Affine Diffusion Models*

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1 Affine Diffusions

A diffusion process is a Markov process solving the stochastic differential equation

$$dY_t = \mu(Y_t, \theta_0)\,dt + \sigma(Y_t, \theta_0)\,dW_t.$$

A discount-rate function $R : D \to \mathbb{R}$ is an affine function of the state

$$R(Y) = \rho_0 + \rho_1 \cdot Y,$$

for $\rho = (\rho_0, \rho_1) \in \mathbb{R} \times \mathbb{R}^N$. The affine dependence of the drift and diffusion coefficients of Y are determined by coefficients (K, H) defined by:

- $\mu(Y) = K_0 + K_1 Y$, for $K = (K_0, K_1) \in \mathbb{R}^N \times \mathbb{R}^{N\times N}$,
- $\left[\sigma(Y)\,\sigma(Y)'\right]_{ij} = [H_0]_{ij} + [H_1]_{ij} \cdot Y$, for $H = (H_0, H_1) \in \mathbb{R}^{N\times N} \times \mathbb{R}^{N\times N\times N}$.

Here

$$[H_1]_{ij} \cdot Y = \sum_{k=1}^{N} [H_1]_{ijk}\, Y_k.$$

A characteristic $\chi = (K, H, \rho)$ captures both the distribution of Y as well as the effects of any discounting.

Suppose we have "intraday" observations $Y_{t+\frac{j-1}{n}h,\,t+\frac{j}{n}h}$ for $j = 1, \ldots, n$. These observations can be used to compute realized variance (RV) that approximates true integrated variance better than just squared "daily" observations $Y_{t,t+h}^2$:

$$RV_{t,h} \equiv \frac{1}{h}\sum_{j=1}^{n} Y^2_{t+\frac{j-1}{n}h,\,t+\frac{j}{n}h} \xrightarrow{a.s.} \frac{1}{h}\int_t^{t+h} \sigma^2(Y_s, \theta_0)\,ds = \frac{1}{h}\int_t^{t+h} d\,[Y, Y]_s \equiv \mathcal{V}_{t,h}.$$

*Python implementation can be found here: https://github.com/khrapovs/diffusions

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2 Geometric Brownian Motion

2.1 The model

Suppose that S_t evolves according to

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t.$$

In logs:

$$d \log S_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t.$$

2.2 AJD representation

Since the model belongs to affine class, we have

$$K_0 = \mu - \frac{1}{2}\sigma^2, \quad K_1 = 0, \quad H_0 = \sigma^2, \quad H_1 = 0.$$

2.3 Exact solution

The solution is

$$r_{t,h} = \frac{1}{h} \log \frac{S_{t+h}}{S_t} = \left(\mu - \frac{1}{2}\sigma^2\right) + \frac{1}{h}\sigma \left(W_{t+h} - W_t\right).$$

In other words,

$$r_{t,h} \sim N\left(\mu - \frac{1}{2}\sigma^2, \sigma^2\right).$$

2.4 Moments

After integration on the interval $[t, t+H]$:

$$r_{t,H} = \frac{1}{H} \log \frac{S_{t+H}}{S_t} = \mu - \frac{1}{2}\sigma^2 + \frac{\sigma}{\sqrt{H}}\varepsilon_{t+H},$$

where $\varepsilon_t \sim N(0,1)$. The first conditional moment is

$$E_t\left[r_{t,H}\right] = \mu - \frac{1}{2}\sigma^2.$$

The second conditional moment is

$$V_t\left[r_{t,H}\right] = E_t\left[r_{t,H}^2\right] - \left(E_t\left[r_{t,H}\right]\right)^2 = \sigma^2.$$

Hence, the moment function is

$$g\left(r_{t,H};\theta\right) = \begin{bmatrix} r_{t,H} - \left(\mu - \frac{1}{2}\sigma^2\right) \\ r_{t,H}^2 - \sigma^2 - \left(\mu - \frac{1}{2}\sigma^2\right)^2 \end{bmatrix}$$

with

$$E_t\left[g\left(r_{t,H};\theta\right)\right]=0.$$

The derivative of the moment function with respect to model parameters $\theta=[\mu,\sigma]$ is

$$\frac{\partial g}{\partial\theta}\left(r_{t,H};\theta\right)=\left[\begin{array}{cc}-1 & \sigma\\ -2\left(\mu-\frac{1}{2}\sigma^2\right) & -2\sigma+2\left(\mu-\frac{1}{2}\sigma^2\right)\sigma\end{array}\right].$$

2.5 Integrated moments

It is trivial to deduce that the integrated variance is constant

$$\mathcal{V}_{t,H}=\sigma^2.$$

Hence, the moment function is instead

$$g\left(r_{t,H},RV_{t,H};\theta\right)=X_{t,H}-C=\left[\begin{array}{c}r_{t,H}\\ RV_{t,H}\\ RV_{t,H}^2\end{array}\right]-\left[\begin{array}{c}\mu-\frac{1}{2}\sigma^2\\ \sigma^2\\ \sigma^4\end{array}\right].$$

Here we used the fact that $V_t\left[\mathcal{V}_{t,H}\right]=0$ and $\left(E_t\left[\mathcal{V}_{t,H}\right]\right)^2=\sigma^4$.

3 Vasicek model

3.1 The model

Consider

$$dr_t=\kappa\left(\mu-r_t\right)dt+\sigma dW_t.$$

3.2 AJD representation

Since the model belongs to affine class, we have

$$K_0=\kappa\mu,\quad K_1=-\kappa,\quad H_0=\sigma^2,\quad H_1=0.$$

3.3 Exact solution

Using Ito's lemma for $r_te^{\kappa t}$ we have

$$d\left(r_te^{\kappa t}\right)=\left(\kappa r_te^{\kappa t}+\kappa\left(\mu-r_t\right)e^{\kappa t}\right)dt+\sigma e^{\kappa t}dW_t,$$

or

$$d\left(r_te^{\kappa t}\right)=\kappa\mu e^{\kappa t}dt+\sigma e^{\kappa t}dW_t.$$

After integration we have

$$r_{t+h}e^{\kappa(t+h)} = r_t e^{\kappa t} + \kappa\mu \int_t^{t+h} e^{\kappa u} du + \sigma \int_t^{t+h} e^{\kappa u} dW_u,$$

or

$$\begin{aligned} r_{t+h} =& r_t e^{-\kappa h} + \kappa\mu \int_t^{t+h} e^{-\kappa(t+h-u)} du + \sigma \int_t^{t+h} e^{-\kappa(t+h-u)} dW_u \\ =& r_t e^{-\kappa h} + \mu\left(1 - e^{-\kappa h}\right) + \sigma \int_t^{t+h} e^{-\kappa(t+h-u)} dW_u. \end{aligned}$$

This implies that

$$r_{t+h}|r_t \sim N\left(r_t e^{-\kappa h} + \mu\left(1 - e^{-\kappa h}\right), \sigma^2 \int_t^{t+h} e^{-2\kappa(t+h-u)} du = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa h}\right)\right).$$

4 Cox-Ingersoll-Ross model

4.1 The model

Cox, Ingersoll, and Ross (1985; Ecta) developed a theory of the term structure of interest rates in which the instantaneous short-term rate of interest, r, follows the mean reverting diffusion

$$dr_t = \kappa(\mu - r_t)\, dt + \sigma\sqrt{r_t} dW_t.$$

Feller condition for positivity of the process is $\kappa\mu > \frac{1}{2}\sigma^2$.

4.2 AJD representation

Since the model belongs to affine class, we have

$$K_0 = \kappa\mu, \quad K_1 = -\kappa, \quad H_0 = 0, \quad H_1 = \sigma^2.$$

4.3 The solution

Using Ito's lemma for $r_t e^{\kappa t}$ we have

$$r_{t+h} = r_t e^{-\kappa h} + \mu\left(1 - e^{-\kappa h}\right) + \sigma \int_t^{t+h} e^{-\kappa(t+h-u)} \sqrt{r_u} dW_u.$$

It can be shown that the transition density is known in closed form:

$$f\left(r_{t+h}\middle| r_t; \theta\right) = ce^{-u-v}\left(\frac{u}{v}\right)^{q/2} I_q\left(2\sqrt{uv}\right),$$

where

$$c = \frac{2\kappa}{\sigma^2\left(1 - e^{-\kappa h}\right)}, \quad u = cr_t e^{-\kappa h}, \quad v = cr_{t+h}, \quad q = \frac{2\kappa\mu}{\sigma^2} - 1,$$

and I_q is the modified Bessel function of the first kind of order q. Sometimes, it is useful to work with a transformation $x_{t+h} = 2cr_{t+h}$. We can derive that the transition density of x_{t+h} is

$$f\left(x_{t+h}\middle|\, x_t;\theta\right) = f\left(2cr_{t+h}\middle|\, 2cr_t;\theta\right) = \frac{1}{2c} f\left(r_{t+h}\middle|\, r_t;\theta\right),$$

which is a noncentral χ^2 with $2q+2$ degrees of freedom and noncentrality parameter $2u$.

4.4 Moments

This implies that

$$\begin{aligned}
E_t\left[r_{t+h}\right] &= r_t e^{-\kappa h} + \mu\left(1 - e^{-\kappa h}\right), \\
V_t\left[r_{t+h}\right] &= \frac{\sigma^2}{\kappa} r_t e^{-\kappa h}\left(1 - e^{-\kappa h}\right) + \frac{\sigma^2}{2\kappa}\mu\left(1 - e^{-\kappa h}\right)^2,
\end{aligned}$$

since

$$\begin{aligned}
V_t\left[r_{t+h}\right] =& \sigma^2 \int_t^{t+h} e^{-2\kappa(t+h-u)} E_t\left[r_u\right] du \\
=& \sigma^2 \int_t^{t+h} e^{-2\kappa(t+h-u)} \left(r_t e^{-\kappa(u-t)} + \mu\left(1 - e^{-\kappa(u-t)}\right)\right) du.
\end{aligned}$$

5 Heston model

5.1 The model

Consider

$$\begin{aligned}
dp_t &= \left(r + \left(\lambda_r - \frac{1}{2}\right)\sigma_t^2\right) dt + \sigma_t dW_t^r, \\
d\sigma_t^2 &= \kappa\left(\mu - \sigma_t^2\right) dt + \eta\sigma_t dW_t^\sigma,
\end{aligned}$$

with $p_t = \log S_t$, and $Corr\left[dW_s^r, dW_s^\sigma\right] = \rho$, or in other words $W_t^\sigma = \rho W_t^r + \sqrt{1-\rho^2} W_t^v$. Also let $R\left(Y_t\right) = r$. Feller condition is

$$2\kappa\mu > \eta^2.$$

5.2 Risk-neutral model

Let the log stochastic discount factor (SDF) process $m_t = \log M_t$ be represented by the following SDE:

$$dm_t = -r dt - \zeta_r dW_t^r - \zeta_v dW_t^v,$$

with

$$\left[\begin{array}{c} \zeta_r \\ \zeta_v \end{array}\right] = \left[\begin{array}{c} \lambda_r \sigma_t \\ -\frac{1}{\sqrt{1-\rho^2}}\left(\rho\lambda_r + \lambda_\sigma\right)\sigma_t \end{array}\right].$$

The risk-neutral innovations are then

$$
\begin{aligned}
d\tilde{W}_t^r &= dW_t^r + \zeta_r dt, \\
d\tilde{W}_t^v &= dW_t^v + \zeta_v dt.
\end{aligned}
$$

After the substitution

$$
\begin{aligned}
dp_t &= \left(r - \frac{1}{2}\sigma_t^2\right) dt + \sigma_t d\tilde{W}_t^r, \\
d\sigma_t^2 &= \tilde{\kappa}\left(\tilde{\mu} - \sigma_t^2\right) dt + \eta\sigma_t d\tilde{W}_t^\sigma,
\end{aligned}
$$

with $\tilde{W}_t^\sigma = \rho\tilde{W}_t^r + \sqrt{1-\rho^2}\tilde{W}_t^v$, and the modified parameters are

$$
\tilde{\kappa} = \kappa - \lambda_\sigma\eta, \quad \tilde{\mu} = \mu\frac{\kappa}{\tilde{\kappa}}.
$$

Feller condition $2\tilde{\kappa}\tilde{\mu} > \eta^2$ is equivalent to the one for physical measure, $2\mu\kappa > \eta^2$. Still, we need $\tilde{\kappa} > 0$ which is equivalent to $\lambda_\sigma < \kappa/\eta$.

5.3 AJD representation

The drift and diffusion functions are

$$
\mu\left(Y_t, \theta_0\right) = \left[\begin{array}{c} r \\ \kappa\mu \end{array}\right] + \left[\begin{array}{cc} 0 & \lambda_r - \frac{1}{2} \\ 0 & -\kappa \end{array}\right]\left[\begin{array}{c} p_t \\ \sigma_t^2 \end{array}\right], \quad \sigma\left(Y_t, \theta_0\right) = \left[\begin{array}{cc} \sigma_t & 0 \\ \eta\rho\sigma_t & \eta\sqrt{1-\rho^2}\sigma_t \end{array}\right].
$$

It follows that

$$
\sigma\left(Y_t, \theta_0\right)\sigma\left(Y_t, \theta_0\right)' = \left[\begin{array}{cc} \sigma_t^2 & \eta\rho\sigma_t^2 \\ \eta\rho\sigma_t^2 & \eta^2\sigma_t^2 \end{array}\right] = \left[\begin{array}{cc} 1 & \eta\rho \\ \eta\rho & \eta^2 \end{array}\right]\sigma_t^2.
$$

So we have

$$
\rho_0 = r, \quad \rho_1 = [0, 0],
$$

and

$$
K_0 = \left[\begin{array}{c} r \\ \kappa\mu \end{array}\right], \quad K_1 = \left[\begin{array}{cc} 0 & \lambda_r - \frac{1}{2} \\ 0 & -\kappa \end{array}\right],
$$

and

$$
H_0 = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right], \quad H_{1,1} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right], \quad H_{1,2} = \left[\begin{array}{cc} 1 & \eta\rho \\ \eta\rho & \eta^2 \end{array}\right].
$$

5.4 Integrated moments

Define average excess return as

$$
\tilde{r}_{t,H} \equiv \frac{1}{H}\int_t^{t+H} dp_s - r.
$$

Given observations on average excess returns $\tilde{r}_{t,H}$ and realized volatility $RV_{t,H}$ we have the following four integrated moments (the proof is given in section §A):

$$\begin{aligned} E_t\left[\left(1-A_hL\right)\mathcal{V}_{t+h,H}\right] &= C_1, \\ E_t\left[\left(1-A_hL\right)\left(1-A_h^2L\right)\mathcal{V}_{t+2h,H}^2\right] &= C_2, \\ E_t\left[\tilde{r}_{t,H}-\left(\lambda_r-\frac{1}{2}\right)\mathcal{V}_{t,H}\right] &= C_3 = 0, \\ E_t\left[\left(1-A_hL\right)\left(\tilde{r}_{t+h,H}\mathcal{V}_{t+h,H}-\left(\lambda-\frac{1}{2}\right)\mathcal{V}_{t+h,H}^2\right)\right] &= C_4, \end{aligned}$$

or in matrix notation

$$E_{t-2h}\left[\begin{pmatrix} \left(1-A_hL\right)L & \cdot & \cdot & \cdot \\ \cdot & \left(1-A_hL\right)\left(1-A_h^2L\right) & \cdot & \cdot \\ \left(\frac{1}{2}-\lambda_r\right)L^2 & \cdot & L^2 & \cdot \\ \cdot & \left(\frac{1}{2}-\lambda_r\right)\left(1-A_hL\right)L & \cdot & \left(1-A_hL\right)L \end{pmatrix}\begin{pmatrix} \mathcal{V}_{t,H} \\ \mathcal{V}_{t,H}^2 \\ \tilde{r}_{t,H} \\ \tilde{r}_{t,H}\mathcal{V}_{t,H} \end{pmatrix}\right] = C.$$

Define the data vector as $X_{t,H}=\left[\mathcal{V}_{t,H},\mathcal{V}_{t,H}^2,\tilde{r}_{t,H},\tilde{r}_{t,H}\mathcal{V}_{t,H}\right]$. Then, the moment conditions can be written more compactly as

$$E_{t-2h}\left[A\left(L\right)X_{t,H}\right] = C,$$

with the lag polynomial

$$A\left(L\right) = A_0 + A_1L + A_2L^2,$$

and

$$A_0 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & -A_h\left(1+A_h\right) & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2}-\lambda_r & \cdot & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -A_h & \cdot & \cdot & \cdot \\ \cdot & A_h^3 & \cdot & \cdot \\ \frac{1}{2}-\lambda_r & \cdot & 1 & \cdot \\ \cdot & \left(\lambda-\frac{1}{2}\right)A_h & \cdot & -A_h \end{pmatrix}.$$

To make this one single matrix product, we can write

$$E_{t-2h}\left[\begin{pmatrix} A_0 & A_1 & A_2 \end{pmatrix}\begin{pmatrix} X_{t,H} \\ LX_{t,H} \\ L^2X_{t,H} \end{pmatrix}\right] = C,$$

where

$$C = \left(A_0 + A_1 + A_2\right)E\left[X_{t,H}\right],$$

and

$$
\begin{aligned}
E\left[\mathcal{V}_{t,H}\right] &= \mu, \\
E\left[\mathcal{V}_{t,H}^2\right] &= \frac{\eta^2}{\kappa^2}\frac{c_H}{H} + \mu^2, \\
E\left[\tilde{r}_{t,H}\right] &= \left(\lambda_r - \frac{1}{2}\right)\mu, \\
E\left[\tilde{r}_{t,H}\mathcal{V}_{t,H}\right] &= \rho\frac{\eta}{\kappa}\frac{c_H}{H} + \left(\lambda_r - \frac{1}{2}\right)E\left[\mathcal{V}_{t,H}^2\right].
\end{aligned}
$$

6 Central Tendency

6.1 The model

Consider

$$
\begin{aligned}
dp_t &= \left(r + \left(\lambda_r - \frac{1}{2}\right)\sigma_t^2\right)dt + \sigma_t dW_t^r, \\
d\sigma_t^2 &= \kappa_\sigma\left(y_t^2 - \sigma_t^2\right)dt + \eta_\sigma\sigma_t dW_t^\sigma, \\
dy_t^2 &= \kappa_y\left(\mu - y_t^2\right)dt + \eta_y v_t dW_t^y,
\end{aligned}
$$

with $p_t = \log S_t$, and $Corr\left[dW_s^r, dW_s^\sigma\right] = \rho$, or in other words $W_t^\sigma = \rho W_t^r + \sqrt{1-\rho^2}W_t^y$. Also let $R\left(Y_t\right) = r$.

6.2 Risk-neutral model

Let the log stochastic discount factor (SDF) process $m_t = \log M_t$ be represented by the following SDE:

$$
dm_t = -rdt - \zeta_r dW_t^r - \zeta_v dW_t^v - \zeta_y dW_t^y.
$$

with

$$
\begin{bmatrix} \zeta_r \\ \zeta_v \\ \zeta_y \end{bmatrix} = \begin{bmatrix} \lambda_r\sigma_t \\ -\frac{1}{\sqrt{1-\rho^2}}\left(\rho\lambda_r + \lambda_\sigma\right)\sigma_t \\ -\lambda_y y_t \end{bmatrix}.
$$

The risk-neutral innovations are then

$$
\begin{aligned}
d\tilde{W}_t^r &= dW_t^r + \zeta_r dt, \\
d\tilde{W}_t^v &= dW_t^v + \zeta_v dt, \\
d\tilde{W}_t^y &= dW_t^y + \zeta_y dt.
\end{aligned}
$$

After the substitution

$$\begin{aligned} dp_t &= \left(r - \frac{1}{2}\sigma_t^2\right) dt + \sigma_t d\tilde{W}_t^r, \\ d\sigma_t^2 &= \tilde{\kappa}_\sigma \left(\frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} y_t^2 - \sigma_t^2\right) dt + \eta_\sigma \sigma_t d\tilde{W}_t^\sigma, \\ dy_t^2 &= \tilde{\kappa}_y \left(\tilde{\mu} - y_t^2\right) dt + \eta_y y_t d\tilde{W}_t^y, \end{aligned}$$

with $\tilde{W}_t^\sigma = \rho \tilde{W}_t^r + \sqrt{1-\rho^2}\tilde{W}_t^v$, and the modified parameters are

$$\tilde{\kappa}_\sigma = \kappa_\sigma - \lambda_\sigma \eta_\sigma, \quad \tilde{\kappa}_y = \kappa_y - \lambda_y \eta_y, \quad \tilde{\mu} = \mu \frac{\kappa_y}{\tilde{\kappa}_y}.$$

Feller condition $2\tilde{\kappa}\tilde{\mu} > \eta^2$ is equivalent to the one for physical measure, $2\mu\kappa > \eta^2$. Still, we need $\tilde{\kappa} > 0$ which is equivalent to $\lambda_\sigma < \kappa/\eta$.

Note that the unconditional means of the processes at hand under the physical measure are the same:

$$E^P\left[\sigma_t^2\right] = E^P\left[y_t^2\right] = \mu,$$

while under the risk-neutral measure, they are different:

$$E^Q\left[y_t^2\right] = \tilde{\mu} = \mu \frac{\kappa_y}{\kappa_y - \lambda_y \eta_y} > \mu,$$

and

$$E^Q\left[\sigma_t^2\right] = E^Q\left[y_t^2\right] \frac{\kappa_\sigma}{\tilde{\kappa}_\sigma} = \mu \frac{\kappa_y}{\kappa_y - \lambda_y \eta_y} \frac{\kappa_\sigma}{\kappa_\sigma - \lambda_\sigma \eta_\sigma} > \tilde{\mu}.$$

Given that all of the parameters above are non-negative, we can write the following inequality:

$$E^P\left[\sigma_t^2\right] = E^P\left[y_t^2\right] \leq E^Q\left[y_t^2\right] \leq E^Q\left[\sigma_t^2\right].$$

6.3 AJD representation

The drift and diffusion functions are

$$\mu\left(Y_t, \theta_0\right) = \begin{bmatrix} r \\ 0 \\ \kappa_y \mu_y \end{bmatrix} + \begin{bmatrix} 0 & \lambda_r - \frac{1}{2} & 0 \\ 0 & -\kappa_\sigma & \kappa_\sigma \\ 0 & 0 & -\kappa_y \end{bmatrix} \begin{bmatrix} p_t \\ \sigma_t^2 \\ y_t^2 \end{bmatrix}, \quad \sigma\left(Y_t, \theta_0\right) = \begin{bmatrix} \sigma_t & 0 & 0 \\ \eta_\sigma \rho \sigma_t & \eta_\sigma \sqrt{1-\rho^2}\sigma_t & 0 \\ 0 & 0 & \eta_y y_t \end{bmatrix}.$$

It follows that

$$\sigma\left(Y_t, \theta_0\right) \sigma\left(Y_t, \theta_0\right)' = \begin{bmatrix} \sigma_t^2 & \eta_\sigma \rho \sigma_t^2 & 0 \\ \eta_\sigma \rho \sigma_t^2 & \eta_\sigma^2 \sigma_t^2 & 0 \\ 0 & 0 & \eta_y^2 y_t^2 \end{bmatrix} = \begin{bmatrix} 1 & \eta_\sigma \rho & 0 \\ \eta_\sigma \rho & \eta_\sigma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sigma_t^2 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \eta_y^2 \end{bmatrix} y_t^2.$$

So we have

$$\rho_0 = r, \quad \rho_1 = [0,0,0]\,,$$

and

$$K_0 = \begin{bmatrix} r \\ 0 \\ \kappa_y \mu_y \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0 & \lambda_r - \frac{1}{2} & 0 \\ 0 & -\kappa_\sigma & \kappa_\sigma \\ 0 & 0 & -\kappa_y \end{bmatrix},$$

and

$$H_0 = [\mathbf{0}_{3\times 3}]\,, \quad H_{1,1} = [\mathbf{0}_{3\times 3}]\,, \quad H_{1,2} = \begin{bmatrix} 1 & \eta_\sigma \rho & 0 \\ \eta_\sigma \rho & \eta_\sigma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H_{1,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \eta_y^2 \end{bmatrix}.$$

6.4 Integrated moments

Define average excess return as

$$\tilde{r}_{t,H} \equiv \frac{1}{H} \int_t^{t+H} dp_s - r.$$

Given observations on average excess returns $\tilde{r}_{t,H}$ and realized volatility $RV_{t,H}$ we have the following four integrated moments (the proof is given in section §B):

$$\begin{aligned}
E_t\left[\left(1 - A_h^y L\right)\left(1 - A_h^\sigma L\right)\mathcal{V}_{t+2h,H}\right] &= C_1, \\
E_t\left[\left(1 - A_{2h}^\sigma L\right)\left(1 - A_{2h}^y L\right)\left(1 - A_h^\sigma A_h^y L\right)\left(1 - A_h^y L\right)\left(1 - A_h^\sigma L\right)\mathcal{V}_{t+5h,H}^2\right] &= C_2, \\
E_t\left[\tilde{r}_{t,H} - \left(\lambda_r - \frac{1}{2}\right)\mathcal{V}_{t,H}\right] &= C_3 = 0, \\
E_t\left[\left(1 - A_h^y L\right)\left(1 - A_h^\sigma L\right)\left(\tilde{r}_{t+h,H}\mathcal{V}_{t+h,H} - \left(\lambda - \frac{1}{2}\right)\mathcal{V}_{t+h,H}^2\right)\right] &= C_4,
\end{aligned}$$

or in matrix notation

$$E_{t-5h}\left[\begin{pmatrix} M_1 L^3 & \cdot & \cdot & \cdot \\ \cdot & M_2 & \cdot & \cdot \\ \left(\frac{1}{2} - \lambda_r\right) L^5 & \cdot & L^5 & \cdot \\ \cdot & \left(\frac{1}{2} - \lambda_r\right) M_1 L^3 & \cdot & M_1 L^3 \end{pmatrix}\begin{pmatrix} \mathcal{V}_{t,H} \\ \mathcal{V}_{t,H}^2 \\ \tilde{r}_{t,H} \\ \tilde{r}_{t,H}\mathcal{V}_{t,H} \end{pmatrix}\right] = C,$$

where

$$\begin{aligned}
M_1 &= \left(1 - A_h^y L\right)\left(1 - A_h^\sigma L\right), \\
M_2 &= M_1\left(1 - A_h^\sigma A_h^y L\right)\left(1 - A_{2h}^y L\right)\left(1 - A_{2h}^\sigma L\right),
\end{aligned}$$

Define the data vector as $X_{t,H} = \left[\mathcal{V}_{t,H}, \mathcal{V}_{t,H}^2, \tilde{r}_{t,H}, \tilde{r}_{t,H}\mathcal{V}_{t,H}\right]$. Then, the moment conditions can be written more compactly as

$$E_{t-5h}\left[A\left(L\right)X_{t,H}\right] = C,$$

with the lag polynomial

$$A(L) = A_0 + A_1 L + A_2 L^2 + A_3 L^3 + A_4 L^4 + A_5 L^5,$$

and

$$A_0 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & m_{2,0} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad A_1 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & m_{2,1} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad A_2 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & m_{2,2} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$A_3 = \begin{pmatrix} m_{1,0} & \cdot & \cdot & \cdot \\ \cdot & m_{2,3} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} - \lambda_r & \cdot & m_{1,0} \end{pmatrix}, \quad A_4 = \begin{pmatrix} m_{1,1} & \cdot & \cdot & \cdot \\ \cdot & m_{2,4} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \left(\frac{1}{2} - \lambda_r\right) m_{1,1} & \cdot & m_{1,1} \end{pmatrix},$$

$$A_5 = \begin{pmatrix} m_{1,2} & \cdot & \cdot & \cdot \\ \cdot & m_{2,5} & \cdot & \cdot \\ \frac{1}{2} - \lambda_r & \cdot & 1 & \cdot \\ \cdot & \left(\frac{1}{2} - \lambda_r\right) m_{1,2} & \cdot & m_{1,2} \end{pmatrix}.$$

Here

$$m_{1,0} = 1, \quad m_{1,1} = -\left(A_h^y + A_h^\sigma\right), \quad m_{1,2} = A_h^y A_h^\sigma,$$

and for the second polynomial the coefficients are found accordingly to the relation between polynomial roots and coefficients[1]. To make this one single matrix product, we can write

$$E_{t-2h}\left[\begin{pmatrix} A_0 & A_1 & A_2 & A_3 & A_4 & A_5 \end{pmatrix} \begin{pmatrix} X_{t,H} \\ LX_{t,H} \\ L^2 X_{t,H} \\ L^3 X_{t,H} \\ L^4 X_{t,H} \\ L^5 X_{t,H} \end{pmatrix}\right] = C,$$

where

$$C = (A_0 + A_1 + A_2 + A_3 + A_4 + A_5)\, E\left[X_{t,H}\right],$$

[1] See e.g.: http://math.stackexchange.com/questions/88917/relation-betwen-coefficients-and-roots-of-a-polynomial

and

$$\begin{aligned}
E\left[\mathcal{V}_{t,H}\right] &= \mu, \\
E\left[\mathcal{V}_{t,H}^2\right] &= \mu^2 + \left(a_H^\sigma\right)^2 V\left[\sigma_t^2\right] + \left(b_H^\sigma\right)^2 V\left[y_t^2\right] + V\left[\frac{1}{H}\int_0^H \epsilon_{t,s}^\sigma ds\right], \\
E\left[\tilde{r}_{t,H}\right] &= \left(\lambda_r - \frac{1}{2}\right)\mu, \\
E\left[\tilde{r}_{t,H}\mathcal{V}_{t,H}\right] &= \rho\frac{\mu\eta_\sigma}{\kappa_\sigma}\frac{1-a_H^\sigma}{H} + \left(\lambda_r - \frac{1}{2}\right)E\left[\mathcal{V}_{t,H}^2\right].
\end{aligned}$$

Appendix

A Integrated moments for Heston model

The spot volatility equation can be integrated on the interval $[t, t+h]$ with h possibly different from H to get

$$\sigma_{t+h}^2 = C_h + A_h \sigma_t^2 + \epsilon_{t,h}^\sigma, \tag{A.1}$$

where I define coefficients and the integrated innovation as

$$A_h = \exp\left(-\kappa h\right), \quad C_h = \mu\left(1 - A_h\right), \quad \epsilon_{t,h}^\sigma = \eta_\sigma \int_t^{t+h} \sigma_u A_{t+h-u} dW_u^\sigma.$$

The spot volatility first moment is

$$E_t\left[\sigma_{t+h}^2\right] = C_h + A_h \sigma_t^2,$$

or, using lag operator L,

$$E_t\left[\left(1 - A_h L\right)\sigma_{t+h}^2\right] = \left(1 - A_h\right) E\left[\sigma_t^2\right] = \left(1 - A_h\right)\mu.$$

Given the above definitions it is also not hard to guess that σ_t^4 is autoregressive:

$$E_t\left[\left(1 - A_h L\right)\left(1 - A_h^2 L\right)\sigma_{t+2h}^4\right] = C_1 = \left(1 - A_h\right)\left(1 - A_h^2\right) E\left[\sigma_t^4\right]$$

with

$$E\left[\sigma_t^4\right] = V\left[\sigma_t^2\right] + \left(E\left[\sigma_t^2\right]\right)^2 = \frac{\mu\eta^2}{2\kappa} + \mu^2.$$

After repeated integration of the volatility equation (A.1) over t as a dummy of the integration on the interval $[t, t+H]$ we obtain

$$\mathcal{V}_{t+h,H} = C_h + A_h \mathcal{V}_{t,H} + \frac{1}{H}\int_0^H \epsilon_{t+s,h}^\sigma ds.$$

Hence, the integrated volatility first moment is

$$E_t\left[\left(1 - A_h L\right)\mathcal{V}_{t+h,H}\right] = \left(1 - A_h\right) E\left[\mathcal{V}_{t,H}\right] = \left(1 - A_h\right)\mu.$$

We can also integrate the volatility equation (A.1) using h as a dummy of integration on the interval from $[0, H]$:

$$\mathcal{V}_{t,H} = c_H + a_H \sigma_t^2 + \frac{1}{H}\int_0^H \epsilon_{t,s}^\sigma ds, \tag{A.2}$$

with

$$a_H = \frac{1}{H}\int_0^H A_s ds = \frac{1}{\kappa H}\left(1 - A_H\right), \quad c_H = \mu\left(1 - a_H\right).$$

The error term may be simplified:

$$
\begin{aligned}
\frac{1}{H}\int_0^H \epsilon^\sigma_{t,s} ds =& \frac{1}{H}\int_0^H \left(\eta \int_t^{t+s} \sigma_u A_{t+s-u} dW^\sigma_u\right) ds \\
=& \frac{\eta}{H}\int_t^{t+H} \sigma_u \left(\int_{u-t}^{H} A_{t+s-u} ds\right) dW^\sigma_u \\
=& \frac{\eta}{H}\int_t^{t+H} \sigma_u \left(\int_0^{t+H-u} A_s ds\right) dW^\sigma_u \\
=& \eta \int_t^{t+H} \sigma_u a_{t+H-u} dW^\sigma_u .
\end{aligned}
$$

From this we can guess that

$$
E_t\left[(1-A_h L)\left(1-A_h^2 L\right)\mathcal{V}^2_{t+2h,H}\right] = C_2 = (1-A_h)\left(1-A_h^2\right)E\left[\mathcal{V}^2_{t,H}\right].
$$

From the equation (A.2) for $\mathcal{V}_{t,H}$ we can derive

$$
\begin{aligned}
V\left[\mathcal{V}_{t,H}\right] =& a_H^2 V\left[\sigma_t^2\right] + V\left[\frac{1}{H}\int_0^H \epsilon^\sigma_{t,s} ds\right] = a_H^2 \frac{\mu\eta^2}{2\kappa} + V\left[\eta\int_t^{t+H}\sigma_u a_{t+H-u} dW^\sigma_u\right] \\
=& a_H^2 \frac{\mu\eta^2}{2\kappa} + E\left[\eta^2 \int_t^{t+H} \sigma_u^2 a_{t+H-u}^2 du\right] = a_H^2 \frac{\mu\eta^2}{2\kappa} + \mu\eta^2 \int_0^H a_{H-u}^2 du \\
=& a_H^2 \frac{\mu\eta^2}{2\kappa} + \frac{\mu\eta^2}{\kappa^2 H^2}\int_0^H \left(1-A_{H-u}\right)^2 du \\
=& \left(1-A_H\right)^2 \frac{\mu\eta^2}{2\kappa^3 H^2} + \frac{\mu\eta^2}{\kappa^2 H^2}\left(H - \frac{2}{\kappa}\left(1-A_H\right) + \frac{1}{2\kappa}\left(1-A_{2H}\right)\right) \\
=& \frac{\mu\eta^2}{\kappa^2 H^2}\left[\left(1-2A_H+A_{2H}\right)\frac{1}{2\kappa} + H - \frac{2}{\kappa}\left(1-A_H\right) + \frac{1}{2\kappa}\left(1-A_{2H}\right)\right] \\
=& \frac{\mu\eta^2}{\kappa^2 H^2}\left[\left(1-A_H\right)\frac{1}{\kappa} + H - \frac{2}{\kappa}\left(1-A_H\right)\right] \\
=& \frac{\mu\eta^2}{\kappa^2 H^2}\left[H - \frac{1}{\kappa}\left(1-A_H\right)\right] = \frac{\mu\eta^2}{\kappa^2 H}\left(1-a_H\right) = \frac{\eta^2}{\kappa^2}\frac{c_H}{H}.
\end{aligned}
$$

Hence,

$$
E\left[\mathcal{V}^2_{t,H}\right] = V\left[\mathcal{V}_{t,H}\right] + \left(E\left[\mathcal{V}_{t,H}\right]\right)^2 = \frac{\eta^2}{\kappa^2}\frac{c_H}{H} + \mu^2.
$$

The return equation can be integrated on $[t, t+H]$ to obtain

$$
\tilde{r}_{t,h} \equiv \frac{1}{H}\int_t^{t+H} dp_s - r = \left(\lambda - \frac{1}{2}\right)\mathcal{V}_{t,H} + \frac{1}{H}\int_t^{t+H}\sigma_s dW^r_s,
$$

The return first conditional moment is

$$
E_t\left[\tilde{r}_{t,H}\right] = \left(\lambda - \frac{1}{2}\right)E_t\left[\mathcal{V}_{t,H}\right],
$$

or\frac{\kappa_{\sigma}}{\kappa_{\sigma}-\kappa_{y}}

$$E_t\left[\tilde{r}_{t,H}-\left(\lambda-\frac{1}{2}\right)\mathcal{V}_{t,H}\right]=C_3=0.$$

Now multiply return by integrated volatility and take the expectation:

$$E_t\left[\tilde{r}_{t,H}\mathcal{V}_{t,H}-\left(\lambda-\frac{1}{2}\right)\mathcal{V}_{t,H}^2\right]=E_t\left[\mathcal{V}_{t,H}\frac{1}{H}\int_t^{t+H}\sigma_s dW_s^r\right].$$

We can guess that

$$E_t\left[\left(1-A_hL\right)\left(\tilde{r}_{t,H}\mathcal{V}_{t,H}-\left(\lambda-\frac{1}{2}\right)\mathcal{V}_{t,H}^2\right)\right]=\left(1-A_h\right)E\left[\mathcal{V}_{t,H}\frac{1}{H}\int_t^{t+H}\sigma_s dW_s^r\right].$$

The last term is

$$\begin{aligned}
E\left[\mathcal{V}_{t,H}\frac{1}{H}\int_t^{t+H}\sigma_s dW_s^r\right]=&E\left[\frac{1}{H^2}\int_0^H\epsilon_{t,s}^\sigma ds\int_t^{t+H}\sigma_s dW_s^r\right]\\
=&\frac{\eta}{H}E\left[\left(\int_t^{t+H}\sigma_u a_{t+H-u}dW_u^\sigma\right)\left(\int_t^{t+H}\sigma_s dW_s^r\right)\right]\\
=&\rho\frac{\eta}{H}E\left[\int_t^{t+H}\sigma_u^2 a_{t+H-u}du\right]=\rho\frac{\eta}{H}\int_t^{t+H}E\left[\sigma_u^2\right]a_{t+H-u}du.\\
=&\rho\frac{\mu\eta}{H}\int_0^H a_{H-u}du=\rho\frac{\mu\eta}{H^2}\int_0^H\frac{1}{\kappa}\left(1-A_{H-u}\right)du\\
=&\rho\frac{\mu\eta}{H^2}\left(H-\frac{1}{\kappa}\left(1-A_H\right)\right)=\rho\frac{\mu\eta}{\kappa H}\left(1-a_H\right)=\rho\frac{\eta}{\kappa}\frac{c_H}{H}.
\end{aligned}$$

B Integrated moments for Central Tendency model

The volaitlity processes can be discretized to obtain

$$\begin{aligned}
\sigma_{t+h}^2=&\quad A_h^\sigma\sigma_t^2+B_h^\sigma y_t^2+C_h^\sigma+\epsilon_{t,h}^\sigma,\\
y_{t+h}^2=&\qquad\qquad A_h^y y_t^2+C_h^y+\epsilon_{t,h}^y.
\end{aligned}\tag{B.1}$$

with

$$A_h^\sigma=\exp\left(-\kappa_\sigma h\right),\quad B_h^\sigma=\frac{\kappa_\sigma}{\kappa_\sigma-\kappa_y}\left(A_h^y-A_h^\sigma\right),\quad C_h^\sigma=\mu\left(1-A_h^\sigma-B_h^\sigma\right),$$

and

$$A_h^y=\exp\left(-\kappa_y h\right),\quad C_h^y=\mu\left(1-A_h^y\right).$$

The same system can be written using lag operators as

$$\begin{aligned}
\left(1-A_h^\sigma L\right)\sigma_{t+h}^2=&\quad B_h^\sigma y_t^2+C_h^\sigma+\epsilon_{t,h}^\sigma,\\
\left(1-A_h^y L\right)y_{t+h}^2=&\qquad\qquad C_h^y+\epsilon_{t,h}^y.
\end{aligned}$$

Note that A_h^y and A_h^σ are multiplicative functions of time interval, that is $A_{h_1}^y A_{h_2}^y = A_{h_1+h_2}^y$.The error structure is represented by

$$\begin{aligned}
\epsilon_{t,h}^\sigma &= \eta_\sigma \int_t^{t+h} \sigma_u A_{t+h-u}^\sigma dW_u^\sigma + \eta_y \int_t^{t+h} y_u B_{t+h-u}^\sigma dW_u^y, \\
\epsilon_{t,h}^v &= \eta_y \int_t^{t+h} y_u A_{t+h-u}^y dW_u^y.
\end{aligned}$$

Clearly, $E_t^P\left[\epsilon_{t,t+h}^\sigma\right] = 0$, and $E_t^P\left[\epsilon_{t,t+h}^y\right] = 0$. Note that the same processes may be represented as infinite stochastic integrals with respect to Brownian motion only:

$$\begin{aligned}
y_t^2 &= \mu + \eta_y \int_{-\infty}^t y_u A_{t-u}^y dW_u^y, \\
\sigma_t^2 &= \mu + \eta_y \int_{-\infty}^t y_u B_{t-u}^\sigma dW_u^y + \eta_\sigma \int_{-\infty}^t \sigma_u A_{t-u}^\sigma dW_u^\sigma.
\end{aligned} \tag{B.2}$$

Next I define integrated variance and central tendency as

$$\mathcal{V}_{t,H} \equiv \frac{1}{H}\int_t^{t+H} \sigma_u^2 du, \quad \mathcal{Y}_{t,H} \equiv \frac{1}{H}\int_t^{t+H} y_u^2 du, \tag{B.3}$$

where the first subscripted value denotes the beginning of the time interval, and the second denotes the length of this interval.

In order to move from instantaneous vector (σ_t^2, y_t) to integrated analog $(\mathcal{V}_{t,H}, \mathcal{Y}_{t,H})$, I integrate the linear system in equation (B.1) over t as a dummy of the integration in the interval $[0,H]$ with the following result

$$\begin{aligned}
\mathcal{V}_{t+h,H} &= A_h^\sigma \mathcal{V}_{t,H} + B_h^\sigma \mathcal{Y}_{t,H} + C_h^\sigma + \frac{1}{H}\int_0^H \epsilon_{t+s,h}^\sigma ds, \\
\mathcal{Y}_{t+h,H} &= A_h^y \mathcal{Y}_{t,H} + C_h^y + \frac{1}{H}\int_0^H \epsilon_{t+s,h}^y ds.
\end{aligned} \tag{B.4}$$

Using the lag operator L and taking the conditional expectation, this system may be written as

$$\begin{aligned}
E_t^P\left[(1 - A_h^\sigma L)\, \mathcal{V}_{t+h,H}\right] &= B_h^\sigma E_t^P\left[\mathcal{Y}_{t,H}\right] + C_h^\sigma, \\
E_t^P\left[(1 - A_h^y L)\, \mathcal{Y}_{t+h,H}\right] &= C_h^y.
\end{aligned}$$

Multiply the first equation by $(1 - A_h^y L)$, shift the time by h, and make a substitution from the second equation to obtain

$$E_t^P\left[(1 - A_h^y L)\,(1 - A_h^\sigma L)\, \mathcal{V}_{t+2h,H}\right] = C_1,$$

where

$$C_2 = (1 - A_h^y)\,(1 - A_h^\sigma)\, E^P\left[\mathcal{V}_{t,H}\right] = (1 - A_h^y)\,(1 - A_h^\sigma)\,\mu.$$

In equation (B.1) replace h by another time indicator s and integrate from 0 to H which leads to the following expression for integrated volatility in terms of spot variables

$$\mathcal{V}_{t,H} = c^\sigma_H + a^\sigma_H \sigma^2_t + b^\sigma_H y^2_t + \frac{1}{H}\int_0^H \epsilon^\sigma_{t,s} ds,$$

where I denote

$$\begin{aligned}
a^\sigma_H &= \frac{1}{H}\int_0^H A^\sigma_s ds = \frac{1}{\kappa_\sigma H}\left(1 - A^\sigma_H\right), \\
b^\sigma_H &= \frac{1}{H}\int_0^H B^\sigma_s ds = \frac{\kappa_\sigma}{\kappa_\sigma - \kappa_y}\frac{1}{H}\int_0^H \left(A^y_s - A^\sigma_s\right) ds = \frac{\kappa_\sigma}{\kappa_\sigma - \kappa_y}\left(a^y_H - a^\sigma_H\right), \\
c^\sigma_H &= \mu\left(1 - a^\sigma_H - b^\sigma_H\right).
\end{aligned}$$

Clearly, the second moment of $\mathcal{V}_{t,H}$ will be a function of σ^2_t, y^2_t, σ^4_t, y^4_t, and $\sigma^2_t y^2_t$. In order to eliminate the first two, we will need to apply $(1 - A^\sigma_h L)$ and $(1 - A^y_h L)$. The squared central tendency y^4_t is a function of itself in the past and itself squared, so it can be eliminated using $(1 - A^y_h L)\left(1 - \left(A^y_h\right)^2 L\right)$. The square volaitlity will be a function of all the above and can be eliminated using $\left(1 - \left(A^\sigma_h\right)^2 L\right)\left(1 - A^\sigma_h A^y_h L\right)\left(1 - \left(A^y_h\right)^2 L\right)\left(1 - A^y_h L\right)\left(1 - A^\sigma_h L\right)$. Hence,

$$E^P_t\left[\left(1 - \left(A^\sigma_h\right)^2 L\right)\left(1 - \left(A^y_h\right)^2 L\right)\left(1 - A^\sigma_h A^y_h L\right)\left(1 - A^y_h L\right)\left(1 - A^\sigma_h L\right)\mathcal{V}^2_{t+5h,H}\right] = C_2,$$

where

$$C_3 = \left(1 - \left(A^\sigma_h\right)^2\right)\left(1 - A^\sigma_h A^y_h\right)\left(1 - \left(A^y_h\right)^2\right)\left(1 - A^y_h\right)\left(1 - A^\sigma_h\right) E^P\left[\mathcal{V}^2_{t,H}\right].$$

The unconditional variance of integrated volatility is

$$V\left[\mathcal{V}_{t,H}\right] = \left(a^\sigma_H\right)^2 V\left[\sigma^2_t\right] + \left(b^\sigma_H\right)^2 V\left[y^2_t\right] + V\left[\frac{1}{H}\int_0^H \epsilon^\sigma_{t,s} ds\right].$$

Directly from (B.2) we see that

$$E\left[y^4_t\right] = \mu\eta^2_y \int_{-\infty}^t \left(A^y_{t-u}\right)^2 du = \mu\eta^2_y \int_{-\infty}^t e^{-2\kappa_y(t-u)} du = \frac{\mu\eta^2_y}{2\kappa_y},$$

and

$$
\begin{aligned}
E\left[\sigma_t^4\right] =&\mu\eta_y^2 \int_{-\infty}^{t} \left(B_{t-u}^{\sigma}\right)^2 du + \mu\eta_\sigma^2 \int_{-\infty}^{t} \left(A_{t-u}^{\sigma}\right)^2 du \\
=&\mu\eta_y^2 \left(\frac{\kappa_\sigma}{\kappa_\sigma - \kappa_y}\right)^2 \int_{-\infty}^{t} \left(A_{2(t-u)}^{y} - 2A_{t-u}^{y}A_{t-u}^{\sigma} + A_{2(t-u)}^{\sigma}\right) du + \mu\eta_\sigma^2 \int_{-\infty}^{t} A_{2(t-u)}^{\sigma} du \\
=&\mu\eta_y^2 \left(\frac{\kappa_\sigma}{\kappa_\sigma - \kappa_y}\right)^2 \left(\frac{1}{2\kappa_y} - \frac{2}{\kappa_y + \kappa_\sigma} + \frac{1}{2\kappa_\sigma}\right) + \frac{\mu\eta_\sigma^2}{2\kappa_\sigma} \\
=&\frac{\mu\eta_y^2}{2\kappa_y}\frac{\kappa_\sigma}{\kappa_y + \kappa_\sigma} + \frac{\mu\eta_\sigma^2}{2\kappa_\sigma} \\
=&E\left[y_t^4\right]\frac{\kappa_\sigma}{\kappa_y + \kappa_\sigma} + \frac{\mu\eta_\sigma^2}{2\kappa_\sigma}.
\end{aligned}
$$

Rewrite the error:

$$
\begin{aligned}
\frac{1}{H}\int_0^H \epsilon_{t,s}^{\sigma} ds =&\eta_\sigma \frac{1}{H}\int_0^H \int_t^{t+s} \sigma_u A_{t+s-u}^{\sigma} dW_u^{\sigma} ds + \eta_y \frac{1}{H}\int_0^H \int_t^{t+s} y_u B_{t+s-u}^{\sigma} dW_u^{y} ds \\
=&\eta_\sigma \int_t^{t+H} \sigma_u a_{t+H-u}^{\sigma} dW_u^{\sigma} + \eta_y \int_t^{t+H} y_u b_{t+H-u}^{\sigma} dW_u^{y} \\
=&\eta_\sigma \int_0^{H} \sigma_u a_{H-u}^{\sigma} dW_u^{\sigma} + \eta_y \int_0^{H} y_u b_{H-u}^{\sigma} dW_u^{y}.
\end{aligned}
$$

Hence,

$$
V\left[\frac{1}{H}\int_0^H \epsilon_{t,s}^{\sigma} ds\right] = \mu\eta_\sigma^2 \int_0^H \left(a_{H-u}^{\sigma}\right)^2 du + \mu\eta_y^2 \int_0^H \left(b_{H-u}^{\sigma}\right)^2 du
$$

The return equation can be integrated on $[t, t+H]$ to obtain

$$
\tilde{r}_{t,h} \equiv \frac{1}{H}\int_t^{t+H} dp_s - r = \left(\lambda_r - \frac{1}{2}\right)\mathcal{V}_{t,H} + \frac{1}{H}\int_t^{t+H} \sigma_s dW_s^r,
$$

The return first conditional moment is

$$
E_t\left[\tilde{r}_{t,H}\right] = \left(\lambda - \frac{1}{2}\right) E_t\left[\mathcal{V}_{t,H}\right],
$$

or

$$
E_t\left[\tilde{r}_{t,H} - \left(\lambda_r - \frac{1}{2}\right)\mathcal{V}_{t,H}\right] = C_3 = 0.
$$

Now multiply return by integrated volatility and take the expectation:

$$
E_t\left[\tilde{r}_{t,H}\mathcal{V}_{t,H} - \left(\lambda - \frac{1}{2}\right)\mathcal{V}_{t,H}^2\right] = E_t\left[\mathcal{V}_{t,H}\frac{1}{H}\int_t^{t+H} \sigma_s dW_s^r\right].
$$

We can guess that

$$E_t\left[\left(1-A_h^y L\right)\left(1-A_h^\sigma L\right)\left(\tilde{r}_{t,H}\mathcal{V}_{t,H}-\left(\lambda-\frac{1}{2}\right)\mathcal{V}_{t,H}^2\right)\right]=C_4,$$

where

$$C_4=\left(1-A_h^y\right)\left(1-A_h^\sigma\right)E\left[\mathcal{V}_{t,H}\frac{1}{H}\int_t^{t+H}\sigma_s dW_s^r\right].$$

The last term is

$$\begin{aligned}
E^P\left[\mathcal{V}_{t,H}\frac{1}{H}\int_t^{t+H}\sigma_s dW_s^r\right]=&\frac{1}{H^2}E^P\left[\int_0^H \epsilon_{t,s}^\sigma ds\int_t^{t+H}\sigma_s dW_s^r\right]\\
=&\eta_\sigma\frac{1}{H^2}E^P\left[\left(\int_0^H\int_t^{t+s}\sigma_u A_{t+s-u}^\sigma dW_u^\sigma ds\right)\left(\int_t^{t+H}\sigma_s dW_s^r\right)\right]\\
=&\eta_\sigma\frac{1}{H^2}E^P\left[\left(\int_t^{t+H}\sigma_u\left(\int_{u-t}^h A_{t+s-u}^\sigma ds\right)dW_u^\sigma\right)\left(\int_t^{t+H}\sigma_u dW_u^r\right)\right]\\
=&\rho\frac{\eta_\sigma}{\kappa_\sigma}\frac{1}{H^2}E^P\left[\int_t^{t+H}\sigma_u^2\left(1-A_{t+H-u}^\sigma\right)du\right]\\
=&\rho\frac{\mu\eta_\sigma}{\kappa_\sigma}\frac{1}{H^2}\int_0^H\left(1-A_{H-u}^\sigma\right)du\\
=&\rho\frac{\mu\eta_\sigma}{\kappa_\sigma}\frac{1}{H^2}\left(H-\frac{1}{\kappa^\sigma}\left(1-A_H^\sigma\right)\right)\\
=&\rho\frac{\mu\eta_\sigma}{\kappa_\sigma}\frac{1-a_H^\sigma}{H}.
\end{aligned}$$