Lossless compression
and
cumulative log loss

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Outline

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- A natural way for describing a distribution.

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- To decode use the same prediction algorithm

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- Widely used in practice.

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- Code = discriminating binary expansion of a point in $[l_t, u_t)$.

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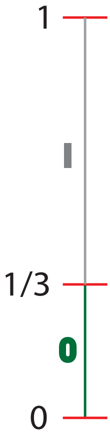

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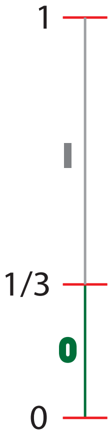

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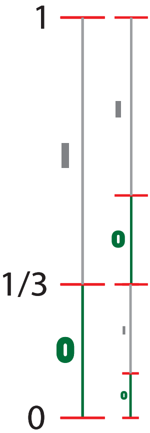

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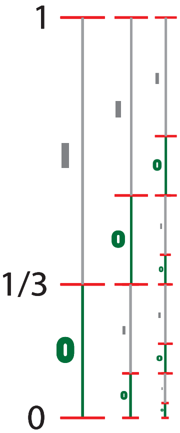

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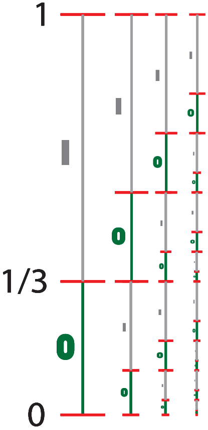

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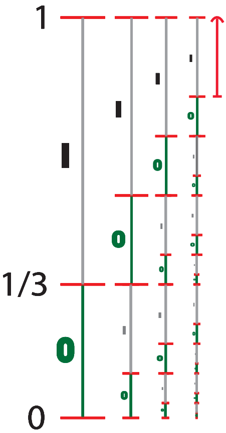

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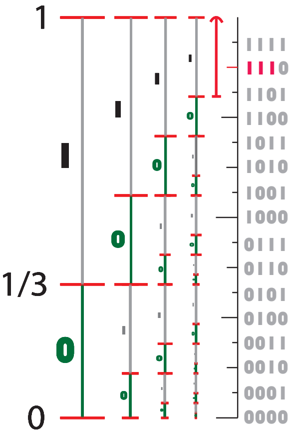

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- Holds for all sequences.

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- The proof of Shannon's lower bound is not trivial (Can be a student lecture).

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- There are other losses with this property, for example, square loss.

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- High risk prediction: Setting $p_t \in \{0, 1\}$ results in Bonus $b_T = 2^T$ if always correct, zero otherwise.
- If forecaster predicts with the true probabilities then

$$E(\log b_T) = T - H(p_T)$$

and that is the maximal expected value for $E(\log b_T)$

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- Taking logs, we get cumulative log loss.

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  - Good prediction model = model that minimizes the total code length
- Often inappropriate because based on lossless coding. **Lossy** coding often more appropriate.

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- **Goal:** Total loss of algorithm minus loss of best predictor should be at most $\log_2 N$

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- $\lceil {L_A}^T \rceil$ is the code length if A is combined with arithmetic coding.

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- **Goal:** minimize regret:

$$-\sum_{t=1}^{T} \log p_A^t(c^t) + \min_{i=1,\ldots,N} \left(-\sum_{t=1}^{T} \log p_i^t(c^t) \right)$$

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$$

- Dividing by T we get $\frac{L_A^T}{T} = \min_i \frac{L_i^T}{T} + \frac{\log N}{T}$

Bound better than for two part codes

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  - For number of mistakes - Bayesian method cannot be "fixed". Requires variable learning rate. Regret bounds are

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- technical details: On iteration t, $|\vec{X}| = t$. Use the predictions of programs $\vec{b}$ such that $|\vec{b}| \leq t$ and for which $V(\vec{b}, \vec{X}, 2^t) = 1$. Assing the remaining mass the prediction $1/2$ (insuring a loss of 1)

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- Can we still get a meaningful bound?

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- We need a new lower bound on the final total weight

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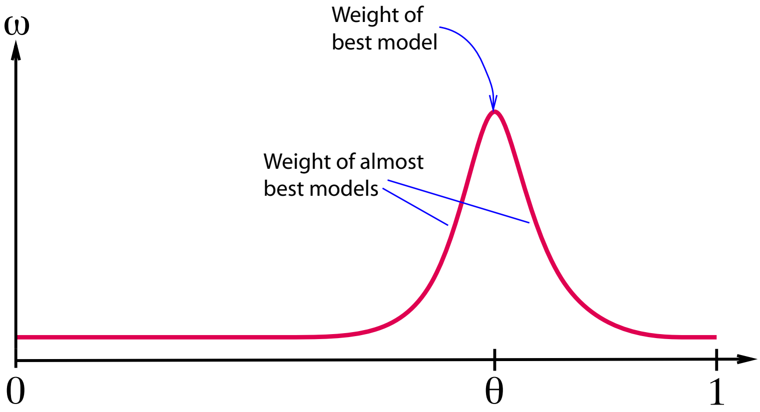

Expanding the exponent around the peak

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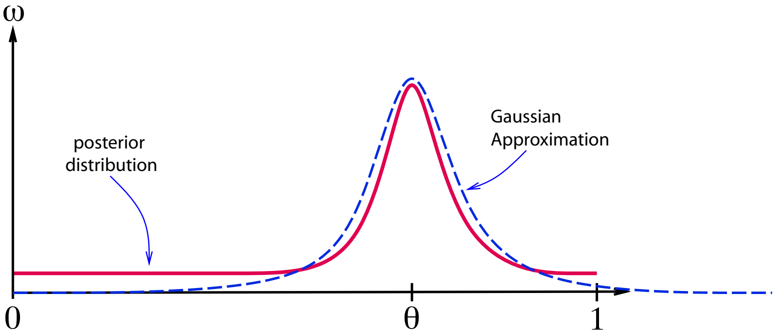

Laplace Approximation (details)

$$\int_0^1 w(\theta) e^{T(g(\hat{\theta},\theta)-g(\hat{\theta},\hat{\theta}))} d\theta$$

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$$\int_0^1 w(\theta) e^{T(g(\hat{\theta},\theta) - g(\hat{\theta},\hat{\theta}))} d\theta$$

$$= \quad w(\hat{\theta}) \sqrt{\frac{-2\pi}{T \left. \frac{d^2}{d\theta^2} \right|_{\theta=\hat{\theta}} (g(\hat{\theta},\theta) - g(\hat{\theta},\hat{\theta}))}} + O(T^{-3/2})$$

Choosing the optimal prior

- Choose $w(\theta)$ to maximize the worst-case final total weight

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- Make bound equal for all $\hat{\theta} \in [0,1]$ by choosing

$$w^*(\hat{\theta}) = \frac{1}{Z} \sqrt{\frac{\left.\frac{d^2}{d\theta^2}\right|_{\theta=\hat{\theta}} (g(\hat{\theta},\theta) - g(\hat{\theta},\hat{\theta}))}{-2\pi}} ,$$

where Z is the normalization factor:

$$Z = \sqrt{\frac{1}{2\pi}} \int_0^1 \sqrt{\left.\frac{d^2}{d\theta^2}\right|_{\theta=\hat{\theta}} (g(\hat{\theta},\hat{\theta}) - g(\hat{\theta},\theta))} \; d\hat{\theta}$$

The bound for the optimal prior

- Plugging in we get

$$
\begin{aligned}
L_A - L_{\min} &\leq \ln \int_0^1 w^*(\theta) e^{T(g(\hat{\theta},\theta) - g(\hat{\theta},\hat{\theta}))} d\theta \\
&= \ln \left(\sqrt{\frac{2\pi Z}{T}} + O(T^{-3/2}) \right) \\
&= \frac{1}{2} \ln \frac{T}{2\pi} - \frac{1}{2} \ln Z + O(1/T) .
\end{aligned}
$$

Solving for log-loss

- The exponent in the integral is

$$g(\hat{\theta}, \theta) - g(\hat{\theta}, \hat{\theta}) = \hat{\theta} \ln \frac{\hat{\theta}}{\theta} + (1 - \hat{\theta}) \ln \frac{1 - \hat{\theta}}{1 - \theta} = D_{KL}(\hat{\theta} || \theta)$$

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- The optimal prior:

$$w^*(\hat{\theta}) = \frac{1}{\pi\sqrt{\hat{\theta}(1-\hat{\theta})}}$$

  Known in general as Jeffrey's prior. And, in this case, the Dirichlet-$(1/2, 1/2)$ prior.

The cumulative log loss of Bayes using Jeffrey's prior

- $$L_A - L_{\min} \leq \frac{1}{2}\ln(T+1) + \frac{1}{2}\ln\frac{\pi}{2} + O(1/T)$$

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- This is called the Trichevsky Trofimov prediction rule.

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- Suffers larger regret when $\hat{\theta}$ is far from $1/2$

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- $$L_*^T - \min_\theta L_\theta^T \geq \frac{1}{2}\ln(T+1) + \frac{1}{2}\ln\frac{\pi}{2} - O(\frac{1}{\sqrt{T}})$$

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- The constant C is optimal.

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