CISS362: Introduction to Automata Theory, Languages, and Computation
Assignment 20

The following is the version of the lemma that you must use:

Pumping Lemma (for regular language). Let L be a language (over Σ). Suppose that for each integer $n \geq 0$, there is some $x \in L$ such that $|x| \geq n$ and, if $x = uvw$ where $u, v, w \in \Sigma^*$, then there is some integer $i_0 \geq 0$ such that

$$uv^{i_0}w \notin L$$

Then L is not regular. $\square$

I expect all proofs to be properly written. This means that not only must the proof be logically correct, it must for instance be grammatically correct. Proper mathematical notation must be used. Make sure you check the spelling of every word.

Q1. Textbook Q 1.29 and (d) below.

(d) $A_4 = \{a^{42^n} \mid n \geq 0\}$

[The answers for (a) and (c) are already given in the book. Make sure you study it. You only need to complete (b) and (d).]

SOLUTION.

1(a) Let $n \geq 0$. We choose $x = 0^n 1^n 2^n$ Note that $x \in A_1$ and $|x| = n + n + n = 3n \geq n$.

Let $x = uvw$ where

$$|uv| \leq n, \quad |v| > 0$$

Note that since uv is a prefix of $x = 0^n 1^n 2^n$ and $|uv| \leq n$, the only characters appearing in uv is 0. Hence

$$u = 0^{|u|}, \quad v = 0^{|v|}, \quad w = 0^{n-|u|-|v|} 1^n 2^n$$

Therefore if we choose $i_0 = 2$, we obtain

$$\begin{aligned} uv^{i_0}w &= uv^2w \\ &= 0^{|u|} \cdot 0^{2|v|} \cdot 0^{n-|u|-|v|} 1^n 2^n \\ &= 0^{n+|v|} 1^n 2^n \end{aligned}$$

Since $|v| > 0$, $n + |v| \neq n$. Hence $uv^{i_0}w \notin A_1$.

By the pumping lemma for regular language, we conclude that A_1 is not regular.

1(b)

1(c) Let $n \geq 0$. We choose $x = a^{2^n}$. Note that $x \in A_3$ and $|x| = 2^n \geq n$.

Let $x = uvw$ with $|uv| \leq n$ and $|v| > 0$. We want to choose i_0 such that

$$uv^{i_0}w \notin A_3$$

[The strategy that we hope will work is to show that $2^n < |uv^{i_0}w| < 2^{n+1}$.]

Note that

$$|uv^{i_0}w| = 2^n + (i_0 - 1)|v|$$

Since $|v| > 0$ if we choose i_0 such that

$$i_0 - 1 \geq 1$$

then we must have

$$2^n < 2^n + (i_0 - 1)|v|$$

We also have

$$|v| \leq |uv|$$

Therefore

$$2^n < 2^n + (i_0 - 1)|v| \leq 2^n + (i_0 - 1)|uv|$$

We also know that $|uv| \leq n$. Therefore

$$2^n < 2^n + (i_0 - 1)|v| \leq 2^n + (i_0 - 1)|uv| \leq 2^n + (i_0 - 1)n$$

Therefore if we choose i_0 such that

$$(i_0 - 1)n < 2^n$$

then we have

$$2^n < 2^n + (i_0 - 1)|v| \leq 2^n + (i_0 - 1)n < 2^n + 2^n = 2 \cdot 2^n = 2^{n+1}$$

Altogether we have to choose i_0 such that

1. $i_0 - 1 \geq 1$
2. $(i_0 - 1)n < 2^n$

If we choose $i_0 = 2$ both conditions are indeed satisfied:

1. $i_0 - 1 = 2 - 1 \geq 1$
2. $(i_0 - 1)n = (2 - 1)n = n < 2^n$

[The fact that $n < 2^n$ (for $n \geq 0$) can be shown easily.] In that case we have

$$2^n < 2^n + (i_0 - 1)|v| < 2^{n+1}$$

and therefore $uv^{i_0}w \underline{\notin} A_3$

By the pumping lemma for regular languages, A_3 is not regular.

1(d)

Q2. Textbook Q 1.30. [This is optional and is not graded.]

Q3. Textbook Q 1.46.

[I've provided the solution for (b). Make sure you study it.]

3(a)

3(b) Let $L = \{0^m 1^n \mid m \neq n\}$.

We will prove this by contradiction. Assume that L is regular.

We already know that complementation is a regular operator, i.e. if L' is a regular language, then $\overline{L'}$ is also regular. Since, according to our assumption, L is regular,

$$\overline{L} = \{x \in \{0,1\}^* \mid x \text{ is not of the form } 0^m 1^n \text{ for } m \neq n\}$$

is also regular Note that $\{0^n 1^n \mid n \geq 0\} \subseteq \overline{L}$. (WARNING: $\{0^n 1^n \mid n \geq 0\} \neq \overline{L}$ since for instance $1010 \in \overline{L}$.)

Note that intersection is a regular operator, i.e. if L', L'' are regular, then $L' \cup L''$ is also regular. $L(0^*1^*)$ is regular since it is the language accepted by the regular expression 0^*1^*. Therefore, since from the above $\overline{L}$ is regular,

$$\overline{L} \sqcap L(0^*1^*) = \{0^n 1^n \mid n \geq 0\}$$

is also regular.

This is a contradiction since we have already proven that $\{0^n 1^n \mid n \geq 0\}$ is not regular. Hence our assumption that $L = \{0^m 1^n \mid m \neq n\}$ is regular cannot hold. Therefore L must be non-regular.

3(c)

3(d)

Q4. Either prove

$$L = \{ab^n ab^n \mid n \geq 0\}$$

is regular by constructing a DFA or NFA or regular expression for L or prove it is not regular using the pumping lemma for regular languages.

[HINT: It's "intuitively" clear that it's not regular. However intuition can be used as a guide, not as a proof. WARNING: If you're using the pumping lemma, note that the uv might contains two types of characters, both a and b.]

SOLUTION.

Q5. Textbook Q 1.53.

SOLUTION.

WARNING!!! SPOILERS ON NEXT PAGE!!!

HINT FOR 1.53: In binary addition we have the following:

$$
\begin{aligned}
1 &= 0 + 1 \\
11 &= 0 + 11 \\
111 &= 0 + 111 \\
1111 &= 0 + 1111
\end{aligned}
$$