Lossless compression and cumulative log loss

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Outline

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- Arithmetic coding
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  - Two part codes
  - Combining expert advice for cumulative log loss
- Combining experts in the log loss framework
- The online Bayes Algorithm
- The performance bound
- Comparison with Bayesian Statistics
- Computational issues

The source compression problem

- **Example:** "There are no people like show people"
  $\xrightarrow{\text{encode}} x \in \{0,1\}^n$
  $\xrightarrow{\text{decode}}$ "there are no people like show people"
- **Lossless:** Message reconstructed perfectly.
- **Goal:** minimize expected length $E(n)$ of coded message.
- Can we do better than $\lceil \log_2(26) \rceil = 5$ bits per character?
- **Basic idea:** Use short codes for common messages.
- **Stream compression:**
  - Message revealed one character at a time.
  - Code generated as message is revealed.
  - Decoded message is constructed gradually.
- Easier than block codes when processing long messages.
- A natural way for describing a distribution.

The Guessing game

- Message reverealed one character at a time
- An algorithm predicts the next character from the revealed part of the message.
- If algorithm wrong - ask for next guess.
- **Example**

t	h	e	r	e		a	r	e		n	o		p	e
6	2	1	2	1	1	5	2	1	1	4	1	1	5	3

- Code = sequence of number of mistakes.
- To decode use the same prediction algorithm

Arithmetic Coding (background)

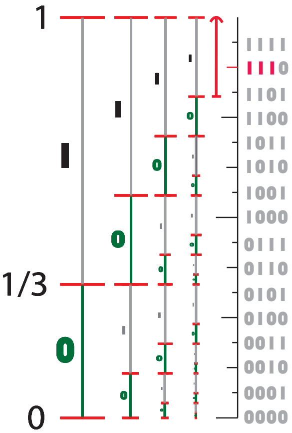
- Refines the guessing game:
  - In guessing game the predictor chooses order over alphabet.
  - In arithmetic coding the predictor chooses a Distribution over alphabet.
- First discovered by Elias (MIT).
- Invented independently by Rissanen and Pasco in 1976.
- Widely used in practice.

Arithmetic Coding (basic idea)

- Easier notation: represent characters by numbers $1 \leq c_t \leq |\Sigma|$. (English: $N \doteq |\Sigma| = 26$)
- message-prefix $c_1, c_2, \ldots, c_{t-1}$ represented by line segment $[l_{t-1}, u_{t-1})$
- Initial segment $[l_0, u_0) = [0, 1)$
- After observing $c_1, c_2, \ldots, c_{t-1}$, predictor outputs $p(c_t = 1 | c_1, c_2, \ldots, c_{t-1}), \ldots, p(c_t = |\Sigma| \; | c_1, c_2, \ldots, c_{t-1})$,
- Distribution is used to partition $[l_{t-1}, u_{t-1})$ into $|\Sigma|$ sub-segments.
- next character c_t determines $[l_t, u_t)$
- Code = discriminating binary expansion of a point in $[l_t, u_t)$.

Arithmetic Coding (sequence example)

- Simplest case.
- $\Sigma = \{0, 1\}$
- $\forall t,$
  $p(c_t = 0) = 1/3$
  $p_t(c_t = 1) = 2/3$
- Message = 1111
- Code = 111
- **Technical:**
  Assume decoder knows that length of message is 4.



The code length for arithmetic coding

- Given m bits of binary expansion we assume the rest are all zero.
- Distance between two m bit expansions is 2^{-m}
- If $l_T - u_T \geq 2^{-m}$ then there must be a point x described by m expansion bits such that $l_T \leq x < u_T$
- Required number of bits is $\lceil -\log_2(u_T - l_T) \rceil$.
- $u_T - l_T = \prod_{t=1}^{T} p\left(c_t \,|\, c_1, c_2, \ldots, c_{t-1}\right) \doteq p(c_1, \ldots c_T)$
- Number of bits required to code $c_1, c_2, \ldots, c_T$ is $\lceil -\sum_{t=1}^{T} \log_2 p_t(c_t) \rceil$.
- We call $-\sum_{t=1}^{T} \log_2 p_t(c_t) = -log_2 p(c_1, \ldots c_T)$ the Cumulative log loss
- Holds for all sequences.

Expected code length

- Fix the messsage length T
- Suppose the message is generated at random according to the distribution $p(c_1, \dots c_T)$
- Then the expected code length is

$$\sum_{c_1, \dots c_T} p(c_1, \dots c_T) \lceil -\log_2 p(c_1, \dots, c_T) \rceil$$

$$\leq \quad 1 - \sum_{c_1, \dots c_T} p(c_1, \dots c_T) \log_2 p(c_1, \dots, c_T) \doteq 1 + H(p_T)$$

- $H(p_T)$ is the **entropy** of the distribution over sequences of length T:

$$H(p_T) \doteq \sum_{(c_1, \dots, c_T)} p(c_1, \dots, c_T) \log \frac{1}{p(c_1, dots, c_T)}$$

- Entropy is the expected value of the cumulative log loss

Shannon's lower bound

- Assume p_T is "well behaved". For example, IID.
- Let $T \to \infty$
- $H(p) \doteq \lim_{T\to\infty} \frac{H(p_T)}{T}$ exists and is called the per character entropy of the source p
- The expected code length for any coding scheme is at least
$$(1 - o(1))H(p_T) = (1 - o(1))\ T\ H(p)$$
- The proof of Shannon's lower bound is not trivial (Can be a student lecture).

log loss encourages unbiased prediction

- Suppose the source is random and the probability of the next outcome is $p(c_t | c_1, c_2, \ldots, c_{t-1})$
- Then the prediction that minimizes the log loss is $p(c_t | c_1, c_2, \ldots, c_{t-1})$.
- Note that when minimizing expected number of mistakes, the best prediction in this situation is to put all of the probability on the most likely outcome.
- There are other losses with this property, for example, square loss.

Monthly bonuses for a weather forecaster

- Before the first of the month assign one dollar to the forecaster's bonus. $b_0 = 1$
- Forecaster assigns probability p_t to rain on day t.
- If it rains on day t then $b_t = 2b_{t-1}p_t$
- If it does not rain on day t then $b_t = 2b_{t-1}(1 - p_t)$
- At the end of the month, give forecaster b_T
- Risk averse strategy: Setting $p_t = 1/2$ for all days, guarantees $b_T = 1$
- High risk prediction: Setting $p_t \in \{0, 1\}$ results in Bonus $b_T = 2^T$ if always correct, zero otherwise.
- If forecaster predicts with the true probabilities then

$$E(\log b_T) = T - H(p_T)$$

and that is the maximal expected value for $E(\log b_T)$

Horse-race betting

- You go to the horse races with one dollar $b_0 = 1$
- m horses compete in each race.
- Before each race, the odds for each horse are announced: $o_t(1), \ldots o_t(m)$ (arbitrary positive numbers)
- You have to divide *all* your money among the different horses. $\sum_{j=1}^{t} \hat{p}_t(j) = 1$
- The horse $1 \leq y_t \leq m$ is winner of the tth race.
- After iteration t, you have $b_t = b_{t-1} \hat{p}_t(y_t) o_t(y_t)$ dollars
- After n races, you have $b_n = \prod_{t=1}^{n} \hat{p}_t(y_t) o_t(y_t)$ dollars.
- Taking logs, we get cumulative log loss.

“Universal” coding

- Suppose there are N alternative predictors / experts.
- We would like to code almost as well as the best predictor.
- We would like to make almost as much money as the best expert in hind-site.

Two part codes

- Send the index of the coding algorithm before the message.
- Requires $\log_2 N$ additional bits.
- Requires the encoder to make **two** passes over the data.
- Is the key idea of MDL (Minimal Description Length) modeling.
  - Good prediction model = model that minimizes the total code length
- Often inappropriate because based on lossless coding. **Lossy** coding often more appropriate.

Combining predictors adaptively

- Treat each of the predictors as an “expert”.
- Assign a weight to each expert and reduce it if expert performs poorly.
- Combine expert predictions according to their weights.
- Would require only a single pass. Truly online.
- **Goal:** Total loss of algorithm minus loss of best predictor should be at most $\log_2 N$

The log-loss framework

- Algorithm A predicts a sequence $c^1, c^2, \ldots, c^T$ over alphabet $\Sigma = \{1, 2, \ldots, k\}$
- The prediction for the c^tth is a distribution over Σ: $\mathbf{p}_A^t = \langle p_A^t(1), p_A^t(2), \ldots, p_A^t(k) \rangle$
- When c^t is revealed, the loss we suffer is $-\log p_A^t(c^t)$
- The cumulative log loss, which we wish to minimize, is $L_A{}^T = -\sum_{t=1}^T \log p_A^t(c^t)$
- $\lceil L_A{}^T \rceil$ is the code length if A is combined with arithmetic coding.

The game

- Prediction algorithm A has access to N experts.
- The following is repeated for $t = 1, \ldots, T$
  - Experts generate predictive distributions: $\mathbf{p}_1^t, \ldots, \mathbf{p}_N^t$
  - Algorithm generates its own prediction $\mathbf{p}_A^t$
  - c^t is revealed.
- **Goal:** minimize regret:

$$-\sum_{t=1}^{T} \log p_A^t(c^t) + \min_{i=1,\ldots,N} \left(-\sum_{t=1}^{T} \log p_i^t(c^t) \right)$$

The online Bayes Algorithm

- Total loss of expert i

$$L_i^t = -\sum_{s=1}^{t} \log p_i^s(c^s); \quad L_i^0 = 0$$

- Weight of expert i

$$w_i^t = w_i^1 e^{-L_i^{t-1}} = w_i^1 \prod_{s=1}^{t-1} p_i^s(c^s)$$

- Freedom to choose initial weights.
  $w_t^1 \geq 0, \sum_{i=1}^n w_i^1 = 1$
- Prediction of algorithm A

$$\mathbf{p}_A^t = \frac{\sum_{i=1}^N w_i^t \mathbf{p}_i^t}{\sum_{i=1}^N w_i^t}$$

Cumulative loss vs. Final total weight

Total weight: $W^t \doteq \sum_{i=1}^N w_i^t$

$$\frac{W^{t+1}}{W^t} = \frac{\sum_{i=1}^N w_i^t e^{\log p_i^t(c^t)}}{\sum_{i=1}^N w_i^t} = \frac{\sum_{i=1}^N w_i^t p_i^t(c^t)}{\sum_{i=1}^N w_i^t} = p_A^t(c^t)$$

$$-\log \frac{W^{t+1}}{W^t} = -\log p_A^t(c^t)$$

$$-\log W^{T+1} = -\log \frac{W^{T+1}}{W^1} = -\sum_{t=1}^T \log p_A^t(c^t) = L_A^T$$

EQUALITY not bound!

Simple Bound

- Use uniform initial weights $w_i^1 = 1/N$
- Total Weight is at least the weight of the best expert.

$$
\begin{aligned}
L_A^T &= -\log W^{T+1} = -\log \sum_{i=1}^{N} w_i^{T+1} \\
&= -\log \sum_{i=1}^{N} \frac{1}{N} e^{-L_i^T} = \log N - \log \sum_{i=1}^{N} e^{-L_i^T} \\
&\leq \log N - \log \max_i e^{-L_i^T} = \log N + \min_i L_i^T
\end{aligned}
$$

- Dividing by T we get $\frac{L_A^T}{T} = \min_i \frac{L_i^T}{T} + \frac{\log N}{T}$

Bound better than for two part codes

- Simple bound as good as bound for two part codes (MDL) but enables online compression
- Suppose we have K copies of each expert.
- Two part code has to point to one of the KN experts
  $L_A \leq \log NK + \min_i L_i^T = \log NK + \min_i L_i^T$
- If we use Bayes predictor + arithmetic coding we get:

$$L_A = -\log W^{T+1} \leq \log K \max_i \frac{1}{NK} e^{-L_i^T} = \log N + \min_i L_i^T$$

- We don't pay a penalty for copies.
- More generally, the regret is smaller if many of the experts perform well.

How to choose the initial weights?

- When experts are similar - you want to assign each of them less weight.
- The min-max prior.
- Priors that allow efficient computation.
- Conjugate priors.

Comparison with standard Bayesian statistics

- The weight update rule is the same.
- Normalized weights = **posterior probability distribution**.
- Bayesian analysis interested in the **final** posterior.
- Bayesian analysis assumes the data is **generated** by a distribution in the support of the prior.
- Goal of Bayesian is to **estimate true distribution**, goal of online learning is to **minimize regret**.
- Optimality of algorithm is **axiom** of Bayesian statistics.
- Bayesian methods perform poorly when the loss is not log loss and the data not generated by a distribution in the support.
  - Loss can sometimes be defined through the noise distribution: square loss is equivalent to assuming guassian noise.
  - For number of mistakes - Bayesian method cannot be "fixed". Requires variable learning rate. Regret bounds are

Computational Issues

- Naive implementation: calculate the prediction of each of the N experts.
- Puts severe limit on number of experts.
- What if set of experts is uncountably infinite.
- Bayesian tricks:
  - **Conjugate priors**: A prior over a continuous domain whose functional form does not change with when updated. Number of parameters defining posterior is constant. Update rule translates into update of parameters. parameters correspond to “sufficient statistics”. exists for the famiity of exponential distributions.
  - **Markov Chain Monte Carlo** Sample the posterior. Can sometimes be done efficiently. Efficient sampling relates to mixing rate of markov chain whose limit dist is the posterior dist.

Standardizing online prediction algorithms

- Fix a universal Turing machine U.
- An online prediction algorithm E is a program that
  - given as input The past $\vec{X} \in \{0,1\}^t$
  - runs finite time and outputs
  - A prediction for the next bit $p(\vec{X}) \in [0,1]$.
  - To ensure p has a finite description. Restrict to rational numbers n/m
- Any online prediction algorithm can be represented as code $\vec{b}(E)$ for U. The code length is $|\vec{b}(E)|$.
- Most sequences do not correspond to valid prediction algorithms.
- $V(\vec{b}, \vec{X}, t) = 1$ if the program $\vec{b}$, given $\vec{X}$ as input, halts within t steps and outputs a well-formed prediction. Otherwise $V(\vec{b}, \vec{X}, t) = 0$
- $V(\vec{b}, \vec{X}, t)$ is computable (recursively enumerable).

A universal prediction machine

- Assign to the code $\vec{b}$ the initial weight $w^1_{\vec{b}} = 2^{-|\vec{b}|-\log_2|\vec{b}|}$.
- The total initial weight over all finite binary sequences is one.
- Run the Bayes algorithm over "all" prediction algorithms.
- technical details: On iteration t, $|\vec{X}| = t$. Use the predictions of programs $\vec{b}$ such that $|\vec{b}| \leq t$ and for which $V(\vec{b}, \vec{X}, 2^t) = 1$. Assing the remaining mass the prediction $1/2$ (insuring a loss of 1)

Performance of the universal prediction algorithm

- Using $L_A \leq \min_i \left(L_i - \log w_i^1\right)$
- Assume E is a prediction algorithm which generates the tth prediction in time smaller than 2^t
- When $t \leq |\vec{b}(E)|$ the algorithm is not used and thus it's loss is 1
- We get that the loss of the Universal algorithm is at most $2|\vec{b}(E)| + \log_2 |\vec{b}(E)| + L_E$
- More careful analysis can reduce $2|\vec{b}(E)| + \log_2 |\vec{b}(E)|$ to $|\vec{b}(E)|$
- Code length is arbitrarily close to the Kolmogorov Complexity of the sequence.
- Ridiculously bad running time.

Bayes coding is better than two part codes

- Simple bound as good as bound for two part codes (MDL) but enables online compression
- Suppose we have K copies of each expert.
- Two part code has to point to one of the KN experts
  $L_A \leq \log NK + \min_i L_i^T = \log NK + \min_i L_i^T$
- If we use Bayes predictor + arithmetic coding we get:

$$L_A = -\log W^{T+1} \leq \log K \max_i \frac{1}{NK} e^{-L_i^T} = \log N + \min_i L_i^T$$

- We don't pay a penalty for copies.
- More generally, the regret is smaller if many of the experts perform well.

The biased coins set of experts

- Each expert corresponds to a biased coin, predicts with a fixed $\theta \in [0, 1]$.
- Set of experts is uncountably infinite.
- Only countably many experts can be assigned non-zero weight.
- Instead, we assign the experts a Density Measure.
- $L_A \leq \min_i \left(L_i - \log w_i^1\right)$ is meaningless.
- Can we still get a meaningful bound?

Bayes Algorithm for biased coins

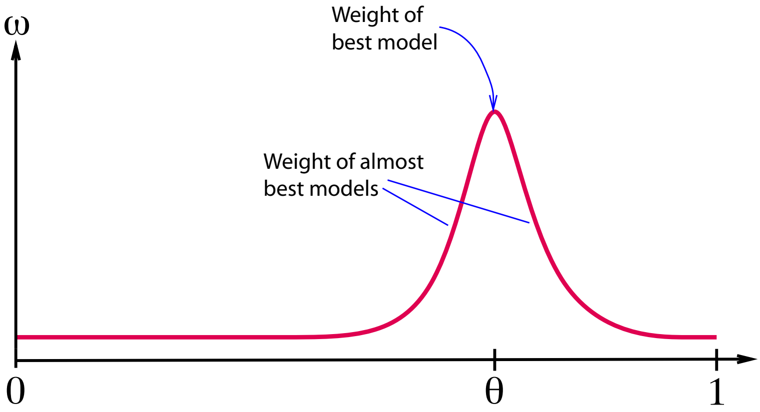
- Replace the initial weight by a density measure $w(\theta) = w^1(\theta)$, $\int_0^1 w(\theta) d\theta = 1$
- Relationship between final total weight and total log loss remains unchanged:

$$L_A = \ln \int_0^1 w(\theta) e^{-L_\theta^{T+1}} d\theta$$

- We need a new lower bound on the final total weight

Main Idea

If $w^t(\theta)$ is large then $w^t(\theta + \epsilon)$ is also large.



Expanding the exponent around the peak

- For log loss the best θ is empirical distribution of the seq.

$$\hat{\theta} = \frac{\#\{x^t = 1; \;\; 1 \leq t \leq T\}}{T}$$

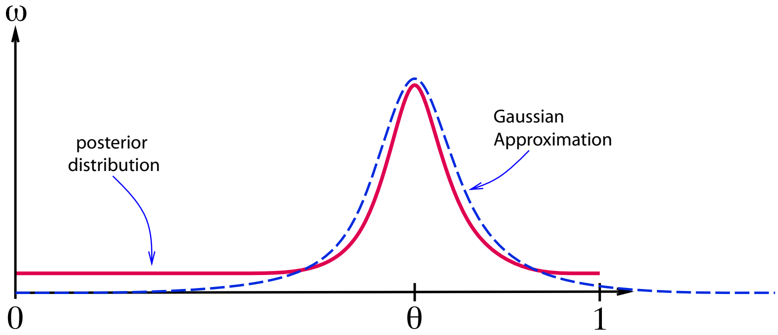
- The total loss scales with T

$$L_\theta = T \cdot (\hat{\theta}\ell(\theta, 1) + (1 - \hat{\theta})\ell(\theta, 0)) \doteq T \cdot g(\hat{\theta}, \theta)$$

$$\begin{aligned}
L_A - L_{\min} &\leq \ln \int_0^1 w(\theta) e^{-L_\theta} d\theta - \ln e^{L_{\min}} \\
&= \ln \int_0^1 w(\theta) e^{-(L_\theta - L_{\min})} d\theta \\
&= \ln \int_0^1 w(\theta) e^{T(g(\hat{\theta},\theta) - g(\hat{\theta},\hat{\theta}))} d\theta
\end{aligned}$$

Laplace approximation (idea)

- Taylor expansion of $g(\hat{\theta}, \theta) - g(\hat{\theta}, \hat{\theta})$ around $\theta = \hat{\theta}$.
- First and second terms in the expansion are zero.
- Third term gives a quadratic expression in the exponent
- $\Rightarrow$ a gaussian approximation of the posterior.



Laplace Approximation (details)

$$\int_0^1 w(\theta) e^{T(g(\hat{\theta},\theta)-g(\hat{\theta},\hat{\theta}))} d\theta$$

$$= \quad w(\hat{\theta}) \sqrt{\frac{-2\pi}{T \left. \frac{d^2}{d\theta^2} \right|_{\theta=\hat{\theta}} (g(\hat{\theta},\theta) - g(\hat{\theta},\hat{\theta}))}} + O(T^{-3/2})$$

Choosing the optimal prior

- Choose $w(\theta)$ to maximize the worst-case final total weight

$$\min_{\hat{\theta}} w(\hat{\theta}) \sqrt{\frac{-2\pi}{T \left. \frac{d^2}{d\theta^2} \right|_{\theta=\hat{\theta}} (g(\hat{\theta}, \theta) - g(\hat{\theta}, \hat{\theta}))}}$$

- Make bound equal for all $\hat{\theta} \in [0, 1]$ by choosing

$$w^*(\hat{\theta}) = \frac{1}{Z} \sqrt{\frac{\left. \frac{d^2}{d\theta^2} \right|_{\theta=\hat{\theta}} (g(\hat{\theta}, \theta) - g(\hat{\theta}, \hat{\theta}))}{-2\pi}} ,$$

where Z is the normalization factor:

$$Z = \sqrt{\frac{1}{2\pi}} \int_0^1 \sqrt{\left. \frac{d^2}{d\theta^2} \right|_{\theta=\hat{\theta}} (g(\hat{\theta}, \hat{\theta}) - g(\hat{\theta}, \theta))} \; d\hat{\theta}$$

The bound for the optimal prior

- Plugging in we get

$$
\begin{aligned}
L_A - L_{\min} &\leq \ln \int_0^1 w^*(\theta) e^{T(g(\hat{\theta},\theta) - g(\hat{\theta},\hat{\theta}))} d\theta \\
&= \ln \left(\sqrt{\frac{2\pi Z}{T}} + O(T^{-3/2}) \right) \\
&= \frac{1}{2} \ln \frac{T}{2\pi} - \frac{1}{2} \ln Z + O(1/T) \, .
\end{aligned}
$$

Solving for log-loss

- The exponent in the integral is

$$g(\hat{\theta},\theta) - g(\hat{\theta},\hat{\theta}) = \hat{\theta}\ln\frac{\hat{\theta}}{\theta} + (1-\hat{\theta})\ln\frac{1-\hat{\theta}}{1-\theta} = D_{KL}(\hat{\theta}||\theta)$$

- The second derivative

$$\left.\frac{d^2}{d\theta^2}\right|_{\theta=\hat{\theta}} D_{KL}(\hat{\theta}||\theta) = \hat{\theta}(1-\hat{\theta})$$

  Is called the empirical Fisher information

- The optimal prior:

$$w^*(\hat{\theta}) = \frac{1}{\pi\sqrt{\hat{\theta}(1-\hat{\theta})}}$$

  Known in general as Jeffrey's prior. And, in this case, the Dirichlet-$(1/2, 1/2)$ prior.

The cumulative log loss of Bayes using Jeffrey's prior

- $$L_A - L_{\min} \leq \frac{1}{2}\ln(T+1) + \frac{1}{2}\ln\frac{\pi}{2} + O(1/T)$$

But what is the prediction rule?

- As luck would have it the Dirichlet prior is the conjugate prior for the Binomial distribution.
- Observed t bits, n of which were 1. The posterior is:

$$\frac{1}{Z\sqrt{\theta(1-\theta)}}\theta^{n}(1-\theta)^{t-n}=\frac{1}{Z}\theta^{n-1/2}(1-\theta)^{t-n-1/2}$$

- The posterior average is:

$$\frac{\int_0^1 \theta^{n+1/2}(1-\theta)^{t-n-1/2}d\theta}{\int_0^1 \theta^{n-1/2}(1-\theta)^{t-n-1/2}d\theta}=\frac{n+1/2}{t+1}$$

- This is called the Trichevsky Trofimov prediction rule.

Laplace Rule of Succession

- Laplace suggested using the uniform prior, which is also a conjugate prior.
- In this case the posterior average is:

$$\frac{\int_0^1 \theta^{n+1}(1-\theta)^{t-n} d\theta}{\int_0^1 \theta^{n}(1-\theta)^{t-n} d\theta} = \frac{n+1}{t+2}$$

- The bound on the cumulative log loss is worse:

$$L_A - L_{\min} = \ln T + O(1)$$

- Suffers larger regret when $\hat{\theta}$ is far from $1/2$

Shtarkov Lower bound

- What is the optimal prediction when T is know in advance?
- $$L_*^T - \min_\theta L_\theta^T \geq \frac{1}{2}\ln(T+1) + \frac{1}{2}\ln\frac{\pi}{2} - O(\frac{1}{\sqrt{T}})$$

Multinomial Distributions

- For a distribution over k elements (Multinomial) [Xie and Barron]
- Use the add 1/2 rule (KT).

$$p(i) = \frac{n_i + 1/2}{t + k/2}$$

- Bound is

$$L_A - L_{\min} \leq \frac{k-1}{2} \ln T + C + o(1)$$

- The constant C is optimal.

Exponential Distributions

- For any set of distributions from the exponential family defined by k parameters (constituting an open set) [Rissanen96]
- Use Bayes Algorithm with Jeffrey's prior:

$$w^*(\hat{\theta}) = \frac{1}{Z} \frac{1}{\sqrt{\left| \mathbf{H}\left(D_{KL}(\hat{\theta} || \theta) \right) \Big|_{\theta=\hat{\theta}} \right|}}$$

H denotes the Hessian.

- $$L_A - L_{\min} \leq \frac{k-1}{2} \ln T - \ln Z + o(1)$$