

The last, the least and the urgent...

Fluid modeling and performance equivalence for scheduling policies in partial service queues with abandonment

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IFIP Performance 2025

Outline

Introduction

Partial service queues and Earliest-Deadline-First

Deadline-oblivious policies

Simulations

Final remarks

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Motivation

A bit of history...

- Several queueing systems have service and **timing** requirements.
- Examples:
 - Computing tasks with real-time constraints.
 - Item delivery problems in logistics.
 - Emergency response.
 - etc. etc. etc.
- This has led to a long and rich history of research about **queues with abandonments** [Barrer, 1957; Stanford, 1979; Baccelli et al., 1984].

Motivation

Recent developments...

One of the most used policies is **Earliest-Deadline-First (EDF)**

- Give priority to tasks with more urgent deadlines.

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Through fluid limits and diffusion approximations, establish performance:

- [Decreusefond and Moyal, 2008] establish EDF fluid limits in the single server case.
- [Kruk et al., 2011] provides diffusion approximations.
- [Moyal, 2013] establish some optimality properties of EDF.
- [Kang and Ramanan, 2010, 2012] analyze the many-server case.
- [Atar et al., 2018, 2023] establish asymptotic performance.

and many others...

Common assumption

Customers renege *only* in the queue, and not during service.

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We call this the *call-center scenario*:

- Akin to waiting for the customer-help line to pick your call while you listen to annoying music.
- The underlying idea is that when a task reaches service, it will stay until completion.

Key performance metric: number of satisfied tasks (or reneging probability).

Motivation

Partial service queues

In several queueing systems:

- Tasks may abandon during service.
- More importantly, **all service provided may be useful.**

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Some examples:

- Electrical vehicle charging: customers leave the system with a *partial charge*.
- LLM inference: longer computation times lead to better answers, but these may be interrupted to deliver a quick response.
- File transfers over the Internet, that can be resumed later.

Key points of this talk

- Provide some suitable representation of the state space and dynamics of these partial service queues.
- Analyze several interesting policies under a suitable fluid model.
- Compute the main performance metric here: [attained work](#).
- *Last but not least*: show that the simple LCFS policy [exhibits the same performance](#) than EDF in this setting, without using deadline information.

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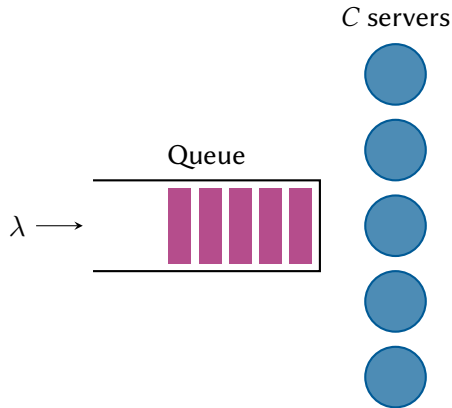
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Partial service queues

Setting

Consider an $M/G/C$ system where:

- Tasks arrive as a Poisson process of intensity λ .

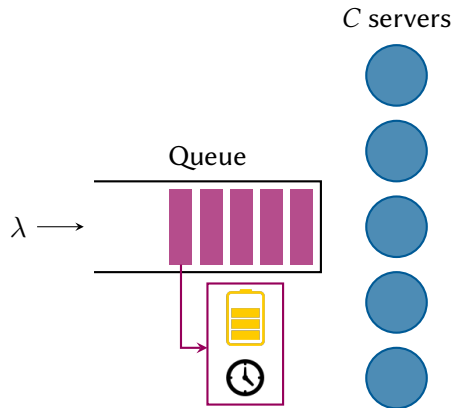


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- Each task i has two characteristics (marks):
 - S_i : service time (at rate 1).
 - T_i : sojourn time or deadline.

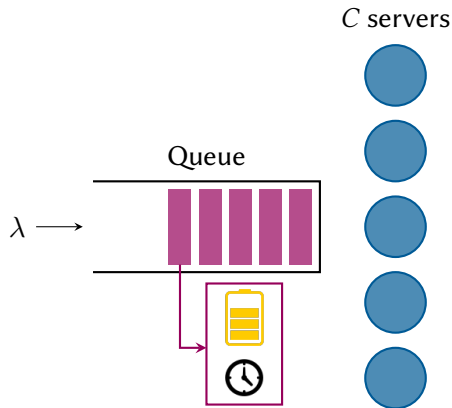


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- Tasks arrive as a Poisson process of intensity λ .
- Each task i has two characteristics (marks):
 - S_i : service time (at rate 1).
 - T_i : sojourn time or deadline.
- (S_i, T_i) are independent across jobs.
- Follow a common distribution $G(\sigma, \tau)$, possibly correlated.



Partial service queues

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Customers depart whenever S_i is attained or the timer T_i expires.

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 - Equivalently, $S_r := S - S_a$, amount of service **reneged**.

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- In particular, they may leave **during service**.
- Key performance metrics:
 - S_a : amount of service **attained**.
 - Equivalently, $S_r := S - S_a$, amount of service **reneged**.
- **Problem**: we have to keep track of remaining service and deadlines simultaneously!

- Before proceeding, it is useful to define the **system load**:

$$\rho := \lambda E[\min\{S, T\}].$$

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- **Interpretation**: the mean number of customers on a system with $C = \infty$.
- What we expect in a large scale fluid model:
 - If $\rho < C$ (underload), all tasks can be served, $S_a = \min\{S, T\}$.
 - If $\rho > C$ (overload), demand *curtailing* will occur. How? It depends on the policy...

System evolution

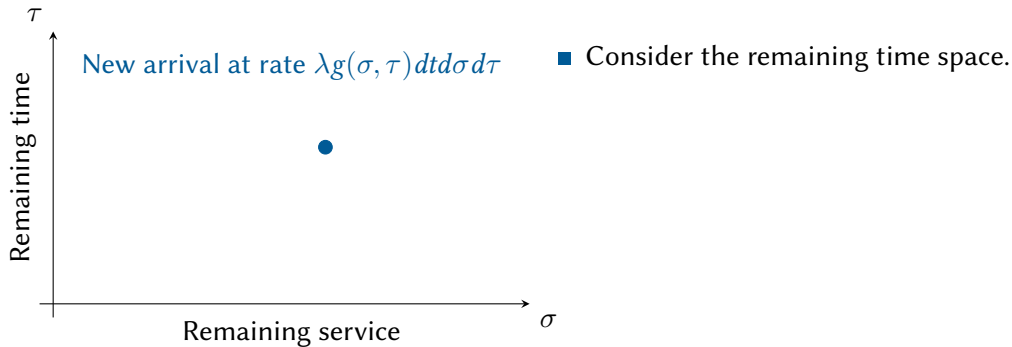
Remaining service times



- Consider the remaining time space.

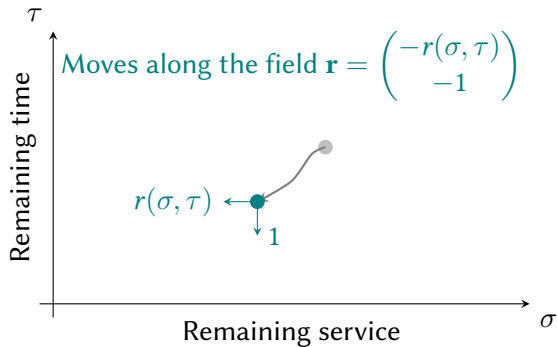
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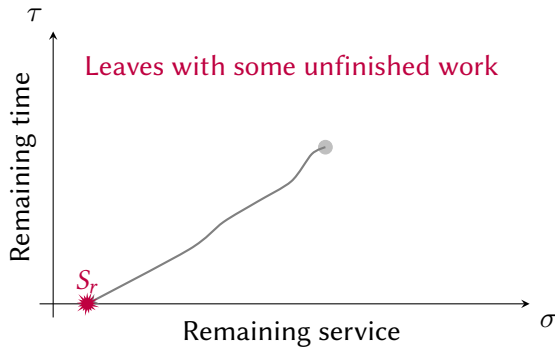
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- May depend on any combination of (σ, τ) .

System evolution

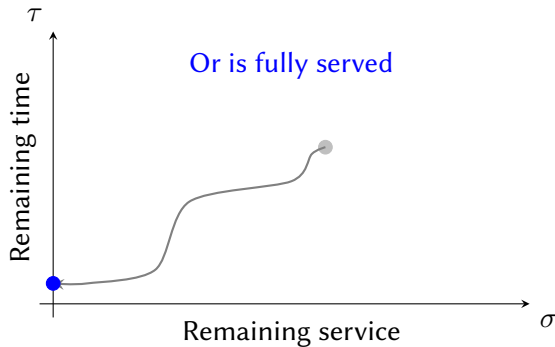
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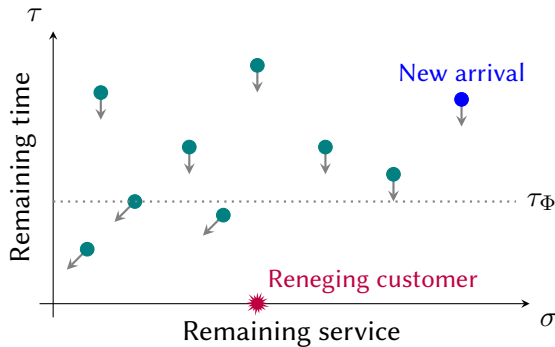
Remaining service times



- Consider the remaining time space.
- **Policy** defines how tasks are served.
- May depend on any combination of (σ, τ) .
- State descriptor:

$$\Phi_t = \sum_i \delta_{(\sigma_i(t), \tau_i(t))}$$

Example: Earliest-deadline-first



New arrival

- Serve the C most urgent customers.
- Corresponds to taking:

$$r_\Phi(\sigma, \tau) = \mathbf{1}_{\{\tau \leq \tau_\Phi\}}$$

with

$$\tau_\Phi := \sup\{\tau \geq 0 : \Phi(\mathbb{R}_+ \times (0, \tau]) < C\}.$$

- Replace Φ_t by a (fluid) measure μ_t .
- Now mass drifts along the field:

$$\mathbf{r}_\mu(\sigma, \tau) = \begin{pmatrix} -r_\mu(\sigma, \tau) \\ -1 \end{pmatrix}$$

- With r_μ satisfying:

$$0 \leq r_\mu \leq 1$$
$$\iint r_\mu(\sigma, \tau) \mu(d\sigma, d\tau) \leq \min\{\mu(\mathbb{R}_{++}^2), C\}.$$

We will describe these dynamics in terms of the projections

$$\langle \varphi, \mu \rangle := \iint \varphi(\sigma, \tau) \mu(d\sigma, d\tau)$$

of the state measure with respect to a test function $\varphi : \mathbb{R}_{++}^2 \rightarrow \mathbb{R}$, with continuous derivatives and compact support, i.e. $\varphi \in \mathcal{C}_c^1(\mathbb{R}_{++}^2)$.

We have:

$$\langle \varphi, \mu_{t+dt} \rangle = \iint \varphi(\sigma - r_{\mu_t} dt, \tau - dt) \mu_t(d\sigma, d\tau) + \lambda dt \iint \varphi(\sigma, \tau) g(\sigma, \tau) d\sigma d\tau + o(dt).$$

Fluid model dynamics

Weak formulation

$$\begin{aligned}\frac{\partial}{\partial t} \langle \varphi, \mu_t \rangle &= \lim_{dt \rightarrow 0} \iint \frac{1}{dt} [\varphi(\sigma - r_{\mu_t} dt, \tau - dt) - \varphi(\sigma, \tau)] \mu_t(d\sigma, d\tau) \\ &\quad + \lambda \iint \varphi(\sigma, \tau) g(\sigma, \tau) d\sigma d\tau \\ &= - \iint [r_{\mu_t}(\sigma, \tau) \varphi_\sigma(\sigma, \tau) + \varphi_\tau(\sigma, \tau)] \mu_t(d\sigma, d\tau) + \lambda \iint \varphi(\sigma, \tau) g(\sigma, \tau) d\sigma d\tau,\end{aligned}$$

Equivalently:

$$\begin{aligned} \langle \varphi, \mu_t \rangle = \langle \varphi, \mu_0 \rangle + \int_0^t \left[- \iint [r_{\mu_s}(\sigma, \tau) \varphi_\sigma(\sigma, \tau) + \varphi_\tau(\sigma, \tau)] \mu_t(d\sigma, d\tau) \right. \\ \left. + \lambda \iint \varphi(\sigma, \tau) g(\sigma, \tau) d\sigma d\tau \right] ds, \end{aligned}$$

for any $\varphi \in \mathcal{C}_c^1(\mathbb{R}_{++}^2)$.

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Looks daunting, but is not that bad...

If μ_t admits a density $f(\sigma, \tau; t)$ with respect to the Lebesgue measure, it corresponds to:

$$\frac{\partial f}{\partial t} + \nabla \cdot [\mathbf{r}_{\mu_t} f] = \lambda g$$

a transport equation.

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Example: EDF

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \sigma} \mathbf{1}_{\{\tau < \tau_{\mu_t}\}} + \frac{\partial f}{\partial \tau} + \lambda g$$

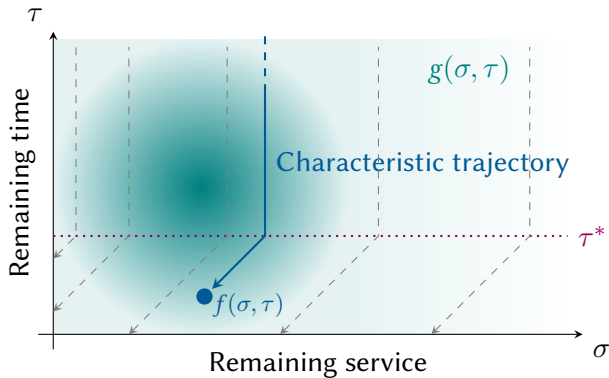
Imposing equilibrium we get:

- $\tau_{\mu^*} = \tau^*$ becomes a constant.
- The measure μ^* must satisfy:

$$\frac{\partial f}{\partial \sigma} \mathbf{1}_{\{\tau < \tau^*\}} + \frac{\partial f}{\partial \tau} + \lambda g = 0.$$

- Linear PDE that can be easily solved by the method of characteristics.

Solving the EDF transport equation



Theorem

Assume that $\rho > C$ and the equation

$$\lambda E[\min\{S, T, \tau^*\}] = C$$

has a unique solution $\tau^ > 0$. Consider the measure μ^* given by the following density:*

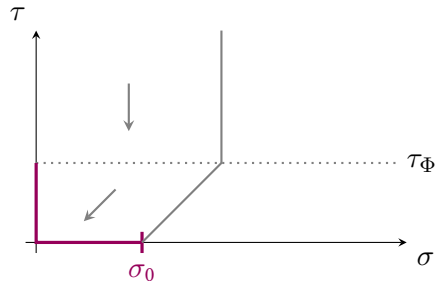
$$f(\sigma, \tau) = \lambda \left[\int_0^{(\tau^* - \tau)^+} g(\sigma + u, \tau + u) du + \int_{(\tau^* - \tau)^+}^{\infty} g(\sigma + (\tau^* - \tau)^+, \tau + u) du \right].$$

This measure is a fluid equilibrium for the EDF policy, and

$$\tau^* = \sup \{ \tau \geq 0 : \mu^*(\mathbb{R}_{++} \times (0, \tau]) \leq C \}.$$

EDF performance in equilibrium

- Let us compute the rate at which work is **reneged**.
- Compute the rate at which mass exits with $S_r < \sigma_0$.



Proposition

$$\int_0^{\tau^*} f(0, \tau) d\tau + \int_0^{\sigma_0} f(\sigma, 0) d\sigma = \lambda P(S - \min\{S, T, \tau^*\} < \sigma_0).$$

i.e. $S_a = S - S_r = \min\{S, T, \tau^*\}.$

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What if we do not know the deadlines?

- Deadlines are often hard to estimate in practice.
- Moreover, tasks may under-report their deadline to get priority!
- What about **deadline-oblivious** policies?
 - Can we model them?
 - What is their performance?

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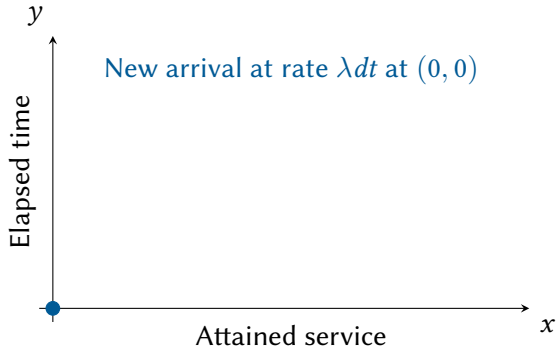
Problem: we need a new state-space...

Attained service state descriptor



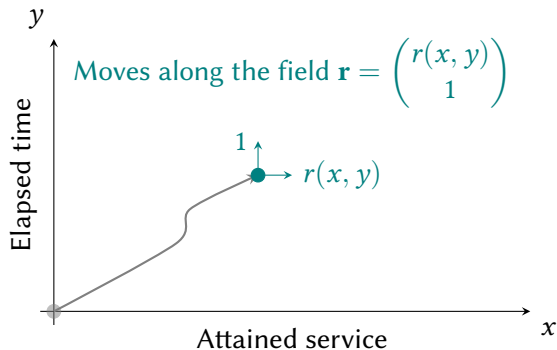
- Consider the elapsed time space.

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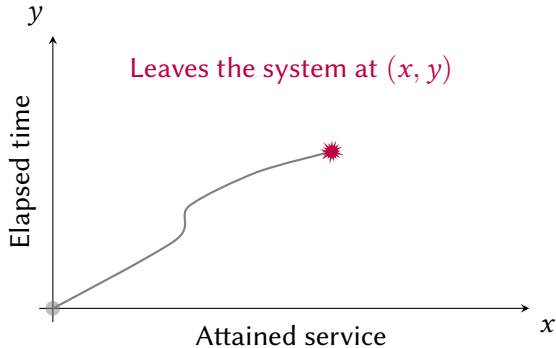
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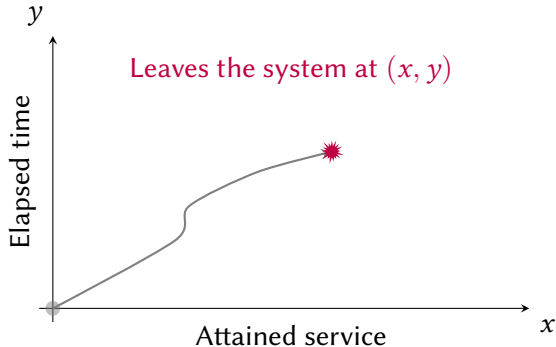
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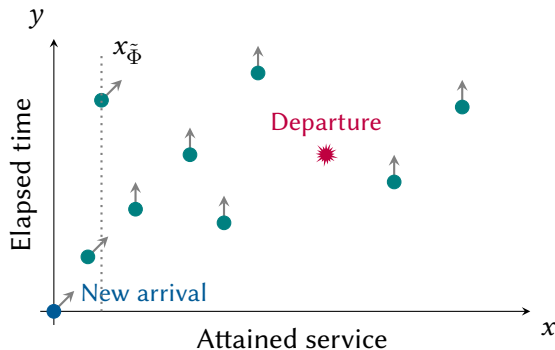
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$$\tilde{\Phi}_t = \sum_i \delta_{(x_i(t), y_i(t))}$$

Example: Least-Attained-Service policy



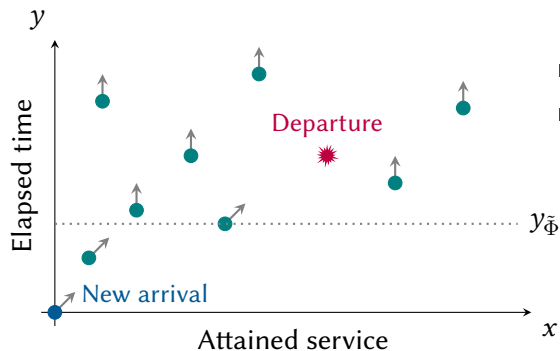
- Serve the C least-served tasks.
- Corresponds to taking:

$$r_{\tilde{\Phi}}(x, y) = \mathbf{1}_{\{x \leq x_{\tilde{\Phi}}\}}$$

with

$$x_{\tilde{\Phi}} := \sup\{x : \tilde{\Phi}([0, x] \times \mathbb{R}_+) \leq C\}.$$

Example: Last-Come-First-Served policy



■ Serve the C more recent tasks.

■ Corresponds to taking:

$$r_{\tilde{\Phi}}(x, y) = \mathbf{1}_{\{y \leq y_{\tilde{\Phi}}\}}$$

with

$$y_{\tilde{\Phi}} := \sup\{y : \tilde{\Phi}(\mathbb{R}_+ \times [0, y]) \leq C\}$$

The hazard rate field

We have a new problem: what is the rate at which users **leave** the system?

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Let $\bar{G}(x, y) = P(S > x, T > y)$ and define:

Definition (Hazard rate field)

$$\mathbf{h}(x, y) = -\nabla \log \bar{G}(x, y) \quad \text{i.e.}$$

- $h^x(x, y) = P(S \in [x, x + dx], T > S \mid S > x, T > y)$
- $h^y(x, y) = P(T \in [y, y + dy], S > T \mid S > x, T > y)$

Interpretation: \mathbf{h} stores the rate at which $\min\{S, T\}$ is attained due to S or T expiring.

- Replace $\tilde{\Phi}_t$ by a (fluid) measure ν_t .
- Now mass arrives at $(0, 0)$ at rate λ .
- Drifts along the field:

$$\mathbf{r}_\nu(x, y) = \begin{pmatrix} r_\nu(x, y) \\ 1 \end{pmatrix}$$

- With r_ν satisfying:

$$0 \leq r_\nu \leq 1$$

$$\iint r_\nu(x, y) \nu(dx, dy) \leq \min\{\nu(\mathbb{R}_+^2), C\}.$$

Now we have to compute the departure rate $\eta_\nu(x, y)$:

$$\eta_\nu(x, y) := \lim_{dt \rightarrow 0} \frac{1}{dt} P(\{S \in (x, x + r_{\tilde{\Phi}} dt)\} \cup \{T \in (y, y + dt)\} \mid S > x, T > y)$$

By the chain rule and some computations:

$$\eta_\nu(x, y) = \frac{1}{\bar{G}(x, y)} \left[-\frac{\partial}{\partial x} \bar{G}(x, y) r_{\tilde{\Phi}}(x, y) - \frac{\partial}{\partial y} \bar{G}(x, y) \right]$$

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Therefore:

$$\eta_\nu(x, y) = h^x(x, y) r_\nu(x, y) + h^y(x, y) = \mathbf{r}_\nu(x, y) \cdot \mathbf{h}(x, y).$$

Attained service transport equation

- We now have all ingredients to formulate the dynamics of the system.
- The transport equation in the elapsed service space is (informally):

$$\frac{\partial \bar{f}}{\partial t} + \nabla \cdot [\mathbf{r}_{\nu_t} \bar{f}] + [\mathbf{r}_{\nu_t} \cdot \mathbf{h}] \bar{f} = \lambda \delta_{(0,0)}.$$

where \tilde{f} is the density of ν_t .

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where \tilde{f} is the density of ν_t .

- The above equation must be treated in weak form:
 - To account for the impulse mass at $(0, 0)$ driving the system.
 - To allow solutions without a density as we shall see.

Last come first served

Fluid equilibrium

Recall that LCFS can be modeled by:

$$r_\nu(x, y) = \mathbf{1}_{\{y < y_\nu\}}$$

with

$$y_\nu = \sup \{y \geq 0 : \nu(\mathbb{R}_+ \times [0, y]) \leq C\}.$$

Last come first served

Fluid equilibrium

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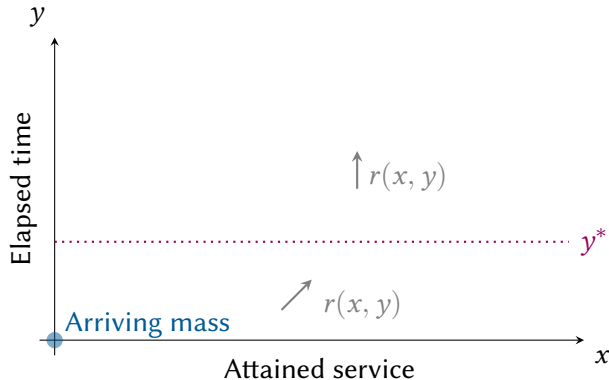
$$y_\nu = \sup \{y \geq 0 : \nu(\mathbb{R}_+ \times [0, y]) \leq C\}.$$

Imposing equilibrium, ν^* , y^* fixed, we have to solve:

$$\nabla \cdot [\mathbf{r}_{\nu^*} \bar{f}] + [\mathbf{r}_{\nu^*} \cdot \mathbf{h}] \bar{f} = \lambda \delta_{(0,0)}.$$

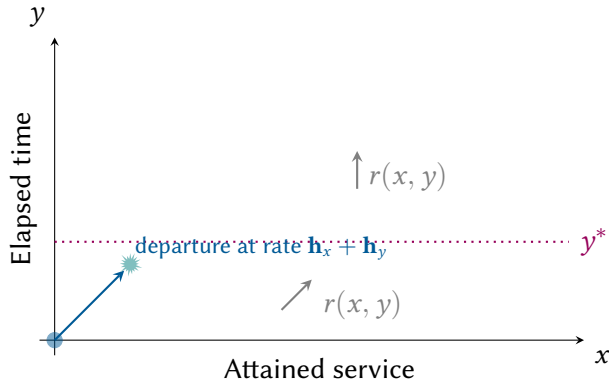
Solving the transport equation

Last come first served case



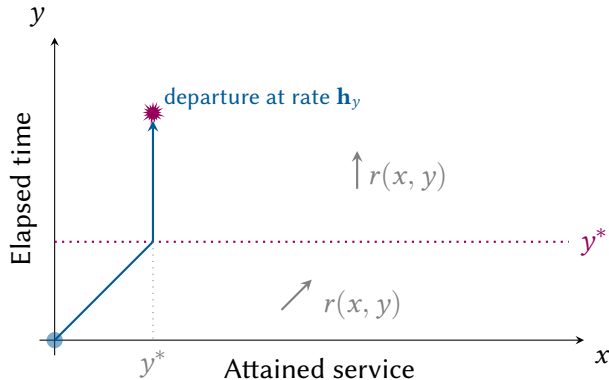
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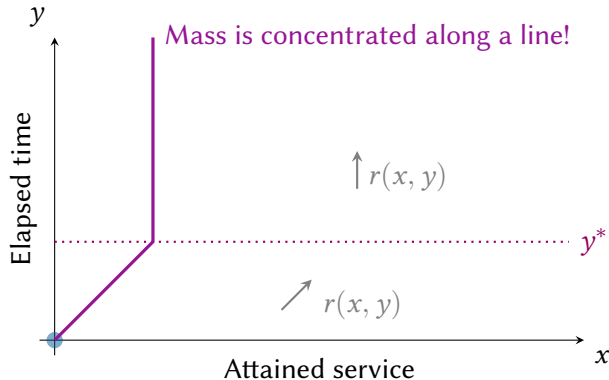
Solving the transport equation

Last come first served case



Solving the transport equation

Last come first served case



Deadline-oblivious policies in overload

Theorem

Assume that $\rho > C$ and the equation

$$\lambda E[\min\{S, T, z^*\}] = C$$

has a unique solution $z^ > 0$. Consider the measure ν^* given by:*

$$\langle \varphi, \nu^* \rangle = \lambda \left[\int_0^{z^*} \varphi(u, u) \bar{G}(u, u) du + \int_{z^*}^{\infty} \varphi(z^*, u) \bar{G}(z^*, u) du \right],$$

for all $\varphi \in C_c(\mathbb{R}_+^2)$. Then this measure is the equilibrium measure for both the Least-Attained-Service and Last-Come-First-Served policies.

LAS/LCFS performance in equilibrium

Compute the rate at which mass leaves the system with less than x_0 attained service:

$$\iint_{[0, x_0] \times \mathbb{R}_+} \eta_{\nu^*}(x, y) \nu^*(dx, dy).$$

LAS/LCFS performance in equilibrium

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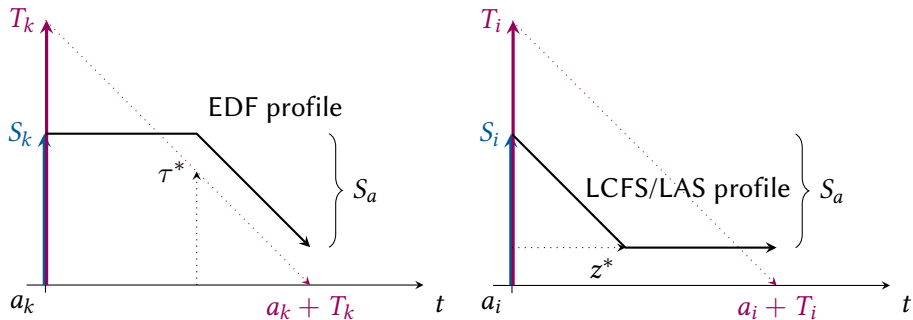
Proposition

Assume that $\rho > C$. Then

$$\int_{[0, x_0] \times \mathbb{R}_+} [h^x(x, y) \mathbf{1}_{\{y < z^*\}} + h^y(x, y)] \nu^*(dx, dy) = \lambda P(\min\{S, T, z^*\} \leq x_0).$$

So again the attained work is $S_a = \min\{S, T, z^*\}!!$

Graphical explanation



Since $\tau^* = x^* = y^* = z^*$, performance is the same in all three policies!!!

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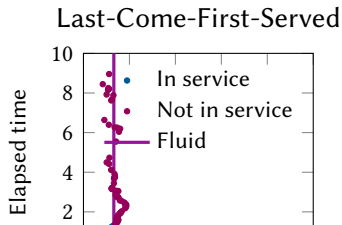
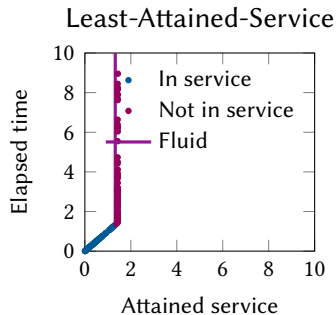
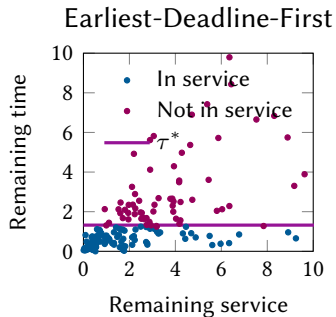
Simulations with correlated S and T

- We finally validate our fluid approximation by stochastic simulations
- In order to account for correlations, we take:

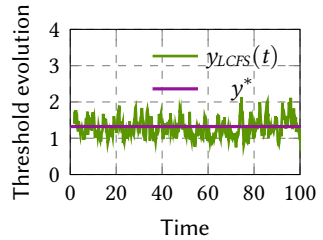
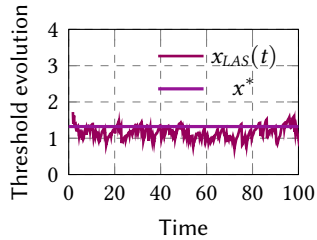
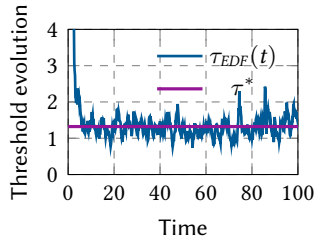
$$S = e^U \quad \text{and} \quad T = e^V \quad \text{with} \quad (U, V) \sim \mathcal{N} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0.9 \\ 0.9 & 1 \end{pmatrix} \right).$$

- In particular, the random variables U and V are correlated with normal distributions, and therefore S and T are correlated with log-normal distributions.
- In this case, $E[\min\{S, T\}] \approx 1.37$ can only be numerically estimated.
- We choose $\lambda = 200$ and $C = 100$, then $z^* \approx 0.593$.

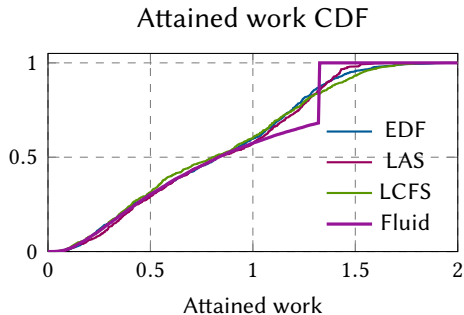
State space snapshots



Stochastic threshold evolution

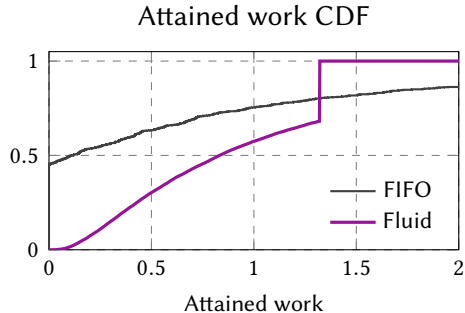


Attained work empirical CDF



Even in the pre-limit system, performance is similar!

Comparison with FIFO



Almost 80% of the jobs leave without service.

Outline

Introduction

Partial service queues and Earliest-Deadline-First

Deadline-oblivious policies

Simulations

Final remarks

- Measure-valued processes are a powerful tool to model general service queues.
- Partial service queues require two-dimensional measures.
- Our proposed dynamics for fluid models are tractable and approximate the real system.
- Last-but-not-least: in this setting, **deadline-oblivious** policies can be used without performance penalty!

- Analyze further policies using these tools (FCFS is easy for instance).
- Establish process-level convergence to the fluid models (long work...help needed...)
- Devise new policies and/or analyze different settings:
 - Tasks stay until service completion, but we want to measure the average *tardiness*, i.e. how late they depart.

Thank you!

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<https://aferragu.github.io>

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