continued...

Please

- Register on twiki (follow directions on my home page)
- Follow link from main twiki page to “Online learning course”
- Add yourself to the list and the table on `ClassParticipants`

To be continued...

Please

- Register on twiki (follow directions on my home page)
- Follow link from main twiki page to “Online learning course”
- Add yourself to the list and the table on `ClassParticipants`
- Go to `CoursePlan/LessonNo1` to see slides and to post questions and answers.

To be continued...

Please

- Register on twiki (follow directions on my home page)
- Follow link from main twiki page to “Online learning course”
- Add yourself to the list and the table on `ClassParticipants`
- Go to `CoursePlan/LessonNo1` to see slides and to post questions and answers.
- See you on Thursday!