The **Hedge**(η)Algorithm

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Outline

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Hedge(η)Algorithm

- Hedging vs. Halving

Bound on total loss

- Upper bound on $\sum_{i=1}^{N} w_i^{T+1}$
- Lower bound on $\sum_{i=1}^{N} w_i^{T+1}$
- Combining Upper and Lower bounds

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Lower Bounds

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- Prediction is a **game** played between algorithm and nature, in which the goal of the algorithm is to minimize **regret**.

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  3. **Loss** of algorithm is expected loss wrt to chosen distribution.

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- Basic idea - reduce probability of lossy actions, but not all the way to zero.
- **Modified Goal:** minimize **Regret=**difference between expected total loss
  and
  minimal total loss of repeating one action.

$$\sum_{t=1}^{T} \mathbf{p}^t \cdot \ell^t - \min_i \left(\sum_{t=1}^{T} \ell_i^t \right)$$

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- Cumulative loss of action i is $L_i^t = \sum_{t=1}^T \ell_i^t$
- Our goal is to minimize the **regret:** $L_A^t - \min_i L_i^t$ for all t and all possible loss sequences.

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- Theorem (main theorem)
  For any sequence of loss vectors $\ell^1, \ldots, \ell^T$, *and for any* $i \in \{1, \ldots, N\}$, *we have*

$$L_{\textbf{Hedge}(\eta)} \leq \frac{\ln(N) + \eta L_i}{1 - e^{-\eta}}.$$

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  - Proof: by combining upper and lower bounds on $\sum_{i=1}^{N} w_i^{T+1}$

Upper bound on $\sum_{i=1}^{N} w_i^{T+1}$

Lemma (upper bound)

For any sequence of loss vectors $\ell^1, \ldots, \ell^T$ *we have*

$$\ln \left(\sum_{i=1}^{N} w_i^{T+1} \right) \leq -(1 - e^{-\eta}) L_{\textbf{Hedge}(\eta)}.$$

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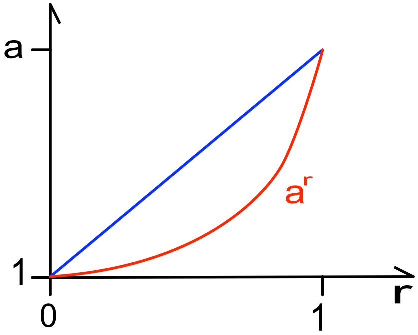

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Proof of upper bound (slide 2)

Applying $a^r \leq 1 - (1-a)^r$ where $a = e^{-\eta}, r = \ell_i^t$

$$\sum_{i=1}^{N} w_i^{t+1} = \sum_{i=1}^{N} w_i^t e^{-\eta \ell_i^t}$$

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$$\begin{aligned}\sum_{i=1}^{N} w_i^{t+1} &= \sum_{i=1}^{N} w_i^t e^{-\eta \ell_i^t} \\ &\leq \sum_{i=1}^{N} w_i^t \left(1-(1-e^{-\eta})\ell_i^t\right)\end{aligned}$$

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&= \left(\sum_{i=1}^N w_i^t\right)\left(1-(1-e^{-\eta})\frac{\mathbf{w}^t}{\sum_{i=1}^N w_i^t}\cdot \ell^t\right)
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&= \left(\sum_{i=1}^N w_i^t\right) \left(1 - (1 - e^{-\eta}) \mathbf{p}^t \cdot \ell^t\right)
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- for $t = 1, \ldots, T$

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$$\begin{aligned}\sum_{i=1}^N w_i^{T+1} &\leq N\prod_{t=1}^T(1-(1-e^{-\eta})\mathbf{p}^t\cdot\ell^t) \\ &\leq \exp\left(\ln N-(1-e^{-\eta})\sum_{t=1}^T\mathbf{p}^t\cdot\ell^t\right)\end{aligned}$$

since $1+x\leq e^x$ for $x = -(1-e^{-\eta})$.

Lower bound on $\sum_{i=1}^{N} w_i^{T+1}$

For any $j = 1, \ldots, N$:

$$\sum_{i=1}^{N} w_i^{T+1} \geq w_j^{T+1} = e^{-\eta L_j}$$

Combining Upper and Lower bounds

- Combining bounds on $\ln\left(\sum_{i=1}^{N} w_i^{T+1}\right)$

$$-\eta L_j \le \ln \sum_{i=1}^{N} w_i^{T+1} \le \ln N - (1 - e^{-\eta}) \sum_{t=1}^{T} \mathbf{p}^t \cdot \ell^t$$

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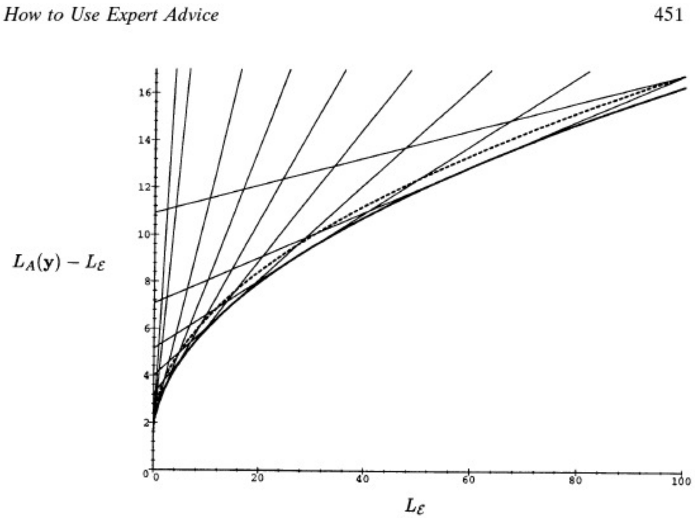
- Reversing signs, using $L_{\mathbf{Hedge}(\eta)} = \sum_{t=1}^{T} \mathbf{p}^t \cdot \ell^t$ and reorganizing we get

$$L_{\mathbf{Hedge}(\eta)} \leq \frac{\ln N + \eta L_i}{1 - e^{-\eta}}$$

Tuning η


How to Use Expert Advice
451
$L_A(\mathbf{y}) - L_\mathcal{E}$
16
14
12
10
8
6
4
2
0
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- Then

$$L_{\textbf{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}} \leq \min_i L_i + \sqrt{2\tilde{L}\ln N} + \ln N$$

Tuning η as a function of T

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- per iteration we get:

$$\frac{L_{\textbf{Hedge}(\eta)}}{T} \leq \min_i \frac{L_i}{T} + \sqrt{\frac{2 \ln N}{T}} + \frac{\ln N}{T}$$

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- The adversarial strategy is random, extremely simple, and does not depend on the hedging strategy!

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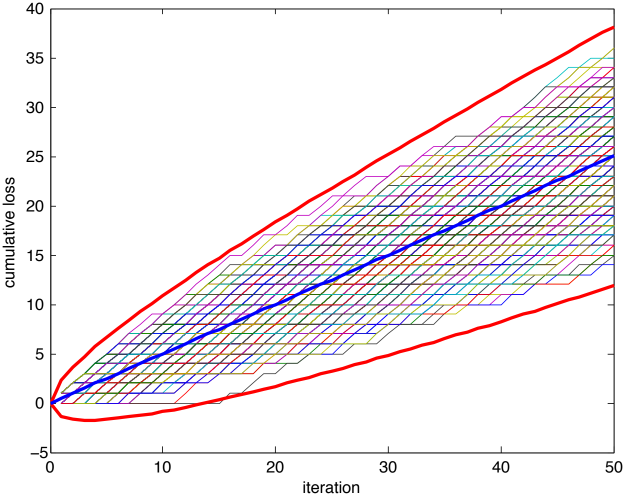
- On the other hand $\min_i L_i \approx T/2 - \sqrt{2T \ln N}$
- The difference $L_{\mathbf{S}} - \min_i L_i$ is due to unlearnable random fluctuations!

The adversarial strategy

- Adversary sets each loss ℓ_i^t indepedently at random to 0 or 1 with equal probabilities $(1/2, 1/2)$.
- Obviously, nothing to learn !
  $L_{\mathbf{S}} \approx T/2$.
- On the other hand $\min_i L_i \approx T/2 - \sqrt{2T \ln N}$
- The difference $L_{\mathbf{S}} - \min_i L_i$ is due to unlearnable random fluctuations!
- Detailed proof in "Adaptive Game playing using multiplicative weights" Freund and Schapire.

The adversarial construction



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- Given learning rate η the **Hedge**(η)algorithm satisfies

$$L_{\textbf{Hedge}(\eta)} \leq \frac{\ln N + \eta L_i}{1 - e^{-\eta}}$$

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- A trivial random data, for which there is nothing to be learned forces any algorithm to suffer this total regret.