The Context Algorithm

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January 26, 2006

Outline

Review

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- Review
- Fixed Length Markov Models
- Variable Length Markov Model (VMM)
- Assigning weights to trees

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Variable Length Markov Model (VMM)

Assigning weights to trees

Learning the structure, an inefficient solution

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The online Bayes Algorithm

- Total loss of expert i

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- Prediction of algorithm A

$$\mathbf{p}_A^t = \frac{\sum_{i=1}^{N} w_i^t \mathbf{p}_i^t}{\sum_{i=1}^{N} w_i^t}$$

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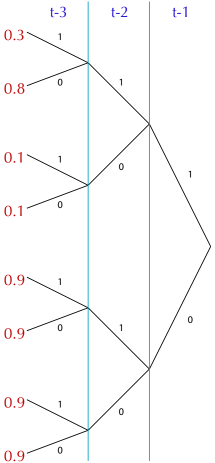

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$$p(x_t = 1 | x_{t-1} = y_1, \ldots, x_{t-k} = y_k) = \frac{b_{y_1,\ldots,y_k} + 1/2}{a_{y_1,\ldots,y_k} + b_{y_1,\ldots,y_k} + 1}$$

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- Total regret is at most $2^{k-1} \log T$

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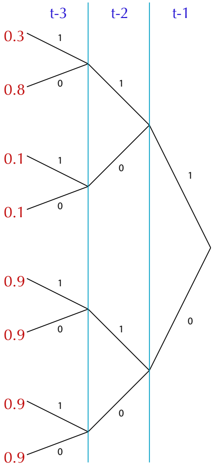

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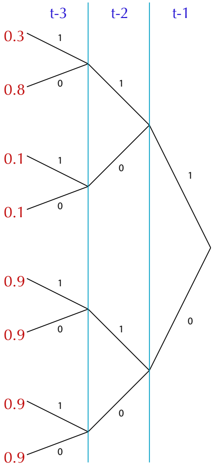

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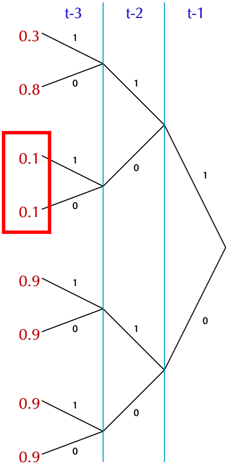

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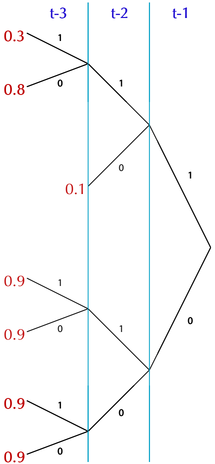

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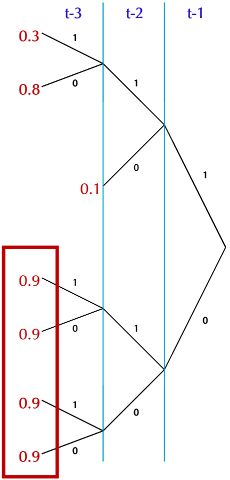

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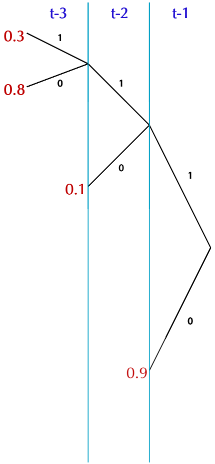

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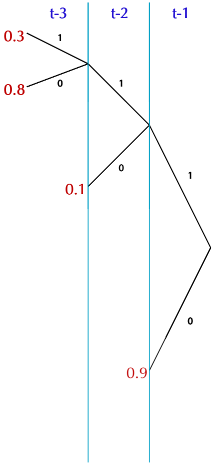

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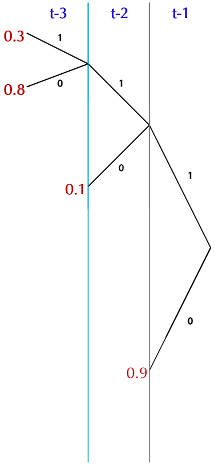

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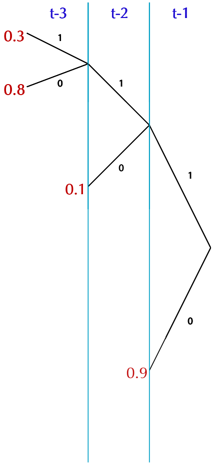

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- English example:
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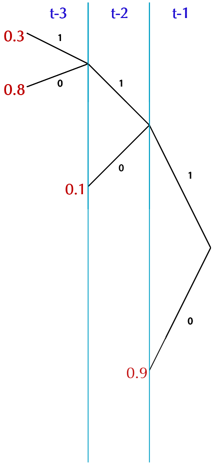

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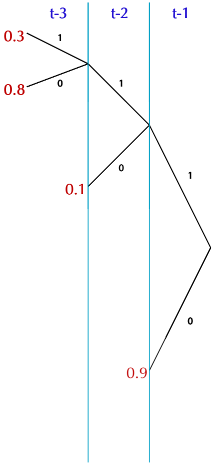

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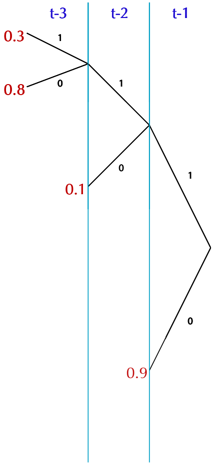

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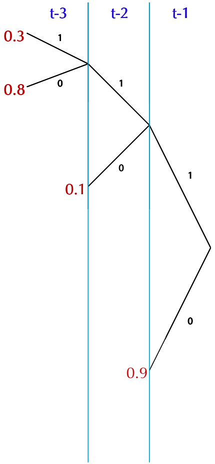

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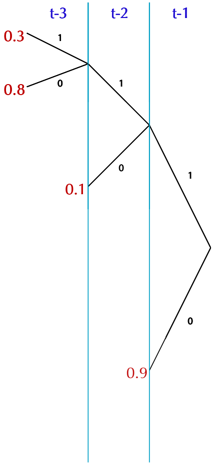

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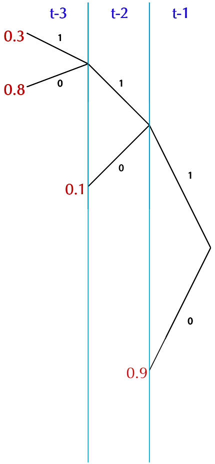

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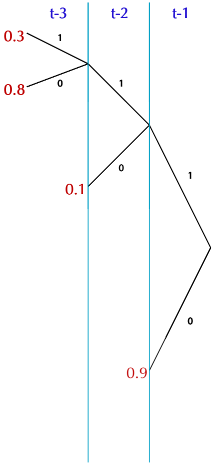

- Reducing number of leaves from 8 to 4 means
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- When we have little data, we can get better prediction even if the children are not Exactly the same

Prefix trees

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- A node with 1 child means that some past histories are not covered.
- A variable length markov model corresponds to a prefix tree.
- But we don't know which prefix tree to use!

Assigning probabilities to complete sequences

- Using the chain rule, we can use a prediction rule to assign probabilities to a complete sequence.

$$P(x_1 = y_1, \ldots, x_T = y_T) = p(x_1 = y_1)p(x_2 = y_2 | x_1 = y_1) \ldots$$

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- We can translate probabilities for complete sequences back into predictions.

$$p(x_t = 1 | x_1 = y_1, \ldots, x_{t-1} = y_{t-1}) = \frac{p(x_1 = y_1, \ldots, x_{t-1} = y_{t-1}, x_t = 1)}{p(x_1 = y_1, \ldots, x_{t-1} = y_{t-1})}$$

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- Will come in handy soon!

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- This algorithm maintains a weight for each tree.
- Requires maintaining $O(2^l)$ weights!

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- The prior weights are used for averaging the complete sequence probabilities - they don't need to be updated.
- Second idea: Compute the average over the prior efficiently.

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- Subset corresponding to node is contained in subset corresponding to node's parent.

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