3. Oil is leaking from a pipeline on the surface of a lake and forms an oil slick whose volume increases at a constant rate of 2000 cubic centimeters per minute. The oil slick takes the form of a right circular cylinder with both its radius and height changing with time. (Note: The volume V of a right circular cylinder with radius r and height h is given by $V = \pi r^2 h$.)

(a) At the instant when the radius of the oil slick is 100 centimeters and the height is 0.5 centimeter, the radius is increasing at the rate of 2.5 centimeters per minute. At this instant, what is the rate of change of the height of the oil slick with respect to time, in centimeters per minute?

(b) A recovery device arrives on the scene and begins removing oil. The rate at which oil is removed is $R(t) = 400\sqrt{t}$ cubic centimeters per minute, where t is the time in minutes since the device began working. Oil continues to leak at the rate of 2000 cubic centimeters per minute. Find the time t when the oil slick reaches its maximum volume. Justify your answer.

(c) By the time the recovery device began removing oil, 60,000 cubic centimeters of oil had already leaked. Write, but do not evaluate, an expression involving an integral that gives the volume of oil at the time found in part (b).

(a) $\frac{dV}{dt} = 2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt} \Rightarrow 2000 = 2\pi(100)(.5)(2.5) + \pi(100)^2 \frac{dh}{dt}$

$\frac{dh}{dt} = 0.03866 \ \text{cm/min}$

(b) $\frac{dV}{dt} = 2000 - R(t)$ C.V $\Rightarrow R(t) = 2000$ $400\sqrt{t} = 2000$ $t = 25$

$\frac{d^2V}{dt^2} = \cancel{2000} - R'(25) = \frac{-200}{\sqrt{25}} = -40$ (max)

(c) $60000 + \int_0^{25} (2000 - R(t))\,dt$

5. The derivative of a function f is given by $f'(x) = (x-3)e^x$ for $x > 0$, and $f(1) = 7$.

(a) The function f has a critical point at $x = 3$. At this point, does f have a relative minimum, a relative maximum, or neither? Justify your answer.

(b) On what intervals, if any, is the graph of f both decreasing and concave up? Explain your reasoning.

(c) Find the value of $f(3)$.

(a) $f''(x) = (x-3)e^x + e^x = e^x(x-2) \rightarrow f''(3) = e^3$ (min)

(b) $\overset{-}{}\big|_{3}\overset{+}{}$ f' $\overset{-}{}\big|_{2}\overset{+}{}$ f'' $f' < 0$ when $x < 3$ and $f'' > 0$ when $x > 2$

so $2 < x < 3$

(c) $f(3) = f(1) + \int_1^3 (x-3)e^x\,dx$

$x-3$ e^x
1 e^x
0 e^x

$f(3) = 7 + \left((x-3)e^x - e^x\right)\Big|_1^3$

$= 7 + \left((0e^3 - e^3) - (-2e - e)\right)$

$= 7 - e^3 + 3e$

4. The functions f and g are given by $f(x) = \int_0^{3x} \sqrt{4+t^2}\,dt$ and $g(x) = f(\sin x)$.

(a) Find $f'(x)$ and $g'(x)$.

(b) Write an equation for the line tangent to the graph of $y = g(x)$ at $x = \pi$.

(c) Write, but do not evaluate, an integral expression that represents the maximum value of g on the interval $0 \le x \le \pi$. Justify your answer.

(a) $f'(x) = 3\sqrt{4+9x^2}$ $\qquad g'(x) = f'(\sin x) = 3\cos x\sqrt{4+9\sin^2 x}$

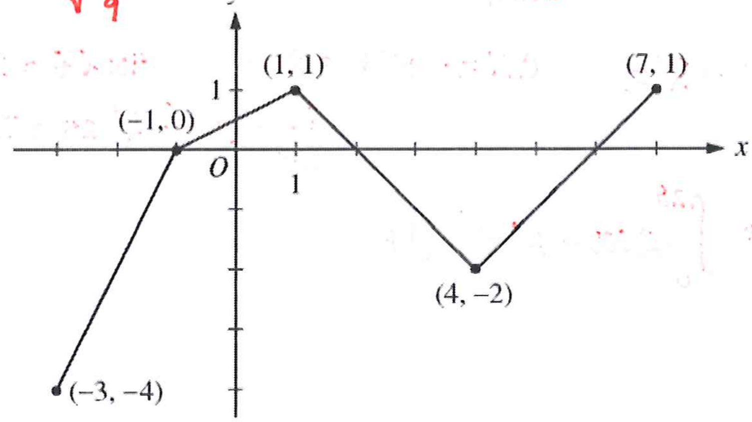
(b) $g(\pi) = f(\sin \pi) = \int_0^{3\sin\pi} \sqrt{4+t^2}\,dt = \int_0^0 \sqrt{4+t^2}\,dt = 0$

$g'(\pi) = 3\cos\pi\sqrt{4+9\sin^2\pi} = -3\sqrt{4} = -6$ $\qquad y - 0 = -6(x-\pi)$

(c) $g'(x) = 0$

$0 = 3\cos x$ $\qquad 0 = \sqrt{4+9\sin^2 x}$ $\qquad$ since $g(0) = g(\pi) = 0$ and $\sqrt{\ }$ have to be +

$x = \frac{\pi}{2}$ $\qquad \sqrt{-\frac{4}{9}} = \sin x$ $\qquad$ then $\int_0^3 \sqrt{4+t^2}\,dt > 0$



Graph of g'

5. Let g be a continuous function with $g(2) = 5$. The graph of the piecewise-linear function g', the derivative of g, is shown above for $-3 \le x \le 7$.

(a) Find the x-coordinate of all points of inflection of the graph of $y = g(x)$ for $-3 < x < 7$. Justify your answer.

(b) Find the absolute maximum value of g on the interval $-3 \le x \le 7$. Justify your answer.

(c) Find the average rate of change of $g(x)$ on the interval $-3 \le x \le 7$. $\quad \frac{g(7) - g(-3)}{7 - (-3)}$

(d) Find the average rate of change of $g'(x)$ on the interval $-3 \le x \le 7$. Does the Mean Value Theorem applied on the interval $-3 \le x \le 7$ guarantee a value of c, for $-3 < c < 7$, such that $g''(c)$ is equal to this average rate of change? Why or why not? $\quad \frac{g'(7) - g'(-3)}{7 - (-3)}$

(a) CV of g' are $x=-1, x=1, x=4$, IP happen at $x=1$ and $x=4$, slope g changes sign.

(b) since graph is g' then zeros are CV of g. Only $x=2$ has g' changing positive to neg.

$g(2) - g(-3) = \int_{-3}^{2} g'\,dx$ $\qquad g(7) = g(2) + \int_2^7 g'\,dx$ $\qquad g(2) = 5$ local max

$g(-3) = g(2) - \int_{-3}^{2} g'\,dx$ $\qquad g(7) = 5 + \left(-4 + \frac{1}{2}\right)$

endpoint $\quad = 5 - \left(-4 + \frac{3}{2}\right)$ $\qquad g(7) = 1.5$

$= 7.5$ absolute max $\qquad$ endpoint $\qquad$ local max