Lossless compression and cumulative log loss

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Outline

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- The guessing game
- Arithmetic coding
- The performance of arithmetic coding
- Source entropy
- Other properties of log loss
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  - Other examples for using log loss
- universal coding
  - Two part codes
  - Combining expert advice for cumulative log loss
- Combining experts in the log loss framework
- The online Bayes Algorithm
- The performance bound

The source compression problem

- **Example:** “There are no people like show people”
  $\xrightarrow{\text{encode}} x \in \{0,1\}^n$
  $\xrightarrow{\text{decode}}$ “there are no people like show people”
- **Lossless:** Message reconstructed perfectly.
- **Goal:** minimize expected length $E(n)$ of coded message.
- Can we do better than $\lceil \log_2(26) \rceil = 5$ bits per character?
- **Basic idea:** Use short codes for common messages.
- **Stream compression:**
  - Message revealed one character at a time.
  - Code generated as message is revealed.
  - Decoded message is constructed gradually.
- Easier than block codes when processing long messages.
- A natural way for describing a distribution.

The Guessing game

- Message reverealed one character at a time
- An algorithm predicts the next character from the revealed part of the message.
- If algorithm wrong - ask for next guess.
- **Example**

t	h	e	r	e		a	r	e		n	o		p	e
6	2	1	2	1	1	5	2	1	1	4	1	1	5	3

- Code = sequence of number of mistakes.
- To decode use the same prediction algorithm

Arithmetic Coding (background)

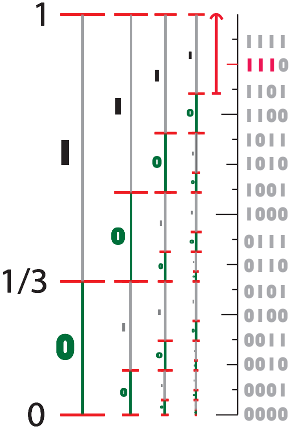
- Refines the guessing game:
  - In guessing game the predictor chooses order over alphabet.
  - In arithmetic coding the predictor chooses a Distribution over alphabet.
- First discovered by Elias (MIT).
- Invented independently by Rissanen and Pasco in 1976.
- Widely used in practice.

Arithmetic Coding (basic idea)

- Easier notation: represent characters by numbers $1 \leq c_t \leq |\Sigma|$. (English: $N \doteq |\Sigma| = 26$)
- message-prefix $c_1, c_2, \ldots, c_{t-1}$ represented by line segment $[l_{t-1}, u_{t-1})$
- Initial segment $[l_0, u_0) = [0, 1)$
- After observing $c_1, c_2, \ldots, c_{t-1}$, predictor outputs $p(c_t = 1 \,|c_1, c_2, \ldots, c_{t-1}), \ldots, p(c_t = |\Sigma| \;|c_1, c_2, \ldots, c_{t-1})$,
- Distribution is used to partition $[l_{t-1}, u_{t-1})$ into $|\Sigma|$ sub-segments.
- next character c_t determines $[l_t, u_t)$
- Code = discriminating binary expansion of a point in $[l_t, u_t)$.

Arithmetic Coding (sequence example)

- Simplest case.
- $\Sigma = \{0, 1\}$
- $\forall t,$
  $p(c_t = 0) = 1/3$
  $p_t(c_t = 1) = 2/3$
- Message = 1111
- Code = 111
- **Technical:**
  Assume decoder knows that length of message is 4.



The code length for arithmetic coding

- Given m bits of binary expansion we assume the rest are all zero.
- Distance between two m bit expansions is 2^{-m}
- If $l_T - u_T \geq 2^{-m}$ then there must be a point x described by m expansion bits such that $l_T \leq x < u_T$
- Required number of bits is $\lceil -\log_2(u_T - l_T) \rceil$.
- $u_T - l_T = \prod_{t=1}^{T} p\,(c_t \,|\, c_1, c_2, \ldots, c_{t-1}\,) \doteq p(c_1, \ldots c_T)$
- Number of bits required to code $c_1, c_2, \ldots, c_T$ is $\lceil -\sum_{t=1}^{T} \log_2 p_t(c_t) \rceil$.
- We call $-\sum_{t=1}^{T} \log_2 p_t(c_t) = -log_2 p(c_1, \ldots c_T)$ the Cumulative log loss
- Holds for all sequences.

Expected code length

- Fix the messsage length T
- Suppose the message is generated at random according to the distribution $p(c_1, \ldots c_T)$
- Then the expected code length is

$$\sum_{c_1,\ldots c_T} p(c_1, \ldots c_T) \lceil -\log_2 p(c_1, \ldots, c_T) \rceil$$

$$\leq \quad 1 - \sum_{c_1,\ldots c_T} p(c_1, \ldots c_T) \log_2 p(c_1, \ldots, c_T) \doteq 1 + H(p_T)$$

- $H(p_T)$ is the **entropy** of the distribution over sequences of length T:

$$H(p_T) \doteq \sum_{(c_1,\ldots,c_T)} p(c_1, \ldots, c_T) \log \frac{1}{p(c_1, dots, c_T)}$$

- Entropy is the expected value of the cumulative log loss

Shannon's lower bound

- Assume p_T is "well behaved". For example, IID.
- Let $T \to \infty$
- $H(p) \doteq \lim_{T\to\infty} \frac{H(p_T)}{T}$ exists and is called the per character entropy of the source p
- The expected code length for any coding scheme is at least
$$(1 - o(1))H(p_T) = (1 - o(1))\ T\ H(p)$$
- The proof of Shannon's lower bound is not trivial (Can be a student lecture).

log loss encourages unbiased prediction

- Suppose the source is random and the probability of the next outcome is $p(c_t | c_1, c_2, \ldots, c_{t-1})$
- Then the prediction that minimizes the log loss is $p(c_t | c_1, c_2, \ldots, c_{t-1})$.
- Note that when minimizing expected number of mistakes, the best prediction in this situation is to put all of the probability on the most likely outcome.
- There are other losses with this property, for example, square loss.

Monthly bonuses for a weather forecaster

- Before the first of the month assign one dollar to the forecaster's bonus. $b_0 = 1$
- Forecaster assigns probability p_t to rain on day t.
- If it rains on day t then $b_t = 2b_{t-1}p_t$
- If it does not rain on day t then $b_t = 2b_{t-1}(1 - p_t)$
- At the end of the month, give forecaster b_T
- Risk averse strategy: Setting $p_t = 1/2$ for all days, guarantees $b_T = 1$
- High risk prediction: Setting $p_t \in \{0, 1\}$ results in Bonus $b_T = 2^T$ if always correct, zero otherwise.
- If forecaster predicts with the true probabilities then

$$E(\log b_T) = T - H(p_T)$$

and that is the maximal expected value for $E(\log b_T)$

Horse-race betting

- You go to the horse races with one dollar $b_0 = 1$
- m horses compete in each race.
- Before each race, the odds for each horse are announced: $o_t(1), \ldots o_t(m)$ (arbitrary positive numbers)
- You have to divide *all* your money among the different horses. $\sum_{j=1}^{t} \hat{p}_t(j) = 1$
- The horse $1 \leq y_t \leq m$ is winner of the tth race.
- After iteration t, you have $b_t = b_{t-1}\hat{p}_t(y_t)o_t(y_t)$ dollars
- After n races, you have $b_n = \prod_{t=1}^{n} \hat{p}_t(y_t)o_t(y_t)$ dollars.
- Taking logs, we get cumulative log loss.

“Universal” coding

- Suppose there are N alternative predictors / experts.
- We would like to code almost as well as the best predictor.
- We would like to make almost as much money as the best expert in hind-site.

Two part codes

- Send the index of the coding algorithm before the message.
- Requires $\log_2 N$ additional bits.
- Requires the encoder to make **two** passes over the data.
- Is the key idea of MDL (Minimal Description Length) modeling.
  - Good prediction model = model that minimizes the total code length
- Often inappropriate because based on lossless coding. **Lossy** coding often more appropriate.

Combining predictors adaptively

- Treat each of the predictors as an "expert".
- Assign a weight to each expert and reduce it if expert performs poorly.
- Combine expert predictions according to their weights.
- Would require only a single pass. Truly online.
- **Goal:** Total loss of algorithm minus loss of best predictor should be at most $\log_2 N$

The log-loss framework

- Algorithm A predicts a sequence $c^1, c^2, \ldots, c^T$ over alphabet $\Sigma = \{1, 2, \ldots, k\}$
- The prediction for the c^tth is a distribution over Σ: $\mathbf{p}_A^t = \langle p_A^t(1), p_A^t(2), \ldots, p_A^t(k) \rangle$
- When c^t is revealed, the loss we suffer is $-\log p_A^t(c^t)$
- The cumulative log loss, which we wish to minimize, is $L_A{}^T = -\sum_{t=1}^T \log p_A^t(c^t)$
- $\lceil L_A{}^T \rceil$ is the code length if A is combined with arithmetic coding.

The game

- Prediction algorithm A has access to N experts.
- The following is repeated for $t = 1, \ldots, T$
  - Experts generate predictive distributions: $\mathbf{p}_1^t, \ldots, \mathbf{p}_N^t$
  - Algorithm generates its own prediction $\mathbf{p}_A^t$
  - c^t is revealed.
- **Goal:** minimize regret:

$$-\sum_{t=1}^{T} \log p_A^t(c^t) + \min_{i=1,\ldots,N} \left(-\sum_{t=1}^{T} \log p_i^t(c^t) \right)$$

The online Bayes Algorithm

- Total loss of expert i

$$L_i^t = -\sum_{s=1}^{t} \log p_i^s(c^s); \quad L_i^0 = 0$$

- Weight of expert i

$$w_i^t = w_i^1 e^{-L_i^{t-1}} = w_i^1 \prod_{s=1}^{t-1} p_i^s(c^s)$$

- Freedom to choose initial weights.
  $w_t^1 \geq 0, \sum_{i=1}^{n} w_i^1 = 1$
- Prediction of algorithm A

$$\mathbf{p}_A^t = \frac{\sum_{i=1}^{N} w_i^t \mathbf{p}_i^t}{\sum_{i=1}^{N} w_i^t}$$

The **Hedge**(η)Algorithm

Consider action i at time t

- Total loss:

$$L_i^t = \sum_{s=1}^{t-1} \ell_i^s$$

- Weight:

$$w_i^t = w_i^1 e^{-\eta L_i^t}$$

Note freedom to choose initial weight (w_i^1) $\sum_{i=1}^n w_i^1 = 1$.

- $\eta > 0$ is the learning rate parameter. Halving: $\eta \to \infty$
- Probability:

$$p_i^t = \frac{w_i^t}{\sum_{j=1}^N w_i^t}, \quad \mathbf{p}^t = \frac{\mathbf{w}^t}{\sum_{j=1}^N w_i^t}$$

Cumulative loss vs. Final total weight

Total weight: $W^t \doteq \sum_{i=1}^N w_i^t$

$$\frac{W^{t+1}}{W^t} = \frac{\sum_{i=1}^N w_i^t e^{\log p_i^t(c^t)}}{\sum_{i=1}^N w_i^t} = \frac{\sum_{i=1}^N w_i^t p_i^t(c^t)}{\sum_{i=1}^N w_i^t} = p_A^t(c^t)$$

$$-\log \frac{W^{t+1}}{W^t} = -\log p_A^t(c^t)$$

$$-\log W^{T+1} = -\log \frac{W^{T+1}}{W^1} = -\sum_{t=1}^T \log p_A^t(c^t) = L_A^T$$

EQUALITY not bound!

Simple Bound

- Use uniform initial weights $w_i^1 = 1/N$
- Total Weight is at least the weight of the best expert.

$$\begin{aligned} L_A^T &= -\log W^{T+1} = -\log \sum_{i=1}^{N} w_i^{T+1} \\ &= -\log \sum_{i=1}^{N} \frac{1}{N} e^{-L_i^T} = \log N - \log \sum_{i=1}^{N} e^{-L_i^T} \\ &\leq \log N - \log \max_i e^{-L_i^T} = \log N + \min_i L_i^T \end{aligned}$$

- Dividing by T we get $\frac{L_A^T}{T} = \min_i \frac{L_i^T}{T} + \frac{\log N}{T}$

Upper bound on $\sum_{i=1}^{N} w_i^{T+1}$ for **Hedge**(η)

Lemma (upper bound)

For any sequence of loss vectors $\ell^1, \ldots, \ell^T$ *we have*

$$\ln\left(\sum_{i=1}^{N} w_i^{T+1}\right) \leq -(1 - e^{-\eta}) L_{\mathbf{Hedge}(\eta)}.$$

Tuning η as a function of T

- trivially $\min_i L_i \leq T$, yielding

$$L_{\mathbf{Hedge}(\eta)} \leq \min_i L_i + \sqrt{2T\ln N} + \ln N$$

- per iteration we get:

$$\frac{L_{\mathbf{Hedge}(\eta)}}{T} \leq \min_i \frac{L_i}{T} + \sqrt{\frac{2\ln N}{T}} + \frac{\ln N}{T}$$

Bound better than for two part codes

- Simple bound as good as bound for two part codes (MDL) but enables online compression
- Suppose we have K copies of each expert.
- Two part code has to point to one of the KN experts
  $L_A \leq \log NK + \min_i L_i^T = \log NK + \min_i L_i^T$
- If we use Bayes predictor + arithmetic coding we get:

$$L_A = -\log W^{T+1} \leq \log K \max_i \frac{1}{NK} e^{-L_i^T} = \log N + \min_i L_i^T$$

- We don't pay a penalty for copies.
- More generally, the regret is smaller if many of the experts perform well.