# Chain Rule and proof

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**Chain Rule.**

Assume:

$$f(x) = g(h(x))$$

Then:

$$f'(x) = g'(x) \cdot h'(g(x))$$

*Proof.*

Let

$$y = f(g(x))$$

so,

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(g(x+h) - f(g(x)))}{h}$$

Now let

$$k = g(x+h) - g(x) \Rightarrow g(x+h) = k + g(x)$$

then

$$\frac{f(g(x+h)) - f(g(x))}{h} \Rightarrow \frac{f(g(x)+k) - f(g(x))}{h} \Rightarrow$$

$$\frac{f(g(x)+k) - f(g(x))}{k} \cdot \frac{k}{h} \Rightarrow \frac{f(g(x)+k) - f(g(x))}{k} \cdot \frac{g(x+h) - g(x)}{h}$$

Now let

$$u = g(x)$$

so now

$$\frac{f(u+k) - f(u)}{k} \cdot \frac{g(x+h) - u}{h}$$

Since g is differentiable at x, it is continuous there:

$$as \text{ h} \to 0, \text{ g}(x+h) \to (x) \text{ and we must have } k \to 0$$

That is

$$\lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$
$$= \lim_{k \to 0} \frac{f(u+k) - f(u)}{k} \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

So, since

$$u = g(x) \text{ we have } \frac{df(g(x))}{dg(x)} \cdot \frac{d(g(x))}{dx}$$

which is the Chain Rule. □