Probability without Measure!

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Overview

A gambler's perspective

- This was the day before probability theory was even a field in mathematics, a field without foundations. Pascal and Fermat simply wanted to win a ton of money betting on horses and wanted to first see what it meant for a game to be fair.

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    - If I pay a\$
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    - The winner gets paid $2a$\$
- This is what is referred to as *inter alia* (equal terms)
- $P[E] =$ how much money you're willing to put on a game where you could win 1\$

Looking at the real world

Bernoulli was the first to suggest that probability can be measured from observation

$$P\{|y/N - p| < \epsilon\} > 1 - \delta$$

Now it seems that there could be a more mathematical treatment of probability..

Kolmogorov's axioms

- The axioms and definitions below relate a set Ω called the sample space and the set of subsets of Ω, $\mathcal{F}$. Every element in $E \in \mathcal{F}$ is called an event
  1. If $E, F \in \mathcal{F}$ then $E \cup F, E \cap F, E \setminus F \in \mathcal{F}$. Or more concisely we say that $\mathcal{F}$ is a field of sets.
  2. $\Omega \subset F$ which with the first axiom means that $\mathcal{F}$ is an algebra of sets
  3. Every set $E \in \mathcal{F}$ is assigned a probability which is a non-negative real value using the function $P : E \to [0, 1]$
  4. $P[\Omega] = 1$
  5. If $E \cap F = \Phi$ then $P[E \cap F] = P[E] + P[F]$, more generally we get what is called the union bound when E and F are not disjoint then $P[E \cap F] \leq P[E] + P[F]$
  6. If $\cap_{n=1}^{\infty} E_n = \Phi$ where $E_n \subseteq E_{n-1} \cdots \subseteq E_1$ we have that $\lim_{n\to\infty} P[E_n] = 0$. This axiom with axiom 2 allows us to call $\mathcal{F}$ a σ-algebra
- A random variable x is then understood as a mapping from the size of elements of $\mathcal{F}$ with respect to the probability measure P

Some Set Theory

Kolmogorov's 6th axiom needs the axiom of choice

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Weird things also happen: We get that there is no such thing as non-measurable sets

Sequential Learning

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- Predict the odds of a 1 (number shouldn't change much if we look at a subsequence called a collective)

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- Given a bit string 001111
- Predict the odds of a 1 (number shouldn't change much if we look at a subsequence called a collective)
- Fortunately we have a method of quantifying how difficult it is to predict the next bit in a string: Kolmogorov complexity!

Martingales

Originally Martingales are a gambling strategy that can guarantee a win of 1$ given an infinite supply of money

$$\alpha + 2\alpha + \cdots + 2^i\alpha$$

stop as soon as you win once

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Given a sequence of outcomes $x_1, \ldots, x_n$ we call a capital process L if

$$E[L(x_1, \ldots, x_n) \mid x_1, \ldots, x_{n-1}] = L(x_1, \ldots, x_n)$$

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$L(E) \to \infty$ if E has probability 0 (more on this next slide)
Now we define the probability of an event E as

$$P(E) = \inf\{L_0 \mid \lim_{n\to\infty} L_n \geq I\}$$

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Theorem (Doob's inequality)

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Other Chernoeff bounds can be derived in this way as well.

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- K_i is the skeptic's capital at time i
- M_n is the amount of tickets that the skeptic purchases
- x_n is the value of a ticket (determined by nature)

Bounded Fair Coin Game

$\mathcal{K}_0 = 1.$
FOR $n = 1, 2, \ldots$:
  Skeptic announces $M_n \in \mathbb{R}$.
  Reality announces $x_n \in \{-1, 1\}$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.

Theorem

There exists a winning strategy for skeptic but let's formally define what we mean by winning

Winning conditions

We claim that the skeptic wins if $K_n > 0 \forall n$ and if one two things happen, either

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i = 0$$

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Law of Large Numbers.

Skeptic bets ϵ on heads, this forces nature not to play heads often or else skeptic will become infinitely rich. So nature will start playing tails, when that happens skeptic puts an ϵ on tails. $\square$

Bounded Fair Coin Game

What if $x_n \in [-1, 1]$ instead of $\{-1, 1\}$?

Players: Skeptic, Reality

Protocol:

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Theorem

There exists a winning strategy for skeptic

Proof of Bounded Fair Coin Game

We will need some terminology to tackle this problem we define a real valued function on Ω called P which is a strategy that takes situations $s = x_1, x_2, \ldots, x_n$ and decides the number of tickets to buy $P(s)$.

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$$K^P(x_1x_2\ldots x_n) = K^P(x_1x_2\ldots x_{n-1}) + P(x_1x_2\ldots x_n)x_n$$

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Definition

Skeptic forces an event E if $K^P(s) = \infty \forall s \in E^c$

Proof of Bounded Fair Coin Game

Lemma

The skeptic can force

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} x_i \leq \epsilon$$

and

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} x_i \geq \epsilon$$

Proof of Bounded Fair Coin Game

Proof.

take 1 as the starting capital

$$1 + K^P(x_1 x_2 \dots x_n) = (1 + K^P(x_1 \dots x_{n-1}))(1 + \epsilon x_n) = \prod_{i=1}^{n}(1 + \epsilon x_i) < C$$

where C is a constant so take the log on both sides

$$\sum_{i=1}^{n} \ln(1 + \epsilon x_i) \leq D$$

Now use $\ln(1 + t) \geq t - t^2$ when $t \geq -1/2$

$$\frac{1}{n}\sum_{i=1}^{n} x_i \leq \frac{D}{\epsilon n} + \epsilon$$

and we get the top part of the lemma. Replace by $-\epsilon$ to get the second

Bounded Forecast games

Somebody has got to be setting the prices, let a forecaster announce price of ticket at iteration n as m_n

Parameter: $C > 0$
Players: Forecaster, Skeptic, Reality
Protocol:

$\mathcal{K}_0 := 1$.
FOR $n = 1, 2, \ldots$:
Forecaster announces $m_n \in [-C, C]$.
Skeptic announces $M_n \in \mathbb{R}$.
Reality announces $x_n \in [-C, C]$.
$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - m_n)$.

Theorem

There exists a winning strategy for skeptic by reduction to the bounded fair coin game.

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Proof.

First divide all prices by C to normalize prices to $[-1, 1]$ then set $m_n = 0$ and we recover the previous game. Note we also need to change the first condition to $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n}(x_i - m_i) = 0$ □

Measure theoretic law of large numbers

Assuming X_i are i.i.d random variables with mean μ and variance σ^2 we define $A_n = \frac{X_1+X_2+\cdots+X_n}{n}$ then $E[A_n] = \frac{n\mu}{n} = \mu$ and similarly $Var[A_n] = \frac{n\sigma^2}{n^2} = \sigma^2/n$. By Chebyshev's inequality we get the weak law of large numbers

$$P(\ |A_n - \mu| \geq \epsilon) \leq Var[A_n]/\epsilon^2 = \frac{\sigma^2}{n\epsilon^2}$$

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To prove Chebyshev's we define $A = \{w \in \Omega \mid |X(w)| \geq \alpha\}$. $X(w) \geq \alpha I_A(w)$. Take the expectation on both sides to get $E(|X|) \geq \alpha E(I_A) = \alpha P(A)$

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In game theoretic proof we don't need i.i.d assumption we don't even to assume a distribution exists!

Unbounded game

Players: Forecaster, Skeptic, Reality

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Theorem

If $\sum_{n=1}^{\infty} \frac{v_n}{n^2} < \infty$ *then the skeptic has a winning strategy*

Proof.

Similar in nature to proof of the bounded fair coin game. Main idea is that the skeptic's capital is a supermartingale (a sequence that decreases in expectation) $\square$

What about an application

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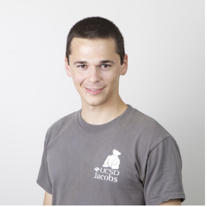

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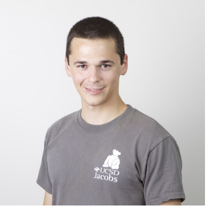

A natural next step is to make an infinitely large amount of money off the stock market

Unfortunately it seems that its difficult to have consistently better returns than the market and we will prove this. We make two assumptions that transaction costs are neglible (not as controversial as it sounds) and that the capital of a specific investor isn't too big relative to the market.

Securities Market Protocol

Parameters: $\mathcal{K}_0 > 0$, natural number $K > 1$
Players: Opening Market, Investor, Skeptic, Closing Market
Protocol:

FOR $n = 1, 2, \ldots$:

Opening Market selects $m_n \in [0,1]^K$ such that $\sum_{k=1}^K m_n^k = 1$.
Investor selects $g_n \in \mathbb{R}^K$.
Skeptic selects $h_n \in \mathbb{R}^K$.
Closing Market selects $x_n \in [-1, \infty)^K$ such that $m_n \cdot x_n = 0$.
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Proof.

Maybe next time, Finance theory might need its own talk :) $\square$

References

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Stochastic Calculus

Peter Clark
All I ever needed to know from Set Theory

Let's think about how this could change machine learning, talk to me and let's write a paper about it!