Blackwell Approachability and Forcing Halfspaces

CSE 254: Online Learning

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Basic Results

Classic minimax theorem for two-player zero-sum games:

Theorem (von Neumann, 1947)

If the players have discrete strategy spaces $[n], [m]$ and the game has payoff function $u : [n] \times [m] \mapsto \mathbb{R}$,

$$\max_{p \in \Delta_n} \min_{q \in \Delta_m} \sum_{i \in [n], j \in [m]} p_i q_j u(i,j) = \min_{q \in \Delta_m} \max_{p \in \Delta_n} \sum_{i \in [n], j \in [m]} p_i q_j u(i,j)$$

Basic Results

- "Optimization" variant

Theorem (Sion, 1958)

If the players have convex compact strategy spaces $\mathcal{X}, \mathcal{Y}$ and the game has loss function $f(x, y) : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}$, convex in x and concave in y,

$$\inf_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} f(x, y) = \sup_{y \in \mathcal{Y}} \inf_{x \in \mathcal{X}} f(x, y)$$

- Analogues for other games, nothing as powerful
- Rich equilibrium structure impossible with more players
- But can we go beyond scalar payoff functions?

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Preliminaries

- Two-player zero-sum game
  - Player X plays against nameless adversary Y (Nature)
  - X plays $x \in \mathcal{X}$, Y plays $y \in \mathcal{Y}$
  - X loses $u(x, y)$, Y wins $u(x, y)$
- Minimax value $V = \min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} u(x, y)$ when U is scalar
- Vector-valued games
  - Natural to model utility of mutually independent factors
  - What can we say when u is vector-valued? Minimax impossible

Explicit Quantification

- Minimax (strong) duality is the conjunction of two statements involving value V:
  1. $\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} u(x,y) \leq V \iff \exists x \in \mathcal{X} : \forall y \in \mathcal{Y} : u(x,y) \leq V$
  2. $\max_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} u(x,y) \geq V \iff \exists y \in \mathcal{Y} : \forall x \in \mathcal{X} : u(x,y) \geq V$

  and weak duality $\max_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} u(x,y) \leq \min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} u(x,y)$.
- Each player can *force* the other into playing in a way that guarantees the payoff in a half-line.
- In this worst-case scenario, the only meaningful control is a uniform guarantee over adversary strategies.

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Setup

- What happens when the payoff is vector-valued?
- What payoffs can X force the adversary into settling for?
- Can X force payoffs in some target set? [1]

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Outline

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Blackwell's Game

- A two-player zero-sum repeated game with vector-valued payoff $u(x, y)$
- On iteration t, X plays x_t first, then Y plays y_t
- **Goal of player** X: "Approach" target set S regardless of Y's actions
- Assumptions
  - Any projection $u_\theta(x, y) = \langle \theta, u(x, y) \rangle$ for any vector θ satisfies minimax conditions (e.g. if u is bilinear)
  - $S, \mathcal{X}, \mathcal{Y}$ are convex, compact
  - These are unnecessary in many cases

Definition: Approachability

Consider a set[2] S. Define S to be *approachable* if there exists a possibly adaptive strategy $x_1, x_2, x_3, \cdots \in \mathcal{X}$ such that for any sequence $y_1, y_2, \cdots \in \mathcal{Y}$,

$$\lim_{T \to \infty} d\left(\frac{1}{T}\sum_{t=1}^{T} u(x_t, y_t), S\right) = 0$$

where d is the distance in Euclidean norm. In other words, if $\bar{u}_T = \frac{1}{T}\sum_{t=1}^{T} u(x_t, y_t)$,

$$\lim_{T \to \infty} \inf_{z \in S} \|\bar{u}_T - z\| = 0$$

[2] For simplicity, throughout only consider subsets of $\mathbb{R}^d$ for finite d.

Outline

Some Related Notions

A set S is:

- **Satisfiable** (by X) if $\exists x \in \mathcal{X} : \forall y \in \mathcal{Y} : u(x, y) \in S$ (player can force S playing first)
- **Response-satisfiable** (by X) if $\forall y \in \mathcal{Y} : \exists x \in \mathcal{X} : u(x, y) \in S$ (player can force S playing second)
- Satisfiability $\implies$ response-satisfiability
- When does response-satisfiability $\implies$ satisfiability?
- Other relations hold (S is satisfiable by $X \iff S^c$ is response-satisfiable by Y)

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Satisfiability for Halfspaces

- Minimax theorem: $(-\infty, c]$ is approachable $\iff c \geq V$
- Consider any halfspace $H = \{s : \langle \theta, s \rangle \leq c\}$
- This induces scalar game with payoff $u_\theta(x, y) = \langle \theta, u(x, y) \rangle$
- H is approachable

$\iff (-\infty, c]$ is approachable in scalar game

$\iff c \geq \min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} u_\theta(x, y)$

$\iff \exists x \in \mathcal{X} : \forall y \in \mathcal{Y} : u(x, y) \in H$

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$$\begin{aligned} &\iff (-\infty, c] \text{ is approachable in scalar game} \\ &\iff c \geq \min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} u_\theta(x, y) \\ &\iff \exists x \in \mathcal{X} : \forall y \in \mathcal{Y} : u(x, y) \in H \\ &\iff H \text{ is satisfiable} \end{aligned}$$

Response-Satisfiability $\iff$ Halfspace-Satisfiability

Theorem

S is response-satisfiable $\iff$ every halfspace $H \supseteq S$ is satisfiable.

Proof.

$(\implies)$
Take any halfspace $H_0 = \{s : \langle \theta_0, s \rangle \leq c_0\} \supseteq S$.
Now S is response-satisfiable $\implies \forall y : \exists x_y : u(x_y, y) \in S \implies u(x_y, y) \in H_0 \implies u_{\theta_0}(x_y, y) \leq c_0$. Thus
$c_0 \geq \max_{y \in \mathcal{Y}} u_{\theta_0}(x_y, y) \geq \max_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} u_{\theta_0}(x, y) = \min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} u_{\theta_0}(x, y)$.
If x^* is the minimizer here, we have
$\forall y \in \mathcal{Y} : c_0 \geq u_{\theta_0}(x^*, y) \implies u(x^*, y) \in H_0$. □

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Response-Satisfiability $\iff$ Halfspace-Satisfiability

Proof.

($\Longleftarrow$)
S not response-satisfiable $\implies \exists y_0 \in \mathcal{Y} : \forall x : u(x, y_0) \notin S$.
The set $U = \{\forall x \in \mathcal{X} : u(x, y_0)\}$ is convex, but $S \cap U = \emptyset$ by assumption. So there is a hyperplane H separating S and U, defining a halfspace $H \supseteq S$. We have for all x that $u(x, y_0) \notin S \implies u(x, y_0) \notin H$, so H is not satisfiable. $\square$

Halfspace-Satisfiability $\iff$ Approachability

Theorem

Every halfspace $H \supseteq S$ is satisfiable $\iff$ S is approachable.

Proof.

($\implies$) Constructive; the algorithm that approaches S relies on a halfspace oracle $O(H)$ for any $H \supseteq S$, with $O(H) = \{x \in \mathcal{X} : \forall y \in \mathcal{Y} : u(x, y) \in H\}$.

($\impliedby$) $\exists H \supseteq S$ not satisfiable $\implies$ $\exists H \supseteq S$ not approachable $\implies$ S is not approachable □

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Summary of Equivalent Notions

The following are equivalent characterizations of S:

1. Response-satisfiable
2. Halfspace-satisfiable
3. Approachable

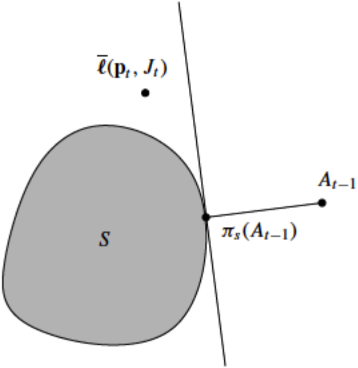
The first is often used to derive the third.

Outline

An Approachability Algorithm

- Assume oracle $O(H)$ for any $H \supseteq S$, with $O(H) = \{x \in \mathcal{X} : \forall y \in \mathcal{Y} : u(x, y) \in H\}$
- Write $A_T = \frac{1}{T} \sum_{t=1}^{T} u(x_t, y_t)$, and the projection $\pi_S(A_t) = \arg\min_{v \in S} \|A_t - v\|$
- Algorithm: On iteration t, if $A_{t-1} \notin S$, play $O(H_{t-1})$, where $H_{t-1} = \left\{ x : \forall y \in \mathcal{Y} : \left\langle \frac{A_{t-1} - \pi_S(A_{t-1})}{\|A_{t-1} - \pi_S(A_{t-1})\|}, u(x, y) - \pi_S(A_{t-1}) \right\rangle \leq 0 \right\}$
- Assumptions: $\|u(x, y)\| \leq 1 \forall x, y$; S is contained in the unit ball also

An Approachability Algorithm



Proof of Approachability (Algorithm)

$$
\begin{aligned}
\|A_t - \pi_S(A_t)\|^2 &\leq \|A_t - \pi_S(A_{t-1})\|^2 \\
&= \|A_t - A_{t-1}\|^2 + \|A_{t-1} - \pi_S(A_{t-1})\|^2 + 2\langle A_{t-1} - \pi_S(A_{t-1}), A_t - A_{t-1}\rangle \\
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Now $A_t - A_{t-1} = \frac{u(x_t, y_t) - A_{t-1}}{t} = \frac{1}{t}\left((u(x_t, y_t) - \pi_S(A_{t-1})) - (A_{t-1} - \pi_S(A_{t-1}))\right)$, so

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Therefore $\|A_t - \pi_S(A_t)\|^2 \leq \left(1 - \frac{2}{t}\right)\|A_{t-1} - \pi_S(A_{t-1})\|^2 + \frac{4}{t^2}$

$\implies \|A_t - \pi_S(A_t)\|^2 \leq \mathcal{O}\left(\frac{1}{t}\right)$.

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Outline

Generalizing Blackwell's Strategy

- Keep track of *potential function* $\Phi(s)$ that measures distance to set S ($\Phi(s) = 0\ \forall s \in S$)
- Want to minimize $\Phi(R_t)$ whenever possible
- Idea: Force halfspace in the direction of $\nabla\Phi(A_{t-1})$, but translated to intersect $\pi_S(A_{t-1})$
- Blackwell strategy: $\Phi(x) = \inf_{y \in S} \|x - y\|^2$
- Loss bound $\Phi(A_t) \in \mathcal{O}(\ln t / t)$

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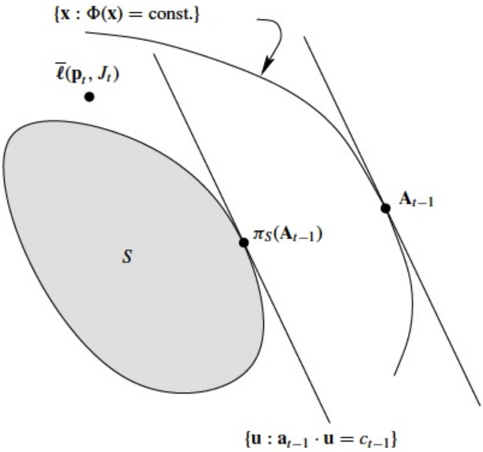
Potential-Based Approachability



Outline

Potential-Based Prediction with Experts

- Prediction with expert advice
  - On iteration t, N experts each predict in decision space $\mathcal{Z}$
  - Algorithm predicts $z_{A,t} \in \mathcal{Z}$, Nature reveals outcome y_t
  - Expert i incurs loss $l_{i,t}$, algorithm incurs $l_{A,t}$
  - Instantaneous regret $r_{i,t} = l_{A,t} - l_{i,t}$ to expert i
- ...As a game with losses in $\mathbb{R}^N$, one expert per coordinate
  - $r_t \in \mathbb{R}^N$ is vector with components $r_{i,t}$; $R_t = \sum_{i=1}^{t} r_i$
  - Game loss at time t is $u_t = r_t$
- ...Solved with a potential $\Phi(u) = \psi\left(\sum_{i=1}^{N} \phi(u_i)\right)$
  - ϕ nonnegative, increasing, twice-diff.
  - ψ **concave**, nonnegative, strictly increasing, twice-diff.
  - Relaxing additivity changes little (*unlike* drifting games)

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  - Relaxing additivity changes little (*unlike* drifting games)

Approachability in the Experts Setting

$$\Phi(R_t) \approx \Phi(R_{t-1}) + \langle \nabla\Phi(R_{t-1}), R_t - R_{t-1} \rangle = \Phi(R_{t-1}) + \langle r_t, \nabla\Phi(R_{t-1}) \rangle$$

- To try to keep $\Phi(R_t)$ decreasing, control $\langle r_t, \nabla\Phi(R_{t-1}) \rangle$
- Generalized Blackwell condition: $\sup\limits_{y_t \in \mathcal{Y}} \langle r_t, \nabla\Phi(R_{t-1}) \rangle \leq 0$

Theorem

Let $C(r_t) = \sup_{u \in \mathbb{R}^N} \psi' \left(\sum_{i=1}^N \phi(u_i) \right) \sum_{i=1}^N \phi''(u_i) r_{i,t}^2$. *Then for all* $n \geq 1$,

$$\Phi(R_n) \leq \Phi(0) + \frac{1}{2} \sum_{t=1}^{n} C(r_t)$$

Generalized Blackwell Condition

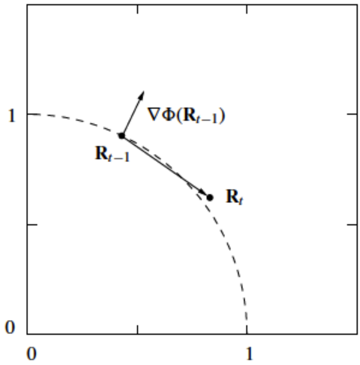

Figure 2.1. An illustration of the Blackwell condition with $N = 2$. The dashed line shows the points in regret space with potential equal to 1. The prediction at time t changed the potential from $\Phi(\mathbf{R}_{t-1}) = 1$ to $\Phi(\mathbf{R}_t) = \Phi(\mathbf{R}_{t-1} + \mathbf{r}_t)$. Though $\Phi(\mathbf{R}_t) > \Phi(\mathbf{R}_{t-1})$, the inner product between $\mathbf{r}_t$ and the gradient $\nabla\Phi(\mathbf{R}_{t-1})$ is negative, and thus the Blackwell condition holds.

Proof of Loss Bound (Potential-Based Forecaster)

Using Taylor's Theorem and denoting ξ as some vector $\in \mathbb{R}^N$,

$$\begin{aligned}
\Phi(R_t) &= \Phi(R_{t-1}) + \langle r_t, \nabla\Phi(R_{t-1})\rangle + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left[\frac{\partial^2\Phi}{\partial u_i \partial u_j}\right]_{\xi} r_{i,t} r_{j,t} \\
&\leq \Phi(R_{t-1}) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left[\frac{\partial^2\Phi}{\partial u_i \partial u_j}\right]_{\xi} r_{i,t} r_{j,t} \\
&\leq \Phi(R_{t-1}) + \frac{1}{2}\left(\psi''\left(\sum_{i=1}^{N}\phi(\xi_i)\right)\left(\sum_{i=1}^{N}\phi'(\xi_i) r_{i,t}\right)^2 + \psi'\left(\sum_{i=1}^{N}\phi(\xi_i)\right)\sum_{i=1}^{N}\phi''(\xi_i) r_{i,t}^2\right)
\end{aligned}$$

Using the concavity of ψ, we therefore have

$$\Phi(R_t) \leq \Phi(R_{t-1}) + \frac{1}{2}\left(\psi'\left(\sum_{i=1}^{N}\phi(\xi_i)\right)\sum_{i=1}^{N}\phi''(\xi_i) r_{i,t}^2\right) \leq \Phi(R_{t-1}) + \frac{1}{2}C(r_t)$$

Induction then gives the result.

Applications of Potential-Based Prediction

- What algorithms obey Blackwell condition and conditions on Φ?
- Weighted average predictors
  - Predict with a weighted average of experts, $w_{i,t} \propto \nabla_i \Phi(R_{t-1})$
  - Always satisfies Blackwell condition
  - Hedge ($\Phi(u) = \sum_{i=1}^{N} e^{\eta u_i}$), Blackwell's strategy ($\Phi(u) = \sum_{i=1}^{N} (u_i)_+^2$)
- Perceptron/Winnow (special mirror descent)
- Adaboost, polynomial potential, various forms of regret, specialists...

Recap: Potential-Based Approachability

- To try to keep $\Phi(R_t)$ decreasing, control $\langle r_t, \nabla\Phi(R_{t-1})\rangle$
- Only very relaxed halfspace control possible, so potential can still increase
- But master loss bound is still very useful

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Outline

Connections to Drifting Games and Online Learning

- Blackwell approachability is intimately tied with the question: what can be done by forcing halfspaces?
- Drifting games deal with this as well
  - Halfspace forcing is a constraint on adversary, by definition satisfying Blackwell condition
  - Drifting games set weights = “derivative" of potential
  - Boosting, hedging (NormalHedge) are examples
- Game-theoretic supermartingales
  - Vovk’s algorithms, markets involve forcing a function to lie on a half-line

Approachability Implies No-Regret Strategies

- Potential-based approachability algorithms can be used to play games (experts = finite strategy set)
- Want to keep regrets (payoffs) low, i.e. approach $S = \{s : s_i \leq 0 \ \forall i \leq N\}$
- S is response-satisfiable (put all weight on best expert) $\implies$ approachable
- So there exists a set of player moves such that

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \max_{i\in[N]} r_{i,t} = 0 \implies \lim_{T\to\infty} \max_{i\in[N]} \frac{1}{T} \sum_{t=1}^{T} r_{i,t} = 0$$

- This verifies the existence of an algorithm with asymptotically vanishing regret - *Hannan consistency*.

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Approachability and No-Regret Strategies

- Approachability games lead to no-regret learning algorithms (potential-based)
- Natural problem considered: online linear optimization (experts setting)
- Generic hammer to apply approachability?
  - Abernethy et al. (2011) produce calibrated probability predictions in $\{0, \frac{1}{m}, \ldots, 1\}$ with it
  - Payoff space $\mathbb{R}^{m+1}$, Φ measures discrepancy between actual and predicted probabilities for each bin
  - S is a small ball around the origin, response-satisfiable
  - Construction: Halfspace oracle possible to implement efficiently, approachability algorithm: GD
  - Other such examples?

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Summary

- Blackwell approachability: generalization of minimax to vector-valued games
- Can be viewed as minimizing a potential (moving down a conservative force field)
- Framework to study halfspace-forcing phenomena in algorithms

Many thanks! Questions?

Sources

- Abernethy, Bartlett, Hazan: Blackwell Approachability and No-Regret Learning are Equivalent. *COLT*, 2011.
- Cesa-Bianchi and Lugosi: Prediction, Learning and Games. 2006. (Ch. 7)
- Cesa-Bianchi and Lugosi: Potential-Based Algorithms in On-Line Prediction and Game Theory. *Machine Learning*, 2003.