Allison Fenichel
ECBM 6040
Homework 1

Part II

Problem C

i) $p_x(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$

$$y = -\frac{1}{\lambda}\ln(x) = g(x)$$

$$-\lambda y = \ln x$$
$$x = e^{-\lambda y} = g^{-1}(y)$$

$$F_x(x) = \int_0^x 1\, dx = \begin{cases} x & \text{for } 0 \le x \le 1 \\ 0 & \text{for } x < 0 \\ 1 & \text{for } x > 1 \end{cases}$$

$$p_Y(y) = p_x(g^{-1}(y))\frac{dg^{-1}(y)}{dy} = p_x(e^{-\lambda y})\left[-\lambda e^{-\lambda y}\right] = -\lambda e^{-\lambda y}$$

ii) $p(x,y) = \begin{cases} 3(xy^2 + yx^2) & \text{for } x, y \in [0,1] \\ 0 & \text{else} \end{cases}$

$$p_x(x) = 3\int_0^1 xy^2 + yx^2\, dy = \left[xy^3 + \frac{3}{2}x^2y^2\right]_{y=0}^1 = x + \frac{3}{2}x^2$$

$$p_Y(y) = 3\int_0^1 xy^2 + yx^2\, dx = \left[\frac{3}{2}x^2y^2 + yx^3\right]_{x=0}^1 = \frac{3}{2}y^2 + y$$

$$E(x) = 3\int_0^1\int_0^1 x^2y^2 + yx^3\, dy dx = \int_0^1\left[x^2y^3 + \frac{3}{2}y^2x^3\right]_{y=0}^1 dx$$
$$= \int_0^1 x^2 + \frac{3}{2}x^3\, dx = \frac{1}{3}x^3 + \frac{3}{8}x^4\Big|_0^1 = \frac{1}{3} + \frac{3}{8} = \frac{17}{24}$$

$$E(y) = 3\int_0^1\int_0^1 xy^3 + y^2x^2\, dx dy = \int_0^1\left[\frac{3}{2}x^2y^3 + y^2x^3\right]_0^1 dy$$
$$= \int_0^1 \frac{3}{2}y^3 + y^2\, dy = \left[\frac{3}{8}y^4 + \frac{1}{3}y^3\right]_0^1 = \frac{3}{8} + \frac{1}{3} = \frac{17}{24}$$

$$E(xy) = 3\int_0^1\int_0^1 x^2y^3 + y^2x^3\, dx\, dy = \int_0^1 \left[x^3y^3 + \frac{3}{4}y^2x^4\right]_{x=0}^1 dy$$

$$= \int_0^1 y^3 + \frac{3}{4}y^2\, dy = \frac{1}{4}y^4 + \frac{1}{4}y^3\Big|_0^1 = \frac{1}{2}$$

$$E(x)\,E(y) = \left(\frac{17}{24}\right)^2 > E(xy) = \frac{1}{2}$$

so x and y are not independent

Problem d

i) $X = \{x^{(1)}, \ldots, x^{(m)}\}$, $x^{(i)} \in \mathbb{R}^n$

$$f(x^{(i)}; \mu, \Sigma) = (2\pi)^{-n/2} |\Sigma|^{-1/2} e^{-\frac{1}{2}[(x^{(i)}-\mu)^T \Sigma^{-1}(x^{(i)}-\mu)]}$$

$$\mathcal{L}(\mu, \Sigma; x^{(1)}, \ldots, x^{(m)}) = (2\pi)^{-\frac{mn}{2}} |\Sigma|^{-\frac{m}{2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^m (x^{(i)}-\mu)^T \Sigma^{-1}(x^{(i)}-\mu)\right\}$$

$$l(\mu, \Sigma; x^{(1)}, \ldots, x^{(m)}) = -\frac{mn}{2}\log 2\pi - \frac{m}{2}\log|\Sigma| - \frac{1}{2}\sum_{i=1}^m (x^{(i)}-\mu)^T \Sigma^{-1}(x^{(i)}-\mu)$$

$$\frac{dl}{d\mu} = \sum_{i=1}^m \Sigma^{-1}(x^{(i)}-\mu) = 0$$

$$\sum_{i=1}^m (x^{(i)}-\mu) = 0$$

$$\sum_{i=1}^m x^{(i)} = m\mu$$

$$\boxed{\hat{\mu} = \sum_{i=1}^m x^{(i)}/m = \bar{X}}$$

$$\frac{dl}{d\Sigma^{-1}} = \frac{m}{2}\frac{d}{d\Sigma^{-1}}\log|\Sigma|^{-1} - \frac{1}{2}\sum_{i=1}^m \frac{d}{d\Sigma^{-1}} tr\left(\Sigma^{-1}(x^{(i)}-\mu)(x^{(i)}-\mu)^T\right)$$

$$= \frac{m}{2}\Sigma^T - \frac{1}{2}\sum_{i=1}^m \frac{d}{d\Sigma^{-1}} tr((x^{(i)}-\mu)^T\Sigma^{-1}(x^{(i)}-\mu))$$

$$= \frac{m}{2}\Sigma^T - \frac{1}{2}\sum_{i=1}^m (x^{(i)}-\mu)(x^{(i)}-\mu)^T$$

$$\frac{m}{2}\Sigma^T = \frac{1}{2}\sum_{i=1}^{m}(x^{(i)}-\mu)(x^{(i)}-\mu)^T \rightarrow \Sigma = \Sigma^T$$

$$\boxed{\hat{\Sigma} = \frac{1}{m}\sum_{i=1}^{m}(x^{(i)}-\hat{\mu})(x^{(i)}-\hat{\mu})^T}$$

ii) $E(\hat{\mu}) = E\left[\sum_{i=1}^{m}\frac{x^{(i)}}{m}\right] = \sum_{i=1}^{m}\frac{E(x^{(i)})}{m} = \frac{m\mu}{m} = \mu$

$E(\hat{\mu}) - \mu = 0$ so $\hat{\mu}$ is an unbiased estimator of μ

$$E(\hat{\Sigma}) = E\left(\frac{1}{m}\sum_{i=1}^{m}(x^{(i)}-\hat{\mu})(x^{(i)}-\hat{\mu})^T\right)$$

$$= \frac{1}{m}\left[\sum_{i=1}^{m}E(x^{(i)}x^{(i)T})\right] - 2E\left(\sum_{i=1}^{m}\frac{x^{(i)}}{m}\hat{\mu}^T\right) + E(\hat{\mu}\hat{\mu}^T)$$

$$E(x^{(i)}x^{(i)T}) = \Sigma + \mu\mu^T$$

$$E(\hat{\mu}\hat{\mu}^T) = \frac{\Sigma}{m} + \mu\mu^T$$

$$E(\hat{\Sigma}) = \Sigma + \mu\mu^T - 2E(\hat{\mu}\hat{\mu}^T) + E(\hat{\mu}\hat{\mu}^T)$$

$$= \Sigma + \mu\mu^T - E(\hat{\mu}\hat{\mu}^T) = \Sigma + \mu\mu^T - \frac{\Sigma}{m} - \mu\mu^T$$

$$= \Sigma - \Sigma/m = \frac{m-1}{m}\Sigma$$

mle $\hat{\Sigma}$ is a biased estimator of Σ