Exponential Weights Algorithms for Online Learning

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Outline

The hedging problem

- N possible actions
- At each time step $t = 1, 2, \ldots, T$:
  - Algorithm chooses a distribution $\mathbf{p}^t$ over actions.
  - Losses $0 \leq \ell_i^t \leq 1$ of all actions $i = 1, \ldots, N$ are revealed.
  - Algorithm suffers **expected** loss $\mathbf{p}^t \cdot \boldsymbol{\ell}^t$
- **Goal:** minimize total expected loss
- Here we have stochasticity - but only in **algorithm**, not in **outcome**
- Fits nicely in game theory

Hedging vs. Halving

- Like halving - we want to zoom into best action (expert).
- Unlike halving - no action is perfect.
- Basic idea - reduce probability of lossy actions,
  but not all the way to zero.
- **Modified Goal:** minimize difference between
  expected total loss
  and
  minimal total loss of repeating one action.

$$\sum_{t=1}^{T} \mathbf{p}^t \cdot \ell^t - \min_i \left(\sum_{t=1}^{T} \ell_i^t \right)$$

Using hedge to generalize halving alg.

- Suppose that there is no perfect expert.
- action i = use prediction of expert i
- Now each iteration of game consistst of three steps:
  - Experts make predictions $e_i^t \in \{0, 1\}$
  - Algorithm predicts 1 with probability $\sum_{i:e_i^t=1} p_i^t$.
  - outcome o_i^t is revealed. $\ell_i^t = 0$ if $e_i^t = o_i^t$, $\ell_i^t = 1$ otherwise.

The **Hedge**(η)Algorithm

Consider action i at time t

- Total loss:

$$L_i^t = \sum_{s=1}^{t-1} \ell_i^s$$

- Weight:

$$w_i^t = w_i^1 e^{-\eta L_i^t}$$

Note freedom to choose initial weight (w_i^1) $\sum_{i=1}^{n} w_i^1 = 1$.

- $\eta > 0$ is the learning rate parameter. Halving: $\eta \to \infty$
- Probability:

$$p_i^t = \frac{w_i^t}{\sum_{j=1}^{N} w_i^t}, \quad \mathbf{p}^t = \frac{\mathbf{w}^t}{\sum_{j=1}^{N} w_i^t}$$

Choosing the initial weights

- Giving an action high initial weight makes alg perform well if that action performs well.
- If good action has low initial weight, our total loss will be larger.
- As $\sum_{i=1}^{n} w_i^1 = 1$ increasing one weight implies decreasing some others.
- Plays a similar role to prior distribution in Bayesian algorithms.

Bound on the loss of **Hedge**(η)Algorithm

- Theorem (main theorem)
  For any sequence of loss vectors $\ell^1, \ldots, \ell^T$*, and for any* $i \in \{1, \ldots, N\}$*, we have*

$$L_{\textbf{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}}.$$

  - Proof: by combining upper and lower bounds on $\sum_{i=1}^{N} w_i^{T+1}$

Upper bound on $\sum_{i=1}^N w_i^{T+1}$

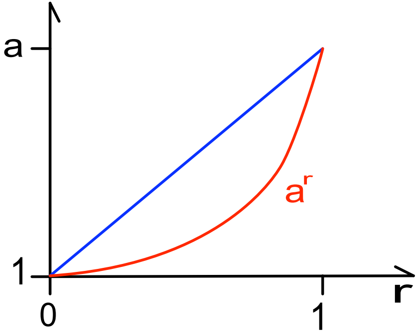
Lemma (upper bound)

For any sequence of loss vectors $\ell^1, \ldots, \ell^T$ we have

$$\ln\left(\sum_{i=1}^N w_i^{T+1}\right) \le -(1-e^{-\eta}) L_{\mathbf{Hedge}(\eta)}.$$

Proof of upper bound (slide 1)

- If $a \geq 0$ then a^r is convex.
- For $r \in [0, 1]$, $a^r \leq 1 - (1 - a)r$



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Proof of upper bound (slide 2)

Applying $a^r \leq 1-(1-a)^r$ where $a = e^{-\eta}, r = \ell_i^t$

$$\begin{aligned}
\sum_{i=1}^N w_i^{t+1} &= \sum_{i=1}^N w_i^t e^{-\eta \ell_i^t} \\
&\leq \sum_{i=1}^N w_i^t \left(1-(1-e^{-\eta})\ell_i^t\right) \\
&= \left(\sum_{i=1}^N w_i^t\right)\left(1-(1-e^{-\eta})\frac{\mathbf{w}^t}{\sum_{i=1}^N w_i^t}\cdot \ell^t\right) \\
&= \left(\sum_{i=1}^N w_i^t\right)\left(1-(1-e^{-\eta})\mathbf{p}^t \cdot \ell^t\right)
\end{aligned}$$

Proof of upper bound (slide 3)

- Combining

$$\sum_{i=1}^{N} w_i^{t+1} \leq \left(\sum_{i=1}^{N} w_i^t\right)\left(1-(1-e^{-\eta})\mathbf{p}^t\cdot\ell^t\right)$$

- for $t = 1, \ldots, T$
- yields

$$\begin{aligned}\sum_{i=1}^{N} w_i^{T+1} &\leq \prod_{t=1}^{T}\left(1-(1-e^{-\eta})\mathbf{p}^t\cdot\ell^t\right)\\ &\leq \exp\left(-(1-e^{-\eta})\sum_{t=1}^{T}\mathbf{p}^t\cdot\ell^t\right)\end{aligned}$$

since $1+x \leq e^x$ for $x = -(1-e^{-\eta})$.

Lower bound on $\sum_{i=1}^N w_i^{T+1}$

For any $j = 1, \ldots, N$:

$$\sum_{i=1}^{N} w_i^{T+1} \geq w_j^{T+1} = w_j^1 e^{-\eta L_j}$$

Combining Upper and Lower bounds

- Combining bounds on $\ln\left(\sum_{i=1}^{N} w_i^{T+1}\right)$

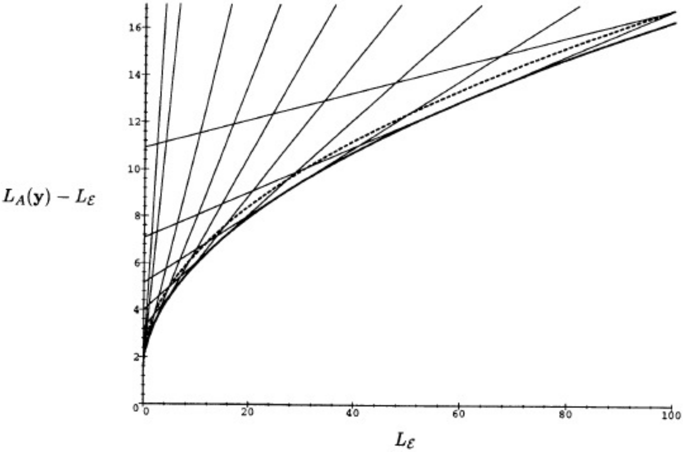
$$\ln w_j^1 - \eta L_j \le \ln \sum_{i=1}^{N} w_i^{T+1} \le -(1-e^{-\eta}) \sum_{t=1}^{T} \mathbf{p}^t \cdot \ell^t$$

- Reversing signs, using $L_{\mathbf{Hedge}(\eta)} = \sum_{t=1}^{T} \mathbf{p}^t \cdot \ell^t$ and reorganizing we get

$$L_{\mathbf{Hedge}(\eta)} \le \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}}$$

Tuning η

How to Use Expert Advice 451



Tuning η

- Suppose $\min_i L_i \le \tilde{L}$
- set

$$\eta = \ln\left(1 + \sqrt{\frac{2\ln N}{\tilde{L}}}\right) \approx \sqrt{\frac{2\ln N}{\tilde{L}}}$$

- use uniform initial weights $\mathbf{w}^1 = \langle 1/N, \ldots, 1/N \rangle$
- Then

$$L_{\mathbf{Hedge}(\eta)} \le \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}} \le \min_i L_i + \sqrt{2\tilde{L}\ln N} + \ln N$$

Tuning η as a function of T

- trivially $\min_i L_i \leq T$, yielding

$$L_{\textbf{Hedge}(\eta)} \leq \min_i L_i + \sqrt{2T \ln N} + \ln N$$

- per iteration we get:

$$\frac{L_{\textbf{Hedge}(\eta)}}{T} \leq \min_i \frac{L_i}{T} + \sqrt{\frac{2 \ln N}{T}} + \frac{\ln N}{T}$$

How good is this bound?

- Very good! There is a closely matching lower bound!
- There exists a stochastic adversarial strategy such that with high probability for any hedging strategy **S** after T trials

$$L_{\mathbf{S}} - \min_i L_i \geq (1 - o(1))\sqrt{2T \ln N}$$

- The adversarial strategy is random, extremely simple, and does not depend on the hedging strategy!

The adversarial strategy

- Adversary sets each loss ℓ_i^t indepedently at random to 0 or 1 with equal probabilities $(1/2, 1/2)$.
- Obviously, nothing to learn !
  $L_{\mathbf{S}} \approx T/2$.
- On the other hand $\min_i L_i \approx T/2 - \sqrt{2T \ln N}$
- The difference $L_{\mathbf{S}} - \min_i L_i$ is due to unlearnable random fluctuations!
- Detailed proof quite involved. See games paper.

Summary

- Given learning rate η the **Hedge**(η)algorithm satisfies

$$L_{\textbf{Hedge}(\eta)} \leq \frac{\ln N + \eta L_i}{1 - e^{-\eta}}$$

- Setting $\eta \approx \sqrt{\frac{2 \ln N}{T}}$ guarantees

$$L_{\textbf{Hedge}(\eta)} \leq \min_i L_i + \sqrt{2T \ln N} + \ln N$$

- A trivial random data, in which there is nothing to be learned forces any algorithm to suffer this total loss

Some loose threads

- Total Loss of best action usually scales linearly with time T, but we need to know the **horizon** T ahead of time to choose η correctly.
- T is tight only when the loss of experts at each iteration is either 0 or 1. If the loss of the best expert is $o(T)$ then we would like to have a tighter bound.
- Observing only the loss of chosen action - the multi-armed bandit problem. Will get to that later in the course.
- Register to the class on the google drive.