Prediction and Playing Games

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K-Person Normal Form Games

- Each player k $(k = 1, ..., K)$ has N_k possible actions - $i_k \in 1, ..., N_K$
- K-tuple of all the players' actions $\mathbf{i} = (i_1, ..., Ii_K) \in \bigotimes_{k=1}^{K} \{1, ..., N_k\}$
- Loss suffered by player k is $\ell^{(k)}$, where $\ell^{(k)} : \mathbf{i} \rightarrow [0, 1]$

Mixed strategy

- Mixed strategy for player k is a probability distribution $\mathbf{p}^k = (p_1^{(k)}, ..., p_{N_k}^{(k)})$
- Action played by player k, $I^{(k)}$ is a random variable taking values in the set $\{1, ..., N_k\}$ according to the distribution $\mathbf{p}^{(k)}$
- K-tuple of actions played by all players, $\mathbf{I} = (I^{(1)}, ..., I^{(K)})$
- Mixed strategy profile,

$$\pi(i) = \mathbb{P}[\mathbf{I} = \mathbf{i}] = p_{i_1}^{(1)} \; x \; . \; . \; . \; x \; p_{i_K}^{(K)}$$

  for all $\mathbf{i} = (i_1, ... i_k) \in \bigotimes_{k=1}^{K} \{1, ..., N_k\}$

Nash Equilibrium

- The expected loss of player is

$$\pi\ell^{(k)} \equiv \mathbb{E}\,\ell^{(k)}(\mathbf{I})$$
$$= \sum_{i_1=1}^{N_1} \dots \sum_{i_K=1}^{N_K} p_{i_1}^{(1)}\, x \dots x\, p_{i_K}^{(K)} \ell^{(k)}(i_1, \dots, i_K)$$

- A mixed strategy profile $\pi = \mathbf{p}^{(1)}\ x \dots x\,\mathbf{p}^{(K)}$ is called a Nash Equilibrium if

$$\pi\,\ell^k \leq \pi'_k\,\ell^k$$

  for all $k = 1, \dots, K$ and mixed strategies $\mathbf{q}^{(k)}$,
  $\pi'_k = \mathbf{p}^{(1)}\ x \dots x\,\mathbf{q}^{(k)}\ x \dots x\,\mathbf{p}^{(K)}$
- **Nash Theorem** : Every finite game has a mixed strategy Nash equilibrium

Two-Person Zero-Sum Games

- For each pair of actions $\mathbf{i} = (i_1, i_2)$, where $i_1 \in \{1, ..., N_1\}$ and $i_2 \in \{1, ..., N_2\}$, the losses of the two players satisfy
$$\ell^{(1)}(\mathbf{i}) = -\ell^{(2)}(\mathbf{i})$$
- Simplifying notation - replace $\ell^{(1)}, N_1, N_2$ by ℓ, N, M
- Mixed strategy profile $\pi = \mathbf{p} \, x \, \mathbf{q}$, where $\mathbf{p} = (p_1, ..., p_N)$ and $\mathbf{q} = (q_1, ..., q_M)$, is a Nash equilibrium if and only if for all $\mathbf{p}' = (p'_1, ..., p'_N)$ and $\mathbf{q}' = (q'_1, ..., q'_M)$,

$$\sum_{i=1}^{N} \sum_{j=1}^{M} p_i \, q'_j \, \ell(i,j) \, \leq \, \sum_{i=1}^{N} \sum_{j=1}^{M} p_i \, q_j \, \ell(i,j) \, \leq \, \sum_{i=1}^{N} \sum_{j=1}^{M} p'_i \, q_j \, \ell(i,j)$$

Two-Person Zero-Sum Games

- Introduce notation $\bar{\ell}(\mathbf{p},\mathbf{q}) = \sum_{i=1}^{N} \sum_{j=1}^{M} p_i \, q_j \, \ell(i,j)$

$$\max_{\mathbf{q}'} \bar{\ell}(\mathbf{p},\mathbf{q}') = \bar{\ell}(\mathbf{p},\mathbf{q}) = \min_{\mathbf{p}'} \bar{\ell}(\mathbf{p}',\mathbf{q})$$

$$\max_{\mathbf{q}'} \bar{\ell}(\mathbf{p},\mathbf{q}') \leq \max_{\mathbf{q}'} \min_{\mathbf{p}'} \bar{\ell}(\mathbf{p}',\mathbf{q}')$$

$$\min_{\mathbf{p}'} \max_{\mathbf{q}'} \bar{\ell}(\mathbf{p}',\mathbf{q}') \leq \max_{\mathbf{q}'} \min_{\mathbf{p}'} \bar{\ell}(\mathbf{p}',\mathbf{q}') \rightarrow (1)$$

- Also, for all $\mathbf{p}$ and $\mathbf{q}'$, $\bar{\ell}(\mathbf{p},\mathbf{q}') \geq \min_{\mathbf{p}'} \bar{\ell}(\mathbf{p}',\mathbf{q}')$

$$\max_{\mathbf{q}'} \bar{\ell}(\mathbf{p},\mathbf{q}') \geq \max_{\mathbf{q}'} \min_{\mathbf{p}'} \bar{\ell}(\mathbf{p}',\mathbf{q}') \text{ for all } \mathbf{p}$$

$$\min_{\mathbf{p}'} \max_{\mathbf{q}'} \bar{\ell}(\mathbf{p}',\mathbf{q}') \geq \max_{\mathbf{q}'} \min_{\mathbf{p}'} \bar{\ell}(\mathbf{p}',\mathbf{q}') \rightarrow (2)$$

von Neumann's minimax theorem

- From (1) & (2), the existence of Nash equilibrium **p x q** implies that

$$\min_{\mathbf{p}'} \max_{\mathbf{q}'} \bar{\ell}(\mathbf{p}', \mathbf{q}') = \max_{\mathbf{q}'} \min_{\mathbf{p}'} \bar{\ell}(\mathbf{p}', \mathbf{q}')$$

- The common value is called the value of the game, V
- Nash equilibrium, $\mathbf{p} \, x \, \mathbf{q} \; \Leftrightarrow \bar{\ell}(\mathbf{p}, \mathbf{q}) = V$

"*As far as I can see, there could be no theory of games ... without that theorem ... I thought there was nothing worth publishing until the Minimax Theorem was proved*" - John von Neumann, 1928

Correlated Equilibrium

A probability distribution P over the set $\bigotimes_{k=1}^{K}\{1,...,N_k\}$ of all possible K-tuples of actions is called a correlated equilibrium if for all $k = 1,...,K$,

$$\mathbb{E}\,\ell^{(k)}(\mathbf{I}) \leq \mathbb{E}\,\ell^{(k)}(\mathbf{I}^{-},\tilde{I}^{(k)}),$$

where the r.v. $\mathbf{I} = (I^{(1)},...,I^{(K)})$ is distributed according to P and $(\mathbf{I}^{-},\tilde{I}^{(k)}) = (I^{(1)},...,I^{(k-1)},\tilde{I}^{(k)},I^{(k+1)},...,I^{(K)})$, where $\tilde{I}^{(k)}$ is an arbitrary $\{1,...,N_k\}$-valued r.v. that is a function of $I^{(k)}$

Correlated Equilibrium

Lemma: A probability distribution P over the set of all K-tuples $\mathbf{i} = (i_1, ..., i_K)$ of actions is a correlated equilibrium if and only if, for every player $k \in \{1, ..., K\}$ and actions $j, j' \in \{1, ..., N_k\}$, we have

$$\sum_{\mathbf{i}: i_k = j} P(\mathbf{i}) \left(\ell^{(k)}(\mathbf{i}) - \ell^{(k)}(\mathbf{i}^-, j')\right) \leq 0$$

where $(\mathbf{i}^-, j') = (i_1, ..., i_{k-1}, j', i_{k+1}, ..., i_K)$.

Repeated Two-Player Zero-Sum Games

- At each time instant $t = 1, 2, ...$ player $k(k = 1, ..., K)$ selects a mixed strategy $\mathbf{p}_t^{(k)} = (p_{1,t}^{(k)}, ..., p_{N_k,t}^{(k)})$ over the set $1, ..., N_k$ of his actions and draws an action $I_t^{(k)}$ according to the distribution.
- Mixed strategy $\mathbf{p}_t^{(k)}$ may depend on the sequence of random variables $\mathbf{I}_1, ..., \mathbf{I}_{t-1}$

Repeated Two-Player Zero-Sum Games

- If all players play to keep their internal regret small, then the joint empirical frequencies of play converge to the set of correlated equilibria
- If every player uses a well-calibrated forecasting strategy to predict the K-tuple of actions $\mathbf{I}_t$ and chooses an action that is the best reply to the forecasted distibution, the same convergence is also achieved.

Regret based strategies

- At each round t, based on the past plays of both players, the row player chooses an action $I_t \in \{1, ..., N\}$ according to the mixed strategy $\mathbf{p}_t$ and the column player chooses an action $J_t \in \{1, ..., M\}$ according to the mixed strategy $\mathbf{q}_t$.
- If the row player knew the column player's actions $J_1, ..., J_n$ in advance, he would choose $I_t = argmin_{i=1,...,N} \ell(i, J_t)$ invoking a total loss $\sum_{t=1}^{n} min_{i=1,...,N} \ell(i, J_t)$.
- A meaningful objective is to minimize the difference between the row player's cumulative loss and the cumulative loss of the best constant strategy,

$$\text{minimize} \sum_{t=1}^{n} \ell(I_t, J_t) - \min_{i=1,...,N} \sum_{t=1}^{n} \ell(i, J_t)$$

Hannan consistent strategy

- A strategy is Hannan consistent if the regret is $o(1)$, regardless of how the column player behaves.
- Assuming row player chooses his actions I_t, regardless of what the column player does,

$$\limsup_{n\to\infty}\left(\frac{1}{n}\sum_{t=1}^{n}\ell(I_t,J_t)-\min_{i=1,\dots,N}\frac{1}{n}\sum_{t=1}^{n}\ell(i,J_t)\right)\le\ 0$$

almost surely.

- For example, this may be achieved by the exponentially weighted average mixed strategy

$$p_{i,t}=\frac{\exp\left(-\eta\sum_{s=1}^{t-1}\ell(i,J_s)\right)}{\sum_{k=1}^{N}\exp\left(-\eta\sum_{s=1}^{t-1}\ell(k,J_s)\right)}\qquad i=\{1,..,N\},\eta>0$$

Hannan consistent strategy

Notation :

$$\bar{\ell}(\mathbf{p}, j) = \sum_{i=1}^{N} p_i \ell(i, j) \qquad \textit{and} \qquad \bar{\ell}(i, \mathbf{q}) = \sum_{i=1}^{M} q_j \ell(i, j)$$

Theorem: Assume that in a two-person zero-sum game the row player plays according to a Hannan-consistent strategy. Then

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} \ell(I_t, J_t) \leq V \quad \textit{almost surely}.$$

Assuming Hannan consistency, it suffices to show that

$$\min_{i=1,\ldots,N} \frac{1}{n} \sum_{t=1}^{n} \ell(i, J_t) \leq V$$

$$\min_{i=1,\ldots,N} \frac{1}{n} \sum_{t=1}^{n} \ell(i, J_t) = \min_{\mathbf{p}} \frac{1}{n} \sum_{t=1}^{n} \bar{\ell}(\mathbf{p}, J_t)$$

Hannan consistent strategy

Then, letting $\hat{q}_{j,n} = \frac{1}{n}\sum_{t=1}^{n} \mathbb{I}_{J_t=j}$ be the emperical probability of the row player's action being j,

$$\begin{aligned}
\min_{\mathbf{p}} \frac{1}{n}\sum_{t=1}^{n} \overline{\ell}(\mathbf{p}, J_t) &= \min_{\mathbf{p}} \sum_{j=1}^{M} \hat{q}_{j,n}\overline{\ell}(\mathbf{p}, j) \\
&= \min_{\mathbf{p}} \overline{\ell}(\mathbf{p}, \hat{\mathbf{q}}_n) \\
&\leq \max_{\mathbf{q}} \ \min_{\mathbf{p}} \ \overline{\ell}(\mathbf{p}, \mathbf{q}) = V.
\end{aligned}$$

Corollary : Assume that in a two-person zero-sum game, both players play according to some Hannan consistent strategy. Then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \ell(I_t, J_t) = V \quad \textit{almost surely}.$$