Linear Pattern Recognition

Prediction Learning and Games: Chapter 11

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Agenda

Agenda

1. Motivation
   - Sequence Prediction with Experts
   - Sequence Prediction with Side Data
2. Mathematical Background
3. Review of Expert Prediction
4. Linear Pattern Recognition
5. Conclusion

Motivation

Typical Expert Setting

Decision Space $f_{i,t}, \hat{p}_t \in \mathcal{D}$

Outcome Space $y_t \in \mathcal{Y}$

Loss Function $\ell : \mathcal{D} \times \mathcal{Y} \mapsto \mathbb{R}$

1. Environment reveals n expert values $f_{i,t}$ for $i \in \{1, ..., n\}$
2. Forecaster make prediction $\hat{p}_t$ using expert values
3. Environment reveals truth y_t
4. Every expert and the forecaster suffer loss via loss function ℓ
5. Regret is $\max_i \sum_t \ell(\hat{p}_t, y_t) - \ell(f_{i,t}, y_t)$

Motivation

Prediction with Side Information Setting

Decision Space $\hat{p}_t \in \mathbb{R}$,

Outcome Space $y_t \in \mathbb{R}$

Loss Function $\ell : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$

Information Space $x_t \in \mathbb{R}^d$

1. Environment reveals side information x_t
2. Forecaster make prediction $\hat{p}_t = w_t \cdot x_t$ with $w_t \in \mathbb{R}^d$
3. Environment reveals truth y_t
4. Forecaster suffers loss $\ell(\hat{p}_t, y_t)$
5. Regret is $\max_u \sum_t \ell(\hat{p}_t, y_t) - \ell(u \cdot x_t, y_t)$

Each possible weight vector $w \in \mathbb{R}^d$ is an "expert"

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Legendre Functions

- A function $F : \mathcal{A} \mapsto \mathbb{R}$ is Legendre if 3 properties hold:
  1. $\mathcal{A} \subseteq \mathbb{R}^d$ is nonempty and $\mathrm{int}(\mathcal{A})$ is convex
  2. F is strictly convex and is continuously differentiable
  3. As x approaches a boundary of $\mathcal{A}$, $||\nabla F(x)|| \to \infty$
- For all Legendre functions, F there is a dual $F^\star : \mathcal{A}^\star \mapsto \mathbb{R}$
  - Defined as: $F^\star(u) = \sup_{v \in \mathcal{A}} (u \cdot v - F(v))$
  - $\mathcal{A}^\star$ is the range of $\nabla F : \mathrm{int}(\mathcal{A}) \mapsto \mathbb{R}^d$
  - $(F^\star)^\star = F$
  - Lemma 11.5: $\nabla F^\star = (\nabla F)^{-1}$

Legendre Function: Example

- The squared p-norm ($\frac{1}{2}||u||_p^2$, $p \geq 2$) is Legendre
  - $\mathcal{A} = \mathbb{R}^d$, (obviously non empty and convex)
  - All norm functions are convex
  - $(\nabla F(x))_i = \frac{\text{sign}(x_i)|x_i|^{p-1}}{||x||_p^{p-2}}$ which goes to ∞ as x_i does
- $F^\star = \frac{1}{2}||u||_q^2$ such that $\frac{1}{p} + \frac{1}{q} = 1$
  - $(\nabla F(x))^{-1} = \nabla \frac{1}{2}||x||_q^2$

Bregman Divergence

- A Bregman divergence is a way of defining a distance measure using a Legendre function

Bregman Divergence on F

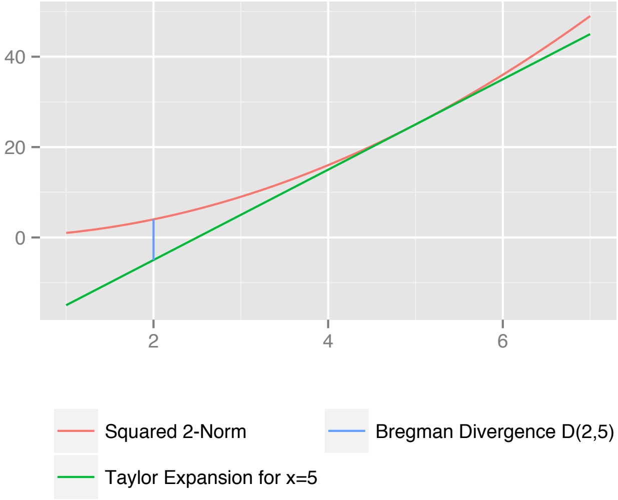
Let F be a Legendre function, then the Bregman divergence induced by F is:

$$D_F(u, v) = F(u) - F(v) - (u - v)\nabla F(v)$$

- The difference between $F(u)$ and its first order Taylor approximation about v
- Lemma 11.1:

$$D_F(u, v) + D_F(v, w) = D_F(u, w) + (u - v)(\nabla F(w) - \nabla F(v))$$

Bregman Divergence: Visual Intuition



Bregman Projections

Bregman Projection

A Bregman projection of v onto a convex set S is defined as:

$$\mathcal{P}_F(v) = \text{argmin}_{u \in S} D_F(u, v)$$

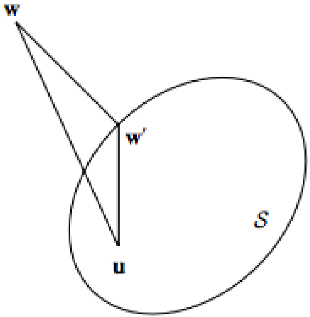
This is the point in S closest to v, as defined by D_F

Generalized Pythagorean Inequality

Generalized Pythagorean Inequality

For all $w \in \text{int}(\mathcal{A})$, and all convex and closed sets $S \subseteq \mathbb{R}^d$ with $S \cap \mathcal{A} \neq \emptyset$, and $w' = \mathcal{P}_F(w)$:

$$D_F(u, w) \geq D_F(u, w') + D_F(w', w) \; \forall u \in S$$



Proof of Generalized Pythagorean Inequality

- Define $G(x) = D(x, w) - D(x, w')$, expanding shows this is linear
- Let $x_\alpha = \alpha u + (1 - \alpha) w'$
- Due to linearity we get: $G(x_\alpha) = \alpha G(u) + (1 - \alpha) G(w')$
- Expanding:
  $D(x_\alpha, w) - D(x_\alpha, w') = \alpha(D(u, w) - D(u, w')) + (1 - \alpha) D(w', w)$
- Rearranging and assuming $\alpha > 0$:

$$\frac{D(x_\alpha, w) - D(x_\alpha, w') - D(w', w)}{\alpha} = D(u, w) - D(u, w') - D(w', w)$$

- Since w' was chosen to be point closest to w in S, $D(x_\alpha, w) \geq D(w', w)$ thus:

$$\frac{D(x_\alpha, w) - D(x_\alpha, w') - D(w', w)}{\alpha} \geq \frac{-D(x_\alpha, w')}{\alpha}$$

Proof of Generalized Pythagorean Inequality

- Rearranging gives:

$$D(u, w) + \frac{-D(x_\alpha, w')}{\alpha} \geq D(u, w') + D(w', w)$$

- Thus we must show an $\alpha > 0$ exists st:

$$\frac{-D(x_\alpha, w')}{\alpha} = 0$$

- At the limit this is true:

$$\lim_{\alpha \to 0^+} \frac{-D(x_\alpha, w')}{\alpha} = \lim_{\alpha \to 0^+} \frac{-D(w' + \alpha(u - w'), w') - D(w')}{\alpha}$$

- The rhs is the derivative of D at w' in the direction $u - w'$
- Since $D(w', w') = 0$, and D is non-negative, D' must be 0
- Hence the Generalized Pythagorean Inequality is true

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Weighted Average Predictor

- Predict the weighted average of the expert advice:

$$\hat{p}_t = \frac{\sum_{i=1}^{N} w_{i,t-1} f_{i,t}}{\sum_{j=1}^{N} w_{j,t-1}}$$

- Define w as the derivative of a *potential function*(Φ) of regret.

$$\Phi(u) = \psi\left(\sum_{i=1}^{N} \phi(u_i)\right)$$

  - $\phi : \mathbb{R} \mapsto \mathbb{R}$ is non-negative, increasing, and twice differentiable
  - $\psi : \mathbb{R} \mapsto \mathbb{R}$ is non-negative, strictly increasing, concave, and twice differentiable

- Define weights with this potential function:

$$w_{t-1} = \nabla\Phi(R_{t-1})$$

Weighted Average Predictor

- The "Blackwell Condition" states:

$$\sup_{y_t \in \mathcal{Y}} r_t \cdot \nabla \Phi(R_{t-1}) \leq 0$$

- The potential gradient and instantaneous regret point away from each other
- Thus the potential stays near its minimum
- Theorem 2.1:

$$\Phi(R_n) \leq \Phi(0) + \frac{1}{2} \sum_{t=1}^{n} C(r_t)$$

$$C(r_t) = \sup_{u \in \mathbb{R}^N} \psi' \left(\sum_{i=1}^{N} \phi(u_i) \right) \sum_{i=1}^{N} \phi''(u_i) r_{i,t}^2$$

Exponentially Weighted Average Forecaster

- Setting the potential to be:

$$\Phi_\eta(u) = \frac{1}{\eta} \log \sum_{i=1}^{N} \exp(\eta u_i)$$

- Plugging this potential into theorem 2.1 leads to this regret bound:

$$\max_i R_{i,n} \leq \frac{\log N}{\eta} + \frac{n\eta}{2}$$

- Tighter bounds are possible with specific loss functions

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Linear Pattern Recognition as Experts

The linear pattern recognition problem is very similar to experts, can we use the same algorithm?

NO

- In experts we measure regret vs best expert (finite number of experts)
- In linear pattern recognition we compare to best weight vector (infinite set)

However, using potentials and Legendre dual we can come up with a modified algorithm.

Potential-Based Gradient Descent

- Choose a potential Φ meeting previous requirements *AND* that is *Legendre*
- Then define $w_{t-1} = \nabla\Phi(R_{t-1})$
- Since Φ is Legendre, we get the following:

$$R_{t-1} = \nabla\Phi^{\star}(w_t)$$

Key Idea

Since we can't easily search for the w_t which did the best in the past, this dual formulation allows us to directly minimize our increase in regret.

Regret Updating

- Define $\theta_t = R_t$ to reinforce the notion that regret is minimized

Regret Update Rules

Primal	*Dual*
$\theta_t = \theta_{t-1} + r_t$	$\nabla\Phi^\star(w_t) = \nabla\Phi^\star(w_{t-1}) + r_t$

- After updating regret, we then use duality to update weights

$$w_t = \nabla\Phi(\theta_t)$$

Potential-Based Gradient Descent:Visual Intuition

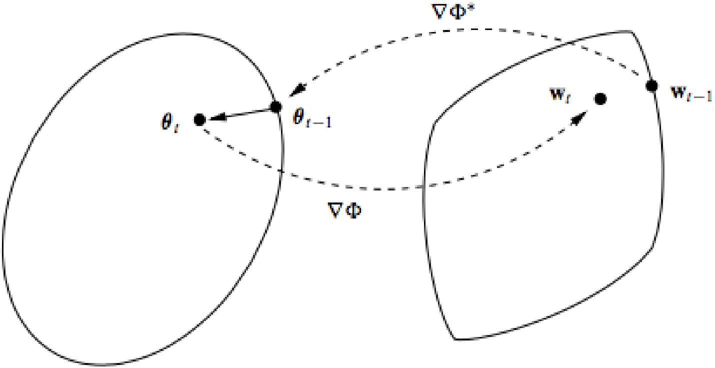

Figure: Duality of regret(left) and weight(right) space

Weight Updating

- This duality argument implies the weight update rule:

$$w_t = \nabla\Phi\left(\nabla\Phi^\star(w_{t-1}) + r_t\right)$$

- However, r_t still depends on the optimal weight
- r_t can be approximated by the *loss gradient*

Final Update Rule

$$w_t = \nabla\Phi\left(\nabla\Phi^\star(w_{t-1}) - \lambda\nabla\ell(x_t \cdot w_{t-1}, y_t)\right)$$

With $\lambda \geq 0$ being an arbitrary scale factor

Final Algorithm

1. Receive x_t from environment
2. Make prediction $\hat{p}_t = w_{t-1} \cdot x_t$
3. Receive y_t from environment
4. Incur loss $\ell(w_{t-1}) = \ell_t(\hat{p}_t, y_t)$
5. Update weights $w_t = \nabla\Phi\left(\nabla\Phi^\star(w_{t-1}) - \lambda\nabla\ell(x_t \cdot w_{t-1}, y_t)\right)$

Bound for Arbitrary Potential

Theorem 11.1

$$R_n(u) \le \frac{1}{\lambda} D_{\Phi^\star}(u, w_0) + \frac{1}{\lambda} \sum_{t=1}^{n} D_{\Phi^\star}(w_{t-1}, w_t)$$

Proof:

$$\begin{aligned}
\ell_t(w_{t-1}) &\le \ell_t(u) - (u - w_{t-1}) \cdot \nabla \ell_t(w_{t-1}) \\
&= \ell_t(u) + \frac{1}{\lambda}(u - w_{t-1}) \cdot (\nabla \Phi^\star(w_t) - \nabla \Phi^\star(w_{t-1})) \\
&= \ell_t(u) + \frac{1}{\lambda}(D_{\Phi^\star}(u, w_{t-1}) - D_{\Phi^\star}(u, w_t) + D_{\Phi^\star}(w_{t-1}, w_t))
\end{aligned}$$

We then sum over t and drop $-D_{\Phi^\star}(u, w_n)$ to complete to proof

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Conclusion

- This covered the basics of Linear Pattern Recognition
- Much more in the book:
  - Using transfer functions
  - Tracking weight vectors
  - Time varying potentials
  - etc
- Also chapter 12 extends this to linear classification

Useful Resources

Prediction Learning and Games Nicolò Cesa-Bianchi and Gábor Lugosi

Bregman Divergence http://mark.reid.name/blog/meet-the-bregman-divergences.html