Biased coins and the Context algorithm

Yoav Freund

January 15, 2014

Outline

Combining experts in the log loss framework

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- $\lceil L_A{}^T \rceil$ is the code length if A is combined with arithmetic coding.

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- **Goal:** minimize regret:

$$-\sum_{t=1}^{T} \log p_A^t(c^t) + \min_{i=1,\ldots,N} \left(-\sum_{t=1}^{T} \log p_i^t(c^t) \right)$$

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- Dividing by T we get $\frac{L_A^T}{T} = \min_i \frac{L_i^T}{T} + \frac{\log N}{T}$

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- More generally, the regret is smaller if many of the experts perform well.

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  - For number of mistakes - Bayesian method cannot be “fixed”. Requires variable learning rate.

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- $V(\vec{b}, \vec{X}, t)$ is computable (recursively enumerable).

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- Run the Bayes algorithm over “all” prediction algorithms.

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- Run the Bayes algorithm over "all" prediction algorithms.
- technical details: On iteration t, $|\vec{X}| = t$. Use the predictions of programs $\vec{b}$ such that $|\vec{b}| \leq t$ and for which $V(\vec{b}, \vec{X}, 2^t) = 1$. Assing the remaining mass the prediction $1/2$ (insuring a loss of 1)

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- We don't pay a penalty for copies.
- More generally, the regret is smaller if many of the experts perform well.

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- Can we still get a meaningful bound?

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- We need a new lower bound on the final total weight

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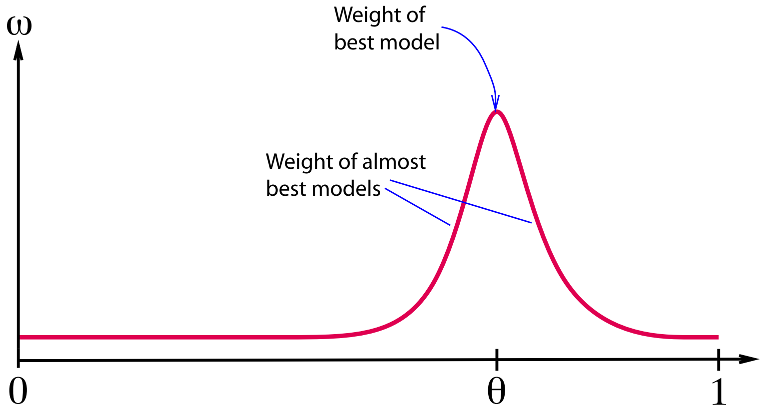

Expanding the exponent around the peak

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Laplace approximation (idea)

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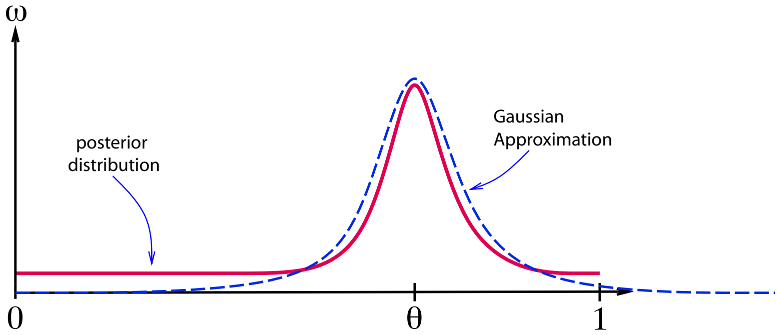

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Choosing the optimal prior

- Choose $w(\theta)$ to maximize the worst-case final total weight

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- Make bound equal for all $\hat{\theta} \in [0,1]$ by choosing

$$w^*(\hat{\theta}) = \frac{1}{Z}\sqrt{\frac{\left.\frac{d^2}{d\theta^2}\right|_{\theta=\hat{\theta}} (g(\hat{\theta},\theta) - g(\hat{\theta},\hat{\theta}))}{-2\pi}},$$

where Z is the normalization factor:

$$Z = \sqrt{\frac{1}{2\pi}} \int_0^1 \sqrt{\left.\frac{d^2}{d\theta^2}\right|_{\theta=\hat{\theta}} (g(\hat{\theta},\hat{\theta}) - g(\hat{\theta},\theta))} \; d\hat{\theta}$$

The bound for the optimal prior

- Plugging in we get

$$
\begin{aligned}
L_A - L_{\min} &\leq \ln \int_0^1 w^*(\theta) e^{T(g(\hat{\theta},\theta) - g(\hat{\theta},\hat{\theta}))} d\theta \\
&= \ln \left(\sqrt{\frac{2\pi Z}{T}} + O(T^{-3/2}) \right) \\
&= \frac{1}{2} \ln \frac{T}{2\pi} - \frac{1}{2} \ln Z + O(1/T) \, .
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Solving for log-loss

- The exponent in the integral is

$$g(\hat{\theta},\theta) - g(\hat{\theta},\hat{\theta}) = \hat{\theta} \ln \frac{\hat{\theta}}{\theta} + (1-\hat{\theta}) \ln \frac{1-\hat{\theta}}{1-\theta} = D_{KL}(\hat{\theta}||\theta)$$

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- The optimal prior:

$$w^*(\hat{\theta}) = \frac{1}{\pi\sqrt{\hat{\theta}(1-\hat{\theta})}}$$

  Known in general as Jeffrey's prior. And, in this case, the Dirichlet-$(1/2, 1/2)$ prior.

The cumulative log loss of Bayes using Jeffrey's prior

- $$L_A - L_{\min} \leq \frac{1}{2}\ln(T+1) + \frac{1}{2}\ln\frac{\pi}{2} + O(1/T)$$

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- This is called the Trichevsky Trofimov prediction rule.

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- Suffers larger regret when $\hat{\theta}$ is far from $1/2$

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- The constant C is optimal.

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For all $\alpha \leq 5/2$.

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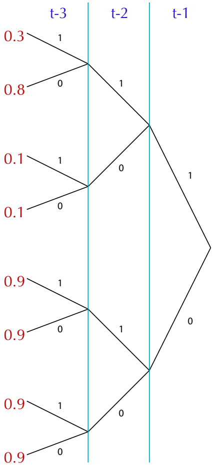

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- Prediction (using Kritchevski Trofimov)

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Learning a markov distribution

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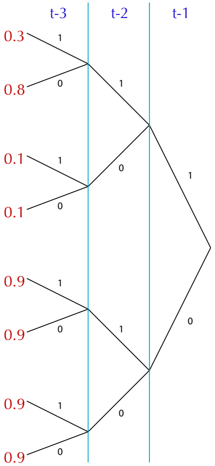
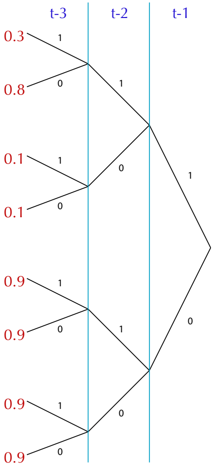
- Total regret is at most $2^{k-1} \log T$

How variable length markov can reduce regret

How variable length markov can reduce regret



How variable length markov can reduce regret



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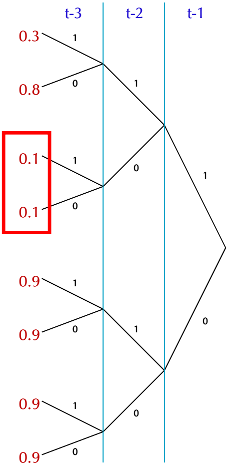

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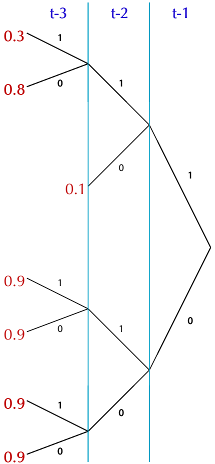

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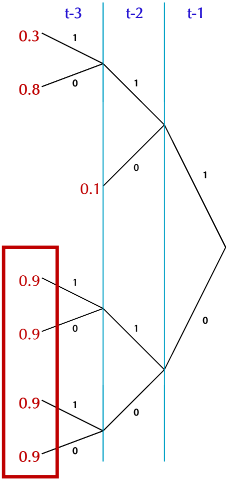

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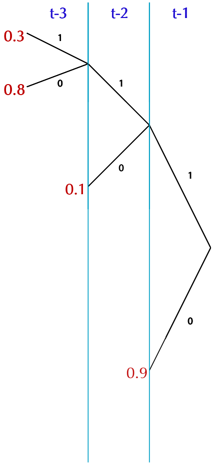

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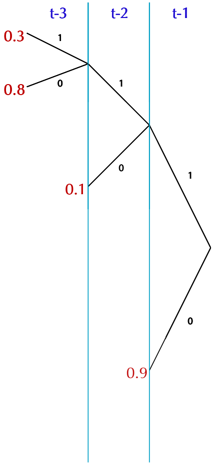
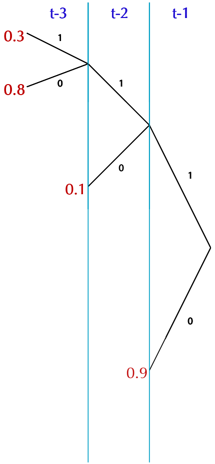


- Reducing number of leaves from 8 to 4 means

How variable length markov can reduce regret



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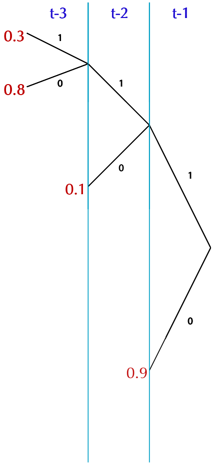

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- English example:
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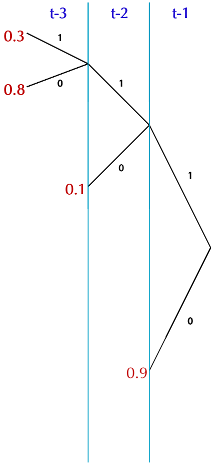

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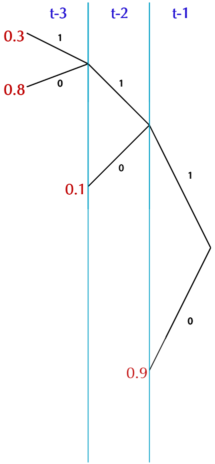

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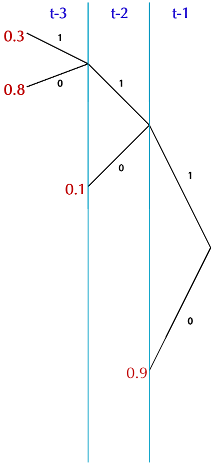

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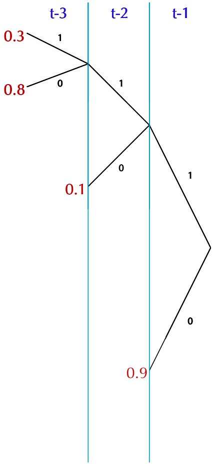

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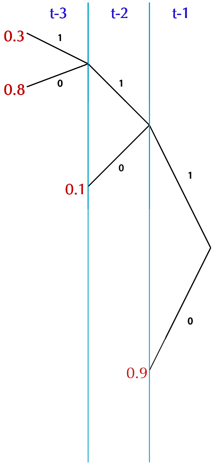

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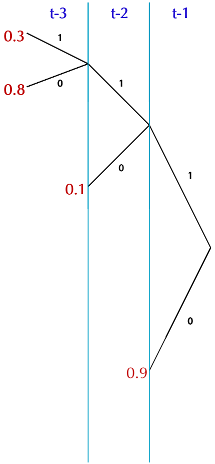

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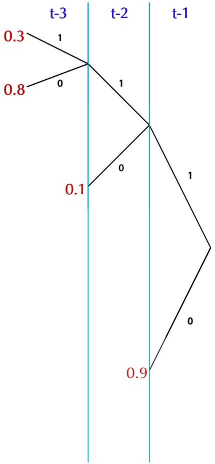

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- When we have little data, we can get better prediction even if the children are not Exactly the same

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- A node with 1 child means that some past histories are not covered.
- A variable length markov model corresponds to a prefix tree.
- But we don't know which prefix tree to use!

Assigning probabilities to complete sequences

- Using the chain rule, we can use a prediction rule to assign probabilities to a complete sequence.

$$P(x_1 = y_1, \ldots, x_T = y_T) = p(x_1 = y_1)p(x_2 = y_2 | x_1 = y_1) \ldots$$

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- Will come in handy soon!

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- Requires maintaining $O(2^l)$ weights!

Efficient implementation

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- Second idea: Compute the average over the prior efficiently.

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- Subset corresponding to node is contained in subset corresponding to node's parent.

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- Average probability assigned by the complete tree is P_w^λ where λ is the root node.

Context-Tree Weighting and Maximizing: Processing Betas

Frans Willems, Tjalling Tjalkens, and Tanya Ignatenko,
Eindhoven University of Technology,
Eindhoven, The Netherlands
Switch to second set of slides.