Exponential Weights Algorithms for Online Learning

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Outline

The Halving Algorithm

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Hedge(η)Algorithm

Hedging vs. Halving

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Bound on total loss

- Upper bound on $\sum_{i=1}^{N} w_i^{T+1}$
- Lower bound on $\sum_{i=1}^{N} w_i^{T+1}$
- Combining Upper and Lower bounds

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- Fits nicely in game theory

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- Basic idea - reduce probability of lossy actions, but not all the way to zero.
- **Modified Goal:** minimize difference between expected total loss and minimal total loss of repeating one action.

$$\sum_{t=1}^{T} \mathbf{p}^t \cdot \ell^t - \min_i \left(\sum_{t=1}^{T} \ell_i^t \right)$$

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  - outcome o_i^t is revealed. $\ell_i^t = 0$ if $e_i^t = o_i^t$, $\ell_i^t = 1$ otherwise.

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- As $\sum_{i=1}^{n} w_i^1 = 1$ increasing one weight implies decreasing some others.
- Plays a similar role to prior distribution in Bayesian algorithms.

Bound on the loss of **Hedge**(η)Algorithm

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- Theorem (main theorem)

  For any sequence of loss vectors $\ell^1, \ldots, \ell^T$, and for any $i \in \{1, \ldots, N\}$, we have

$$L_{\textbf{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}}.$$

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$$L_{\textbf{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}}.$$

  - Proof: by combining upper and lower bounds on $\sum_{i=1}^{N} w_i^{T+1}$

Upper bound on $\sum_{i=1}^{N} w_i^{T+1}$

Lemma (upper bound)

For any sequence of loss vectors $\ell^1, \ldots, \ell^T$ *we have*

$$\ln\left(\sum_{i=1}^{N} w_i^{T+1}\right) \leq -(1 - e^{-\eta}) L_{\mathbf{Hedge}(\eta)}.$$

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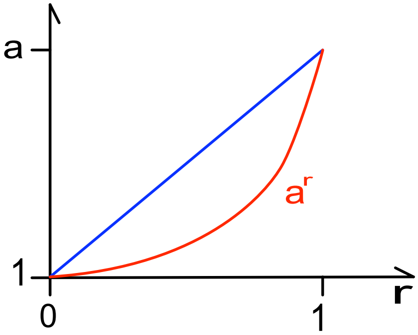

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Proof of upper bound (slide 2)

Applying $a^r \leq 1 - (1 - a)^r$ where $a = e^{-\eta}, r = \ell_i^t$

$$\sum_{i=1}^{N} w_i^{t+1} = \sum_{i=1}^{N} w_i^t e^{-\eta \ell_i^t}$$

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\sum_{i=1}^N w_i^{t+1} &= \sum_{i=1}^N w_i^t e^{-\eta \ell_i^t} \\
&\leq \sum_{i=1}^N w_i^t \left(1-(1-e^{-\eta})\ell_i^t\right)
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&= \left(\sum_{i=1}^{N} w_i^t\right) \left(1 - (1 - e^{-\eta}) \frac{\mathbf{w}^t}{\sum_{i=1}^{N} w_i^t} \cdot \ell^t\right)
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&= \left(\sum_{i=1}^N w_i^t\right)\left(1-(1-e^{-\eta})\mathbf{p}^t\cdot \ell^t\right)
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- Combining

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$$\sum_{i=1}^{N} w_i^{t+1} \leq \left(\sum_{i=1}^{N} w_i^{t} \right) \left(1 - (1 - e^{-\eta}) \mathbf{p}^t \cdot \ell^t \right)$$

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$$\begin{aligned}\sum_{i=1}^N w_i^{T+1} &\leq \prod_{t=1}^T\left(1-(1-e^{-\eta})\mathbf{p}^t\cdot\ell^t\right)\\ &\leq \exp\left(-(1-e^{-\eta})\sum_{t=1}^T\mathbf{p}^t\cdot\ell^t\right)\end{aligned}$$

since $1+x\leq e^x$ for $x=-(1-e^{-\eta})$.

Lower bound on $\sum_{i=1}^{N} w_i^{T+1}$

For any $j = 1, \ldots, N$:

$$\sum_{i=1}^{N} w_i^{T+1} \geq w_j^{T+1} = w_j^1 e^{-\eta L_j}$$

Combining Upper and Lower bounds

- Combining bounds on $\ln\left(\sum_{i=1}^{N} w_i^{T+1}\right)$

$$\ln w_j^1 - \eta L_j \leq \ln \sum_{i=1}^{N} w_i^{T+1} \leq -(1 - e^{-\eta}) \sum_{t=1}^{T} \mathbf{p}^t \cdot \ell^t$$

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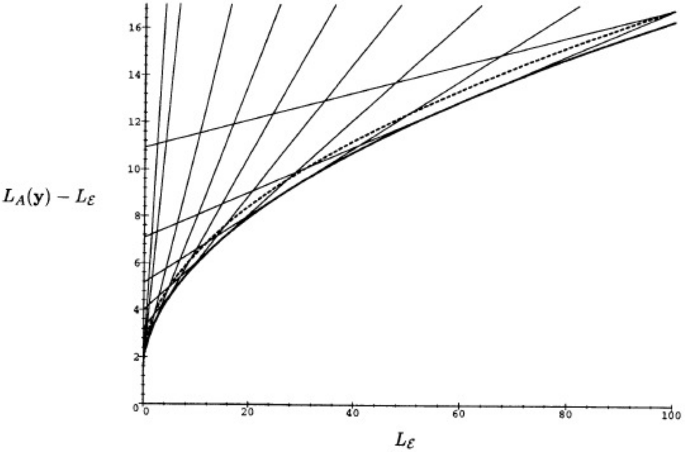
- Reversing signs, using $L_{\mathbf{Hedge}(\eta)} = \sum_{t=1}^{T} \mathbf{p}^t \cdot \ell^t$ and reorganizing we get

$$L_{\mathbf{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1-e^{-\eta}}$$

Tuning η

How to Use Expert Advice 451



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- Then

$$L_{\mathbf{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}} \leq \min_i L_i + \sqrt{2\tilde{L}\ln N} + \ln N$$

Tuning η as a function of T

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- per iteration we get:

$$\frac{L_{\textbf{Hedge}(\eta)}}{T} \leq \min_i \frac{L_i}{T} + \sqrt{\frac{2 \ln N}{T}} + \frac{\ln N}{T}$$

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- The adversarial strategy is random, extremely simple, and does not depend on the hedging strategy!

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- Detailed proof quite involved. See games paper.

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- A trivial random data, in which there is nothing to be learned forces any algorithm to suffer this total loss

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- Register to the class on the google drive.