Timer-based pre-fetching for increasing hazard rates.

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joint work with Matias Carrasco and Fernando Paganini

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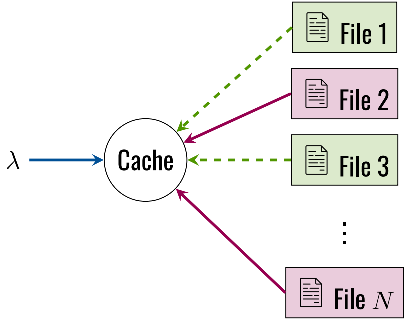
Outline

The caching problem

- Consider a local memory system that handles items from a catalog of N objects.
- Requests for objects arrive as a random process.
- The memory (cache) can locally store $C < N$ of them.
- If item is in cache, we have a hit. Otherwise, it is a miss.

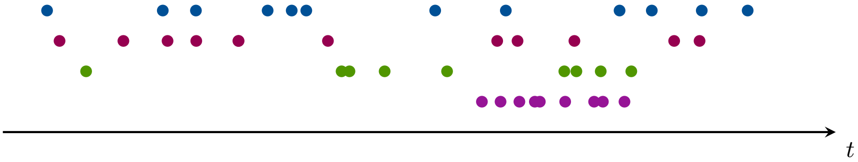


Objective: for a given arrival stream, maximize the steady-state hit rate.

Point process approach

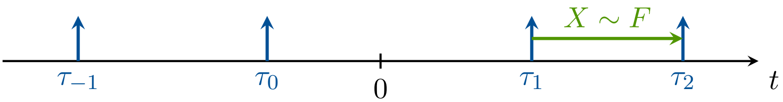
Introduced in [Fofack et al. 2014]

- Assume requests for item i come from a point process of intensity λ_i (popularities).



- At each point in time we must decide which items must be stored locally.

Two important distributions:



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$$X \sim F(t), \quad E[X] = 1/\lambda$$

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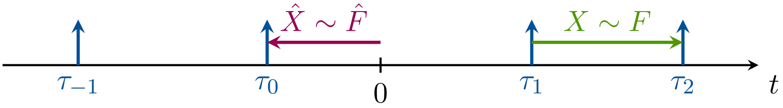

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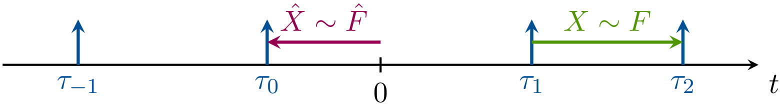

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Note: you can formalize this under the Palm probability framework for stationary point processes.

Populating a cache: timer based (TTL) policies

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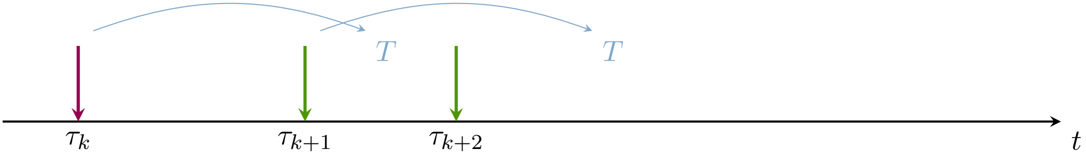

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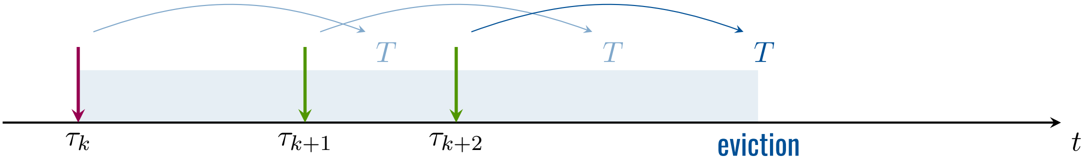

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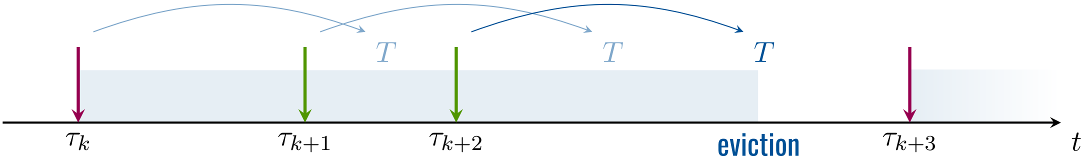
- Keep timers T_i such that average cache occupation is C.



Hit and occupation probabilities

Focus on a single item i:

- Hit probability: prob. that next request is still stored before timer expires:

$$P(X_i < T_i) = F_i(T)$$

- Memory usage: prob. that item is stored at a random point in time, i.e. prob. that timer has not expired since the last request:

$$P(\hat{X}_i < T_i) = \hat{F}_i(T)$$

Choosing the optimal timers

Requests come from independent sources with intensities λ_i and inter-arrival distribution F_i:

Problem (Optimal TTL policy)

Choose timers $T_i \geqslant 0$ such that:

$$\max_{T_i \geqslant 0} \sum_i \lambda_i F_i(T_i)$$

subject to:

$$\sum_i \hat{F}_i(T_i) \leqslant C$$

Remark: non-convex non-linear program. But it can be solved by a change of variables!!! [Ferragut et al. – QUESTA – 2018].

Structure of the optimal caching policy

- The crucial magnitude is the hazard rate of F:

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Theorem (F', Rodriguez, Paganini – 2016)

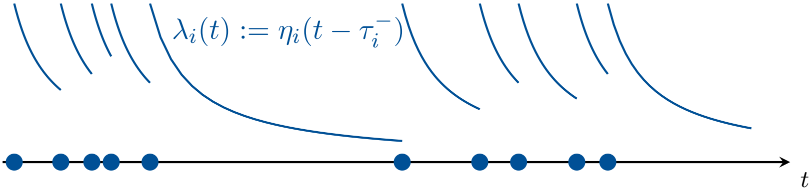
If the F_i have decreasing hazard rates, then the optimal TTL policy satisfies:

$$\eta_i(T_i^*) \geqslant \theta^*,$$

whenever $T_i^* > 0$ (i.e. the item is cached).
Moreover, inequality is strict iff $T_i^* = \infty$ (item always stored).

Why caching helps in this case?

Decreasing hazard rates corresponds to bursty traffic:



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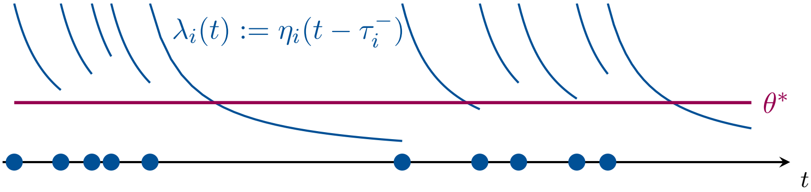

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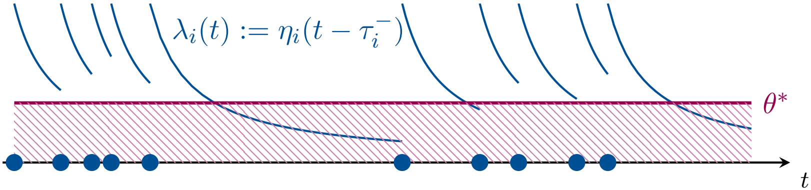

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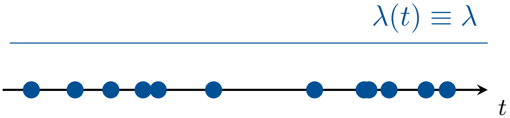
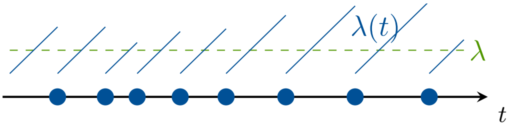
What about other types of traffic?



Constant hazard rate $\to$ Poisson process.



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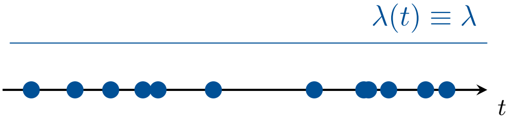

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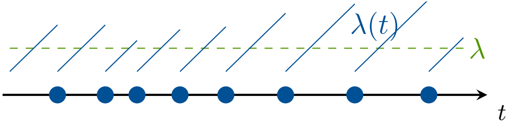

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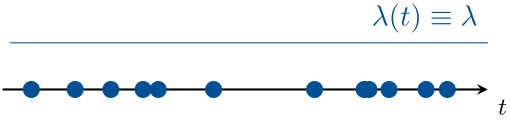

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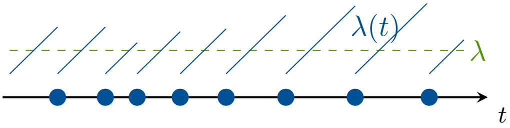

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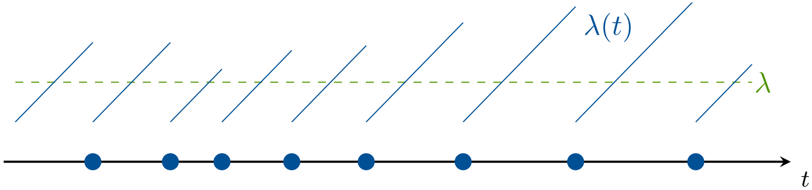
Can we improve upon this?

Thinking about increasing hazard rates...

- Once you have seen a request, it's less likely to see the same item again for a while.



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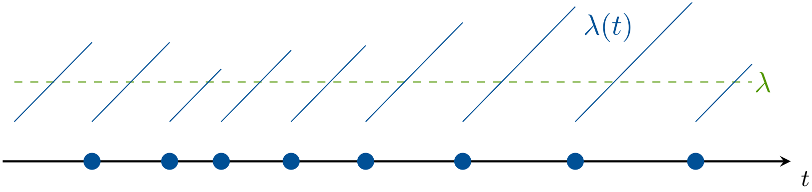

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Key insight

The question now is not how long we should remember something, but instead how long we should forget about it!

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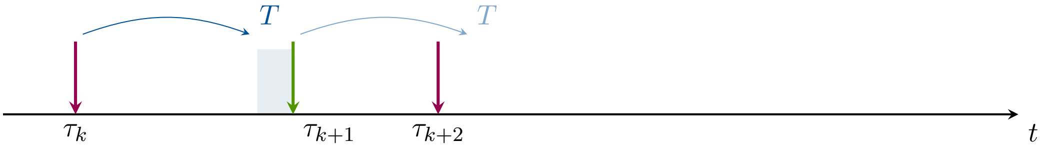

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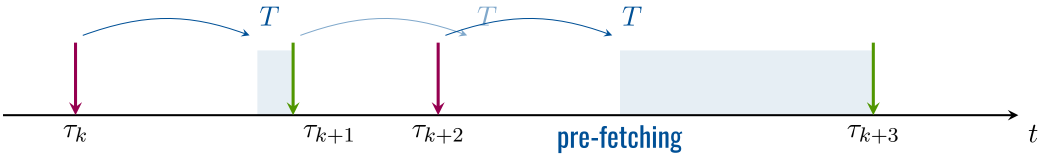

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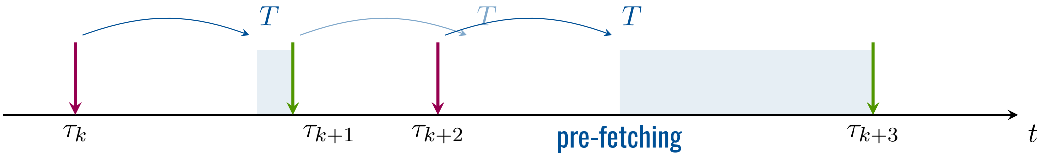
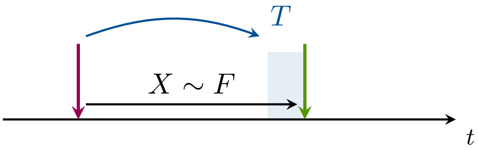


Timer based pre-fetching

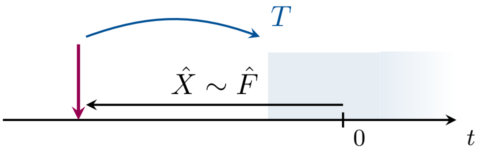
Consider a single item with a timer T and its request process:

Hit probability: next arrival occurs after timer expires.



Hit probability $= 1 - F(T)$

Occupation probability: probability that timer has expired by 0 since last arrival.



Avg. occupation $= 1 - \hat{F}(T)$

Choosing the optimal timers

Requests come from independent sources with intensities λ_i and inter-arrival distribution F_i:

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subject to:

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$$\sum_i \hat{F}(T_i) \geqslant N - C$$

Remark: we can use the same change of variables again!

Choosing the optimal timers

Change of variables

- Apply the change of variables $u_i = \hat{F}_i(T_i)$.
- Note that u_i is the probability of not being stored.
- The problem becomes:

$$\min_{u_i \in [0,1]} \sum_i \lambda_i F_i(\hat{F}_i^{-1}(u_i))$$

subject to:

$$\sum_i u_i \geqslant N - C$$

- Objective gradient:

$$\frac{\partial}{\partial u_i}\lambda_i F_i \circ \hat{F}_i^{-1}(u_i) = \frac{\lambda_i f_i(\hat{F}_i^{-1}(u_i))}{\lambda_i \left(1 - F_i(\hat{F}_i^{-1}(u_i))\right)} = \eta_i(\hat{F}_i^{-1}(u_i))$$

- Increasing! $\rightarrow$ Proper convex optimization problem.

Choosing the optimal timers

Lagrangian duality

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- Increasing! $\rightarrow$ Proper convex optimization problem.
- Lagrangian duality:

$$\begin{aligned}\mathcal{L}(u,\theta) &= \sum_{i=1}^{N} \lambda_i F_i\left(\hat{F}_i^{-1}(u_i)\right) + \theta\left(N - C - \sum_{i=1}^{N} u_i\right)\\ &= \sum_{i=1}^{N}\left[\lambda_i F_i\left(\hat{F}_i^{-1}(u_i)\right) - \theta u_i\right] + \theta(N-C).\end{aligned}$$

Main result

Theorem

If the F_i satisfy the IHR property, there exists a unique threshold $\theta^* \geqslant 0$ such that the optimal timers satisfy:

$$\eta_i(T_i^*) \geqslant \theta^*,$$

whenever $T_i^* < \infty$ (pre-fetching).
The inequality is strict if and only if $T_i^* = 0$, i.e. the content is always stored.

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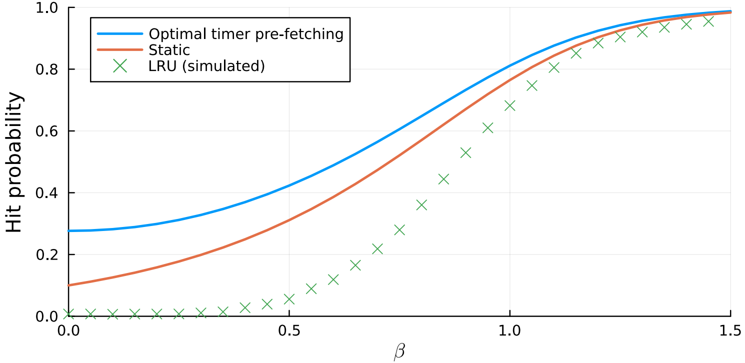
Remark: Again the policy is a threshold policy.

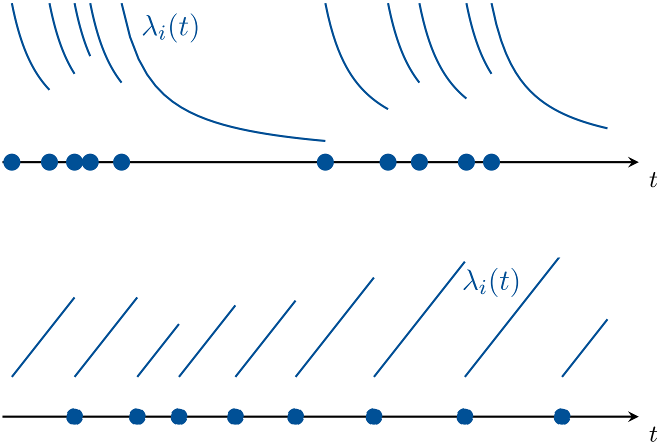
Simulation example

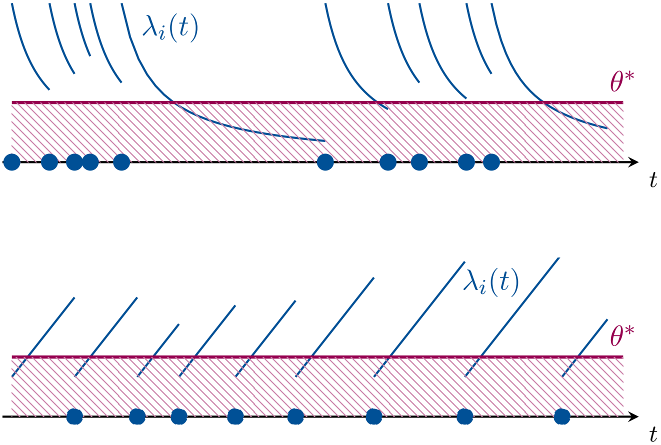
Erlang ($k = 5$) inter-arrival times, Zipf $\propto n^{-\beta}$ popularities, varying β, $N = 10000$, $C = 1000$.



- Pre-fetching improves over the static policy.
- Classical caching (e.g. LRU) is a very bad idea for regular traffic.

A tale of two policies...





Both policies are just the same policy!

- Keep a hazard rate threshold θ for storing a content
- Compute θ^* such that avg. memory occupation is C.

Asymptotic optimality

- Threshold policies in fact are related to recent results by [Panigrahy et al. 2022], about the optimal causal policy.
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Theorem (F', Carrasco, Paganini – In preparation – check ArXiv soon)

Under an appropriate scaling regime, as $N \to \infty$ and $C = cN$, the optimal causal policy converges to a fixed threshold policy. Moreover, the limit threshold θ^* coincides with its timer based counterpart.

Therefore, the above policies give a universal asymptotic upper bound on caching performance.

Final remarks

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Final remarks

- You have to know your traffic before deciding on caching or pre-fetching!
- Classical caching is not well suited to regular traffic.
- We identified the hazard rate as the crucial indicator of regularity, and devised a new policy for IHR, that is also asymptotically optimal among all causal policies.
- A lot of open questions, in particular:
  - How we can learn the hazard rates online?
  - How we can estimate the appropriate threshold?
  - What about mixtures of IHR and DHR traffic?

Tante grazie!

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