The **Hedge**(η)Algorithm

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Outline

Halving Algorithm

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Hedging vs. Halving

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Bound on total loss

- Upper bound on $\sum_{i=1}^{N} w_i^{T+1}$
- Lower bound on $\sum_{i=1}^{N} w_i^{T+1}$
- Combining Upper and Lower bounds

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Lower Bounds

The halving algorithm

- Our goal is to predict a binary sequence:
  $x_1, x_2, \ldots, x_t, \ldots \quad x_t \in \{0, 1\}$

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- We have N experts, at each time each expert makes a binary prediction.
- **We know that one of the experts is perfect**

Example trace for Halving Algorithm

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expert1
expert2
expert3
expert4
expert5
expert6
expert7
expert8

alg.

Example trace for Halving Algorithm

expert1
expert2
expert3
expert4
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outcome

Example trace for Halving Algorithm

	$t = 1$
expert1	1
expert2	1
expert3	0
expert4	1
expert5	1
expert6	0
expert7	1
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expert1	1
expert2	1
expert3	0
expert4	1
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expert6	0
expert7	1
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alg.	1
outcome	1

Example trace for Halving Algorithm

	$t = 1$	$t = 2$
expert1	1	1
expert2	1	0
expert3	0	-
expert4	1	0
expert5	1	0
expert6	0	-
expert7	1	1
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alg.	1	
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	$t = 1$	$t = 2$
expert1	1	1
expert2	1	0
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expert6	0	-
expert7	1	1
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Example trace for Halving Algorithm

	$t = 1$	$t = 2$	$t = 3$
expert1	1	1	1
expert2	1	0	-
expert3	0	-	-
expert4	1	0	-
expert5	1	0	-
expert6	0	-	-
expert7	1	1	1
expert8	1	1	1
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	$t = 1$	$t = 2$	$t = 3$	$t = 4$
expert1	1	1	1	1
expert2	1	0	-	-
expert3	0	-	-	-
expert4	1	0	-	-
expert5	1	0	-	-
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	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
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expert2	1	0	-	-	-
expert3	0	-	-	-	-
expert4	1	0	-	-	-
expert5	1	0	-	-	-
expert6	0	-	-	-	-
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- Each time algorithm makes a mistakes, the pool of perfect experts is halved (at least).
- We assume that at least one expert is perfect.
- Number of mistakes is at most $\log_2 N$.
- No stochastic assumptions whatsoever.
- Proof is based on combining a lower and upper bounds on the number of perfect experts.

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- **Goal:** minimize total expected loss
- Here we have stochasticity - but only in **algorithm**, not in **outcome**

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- Unlike halving - no action is perfect.
- Basic idea - reduce probability of lossy actions, but not all the way to zero.
- **Modified Goal:** minimize difference between expected total loss and minimal total loss of repeating one action.

$$\sum_{t=1}^{T} \mathbf{p}^t \cdot \ell^t - \min_i \left(\sum_{t=1}^{T} \ell_i^t \right)$$

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- Now each iteration of game consistst of three steps:
  - Experts make predictions $e_i^t \in \{0, 1\}$
  - Algorithm predicts 1 with probability $\sum_{i:e_i^t=1} p_i^t$.
  - outcome o_i^t is revealed. $\ell_i^t = 0$ if $e_i^t = o_i^t$, $\ell_i^t = 1$ otherwise.

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$$p_i^t = \frac{w_i^t}{\sum_{j=1}^{N} w_i^t}, \quad \mathbf{p}^t = \frac{\mathbf{w}^t}{\sum_{j=1}^{N} w_i^t}$$

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- As $\sum_{i=1}^{n} w_i^1 = 1$ increasing one weight implies decreasing some others.
- Plays a similar role to prior distribution in Bayesian algorithms.

Bound on the loss of **Hedge**(η)Algorithm

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- Theorem (main theorem)

  For any sequence of loss vectors $\ell^1, \ldots, \ell^T$*, and for any* $i \in \{1, \ldots, N\}$*, we have*

$$L_{\textbf{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}}.$$

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- Theorem (main theorem)
  For any sequence of loss vectors $\ell^1, \ldots, \ell^T$, and for any $i \in \{1, \ldots, N\}$, we have

$$L_{\textbf{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}}.$$

  - Proof: by combining upper and lower bounds on $\sum_{i=1}^{N} w_i^{T+1}$

Upper bound on $\sum_{i=1}^{N} w_i^{T+1}$

Lemma (upper bound)

For any sequence of loss vectors $\ell^1, \ldots, \ell^T$ *we have*

$$\ln\left(\sum_{i=1}^{N} w_i^{T+1}\right) \leq -(1-e^{-\eta})L_{\mathbf{Hedge}(\eta)}.$$

Proof of upper bound (slide 1)

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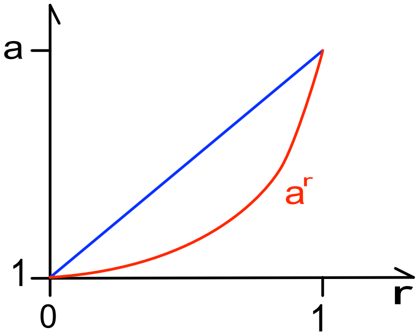

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Proof of upper bound (slide 2)

Applying $a^r \leq 1 - (1-a)^r$ where $a = e^{-\eta}, r = \ell_i^t$

$$\sum_{i=1}^{N} w_i^{t+1} = \sum_{i=1}^{N} w_i^t e^{-\eta \ell_i^t}$$

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$$\begin{aligned}\sum_{i=1}^N w_i^{t+1} &= \sum_{i=1}^N w_i^t e^{-\eta \ell_i^t} \\ &\leq \sum_{i=1}^N w_i^t \left(1 - (1 - e^{-\eta})\ell_i^t\right)\end{aligned}$$

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\sum_{i=1}^N w_i^{t+1} &= \sum_{i=1}^N w_i^t e^{-\eta \ell_i^t} \\
&\leq \sum_{i=1}^N w_i^t \left(1-(1-e^{-\eta})\ell_i^t\right) \\
&= \left(\sum_{i=1}^N w_i^t\right)\left(1-(1-e^{-\eta})\frac{\mathbf{w}^t}{\sum_{i=1}^N w_i^t}\cdot \ell^t\right)
\end{aligned}
$$

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&= \left(\sum_{i=1}^{N} w_i^t\right)\left(1 - (1 - e^{-\eta})\frac{\mathbf{w}^t}{\sum_{i=1}^{N} w_i^t} \cdot \ell^t\right) \\
&= \left(\sum_{i=1}^{N} w_i^t\right)\left(1 - (1 - e^{-\eta})\mathbf{p}^t \cdot \ell^t\right)
\end{aligned}$$

Proof of upper bound (slide 3)

- Combining

$$\sum_{i=1}^{N} w_i^{t+1} \leq \left(\sum_{i=1}^{N} w_i^{t}\right) \left(1 - (1 - e^{-\eta})\mathbf{p}^t \cdot \ell^t\right)$$

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- for $t = 1, \ldots, T$

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- for $t = 1, \ldots, T$
- yields

$$\sum_{i=1}^{N} w_i^{T+1} \leq \prod_{t=1}^{T} (1 - (1 - e^{-\eta})\mathbf{p}^t \cdot \ell^t)$$

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- for $t = 1, \ldots, T$
- yields

$$\begin{aligned} \sum_{i=1}^{N} w_i^{T+1} &\leq \prod_{t=1}^{T}\left(1 - (1 - e^{-\eta})\mathbf{p}^t \cdot \ell^t\right) \\ &\leq \exp\left(-(1 - e^{-\eta})\sum_{t=1}^{T} \mathbf{p}^t \cdot \ell^t\right) \end{aligned}$$

since $1 + x \leq e^x$ for $x = -(1 - e^{-\eta})$.

Lower bound on $\sum_{i=1}^{N} w_i^{T+1}$

For any $j = 1, \ldots, N$:

$$\sum_{i=1}^{N} w_i^{T+1} \geq w_j^{T+1} = w_j^1 e^{-\eta L_j}$$

Combining Upper and Lower bounds

- Combining bounds on $\ln\left(\sum_{i=1}^{N} w_i^{T+1}\right)$

$$\ln w_j^1 - \eta L_j \leq \ln \sum_{i=1}^{N} w_i^{T+1} \leq -(1-e^{-\eta}) \sum_{t=1}^{T} \mathbf{p}^t \cdot \ell^t$$

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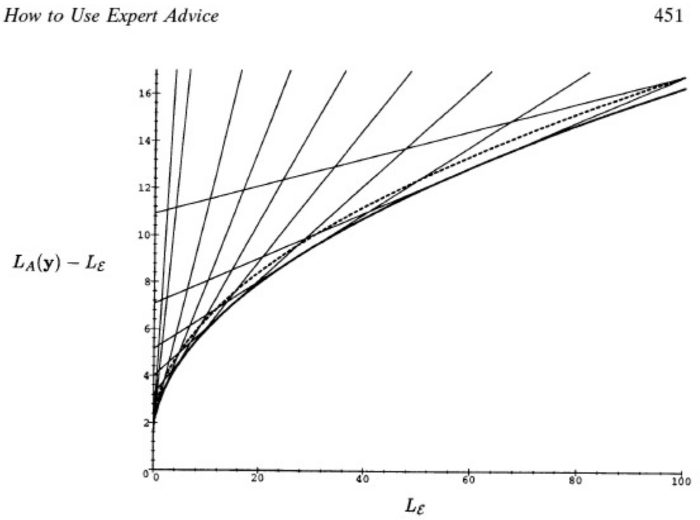
- Reversing signs, using $L_{\textbf{Hedge}(\eta)} = \sum_{t=1}^{T} \mathbf{p}^t \cdot \ell^t$ and reorganizing we get

$$L_{\textbf{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}}$$

Tuning η


How to Use Expert Advice
451
$L_A(\mathbf{y}) - L_{\mathcal{E}}$
$L_{\mathcal{E}}$
0
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- use uniform initial weights $\mathbf{w}^1 = \langle 1/N, \ldots, 1/N \rangle$

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- use uniform initial weights $\mathbf{w}^1 = \langle 1/N, \ldots, 1/N \rangle$
- Then

$$L_{\textbf{Hedge}(\eta)} \leq \frac{-\ln(w_i^1) + \eta L_i}{1 - e^{-\eta}} \leq \min_i L_i + \sqrt{2\tilde{L}\ln N} + \ln N$$

Tuning η as a function of T

- trivially $\min_i L_i \leq T$, yielding

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- per iteration we get:

$$\frac{L_{\textbf{Hedge}(\eta)}}{T} \leq \min_i \frac{L_i}{T} + \sqrt{\frac{2 \ln N}{T}} + \frac{\ln N}{T}$$

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- The adversarial strategy is random, extremely simple, and does not depend on the hedging strategy!

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- Detailed proof quite involved. See games paper.

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- A trivial random data, in which there is nothing to be learned forces any algorithm to suffer this total loss

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- next time: Using experts for estimation and control - a large set of possible projects!