Solution Of Homework 3

January 27, 2011

1. **Exercise 4.8**

   Denote the actions by $\{1, \ldots, N\}$. Internal regret takes the maximum over all pairs of actions. Swap regret takes the maximum over all functions from the set of actions into itself. To prove the desired bound we partition the function σ which achieves the highest smap regret into N action pairs.

   Formally, let $R_{(i,j),n}$ be the regret associated with (not) taking action j each time we took action i, and $R_{\sigma,n}$ be the regret associate with the mapping σ.

   As the sequences of times at which actions $i \neq j$ have been taken are disjoint, we get that As the times at which different actions were taken we get that, for any mapping σ

   $$R_{\sigma,n} = \sum_{i=1}^{N} R_{(i,\sigma(i)),n} \leq N \max_{i,j} R_{(i,j),n}$$

   From which it follows that

   $$\max_{\sigma} R_{\sigma,n} \leq N \max_{i,j} R_{(i,j),n}$$

2. **Exercise 4.9**

   Intuitively, the reason that a deterministic predictor cannot be well calibrated for any sequence is that an adversary can construct the next bit in the sequence y_t as a function of the deterministically predetermined prediction q_t.

   We want to show non ϵ-calibration for $\epsilon < 1/3$.

   We consider the following strategy for the adversary:

   - If $q_t < 2/3$ then $y_t = 1$
   - If $q_t \geq 2/3$ then $y_t = 0$

To prove that the resulting sequence will not be ϵ-calibrated we need to show the existance of x such that $q_t \in [x-\epsilon, x+\epsilon]$ occurs infinitely often and $|\rho_n^\epsilon(x)-x| > \epsilon$

Let m be the number of times t in the infinite sequence for which $q_t < 2/3$ and $y_t = 1$ We consider two cases, conditioned on whether or not this m is finite.

If m is infinite, then there must be an $x \leq 2/3-\epsilon$ such that there are an infinite number of ts such that $q_t \in [x-\epsilon, x+\epsilon]$. For this x, $\rho_n^\epsilon(x) = 1$ and thus $\limsup_{n\to\infty} |x - \rho_n^\epsilon(x)| \geq 1/3 + \epsilon > \epsilon$.

If m is finite, the number of times that $y_t = 1$ is finite. Therefor $\lim_{n\to\infty} \rho_n^\epsilon(x) = 0$. On the other hand, there must be some $x \geq 2/3$ for which $(x-\epsilon, x+\epsilon)$ contains an infinite number of elements, as $\lim_{n\to\infty} \rho_n^\epsilon(x) = 0$ for all x, we get that in this case there is also no calibration.

3. **Exercise 4.10**

We are given that $\lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} y_t = a$. We are using the estimator $q_t = \frac{1}{t-1}\sum_{s=1}^{t-1} y_t$

If the number of times that $q_n \in (x-\epsilon, x+\epsilon)$ is infinite, then $|x-a| \leq \epsilon$. As the estimator q_n converges to a this implies that

$$\limsup_{n\to\infty} |\rho_n^\epsilon(x) - x| = |a - x| \leq \epsilon$$

As ϵ is arbitrary, we get that the estimator is well calibrated.