Lossless compression and cumulative log loss

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- A natural way for describing a distribution.

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t	h	e	r	e		a
6	2	1	2	1	1	5

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t	h	e	r	e		a	r	e		n
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6	2	1	2	1	1	5	2	1	1	4	1

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- To decode use the same prediction algorithm

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- Widely used in practice.

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- After observing $c_1, c_2, \ldots, c_{t-1}$, predictor outputs $p(c_t = 1 \,|c_1, c_2, \ldots, c_{t-1}), \ldots, p(c_t = |\Sigma| \;|c_1, c_2, \ldots, c_{t-1}),$

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- Code = discriminating binary expansion of a point in $[l_t, u_t)$.

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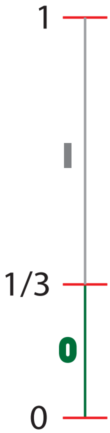

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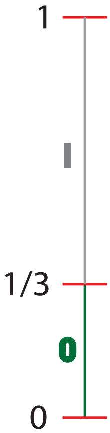

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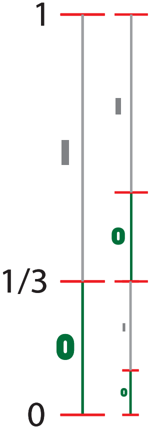

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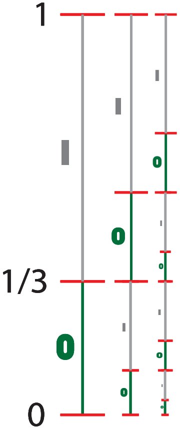

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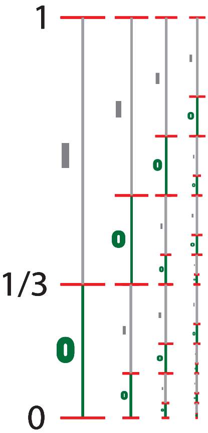

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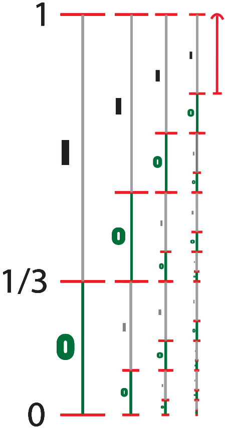

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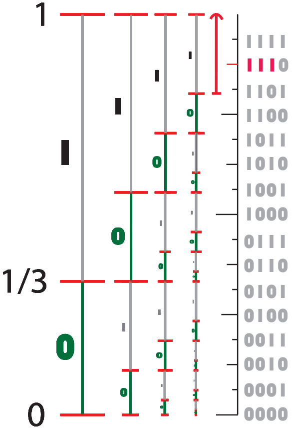

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- Holds for all sequences.

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$$\leq \quad 1 + \sum_{c_1, \ldots c_T} p(c_1, \ldots c_T) - \log_2 p(c_1, \ldots c_T)$$

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- Then the expected code length is

$$
\begin{aligned}
& \sum_{c_1, \ldots c_T} p(c_1, \ldots c_T) \lceil - \log_2 p(c_1, \ldots c_T) \rceil \\
\leq \quad & 1 + \sum_{c_1, \ldots c_T} p(c_1, \ldots c_T) - \log_2 p(c_1, \ldots c_T) \\
\doteq \quad & 1 + H(p_T)
\end{aligned}
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- $H(p)$ is the entropy of the distribution p.

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- The proof of Shannon's lower bound is not trivial (suggested project for 4 unit students).

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- There are other losses with this property, for example, square loss.

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- High risk prediction: Setting $p_t \in \{0, 1\}$ results in Bonus $b_T = 2^T$ if always correct, zero otherwise.
- If forecaster predicts with the true probabilities then

$$E(\log b_T) = T - H(p_T)$$

and that is the maximal expected value for $E(\log b_T)$

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- We would like to code almost as well as the best one.

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- Is the key idea of MDL (Minimal Description Length) modeling.
  - Good prediction model = model that minimizes the total code length
- Often inappropriate because based on lossless coding. **Lossy** coding often more appropriate.

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- Details: next class.