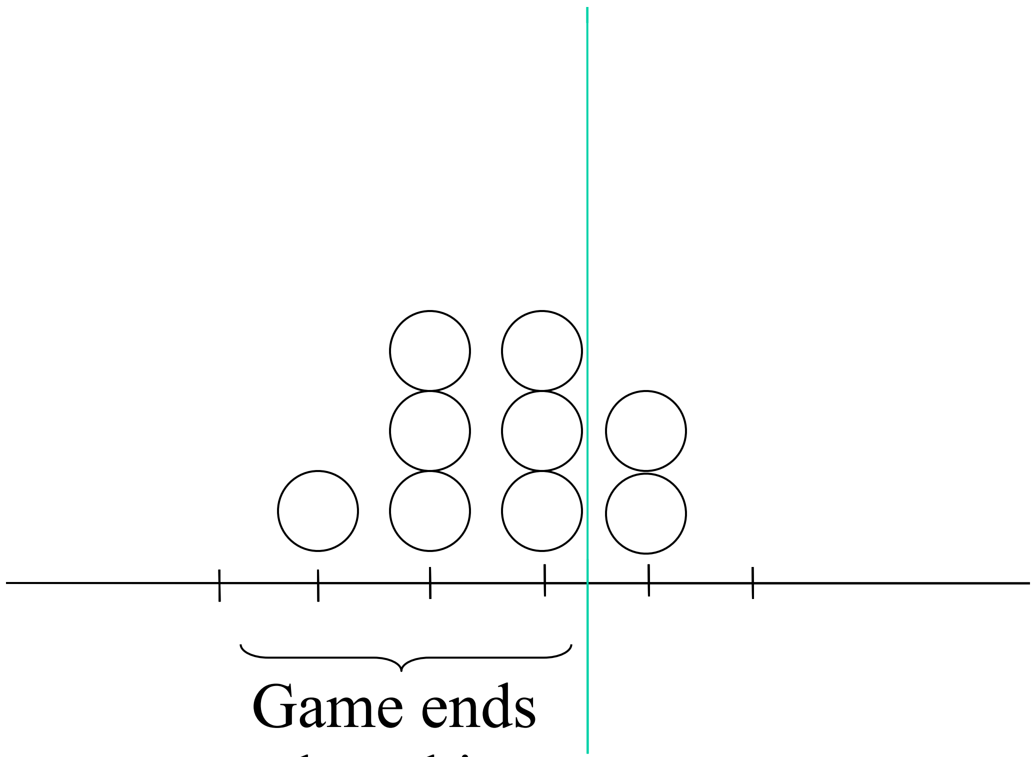
Chip games
Online Learning, Boosting
and
why continuous is easier than discrete

Yoav Freund

UCSD

The Chip game



Plan of the talk

- Why is the chip game is interesting?
- Letting the number of chips go to infinity.
- The boosting game.
- Drifting games (let chip=sheep).
- Letting the number of rounds go to infinity
- Brownian motion and generalized boosting (nice pictures!)

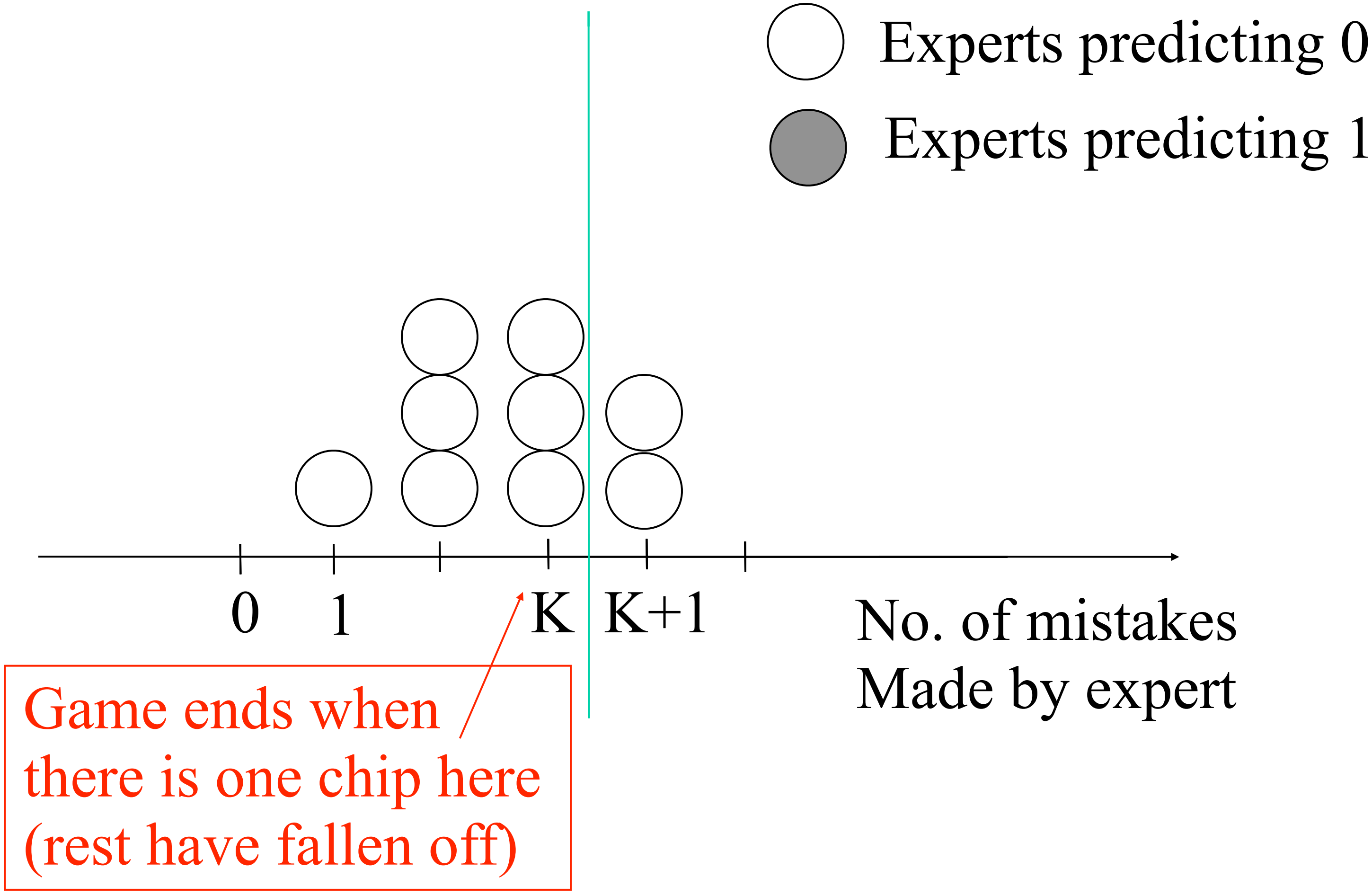
Combining expert advice

- Binary sequence: 1,0,0,1,1,0,?
- N experts make predictions:

expert 1:	1,0,1,0,1,1,1
expert 2:	0,0,0,1,1,0,1
…	
expert N:	1,0,0,1,1,1,0

- Algorithm makes prediction: 1,0,1,1,0,1,1
- Assumption: there exists an expert which makes at most k mistakes
- Goal: make least mistakes under the assumption (no statistics!)
- Chip = expert, bin = number of mistakes made
- Game step = algorithm's prediction is incorrect.

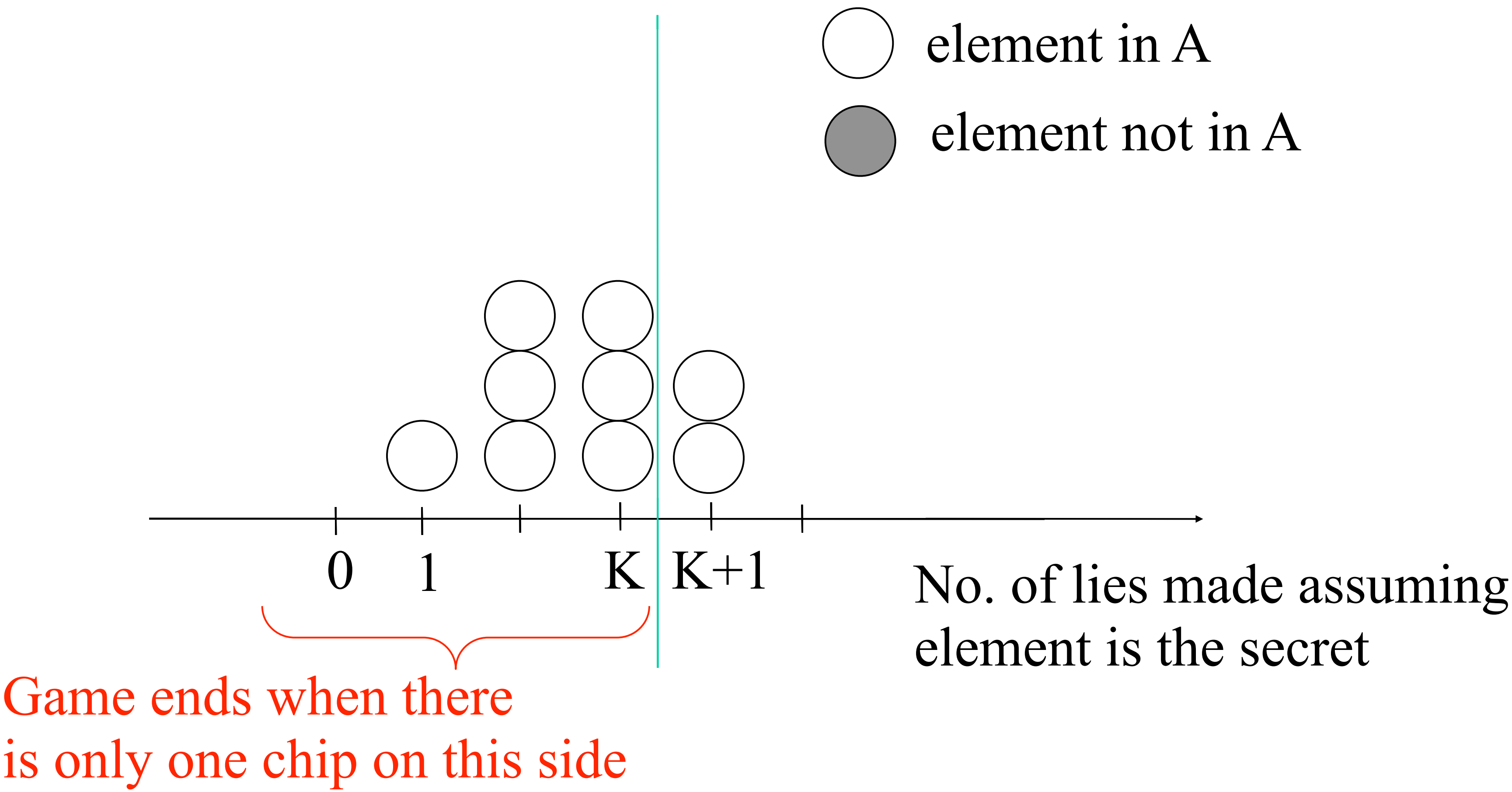
Chip game for expert advice



20 questions with k lies

- Player 1 chooses secret x from S1,S2,…,SN
- Player 2 asks "is x in the set A?"
- Player 1 can lie k times.
- Game ends when player 2 can determine x.
- A smart player 1 aims to have more than one consistent x for as long as possible.
- Chip = element (Si)
- Bin = number of lies made regarding element

Chip game for 20 questions

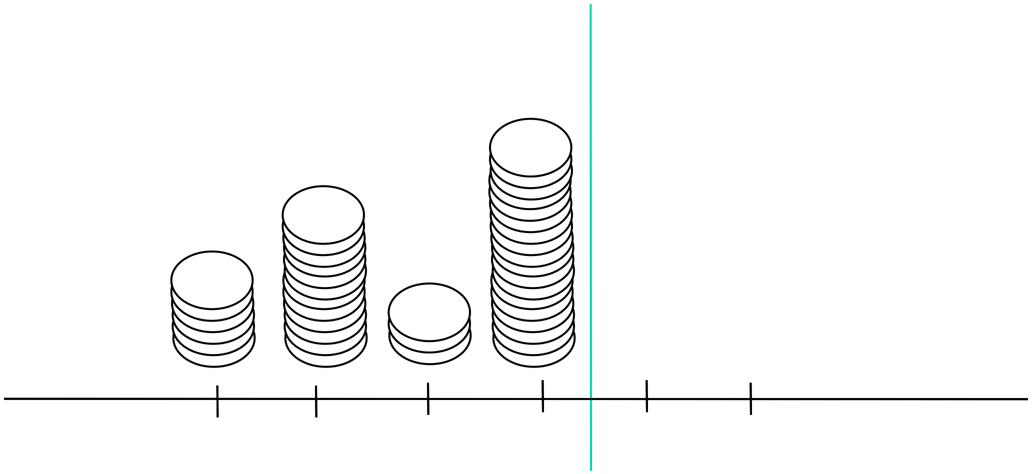


Simple case – all chips in bin 1

- 21 questions without lies
- Combining experts where one is perfect
- Splitting a cookie
- Optimal strategies:
  - Player 1: split chips into two equal parts
  - Player 2: choose larger part
- Note problem when number of chips is odd

Number of chips to infinity

Replace individual chips by chip mass



Optimal splitter strategy:
Split each bin into two equal parts

Binomial weights strategies

- What should chooser do if parts are not equal?
- Assume that on following iterations splitter will play optimally = split each bin into 2 equal parts.
- Future configuration independent of chooser's choice
- Potential: Fraction of bin i that will remain in bins 1..k when m iterations remain $\binom{m}{\leq k-i} \doteq \frac{1}{2^m}\sum_{j=0}^{k-i}\binom{m}{j}$
- Choose part so that next configuration will have maximal (or minimal) potential.
- Solve for m: $m := \max\left\{q \in \mathbb{N} : \sum_{E\in\mathcal{E}} \binom{q}{\leq k-i_E} > 1\right\}$

Optimality of strategy

- If the chips are infinitely divisible then the solution is min/max optimal
- [Spencer95] Enough for optimality if number of chips is at least

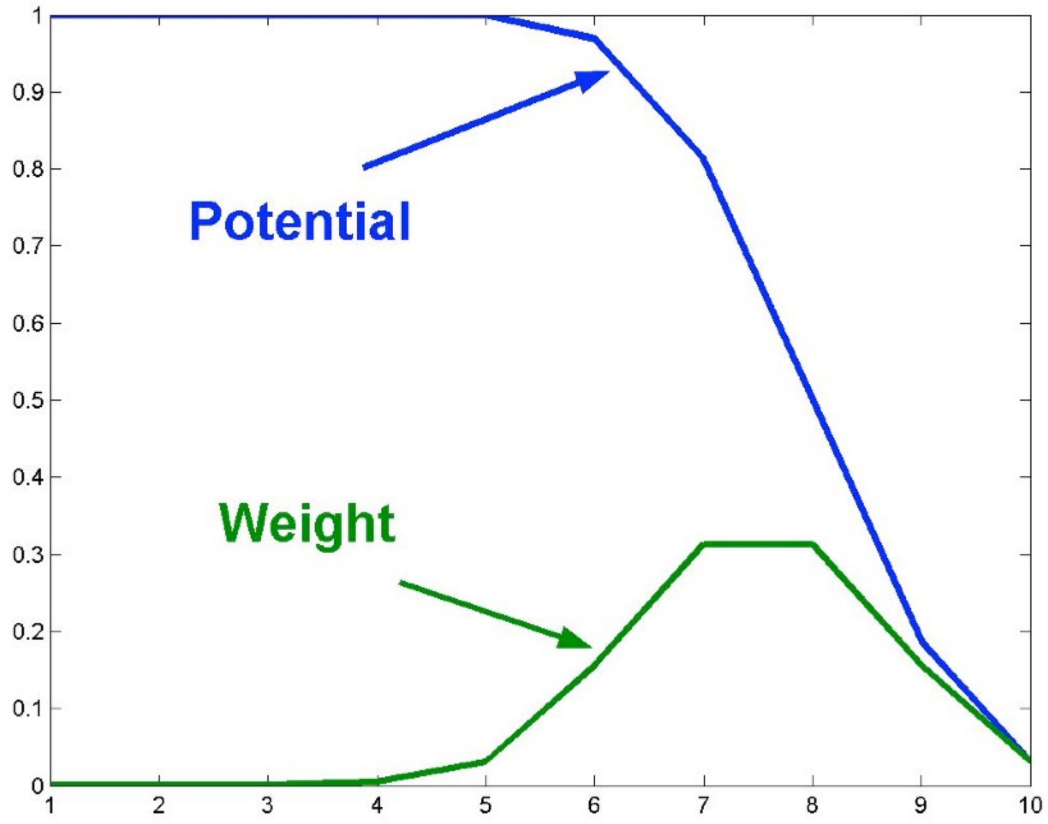
$$\Omega\left(2^{2^k}\right)$$

Equivalence to a random walk

- Both sides playing optimally is equivalent to each chip performing an independent random walk.
- Potential = probability of a chip in bin i ending in bins $1..K$ after m iteration
- Weight = difference between the potentials of a chip in its two possible locations on the following iteration.
- Chooser's optimal strategy: choose set with smaller (larger) weight

Example potential and weight

m=5; k=10



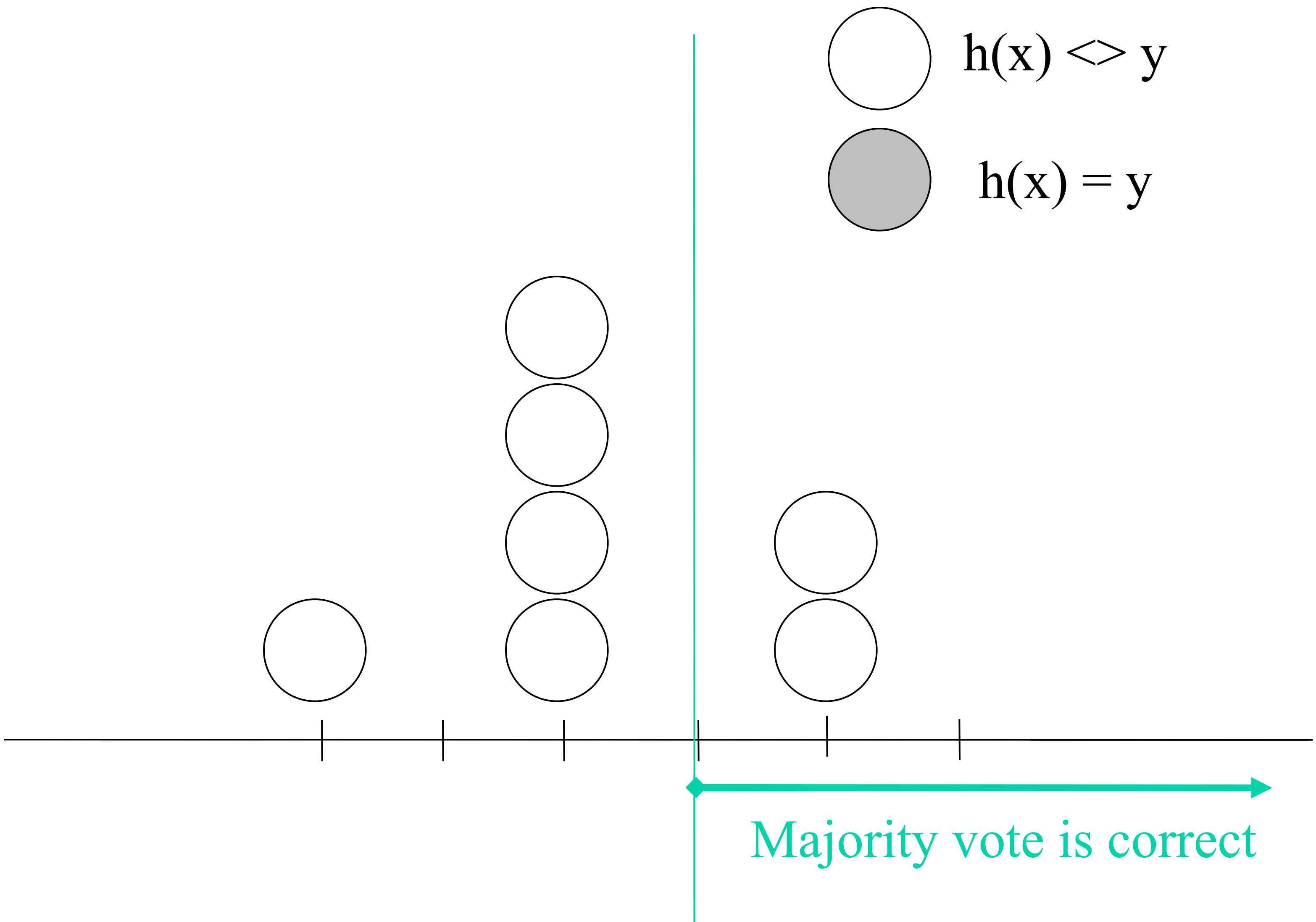
Boosting

- A method for improving classifier accuracy
- Weak Learner: a learning algorithm generating rules slightly better than random guessing.
- Basic idea: re-weight training examples to force weak learner into different parts of the space.
- Combine weak rules by a majority vote.

Boosting as a chip game

- Chip = training example
- Booster assigns a weight to each chip, weights sum to 1.
- Learner selects a subset with weight $\geq \frac{1}{2} + \gamma$
  - Selected set moves a step right (correct)
  - Unselected set moves a step left (incorrect)
- Booster wins examples on right of origin.
  - Implies that majority vote is correct.

The boosting chip game



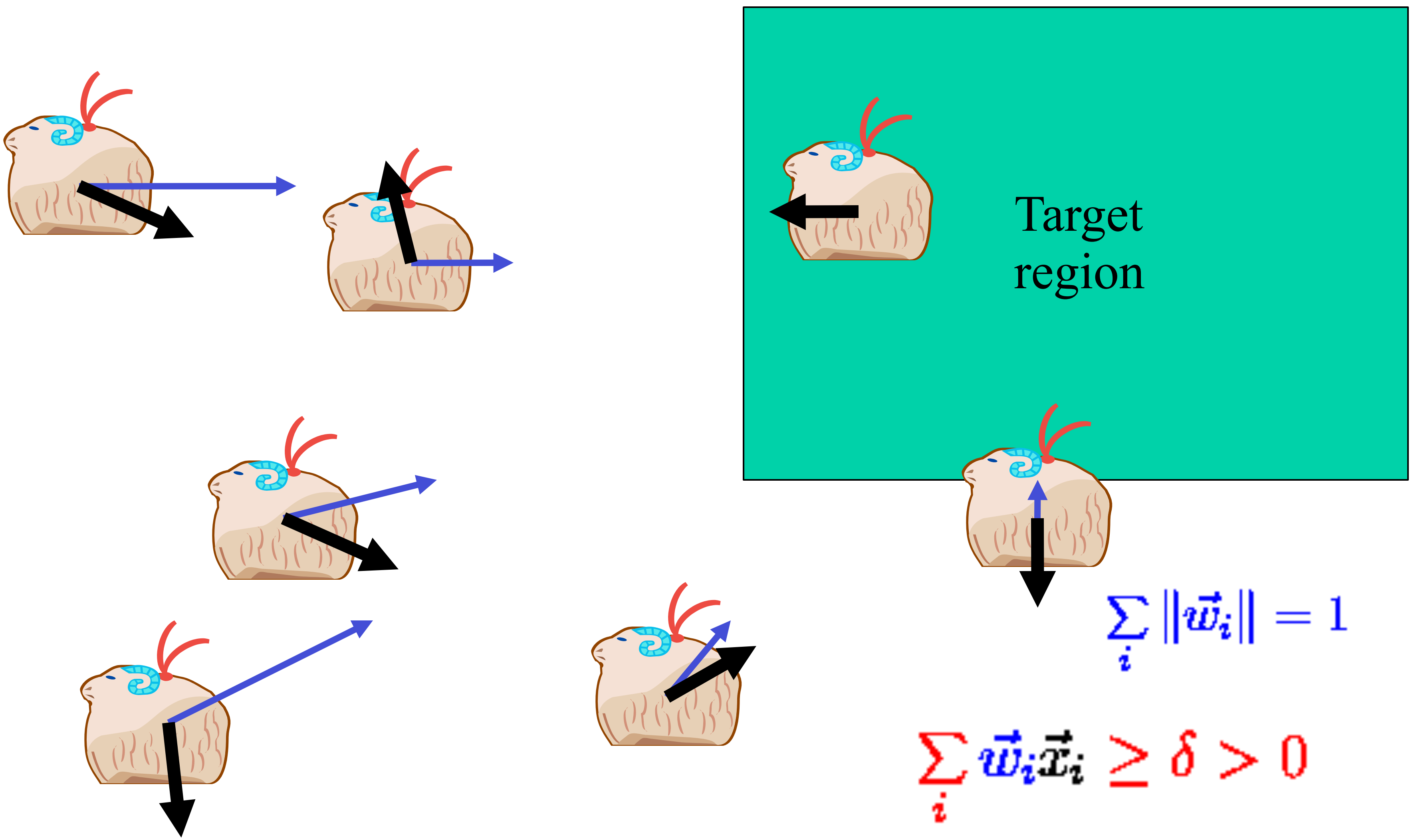
Binomial strategies for boosting

- Learner moves chip right independently with probability $\frac{1}{2}+\gamma$
- The potential of example that is correctly classified r times after i out of k iterations

$$\sum_{j=0}^{\lfloor\frac{k}{2}\rfloor - r} \binom{k-i}{j} \left(\frac{1}{2}+\gamma\right)^{j} \left(\frac{1}{2}-\gamma\right)^{k-i-j}$$

- Booster assigns to each example a weight proportional to the difference between its possible next-step potentials.

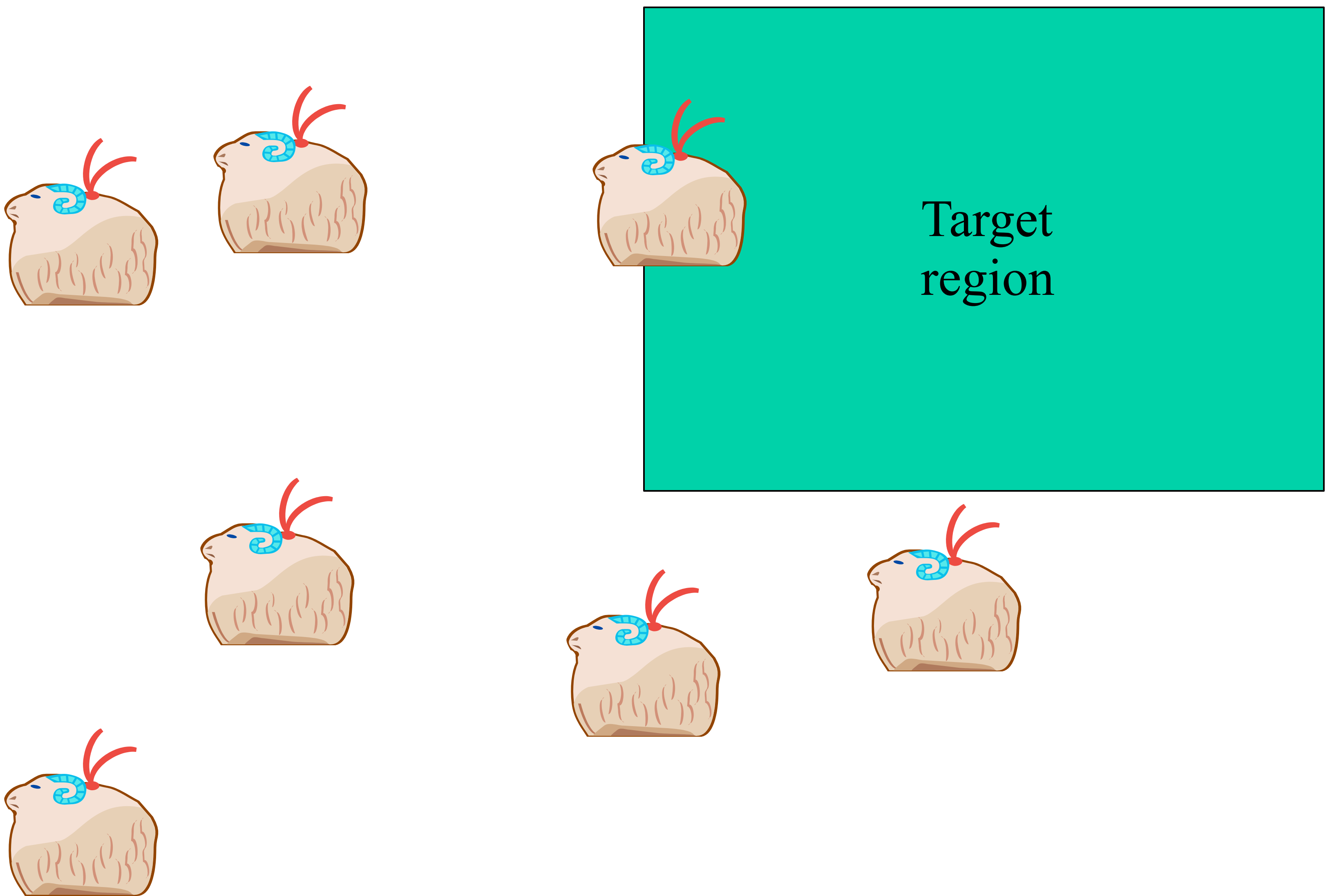
Drifting games (in 2d)



$$\sum_i \|\vec{w}_i\| = 1$$

$$\sum_i \vec{w}_i \vec{x}_i \geq \delta > 0$$

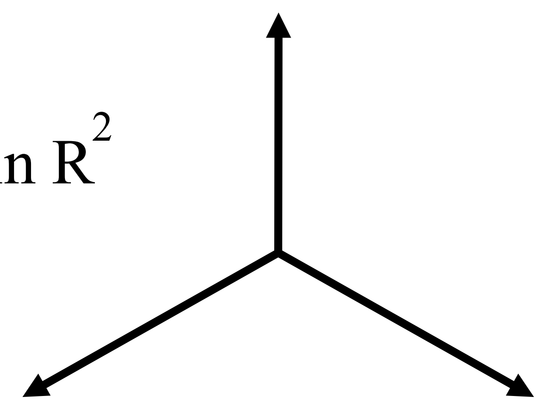
Drifting games (in 2d)



The allowable steps

B = the set of all allowable step
Normal B = minimal set that spans the space. (~basis)
Regular B = a symmetric regular set. (~orthonormal basis)

Regular step set in R^2



The min/max solution

[Schapire99]

- A potential defined by a min/max recursion

$$\phi_T(\mathbf{s}) = L(\mathbf{s})$$
$$\phi_{t-1}(\mathbf{s}) = \min_{\mathbf{w}\in\mathbb{R}^d} \sup_{\mathbf{z}\in B} (\phi_t(\mathbf{s}+\mathbf{z}) + \mathbf{w}\cdot\mathbf{z} - \delta\,\|\mathbf{w}\|)$$

$\phi_0(0) =$ the value of the game

- Shepherd´s strategy

$$\mathbf{w}_i^t = \arg\min_{\mathbf{w}} \sup_{\mathbf{z}\in B} (\phi_t(\mathbf{s}+\mathbf{z}) + \mathbf{w}\cdot\mathbf{z} - \delta\,\|\mathbf{w}\|)$$

The solution simplifies when $\delta \to 0$

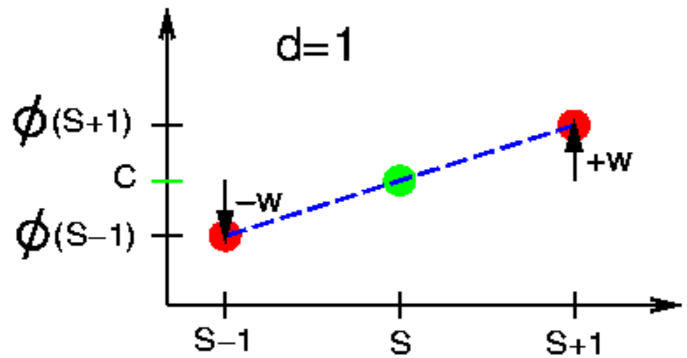
If B is normal, and δ is sufficiently small then $\exists \mathbf{w}^*$ such that

$$\phi_{t-1}(\mathbf{s}) = \phi_t(\mathbf{s} + \mathbf{z}) + \mathbf{w}^* \cdot \mathbf{z} - \delta \|\mathbf{w}^*\|$$

for all $\mathbf{z} \in B$ (and all $t = 1, 2 \ldots$, $\mathbf{s} \in \mathbb{R}^d$)

Implies that: $\mathbf{w}^*$ is the "local slope" at $\phi_t(\mathbf{s})$, i.e.

$$\phi_t(\mathbf{s} + \mathbf{z}_i) = C + \mathbf{w}^* \mathbf{z}_i \; ; \quad C \doteq \frac{\Sigma_{j=0}^{d} \phi_t(\mathbf{s} + \mathbf{z}_j)}{d+1}$$



and that

$$\phi_{t-1}(\mathbf{s}) = C - \delta \|\mathbf{w}^*\|$$

Increasing the number of steps

- Consider T steps in a unit time
- Drift δ should scale like 1/T
- Step size O(1/T) gives game to shepherd
- Step size $O\left(1/\sqrt{T}\right)$ keeps game balanced

The continuous time limit

For $t = 0,1,2,\ldots,T$ instantanous loss is $l_i^t \in \{-1,+1\}$

Instead of $t = 0,1,2,\ldots,T$ let $t = 0,\frac{1}{T},\frac{2}{T},\ldots,1$

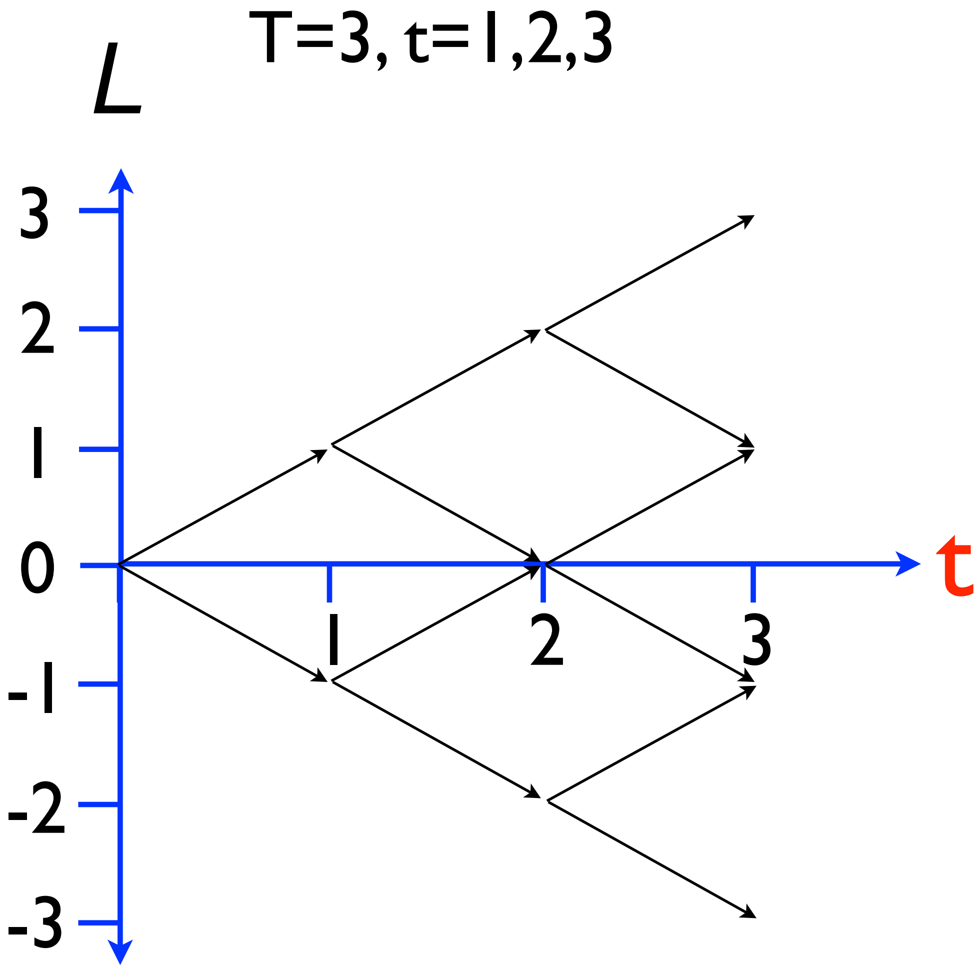
Let $T \to \infty$

How should we set the instantanous loss?

Natural choice: $l_i^t = \pm\frac{1}{T}$?

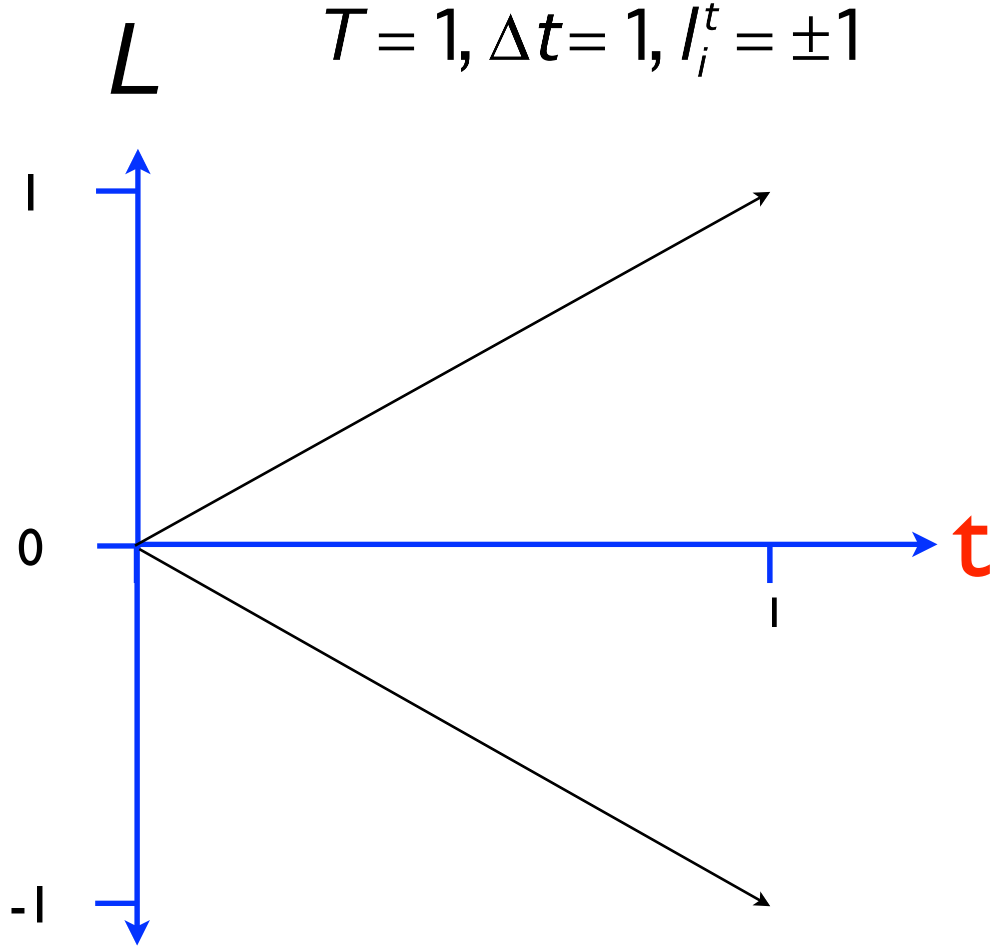
For optimal adversary, cumulative loss of each expert defines a random walk. What is random walk in continuous time?

The game lattice



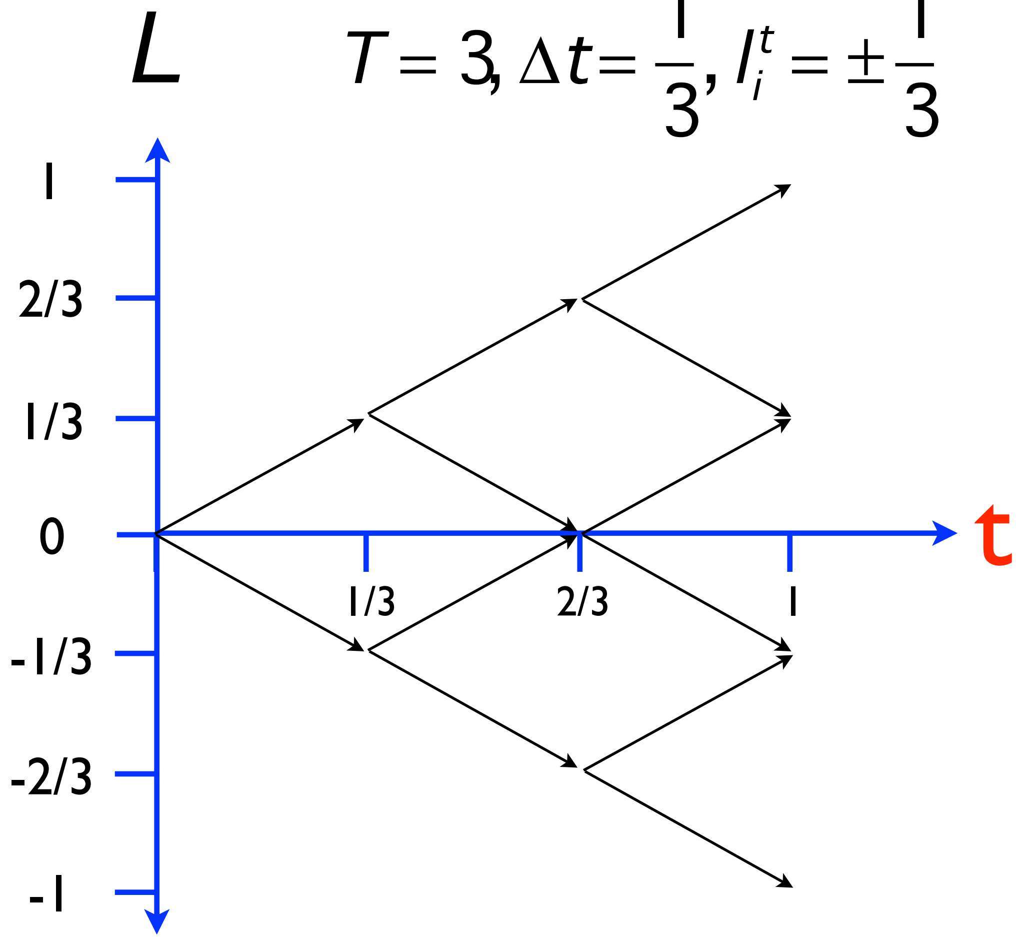
Using step $l_i^t = \pm\frac{1}{T}$

$T = 1, \Delta t = 1, l_i^t = \pm 1$

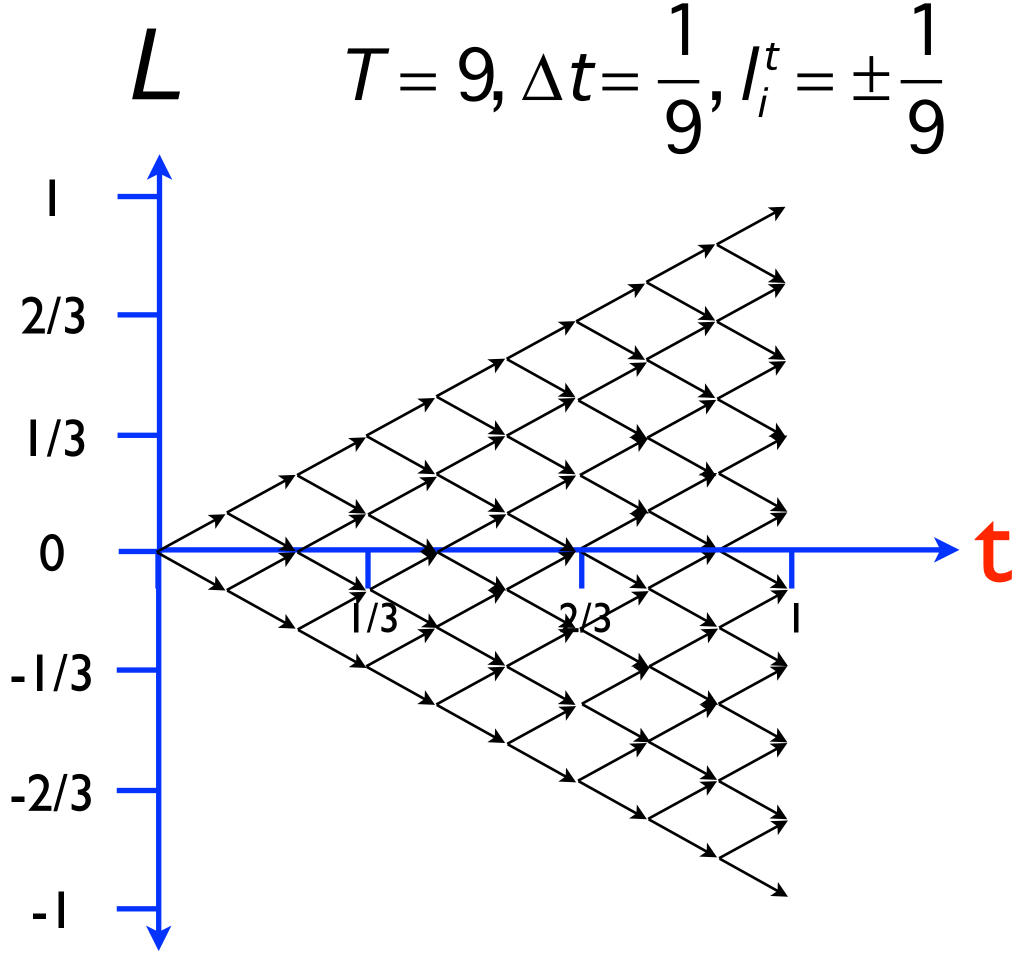


Using step $l_i^t = \pm\frac{1}{T}$

$$T = 3, \Delta t = \frac{1}{3}, l_i^t = \pm\frac{1}{3}$$



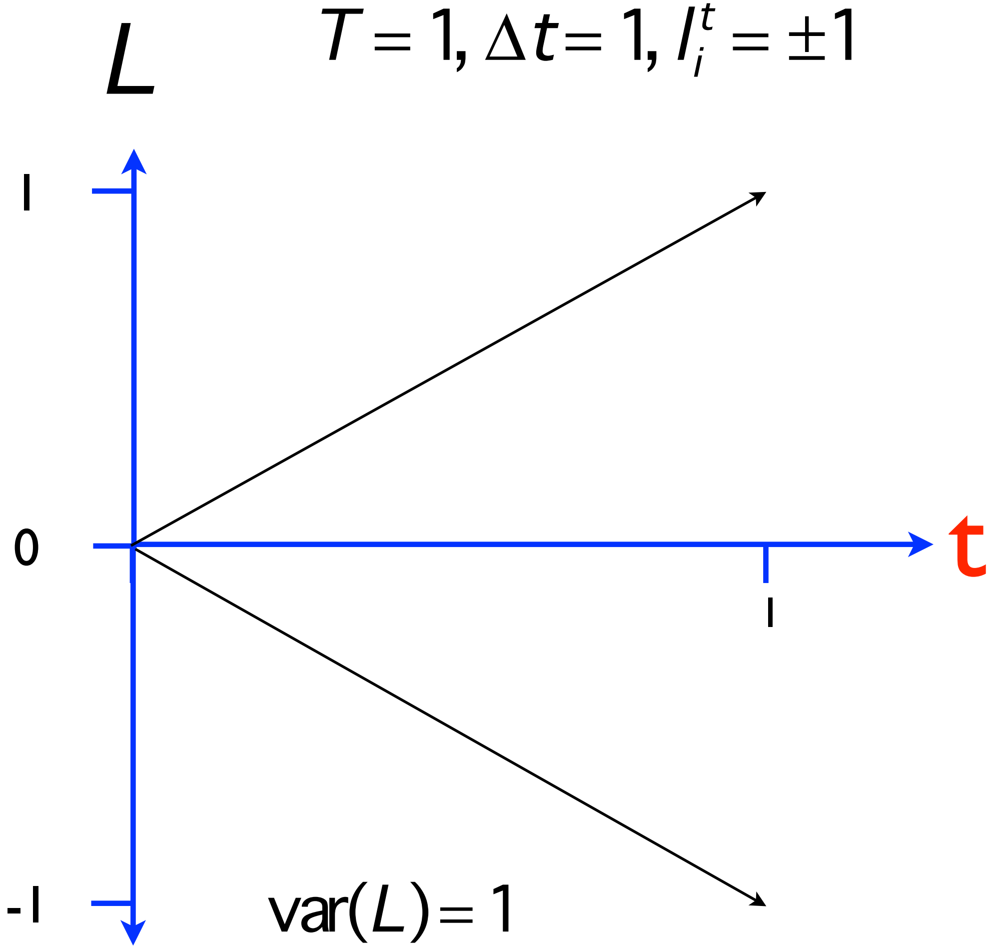
Using step $l_i^t = \pm\frac{1}{T}$



Looks fine but $\mathrm{var}(L) = T\frac{1}{T^2} = \frac{1}{T} \underset{T\to\infty}{\longrightarrow} 0$

Using step $l_i^t = \pm\frac{1}{\sqrt{T}}$

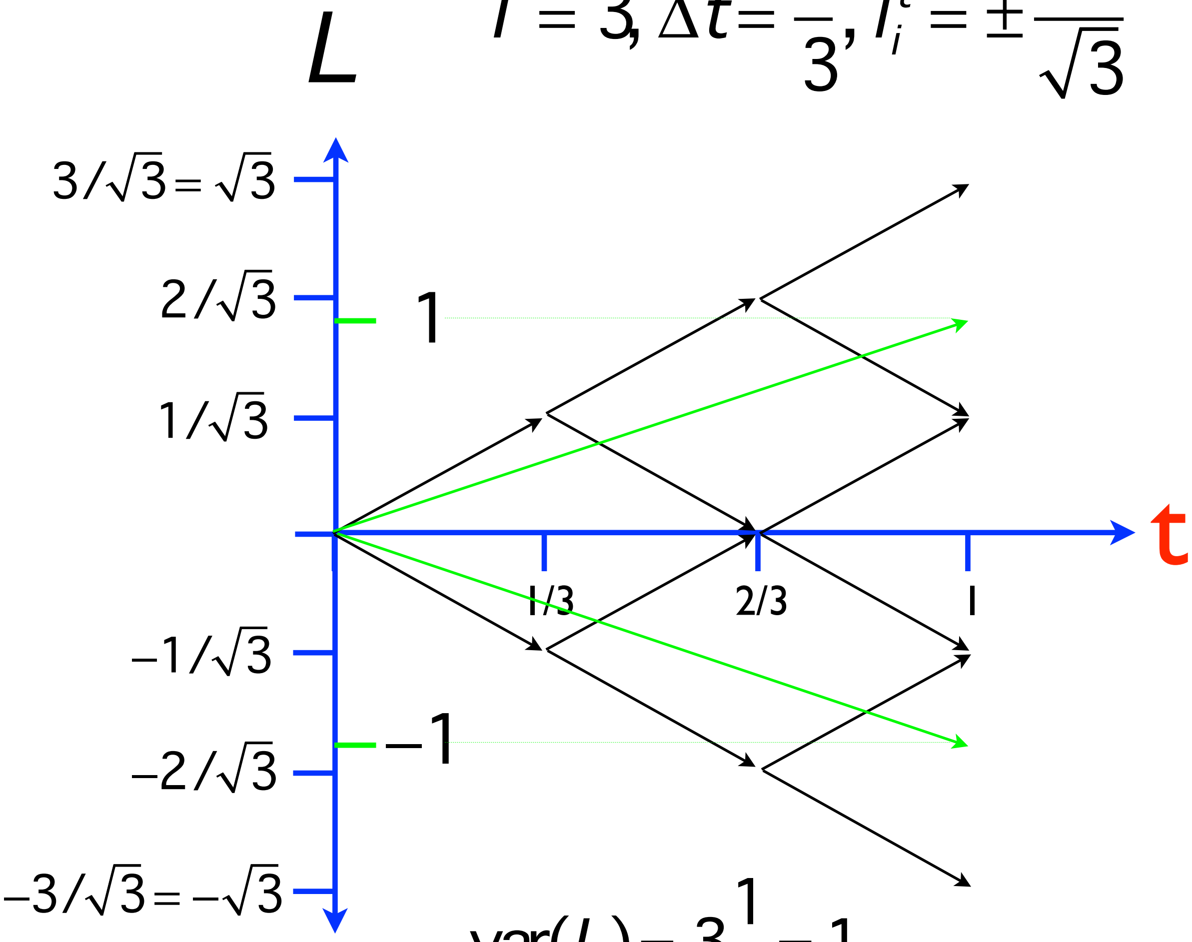
$T = 1, \Delta t = 1, l_i^t = \pm 1$



$\mathrm{var}(L) = 1$

Using step $l_i^t = \pm\frac{1}{\sqrt{T}}$

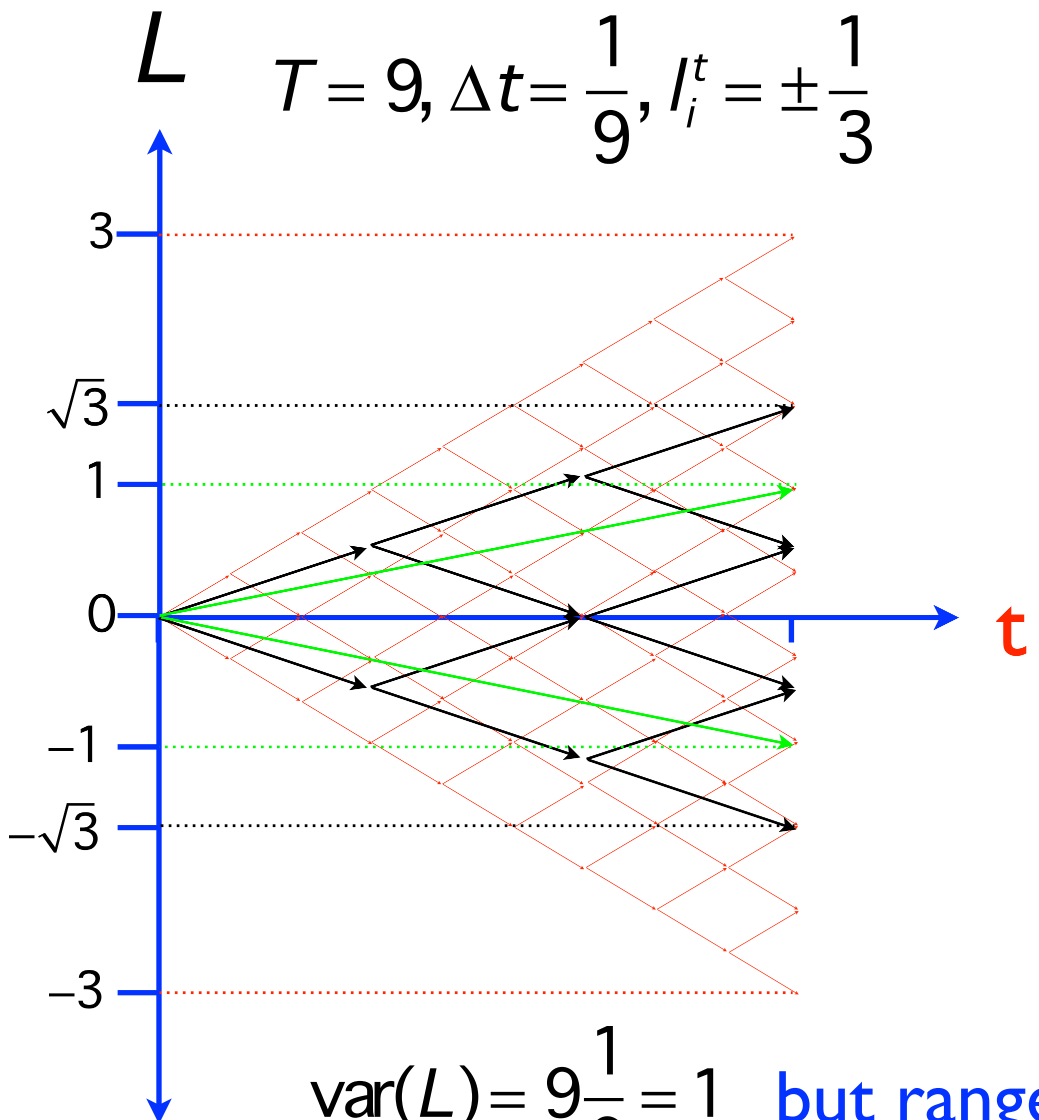
$$T = 3,\ \Delta t = \frac{1}{3},\ l_i^t = \pm\frac{1}{\sqrt{3}}$$



$$\mathrm{var}(L) = 3\frac{1}{3} = 1$$

Using step $l_i^t = \pm\frac{1}{\sqrt{T}}$

$T = 9, \Delta t = \frac{1}{9}, l_i^t = \pm\frac{1}{3}$



$\mathrm{var}(L) = 9\frac{1}{9} = 1$ but range(L) → ∞

The solution when $T \to \infty$

The local slope becomes the gradient

$$\mathbf{w}^* = \nabla \phi_\tau(\mathbf{s})$$

The recursion becomes a PDE

$$\begin{aligned}
\frac{\partial \phi_\tau(\mathbf{s})}{\partial \tau} &= -\frac{1}{2} \sum_{k=1}^{d} \frac{\partial^2 \phi_\tau(\mathbf{s})}{\partial^2 s_k} + \delta \left\| \mathbf{w}^* \right\| \\
&= -\frac{1}{2} \Delta \phi_\tau(\mathbf{s}) + \delta \left\| \nabla \phi_\tau(\mathbf{s}) \right\|
\end{aligned}$$

Same PDE describes time development of Brownian motion with drift

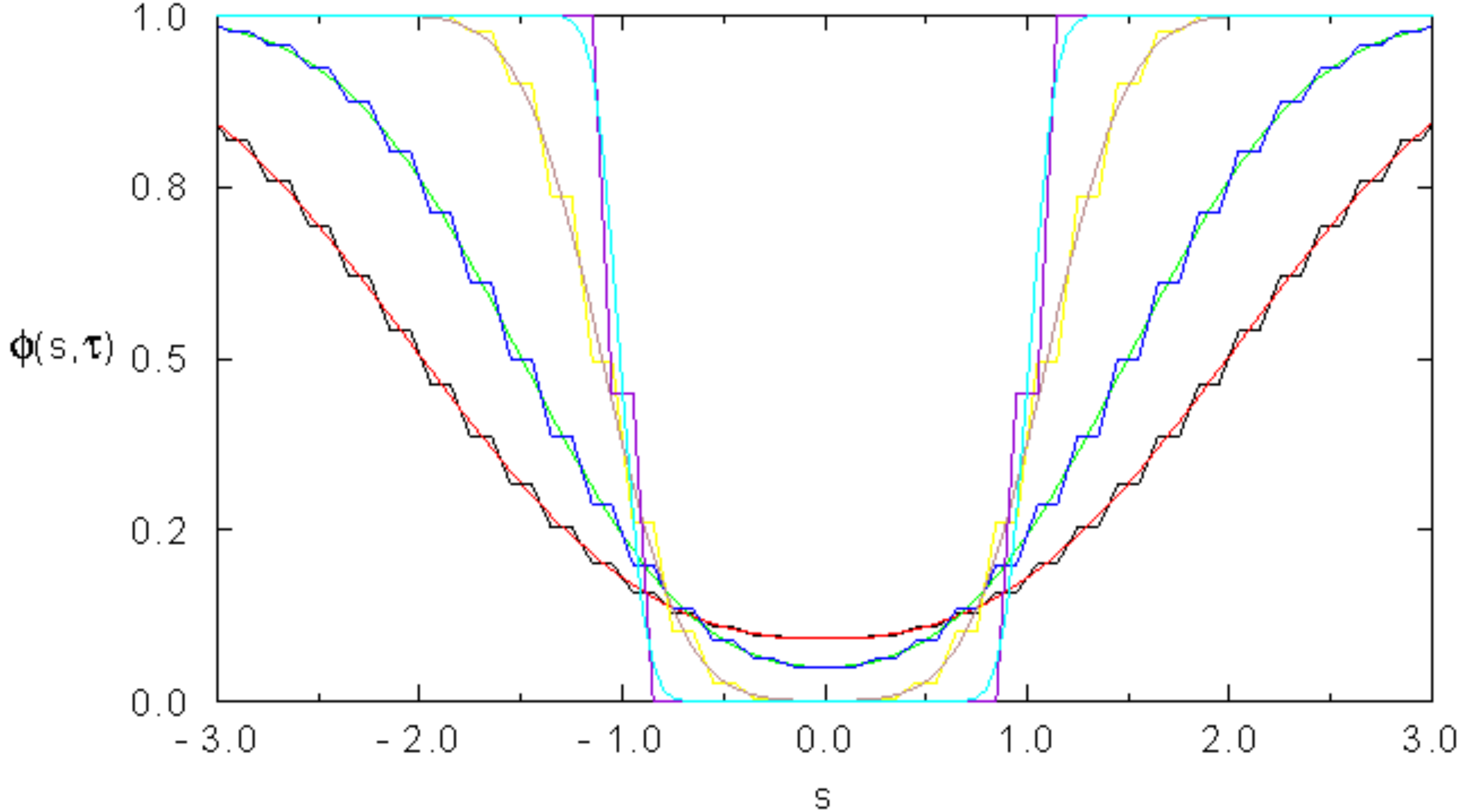
Special Cases

- Target func. = PDE boundary condition at time=1.
- Target func. = exponential
  - potential and weight are exponential at all times
  - Adaboost.
  - Exponential weights algs for online learning.
- Target func. = step function
  - Potential is the error function, weight is the normal distribution.
  - Potential and weight change with time.
  - Brownboost.

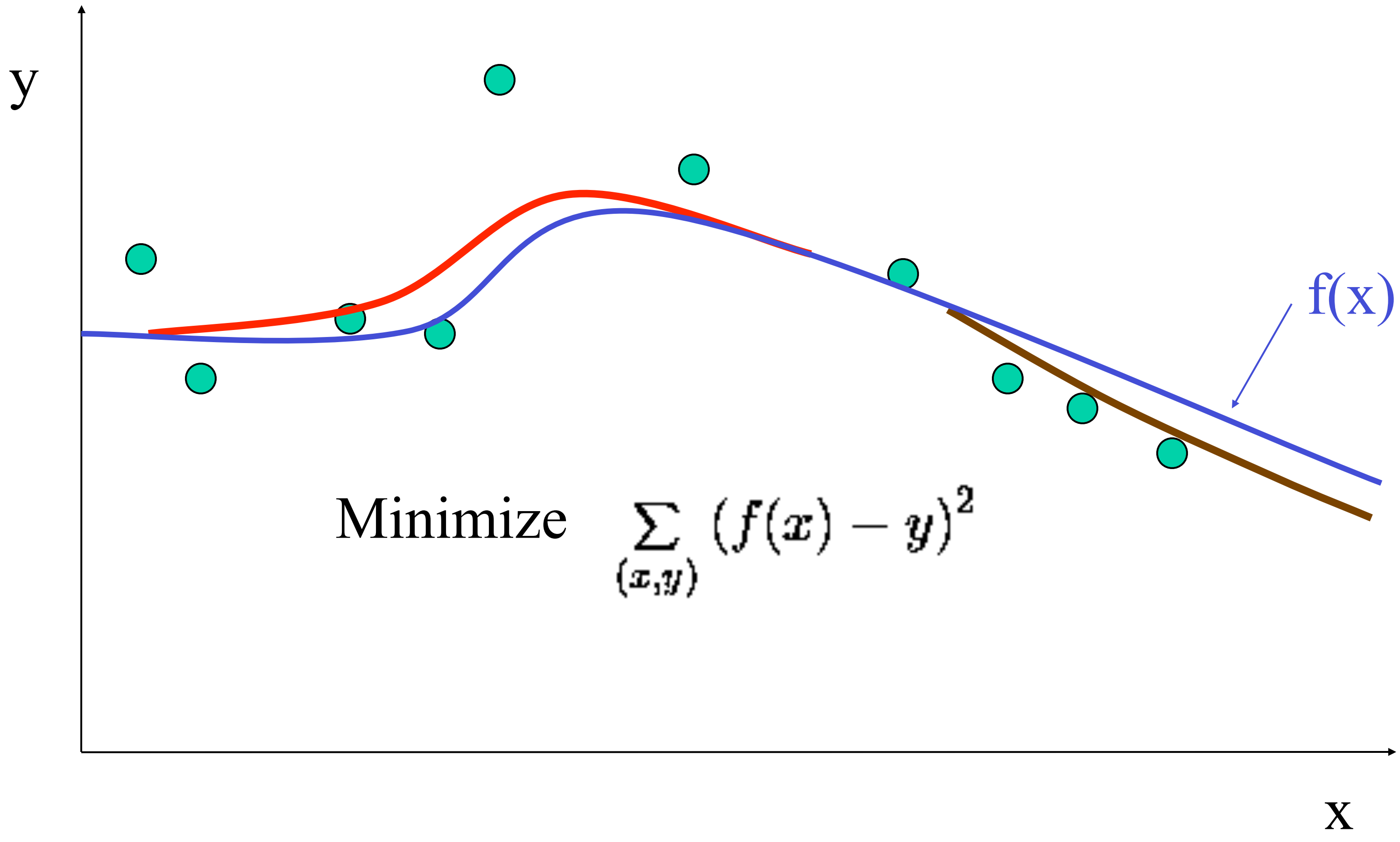
Potential applications

- Generalized boosting
  - Classification for >2 labels
  - Regression
  - Density estimation
  - Variational function fitting
- Generalized online learning
  - Continuous predictions (instead of binary)

solution for $L(s) = I_{|s|>1}$



Boosting for variational optimization



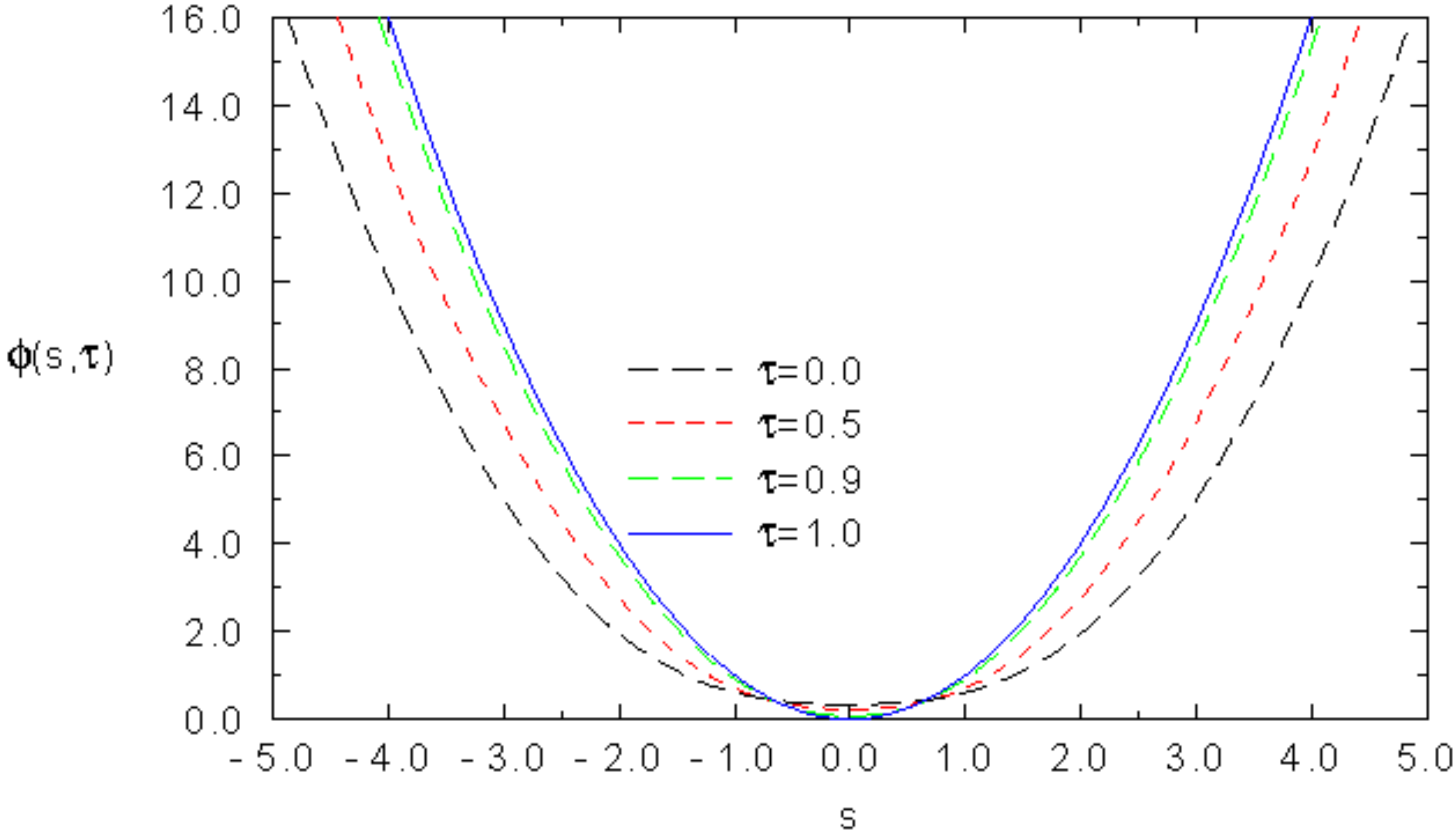


Figure 2: The potential $\phi(\mathbf{s}, t)$ for the square loss $L(y) = y^2$.

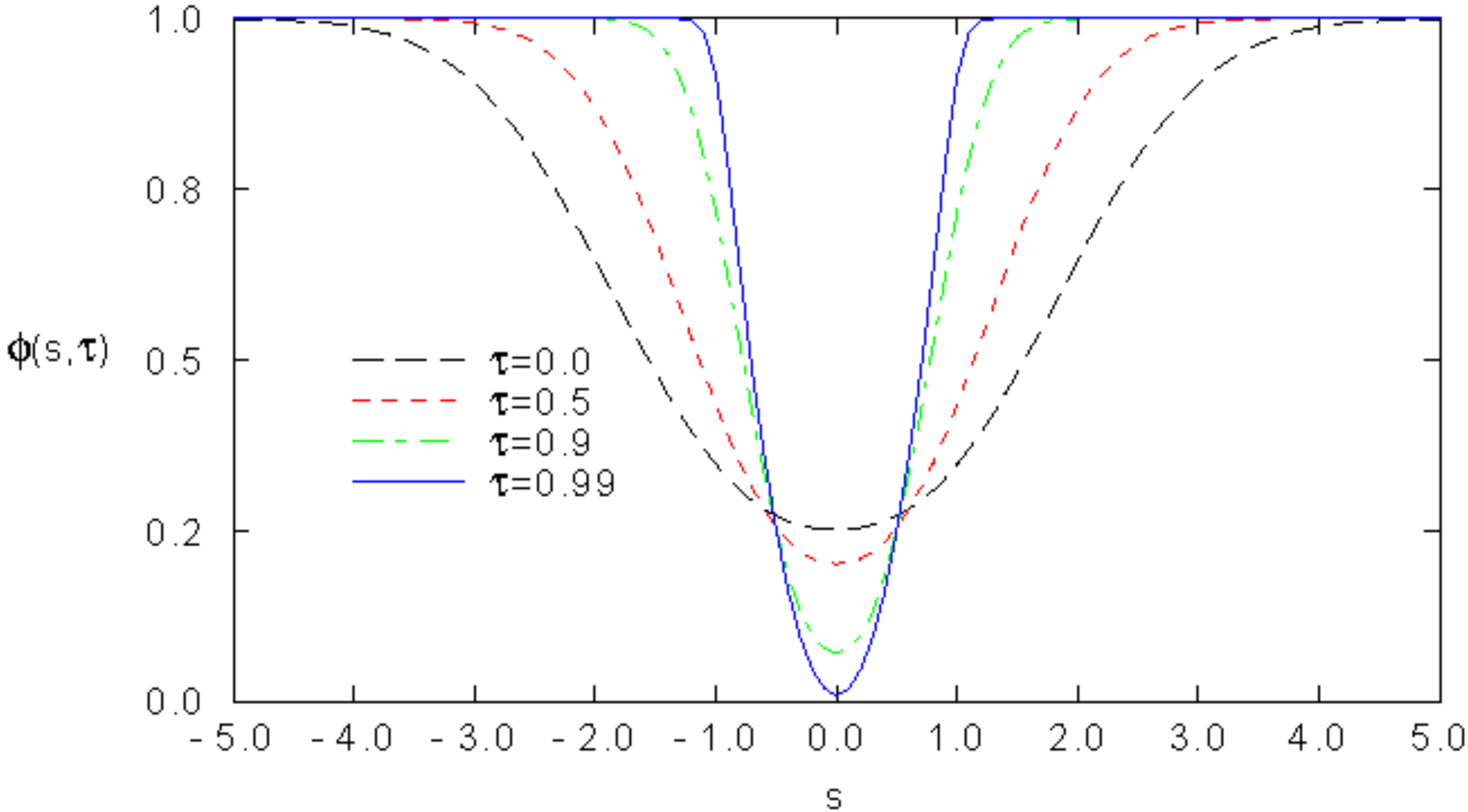


Figure 3: The potential $\phi(\mathbf{s}, \boldsymbol{t})$ for the loss $L(y) = \mathbf{min}(y^2, 1)$.

Plan of talk

- How to use expert advice,
  A quick tour of some old results.
- Decision Theoretic Online Learning
  - Hedge
  - Binomial Weights
  - Normal Hedge
  - Open problem

The Hedge Algorithm

[Freund & Schapire 1997]

based on [Littlestone and Warmuth 1989],[Cesa-Bianchi et al 1997]

Initial weights: $w^1 = \left\langle \frac{1}{N}, \ldots, \frac{1}{N} \right\rangle$

Weights update rule: $w_i^{t+1} = w_i^t e^{-\eta l_i^t} = e^{-\eta L_i^{t+1}}$

Learning rate

Alternatively: $w_i^{t+1} = \frac{1}{N} e^{-\eta L_i^t}$

Posterior probability
(un-normalized)

Prior probability

Potential-based bound

Potential: $W^{t+1} = \sum_{i=1}^{N} w_i^{t+1} = \sum_{i=1}^{N} e^{-\eta L_i^t}$

Large loss of algorithm => small potential $\quad L_A^t \le \frac{-\log W^{t+1}}{1-e^{-\eta}}$

Good expert=> large potential $\quad W^{t+1} \ge w_i^{t+1} = e^{-\eta L_i^t}$

Combining, we get: $\quad \forall i, L_A^T \le \frac{\eta L_i^T + \ln N}{1-e^{-\eta}}$

Tuning the learning rate

$$\forall i, L_A^T \le \frac{\eta L_i^T + \ln N}{1 - e^{-\eta}}$$

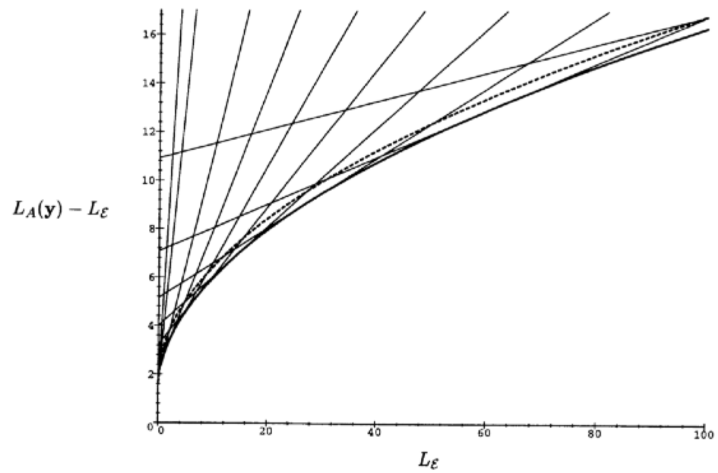
If we set $\eta = \sqrt{\frac{2\ln N}{T}}$

Then we guarantee $\mathrm{L}_A^T \le \min_i L_i^T + \sqrt{2T\ln N} + \ln N$

Equivalently $\forall i, R_i^T \le \sqrt{2T\ln N} + \ln N; \quad \lim_{T\to\infty} \frac{\sqrt{2T\ln N} + \ln N}{T} = 0$

Tuning the learning rate

$$\forall i, L_A^T \le \frac{\eta L_i^T + \ln N}{1-e^{-\eta}}$$



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Is it possible to do better?

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Design of NormalHedge

- BW: potential function depends on loss and number of remaining mistakes
- Normal-Hedge: Potential function based on regret and variance of the positive cumulative regrets

The NormalHedge potential

$$\text{Potential: } \psi(r,c) = \begin{cases} \exp\left(\dfrac{R^2}{2c}\right) & \text{if R} \geq 0 \\ 1 & \text{if R} \leq 0 \end{cases}$$

$$\text{Weight: } w(R,c) = \frac{\partial}{\partial R}\psi(R,c) = \begin{cases} \dfrac{R}{c}\exp\left(\dfrac{R^2}{2c}\right) & \text{if R} \geq 0 \\ 0 & \text{if R} \leq 0 \end{cases}$$

Intuition: If we play against the random walk player, then

the probability that the cumulative regret is R is approximately $e^{-\frac{R^2}{2t}}$,
to have any hope of keeping the potential constant the potential
function cannot increase faster than 1/probability.

NormalHedge algorithm

for t=0,1,2,...

if $\forall i, R_i^t \leq 0$ then $w_i^t = 1/N$

else

set $c(t)$ so that $\frac{1}{N}\sum_{i=1}^{N}\psi\left(R_i^t, c(t)\right) = e$

$w_i^t = w\left(R_i^t, c(t)\right)$

Incur instantanous losses: $\left\langle l_1^t, l_2^t, \ldots, l_N^t \right\rangle$

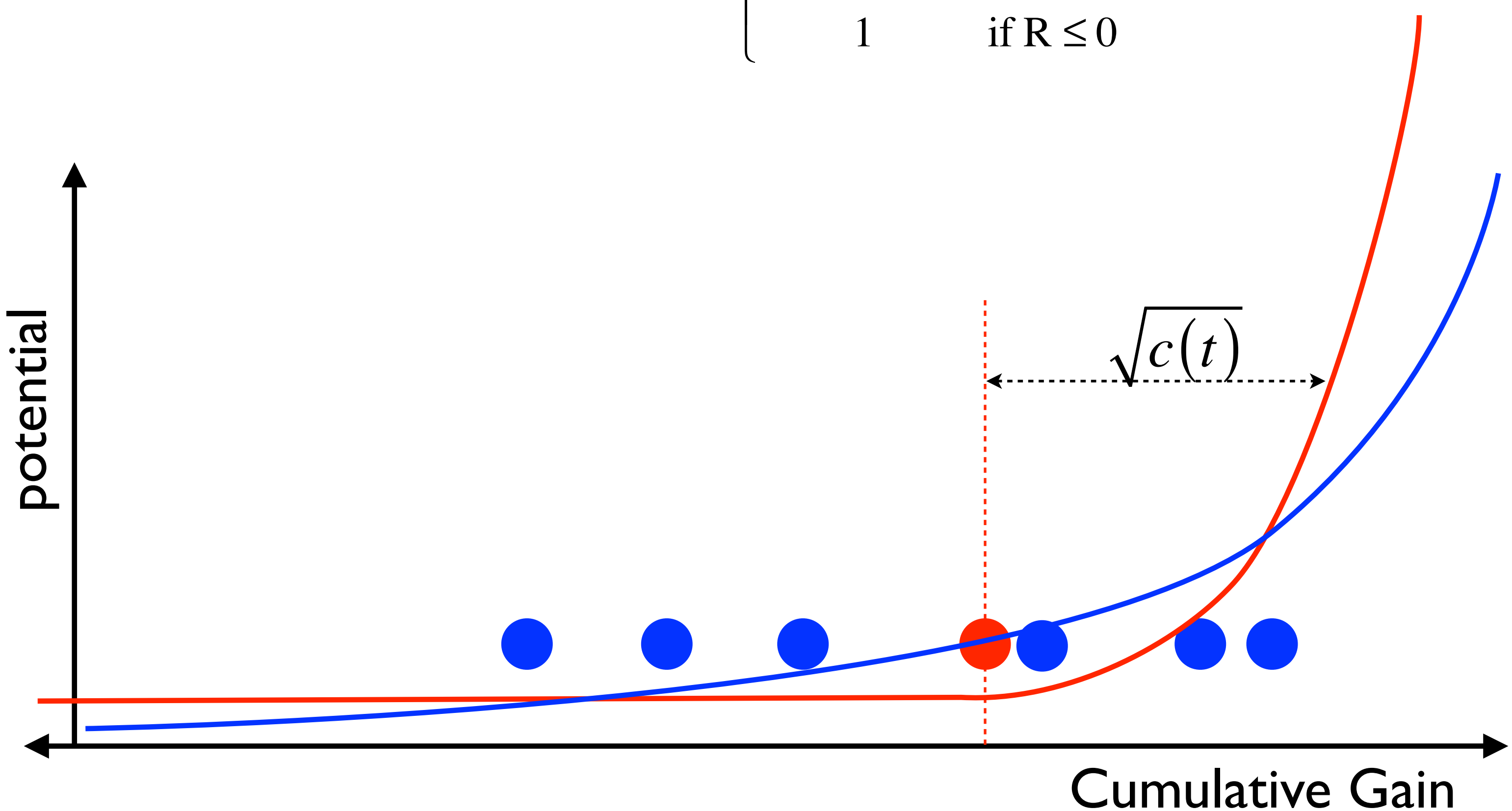
Algorithm loss: $l_A^t = \dfrac{\sum_{i=1}^{N} w_i^t l_i^t}{\sum_{i=1}^{N} w_i^t}$

Update regrets: $R_i^{t+1} = R_i^t + l_A^t - l_i^t$

Illustrative Example

Expert — $\exp(\eta G)$

Algorithm — $\begin{cases} \exp\left(\dfrac{R^2}{2c}\right) & \text{if } \mathrm{R} \geq 0 \\ 1 & \text{if } \mathrm{R} \leq 0 \end{cases}$



Illustrative Example

- Expert
- Algorithm

$\exp(\eta G)$

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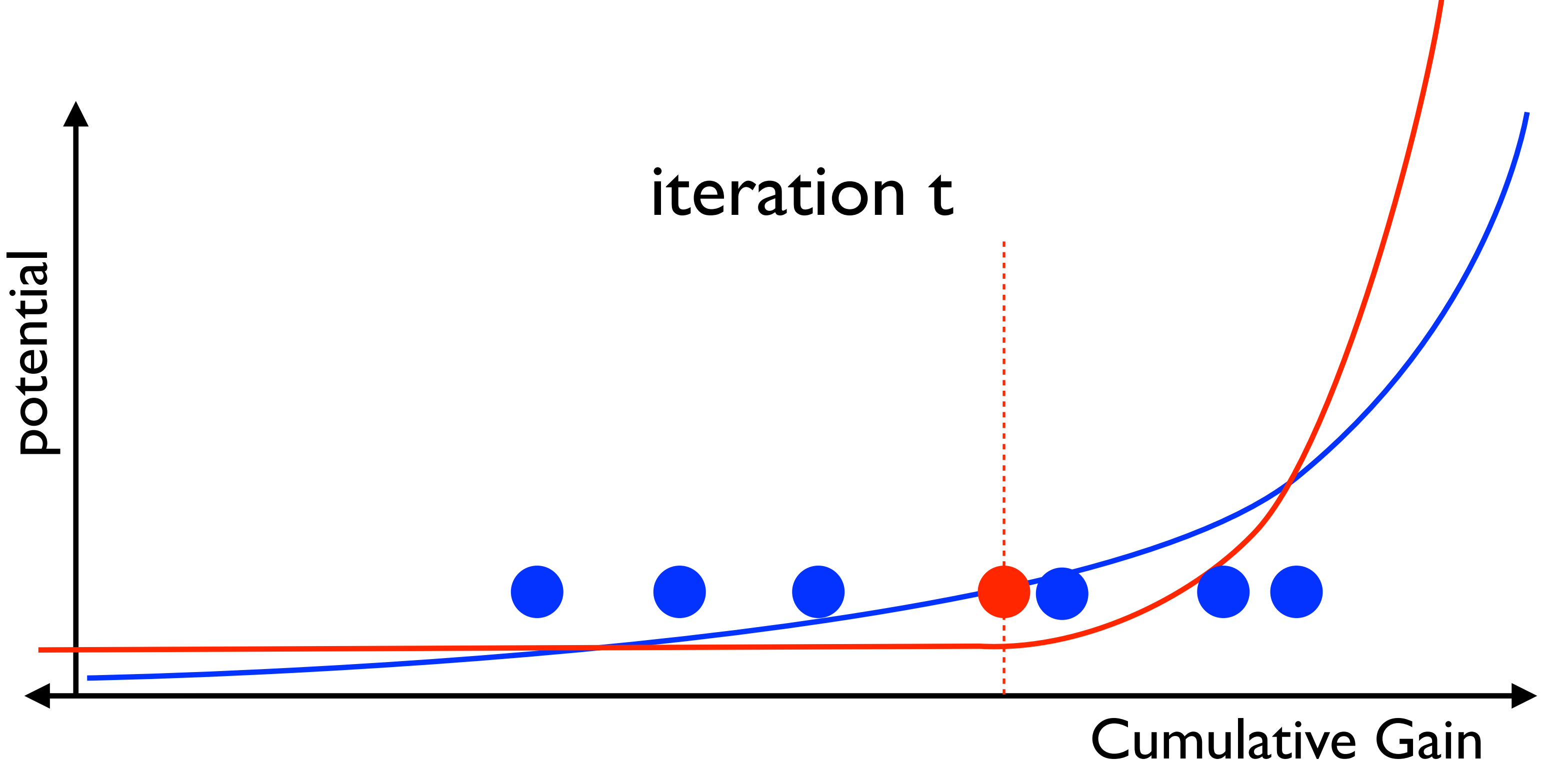

Illustrative Example

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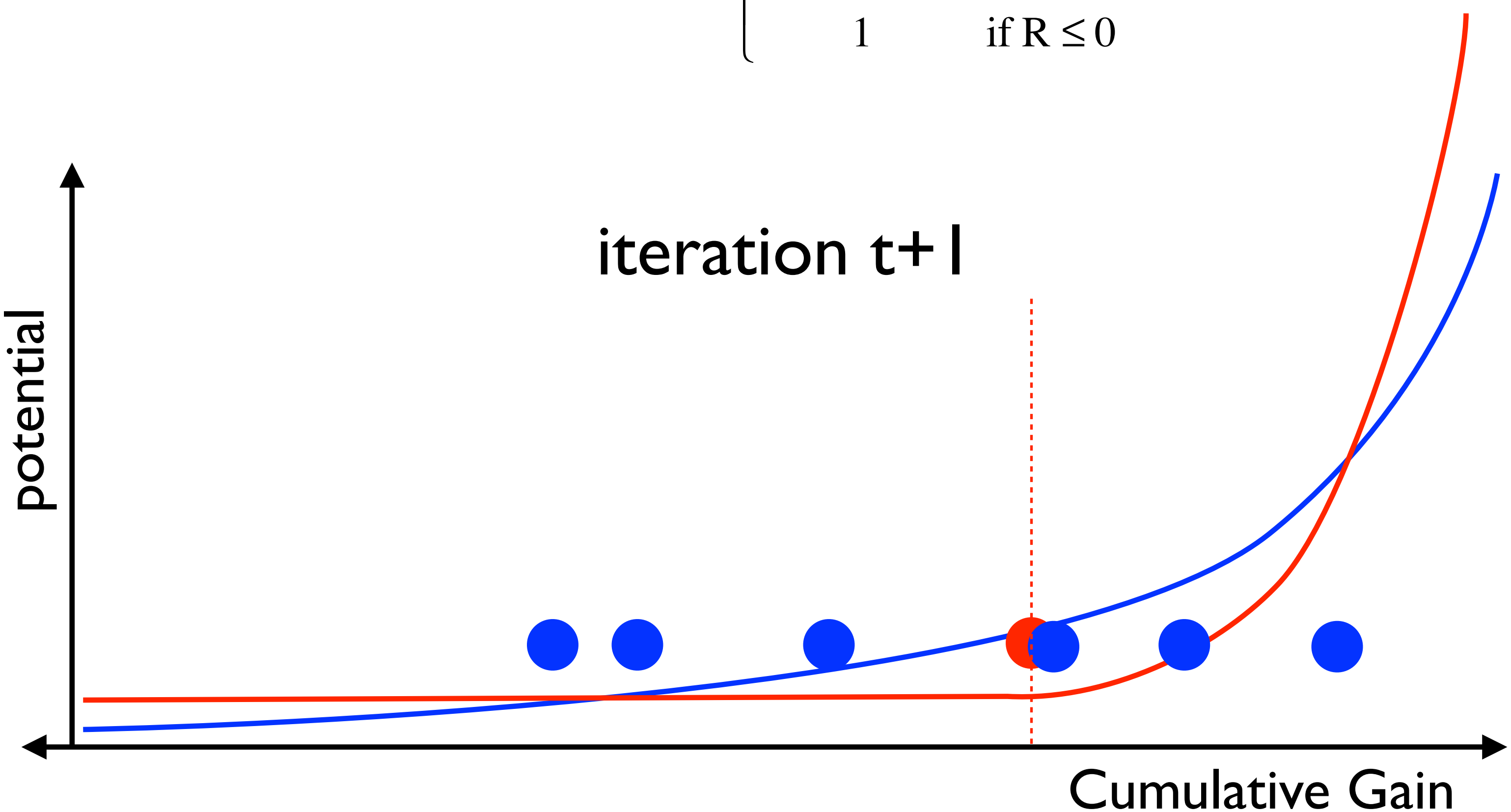

Illustrative Example

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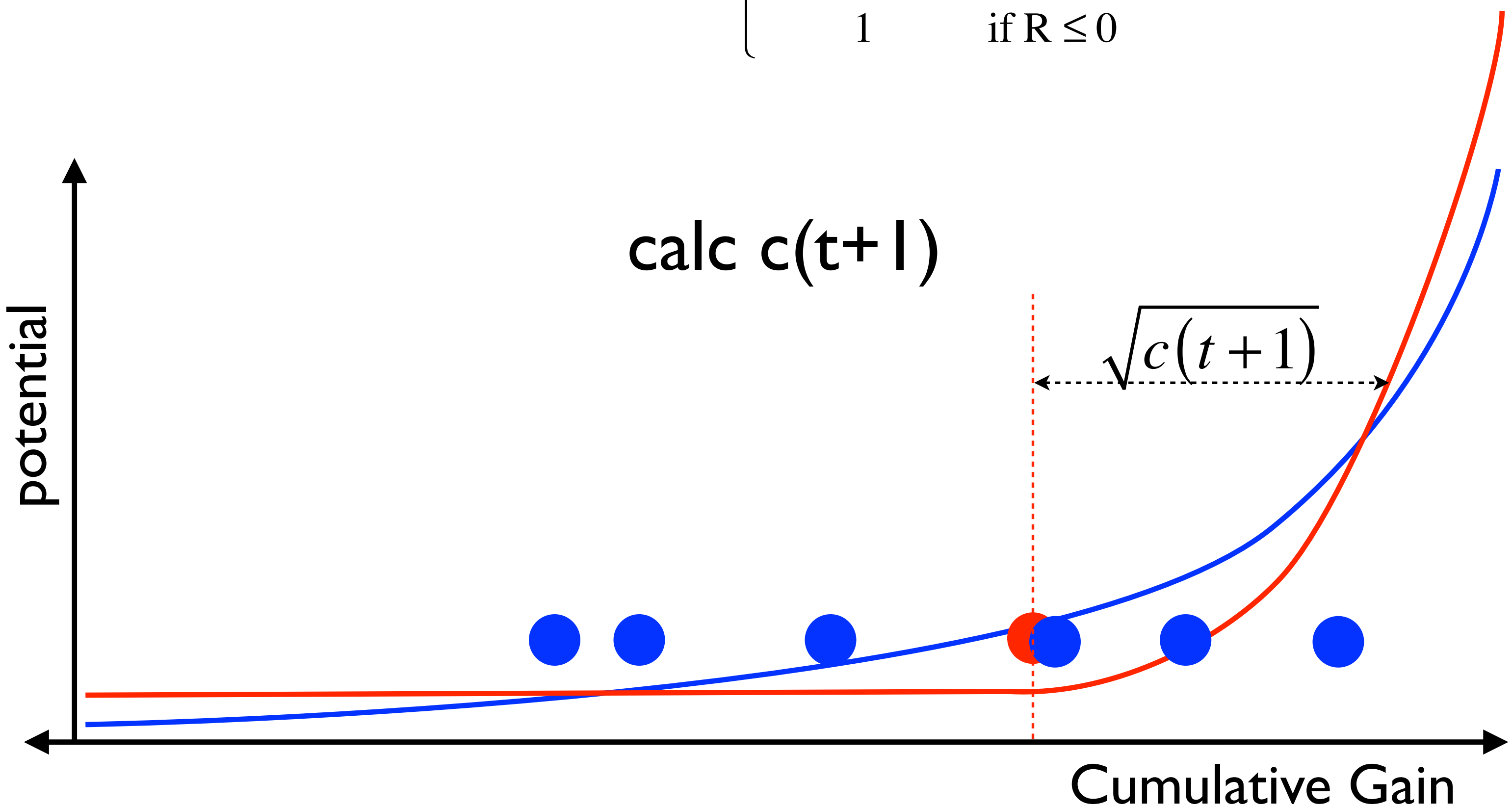


Normal-Hedge Performance bound

[Chaudhuri, Freund & Hsu 2009]

Main Lemma: $c(t) \leq t + C_N$

The regret of NormalHedge is upper bounded by

$$\sqrt{3t\left(1+\ln N\right)+o(t)}$$

The regret to the top ϵ-percentile is upper bounded by

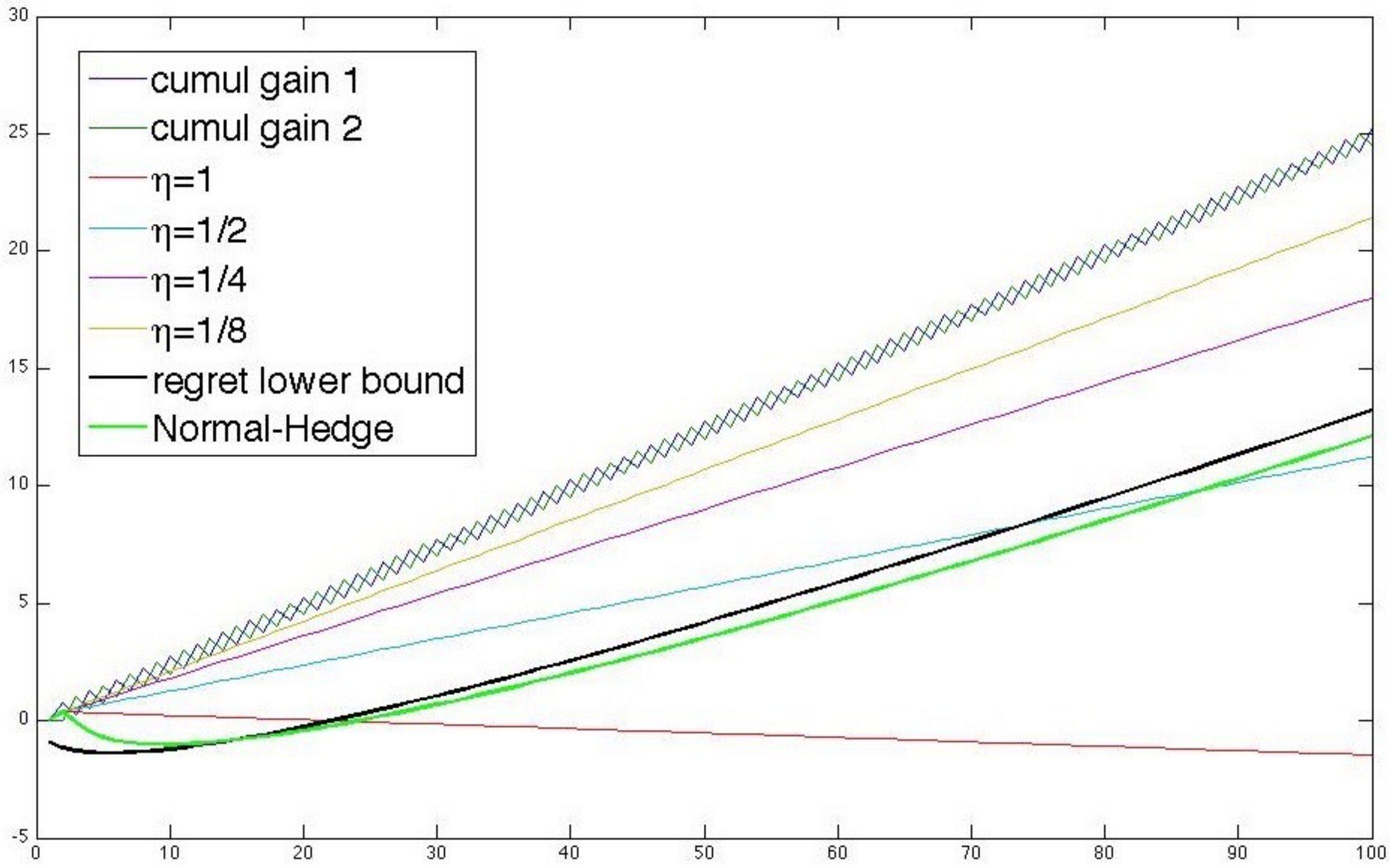
$$O\left(\sqrt{t\ln(1/\epsilon)}+\ln^2 N\right)$$

We failed to get rid of the dependence on N

BW depends only on $1/\epsilon$ not on N

Performance on flip-flop

- Worst case for follow the leader



NormalHedge.DT / Luo and Schapire, 2014

$$\text{Potential: } \psi(r,t) = \begin{cases} \exp\left(\dfrac{r^2}{3t}\right) & \text{if } r \geq 0 \\ 1 & \text{if } r \leq 0 \end{cases}$$

$$\text{Weight: } w(r,t) = \psi(r+1,t+1) - \psi(r+1,t-1)$$

$$\text{Recall NormalHedge weight: } w(r,c) = \frac{\partial}{\partial r}\psi(r,c)$$

A very clean application of drifting games analysis

NormalHedge.DT / analysis

Instead of keeping potential constant,

allow potential to increase like $c\log(t)$

The regret of AdaNormalHedge is bounded by

$$\sqrt{3t\ln\left(\frac{1}{2\epsilon}\left(e^{4/3}-1\right)\left(\ln t+1\right)+1\right)} = O\left(\sqrt{t\ln(1/\epsilon)+t\ln\ln t}\right)$$

for all t and for all $\epsilon > 0$

Algorithm does not depend on ϵ !!!

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  - Open problem

Open Problem

- Luo and Schapire got rid of the dependence on N
- What about the dependence on T?
- Suppose that, although loss is bounded in [0,1], it turns out that the loss is always in [0,1/2].
- If alg. has information in advance, bound is halved
- Equivalently: $T \to \frac{T}{4}$
- Can we achieve this performance without a -priori knowledge?
- Hypothesis: The increase c(t+1)-c(t) for normal hedge can upper bounded by a function of the form

$$\int_\theta \alpha_\theta \left(r_\theta^t\right)^2 d\mu(\theta) \quad \text{(current bound is 1)}$$

- Without degrading current performance.

Recent Progress

Schapire and Liu 2015:

AdaNormalHedge, an adaptive version of NormalHedge.DT

The regret of AdaNormalHedge relative to the ϵ best experts at time T

$$\forall T,\epsilon \quad R(\epsilon,T) \le O\left(\sqrt{T\log\left(\frac{\log T}{\epsilon}\right)}\right)$$

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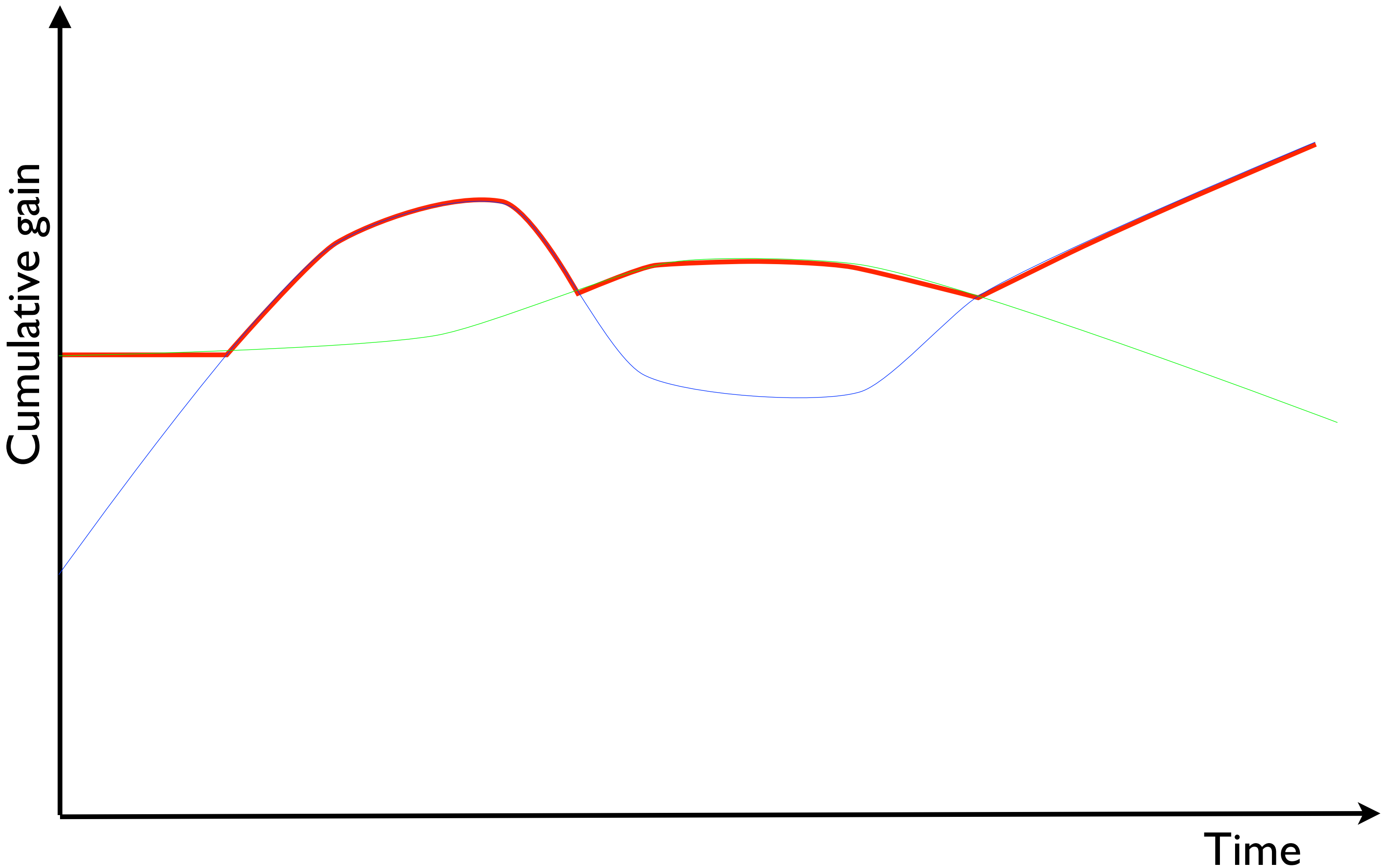
$$\int_\theta \alpha_\theta \left(r_\theta^t\right)^2 d\mu(\theta) \quad \text{(current bound is 1)}$$

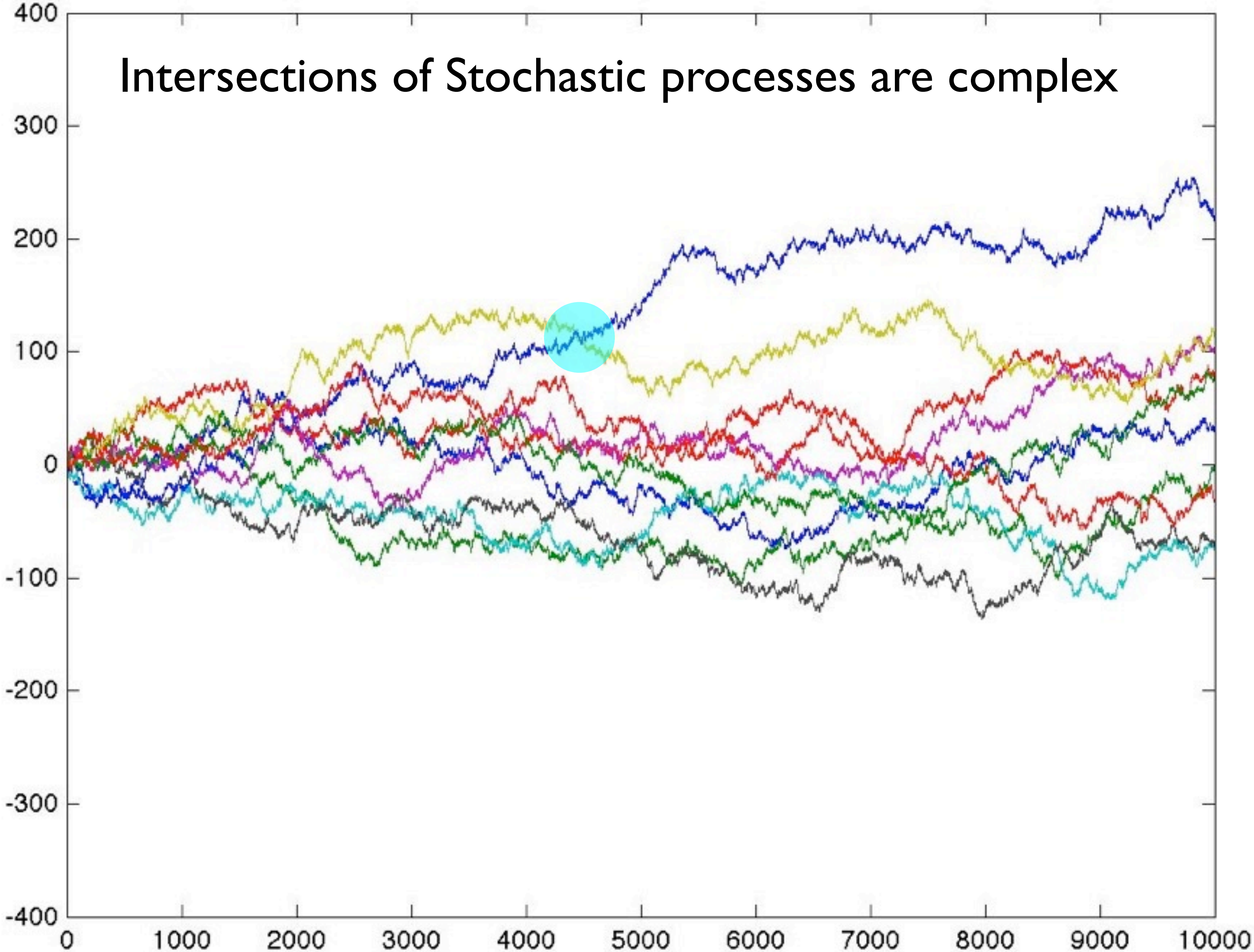
- Without degrading current performance.

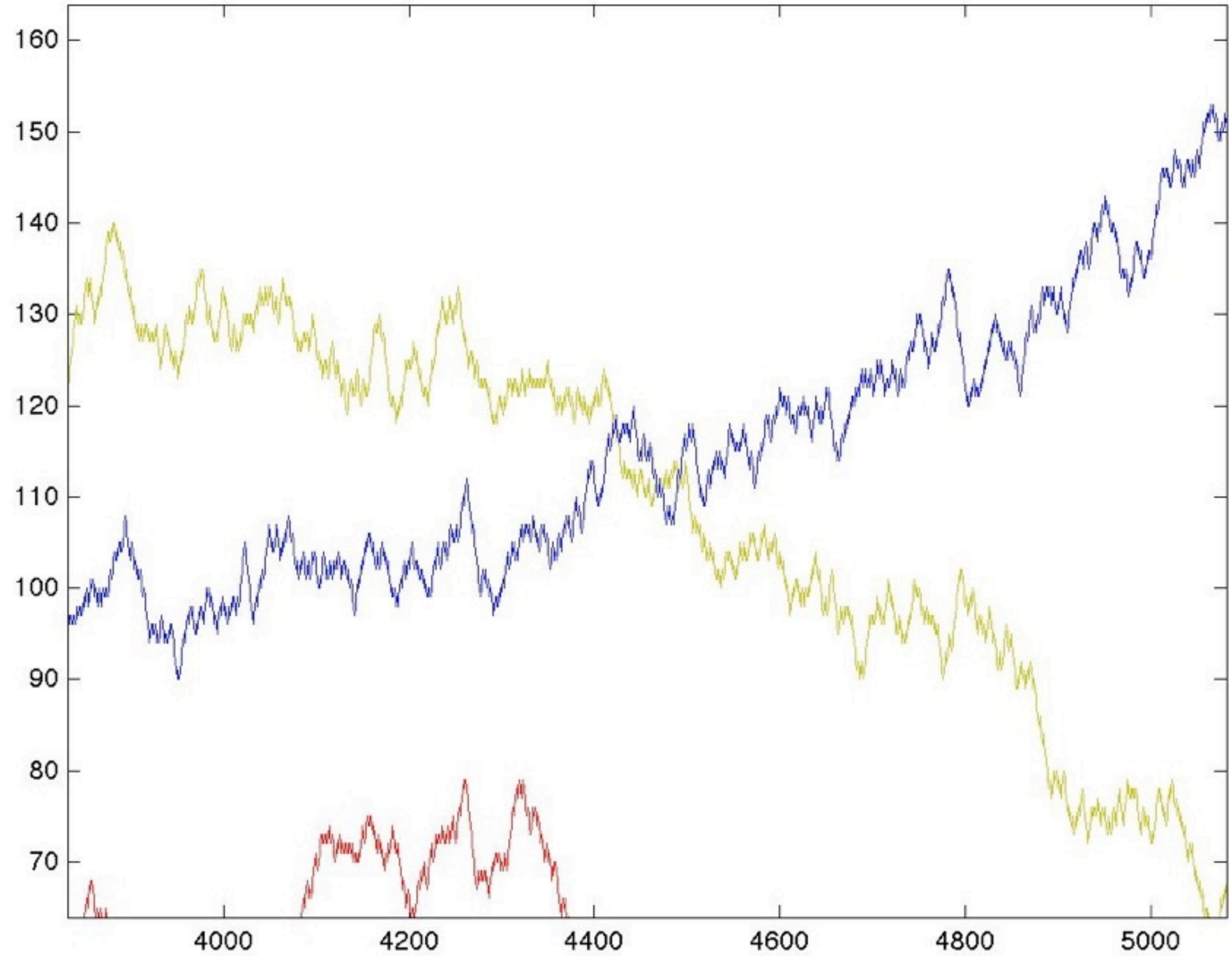
Hypothesis

- In the bound on NormalHedge, T can be replaced by cumulative variance.
- Similar to measuring return vs. volatility in the stock market.

Hedging differentiable processes is trivial




Intersections of Stochastic processes are complex
400
300
200
100
0
-100
-200
-300
-400
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000



160
150
140
130
120
110
100
90
80
70
4000
4200
4400
4600
4800
5000

Thank You!

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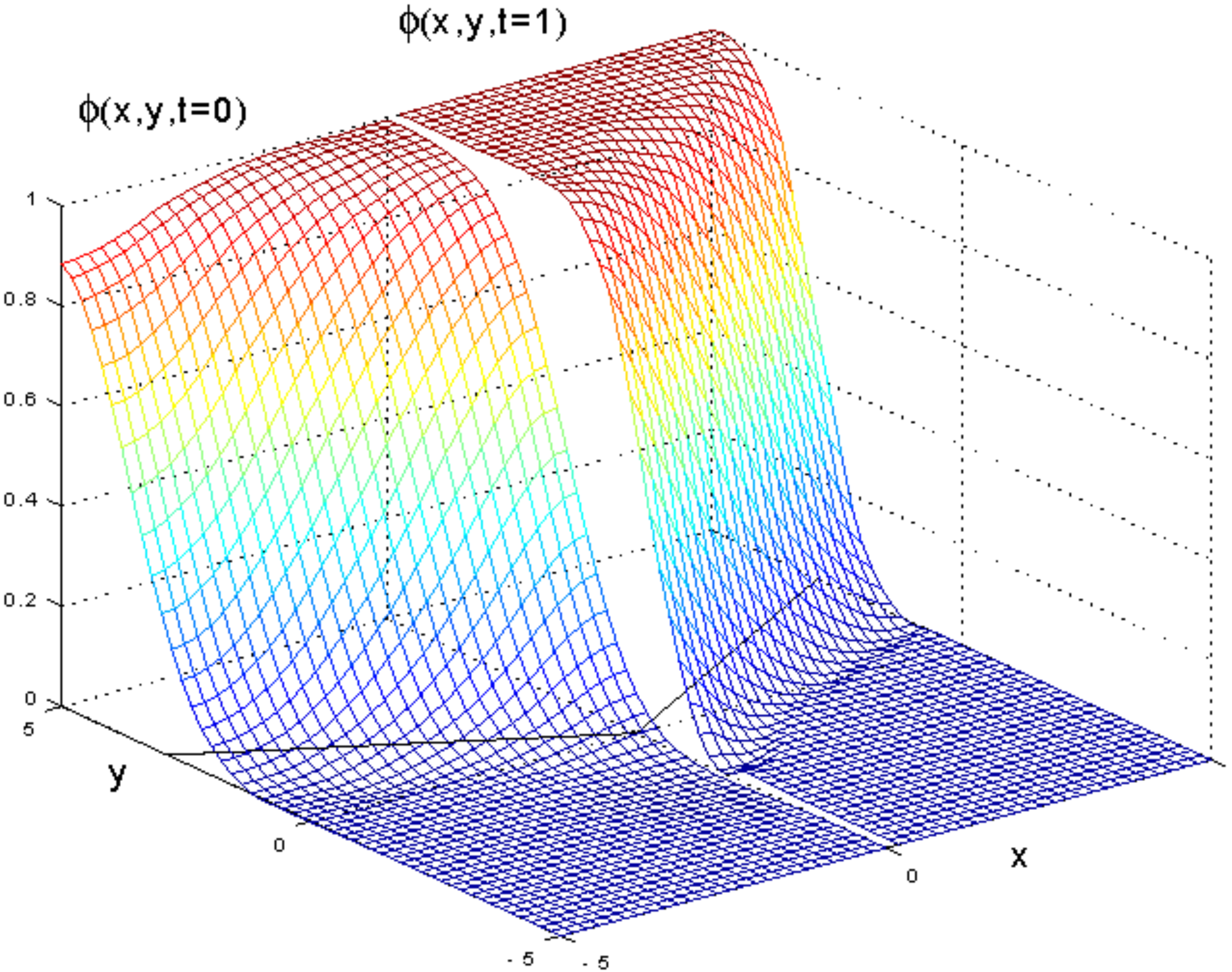
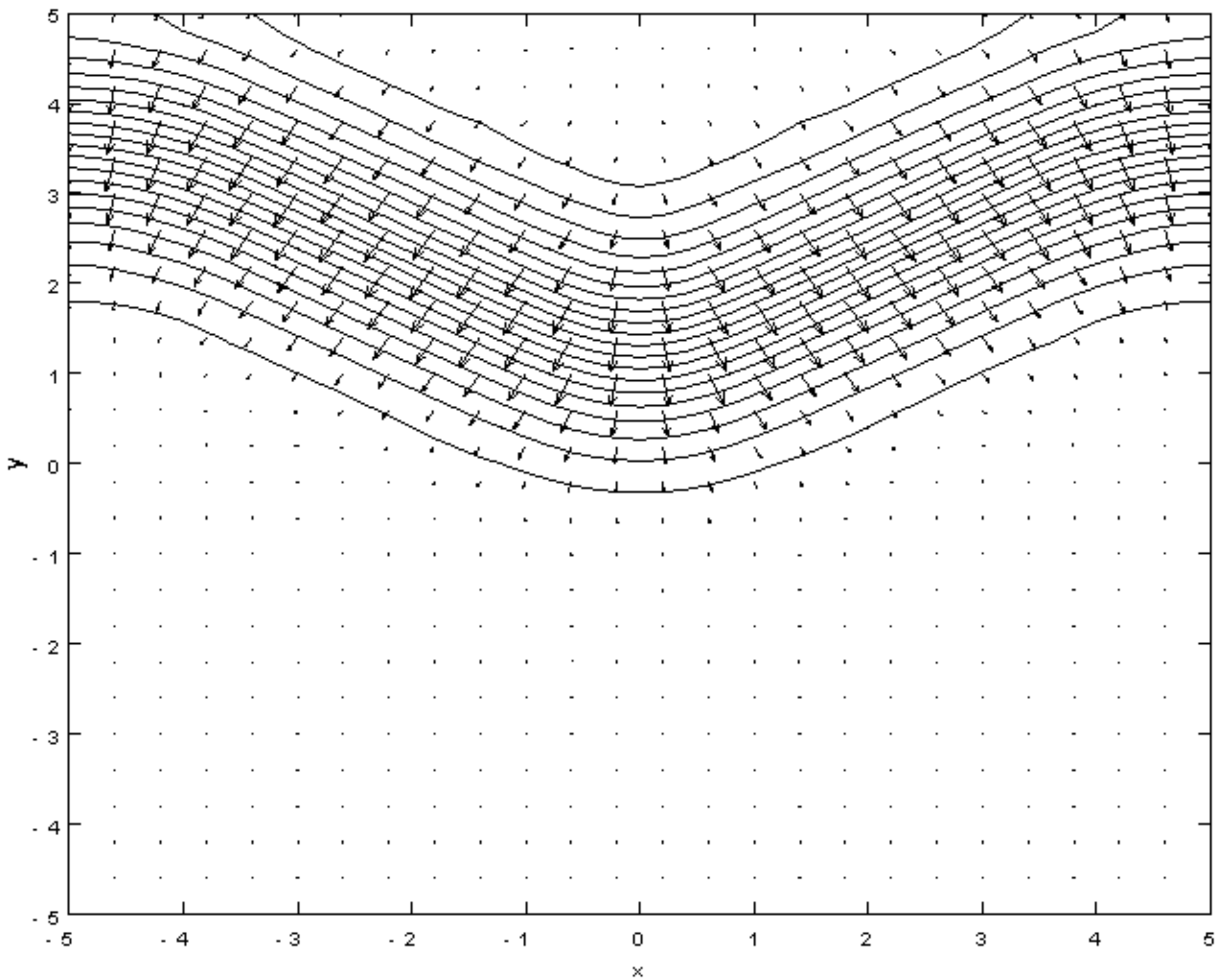
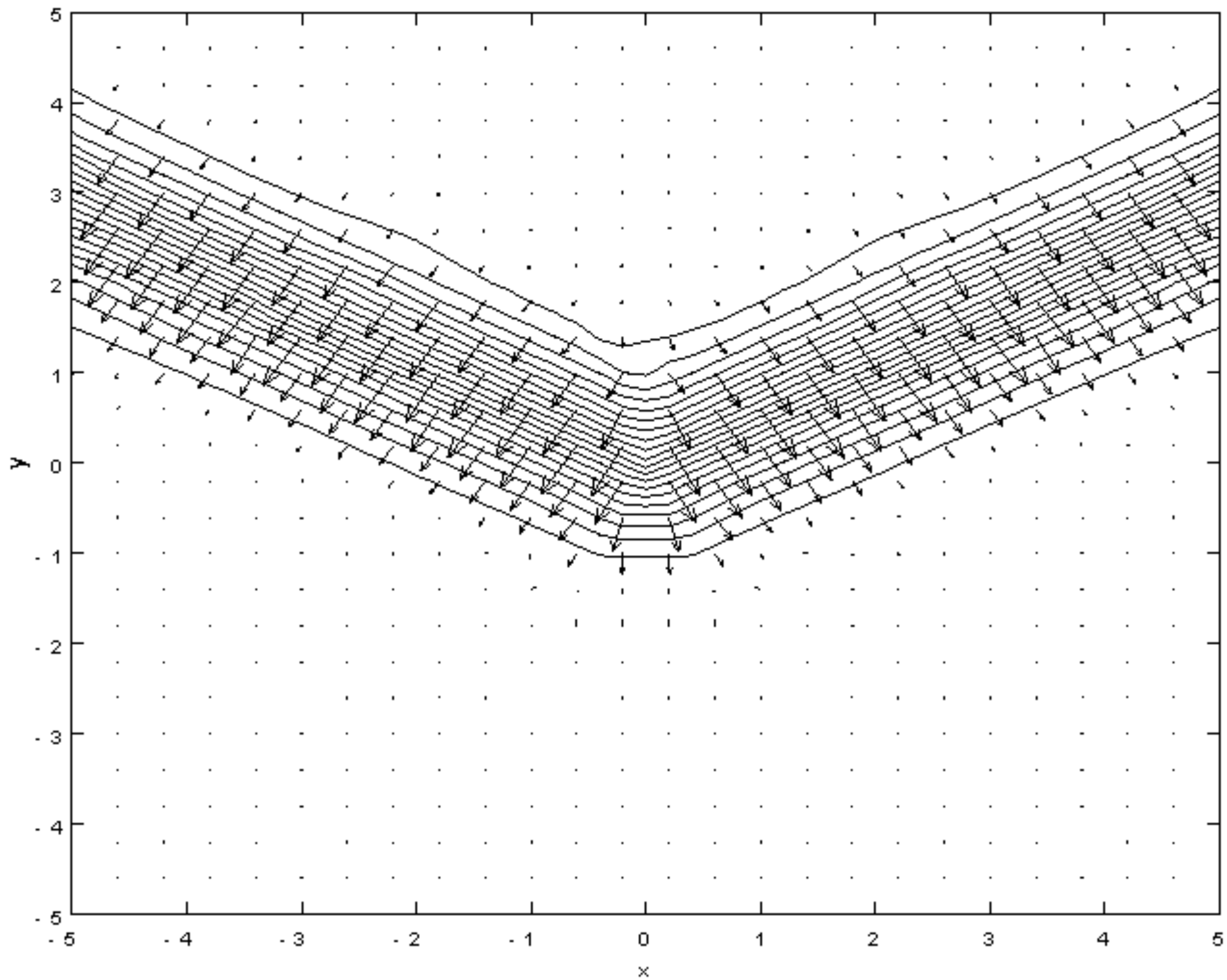


φ(x,y,t=1)
φ(x,y,t=0)
1
0.8
0.6
0.4
0.2
0
5
y
0
-5
-5
0
x
5



y
x
5
4
3
2
1
0
- 1
- 2
- 3
- 4
- 5
- 5
- 4
- 3
- 2
- 1
0
1
2
3
4
5



5
4
3
2
1
0
- 1
- 2
- 3
- 4
- 5
y
- 5
- 4
- 3
- 2
- 1
0
1
2
3
4
5
x