

Introduction to Online Learning Algorithms

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Outline

Example trace for Halving Algorithm

Example trace for Halving Algorithm

expert1
expert2
expert3
expert4
expert5
expert6
expert7
expert8

alg.

Example trace for Halving Algorithm

expert1
expert2
expert3
expert4
expert5
expert6
expert7
expert8

alg.

outcome

Example trace for Halving Algorithm

	$t = 1$
expert1	1
expert2	1
expert3	0
expert4	1
expert5	1
expert6	0
expert7	1
expert8	1

alg.
outcome

Example trace for Halving Algorithm

	$t = 1$
expert1	1
expert2	1
expert3	0
expert4	1
expert5	1
expert6	0
expert7	1
expert8	1
alg.	1
outcome	

Example trace for Halving Algorithm

	$t = 1$
expert1	1
expert2	1
expert3	0
expert4	1
expert5	1
expert6	0
expert7	1
expert8	1
alg.	1
outcome	1

Example trace for Halving Algorithm

	$t = 1$	$t = 2$
expert1	1	1
expert2	1	0
expert3	0	-
expert4	1	0
expert5	1	0
expert6	0	-
expert7	1	1
expert8	1	1
alg.	1	
outcome	1	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$
expert1	1	1
expert2	1	0
expert3	0	-
expert4	1	0
expert5	1	0
expert6	0	-
expert7	1	1
expert8	1	1
alg.	1	0
outcome	1	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$
expert1	1	1
expert2	1	0
expert3	0	-
expert4	1	0
expert5	1	0
expert6	0	-
expert7	1	1
expert8	1	1
alg.	1	0
outcome	1	1

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$
expert1	1	1	1
expert2	1	0	-
expert3	0	-	-
expert4	1	0	-
expert5	1	0	-
expert6	0	-	-
expert7	1	1	1
expert8	1	1	1
alg.	1	0	
outcome	1	1	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$
expert1	1	1	1
expert2	1	0	-
expert3	0	-	-
expert4	1	0	-
expert5	1	0	-
expert6	0	-	-
expert7	1	1	1
expert8	1	1	1
alg.	1	0	1
outcome	1	1	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$
expert1	1	1	1
expert2	1	0	-
expert3	0	-	-
expert4	1	0	-
expert5	1	0	-
expert6	0	-	-
expert7	1	1	1
expert8	1	1	1
alg.	1	0	1
outcome	1	1	1

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$
expert1	1	1	1	1
expert2	1	0	-	-
expert3	0	-	-	-
expert4	1	0	-	-
expert5	1	0	-	-
expert6	0	-	-	-
expert7	1	1	1	1
expert8	1	1	1	0
alg.	1	0	1	
outcome	1	1	1	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$
expert1	1	1	1	1
expert2	1	0	-	-
expert3	0	-	-	-
expert4	1	0	-	-
expert5	1	0	-	-
expert6	0	-	-	-
expert7	1	1	1	1
expert8	1	1	1	0
alg.	1	0	1	1
outcome	1	1	1	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$
expert1	1	1	1	1
expert2	1	0	-	-
expert3	0	-	-	-
expert4	1	0	-	-
expert5	1	0	-	-
expert6	0	-	-	-
expert7	1	1	1	1
expert8	1	1	1	0
alg.	1	0	1	1
outcome	1	1	1	0

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
expert1	1	1	1	1	-
expert2	1	0	-	-	-
expert3	0	-	-	-	-
expert4	1	0	-	-	-
expert5	1	0	-	-	-
expert6	0	-	-	-	-
expert7	1	1	1	1	0
expert8	1	1	1	0	-
alg.	1	0	1	1	
outcome	1	1	1	0	

Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
expert1	1	1	1	1	-
expert2	1	0	-	-	-
expert3	0	-	-	-	-
expert4	1	0	-	-	-
expert5	1	0	-	-	-
expert6	0	-	-	-	-
expert7	1	1	1	1	0
expert8	1	1	1	0	-
alg.	1	0	1	1	0
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expert2	1	0	-	-	-
expert3	0	-	-	-	-
expert4	1	0	-	-	-
expert5	1	0	-	-	-
expert6	0	-	-	-	-
expert7	1	1	1	1	0
expert8	1	1	1	0	-
alg.	1	0	1	1	0
outcome	1	1	1	0	0

Mistake bound for Halving algorithm

- ▶ Each time algorithm makes a mistakes, the pool of perfect experts is halved (at least).

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- ▶ Each time algorithm makes a mistakes, the pool of perfect experts is halved (at least).
- ▶ We assume that at least one expert is perfect.
- ▶ Number of mistakes is at most $\log_2 N$.
- ▶ No stochastic assumptions whatsoever.
- ▶ Proof is based on combining a lower and upper bounds on the number of perfect experts.

The hedging problem

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 - ▶ Algorithm suffers **expected** loss.
- ▶ **Goal:** minimize total expected loss
- ▶ Here we have stochasticity - but only in **algorithm**, not in **outcome**
- ▶ Fits nicely in game theory

Hedging vs. Halving

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- ▶ Basic idea - reduce probability of lossy actions, but **not all the way to zero**.

Hedging vs. Halving

- ▶ Like halving - we want to zoom into best action (expert).
- ▶ Unlike halving - no action is perfect.
- ▶ Basic idea - reduce probability of lossy actions, but **not all the way to zero**.
- ▶ **Modified Goal:** minimize **difference between** expected total loss **and** minimal total loss of repeating one action.

The Hedge Algorithm

Consider action i at time t

- ▶ Total loss:

$$L_i^t = \sum_{s=1}^{t-1} \ell_i^s$$

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- ▶ Weight:

$$W_i^t = e^{-\eta L_i^t}$$

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Consider action i at time t

- ▶ Total loss:

$$L_i^t = \sum_{s=1}^{t-1} \ell_i^s$$

- ▶ Weight:

$$W_i^t = e^{-\eta L_i^t}$$

- ▶ $\eta > 0$ is the learning rate parameter. Halving: $\eta = \frac{1}{\ln 2}$

The Hedge Algorithm

Consider action i at time t

- ▶ Total loss:

$$L_i^t = \sum_{s=1}^{t-1} \ell_i^s$$

- ▶ Weight:

$$W_i^t = e^{-\eta L_i^t}$$

- ▶ $\eta > 0$ is the learning rate parameter. Halving: $\eta = \infty$
- ▶ Probability:

$$P_i^t = \frac{W_i^t}{\sum_{j=1}^N W_j^t}$$

Example trace for Hedge Algorithm

$$\eta = 1$$

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$$\eta = 1$$

expert1

expert2

expert3

expert4

expert5

expert6

expert7

expert8

alg.

Example trace for Hedge Algorithm

$$\eta = 1$$

 \vec{W}^1

expert1

1

expert2

1

expert3

1

expert4

1

expert5

1

expert6

1

expert7

1

expert8

1

alg.

Example trace for Hedge Algorithm

$$\eta = 1$$

	\vec{W}^1	$\vec{\ell}^1$
expert1	1	.1
expert2	1	.8
expert3	1	.3
expert4	1	.1
expert5	1	.9
expert6	1	0
expert7	1	1
expert8	1	.8

alg.

Example trace for Hedge Algorithm

$$\eta = 1$$

	\vec{W}^1	$\vec{\ell}^1$
expert1	1	.1
expert2	1	.8
expert3	1	.3
expert4	1	.1
expert5	1	.9
expert6	1	0
expert7	1	1
expert8	1	.8
alg.		.5

Example trace for Hedge Algorithm

$$\eta = 1$$

	\vec{W}^1	$\vec{\ell}^1$	\vec{W}^2
expert1	1	.1	.90
expert2	1	.8	.45
expert3	1	.3	.74
expert4	1	.1	.90
expert5	1	.9	.41
expert6	1	0	1
expert7	1	1	.37
expert8	1	.8	.45
alg.		.5	

Example trace for Hedge Algorithm

$$\eta = 1$$

	\vec{W}^1	$\vec{\ell}^1$	\vec{W}^2	$\vec{\ell}^2$
expert1	1	.1	.90	.1
expert2	1	.8	.45	.5
expert3	1	.3	.74	.2
expert4	1	.1	.90	.7
expert5	1	.9	.41	1
expert6	1	0	1	.1
expert7	1	1	.37	.5
expert8	1	.8	.45	.2
alg.		.5		

Example trace for Hedge Algorithm

$$\eta = 1$$

	\vec{W}^1	$\vec{\ell}^1$	\vec{W}^2	$\vec{\ell}^2$
expert1	1	.1	.90	.1
expert2	1	.8	.45	.5
expert3	1	.3	.74	.2
expert4	1	.1	.90	.7
expert5	1	.9	.41	1
expert6	1	0	1	.1
expert7	1	1	.37	.5
expert8	1	.8	.45	.2
alg.		.5		.36

Example trace for Hedge Algorithm

$$\eta = 1$$

	\vec{W}^1	$\vec{\ell}^1$	\vec{W}^2	$\vec{\ell}^2$	\vec{W}^3
expert1	1	.1	.90	.1	0.82
expert2	1	.8	.45	.5	0.27
expert3	1	.3	.74	.2	0.61
expert4	1	.1	.90	.7	0.45
expert5	1	.9	.41	1	0.15
expert6	1	0	1	.1	0.91
expert7	1	1	.37	.5	0.22
expert8	1	.8	.45	.2	0.37
alg.		.5		.36	

Example trace for Hedge Algorithm

$$\eta = 1$$

	\vec{W}^1	$\vec{\ell}^1$	\vec{W}^2	$\vec{\ell}^2$	\vec{W}^3	$\vec{\ell}^3$
expert1	1	.1	.90	.1	0.82	0
expert2	1	.8	.45	.5	0.27	.2
expert3	1	.3	.74	.2	0.61	.2
expert4	1	.1	.90	.7	0.45	.8
expert5	1	.9	.41	1	0.15	.8
expert6	1	0	1	.1	0.91	.2
expert7	1	1	.37	.5	0.22	.4
expert8	1	.8	.45	.2	0.37	.6
alg.		.5		.36		

Example trace for Hedge Algorithm

$$\eta = 1$$

	\vec{W}^1	$\vec{\ell}^1$	\vec{W}^2	$\vec{\ell}^2$	\vec{W}^3	$\vec{\ell}^3$
expert1	1	.1	.90	.1	0.82	0
expert2	1	.8	.45	.5	0.27	.2
expert3	1	.3	.74	.2	0.61	.2
expert4	1	.1	.90	.7	0.45	.8
expert5	1	.9	.41	1	0.15	.8
expert6	1	0	1	.1	0.91	.2
expert7	1	1	.37	.5	0.22	.4
expert8	1	.8	.45	.2	0.37	.6
alg.		.5		.36		.30

Example trace for Hedge Algorithm

$$\eta = 1$$

	\vec{W}^1	$\vec{\ell}^1$	\vec{W}^2	$\vec{\ell}^2$	\vec{W}^3	$\vec{\ell}^3$	total
expert1	1	.1	.90	.1	0.82	0	.2
expert2	1	.8	.45	.5	0.27	.2	1.5
expert3	1	.3	.74	.2	0.61	.2	.7
expert4	1	.1	.90	.7	0.45	.8	1.6
expert5	1	.9	.41	1	0.15	.8	2.7
expert6	1	0	1	.1	0.91	.2	.3
expert7	1	1	.37	.5	0.22	.4	1.9
expert8	1	.8	.45	.2	0.37	.6	1.6
alg.		.5		.36		.30	

Example trace for Hedge Algorithm

$$\eta = 1$$

	\vec{W}^1	$\vec{\ell}^1$	\vec{W}^2	$\vec{\ell}^2$	\vec{W}^3	$\vec{\ell}^3$	total
expert1	1	.1	.90	.1	0.82	0	.2
expert2	1	.8	.45	.5	0.27	.2	1.5
expert3	1	.3	.74	.2	0.61	.2	.7
expert4	1	.1	.90	.7	0.45	.8	1.6
expert5	1	.9	.41	1	0.15	.8	2.7
expert6	1	0	1	.1	0.91	.2	.3
expert7	1	1	.37	.5	0.22	.4	1.9
expert8	1	.8	.45	.2	0.37	.6	1.6
alg.		.5		.36		.30	1.16

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- ▶ L_{Hedge}^t : Expected total loss of Hedge algorithm for time $1, 2, \dots, t$

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$$\forall t, i, \quad L_{\text{Hedge}} \leq \frac{\ln N + \eta L_i^t}{1 - e^{-\eta}}$$

Bound for Hedge Algorithm

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$$\forall t, i, \quad L_{\text{Hedge}} \leq \frac{\ln N + \eta L_i^t}{1 - e^{-\eta}}$$

- ▶ Which implies

$$\forall t, \quad L_{\text{Hedge}} \leq \min_i \left(\frac{\ln N + \eta L_i^t}{1 - e^{-\eta}} \right)$$

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- ▶ L_{Hedge}^t : Expected total loss of Hedge algorithm for time $1, 2, \dots, t$



$$\forall t, i, \quad L_{\text{Hedge}} \leq \frac{\ln N + \eta L_i^t}{1 - e^{-\eta}}$$

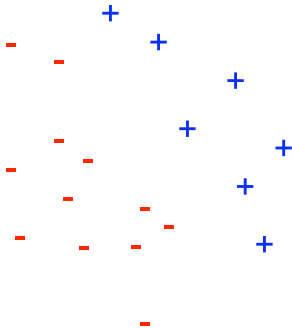
- ▶ Which implies

$$\forall t, \quad L_{\text{Hedge}} \leq \min_i \left(\frac{\ln N + \eta L_i^t}{1 - e^{-\eta}} \right)$$

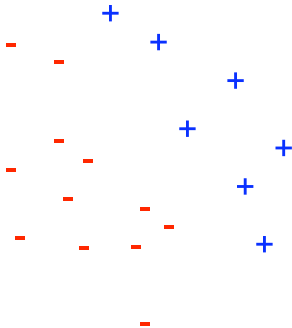
- ▶ Proof and choice of η : next class.

The Perceptron Problem

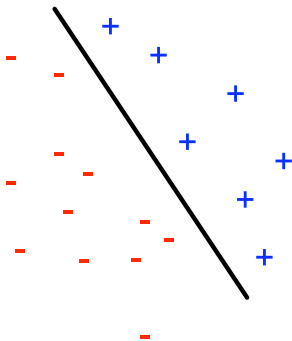
The Perceptron Problem



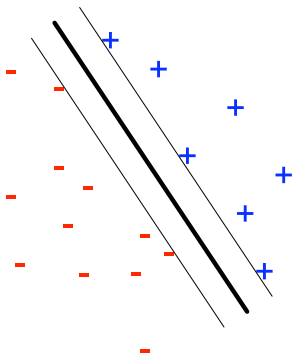
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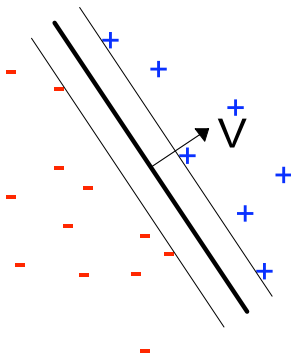
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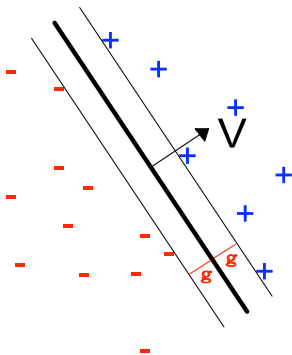
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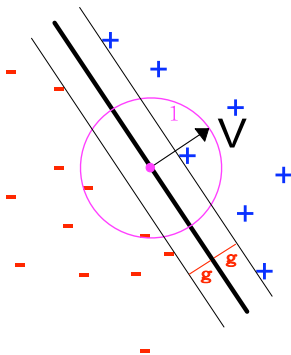
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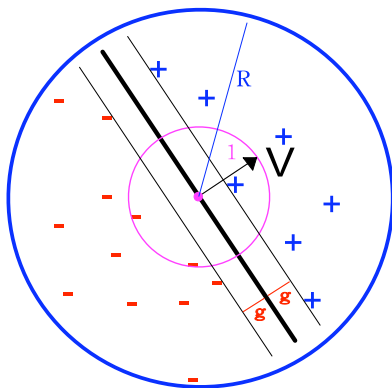
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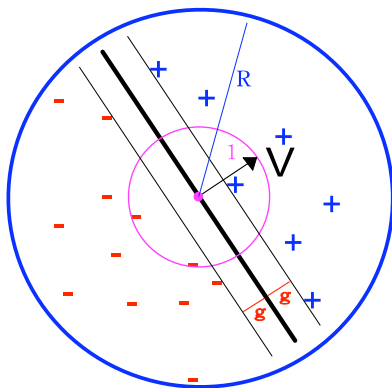
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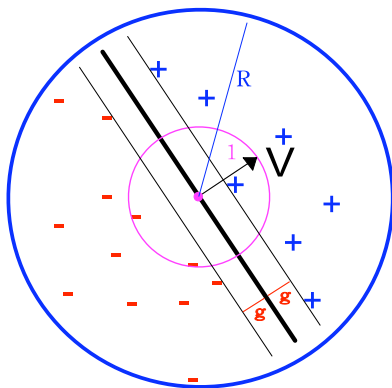


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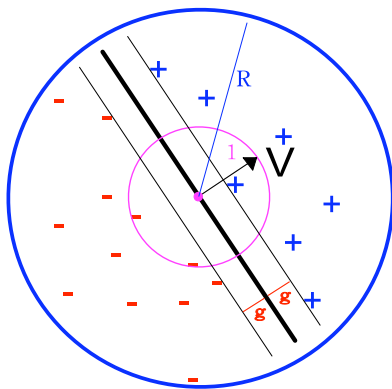
► $\|\vec{V}\| = 1$

The Perceptron Problem



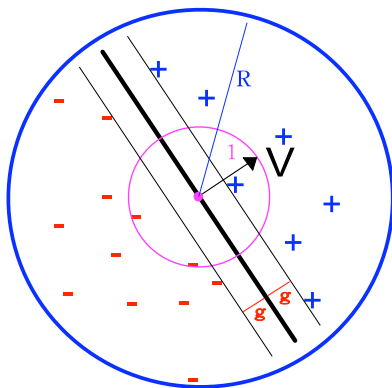
- ▶ $\|\vec{V}\| = 1$
- ▶ Example = (\vec{X}, y) ,
 $y \in \{-1, +1\}$.

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- ▶ Example = (\vec{X}, y) ,
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- ▶ $\forall \vec{X}, \|\vec{X}\| \leq R$.

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- ▶ $\|\vec{V}\| = 1$
- ▶ Example = (\vec{X}, y) ,
 $y \in \{-1, +1\}$.
- ▶ $\forall \vec{X}, \|\vec{X}\| \leq R$.
- ▶ $\forall (\vec{X}, y),$
 $y(\vec{X} \cdot \vec{V}) \geq g$

The Perceptron learning algorithm

- ▶ An online algorithm. Examples presented one by one.

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- ▶ start with $\vec{W}_0 = \vec{0}$.

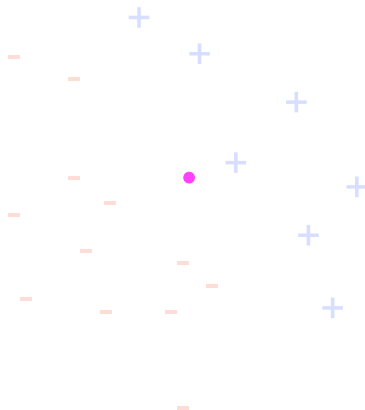
The Perceptron learning algorithm

- ▶ An online algorithm. Examples presented one by one.
- ▶ start with $\vec{W}_0 = \vec{0}$.
- ▶ If mistake: $(\vec{W}_i \cdot \vec{X}_i)y_i \leq 0$

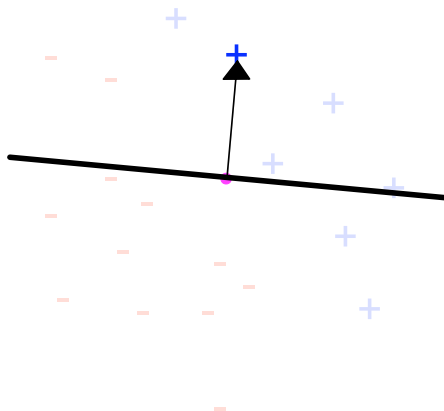
The Perceptron learning algorithm

- ▶ An online algorithm. Examples presented one by one.
- ▶ start with $\vec{W}_0 = \vec{0}$.
- ▶ If mistake: $(\vec{W}_i \cdot \vec{X}_i)y_i \leq 0$
 - ▶ Update $\vec{W}_{i+1} = \vec{W}_i + y_i X_i$.

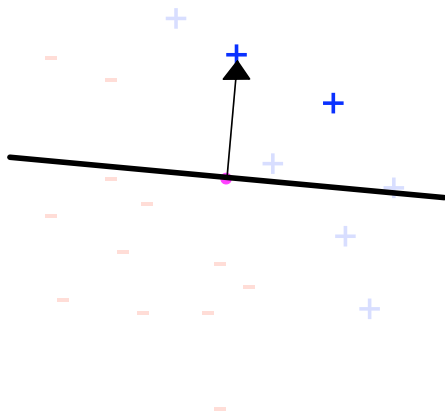
Example trace for the perceptron algorithm



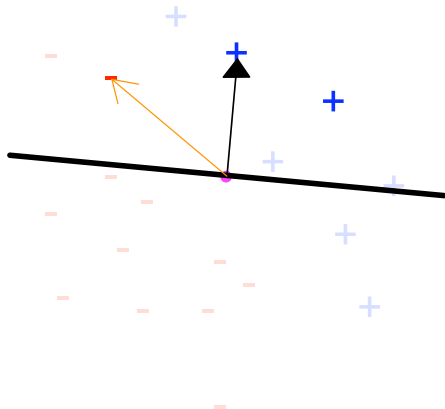
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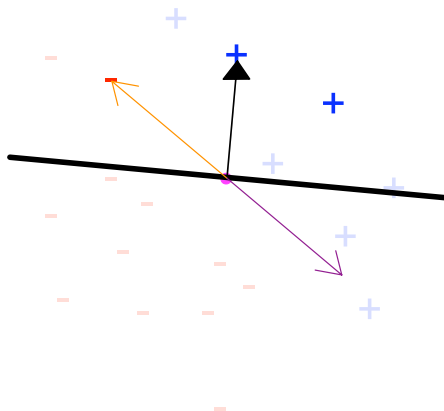
Example trace for the perceptron algorithm



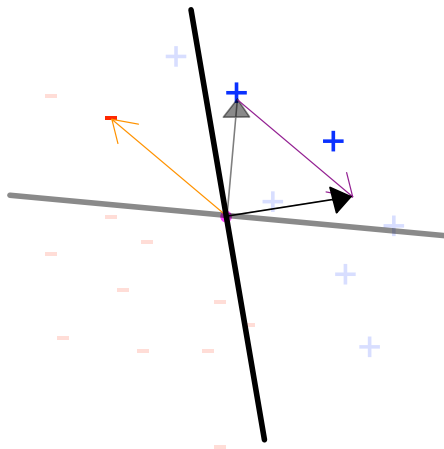
Example trace for the perceptron algorithm



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Example trace for the perceptron algorithm



Bound on number of mistakes

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- ▶ Proof by combining upper and lower bounds on $\|\vec{W}\|$.

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If $(\vec{W}_i \cdot X_i)y < 0$ then

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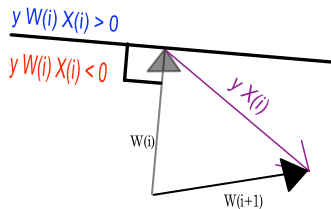
If $(\vec{W}_i \cdot \vec{X}_i)y < 0$ then

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- ▶ Induction step (assume for i and prove for $i + 1$):
$$\begin{aligned}\|\vec{W}_{i+1}\|^2 &\leq \|\vec{W}_i\|^2 + \|\vec{X}_i\|^2 \\ &\leq \|\vec{W}_i\|^2 + R^2 \leq (i + 1)R^2\end{aligned}$$

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$$\begin{aligned}\vec{W}_{i+1} \cdot \vec{V} &= (\vec{W}_i + \vec{X}_i y_i) \cdot \vec{V} = \vec{W}_i \cdot \vec{V} + y_i \vec{X}_i \cdot \vec{V} \\ &\geq ig + g = (i + 1)g\end{aligned}$$

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Thus:

$$i \leq \left(\frac{R}{g}\right)^2$$

Estimating the bias of a coin

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- ▶ Natural Answer:

$$\frac{\#H}{n} = \frac{4}{9}$$

What if the estimation has to be done online?

- ▶ We observe a bf sequence of coin flips, and have to predict the probability of **H**ead at each step:

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Krichevsky and Trofimov, 1981

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- ▶ What does “better” mean?

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- ▶ See you on Thursday!