Online learning in repeated matrix games

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Outline

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Zero sum games in matrix form

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- Game repeated many times.

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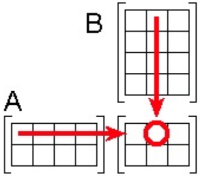
Mixed strategies in matrix notation



$$(A \times B)_{12} = \sum_{r=1}^{4} a_{1r} b_{r2} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} + a_{14}b_{42}$$

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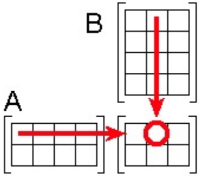

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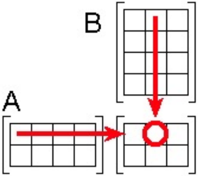

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$$\mathbf{M}(\mathbf{P}, \mathbf{Q}) = \mathbf{P}^T \mathbf{M} \mathbf{Q} = \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbf{P}(i)\mathbf{M}(i,j)\mathbf{Q}(j)$$

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In words: for mixed strategies, choosing second gives no advantage.

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- Learning guarantees no regret with respect to the past.
- If all sides use learning, then game will converge to minmax equilibrium.
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- Our goal is to minimize regret.

Fictitious play

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Fictitious play

- Choose the best action with respect to the sum of past loss vectors.
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- Consider playing the matching coins game against an adversary that alternates HTTHHTTHHTTHH....

Randomized Fictitious play

- Choose the best action with respect to the sum of past loss vectors plus noise.

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- Hannan 1957 Randomized ficticonverge to regret minimizing strategy.

The basic algorithm

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- $$\mathbf{P}_{t+1}(i) = \mathbf{P}_t(i)\frac{e^{-\eta \mathbf{M}(i,\mathbf{Q}_t)}}{Z_t}$$
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- Where $Z_t = \sum_{i=1}^{n} \mathbf{P}_t(i) e^{-\eta \mathbf{M}(i,\mathbf{Q}_t)}$
- $\eta > 0$ is the learning rate.

Generalized regret bound

- Regret relative to the best *pure strategy* i

$$\sum_{t=1}^{T} \mathbf{M}(\mathbf{P}_t, \mathbf{Q}_t) \leq \left(\frac{1}{1 - e^{-\eta}}\right) \min_i \left[\eta \sum_{t=1}^{T} \mathbf{M}(i, \mathbf{Q}_t) - \ln \mathbf{P}_1(i)\right]$$

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- regret with respect the the best *mixed strategy* $\mathbf{P}$:

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- Where

$$\mathrm{RE}\left(1\mathbf{P} \parallel \mathbf{Q}\right) \doteq \sum_{i=1}^{n} \mathbf{P}(i) \ln \frac{\mathbf{P}(i)}{\mathbf{Q}(i)}$$

Main Theorem

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- Any sequence of mixed strat. $\mathbf{Q}_1, \ldots, \mathbf{Q}_T$
- The sequence $\mathbf{P}_1, \ldots, \mathbf{P}_T$ produced by basic alg using $\eta > 0$ satisfies

$$\sum_{t=1}^{T} \mathbf{M}(\mathbf{P}_t, \mathbf{Q}_t) \leq \left(\frac{1}{1 - e^{-\eta}} \right) \min_{\mathbf{P}} \left[\eta \sum_{t=1}^{T} \mathbf{M}(\mathbf{P}, \mathbf{Q}_t) + \mathrm{RE}\left(\mathbf{P} \parallel \mathbf{P}_1 \right) \right]$$

Corollary

- Setting $\eta = \ln\left(1 + \sqrt{\frac{2 \ln n}{T}}\right)$

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$$\frac{1}{T}\sum_{t=1}^{T}\mathbf{M}(\mathbf{P}_t, \mathbf{Q}_t) \le \min_{\mathbf{P}} \frac{1}{T}\sum_{t=1}^{T}\mathbf{M}(\mathbf{P}, \mathbf{Q}_t) + \Delta_{T,n}$$

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- Where

$$\Delta_{T,n} = \sqrt{\frac{2\ln n}{T}} + \frac{\ln n}{T} = O\left(\sqrt{\frac{\ln n}{T}}\right).$$

Main Lemma

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$$\mathrm{RE}\left(\tilde{\mathbf{P}} \ \| \ \mathbf{P}_{t+1}\right) - \mathrm{RE}\left(\tilde{\mathbf{P}} \ \| \ \mathbf{P}_{t}\right) \leq \eta \mathbf{M}(\tilde{\mathbf{P}}, \mathbf{Q}_t) - (1 - e^{-\eta}) \mathbf{M}(\mathbf{P}_t, \mathbf{Q}_t)$$

Visual intuition

$$\mathrm{RE}\left(\tilde{\mathbf{P}} \;\|\; \mathbf{P}_{t+1}\right) - \mathrm{RE}\left(\tilde{\mathbf{P}} \;\|\; \mathbf{P}_{t}\right) \leq \eta \mathbf{M}(\tilde{\mathbf{P}}, \mathbf{Q}_t) - (1 - e^{-\eta})\mathbf{M}(\mathbf{P}_t, \mathbf{Q}_t)$$

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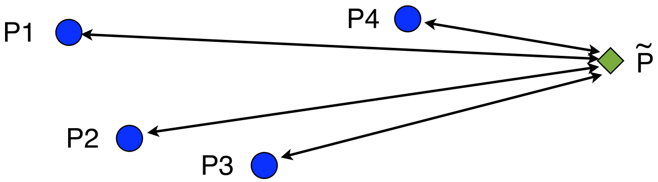

Proof of Lemma (1)

$$\mathrm{RE}\left(\tilde{\mathbf{P}} \;\|\; \mathbf{P}_{t+1}\right) - \mathrm{RE}\left(\tilde{\mathbf{P}} \;\|\; \mathbf{P}_t\right)$$

Proof of Lemma (1)

$$\begin{aligned} &\mathrm{RE}\left(\tilde{\mathbf{P}} \;\|\; \mathbf{P}_{t+1}\right) - \mathrm{RE}\left(\tilde{\mathbf{P}} \;\|\; \mathbf{P}_{t}\right) \\ = &\sum_{i=1}^{n} \tilde{\mathbf{P}}(i) \ln \frac{\tilde{\mathbf{P}}(i)}{\mathbf{P}_{t+1}(i)} - \sum_{i=1}^{n} \tilde{\mathbf{P}}(i) \ln \frac{\tilde{\mathbf{P}}(i)}{\mathbf{P}_{t}(i)} \end{aligned}$$

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&= \sum_{i=1}^{n} \tilde{\mathbf{P}}(i) \ln \frac{\mathbf{P}_{t}(i)}{\mathbf{P}_{t+1}(i)} \\
&= \sum_{i=1}^{n} \tilde{\mathbf{P}}(i) \ln \frac{Z_t}{e^{\eta \mathbf{M}(i, \mathbf{Q}_t)}}
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Row player chooses $\mathbf{P}_t$ using learning alg.

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Let $\overline{\mathbf{P}} \doteq \frac{1}{T}\sum_{t=1}^{T} \mathbf{P}_t$ and $\overline{\mathbf{Q}} \doteq \frac{1}{T}\sum_{t=1}^{T} \mathbf{Q}_t$

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\end{aligned}$$

Proving minmax Theorem using online learning (2)

$$= \frac{1}{T}\sum_{t=1}^{T} \mathbf{P}_t{}^{\mathrm{T}}\mathbf{M}\mathbf{Q}_t \qquad \text{by definition of } \mathbf{Q}_t$$

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$$\leq \min_{\mathbf{P}} \frac{1}{T}\sum_{t=1}^{T}\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{Q}_t + \Delta_{T,n} \qquad \text{by the Corollary}$$

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but $\Delta_{T,n}$ can be set arbitrarily small.

Solving a game

- to solve a game is to find the min-max mixed strategies **P**, **Q**

Solving a game

- to solve a game is to find the min-max mixed strategies $\mathbf{P}, \mathbf{Q}$
- Suppose that **Hedge**(η)is playing $\mathbf{P}_1, \mathbf{P}_2,$ against an adversary that plays $\mathbf{Q}_1, \mathbf{Q}_2, \ldots$ such that

Using average row distribution

- Using the

Using the final row distribution

- XXX