Null space decomposition for commuting operators

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Here are some results I do not remember seeing in textbooks (except for Theorem 1), but which seem to be really useful.

Theorem 1. *Let V be a vector space and $S, T \in \mathcal{L}(V)$ such that $ST = TS$. Then $S(\text{null}(T)) \subseteq \text{null}(T)$ and $S(\text{range}(T)) \subseteq \text{range}(T)$.*

Proof. Let $v \in S(\text{null}(T))$; then $v = S(u)$ for some $u \in V$ such that $T(u) = 0$. We have $T(v) = T(S(u)) = S(T(u)) = 0$, so $v \in \text{null}(T)$.

Let $v \in S(\text{range}(T))$; then $v = S(T(u))$ for some $u \in V$. We have $v = T(S(U))$, so $v \in \text{range}(T)$. □

Theorem 2. *Let V be a finite-dimensional vector space and $S, T \in \mathcal{L}(V)$ such that $ST = TS$ and $\text{null}(S) \cap \text{null}(T) = 0$. Then $S(\text{null}(T)) = \text{null}(T)$.*

Proof. By Theorem 1 we already know that $S(\text{null}(T)) \subseteq \text{null}(T)$. We can see that:

$$\begin{aligned} S(\text{null}(T)) &= \{S(v) \mid v \in V, T(v) = 0\} \\ &= \text{range}(S\big|_{\text{null}\, T}) \end{aligned}$$

And:

$$\begin{aligned} 0 &= \text{null}(S) \cap \text{null}(T) \\ &= \{v | v \in V, S(v) = 0 \wedge T(v) = 0\} \\ &= \text{null}(S\big|_{\text{null}\, T}) \end{aligned}$$

But:

$$\dim(\text{range}(S\big|_{\text{null}\, T})) + \dim(\text{null}(S\big|_{\text{null}\, T})) = \dim(\text{null}\, T)$$

So $\dim(S(\text{null}(T))) = 0 + \dim(\text{null}(T))$ and $S(\text{null}(T)) = \text{null}(T)$. □

Theorem 3. *Let V be a finite-dimensional vector space and $S, T \in \mathcal{L}(V)$ such that $ST = TS$ and $\text{null}(S) \cap \text{null}(T) = 0$. Then $\text{null}(T\big|_{\text{range}\, S}) = \text{null}(T)$.*

Proof. The direction $\text{null}(T\big|_{\text{range}\, S}) \subseteq \text{null}(T)$ is easy. Using Theorem 2:

$$\begin{aligned} \text{null}(T) &= S(\text{null}(T)) \\ &= \{S(v) \mid v \in V, T(v) = 0\} \\ &\subseteq \{S(v) \mid v \in V, S(T(v)) = 0\} \\ &= \{S(v) \mid v \in V, T(S(v)) = 0\} \\ &= \text{null}(T\big|_{\text{range}\, S}) \end{aligned}$$

□

Theorem 4. *Let V be a finite-dimensional vector space and $S, T \in \mathcal{L}(V)$ such that $ST = TS$ and $\text{null}(S) \cap \text{null}(T) = 0$. Then $\text{null}(S) \oplus \text{null}(T) = \text{null}(ST)$.*

Proof. The direction $\text{null}(S) \oplus \text{null}(T) \subseteq \text{null}(ST)$ is easy. We can see that:

$$\begin{aligned} \text{range}(T\big|_{\text{range}\, S}) &= \{T(S(v)) \mid v \in V\} \\ &= \{S(T(v)) \mid v \in V\} \\ &= \text{range}(ST) \end{aligned}$$

And $\text{null}(T|_{\text{range } S}) = \text{null}(T)$ by Theorem 3. We have:

$$\begin{aligned}
\dim(\text{null}(S) \oplus \text{null}(T)) &= \dim(\text{null}(S)) + \dim(\text{null}(T)) \\
&= (\dim(V) - \dim(\text{range}(S))) + (\dim(V) - \dim(\text{range}(T))) \\
&= 2\dim(V) - \dim(\text{range}(S)) - \dim(\text{range}(T)) \\
&= 2\dim(V) - (\dim(\text{range}(T|_{\text{range } S})) + \dim(\text{null}(T|_{\text{range } S}))) - \dim(\text{range}(T)) \\
&= 2\dim(V) - (\dim(\text{range}(ST)) + \dim(\text{null}(T))) - \dim(\text{range}(T)) \\
&= (\dim(V) - \dim(\text{range}(ST))) + (\dim(V) - \dim(\text{range}(T)) - \dim(\text{null}(T))) \\
&= \dim(\text{null}(ST))
\end{aligned}$$

And so $\text{null}(S) \oplus \text{null}(T) = \text{null}(ST)$. □

These properties (here Theorem 4 is taken) could by tested randomly and computationally using a code like the following (using my library for matrix operations):

```
from random import randint

for _ in range(100000000):
    S = Matrix([[randint(-4, 4), randint(-4, 4), randint(-4, 4)],
                [0,              randint(-4, 4), randint(-4, 4)],
                [0,              0,              randint(-4, 4)]])
    T = Matrix([[randint(-4, 4), randint(-4, 4), randint(-4, 4)],
                [0,              randint(-4, 4), randint(-4, 4)],
                [0,              0,              randint(-4, 4)]])
    if S * T == T * S and \
       S.null().append_right(T.null()).rang() == S.null().m + T.null().m != (S * T).null().m:
        print(S)
        print(T)
```

As the linear operators we consider commute, we should always be able to find a common base where the matrices of these operators are triangular, so we only check matrices of this form. Also, coefficients are taken to be fairly small, as non-invertible matrices are rare.