Universal source coding and the Online Bayes algorithm

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January 11, 2011

Outline

Combining experts in the log loss framework

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Generalization to larger sets of distributions

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- $\lceil L_A{}^T \rceil$ is the code length if A is combined with arithmetic coding.

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  - c^t is revealed.
- **Goal:** minimize regret:

$$-\sum_{t=1}^{T} \log p_A^t(c^t) + \min_{i=1,\ldots,N} \left(-\sum_{t=1}^{T} \log p_i^t(c^t) \right)$$

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- Dividing by T we get $\frac{L_A^T}{T} = \min_i \frac{L_i^T}{T} + \frac{\log N}{T}$

Upper bound on $\sum_{i=1}^{N} w_i^{T+1}$ for **Hedge**(η)

Lemma (upper bound)

For any sequence of loss vectors $\ell^1, \ldots, \ell^T$ *we have*

$$\ln\left(\sum_{i=1}^{N} w_i^{T+1}\right) \leq -(1 - e^{-\eta}) L_{\mathbf{Hedge}(\eta)}.$$

Tuning η as a function of T

- trivially $\min_i L_i \leq T$, yielding

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- per iteration we get:

$$\frac{L_{\textbf{Hedge}(\eta)}}{T} \leq \min_i \frac{L_i}{T} + \sqrt{\frac{2 \ln N}{T}} + \frac{\ln N}{T}$$

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- More generally, the regret is smaller if many of the experts perform well.

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  - For number of mistakes - Bayesian method cannot be "fixed". Requires variable learning rate.

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  - **Markov Chain Monte Carlo** Sample the posterior. Can sometimes be done efficiently. Efficient sampling relates to mixing rate of markov chain whose limit dist is the posterior dist.

Standardizing online prediction algorithms

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- $V(\vec{b}, \vec{X}, t)$ is computable (recursively enumerable).

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- technical details: On iteration t, $|\vec{X}| = t$. Use the predictions of programs $\vec{b}$ such that $|\vec{b}| \leq t$ and for which $V(\vec{b}, \vec{X}, 2^t) = 1$. Assing the remaining mass the prediction $1/2$ (insuring a loss of 1)

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- We don't pay a penalty for copies.
- More generally, the regret is smaller if many of the experts perform well.

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- Can we still get a meaningful bound?

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- We need a new lower bound on the final total weight

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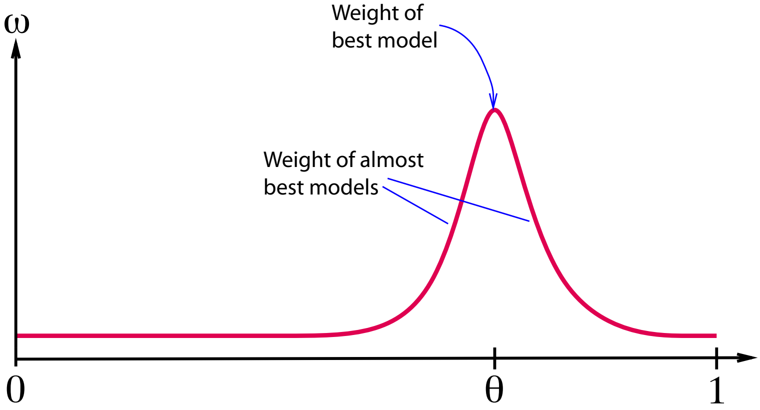

Expanding the exponent around the peak

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Laplace approximation (idea)

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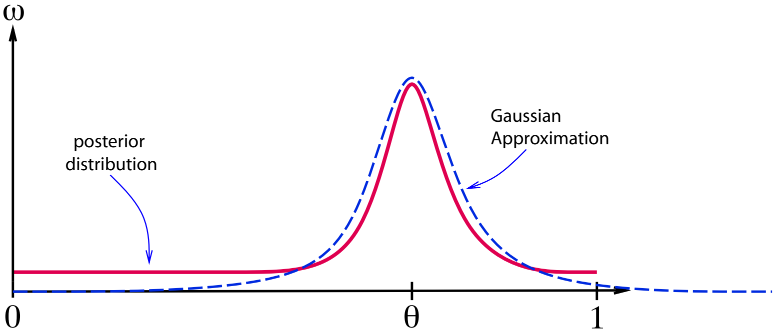

Laplace Approximation (details)

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$$\int_0^1 w(\theta) e^{T(g(\hat{\theta},\theta)-g(\hat{\theta},\hat{\theta}))} d\theta$$

$$= \quad w(\hat{\theta}) \sqrt{\frac{-2\pi}{T \left. \frac{d^2}{d\theta^2} \right|_{\theta=\hat{\theta}} (g(\hat{\theta},\theta) - g(\hat{\theta},\hat{\theta}))}} + O(T^{-3/2})$$

Choosing the optimal prior

- Choose $w(\theta)$ to maximize the worst-case final total weight

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- Make bound equal for all $\hat{\theta} \in [0,1]$ by choosing

$$w^*(\hat{\theta}) = \frac{1}{Z} \sqrt{\frac{\left.\frac{d^2}{d\theta^2}\right|_{\theta=\hat{\theta}} (g(\hat{\theta},\theta) - g(\hat{\theta},\hat{\theta}))}{-2\pi}},$$

where Z is the normalization factor:

$$Z = \sqrt{\frac{1}{2\pi}} \int_0^1 \sqrt{\left.\frac{d^2}{d\theta^2}\right|_{\theta=\hat{\theta}} (g(\hat{\theta},\hat{\theta}) - g(\hat{\theta},\theta))} \; d\hat{\theta}$$

The bound for the optimal prior

- Plugging in we get

$$
\begin{aligned}
L_A - L_{\min} &\leq \ln \int_0^1 w^*(\theta) e^{T(g(\hat{\theta},\theta)-g(\hat{\theta},\hat{\theta}))} d\theta \\
&= \ln \left(\sqrt{\frac{2\pi Z}{T}} + O(T^{-3/2}) \right) \\
&= \frac{1}{2} \ln \frac{T}{2\pi} - \frac{1}{2} \ln Z + O(1/T) \, .
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Solving for log-loss

- The exponent in the integral is

$$g(\hat{\theta},\theta) - g(\hat{\theta},\hat{\theta}) = \hat{\theta}\ln\frac{\hat{\theta}}{\theta} + (1-\hat{\theta})\ln\frac{1-\hat{\theta}}{1-\theta} = D_{KL}(\hat{\theta}||\theta)$$

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- The optimal prior:

$$w^*(\hat{\theta}) = \frac{1}{\pi\sqrt{\hat{\theta}(1-\hat{\theta})}}$$

Known in general as Jeffrey's prior. And, in this case, the Dirichlet-$(1/2, 1/2)$ prior.

The cumulative log loss of Bayes using Jeffrey's prior

- $$L_A - L_{\min} \leq \frac{1}{2}\ln(T+1) + \frac{1}{2}\ln\frac{\pi}{2} + O(1/T)$$

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- This is called the Trichevsky Trofimov prediction rule.

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- Suffers larger regret when $\hat{\theta}$ is far from $1/2$

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- The constant C is optimal.

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$\mathbf{H}$ denotes the Hessian.

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- Use Bayes Algorithm with Jeffrey's prior:

$$w^*(\hat{\theta}) = \frac{1}{Z} \frac{1}{\sqrt{\left| \mathbf{H}\left(D_{KL}(\hat{\theta}||\theta) \right) \Big|_{\theta=\hat{\theta}} \right|}}$$

H denotes the Hessian.

- $$L_A - L_{\min} \leq \frac{k-1}{2} \ln T - \ln Z + o(1)$$

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- [Haussler and Opper] show that the coefficient in front of $\ln T$ is optimal for distribution families where the metric entropy is up to
$$N(1/\epsilon) = O\left(e^{\epsilon^{-\alpha}}\right)$$
For all $\alpha \leq 5/2$.