

# Solution Of Homework 3

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## 1. Exercise 4.8

Denote the actions by  $\{1, \dots, N\}$ . Internal regret takes the maximum over all pairs of actions. Swap regret takes the maximum over all functions from the set of actions into itself. To prove the desired bound we partition the function  $\sigma$  which achieves the highest swap regret into  $N$  action pairs.

Formally, let  $R_{(i,j),n}$  be the regret associated with (not) taking action  $j$  each time we took action  $i$ , and  $R_{\sigma,n}$  be the regret associated with the mapping  $\sigma$ .

As the sequences of times at which actions  $i \neq j$  have been taken are disjoint, we get that As the times at which different actions were taken we get that, for any mapping  $\sigma$

$$R_{\sigma,n} = \sum_{i=1}^N R_{(i,\sigma(i)),n} \leq N \max_{i,j} R_{(i,j),n}$$

From which it follows that

$$\max_{\sigma} R_{\sigma,n} \leq N \max_{i,j} R_{(i,j),n}$$

## 2. Exercise 4.9

Intuitively, the reason that a deterministic predictor cannot be well calibrated for any sequence is that an adversary can construct the next bit in the sequence  $y_t$  as a function of the deterministically predetermined prediction  $q_t$ .

We want to show non  $\epsilon$ -calibration for  $\epsilon < 1/3$ .

We consider the following strategy for the adversary:

- If  $q_t < 2/3$  then  $y_t = 1$
- If  $q_t \geq 2/3$  then  $y_t = 0$

To prove that the resulting sequence will not be  $\epsilon$ -calibrated we need to show the existence of  $x$  such that  $q_t \in [x - \epsilon, x + \epsilon]$  occurs infinitely often and  $|\rho_n^\epsilon(x) - x| > \epsilon$

Let  $m$  be the number of values of  $t$  such that  $q_t < 2/3$  and therefor, by construction  $y_t = 1$ . We consider two cases, conditioned on whether or not  $m$  is finite.

If  $m$  is infinite, then there must be an  $x \leq 2/3 - \epsilon$  such that  $q_t \in [x - \epsilon, x + \epsilon]$  infinitely often. For this  $x$ ,  $\rho_n^\epsilon(x) = 1$  and thus  $\limsup_{n \rightarrow \infty} |x - \rho_n^\epsilon(x)| \geq 1/3 + \epsilon > \epsilon$ .

If  $m$  is finite, the number of times that  $y_t = 1$  is finite. Therefor  $\lim_{n \rightarrow \infty} \rho_n^\epsilon(x) = 0$ . On the other hand, there must be some  $x \geq 2/3$  for which  $q_t \in (x - \epsilon, x + \epsilon)$  infinitely often, as  $\lim_{n \rightarrow \infty} \rho_n^\epsilon(x) = 0$  for all  $x$ , we get that in this case there is also no  $\epsilon$ -calibration.

### 3. Exercise 4.10

We are given that  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n y_t = a$ .

We are using the estimator  $q_t = \frac{1}{t-1} \sum_{s=1}^{t-1} y_s$

We wish to show that the estimator  $q_t$  is well calibrated.

Fix  $\epsilon$ . We consider three cases,  $|x - a| > \epsilon$ ,  $|x - a| < \epsilon$ ,  $|x - a| = \epsilon$ .

If  $|x - a| > \epsilon$  then there comes a point in the sequence after which  $|q_t - a| < |x - a| - \epsilon$  and so  $q_t \notin (x - \epsilon, x + \epsilon)$ . In other words, there are only a finite number of elements assigned to that  $x$  and we don't care whether the estimator is calibrated for that value of  $x$

If  $|x - a| < \epsilon$  then there comes a point in the sequence after which  $|q_t - a| < \epsilon - |x - a|$ . From that it follows that  $q_t \in (x - \epsilon, x + \epsilon)$  for all  $t > t_x$ . Therefor  $\rho_n^\epsilon(x) \rightarrow a$  and  $\limsup |\rho_n^\epsilon(x) - x| \leq \epsilon$ .

However, there seems to be a problem with the proof when  $|x - a| = \epsilon$ . In this case it is possible that  $q_t \in (x - \epsilon, x + \epsilon)$  an infinite number of times and at the same time  $q_t \notin (x - \epsilon, x + \epsilon)$  an infinite number of times.

In fact, it is not hard to construct a sequence for which the claim fails:

Consider the alternating sequence

$$0, 1, 0, 1, 0, 1, \dots$$

This is a legitimate sequence as the average of the prefixes converges to  $1/2$ . Let  $x = 0.49$ ,  $\epsilon = 0.01$ . In this case  $q_t = 1/2$  for odd values of  $t$  and  $q_t$  for even values of  $t$ . Thus the subsequence that corresponds to  $q_t \in (0.48, 0.5)$  corresponds to  $y_t = 1$  and the estimate  $\rho_t^\epsilon(0.49) = 1$ , i.e. the estimator is not calibrated!