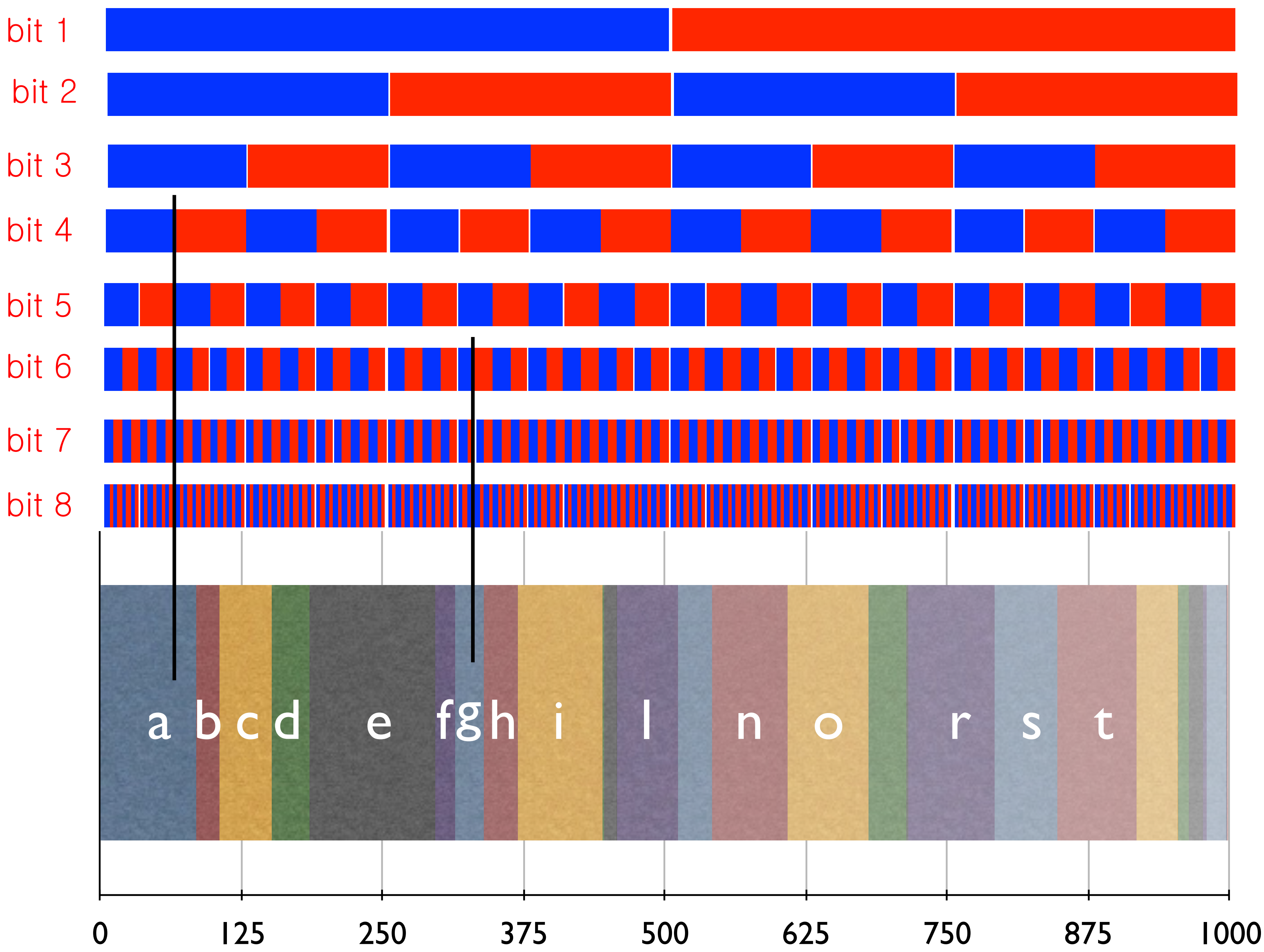
Arithmetic Coding and Adaptive Coding

Review

- Huffman Codes
- Entropy

Arithmetic coding

- Partitioning the unit segment.
- Identifying a part using a binary expansion.



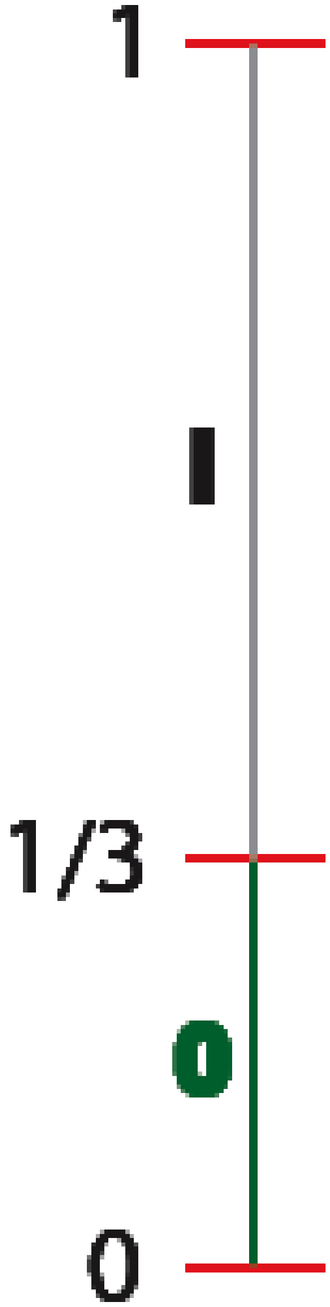
bit 1
bit 2
bit 3
bit 4
bit 5
bit 6
bit 7
bit 8
a b c d e f g h i l n o r s t
0
125
250
375
500
625
750
875
1000

How many bits?

- p - the probability of the character
  - = the length of the segment
- There segment must contain a dyadic number with log (1/p) bits

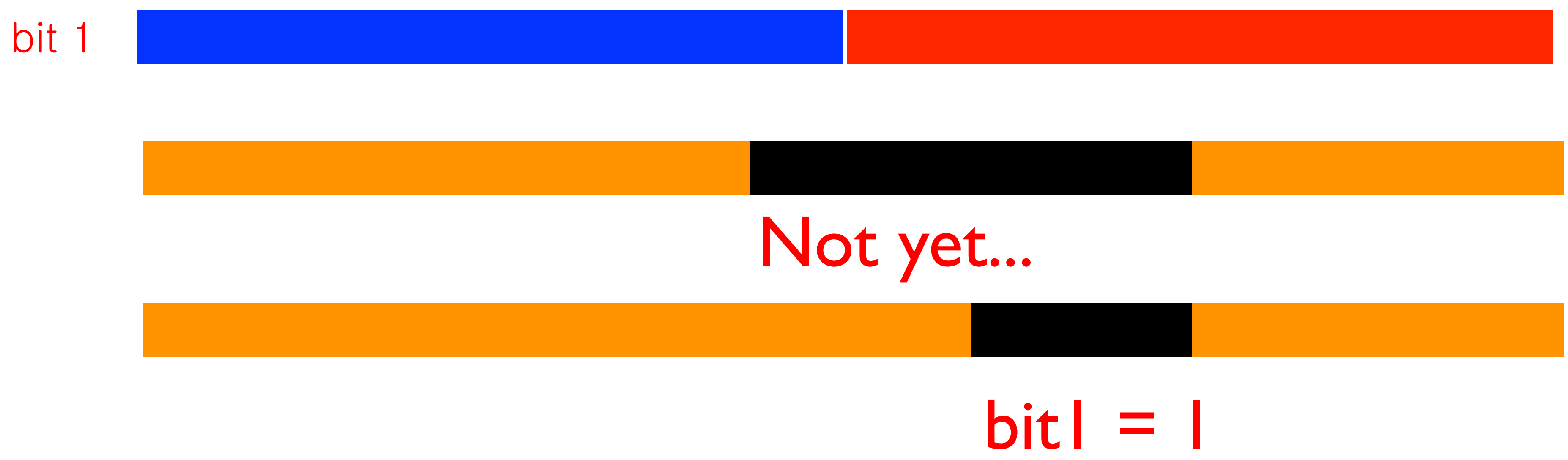
Coding more than one char

- An input stream: $x_1, x_2, x_3, \ldots$
- x_1 chooses a part $[a_1,b_1)$ in partition of $[0,1)$
- x_2 chooses a part $[a_2,b_2)$ in partition of $[a_1,b_1)$
- x_3 chooses ....


1
I
1/3
0
0

When can we send the next bit?

- As soon as we know whether the segment is on the left or on the right of a dyadic partition.
- Unbounded delay ...



Performance of arithmetic codes

The message: $x_1, x_2, x_3, \ldots x_n$

Generated IID according to distribution p

$$\ell = \left\lceil \log_2 \frac{1}{\prod_{i=1}^{n} p(x_i)} \right\rceil = \left\lceil \sum_{i=1}^{n} \log_2 \frac{1}{p(x_i)} \right\rceil < \sum_{i=1}^{n} \log_2 \frac{1}{p(x_i)} + 1$$

$$E(\ell) < n \sum_{x} p(x) \log_2 \frac{1}{p(x)} + 1 = nH(p) + 1$$

At most one bit more than the shannon lower bound <u>for the whole message</u>

Using the wrong distribution

- So far we assumed that we are coding using the correct distribution p. Suppose that we are coding according to a dist q≠p

$$E(\ell) < n \sum_x p(x) \log_2 \frac{1}{q(x)} + 1 =$$

$$= n \left(\sum_x p(x) \log_2 \frac{1}{p(x)} + \sum_x p(x) \log_2 \frac{p(x)}{q(x)} \right) + 1$$

$$= n \left(H(p) + D_{\mathrm{KL}}(p||q) \right) + 1$$

Entropy KL-divergence

Two part codes

- Receiver does not know distribution
- Sender sends two pieces:

1. Distribution parameters (Model)
2. Message, coded using distribution (Data given model)

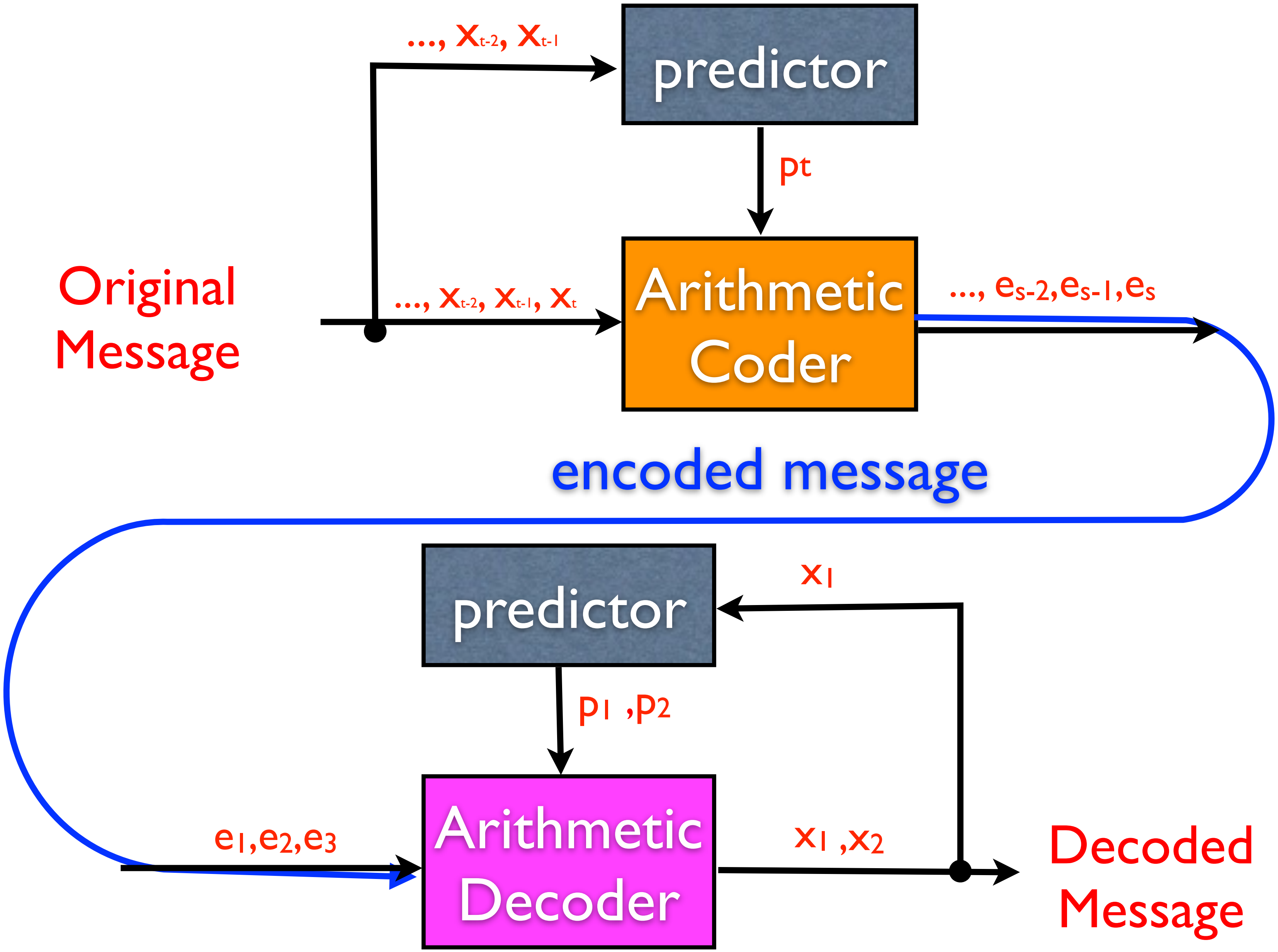
Non IID sources

$$p(x_1, x_2, ..., x_t) \neq \prod_{s=1}^{t} p(x_s)$$

$$p(x_t | x_{t-1}, ..., x_1) \neq p(x_t)$$

Arithmetic coding does not require characters to be IID

Adaptive Coding



performance of adaptive codes.

- Source is IID
- Predictor converges to correct distribution over time.
- Code length: $\ell = \left\lceil \sum_{t=1}^{n} \log_2 \frac{1}{q_t(x_t)} \right\rceil$

$$E(\ell) < \sum_{t=1}^{n} \sum_{x} p(x) \log_2 \frac{1}{q_t(x)} + 1$$

online prediction of probabilities

- A binary input stream: $x_1, x_2, x_3, \ldots$
- Generated IID according to a fixed but unknown distribution $(p, 1-p)$.
- Task: map $x_1, x_2, x_3, \ldots, x_{t-1}$ to q_t so that $q_t \to p$ quickly so as to minimize

$$E_{x_1 \sim p, \ldots x_t \sim p} \left(\sum_{t=1}^{n} \log_2 \frac{1}{q_t(x_t)} \right)$$

Laplace Law Of Succession

$$q_t = \frac{\#1 + 1}{t + 1}$$

#1 = number of 1's in $x_1, x_2, x_3, \ldots, x_{t-1}$

$$E_{x_1 \sim p, \ldots x_t \sim p}\left(\sum_{t=1}^{n} \log_2 \frac{1}{q_t(x_t)}\right) \leq tH(p) + \log_2 t$$

Kritchvski-Trofimov Prediction rule

$$q_t = \frac{\#1 + 1/2}{t}$$

#1 = number of 1's in $x_1, x_2, x_3, \ldots, x_{t-1}$

$$E_{x_1 \sim p, \ldots x_t \sim p}\left(\sum_{t=1}^{n} \log_2 \frac{1}{q_t(x_t)}\right) \leq tH(p) + \boxed{\frac{1}{2}} \log_2 t$$

Best possible factor

Summary

- Arithmetic coding
- Adaptive coding
- Predictive coding
- Laplace Law of succession and the KT prediction rule.