The last, the least and the urgent...

A story of three policies

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Outline

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Motivation

A bit of history...

- Several queueing systems have service and timing requirements.
- Examples:
    - Computing tasks with real-time constraints.
    - Item delivery problems in logistics.
    - Emergency response.
    - etc. etc. etc.
- This has led to a long and rich history of research about queues with abandonments [Barrer, 1957; Stanford, 1979; Baccelli et al., 1984].

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Recent developments...

One of the most used policies is Earliest-Deadline-First (EDF)

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Through fluid limits and diffusion approximations, establish performance:

- [Decreusefond and Moyal, 2008] establish EDF fluid limits in the single server case.
- [Kruk et al., 2011] provides diffusion approximations.
- [Moyal, 2013] establish some optimality properties of EDF.
- [Kang and Ramanan, 2010, 2012] analyze the many-server case.
- [Atar et al., 2018, 2023] establish asymptotic performance.

and many others...

Motivation

Common assumption

Customers renege *only* in the queue, and not during service.

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We call this the *call-center scenario*:

- Akin to waiting for the customer-help line to pick your call while you listen to annoying music.
- The underlying idea is that when a task reaches service, it will stay until completion.

Key performance metric: number of satisfied tasks (or reneging probability).

In several queueing systems:

- Tasks may abandon during service.
- More importantly, all service provided may be useful.

We call this setting queues with partial service.

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Partial service queues

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Some examples:

- Electrical vehicle charging: customers leave the system with a *partial charge*.
- LLM inference: longer computation times lead to better answers, but these may be interrupted to deliver a quick response.
- File transfers over the Internet, that can be resumed later.

Key points of this talk

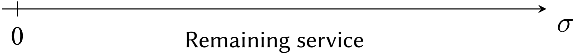
- Provide some suitable representation of the state space and dynamics of these partial service queues.
- Analyze several interesting policies under a suitable fluid model.
- Compute the main performance metric here: attained work.
- *Last but not least:* show that the simple LCFS policy exhibits the same performance than EDF in this setting, without using deadline information.

Outline

Measure valued stochastic processes

Consider the simple $M/G/\infty$ queue:

- Tasks arrive as a Poisson process of intensity λ.
- Each task has a service requirement $S \sim g(\sigma)$.



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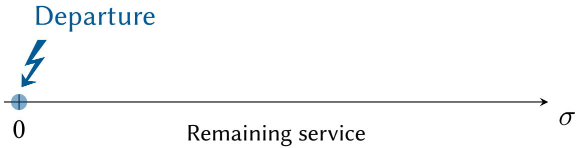

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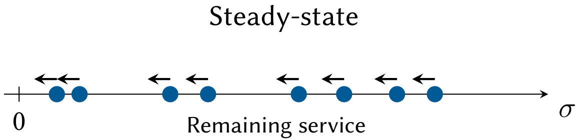

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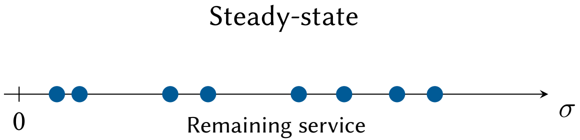

State-descriptor:

$$\Phi_t = \sum_i \delta_{\sigma_i(t)}$$

a Point-process on the positive half-line.

M/G/∞, steady state

- Φ_t is a measure-valued Markov process.
- Its dynamics can be characterized through its generator.
- In steady state:

$$\Phi \sim \text{Poisson Process with mean measure } \mu(d\sigma) = \lambda \bar{G}(\sigma) d\sigma$$

where $\bar{G}$ is the CCDF of S.

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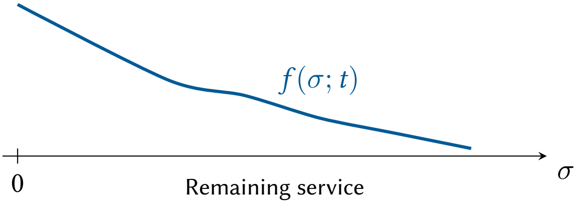
Interpretation:

- Write $\mu(d\sigma) = \rho\left[\frac{1}{E[S]}(1 - G(\sigma))\right] d\sigma$, with $\rho = \lambda E[S]$.
- Then $\left[\frac{1}{E[S]}(1 - G(\sigma))\right] d\sigma$ is the *residual service time distribution* associated to G.
- In steady-state, the total number of customers $\sim$ Poisson(ρ) and distributed in σ as the residual lifetime distribution.

M/G/∞, fluid approximation.

Suppose that we can replace Φ_t by a general measure μ_t with density $f(\sigma; t)$.



- Mass is transported to the left at rate 1.
- New mass arrives at σ with intensity $\lambda g(\sigma) d\sigma dt$.

We can combine this in the following transport equation:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \sigma} + \lambda g(\sigma).$$

$M/G/\infty$, fluid approximation.

Imposing equilibrium and the boundary condition $f(\sigma) \to 0$ as $\sigma \to \infty$ we get:

$$\frac{\partial f}{\partial \sigma} + \lambda g(\sigma) = 0 \Longrightarrow f(\sigma) = \lambda \int_{\sigma}^{\infty} g(u)\, du = \lambda \bar{G}(\sigma),$$

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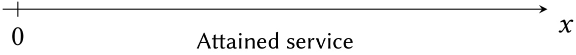
so the fluid approximation recovers the mean measure of Φ.

- This is a deterministic measure, with total mass ρ...
- ...distributed in the real line as the residual service distribution.
- Serves as an approximation of Φ in a large scale system ($\lambda \to \infty$).

M/G/∞: take two

Attained service state descriptor

Here is another approach to model the same system [Kang and Ramanan, 2010]:



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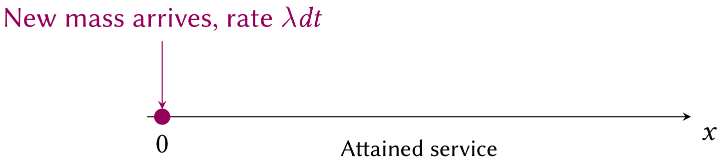

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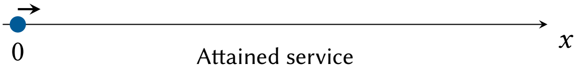

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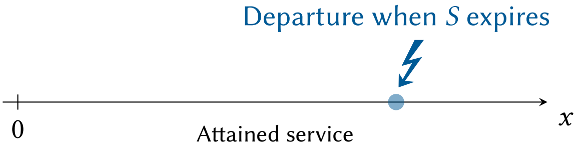

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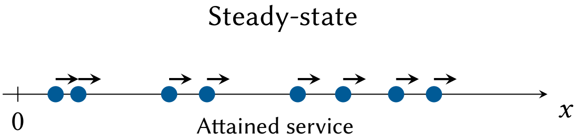

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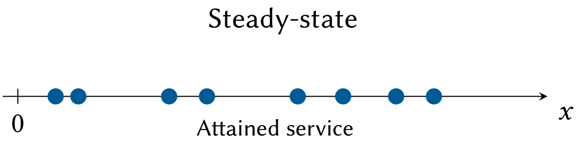

State-descriptor:

$$\tilde{\Phi}_t = \sum_i \delta_{x_i(t)}$$

a Point-process on the positive half-line, where $x_i(t)$ is the elapsed time in the system

M/G/∞, take two

Steady-state

$\tilde{\Phi}_t$ is a measure-valued Markov process.

- Mass always arrive at 0 with rate λdt.
- Transports to the right at rate 1.
- Leaves the system at rate $h(x)$, the hazard rate function:

$$h(x) = \lim_{dt \to 0} P(S \in [x, x+dt] \mid S > x) = \frac{g(x)}{\bar{G}(x)} = -\frac{\partial}{\partial x} \log \bar{G}(x).$$

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Steady-state:

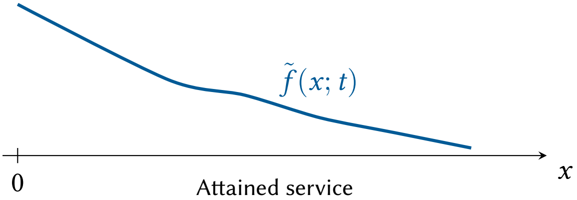
$$\tilde{\Phi} \sim \text{Poisson Process with mean measure } \nu(dx) = \lambda \bar{G}(x) dx$$

So the reversed representation has the same distribution, because in a random point in time the elapsed service and the remaining service have the same distribution.

M/G/∞: take two

Fluid approximation.

Suppose that we can replace $\tilde{\Phi}_t$ by a general measure ν_t with density $\tilde{f}(x;t)$.



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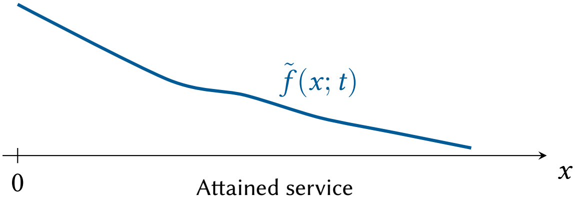

The corresponding transport equation is (informally):

$$\frac{\partial \tilde{f}}{\partial t} = -\frac{\partial \tilde{f}}{\partial x} - h(x)\tilde{f} + \lambda\delta_0.$$

$M/G/\infty$: take two

Fluid equilibrium.

Imposing equilibrium we get:

$$\frac{\partial \tilde{f}}{\partial x} = -h(x)\tilde{f} + \lambda \delta_0.$$

Solving (in a distribution sense) with the boundary condition $\tilde{f}(\infty) = 0$ we get:

$$\tilde{f}(x) = \lambda e^{-\int_0^x h(u)du}.$$

But by definition $\int_0^x h(u)du = -\log \bar{G}(x)$, and thus:

$$\tilde{f}(x) = \lambda \bar{G}(x)$$

So the transport fluid equation recovers again the mean measure of the steady-state.

Lessons learned

- We can model M/G systems by using two state descriptors:
  - The remaining service Φ.
  - The attained service $\tilde{\Phi}$.
- Both admit reasonable fluid approximations, which correspond to transport equations.
- In fact this has been used in the literature to model abandonments (since they operate as $M/G/\infty$ systems in some sense).

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- We can model M/G systems by using two state descriptors:
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Question: can we do more using this machinery of measure-valued processes?

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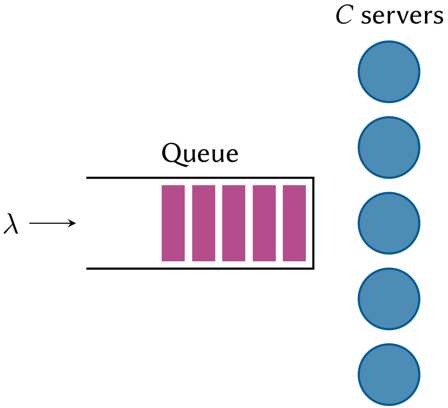
Partial service queues

Setting

Consider an $M/G/C$ system where:

- Tasks arrive as a Poisson process of intensity λ.



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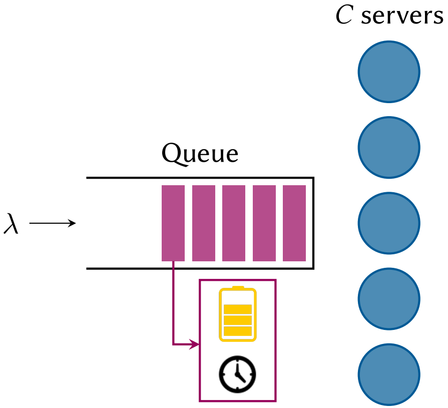
  - T_i: sojourn time or deadline.



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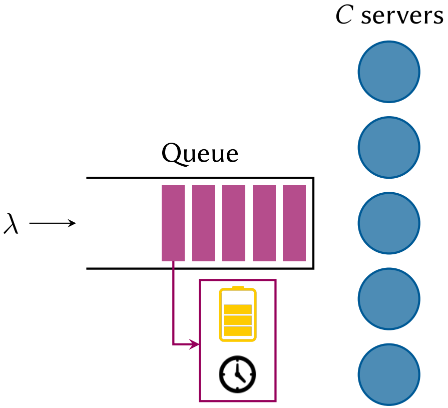
- Tasks arrive as a Poisson process of intensity λ.
- Each task i has two characteristics (marks):
  - S_i: service time (at rate 1).
  - T_i: sojourn time or deadline.
- (S_i, T_i) are independent across jobs.
- Follow a common distribution $G(\sigma, \tau)$, possibly correlated.



Partial service queues

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Customers depart whenever S_i is attained or the timer T_i expires.

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- Key performance metrics:
  - S_a: amount of service attained.
  - Equivalently, $S_r := S - S_a$, amount of service reneged.
- Problem: we have to keep track of remaining service and deadlines simultaneously!

System load

- Before proceeding, it is useful to define the system load:

$$\rho := \lambda E[\min\{S, T\}].$$

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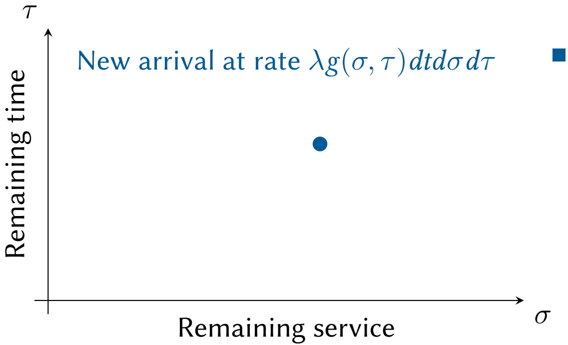
- Interpretation: the mean number of customers on a system with $C = \infty$.

- What we expect in a large scale fluid model:
  - If $\rho < C$ (underload), all tasks can be served, $S_a = \min\{S, T\}$.
  - If $\rho > C$ (overload), demand *curtailing* will occur. How? It depends on the policy...

- Consider the remaining time space.



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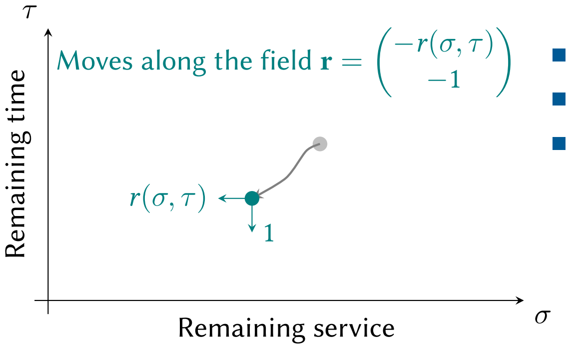
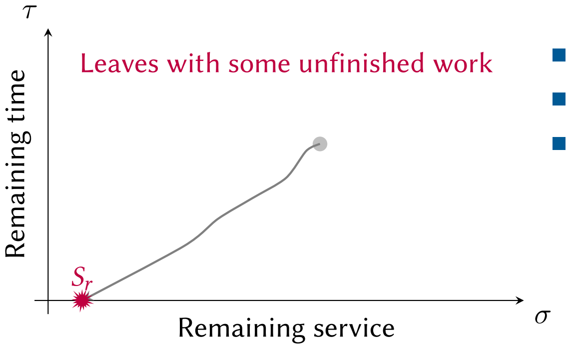


- Consider the remaining time space.
- Policy defines how tasks are served.
- May depend on any combination of (σ, τ).

System evolution

Remaining service times



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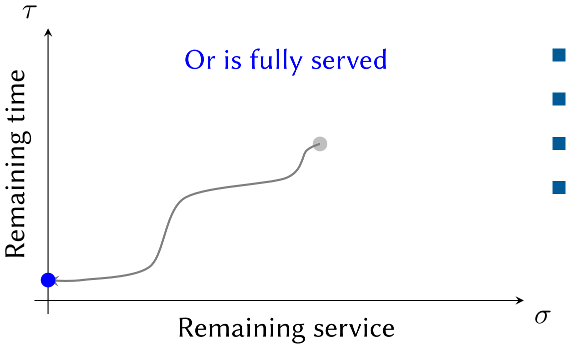
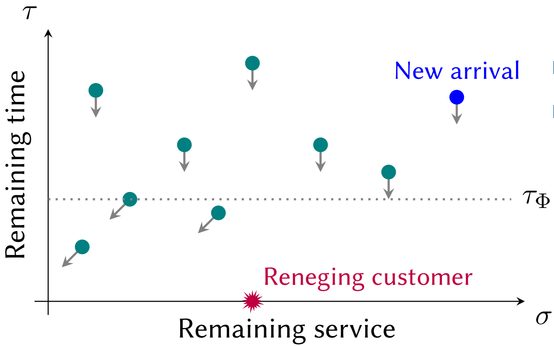


- Consider the remaining time space.
- Policy defines how tasks are served.
- May depend on any combination of (σ, τ).
- State descriptor:

$$\Phi_t = \sum_i \delta_{(\sigma_i(t), \tau_i(t))}$$

Example: Earliest-deadline-first



- Serve the C most urgent customers.
- Corresponds to taking:

$$r_\Phi(\sigma, \tau) = \mathbf{1}_{\{\tau \leqslant \tau_\Phi\}}$$

with

$$\tau_\Phi := \sup\{\tau \geqslant 0 : \Phi(\mathbb{R}_+ \times (0, \tau]) < C\}.$$

Fluid model dynamics

- Replace Φ_t by a (fluid) measure μ_t.
- Now mass drifts along the field:

$$\mathbf{r}_\mu(\sigma,\tau) = \begin{pmatrix} -r_\mu(\sigma,\tau) \\ -1 \end{pmatrix}$$

- With r_μ satisfying:

$$0 \leqslant r_\mu \leqslant 1$$
$$\iint r_\mu(\sigma,\tau)\mu(d\sigma,d\tau) \leqslant \min\{\mu(\mathbb{R}^2_{++}), C\}.$$

We will describe these dynamics in terms of the projections

$$\langle \varphi, \mu \rangle := \iint \varphi(\sigma,\tau)\, \mu(d\sigma, d\tau)$$

of the state measure with respect to a test function $\varphi : \mathbb{R}^2_{++} \to \mathbb{R}$, with continuous derivatives and compact support, i.e. $\varphi \in \mathcal{C}^1_c(\mathbb{R}^2_{++})$.

We have:

$$\langle \varphi, \mu_{t+dt} \rangle = \iint \varphi(\sigma - r_{\mu_t} dt, \tau - dt)\, \mu_t(d\sigma, d\tau) + \lambda dt \iint \varphi(\sigma,\tau) g(\sigma,\tau)\, d\sigma d\tau + o(dt).$$

$$
\begin{aligned}
\frac{\partial}{\partial t}\langle\varphi,\mu_t\rangle &= \lim_{dt\to 0}\iint \frac{1}{dt}\left[\varphi(\sigma - r_{\mu_t}dt, \tau - dt) - \varphi(\sigma,\tau)\right]\mu_t(d\sigma, d\tau) \\
&\quad + \lambda\iint \varphi(\sigma,\tau)g(\sigma,\tau)\,d\sigma\,d\tau \\
&= -\iint \left[r_{\mu_t}(\sigma,\tau)\varphi_\sigma(\sigma,\tau) + \varphi_\tau(\sigma,\tau)\right]\mu_t(d\sigma, d\tau) + \lambda\iint \varphi(\sigma,\tau)g(\sigma,\tau)\,d\sigma\,d\tau,
\end{aligned}
$$

Equivalently:

$$\langle \varphi, \mu_t \rangle = \langle \varphi, \mu_0 \rangle + \int_0^t \Big[- \iint \left[r_{\mu_s}(\sigma,\tau)\varphi_\sigma(\sigma,\tau) + \varphi_\tau(\sigma,\tau) \right] \mu_t(d\sigma, d\tau) + \lambda \iint \varphi(\sigma,\tau) g(\sigma,\tau)\, d\sigma\, d\tau \Big]\, ds,$$

for any $\varphi \in \mathcal{C}_c^1(\mathbb{R}^2_{++})$.

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for any $\varphi \in \mathcal{C}^1_c(\mathbb{R}^2_{++})$.

Looks daunting, but is not that bad...

Fluid model dynamics

Transport PDE

If μ_t admits a density $f(\sigma, \tau; t)$ with respect to the Lebesgue measure, it corresponds to:

$$\frac{\partial f}{\partial t} + \nabla \cdot [\mathbf{r}_{\mu_t} f] = \lambda g$$

a transport equation.

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Example: EDF

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \sigma} \mathbf{1}_{\{\tau < \tau_{\mu_t}\}} + \frac{\partial f}{\partial \tau} + \lambda g$$

EDF Fluid model equilibrium

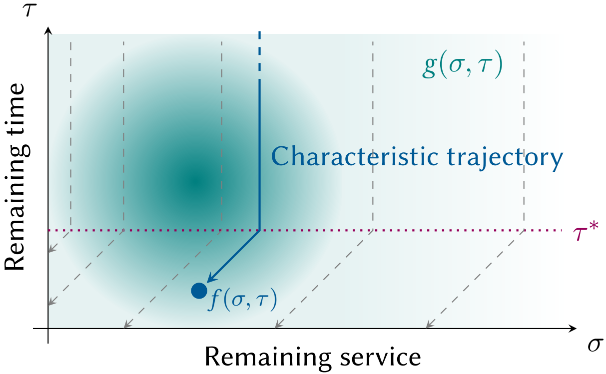
Imposing equilibrium we get:

- $\tau_{\mu^*} = \tau^*$ becomes a constant.
- The measure μ^* must satisfy:

$$\frac{\partial f}{\partial \sigma}\mathbf{1}_{\{\tau<\tau^*\}} + \frac{\partial f}{\partial \tau} + \lambda g = 0.$$

- Linear PDE that can be easily solved by the method of characteristics.

Solving the EDF transport equation



EDF in overload

Fluid model equilibrium

Theorem

Assume that $\rho > C$ and the equation

$$\lambda E[\min\{S, T, \tau^*\}] = C$$

has a unique solution $\tau^* > 0$. Consider the measure μ^* given by the following density:

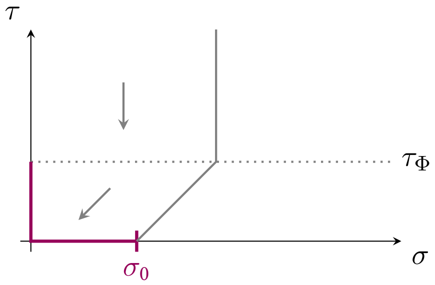
$$f(\sigma, \tau) = \lambda \left[\int_0^{(\tau^* - \tau)^+} g(\sigma + u, \tau + u) du + \int_{(\tau^* - \tau)^+}^{\infty} g\left(\sigma + (\tau^* - \tau)^+, \tau + u\right) du \right].$$

This measure is a fluid equilibrium for the EDF policy, and

$$\tau^* = \sup\left\{\tau \geq 0 : \mu^*(\mathbb{R}_{++} \times (0, \tau]) \leq C\right\}.$$

EDF performance in equilibrium

- Let us compute the rate at which work is reneged.
- Compute the rate at which mass exits with $S_r < \sigma_0$.



Proposition

$$\int_0^{\tau^*} f(0,\tau)d\tau + \int_0^{\sigma_0} f(\sigma,0)d\sigma = \lambda P\left(S - \min\left\{S, T, \tau^*\right\} < \sigma_0\right).$$

i.e. $S_a = S - S_r = \min\{S, T, \tau^*\}$.

Outline

What if we do not know the deadlines?

- Deadlines are often hard to estimate in practice.
- Moreover, tasks may under-report their deadline to get priority!
- What about deadline-oblivious policies?
  - Can we model them?
  - What is their performance?

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Problem: we need a new state-space...

Attained service state descriptor

- Consider the elapsed time space.

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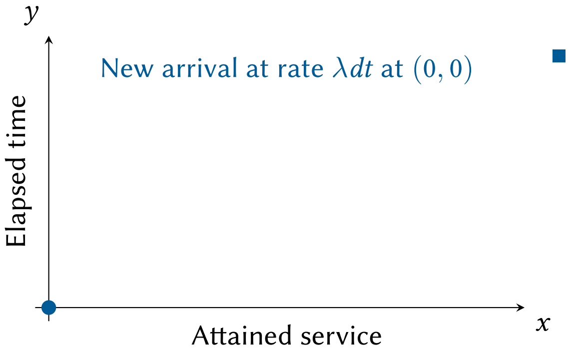

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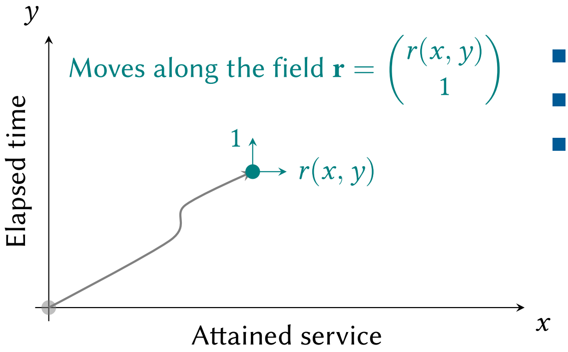
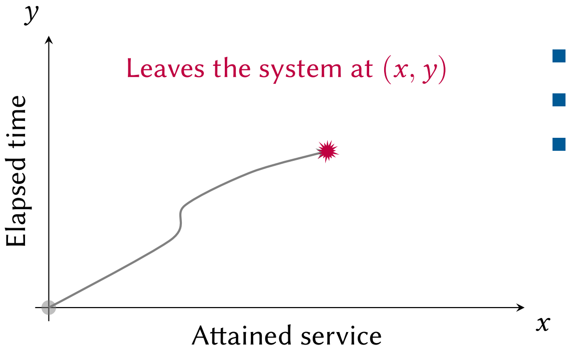


- Consider the elapsed time space.
- Policy again defines how tasks are served.
- May depend on any combination of (x, y).

Attained service state descriptor



- Consider the elapsed time space.
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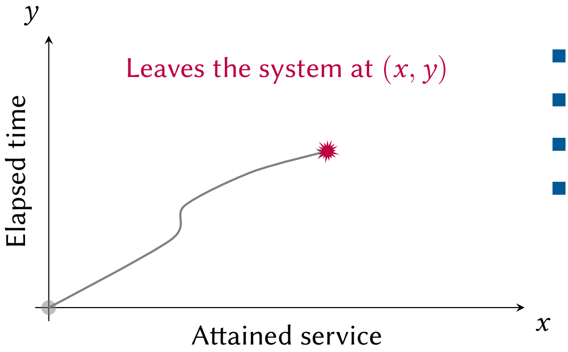
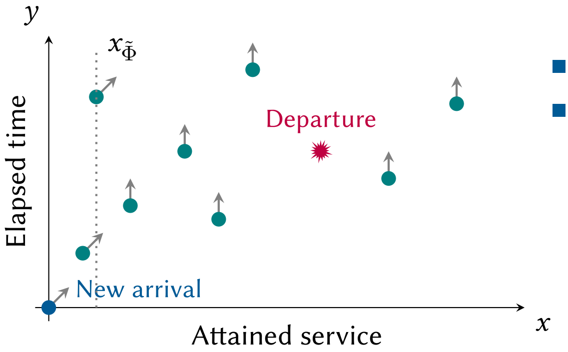
Attained service state descriptor



- Consider the elapsed time space.
- Policy again defines how tasks are served.
- May depend on any combination of (x, y).
- State descriptor:

$$\tilde{\Phi}_t = \sum_i \delta_{(x_i(t), y_i(t))}$$

Example: Least-Attained-Service policy



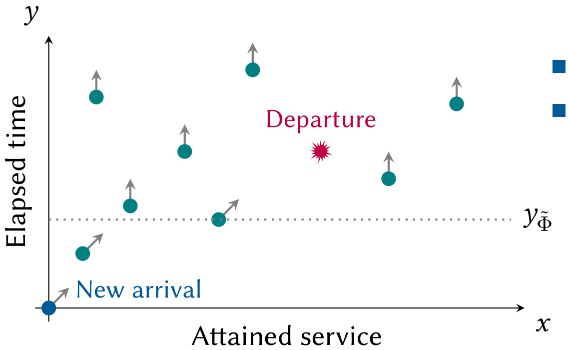
- Serve the C least-served tasks.
- Corresponds to taking:

$$r_{\tilde{\Phi}}(x, y) = \mathbf{1}_{\{x \leqslant x_{\tilde{\Phi}}\}}$$

with

$$x_{\tilde{\Phi}} := \sup\{x : \tilde{\Phi}([0, x] \times \mathbb{R}_+) \leqslant C\}.$$

Example: Last-Come-First-Served policy



- Serve the C more recent tasks.
- Corresponds to taking:

$$r_{\tilde{\Phi}}(x, y) = \mathbf{1}_{\{y \leqslant y_{\tilde{\Phi}}\}}$$

with

$$y_{\tilde{\Phi}} := \sup\{y : \tilde{\Phi}(\mathbb{R}_+ \times [0, y]) \leqslant C\}$$

The hazard rate field

We have a new problem: what is the rate at which users leave the system?

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Let $\bar{G}(x, y) = P(S > x, T > y)$ and define:

Definition (Hazard rate field)

$$\mathbf{h}(x, y) = -\nabla \log \bar{G}(x, y) \quad \text{i.e.}$$

- $h^x(x, y) = P(S \in [x, x + dx], T > S \mid S > x, T > y)$
- $h^y(x, y) = P(T \in [y, y + dy], S > T \mid S > x, T > y)$

Interpretation: $\mathbf{h}$ stores the rate at which $\min\{S, T\}$ is attained due to S or T expiring.

Fluid model dynamics

- Replace $\tilde{\Phi}_t$ by a (fluid) measure ν_t.
- Now mass arrives at $(0,0)$ at rate λ.
- Drifts along the field:

$$\mathbf{r}_\nu(x,y) = \begin{pmatrix} r_\nu(x,y) \\ 1 \end{pmatrix}$$

- With r_ν satisfying:

$$0 \leqslant r_\nu \leqslant 1$$
$$\iint r_\nu(x,y)\nu(dx,dy) \leqslant \min\{\nu(\mathbb{R}^2_+), C\}.$$

Departure rate

Now we have to compute the departure rate $\eta_\nu(x, y)$:

$$\eta_\nu(x, y) := \lim_{dt \to 0} \frac{1}{dt} P\left(\left\{S \in \left(x, x + r_{\tilde{\Phi}} dt\right)\right\} \cup \{T \in (y, y + dt)\} \mid S > x, T > y\right)$$

By the chain rule and some computations:

$$\eta_\nu(x, y) = \frac{1}{\bar{G}(x, y)} \left[-\frac{\partial}{\partial x} \bar{G}(x, y) r_{\tilde{\Phi}}(x, y) - \frac{\partial}{\partial y} \bar{G}(x, y)\right]$$

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Therefore:

$$\eta_\nu(x, y) = h^x(x, y) r_\nu(x, y) + h^y(x, y) = \mathbf{r}_\nu(x, y) \cdot \mathbf{h}(x, y).$$

Attained service transport equation

- We now have all ingredients to formulate the dynamics of the system.

- The transport equation in the elapsed service space is (informally):

$$\frac{\partial \bar{f}}{\partial t} + \nabla \cdot \left[\mathbf{r}_{\nu_t} \bar{f}\right] + [\mathbf{r}_{\nu_t} \cdot \mathbf{h}]\bar{f} = \lambda \delta_{(0,0)}.$$

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  where $\tilde{f}$ is the density of ν_t.

- The above equation must be treated in weak form:
  - To account for the impulse mass at $(0,0)$ driving the system.
  - To allow solutions without a density as we shall see.

Last come first served
Fluid equilibrium

Recall that LCFS can be modeled by:

$$r_\nu(x, y) = \mathbf{1}_{\{y<y_\nu\}}$$

with

$$y_\nu = \sup\{y \geq 0 : \nu(\mathbb{R}_+ \times [0, y]) \leqslant C\}.$$

Recall that LCFS can be modeled by:

$$r_\nu(x,y) = \mathbf{1}_{\{y<y_\nu\}}$$

with

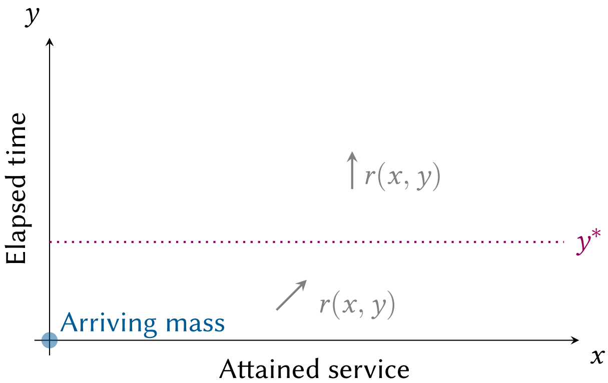
$$y_\nu = \sup\left\{y \geq 0 : \nu(\mathbb{R}_+ \times [0,y]) \leqslant C\right\}.$$

Imposing equilibrium, ν^*, y^* fixed, we have to solve:

$$\nabla \cdot \left[\mathbf{r}_{\nu^*}\bar{f}\right] + [\mathbf{r}_{\nu^*} \cdot \mathbf{h}]\bar{f} = \lambda\delta_{(0,0)}.$$

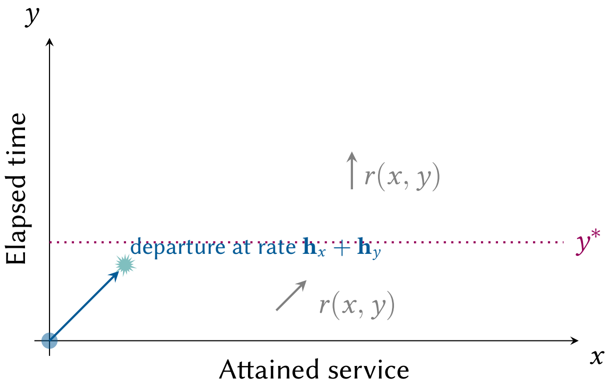
Solving the transport equation

Last come first served case



Solving the transport equation

Last come first served case



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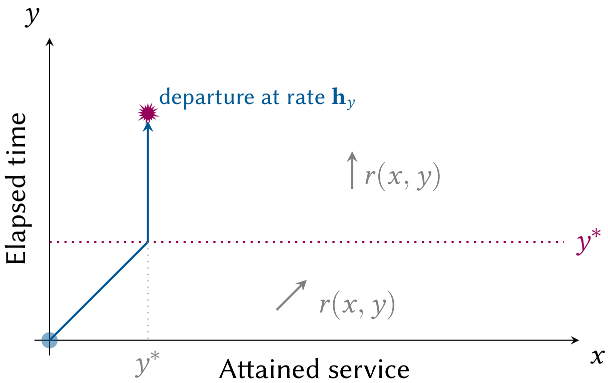

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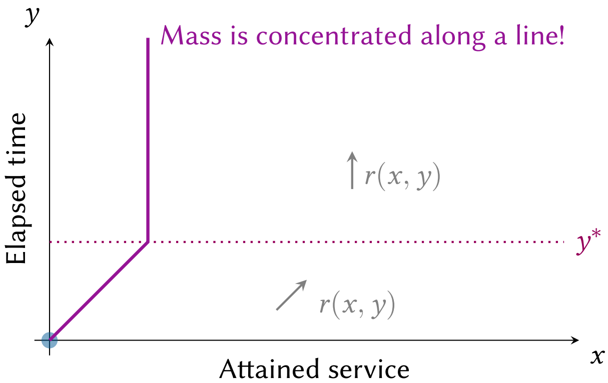


Deadline-oblivious policies in overload

Theorem

Assume that $\rho > C$ *and the equation*

$$\lambda E[\min\{S, T, z^*\}] = C$$

has a unique solution $z^* > 0$. *Consider the measure* ν^* *given by:*

$$\langle \varphi, \nu^* \rangle = \lambda \left[\int_0^{z^*} \varphi(u, u)\bar{G}(u, u)du + \int_{z^*}^{\infty} \varphi\left(z^*, u\right)\bar{G}(z^*, u)du \right],$$

for all $\varphi \in C_c(\mathbb{R}^2_+)$. *Then this measure is the equilibrium measure for both the Least-Attained-Service and Last-Come-First-Served policies.*

LAS/LCFS performance in equilibrium

Compute the rate at which mass leaves the system with less than x_0 attained service:

$$\iint_{[0,x_0]\times\mathbb{R}_+} \eta_{\nu^*}(x,y)\nu^*(dx,dy).$$

LAS/LCFS performance in equilibrium

Compute the rate at which mass leaves the system with less than x_0 attained service:

$$\iint_{[0,x_0]\times\mathbb{R}_+} \eta_{\nu^*}(x,y)\nu^*(dx,dy).$$

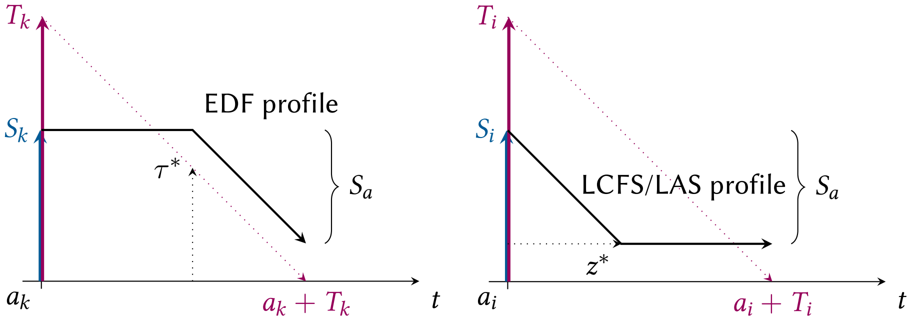
Proposition

Assume that $\rho > C$. Then

$$\int_{[0,x_0]\times\mathbb{R}_+} \left[h^x(x,y)\mathbf{1}_{\{y<z^*\}} + h^y(x,y)\right]\nu^*(dx,dy) = \lambda P\left(\min\{S,T,z^*\} \leqslant x_0\right).$$

So again the attained work is $S_a = \min\{S,T,z^*\}$!!

Graphical explanation



Since $\tau^* = x^* = y^* = z^*$, performance is the same in all three policies!!!

Outline

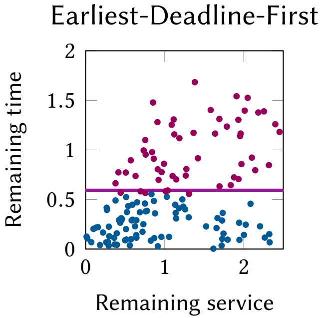
Simulations with correlated S and T

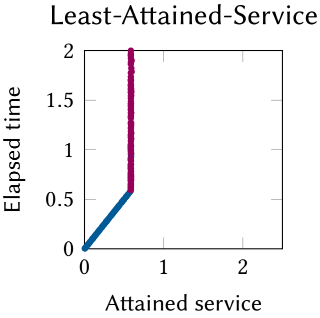
- We finally validate our fluid approximation by stochastic simulations
- In order to account for correlations, we take:

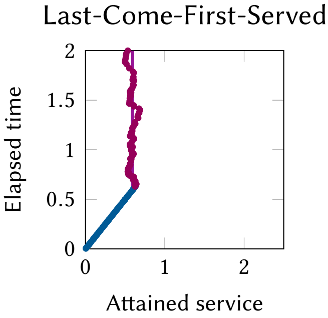
$$S = e^U \quad \text{and} \quad T = e^V \quad \text{with} \quad (U, V) \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0.9 \\ 0.9 & 1 \end{pmatrix} \right).$$

- In particular, the random variables U and V are correlated with normal distributions, and therefore S and T are correlated with log-normal distributions.
- In this case, $E[\min\{S, T\}] \approx 1.37$ can only be numerically estimated.
- We choose $\lambda = 200$ and $C = 100$, then $z^* \approx 0.593$.

State space snapshots

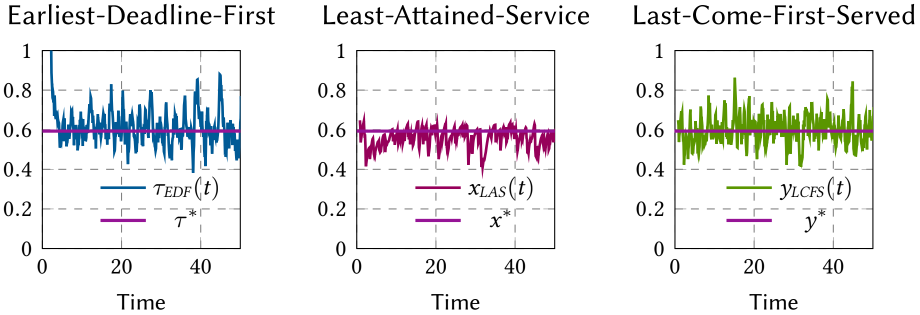




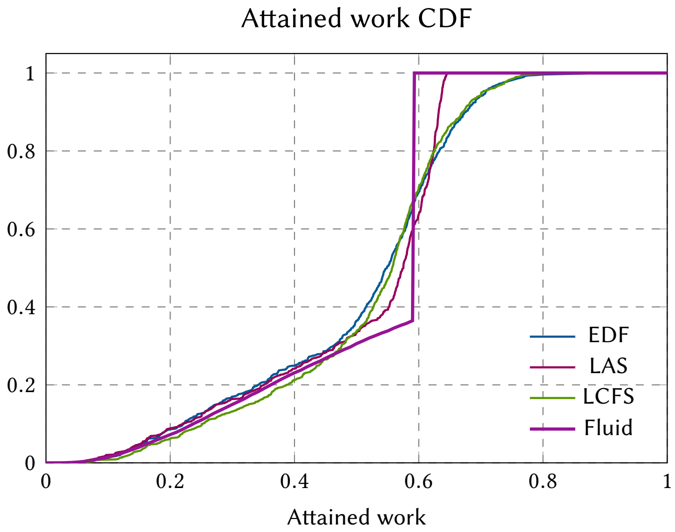


Blue dots are in service, red dots are not in service.

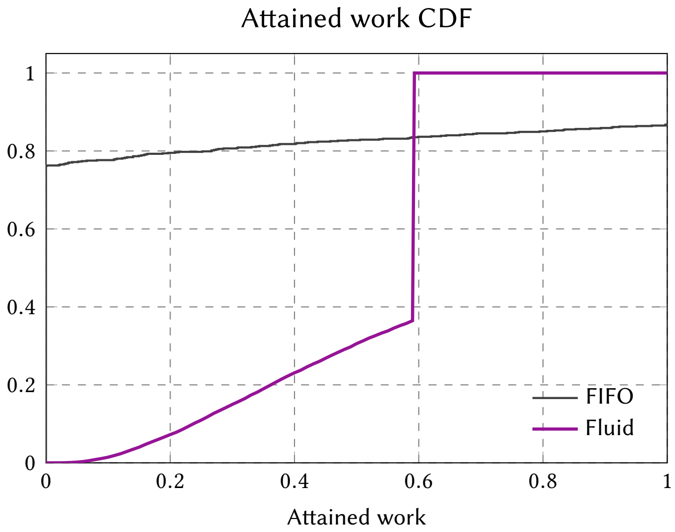
Stochastic threshold evolution



Attained work empirical CDF



Comparison with FIFO



Outline

Messages from the talk

- Measure-valued processes are a powerful tool to model general service queues.
- Partial service queues require two-dimensional measures.
- Our proposed dynamics for fluid models are tractable and approximate the real system.
- Last-but-not-least: in this setting, deadline-oblivious policies can be used without performance penalty!

Future work

- Analyze further policies using these tools (FCFS is easy for instance).
- Establish process-level convergence to the fluid models (long work...help needed...)
- Devise new policies and/or analyze different settings:
  - Tasks stay until service completion, but we want to measure the average *tardiness*, i.e. how late they depart.

Gracias!

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