Vovk's algorithm
Mixable and unmixable loss functions

Yoav Freund

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Outline

Review

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The general prediction game

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Some useful loss functions

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Log loss

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- Square loss
  - Square loss using simple averaging

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- Summary table

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  - c^t is revealed.
- **Goal:** minimize regret:

$$-\sum_{t=1}^{T} \log p_A^t(c^t) + \min_{i=1,\ldots,N} \left(-\sum_{t=1}^{T} \log p_i^t(c^t) \right)$$

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$$\mathbf{p}_A^t = \frac{\sum_{i=1}^N w_i^t \mathbf{p}_i^t}{\sum_{i=1}^N w_i^t}$$

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EQUALITY not bound!

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4. Each expert incurs loss $\ell_i^t = \lambda(\omega^t, \gamma_i^t)$
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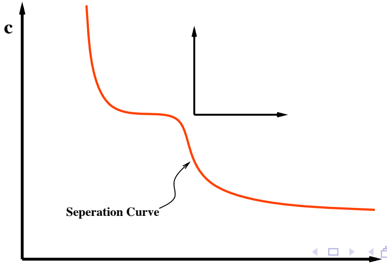

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- No triangle inequality
  $\exists p_1, p_2, p_3 \;\; \lambda(p_1,p_3) > \lambda(p_1,p_2) + \lambda(p_2,p_3)$

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- Corresponds to regression.

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- Defines a metric.
- $\lambda_{\text{hel}}(p, q) \approx \lambda_{\text{ent}}(p, q)$ when $p \approx q$ and $p, q \in (0, 1)$

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- Probability of making a mistake if predicting 0 or 1 using a biased coin
- If $P[\omega^t = 1] = q, \quad P[\omega^t = 0] = 1 - q$, then the optimal prediction is
$$\gamma^t = \begin{cases} 1 & \text{if } q > 1/2, \\ 0 & \text{otherwise} \end{cases}$$

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- **Which losses behave like entropy loss and which behave like hedge loss?**

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- There is a universally reasonable prediction $\exists \gamma \in \Gamma, \forall \omega \in \Omega, \lambda(\omega, \gamma) < \infty$
- There is no universally optimal prediction $\neg \exists \gamma \in \Gamma, \forall \omega \in \Omega, \lambda(\omega, \gamma) = 0$

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$$\sum_t \lambda(\omega^t, \gamma^t) \leq -c \ln \sum_i W_i^{T+1} \leq a L_{\min} + c \ln N$$
- Vovk's result: *yes!* a good choice for γ_t always exists!

Vovk's algorithm is the the highest achiever [Vovk95]

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The separation curve is $\left\{ \left(a(\eta), \frac{a(\eta)}{\eta} \right) \middle| \eta \in [0, \infty] \right\}$

Vovk's algorithm is the the highest achiever [Vovk95]

The pair (a, c) is achieved by some algorithm if and only if it is achieved by Vovk's algorithm.

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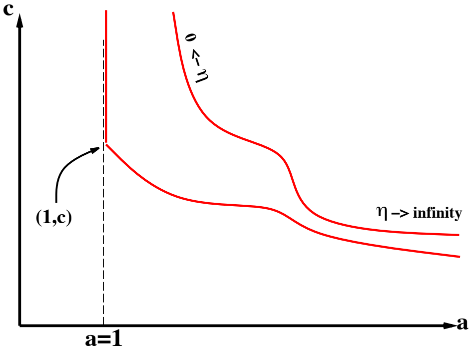

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- Equivalently - the image of the set Γ under the mapping $F(\gamma) = \left\langle e^{-\eta \lambda(\omega, \gamma)} \right\rangle_{\omega \in \Omega}$ is concave.

convexity condition: Pictorially

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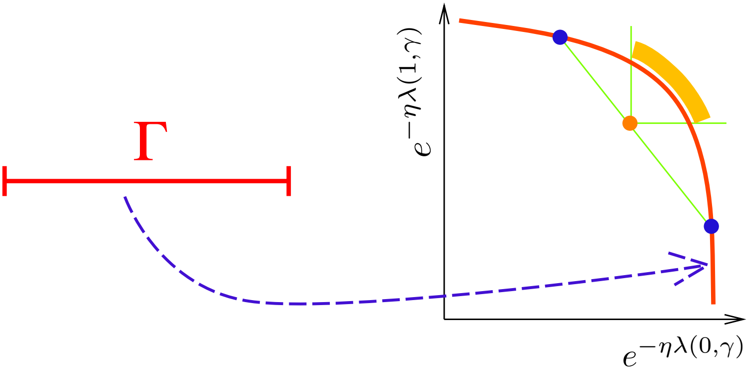

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- We are back to the online Bayes algorithm.

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- Which yields the bound

$$L_A \leq L_{\min} + 2 \ln N$$

Summary of bounds for mixable losses

TRACKING THE BEST EXPERT

Loss Functions:	c values: $(\eta = 1/c)$	
	$\mathbf{pred}_{\text{wmean}}(v, x)$	$\mathbf{pred}_{\text{Vovk}}(v, x)$
$L_{\text{sq}}(p, q)$	2	$1/2$
$L_{\text{ent}}(p, q)$	1	1
$L_{\text{hel}}(p, q)$	1	$1/\sqrt{2}$

Figure 2. $(c, 1/c)$-realizability: c values for loss and prediction function pairin