The caching problem under a point process perspective

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Outline

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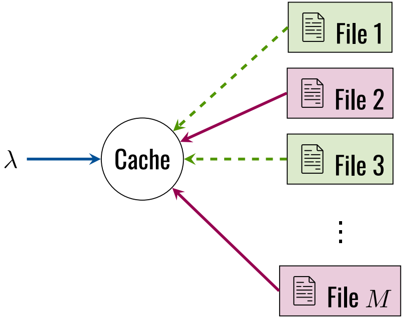
The caching problem

- Consider a cache system with a catalog of M objects.
- Requests for objects arrive at random.
- The cache can locally store $C < M$ of them.
- If item is in cache, we have a hit. Otherwise, it is a miss.



Objective: for a given arrival stream, maximize the steady-state hit rate.

A sequential approach

- Consider a sequence of random variables $Z_1, Z_2, \ldots$ with values in $\{1, \ldots, M\}$.
- Consider also the set:

$$\mathcal{C} = \{\{i_1, \ldots, i_k\} \subset \{1, \ldots, M\}, k \leqslant C\}$$

- A (causal) caching policy would be a sequence of maps π_n deciding which contents to store:

$$\pi_n(Z_1, \ldots, Z_{n-1}) \to \mathcal{C}$$

- In probabilistic terms, let $\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n)$, then π_n is any $\mathcal{C}-$valued $\mathcal{F}_n-$predictable process ($\mathcal{F}_{n-1}$-measurable).

A simple case

The Independent Reference Model (IRM)

- Assume now that Z_n are iid with distribution $p_i = P(Z_n = i)$, where p_i is the popularity of content i. Wlog, we take $p_1 \geqslant p_2 \geqslant \ldots$.

- In this case, $Z_n \mid \mathcal{F}_{n-1} \sim p$, thus the hit probability at time n is:

$$P(Z_n \in \pi_n) = E\left[\mathbf{1}_{Z_n \in \pi_n}\right] = E\left[E\left[\mathbf{1}_{Z_n \in \pi_n} \mid \mathcal{F}_{n-1}\right]\right] = E\left[\sum_{i \in \pi_n} p_i\right] \leqslant \sum_{i=1}^{C} p_i$$

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Conclusion: under iid requests, the static “keep the most popular” policy is optimal.

Practical policies: LFU and LRU

In practice, popularities are not known. This leads to the least-frequently-used (LFU) eviction policy:

- Take π_n as the most requested objects so far (remove the least frequently used).
- In the long range, converges to the static policy.

Another popular eviction policy is least-recently-used (LRU), which treats π_n as a list defined recursively:

- If $Z_n \in \pi_n$, serve the content, move Z_n to the front of the list.
- If $Z_n \notin \pi_n$, fetch the content, put Z_n in the front of the list, remove the last object in the list (which is the least recently requested).

Beyond the IRM

- Typically, requests are correlated, and popularities evolve over time.
- For instance, requests for a file may arrive in bursts.
- LRU adapts to changes in popularity. Is good for bursts of requests. Tons of literature on this policy (also called move-to-front).
- However, performance metrics and optimality results are hard to establish.

The caching problem, take 2

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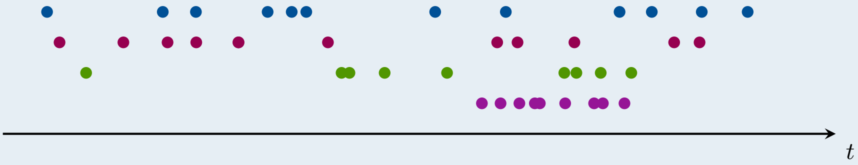
Point process approach [Fofack et al. 2014]:

- Assume requests for item i come from a point process of intensity $\lambda_i := \lambda p_i$.



- At each point in time we must decide which items must be stored locally.

If inter-request times are heavy tailed, this can model burstiness.

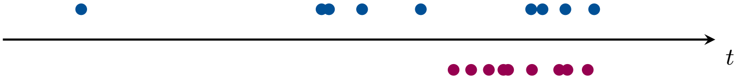
Example: Pareto arrivals

Consider two items, with equal popularity...

- Poisson arrivals:

Homogeneous

- Heavy tailed arrivals (Pareto $\alpha = 2$):



Bursty!

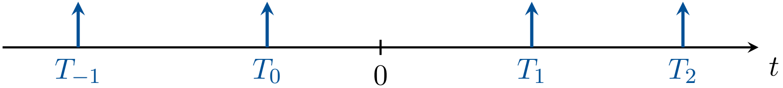
Some open questions...

- What is the optimal causal policy in this framework?
- Can we compute the optimal hit rate/hit probability?
- What is its large scale behavior?
- How typical policies compare to the optimal one?

Outline

A bit of point process theory...

Let $N = \{T_k : k \in \mathbb{Z}\}$ be a stationary point process representing request times:



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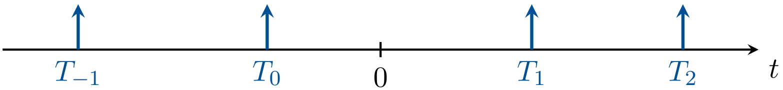

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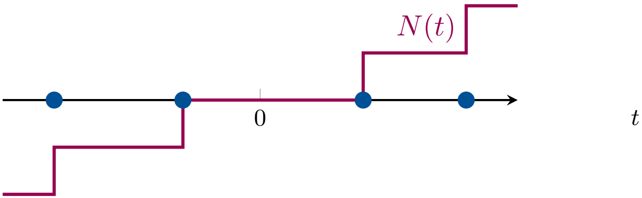
Counting process:

$$N(t) = \begin{cases} N([0,t]) & t \geqslant 0 \\ -N((t,0)) & t < 0 \end{cases}$$



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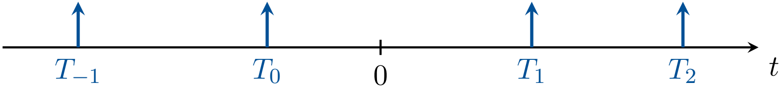

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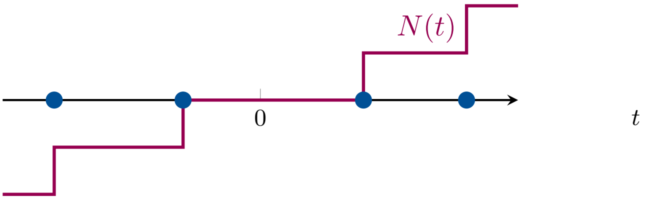
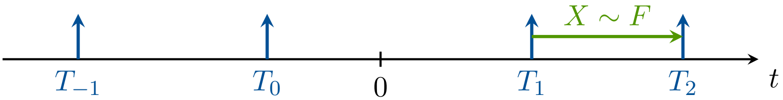


Let $\mathcal{F}_t = \sigma(N(s), s \leqslant t)$ be its internal history.

Two important distributions:



Inter-arrival distribution: $F(t) := P_N^0(T_1 - T_0 \leqslant t), \quad E_N^0[T_1] = 1/\lambda.$

Note: here P_N^0 is the Palm probability of the point process (conditioning on $T_0 = 0$).

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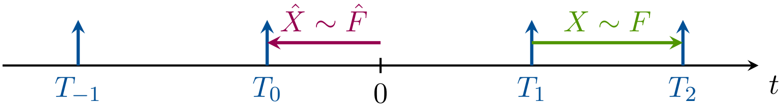

Inter-arrival distribution: $F(t) := P_N^0(T_1 - T_0 \leqslant t), \quad E_N^0[T_1] = 1/\lambda.$

Age distribution: $\hat{F}(t) := P(-T_0 \leqslant t) = \lambda \int_0^t 1 - F(s) ds,$

Note: here P_N^0 is the Palm probability of the point process (conditioning on $T_0 = 0$).

Stochastic intensity

Consider a simple stationary point process N with intensity λ, defined in some probability space $(\Omega, \mathcal{F}, P)$. Let some filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ be a history of the process.

Definition:

The random process $\lambda(t) \geqslant 0$ is a stochastic intensity for the history $\mathcal{F}_t$ iff it is a.s. locally integrable, $\mathcal{F}_t-$adapted and:

$$E\left[N((a,b]) \mid \mathcal{F}_a\right] = E\left[\int_a^b \lambda(t)dt \middle| \mathcal{F}_a\right]$$

for all $a, b \in \mathbb{R}$.

Local interpretation:

$$E[N((t,t+h]) \mid \mathcal{F}_t] = \lambda(t)h + o(h) \quad P-a.s.,$$

So $\lambda(t)$ acts as a local notion of intensity based on previous history.

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Martingale interpretation:

$$M_a(t) = N(t) - N(a) - \int_a^t \lambda(s)ds$$

is a local $(P, \mathcal{F}_t)$ martingale for any $a \in \mathbb{R}$.

Namely, $A(t) = N(a) + \int_a^t \lambda(s)ds$ is the compensator of the counting process.

Stochastic intensity of a Poisson process

- If $N(t)$ is a Poisson process, then we know that

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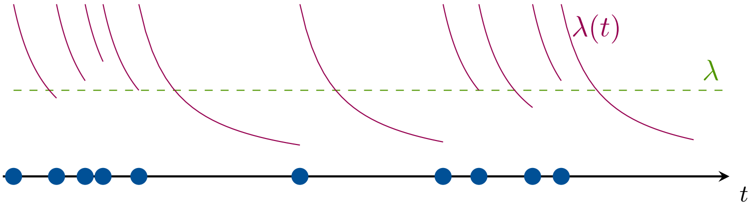
- In fact, this characterizes the Poisson process. The stochastic intensity $\lambda(t)$ is deterministic if and only if N is a Poisson process of (possible time-varying) intensity $\lambda(t)$.

Stochastic intensity

A local notion of intensity...

However, if traffic is bursty, the stochastic intensity rises after arrivals:



Note: for stationary processes, $E[\lambda(t)] = E[\lambda(0)] = \lambda$, the average intensity.

Renewal processes

- Let now N be a stationary renewal process, i.e. inter request times $T_{n+1} - T_n$ are $iid \sim F$.
- Assume that F has a density, and define the hazard rate of F as:

$$\eta(t) = \frac{f(t)}{1 - F(t)}$$

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Theorem (Daley-Vere Jones, Chapter 7)

For a renewal process and its natural history, the stochastic intensity is:

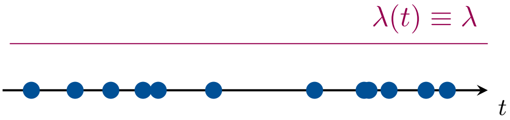
$$\lambda(t) = \eta(t - T^*(t)),$$

where

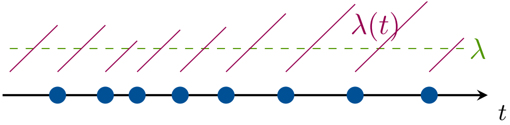
$$T^*(t) = \sup\{T_n : T_n < t\}$$

is the last point before t.

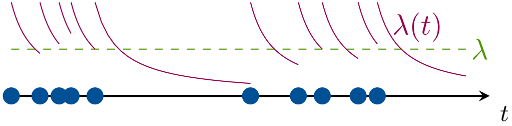
Some examples...



Constant hazard rate $\rightarrow$ Poisson process.



Increasing hazard rate $\rightarrow$ more periodic!



Decreasing hazard rate $\rightarrow$ more bursty!

Outline

The predictable $\sigma-$algebra

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in \mathbb{R}\}, P)$ be a filtered probability space.

Definition

The predictable $\sigma-$algebra $\mathcal{P}(\mathcal{F}.)$ is the $\sigma-$álgebra in $\mathbb{R} \times \Omega$ generated by the sets:

$$(a, b] \times A, \; a < b, \; A \in \mathcal{F}_a,$$

Predictable processes

Definition (Predictable process)

A stochastic process $X(t,\omega)$ taking values on a measurable space $(E,\mathcal{E})$ is $\mathcal{F}_t-$predictable if the mapping $(t,\omega)\mapsto X(t,\omega)$ is $\mathcal{P}(\mathcal{F}.)-$measurable.

- **Key idea:** a process is $\mathcal{F}_t-$predictable if its value at t is completely determined by the information prior to t.

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- Key idea: a process is $\mathcal{F}_t-$predictable if its value at t is completely determined by the information prior to t.
- In particular $\mathcal{F}_t-$adapted $+$ left continuous $\implies$ $\mathcal{F}_t$-predictable.
- Since the stochastic intensity of a point process can be chosen left-continuous, it is $\mathcal{F}_t$-predictable.

Causal caching policies

- Consider again a cache system fed by M independent request processes $N_i(t)$ with stochastic intensities $\lambda_i(t)$.
- Let $\mathcal{F}_t = \sigma(\{\mathcal{F}_t^{(i)} : i = 1, \ldots, M\})$ their aggregate history.

Definition

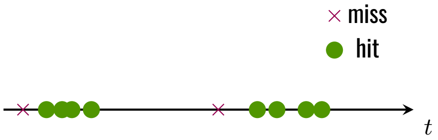
A causal caching policy is an $\mathcal{F}_t$ predictable stochastic process

$$\pi(t) : \Omega \times \mathbb{R} \to \mathcal{C}$$

i.e. $\pi(t) = \{i_1, \ldots, i_k\}$ (with $k \leqslant C$) is the subset kept at time t, and only depends on the past history of item requests.

Focus now on a particular content i, its hit process is the point process given by:

$$H_i(B) = \sum_{n \in \mathbb{Z}} \mathbf{1}_{\{T_n^i \in B\}} \mathbf{1}_{\{i \in \pi(T_n^i)\}}$$



Now $\mathbf{1}_{\{i \in \pi(t)\}}$ is $\mathcal{F}_t$-predictable, so the stochastic intensity of H_i is:

$$h_i(t) = \lambda_i(t) \mathbf{1}_{\{i \in \pi(t)\}}$$

i.e., $h_i(t) = \lambda_i(t)$ while i is cached and otherwise 0.

If we now consider the aggregate of requests, the total hit process is given by:

$$H = \sum_{i=1}^{M} H_i$$

And its stochastic intensity is just:

$$h(t) = \sum_{i=1}^{M} h_i(t) = \sum_{i=1}^{M} \lambda_i(t) \mathbf{1}_{\{i \in \pi(t)\}}$$

The steady state hit rate of the policy is:

$$\text{hit rate} = \lambda_{\text{hit}} := E[h(t)]$$

Maximizing the hit rate

In order to maximize λ_{hit}, consider the causal policy:

$$\pi^*(t) = \{i_1, \ldots, i_C\} \quad \text{such that} \quad \sum_{i \in \{i_1, \ldots, i_C\}} \lambda_i(t) \text{ is maximized.}$$

Then, for any causal policy π and for each realization:

$$h(t) = \sum_{i \in \pi(t)} \lambda_i(t) \leqslant \sum_{i \in \pi^*(t)} \lambda_i(t) = h^*(t).$$

Theorem

The optimal causal policy is to keep in the cache the C objects with the highest stochastic intensity at any time.

Back to the Poisson case

- Assume the N_i are Poisson processes of intensities λ_i.
- We take $\lambda_1 > \lambda_2 > \ldots$ as the popularities.
- The total request process is also Poisson of intensity $\sum_i \lambda_i$.
- In that case, the optimal policy is:

$$\pi^*(t) \equiv \{1, \ldots, C\}$$

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Conclusion: under Poisson arrivals, statically keeping the most popular objects is optimal (compare to the IRM before).

The renewal case

- If now the N_i are renewal processes of (decreasing) intensities λ_i.
- The total request process is no longer renewal, but its intensity is again $\sum_i \lambda_i$.
- Since $\lambda_i(t) = \eta_i(t - T_i^*(t))$, the optimal policy is:
  - Keep track of the current hazard rate of each content i.
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Conclusion: under renewal arrivals, the optimal policy only depends on the current hazard rates since the last request.

An interesting observation

Decreasing hazard rates

- If hazard rates are decreasing, caching makes sense! After an arrival it becomes more likely to get another request.
- After some time, we will evict the content to make room for more recent ones (as in LRU).

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Increasing hazard rates

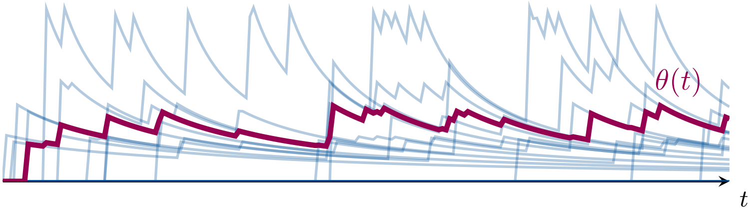
- If instead hazard rates are increasing, then when a request arrives, the item becomes less likely to be requested again!
- It may be better to remove it and make room for other ones (i.e. LRU makes no sense!).
- If we haven't seen it for a while, then we may have to fetch it anticipating the upcoming request.

Outline

We can rewrite this optimal policy as a threshold policy:

$$i \in \pi^*(t) \Leftrightarrow \lambda_i(t) \geqslant \theta(t) := \text{ the } C \text{ largest stochastic intensity}$$

Example: Pareto requests, Zipf popularities, $N = 20, C = 4$.



¿What is the large scale behavior of $\theta(t)$ in steady state?.

The threshold value in steady state

- Now we have M independent renewal processes with intensities $\lambda_i(t)$.
- At time $t = 0$, we have a sample $\{X_1, \ldots, X_M\}$ of independent, but **not identically distributed** random variables, with distribution:

$$X_i \sim \eta_i(-T_0^i), \quad -T_0 \sim \hat{F}_i(t)$$

- The threshold $\theta(0)$ is the $C-$th order statistic (in decreasing order) of the sample.

Problem: for non iid random variables, no closed form $\rightarrow$ Can we say something about the large scale limit?

A useful Theorem

Let $\{X_i\}$ be a sequence of independent random variables with distributions G_i. Define:

$$\hat{G}_M(x) = \frac{1}{M}\sum_{i=1}^{M} \mathbf{1}_{\{X_i \leqslant x\}}$$

the empirical distribution, and let:

$$\bar{G}_M(x) = \frac{1}{M}\sum_{i=1}^{M} G_i(x)$$

Theorem (Shorack)

If the family $\{G_i\}$ is tight, then:

$$||\hat{G}_M - \bar{G}_M||_\infty \to 0 \quad \text{almost surely as } M \to \infty.$$

Assume now that the request processes come from a common scale family, i.e. their inter-arrival distributions satisfy:

$$F_i(t) = F_0(\lambda_i t)$$

where F_0 has mean 1, so F_i has mean $1/\lambda_i$.

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In this case:

- The distribution of $-T_0^i$ is $\hat{F}_i(t) = \hat{F}_0(\lambda_i t)$.
- The hazard-rate of F_i is $\eta_i(t) = \lambda_i \eta_0(t/\lambda_i)$.
- The random variable $X_i \sim G_i(x) := G_0(x/\lambda_i)$

where $G_0(x) = P(\eta_0(-T_0) \leqslant x)$ is the observed hazard rate distribution for the base process.

The distribution of popularities

Consider now the popularities $\lambda_1 > \ldots > \lambda_M$ and define:

$$\phi_M(\lambda) = \frac{1}{M}\sum_{i=1}^{M} \mathbf{1}_{\{\lambda_i \leqslant \lambda\}}$$

their empirical (deterministic) distribution.

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Assumption:

$$\phi_M(\lambda) \to \phi(\lambda) \quad \text{as } M \to \infty$$

where $\phi(\lambda)$ is a probability distribution.

Example: Zipf popularities

- A common model for popularities is the Zipf distribution, where $\lambda_i \propto \frac{1}{i^\beta}$.
- In our framework, take:

$$\lambda_i = \left(\frac{M}{i}\right)^\beta$$

- Then we can show that:

$$\phi_M(\lambda) \to \phi(\lambda) = \left[1 - \lambda^{-1/\beta}\right] \mathbf{1}_{\{\lambda \geqslant 1\}}$$

Remark: note that $\sum_i \lambda_i$ diverges, so the system is scaling up...

Main result

Theorem (Carrasco,F',Paganini)

Consider a caching system fed by M independent and stationary renewal processes, with intensities $\{\lambda_i\}$, and inter-arrival distributions $F_i(t) = F_0(\lambda_i t)$. Let $X_1, \ldots, X_M$ denote the observed hazard-rates at time 0. Then, under the preceding assumption, the empirical distribution:

$$\hat{G}_M(x) = \frac{1}{M}\sum_{i=1}^{M} \mathbf{1}_{\{X_i \leqslant x\}} \to_M G_\infty(x) = \int_0^\infty G_0\left(\frac{x}{\lambda}\right)\phi(d\lambda)$$

Proof sketch

- By Shorack's result:

$$\hat{G}_M(x) = \frac{1}{M}\sum_{i=1}^{M} \mathbf{1}_{\{X_i \leqslant x\}} \approx \bar{G}_M := \frac{1}{M}\sum_{i=1}^{M} G_i(x)$$

- Note that:

$$\frac{1}{M}\sum_{i=1}^{M} G_i(x) = \sum_{i=1}^{M} G_0\left(\frac{x}{\lambda_i}\right)\frac{1}{M} = \int_0^\infty G_0\left(\frac{x}{\lambda}\right)\phi_M(d\lambda)$$

- Use the assumption to show that:

$$\int_0^\infty G_0\left(\frac{x}{\lambda}\right)\phi_M(d\lambda) \to_M \int_0^\infty G_0\left(\frac{x}{\lambda}\right)\phi(d\lambda) = G_\infty(x).$$

A law of large numbers for the threshold

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Then, the optimal policy threshold $\theta_M^*(0)$ is the random variable:

$$\theta_M^* : \sum_{i=1}^{M} \mathbf{1}_{\{X_i \leqslant \theta_M^*\}} = (1-c)M$$

or equivalently θ_M^* is such that $\hat{G}_M(\theta_M^*) = 1 - c$.

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Then, the optimal policy threshold $\theta^*_M(0)$ is the random variable:

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or equivalently θ^*_M is such that $\hat{G}_M(\theta^*_M) = 1 - c$.

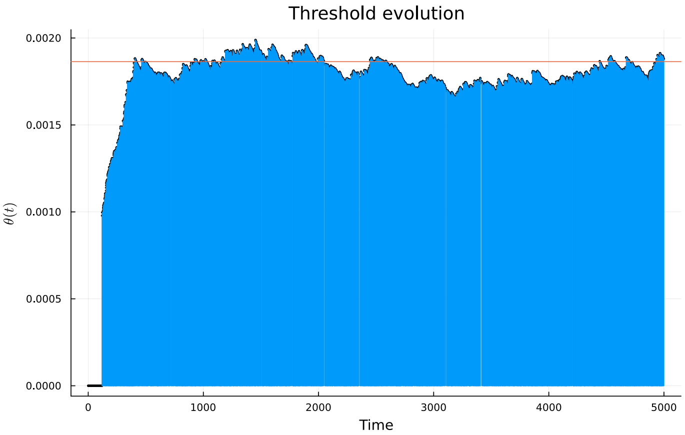
Corollary

If the cache size scales linearly with the catalog as $C_M = cM$, then:

$$\theta^*_M \to \theta^* : \; G_\infty(\theta^*) = 1 - c$$

So the optimal policy becomes a fixed threshold policy.

Simulation example



$M = 1000, C = 100$. Pareto $\alpha = 2$ requests, Zipf $\beta = 0.5$ popularities.

Asymptotic miss probability

Moreover, we can calculate the asymptotic performance:

Theorem

Under all the above assumptions, the asymptotic miss rate verifies:

$$\lambda_{\text{miss},M} \to_M \int_0^\infty \lambda \tilde{G}_0\left(\frac{\theta^*}{\lambda}\right)\phi(d\lambda) = E\left[\Lambda \tilde{G}_0\left(\frac{\theta^*}{\Lambda}\right)\right]$$

where $\Lambda \sim \phi$, and $\tilde{G}_0$ is the distribution of the hazard-rate prior to an arrival:

$$\tilde{G}_0(x) = \int_0^\infty \mathbf{1}_{\{\eta_0(t)\leqslant x\}} F_0(dt).$$

Outline

Final remarks

- The above result characterizes the optimal policy completely in the large-scale scenario.
- For particular distributions of interest (e.g. Pareto requests, Zipf popularities) the threshold can be computed explicitly.
- Once the threshold is computed, we can compute the asymptotic hit probability.
- Therefore, we have a computable absolute performance bound in the limit.

Final remarks

- There is much more to do (students welcome!).
- In particular, in a previous paper we explored timer-based policies.
- Using this result, we can show that the optimal timer-based policy matches the optimal causal policy in the limit, for decreasing hazard-rates.
- For increasing hazard-rates, we have to think about pre-fetching content anticipating future arrivals.

Gracias!

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