Exponential Weights Algorithms for Online Learning

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The Halving Algorithm

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 ${f Hedge}(\eta) {f Algorithm}$ Hedging vs. Halving

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Upper bound on $\sum_{i=1}^{N} w_i^{T+1}$ Lower bound on $\sum_{i=1}^{N} w_i^{T+1}$ Combining Upper and Lower bounds

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Lower Bounds

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- Fits nicely in game theory

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- Modified Goal: minimize difference between expected total loss and minimal total loss of repeating one action.

$$\sum_{t=1}^{T} \mathbf{p}^{t} \cdot \ell^{t} - \min_{i} \left(\sum_{t=1}^{T} \ell_{i}^{t} \right)$$

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 - ▶ Algorithm predicts 1 with probability $\sum_{i:e_i^t=1} p_i^t$.
 - outcome o_i^t is revealed. $\ell_i^t = 0$ if $e_i^t = o_i^t$, $\ell_i^t = 1$ otherwise.

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$$L_i^t = \sum_{s=1}^{t-1} \ell_i^s$$

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Note freedom to choose initial weight $(w_i^1) \sum_{i=1}^n w_i^1 = 1$.

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- If good action has low initial weight, our total loss will be larger.
- As $\sum_{i=1}^{n} w_i^1 = 1$ increasing one weight implies decreasing some others.
- Plays a similar role to prior distribution in Bayesian algorithms.

Bound on the loss of $Hedge(\eta)$ Algorithm

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Theorem (main theorem)
For any sequence of loss vectors ℓ¹,...,ℓ^T, and for any
i ∈ {1,...,N}, we have

$$L_{\mathsf{Hedge}(\eta)} \leq \frac{-\mathsf{In}(w_i^1) + \eta L_i}{1 - e^{-\eta}}.$$

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Theorem (main theorem)
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► Proof: by combining upper and lower bounds on $\sum_{i=1}^{N} w_i^{T+1}$

Upper bound on $\sum_{i=1}^{N} w_i^{T+1}$

Lemma (upper bound)

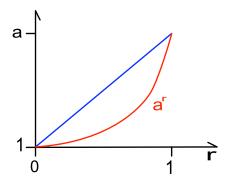
For any sequence of loss vectors ℓ^1, \dots, ℓ^T we have

$$\ln\left(\sum_{i=1}^N w_i^{T+1}\right) \leq -(1-e^{-\eta})L_{\mathsf{Hedge}(\eta)}.$$

▶ If $a \ge 0$ then a^r is convex.

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Upper bound on $\sum_{i=1}^{N} w_i^{T+1}$

Proof of upper bound (slide 2)

Applying
$$a^r \le 1 - (1 - a)^r$$
 where $a = e^{-\eta}, r = \ell_i^t$

$$\sum_{i=1}^{N} w_i^{t+1} = \sum_{i=1}^{N} w_i^t e^{-\eta \ell_i^t}$$

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Combining

$$\sum_{i=1}^N w_i^{t+1} \leq \left(\sum_{i=1}^N w_i^t\right) \left(1 - (1 - e^{-\eta}) \mathbf{p}^t \cdot \ell^t\right)$$

 \blacktriangleright for $t=1,\ldots,T$

$$\sum_{i=1}^{N} w_i^{t+1} \leq \left(\sum_{i=1}^{N} w_i^t\right) \left(1 - (1 - e^{-\eta})\mathbf{p}^t \cdot \ell^t\right)$$

- ightharpoonup for $t = 1, \ldots, T$
- yields

$$\sum_{i=1}^{N} w_i^{T+1} \leq \prod_{t=1}^{T} (1 - (1 - e^{-\eta}) \mathbf{p}^t \cdot \ell^t)$$

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$$\sum_{i=1}^{N} w_i^{T+1} \leq \prod_{t=1}^{I} (1 - (1 - e^{-\eta}) \mathbf{p}^t \cdot \ell^t)$$

$$\leq \exp \left(-(1 - e^{-\eta}) \sum_{t=1}^{T} \mathbf{p}^t \cdot \ell^t \right)$$

since
$$1 + x \le e^x$$
 for $x = -(1 - e^{-\eta})$.

Lower bound on $\sum_{i=1}^{N} w_i^{T+1}$

For any
$$j = 1, \ldots, N$$
:

$$\sum_{i=1}^{N} w_i^{T+1} \ge w_j^{T+1} = w_j^{1} e^{-\eta L_j}$$

Combining Upper and Lower bounds

► Combining bounds on $\ln \left(\sum_{i=1}^{N} w_i^{T+1} \right)$

$$\ln w_j^1 - \eta L_j \le \ln \sum_{i=1}^N w_i^{T+1} \le -(1 - e^{-\eta}) \sum_{t=1}^T \mathbf{p}^t \cdot \ell^t$$

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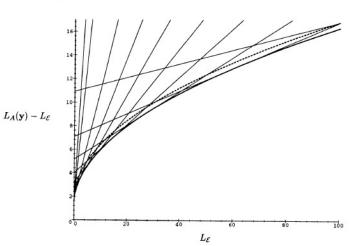
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► Reversing signs, using $L_{\text{Hedge}(\eta)} = \sum_{t=1}^{T} \mathbf{p}^t \cdot \boldsymbol{\ell}^t$ and reorganizing we get

$$L_{\mathsf{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}}$$

How to Use Expert Advice

451



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- Then

$$L_{\mathsf{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}} \leq \min_i L_i + \sqrt{2\tilde{L} \ln N} + \ln N$$

Tuning η as a function of T

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per iteration we get:

$$\frac{L_{\mathsf{Hedge}(\eta)}}{T} \leq \min_{i} \frac{L_{i}}{T} + \sqrt{\frac{2 \ln N}{T}} + \frac{\ln N}{T}$$

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The adversarial strategy is random, extremely simple, and does not depend on the hedging strategy!

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- Detailed proof quite involved. See games paper.

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► A trivial random data, in which there is nothing to be learned forces any algorithm to suffer this total loss

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- Office hour: 2-3pm on tuesdays.