Online learning
in
repeated matrix games

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Outline

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- Player choices can depend on the past.

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- The set of Nash Equilibria.
- The set of Correlated equilibria.

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- Convergence to Nash set is not guaranteed in general.
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- Conforming player can suffer non-diminishing regret.

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- Instead of using "follow the leader" use "follow the perturbed leader", i.e. add a small amount of noise to the cumulative utility of each action, *then* pick the leader.
- **Hannan consistency:** Cumulative regret / Cumulative utility $\rightarrow 0$.

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- Hannan's set contains the set of correlated equilibrium which contains the set of Nash Equilibria.

Reaching correlated equilibrium

- By all players minimizing internal regret.

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- By all players minimizing internal regret.
- By making a calibrated predictions of the opponent's next move and playing best response.

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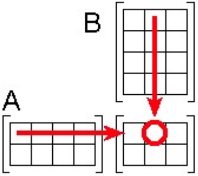
Mixed strategies in matrix notation



$$(A \times B)_{12} = \sum_{r=1}^{4} a_{1r} b_{r2} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} + a_{14}b_{42}$$

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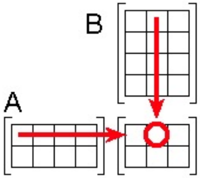

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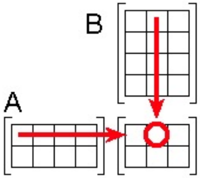

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$$\mathbf{M}(\mathbf{P}, \mathbf{Q}) = \mathbf{P}^T\mathbf{M}\mathbf{Q} = \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{P}(i)\mathbf{M}(i,j)\mathbf{Q}(j)$$

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- $\eta > 0$ is the learning rate.

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- For any game matrix **M**.
- Any sequence of mixed strat. $\mathbf{Q}_1, \ldots, \mathbf{Q}_T$
- The sequence $\mathbf{P}_1, \ldots, \mathbf{P}_T$ produced by basic alg using $\eta > 0$ satisfies

$$\sum_{t=1}^{T} \mathbf{M}(\mathbf{P}_t, \mathbf{Q}_t) \leq \left(\frac{1}{1 - e^{-\eta}} \right) \min_{\mathbf{P}} \left[\eta \sum_{t=1}^{T} \mathbf{M}(\mathbf{P}, \mathbf{Q}_t) + \mathrm{RE}\left(\mathbf{P} \;\|\; \mathbf{P}_1\right) \right]$$

Corollary

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- Where

$$\Delta_{T,n} = \sqrt{\frac{2\ln n}{T}} + \frac{\ln n}{T} = O\left(\sqrt{\frac{\ln n}{T}}\right).$$

Main Lemma

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$$\mathrm{RE}\left(\tilde{\mathbf{P}} \;\|\; \mathbf{P}_{t+1}\right) - \mathrm{RE}\left(\tilde{\mathbf{P}} \;\|\; \mathbf{P}_t\right) \leq \eta \mathbf{M}(\tilde{\mathbf{P}}, \mathbf{Q}_t) - (1 - e^{-\eta})\mathbf{M}(\mathbf{P}_t, \mathbf{Q}_t)$$

Visual intuition

$$\mathrm{RE}\left(\tilde{\mathbf{P}} \parallel \mathbf{P}_{t+1}\right) - \mathrm{RE}\left(\tilde{\mathbf{P}} \parallel \mathbf{P}_t\right) \leq \eta \mathbf{M}(\tilde{\mathbf{P}}, \mathbf{Q}_t) - (1 - e^{-\eta}) \mathbf{M}(\mathbf{P}_t, \mathbf{Q}_t)$$

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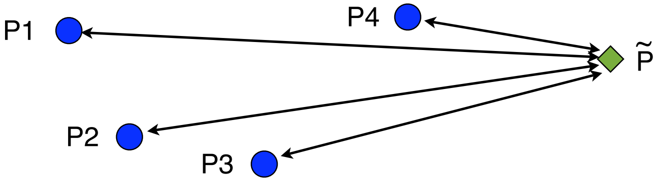

Proof of Lemma (1)

$$\mathrm{RE}\left(\tilde{\mathbf{P}} \;\|\; \mathbf{P}_{t+1}\right) - \mathrm{RE}\left(\tilde{\mathbf{P}} \;\|\; \mathbf{P}_t\right)$$

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&\mathrm{RE}\left(\tilde{\mathbf{P}} \;\|\; \mathbf{P}_{t+1}\right) - \mathrm{RE}\left(\tilde{\mathbf{P}} \;\|\; \mathbf{P}_{t}\right) \\
&= \sum_{i=1}^{n} \tilde{\mathbf{P}}(i) \ln \frac{\tilde{\mathbf{P}}(i)}{\mathbf{P}_{t+1}(i)} - \sum_{i=1}^{n} \tilde{\mathbf{P}}(i) \ln \frac{\tilde{\mathbf{P}}(i)}{\mathbf{P}_{t}(i)}
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&= \sum_{i=1}^{n} \tilde{\mathbf{P}}(i) \ln \frac{\mathbf{P}_t(i)}{\mathbf{P}_{t+1}(i)} \\
&= \sum_{i=1}^{n} \tilde{\mathbf{P}}(i) \ln \frac{Z_t}{e^{\eta \mathbf{M}(i,\mathbf{Q}_t)}}
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In words: for mixed strategies, choosing second gives no advantage.

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Row player chooses $\mathbf{P}_t$ using learning alg.

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Let $\overline{\mathbf{P}} \doteq \frac{1}{T}\sum_{t=1}^{T} \mathbf{P}_t$ and $\overline{\mathbf{Q}} \doteq \frac{1}{T}\sum_{t=1}^{T} \mathbf{Q}_t$

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&\leq \min_{\mathbf{P}} \frac{1}{T}\sum_{t=1}^{T}\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{Q}_t + \Delta_{T,n} && \text{by the Corollary} \\
&= \min_{\mathbf{P}} \mathbf{P}^{\mathrm{T}}\mathbf{M}\overline{\mathbf{Q}} + \Delta_{T,n} && \text{by definition of } \overline{\mathbf{Q}} \\
&\leq \max_{\mathbf{Q}} \min_{\mathbf{P}} \mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{Q} + \Delta_{T,n}.
\end{aligned}
$$

Proving minmax Theorem using online learning (2)

$$
\begin{aligned}
&= \frac{1}{T}\sum_{t=1}^{T} \mathbf{P}_t^{\mathrm{T}} \mathbf{M} \mathbf{Q}_t && \text{by definition of } \mathbf{Q}_t \\
&\leq \min_{\mathbf{P}} \frac{1}{T}\sum_{t=1}^{T} \mathbf{P}^{\mathrm{T}} \mathbf{M} \mathbf{Q}_t + \Delta_{T,n} && \text{by the Corollary} \\
&= \min_{\mathbf{P}} \mathbf{P}^{\mathrm{T}} \mathbf{M} \overline{\mathbf{Q}} + \Delta_{T,n} && \text{by definition of } \overline{\mathbf{Q}} \\
&\leq \max_{\mathbf{Q}} \min_{\mathbf{P}} \mathbf{P}^{\mathrm{T}} \mathbf{M} \mathbf{Q} + \Delta_{T,n}.
\end{aligned}
$$

but $\Delta_{T,n}$ can be set arbitrarily small.

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- Is it realistic to assume that markets are at equilibrium?
- If game is not zero sum (allows incentives to collaborate) and all players use learning then game converges to correlated equilibrium.