Predictors that Specialize

Yoav Freund

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Outline

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The specialists setup

bounding cumulative loss using relative entropy

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Applications of specialists

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- Gives the designer a lot of flexibility.
- Generalizes the switching experts setup.

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On each iteration $t = 1, 2, 3, \ldots$

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- Adversary chooses predictions for specialists in E^t
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- Algorithm suffers loss. Specialists in E^t suffer loss. Sleeping specialists suffer no loss.

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- Average loss w.r.t. **u**: $\ell^t_{\mathbf{u}} \doteq \frac{\sum_{i \in E^t} u_i \ell^t_i}{\sum_{i \in E^t} u_i}$

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- Average loss w.r.t. $\mathbf{u}$: $\ell_{\mathbf{u}}^t \doteq \frac{\sum_{i \in E^t} u_i \ell_i^t}{\sum_{i \in E^t} u_i}$
- Goal: $L_A \leq \min_{\mathbf{u}} \sum_{t=1}^{T} \ell_{\mathbf{u}}^t +$ something small

Applying Vovk-style algs to specialists

- We use normalized weights:

$$v_i^t = \frac{w_i^t}{\sum_{j=1}^{N} w_i^t}, \quad \mathbf{v}^t = \frac{\mathbf{w}^t}{W^t}$$

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- In particular: total weight is always 1.

Bound for log-loss case

- Bound for log loss (Theorem 1), for any distribution $\mathbf{u}$:
$\sum_{t=1} u(E^t)\ell_A^t \leq \sum_{t=1}^{T} \sum_{i \in E^t} u_i \ell_i^t + \mathbf{RE}(\mathbf{u}||\mathbf{v}^1)$

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- $u(E^t) \doteq \sum_{i\in E^t} u_i$
- If we assume that $u(E^t) = U$ is constant, we get

$$L_A \leq \sum_{t=1}^{T} \ell_{\mathbf{u}}^t + \frac{\mathbf{RE}(\mathbf{u}\|\mathbf{v}^1)}{U}$$

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EQUALITY not bound!

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- $\exists \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \quad \mathbf{RE}(\mathbf{u}_1||\mathbf{u}_3) > \mathbf{RE}(\mathbf{u}_1||\mathbf{u}_2) + \mathbf{RE}(\mathbf{u}_2||\mathbf{u}_3)$

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- For the special case $\mathbf{u} = \langle 0, \ldots, 0, 1, 0, \ldots, 0 \rangle$ and $\mathbf{v}^1 = \langle 1/N, \ldots, 1/N \rangle$ we get the old bound:
  $L_A \leq \min_i L_i + \log N$

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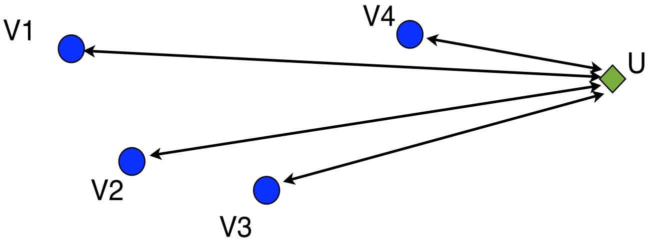

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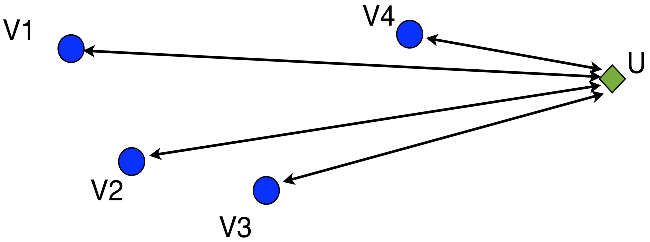

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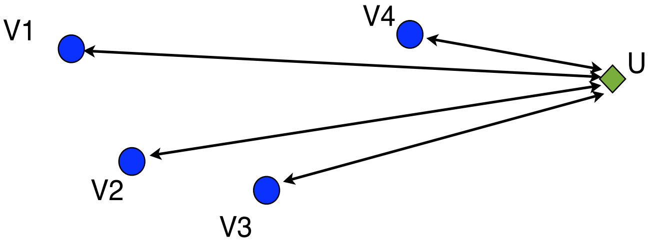

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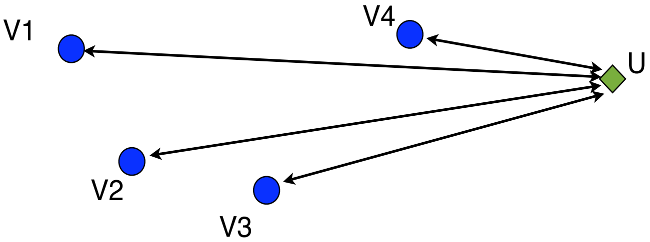

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But Manfred Warmuth does!

Proof of Lemma

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&= \sum_i u_i \log\left(\frac{W^t}{W^{t+1}}\,\frac{w_i^{t+1}}{w_i^t}\right) \\
&= \log\frac{W^t}{W^{t+1}} + \sum_i u_i \log e^{-\ell_i^t} = \ell_A^T - \sum_i u_i \ell_i^t
\end{aligned}$$

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- For any mixable loss, $a = 1$, using
  $\mathbf{u} = \langle 0, \ldots, 0, 1, 0, \ldots, 0\rangle$ and $\mathbf{v}^1 = \langle 1/N, \ldots, 1/N\rangle$ we get
  the old bound: $L_A \leq \min_i L_i + c \log N$

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