

Trofimov [186]. Lemma 9.3 appears in Willems, Shtarkov, and Tjalkens [311]. It is shown in Xie and Barron [312] and Freund [111] that the Krichevsky–Trofimov mixture, in fact, achieves a regret  $\frac{1}{2} \ln n + \frac{1}{2} \ln \frac{\pi}{2} + o(1)$ . This is optimal even in the additive constant for all sequences except for those containing very few 1's or 2's. Xie and Barron [312] refine the mixture further so that it achieves a worst-case cumulative regret of  $\frac{1}{2} \ln n + \frac{1}{2} \ln \frac{\pi}{2} + o(1)$ , matching the performance of the minimax optimal forecaster. Xie and Barron [312] also derive the analog of all these results in the general case  $m \geq 2$ . Theorem 9.5 also appears in [312], where the case of  $m$ -ary alphabet is also treated and the asymptotic constant is determined. Szpankowski [282] develops analytical tools to determine  $V_n(\mathcal{F})$  to arbitrary precision for the class of constant experts; see also Drmota and Szpankowski [90].

The material of Section 9.6 is based on the work of Weinberger, Merhav, and Feder [307], who prove a similar result in a significantly more general setup. In [307] an analog lower bound is shown for classes of experts defined by a finite-state machine with a strongly connected state transition graph. The work of Weinberger, Merhav, and Feder was inspired by a similar result of Rissanen [238] in the model of probabilistic prediction.

Lower bounds for the minimax regret under general metric entropy assumptions may be obtained by noting that lower bounds for the probabilistic counterpart work in the setup of individual sequences as well. We mention the important work of Haussler and Oppé [152].

Theorem 9.8 is due to Cesa-Bianchi and Lugosi [53], who improve an earlier result of Oppé and Haussler [228] for classes of static experts. A general expression for the minimax regret, not described in this chapter, for certain regular parametric classes has been derived by Rissanen [242]. More specifically, Rissanen considers classes  $\mathcal{F}$  of experts  $f_{n,\theta}$  parameterized by an open and bounded set of parameters  $\Theta \subset \mathbb{R}^k$ . It is shown in [242] that under certain regularity assumptions,

$$V_n(\mathcal{F}) = \frac{k}{2} \ln \frac{n}{2\pi} + \ln \int_{\Theta} \sqrt{\det(I(\theta))} d\theta + o(1),$$

where the  $k \times k$  matrix  $I(\theta)$  is the so-called *Fisher information matrix*, whose entry in position  $(i, j)$  is defined by

$$-\frac{1}{n} \sum_{y^n} f_{n,\theta}(y^n) \frac{\partial^2 \ln f_{n,\theta}(y^n)}{\partial \theta_i \partial \theta_j},$$

where  $\theta_i$  is the  $i$ th component of vector  $\theta$ . Yamanishi [313] generalizes Rissanen's results to a wider class of loss functions. The expressions of the minimax regret for the class of Markov experts were determined by Rissanen [242] and Jacquet and Szpankowski [168].

Finally, we mention that the problem of prediction under the logarithmic loss has applications in the study of the general principle of *minimum description length* (MDL), first proposed by Rissanen [237, 238, 241]. For quite exhaustive surveys see Barron, Rissanen, Yu [22], Grünwald [134], and Hansen and Yu [143].

### 9.13 Exercises

- 9.1 Let  $\mathcal{F}$  be the class of all experts such that for each  $f \in \mathcal{F}$ ,  $f_i(j | y^{i-1}) = f(j)$  (with  $f(j) \geq 0$ ,  $\sum_{j=1}^m f(j) = 1$ ) independently of  $i$  and  $y^{i-1}$ . For a particular sequence  $y^n \in \mathcal{Y}^n$ , determine the best expert and its cumulative loss.

- 9.2 Assume that you want to bet in the horse race only once and that you know that the  $j$ th horse wins with probability  $p_j$  and the odds are  $o_1, \dots, o_m$ . How would you distribute your money to maximize your expected winnings? Contrast your result with the setup described in Section 9.3, where the optimal betting strategy is independent of the odds.
- 9.3 Show that there exists a class  $\mathcal{F}$  of experts with cardinality  $|\mathcal{F}| = N$  such that for all  $n \geq \log_2 N$ ,  $V_n(\mathcal{F}) \geq \ln N$ . This exercise shows that the bound  $\ln N$  achieved by the uniform mixture forecaster is not improvable for some classes.
- 9.4 Consider class  $\mathcal{F}$  of all constant experts. Show that the normalized maximum likelihood forecaster  $p_n^*$  is horizon dependent in the sense that if  $p_t'$  denotes the normalized maximum likelihood forecaster for some  $t < n$  (i.e.,  $p_t'$  achieves the minimax regret  $V_t(\mathcal{F})$ ), then it is not true that

$$\sum_{y_{t-1}^n \in \mathcal{Y}^{n-t}} p_n^*(y^n) = p_t'(y^t).$$

- 9.5 Let  $\mathcal{F}$  be a class of experts and let  $q$  and  $\hat{p}$  be arbitrary forecasters (i.e., probability distributions over  $\mathcal{Y}^n$ ). Show that

$$\sum_{y^n \in \mathcal{Y}^n} q(y^n) \ln \frac{\sup_{f \in \mathcal{F}} f_n(y^n)}{\hat{p}_n(y^n)} \geq \sum_{y^n \in \mathcal{Y}^n} q(y^n) \ln \frac{\sup_{f \in \mathcal{F}} f_n(y^n)}{q(y^n)}$$

and that

$$\sum_{y^n \in \mathcal{Y}^n} q(y^n) \ln \frac{\sup_{f \in \mathcal{F}} f_n(y^n)}{q(y^n)} = V_n(\mathcal{F}) - D(q \| p_n^*),$$

where  $p_n^*$  is the normalized maximum likelihood forecaster and  $D$  denotes Kullback–Leibler divergence.

- 9.6 Show that

$$\int_0^1 \frac{1}{\sqrt{x(1-x)}} dx = \pi.$$

*Hint:* Substitute  $x$  by  $\sin^2 \alpha$ .

- 9.7 Show that for every  $n > 1$  there exists a class  $\mathcal{F}_n$  of two static experts such that if  $\hat{p}$  denotes the exponentially weighted average (or mixture) forecaster, then

$$\frac{V_n(\hat{p}, \mathcal{F}_n)}{V_n(\mathcal{F}_n)} \geq c\sqrt{n}$$

for some universal constant  $c$  (see Cesa-Bianchi and Lugosi [53]). *Hint:* Let  $\mathcal{Y} = \{0, 1\}$  and let  $\mathcal{F}_n$  contain the two experts  $f, g$  defined by  $f(1 | y^{t-1}) = \frac{1}{2}$  and  $g(1 | y^{t-1}) = \frac{1}{2} + \frac{1}{2n}$ . Show, on the one hand, that  $V_n(\mathcal{F}_n) \leq c_1 n^{-1/2}$  and on the other hand that  $V_n(\hat{p}, \mathcal{F}_n) \geq c_2$  for appropriate constants  $c_1, c_2$ .

- 9.8 Extend Theorem 9.2 to the case when the outcome space is  $\mathcal{Y} = \{1, \dots, m\}$ . More precisely, show that the minimax regret of the class of constant experts is

$$V_n(\mathcal{F}) = \frac{m-1}{2} \ln \frac{n}{2\pi} + \ln \frac{\Gamma(1/2)^m}{\Gamma(m/2)} + o(1) = \frac{m-1}{2} \ln \frac{ne}{m} + o(1)$$

(Xie and Barron [312]).

- 9.9 Complete the proof of Theorem 9.2 by showing that  $V_n(\mathcal{F}) \geq \frac{1}{2} \ln n + \frac{1}{2} \ln \frac{\pi}{2} + o(1)$ . *Hint:* The proof goes the same way as that of the upper bound, but to get the right constant you need to be a bit careful when  $n_1/n$  or  $n_2/n$  is small.