Vovk's algorithm
Mixable and unmixable loss functions

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Outline

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  - c^t is revealed.
- **Goal:** minimize regret:

$$- \sum_{t=1}^{T} \log p_A^t(c^t) + \min_{i=1,\ldots,N} \left(- \sum_{t=1}^{T} \log p_i^t(c^t) \right)$$

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- Prediction of algorithm A

$$\mathbf{p}_A^t = \frac{\sum_{i=1}^{N} w_i^t \mathbf{p}_i^t}{\sum_{i=1}^{N} w_i^t}$$

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EQUALITY not bound!

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3. Nature chooses an outcome $\omega^t \in \Omega$
4. Each expert incurs loss $\ell_i^t = \lambda(\omega^t, \gamma_i^t)$
   The learner incurs loss $\ell_A^t = \lambda(\omega^t, \gamma^t)$

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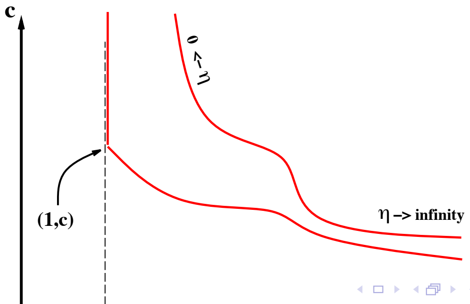

Some useful loss functions

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- No triangle inequality
  $\exists p_1, p_2, p_3 \;\; \lambda(p_1, p_3) > \lambda(p_1, p_2) + \lambda(p_2, p_3)$

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- Corresponds to regression.

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- Defines a metric.
- $\lambda_{\text{hel}}(p, q) \approx \lambda_{\text{ent}}(p, q)$ when $p \approx q$ and $p, q \in (0, 1)$

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- Probability of making a mistake if predicting 0 or 1 using a biased coin
- If $P[\omega^t = 1] = q, \;\; P[\omega^t = 0] = 1 - q$, then the optimal prediction is
$$\gamma^t = \begin{cases} 1 & \text{if } q > 1/2, \\ 0 & \text{otherwise} \end{cases}$$

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- **Which losses behave like entropy loss and which behave like hedge loss?**

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- There is a universally reasonable prediction
  $\exists \gamma \in \Gamma, \forall \omega \in \Omega, \lambda(\omega, \gamma) < \infty$
- There is no universally optimal prediction
  $\neg \exists \gamma \in \Gamma, \forall \omega \in \Omega, \lambda(\omega, \gamma) = 0$

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  Choose γ_t so that, for all $\omega^t \in \Omega$:

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- Vovk's result: *yes!* a good choice for γ_t always exists!

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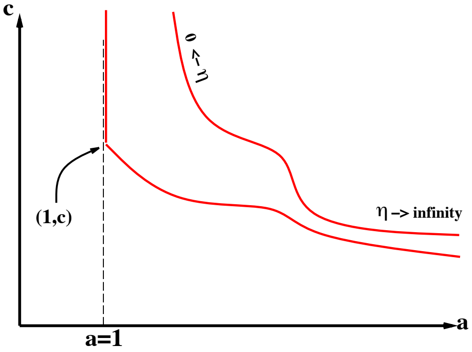

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- Equivalently - the image of the set Γ under the mapping $F(\gamma) = \left\langle e^{-\eta\lambda(\omega,\gamma)} \right\rangle_{\omega\in\Omega}$ is concave.

convexity condition: Pictorially

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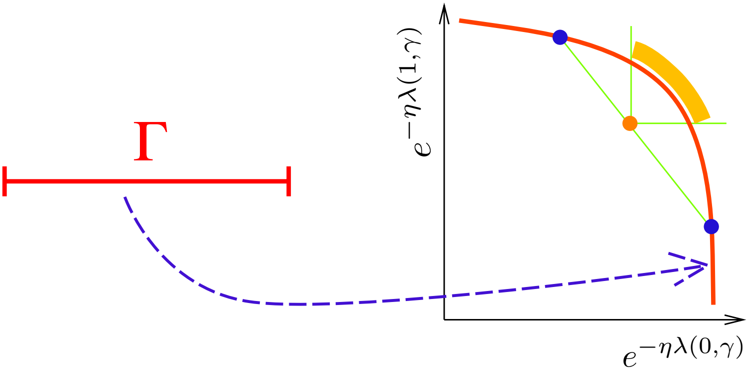

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- We are back to the online Bayes algorithm.

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- Which yields the bound

$$L_A \leq L_{\min} + 2 \ln N$$

Summary of bounds for mixable losses

TRACKING THE BEST EXPERT

Loss Functions:	c values: $(\eta = 1/c)$	
	$\mathbf{pred}_{\mathrm{wmean}}(v, x)$	$\mathbf{pred}_{\mathrm{Vovk}}(v, x)$
$L_{\mathrm{sq}}(p, q)$	2	$1/2$
$L_{\mathrm{ent}}(p, q)$	1	1
$L_{\mathrm{hel}}(p, q)$	1	$1/\sqrt{2}$

Figure 2. $(c, 1/c)$-realizability: c values for loss and prediction function pairin