Game Theory, online learning and boosting

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Outline

The Minimax theorem

Learning games

Matrix Games

1	0	1	0	1
-1	0	0	1	1
1	0	-1	1	0

- A game between the column player and the row player.
- The chosen entry defines the loss of column player = gain of row player.
- If choices made serially, second player to choose has an advantage.

Mixed strategies

	q_1	q_2	q_3	q_4	q_5
p_1	1	0	1	0	1
p_2	-1	0	0	1	1
p_3	1	0	-1	1	0

- pure strategies: each player chooses a single action.
- mixed strategies: each player chooses a distribution over actions.
- Expeted gain/loss: $\vec{p}M\vec{q}^T$

The Minimax theorem

John Von-Neumann, 1928

$$\max_{\vec{p}} \min_{\vec{q}} \vec{p} M \vec{q}^T = \min_{\vec{q}} \max_{\vec{p}} \vec{p} M \vec{q}^T$$

- Unlike pure strategies, the order of choice of mixed strategies does not matter.
- Optimal mixed strateigies: the strategies that achieve the minimax.
- Value of the game: the value of the minimax.
- Finding the minimax strategies when the matrix is known = Linear Programming.

A matrix corresponding to online learning.

$t =$	1	2	3	4	...
expert 1	1	0	1	0	...
expert 2	-1	0	0	1	...
expert 3	1	0	-1	1	...

- The columns are revealed one at a time. strategies does not matter.
- Using Hedge or NormalHedge the row player chooses a mixed strategy over the rows that is almost as good as the best single row in hind-sight.
- The best single row in hind-site is at least as good as any mixed strategy in hind-sight.

A matrix corresponding to online learning.

$t =$	1	2	3	4	...
expert 1	1	0	1	0	...
expert 2	-1	0	0	1	...
expert 3	1	0	-1	1	...

- If the adversary playes optimally, then the row distribution coverges to a minimax optimal mixed strategy.
- But adversary might not play optimally - minimizing regret is a stronger criterion than converging to minimax optimal mixed strategy.

A matrix corresponding to boosting

	ex. 1	ex. 2	ex. 3	ex. 4	...
base rule 1	1	0	1	0	...
base rule 2	0	0	0	1	...
base rule 3	1	0	0	1	...

- 0 mistake, 1 correct.
- A weak learning algorithm: can find a base rule whose weighted error is smaller than $1/2 - \gamma$ or any distribution over the examples.
- There is a distribution over the base rules such that for any example the expected error is smaller $1/2 - \gamma$.
- Implies that the majority vote wrt this distribution over base rules is correct on **all** examples.
- Moreover - the weight of the majority is at least $1/2 + \gamma$, the minority is at most $1/2 - \gamma$.