# Introduction to Online Learning Algorithms

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# OutlineA

Halving Algorithm

Hedge Algorithm

Perceptron

Laplace law of succession

# Example trace for Halving Algorithm

| | $t = 1$ | $t = 2$ | $t = 3$ | $t = 4$ | $t = 5$ |
|---|---|---|---|---|---|
| expert1 | 1 | 1 | 1 | 1 | - |
| expert2 | 1 | 0 | - | - | - |
| expert3 | 0 | - | - | - | - |
| expert4 | 1 | 0 | - | - | - |
| expert5 | 1 | 0 | - | - | - |
| expert6 | 0 | - | - | - | - |
| expert7 | 1 | 1 | 1 | 1 | 0 |
| expert8 | 1 | 1 | 1 | 0 | - |
| | | | | | |
| alg. | 1 | 0 | 1 | 1 | 0 |
| outcome | 1 | 1 | 1 | 0 | 0 |

# Mistake bound for Halving algorithm

- Each time algorithm makes a mistakes, the pool of perfect experts is halved (at least).
- We assume that at least one expert is perfect.
- Number of mistakes is at most $\log_2 N$.
- No stochastic assumptions whatsoever.
- Proof is based on combining a lower and upper bounds on the number of perfect experts.

# The hedging problem

- $N$ possible actions
- At each time step $t$:
  - Algorithm chooses a distribution $\vec{P}^t$ over actions.
  - Losses $0 \leq \ell_i^t \leq 1$ of all actions $i = 1, \ldots, N$ are revealed.
  - Algorithm suffers **expected** loss.
- **Goal:** minimize total expected loss
- Here we have stochasticity - but only in **algorithm**, not in **outcome**
- Fits nicely in game theory

# Hedging vs. Halving

- Like halving - we want to zoom into best action (expert).
- Unlike halving - no action is perfect.
- Basic idea - reduce probability of lossy actions,
  but not all the way to zero.
- **Modified Goal:** minimize difference between
  expected total loss
  and
  minimal total loss of repeating one action.

# The Hedge Algorithm

Consider action $i$ at time $t$

- Total loss:

$$L_i^t = \sum_{s=1}^{t-1} \ell_i^s$$

- Weight:

$$W_i^t = e^{-\eta L_i^t}$$

- $\eta > 0$ is the learning rate parameter. Halving: $\eta = \infty$
- Probability:

$$P_i^t = \frac{W_i^t}{\sum_{j=1}^{N} W_j^t}$$

# Example trace for Hedge Algorithm

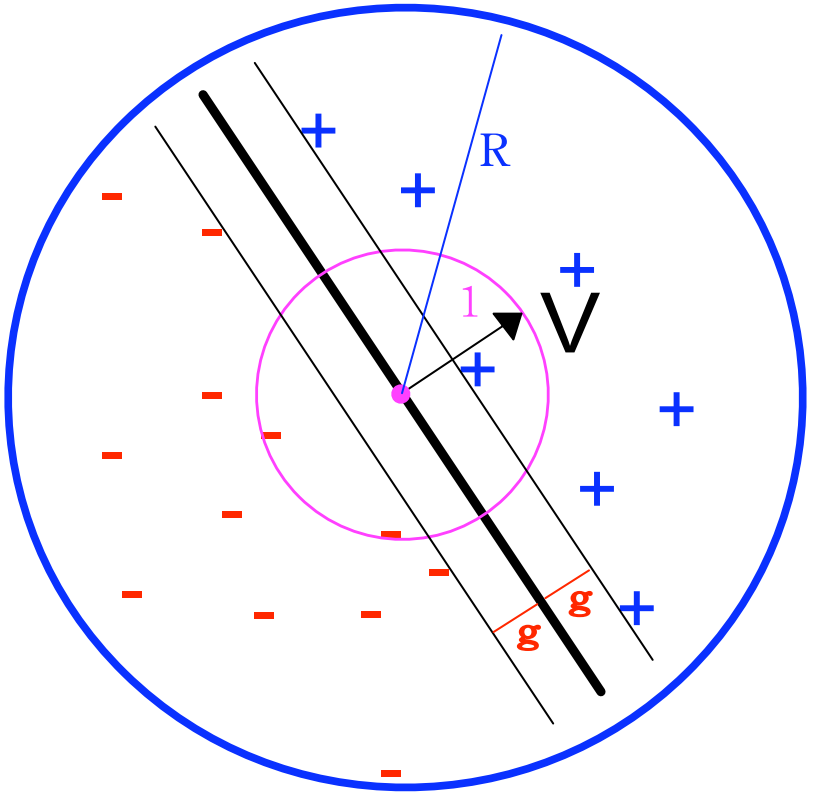
$\eta = 1$

| | $\vec{W}^1$ | $\vec{\ell}^1$ | $\vec{W}^2$ | $\vec{\ell}^2$ | $\vec{W}^3$ | $\vec{\ell}^3$ | total |
|---|---|---|---|---|---|---|---|
| expert1 | 1 | .1 | .90 | .1 | 0.82 | 0 | .2 |
| expert2 | 1 | .8 | .45 | .5 | 0.27 | .2 | 1.5 |
| expert3 | 1 | .3 | .74 | .2 | 0.61 | .2 | .7 |
| expert4 | 1 | .1 | .90 | .7 | 0.45 | .8 | 1.6 |
| expert5 | 1 | .9 | .41 | 1 | 0.15 | .8 | 2.7 |
| expert6 | 1 | 0 | 1 | .1 | 0.91 | .2 | .3 |
| expert7 | 1 | 1 | .37 | .5 | 0.22 | .4 | 1.9 |
| expert8 | 1 | .8 | .45 | .2 | 0.37 | .6 | 1.6 |
| alg. | | .5 | | .36 | | .30 | 1.16 |

# Bound for Hedge Algorithm

- $L^t_{\text{Hedge}}$: Expected total loss of Hedge algorithm for time $1, 2, \ldots, t$
- $$\forall t, i, \quad L_{\text{Hedge}} \leq \frac{\ln N + \eta L_i^t}{1 - e^{-\eta}}$$
- Which implies
$$\forall t, \quad L_{\text{Hedge}} \leq \min_i \left( \frac{\ln N + \eta L_i^t}{1 - e^{-\eta}} \right)$$
- Proof and choice of $\eta$: next class.

# The Perceptron Problem

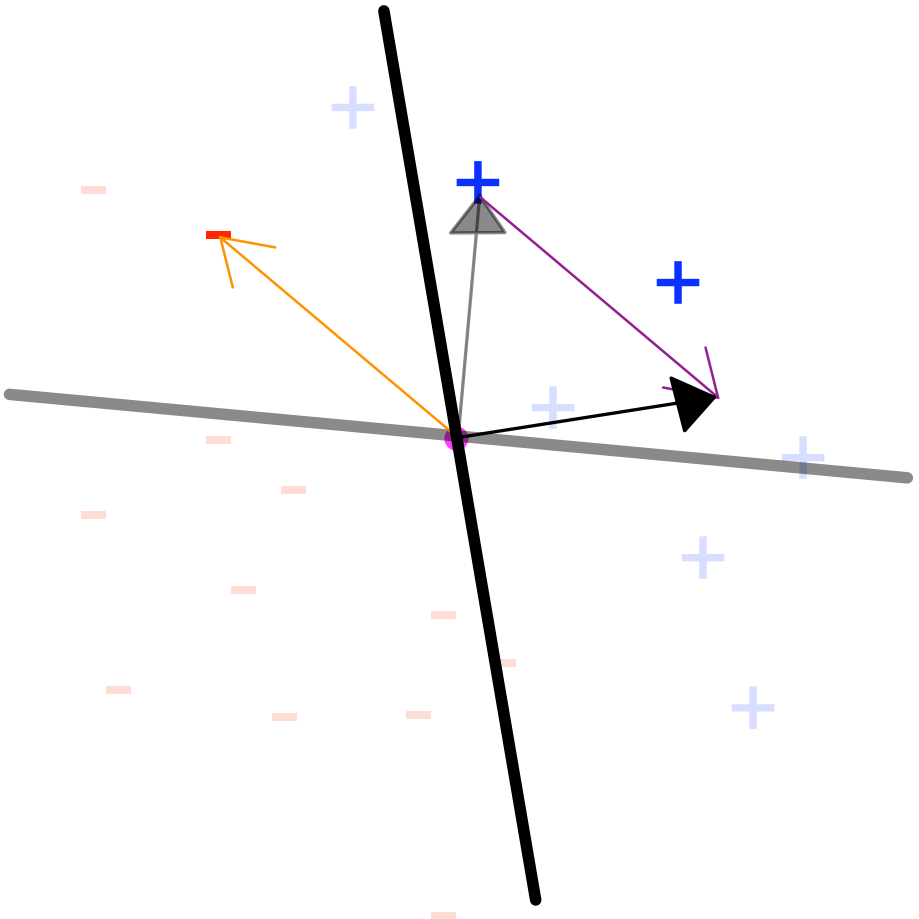



- $\|\vec{V}\| = 1$
- Example = $(\vec{X}, y)$, $y \in \{-1, +1\}$.
- $\forall \vec{X},\ \ \|\vec{X}\| \leq R$.
- $\forall (\vec{X}, y)$, $y(\vec{X} \cdot \vec{V}) \geq g$

# The Perceptron learning algorithm

- An online algorithm. Examples presented one by one.
- start with $\vec{W}_0 = \vec{0}$.
- If mistake: $(\vec{W}_i \cdot \vec{X}_i) y_i \leq 0$
  - Update $\vec{W}_{i+1} = \vec{W}_i + y_i X_i$.

# Example trace for the perceptron algorithm



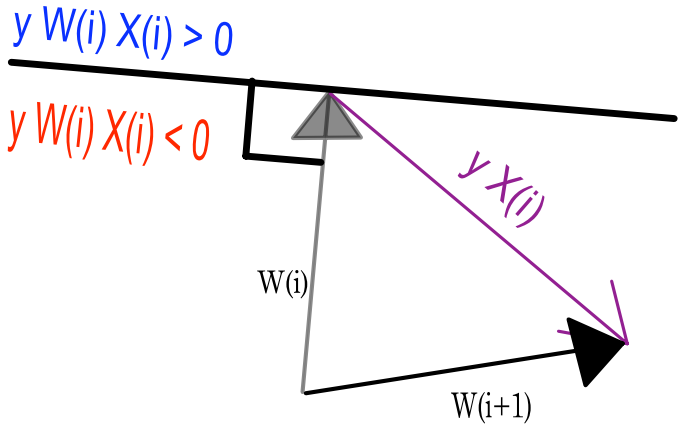
# Bound on number of mistakes

- The number of mistakes that the perceptron algorithm can make is at most $\left(\frac{R}{g}\right)^2$.
- Proof by combining upper and lower bounds on $||\vec{W}||$.

# Pythagorian Lemma

If $(\vec{W}_i \cdot X_i)y < 0$ then

$$\|\vec{W}_{i+1}\|^2 = \|\vec{W}_i + y_i\vec{X}_i\|^2 \leq \|\vec{W}_i\|^2 + \|\vec{X}_i\|^2$$



# Upper bound on $\|\vec{W}_i\|$

Proof by induction

- Claim: $\|\vec{W}_i\|^2 \leq iR^2$
- Base: $i = 0$, $\|\vec{W}_0\|^2 = 0$
- Induction step (assume for $i$ and prove for $i + 1$):
  $\|\vec{W}_{i+1}\|^2 \leq \|\vec{W}_i\|^2 + \|\vec{X}_i\|^2$
  $\leq \|\vec{W}_i\|^2 + R^2 \leq (i + 1)R^2$

# Lower bound on $\|\vec{W}_i\|$

$\|\vec{W}_i\| \geq \vec{W}_i \cdot \vec{V}$ because $\|\vec{V}\| = 1$.
We prove a lower bound on $\vec{W}_i \cdot \vec{V}$ using induction over $i$

- Claim: $\vec{W}_i \cdot \vec{V} \geq ig$
- Base: $i = 0$, $\vec{W}_0 \cdot \vec{V} = 0$
- Induction step (assume for $i$ and prove for $i + 1$):
  $\vec{W}_{i+1} \cdot \vec{V} = \left(\vec{W}_i + \vec{X}_i y_i\right) \vec{V} = \vec{W}_i \cdot \vec{V} + y_i \vec{X}_i \cdot \vec{V}$
  $\geq ig + g = (i + 1)g$

# Combining the upper and lower bounds

$$(ig)^2 \leq \|\vec{W}_i\|^2 \leq iR^2$$

Thus:

$$i \leq \left(\frac{R}{g}\right)^2$$

# Estimating the bias of a coin

- We observe $n$ coin flips:
  **H,T,T,H,H,T,H,T,T**
- We want to estimate the **probability** that the next flip will be **H**ead.
- Natural Answer:

$$\frac{\#\mathbf{H}}{n} = \frac{4}{9}$$

# What if the estimation has to be done online?

- We observe a bf sequence of coin flips, and have to predict the probability of **H**ead at each step:
  $p_0$,**H**,$p_1$,**T**,$p_2$,...
- What would be a good value for $p_0$?
- For $p_1$?
- Laplace Law of succession

$$\frac{\#\mathbf{H}+1}{n+2}$$

- Turns out that a better rule is

$$\frac{\#\mathbf{H}+1/2}{n+1}$$

  Krichevsky and Trofimov, 1981
- Why?
- What does “better” mean?

# To be continued...

Please

- Register on twiki (follow directions on my home page)
- Follow link from main twiki page to “Online learning course”
- Add yourself to the list and the table on `ClassParticipants`
- Go to `CoursePlan/LessonNo1` to see slides and to post questions and answers.
- See you on Thursday!