Minimax Option Pricing Meets Black-Scholes in the Limit

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What is an option

Let us focus on the European call option, parameterized by an asset A, a strike price K and a future expiration data Y. The buyer of an (A, K, T) call option is allowed to purchase 1 share of asset A for a fixed price of K on data T is she chooses.

Pricing Options

A main topic in finance is the problem of pricing options in terms of known properties of the underlying asset and distributional properties of its price fluctuations.

Black-Scholes Pricing Model

Assume the Geometric Brownian motion (GBM) model for price path, then Black-Scholes suggest the fair price of the option should be

$$\mathbb{E}_{P \sim GBM}[max(0, P(T) - K)].$$

Investment is a game

The Black-Scholes model has undergone a reasonable amount of criticism, much of which is due to the GBM characterization of the underlying asset price, a model which heavily discounts tail risk.

Pricing options with less stochastic assumptions?

Pricing options is a game

Imagines a multi-stage game between an investor and Nature. The investor choose an amount of money to invest in the underlying asset, and Nature chooses how the asset's price should change from round to round. The investor would like to exhibit a trading strategy with the ultimate goal of earning almost as the payout of the option againt a worst-case price path.

Investment is a game

The value of options depends on

- the expiration data T;
- a specified strike price K;
- the price $X(T)$ of some underlying asset at time T.

Without loss of generality, we assume that $X(0) = 1$ is known in advance, and let $T = 1$. Let the payoff function is $g(x) = \max(0, x - K)$ where $K \geq 0$ is the strike price.

In the option pricing game we consider, we imagine that Nature chooses a randomized price path for the underlying asset, with the goal of maximizing the expected difference between the payoff of the option and the investor's earnings.

Notations

Let $\mathcal{X}$ denote the set of continuous stochastic process $X : [0,1] \to \mathbb{R}_0$ with $X(0) = 1$, representing the asset prices.

For each $n \in \mathbb{N}$, let $\mathcal{S}^n$ denote the set of sequences of random variable $S_n = (S_{n,0}, \ldots, S_{n,n})$ with $S_{n,0} = 1$ and $S_{n,m} \in \mathbb{R}_+$ for $0 \le m \le n$.

Given $X \in \mathcal{X}$, we write $S_n(X) = (X(0), X(1/n), \ldots, X(n/n)) \in \mathcal{S}^n$.

Given $S_n \in \mathcal{S}^n$, we write $X(S_n)$ to denote the stochastic process $X \in \mathcal{X}$ obtained by linear interpolating the values of S_n in the log space, i.e., for any $t \in [0,1]$ of the form $t = (m + \alpha)/n$ with $m \in \{0, 1, \ldots, n-1\}$ and $0 \le \alpha < 1$, we define

$$\log X(t) = (1 - \alpha) \log S_{n,m} + \alpha \log S_{n,m+1}.$$

Continuous Pricing Game

Investor's strategy

Investor's strategy $A \in \mathcal{A}_n$ is a tuple of functions $A_1, \ldots, A_n$ each having the form $A_m : X|_{[0,(m-1)/n]} \mapsto \Delta_m$ where $\Delta_m \in \mathbb{R}$ and $X|_{[0,t]}$ is the stochastic process $X \in \mathcal{X}$ restricted to the range $[0, t]$. In other words, the investor will choose an amount of money Δ_m to invest in the underlying asset after observing the known information.

Loss function

Assume now that the investor has committed to a strategy A and Nature has committed to a price path X. At round m, the invest has invested Δ_m units of currency in the underlying asset previously, and the price fluctuated from $X((m-1)/n)$ to $X(m/n)$. Hence, in this round the investor has earned exactly $(\frac{X(m/n)}{X((m-1)/n)} - 1)\Delta_m$.

Definition

For a particular trading strategy $A \in \mathcal{A}_n$ and a price path $X \in \mathcal{X}$, we define the loss

$$L_n(A, X) = \mathbb{E}_X[g(X(1)) - \sum_{m=1}^{n}(\frac{X(m/n)}{X((m-1)/n)} - 1)\Delta_m]$$

Notations

Geometric Brownian motion

Let $B : [0, 1] \to \mathbb{R}$ denote the standard Brownian motion with $B(0) = 0$, and let $G : [0, 1] \to \mathbb{R}_+$ denote the geometric Brownian motion with drift 0 and volatility $\sqrt{c}$,

$$G(t) = exp(\sqrt{c}B(t) - \frac{ct}{2}).$$

Also, we can write is as

$$dG(t) = \frac{c}{2}G(t)dt + \sqrt{c}G(t)dB(t)$$

Some constraints

Definition(CVC)

We say that $X \in \mathcal{X}$ satisfies the continuous variance constraint (CVC) if

$$\mathbb{E}[(X(t)-X(s))^2|\mathcal{F}_s] \leq (\exp(c(t-s))-1)X(s)^2 \text{ a.s. for all } 0 \leq s \leq t \leq 1,$$

here $\mathcal{F}_s$ is the filtration generated by X up to time s, and c is a parameter.

Definition(DVC)

We say that $S_n \in \mathcal{S}^n$ satisfies the discrete variance constraint (DVC) if

$$\mathbb{E}[(S_{n,m+1}-S_{n,m})^2|\mathcal{F}_{n,m}] \leq (\exp(c/n)-1)S_{n,m}^2 \text{ a.s. for all } m = 0, \ldots, n-1,$$

where now $\mathcal{F}_{n,m} = \sigma(S_{n,0}, \ldots, S_{n,m})$.

Some constraints

Definition(ZC$_n$)

We say that $S_n \in \mathcal{S}^n$ satisfies the ζ_n constraint if

$$|\frac{S_{n,m+1}}{S_{n,m}} - 1| \leq \zeta_n \text{ a.s. for all } m = 0, 1, \ldots, n-1.$$

Similarly, we say that $X \in \mathcal{X}$ satisfies the ζ_n constraint if $S_n(X)$ does.

We use ζ to denote a sequence of $(\zeta_n : n \in \mathbb{N})$ that approaches to 0 and

$$liminf_{n \to \infty} \frac{n\zeta_n^2}{\log n} > 16c.$$

Some constraints

Definition

Define the following sets:

- $\mathcal{X}_C = \{X \in \mathcal{X} : X \text{ satisfies } CVC\}$.
- $\mathcal{X}_{C,\zeta} = \{X \in \mathcal{X}_C : S_n(X) \text{ satisfies } ZC_n\}$.
- $\mathcal{S}_D^n = \{S_n \in \mathcal{S}^n : S_n \text{ satisfies } DVC\}$.
- $\mathcal{S}_{D,\zeta}^n = \{S_n \in \mathcal{S}_D^n : S_n \text{ satisfies } ZC_n\}$.
- $\mathcal{S}_{D,\zeta,mg}^n = \{S_n \in \mathcal{S}_{D,\zeta}^n : S_n \text{ is martingale respect to filtration } (\mathcal{F}_{m,n})\}$.
- $\mathcal{S}_{D=,\zeta,mg}^n = \{S_n \in \mathcal{S}_{D,\zeta,mg}^n : S_n \text{ satisfies DVC with equality a.s. for all } m\}$.

Main Theorem

Theorem

We have

$$\lim_{n \to \infty} \inf_{A \in \mathcal{A}_n} \sup_{X \in \mathcal{X}_{C,\zeta}^n} L_n(A, X) = \beta,$$

where $\beta = \mathbb{E}[g(G(1))]$ is the Black-Sholes price.

Proof of the theorem

Proof: Step 1

In the first step, we change the quantifiers by the famous minmax theorem, i.e.,

Lemma 1

For every n we have

$$\inf_{A \in \mathcal{A}_n} \sup_{X \in \mathcal{X}_{C,\zeta}^n} L_n(A, X) = \sup_{X \in \mathcal{X}_{C,\zeta}^n} \inf_{A \in \mathcal{A}_n} L_n(A, X).$$

Also, since $S_n(\mathcal{X}_{C,\zeta}^n) \subseteq \mathcal{S}_{D,\zeta}^n$, we have that

$$\sup_{X \in \mathcal{X}_{C,\zeta}^n} \inf_{A \in \mathcal{A}_n} L_n(A, X) \leq \sup_{S_n \in \mathcal{S}_{D,\zeta}^n} \inf_{A \in \mathcal{A}_n} L_n(A, S_n).$$

Proof of the theorem

Step 2

In the second step, we show that the optimal strategy for natural is a martingale, and furthermore, the strategy of the investor does not matter,

$$\sup_{S_n \in \mathcal{S}^n_{D,\zeta}} \inf_{A \in \mathcal{A}_n} L_n(A, S_n) = \sup_{S_n \in \mathcal{S}^n_{D,\zeta,mg}} L_n(S_n).$$

Defining $T_{n,m} = \frac{S_{m,n}}{S_{n,m-1}} - 1$, we can unwind the game round by round:

$$left = \sup_{T_{n,1}} \inf_{\Delta_1} \mathop{\mathbb{E}}_{S_{n,1}} \left[-T_{n,1}\Delta_1 + \cdots + \sup_{T_{n,n}} \inf_{\Delta_n} \mathop{\mathbb{E}}_{(S_{n,n}|S_{n,m-1})}[-T_{n,n}\Delta_n + g(S_{n,n})]\right].$$

To ensure $L_n(A, S_n) > -\infty$, the adversary must set $\mathbb{E}[T_{n,m}|S_{n,m-1}] = 0$ for each m, meaning S_n must be a martingale sequence, and the investor's actions Δ_m are irrelevant.

Proof of the theorem

Step 3

In this step, it shows that the optimal strategy occurs at the boundary, i.e., for sufficiently large n, there exists $S_n^* \in \mathcal{S}_{D=,\zeta,mg}^n$ such that

$$L_n(S_n^*) = \sup_{S_n \in \mathcal{S}_{D,\zeta,mg}^n} L_n(S_n).$$

The key to this lemma is that maximization of a convex function always occurs at the boundary, and we omit the details here.

Proof of the theorem

Step 4

For any sequence $(S_n^*, n \in \mathbb{N})$ with $S_n^* \in \mathcal{S}_{D=,\zeta,mg}^n$,

$$X(S_n^*) \xrightarrow{d} G,$$

where G is the geometric Brownian motion. The proof is based on some techniques from functional analysis, and we omit the proof here.

Then we can show that, for any sequence $(S_n^*, n \in \mathbb{N})$ with $S_n^* \in \mathcal{S}_{D=,\zeta,mg}^n$,

$$\lim_{n \to \infty} L_n(S_n^*) = \beta.$$

Proof of the theorem

Step 5

$$\beta \leq \lim_{n \to \infty} \inf_{A \in \mathcal{A}_n} \sup_{X \in \mathcal{X}_{C,\zeta}^n} L_n(A, X).$$

For each $n \in \mathbb{N}$, define a stochastic process $G_n : [0,1] \to \mathbb{R}_+$ by setting $G_n = G$ is the geometric Brownian motion G satisfies ZC_n, and $G_n = 1$ otherwise. We can verify that $G_n \in \mathcal{X}_{C,\zeta}^n$ and G_n is martingale, thus

$$\inf_{A \in \mathcal{A}_n} \sup_{X \in \mathcal{X}_{C,\zeta}^n} L_n(A, X) \geq \inf_{A \in \mathcal{A}_n} L_n(A, G_n) = L_n(G_n) = \mathbb{E}[g(G_n(1))].$$

Proof of the theorem

Step 5 (cont)

By the requirement of ζ_n and the concentration property of G, we claim that for sufficiently large n,

$$\Pr(G \text{ does not violate } ZC_n) \geq (1 - \frac{1}{n^2})^n.$$

Then we can show that $|\mathbb{E}[g(G(1))] - \mathbb{E}[g(G_n(1))]| \to 0$ as $n \to \infty$.

The End