Specialization, Complexity & Resilience in Supply Chains *

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Abstract

Despite growing policy interest, the determinants of supply chain resilience are still not well understood. We propose a new theory of supply chain formation with compatibility frictions: only compatible inputs can be used in the final production process. Intermediate producers choose the degree of specialization of their goods, trading off higher productivity with a lower share of compatible final producers. The production process is complex, meaning that multiple key inputs need to be sourced for final production to take place. Specialization choices, production complexity, and search frictions jointly determine the resilience of supply chains. If the production process is sufficiently complex, this environment is characterized by over-specialization due to a novel network externality arising from the interplay between frictional markets, endogenous specialization, and complex production. In turn, over-specialization implies that supply chains are less resilient than efficient. We characterize how a social planner can decentralize efficient supply chain resilience through a targeted transaction subsidy.

Keywords: Supply chains, specialization, product design, resilience
JEL Codes: D21, L14, L22, L23

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1 Introduction

In modern economies, production is organized around long and complex supply chains. The substantial progress in information and transportation technologies allows firms to source specialized inputs from a large number of suppliers. The rise of supply chains has brought about enormous productivity gains as well as concerns about the frequency and economic costs of their disruptions. In recent years, policy interest has been focused on whether these supply chains are efficient or if, instead, governments should intervene.[1] In particular, gains from specialization are suspected to carry along risks associated with bottlenecks or over-reliance on single providers of key inputs, which might not be socially desirable.

In this paper, we propose a new theory of frictional supply chain formation where intermediate producers choose the degree of specialization of their goods. The economy features an intuitive technological friction: inputs need to be *compatible* with the production process they are used in. Higher specialization of an input increases its productivity but reduces the share of compatible final producers. Furthermore, production is *complex*: multiple key inputs need to be sourced for final production to take place. Supply chain resilience, defined as the time it takes for firms to return to normal operations following disruptions, is fundamentally shaped by the average specialization of goods, search frictions, and production complexity. Our model provides us with a framework to analyze the sources of inefficiencies that are likely to affect supply chain resilience, as well as to address normative questions related to optimal supply chain organization. The intuitive trade-off that characterizes our economy is that higher specialization increases the value of production in normal times but results in a higher cost of disruptions as replacing specialized inputs takes longer.

We study an economy where final producers source multiple key inputs in frictional markets. Intermediate producers make product design decisions by choosing the degree of specialization of their goods. More specialized products grant a higher surplus upon matching but are compatible with the production process of fewer final producers. To source each input, final producers search the market for the desired intermediate good. Specialization generates endogenous compatibility frictions so that not all intermediate good varieties fit the production process of every final producer. Whenever search is unsuccessful for any of the key inputs, the entire production process is halted.

Our analysis proceeds in four steps. In Section 2, we start by characterizing the equilibrium in a static economy where heterogeneous intermediate producers choose the degree of specialization of their goods and ex ante identical final producers need to source multiple key inputs in the market. Higher specialization increases the value of the intermediate input but reduces the share of compatible final producers. The market for intermediate goods display realistic sourcing frictions.[2] Each final producer contacts a finite number of intermediate producers, as governed

[1]In June 2021, the Biden-Harris Administration instituted the *Supply Chain Disruption Task Force* "to provide a whole-of-government response to address near-term supply chain challenges to the economic recovery" (see White House (2021)).

[2]According to a McKinsey, "*At most organizations [...], hunting for new suppliers is a daunting, manual process. On average, it takes about three months to complete a single supplier search, with a sourcing pro-*

by the extent of search frictions. Within the set of contacted intermediate producers, it selects those compatible with its production process, if any, and request them a quotation. Intermediate producers submit a bid characterized by the surplus (profits) offered to the final producer. Finally, the final producer trade with the intermediate offering the highest surplus.

In this static framework, we can highlight the key inefficiency arising from endogenous specialization when production is complex. Intermediate producers choose their optimal specialization by trading off the surplus they obtain upon trading against the probability of trading. Crucially, they fail to internalize how their specialization choices affect the likelihood that the final good is produced and, as a consequence, the payoffs of other input providers. For concreteness, consider the case of the car industry. Final carmakers can produce city cars and SUVs by sourcing engines and chassis. The production process of city cars requires electric engines, that of SUVs hybrid engines. Engine providers are specialized: they have the capacity to produce either electric or hybrid engines – not both at the same time. Suppose a carmaker wishes to produce a SUV but all the engine producers it is able to locate are specialized in electric engines. Then, the production of SUVs is halted. The key negative externality is that imposed by engine producers on chassis producers. Indeed, the latter do not receive any payoff from SUVs since production does not happen because of the engine providers' specialization choices. This example highlights the presence of a *network externality.* In Proposition 1, we show that this externality entails that the equilibrium displays *over-specialization* relative to the constrained efficient allocation. Intuitively, specialization choices fundamentally interact with the degree of complexity in the production process: if cars were only made of engines – that is, the production process were not complex – the *network externality* would be absent, and the equilibrium would be constrained efficient.

In Section 3, we study a dynamic version of our model to assess the role of specialization decisions in shaping supply chain formation. In particular, we assume long-term relationships between intermediate and final producers, as well as that final producers face a disruption with some positive probability each period. We refer to the inverse of this probability as *robustness*: the ability to maintain operations in the presence of shocks. On top of the network externality, endogenous specialization choices bring about a further *search externality* (Pissarides, 2000): intermediate producers do not account for the effect of their specialization on the meeting probability, as more specialized goods raise the share of searching final producers. This external effect introduces a force pushing the equilibrium towards under-specialization. However, in Proposition 4, we show that, if the production process is sufficiently complex, the equilibrium keeps displaying over-specialization. Importantly, the dynamic framework allows for a model-consistent notion of supply chain resilience, that is, the average time it takes for a final producer to restore production following a disruption. This allows us to single out the key determinants of supply chain resilience: the higher the extent of search frictions, average specialization and

fessional logging more than 40 hours of work – and yet able to consider only a few dozen suppliers from a total population of thousands." (see `https://www.mckinsey.com/capabilities/operations/our-insights/with-artificial-intelligence-find-new-suppliers-in-days-not-months`). We interpret this evidence as indicative of search and compatibility frictions in intermediate markets.

complexity, the lower supply chain resilience. It follows that equilibrium over-specialization entails that supply chains are *less resilient than efficient*. Finally, we show how resilience and robustness of supply chains, together with the expected productivity conditional on matching, determine the value of output and, therefore, welfare. Since the degree of specialization shapes both the value of output and the likelihood that goods are produced, a trade-off arises between *productivity* and *resilience*.

In Section 4, we further extend the model along two important dimensions. First, the existence of critical failures begs the question of how final producers choose the degree of complexity of their production process. Next, the introduction of disruptions motivates us to study investment in redundancies or in own *robustness*. Firms typically take actions targeted to smoothing out their production process, be they sourcing the same input from multiple suppliers or holding inventories. To understand how such prudential investments interplay with specialization choices, we allow final producers to invest resources to reduce their probability of disruption.

On the one hand, we show in Proposition 5 that, in a static economy, final producers' choice of their production complexity always induces *under-resilience*. Intuitively, final producers do not internalize the product design costs of each additional input they employ, which are borne by intermediate producers and sunk at the surplus-sharing stage. On the other hand, our analysis of prudential investment in redundancies suggests that, when production is complex enough, the equilibrium features too much investment in specialization and, therefore, is under-resilient. This is partly offset by excessive investment in redundancies relative to the planner's choice. As complexity increases or search frictions improve, the equilibrium veers away from the efficient allocation.

We conclude in Section 5 by studying the normative implications of our framework. We start by showing that, if a set of firm-specific transaction subsidies is available, a planner can decentralize the constrained efficient allocation in both our static and dynamic economies. In Proposition 9 we show that the planner would like to increase the price of all transactions. This is equivalent to giving more surplus to intermediate producers for them to have *more skin in the (final production) game*, which allows decentralizing the efficient level of specialization. We also show that the optimal subsidy schedule takes a simple form comprising an aggregate component and a targeting element that depends on the average trading probability of lower-productivity intermediate competitors. Decentralizing the efficient allocation in economies with endogenous complexity and robustness requires an additional tax instrument on top of the targeted transaction subsidy just described. In Proposition 11, we show that levying a profit tax from final producers as a function of their complexity addresses the inefficiencies arising from complexity choices. Similarly, a flat subsidy to robustness costs suffices to induce efficient investment in robustness, as established in Proposition 12.

Literature Review This paper is related to several strands of literature. First, our interest in supply chains speaks to the growing literature on production network formation and its role in the economy's shock-absorption capacity. Levine (2012), Elliott et al. (2022), Acemoglu and

Tahbaz-Salehi (2023) and Carvalho et al. (2023) are papers that study the *fragility* properties of economies in which supply chains feature coordination problems due to complementarities or public goods. The closest paper to ours in this literature is Grossman et al. (2023b), which explores market failures in vertical supply chains where firms can invest to reduce their risk of disruptions and the terms of trade in firm-to-firm transactions are bargained over. We contribute to this literature by studying the problem of endogenous specialization in frictional markets and by proposing a dynamic framework that allows separately identifying the fragility of production (*robustness*) and its ability to recover from shocks in a timely manner (*resilience*).

Our work is also closely related to the problem of the socially optimal number of varieties, originally studied in Spence (1976), Dixit and Stiglitz (1977) and Mankiw and Whinston (1986), more recently analyzed in Zhelobodko et al. (2012), Parenti et al. (2017) and Dhingra and Morrow (2019), and applied to supply chains by Grossman et al. (2023a). We show that many of the insights from this literature have tight analogs in our search model with optimal specialization choices and price posting, chief among them the offsetting effects of *appropriability* and *business-stealing* externalities.

A small set of papers has studied the dynamic response of supply chains to disruptions, highlighting the role of time-to-build and inventories (Meier, 2020; Alessandria et al., 2023; Ferrari, 2023; Carreras-Valle, 2023). We add to this discussion by studying how alternative strategies, such as the investment in redundancies and the choice of how complex the production process should be, affect the supply-side response to shocks.

Both methodologically and conceptually, our work is close to the strands of literature on product design (Bar-Isaac et al., 2012, 2023; Menzio, 2023; Albrecht et al., 2023), supply chain formation (Oberfield, 2018; Antràs and De Gortari, 2020; Boehm and Oberfield, 2020; Kopytov et al., 2021; Kim and Shin, 2023; Antràs, 2023) and specialization (Rauch, 1999; Nunn, 2007; Barrot and Sauvagnat, 2016; Martin et al., 2020; Ekerdt and Wu, 2023). Relative to these contributions, we study the effects of specialization choices on the resilience of supply chains. Our key novelty is the study of product design choices when goods are complements in production, which allows us to highlight the key inefficiencies associated with supply chain formation.

Finally, our model is also close in spirit to the large literature on the hold-up problem in firm-to-firm transactions (see Klein et al., 1978; Williamson, 1979; Rogerson, 1992; Hart and Moore, 1990; Tirole, 1999) and the organization of production (Antràs, 2003; Antràs and Helpman, 2006; Antràs and Chor, 2013). We study a context in which firms make sunk investments in specialization, yet our price-posting mechanism still generates efficient outcomes. We show that the standard *hold-up problem* arises in our model whenever final producers can invest in redundancies to make the entire supply chain safer.

2 Static Model

In this section, we propose a static model of sourcing from frictional markets when firms make endogenous product design decisions.

Environment. The economy is populated by a representative household and a measure 1 of perfectly competitive final producers and a measure m of intermediate producers. The final good is the numeraire. The model is static. The household consumes a single unit of the final good. The consumer enjoys higher utility from consuming a higher value final good, where the value of the good is determined by the level of specialization of the inputs used in its production.

Final producers. Final producers operate a production function combining $N > 1$ *key* inputs, each of which is necessary for production to take place, in an additively separable fashion. Exactly one unit of each input is needed for production, and inputs are indivisible. Following Elliott et al. (2022), we refer to the number of inputs sourced in the market, N, as the *complexity* of the production process.[3] We think of the N inputs as defining a recipe for final production in the spirit of Oberfield (2018). [4] The value of the output of a final producer reads:

$$y = \begin{cases} \sum_{i=1}^{N} A_i & \text{if } \min\{y_1, \ldots, y_N\} > 0, \\ 0 & \text{else,} \end{cases}$$

where $A_i \in \mathbb{R}_+$ and $y_i = \{0, 1\}$ represent the sophistication and quantity of input i, respectively. We view this production function as determining the value or quality of the final good – rather than its fixed quantity. For example, a car manufacturer will need an engine, a chassis, and four tires in exact proportions to assemble its car. Yet, all these inputs can be purchased at very different levels of sophistication, which will ultimately determine the value of the car produced. For now, we assume that at the intensive margin, such qualities are perfect substitutes. Keeping with our example, a producer could make a \$ 100,000 car with a high-quality engine and a medium-quality chassis, or with a medium-quality engine and a high-quality chassis. We relax this assumption in Appendix B, where we allow for arbitrary substitutability or complementarity at the intensive margin.

Input value is an increasing and concave function of the specialization s of the corresponding input, *i.e.*, $A = A(s)$, $\frac{\partial A}{\partial s} > 0$, $\frac{\partial^2 A}{\partial s^2} < 0$. The economy features *compatibility frictions*: higher specialized goods are compatible with a lower share of final producers. We denote the probability that a variety of specialization s fits the final production process as $\phi(s)$, where $\phi'(s) < 0$ and $\phi''(s) < 0$.

The payoff of a final good firm transacting with intermediate good producer i is $x_i = \mathbb{1}_{\{y>0\}}(A_i - p_i)$. Hence, final good producers obtain total profits given by:

$$\pi = \begin{cases} y - \sum_{i=1}^{N} p_i & \text{if } y > 0, \\ 0 & \text{else,} \end{cases}$$

[3] As will become apparent, the network structure of our model lends itself both to a "snake" and a "spider" interpretation (Baldwin and Venables, 2013). In the main text, we interpret the production network as spider-like. Alternatively, one can interpret the N key inputs as the number of vertical stages leading to final production.

[4] For simplicity, we think of N as a measure rather than an integer number. None of our results hinges upon this assumption.

depending on whether they can successfully source all key inputs or not.

Intermediate market structure. Intermediate good markets are characterized by search and compatibility frictions. Search frictions are represented by a finite number $n \sim \text{Poisson}(\lambda)$ of intermediate producers that each final producer meets each period, where $\lambda < \infty$. Compatibility frictions are represented by an endogenous distribution of compatibility probability with mean $\bar{\phi}$. Hence, in the presence of search and compatibility frictions, the number of compatible intermediate producers met by a final producer is Poisson-distributed with mean $\lambda\bar{\phi}$.[5] Upon selecting compatible intermediate producers, final producers request them a quotation. Intermediate producers submit a bid characterized by a surplus x offered to the final producer. Let the (endogenous) distribution of offered surplus be $G(x)$. Finally, the final producer trade with the intermediate offering the highest surplus.

To fix ideas, λ is the *average number of meetings*, $\bar{\phi}$ is the *average probability that an input provider is compatible*, and $\lambda\bar{\phi}$ is the *average number of compatible input providers* per final producer. As a consequence, the probability that a final producer finds a compatible input, or *input finding probability*, is given by:

$$f = 1 - \exp\{-\lambda\bar{\phi}\}. \tag{1}$$

Anticipating the discussion in Section (3), we refer to f^N, i.e., the probability that all inputs are sourced successfully and production occurs, as *resilience*.

Intermediate producers. Each intermediate good producer can produce all N inputs through distinct product lines.[6] For each intermediate good, denote market tightness as $\theta = \frac{1}{m}$. An intermediate producer meets an expected number m_k of final producers who are in contact with $k = 0, 1, 2, \ldots$ other *compatible* intermediate producers, where:

$$m_k = \theta \sum_{n=k+1}^{\infty} n \underbrace{\frac{\lambda^n \exp\{-\lambda\}}{n!}}_{\text{Prob. buyer meets n sellers}} \underbrace{\frac{(n-1)!}{k!(n-k-1)!}(\bar{\phi})^k(1-\bar{\phi})^{n-k-1}}_{\text{Prob. k other sellers are suitable}} = \theta\lambda \exp\{-\lambda\bar{\phi}\}\frac{(\lambda\bar{\phi})^k}{k!}.$$

This statistic summarizes the expected number of final customers *binned* by how many other suitable intermediate competitors they are in contact with. For example, an intermediate producer j of good i may expect to meet 3 final producers who are in contact with 5 suitable intermediate producers of good i, namely producer j itself and 4 of its competitors: this amounts to $m_4 = 3$.

Intermediate producers differ in productivity $z \sim \Gamma(z)$, where Γ is a continuous distribution with support $[\underline{z}, \bar{z}]$. Productivity z is associated with marginal cost $c(z)$, where $c'(z) < 0$.

[5] Following Menzio (2023), we pick the average probability of input suitability as the probability of success in the binomial distribution of the number of suitable meetings. Since the probability of input compatibility differs across the productivity distribution, this holds just as a first-order approximation, though. Since results do not hinge upon this approximation – and in the light of the tractability it grants – we assume this throughout.

[6] Since intermediate good producers make positive profits in equilibrium in each product line, they optimally choose to compete in all N products.

Intermediate good producers with productivity z choose a level of specialization $s(z)$ and post a surplus contract $x(z)$ to maximize their expected operating profits:

$$\begin{aligned}\Pi(s,x;z) &= \left(\sum_{k=0}^{\infty} m_k \; \phi(s) \; G(x)^k\right) f^{N-1}\left(A(s)-x-c(z)\right) \\ &= \underbrace{\theta\lambda}_{\text{exp \# meetings}} \quad \underbrace{\phi(s)\;\exp\left\{-\lambda\bar{\phi}(1-G(x))\right\}}_{\text{matching prob.}|\text{meeting}} \quad \underbrace{f^{N-1}}_{\text{prob. other inputs sourced}} \quad \underbrace{(A(s)-x-c(z))}_{\text{unit profit}}.\end{aligned}$$

Operating profits are composed of four elements. First, an intermediate producer with productivity z and specialization s offering surplus x meets an expected number $\theta\lambda$ of final producers. Second, upon meeting a final producer, its input variety is compatible with the production process with probability $\phi(s)$ and, with probability $\exp\{-\lambda\bar{\phi}(1-G(x))\}$, the intermediate producer will be chosen by the final producer over contacted competitors producing other suitable varieties of the same good.[7] At this point, the two firms can successfully trade. However, the final producer can produce only if all other $N-1$ inputs are also procured. This occurs with probability f^{N-1}. If all the inputs are sourced, trade happens, and the final producer obtains a unit profit of $p(s,z)-c(z)$. For ease of notation, we denote $\mathcal{P}(s,x) \equiv \phi(s)\exp\{-\lambda\bar{\phi}(1-G(x))\}f^{N-1}$ the *trading probability*. For convenience, we express operating profits in terms of offered surplus rather than price, with the understanding that choosing either of the two directly implies the other: $p(s,z) = A(s) - x(s,z)$.

2.1 Characterization

We start by studying the problem of an intermediate producer choosing specialization and the offered surplus. We think of these decisions as being made simultaneously, but the same results would equally go through if they were made sequentially. The intermediate producer's problem reads:

$$V(s,x;z) = \max_{s,x} \Pi(s,x;z) - wq(s),$$

where $q(s), q' > 0$ is the labor requirement for specialization. We lighten notation by defining as $\hat{\phi}(z_l,z_u) \equiv \int_{z_l}^{z_u} \phi(s(z))\gamma(z)dz$ the average probability of input suitability for firms with productivity between two generic bounds z_l and z_u, such that $\bar{\phi} = \hat{\phi}(\underline{z},\bar{z})$. The optimal choices of s and x are given by a system of differential equations:

$$\lambda\phi(s(z))\gamma(z)\left(A(s(z))-x(z)-c(z)\right) = x'(z), \tag{2}$$

$$\theta\lambda\mathcal{P}(z)\left[A'(s(z)) - \left(\frac{-\phi'(s(z))}{\phi(s(z))}\right)\left(A(s(z))-x(z)-c(z)\right)\right] = wq'(s(z)). \tag{3}$$

[7] This matching process is similar to the one proposed by Huang et al. (2022). The key difference is that, in our setting, intermediate producers can influence their ranking based on the surplus they offer.

with boundary condition $x(\underline{z}) = 0$. We assume $q(s)$ is such that the SOC holds.[8] We proceed by characterizing how equilibrium specialization changes with the offered surplus. All proofs are relegated to Appendix C.

Lemma 1 (Specialization and Offered Surplus)**.** *The payoff function of intermediate good producers is supermodular in $s(z)$ and $x(z)$. Hence specialization and offered surplus are strategic complements.*

Lemma 1 states that if an intermediate producer were forced to offer a higher surplus to its counterparty ($x \uparrow$), it would find it optimal to specialize more ($s \uparrow$) and viceversa. This result is in contrast to standard models in which an investment is made before the parties bargain over the surplus split (*hold-up problem*). We discuss the relation of our model to this class of problems in detail in Appendix B.4. For now, we clarify the intuition underlying our result. From eq. (3), the marginal benefit of specialization (left-hand side) has two components. First, higher specialization implies higher surplus conditional on trading, as summarized by A'; second, higher specialization reduces the probability of trading, as summarized by $\phi' < 0$. Offering a higher surplus $x(z)$ reduces the cost of additional specialization in terms of foregone trade. Suppose that an intermediate producer offers no surplus to final producers ($x = 0$). Then, the cost of reducing the probability of trading is $A - c > 0$. Consider instead the opposite case, where all the surplus of the transaction is offered to the final producer ($x = A - c$). Then, the cost of losing trading probability is 0. These limit cases highlight how a higher offered surplus decreases the cost of reducing the trading probability upon choosing higher specialization, thus increasing optimal specialization.

Whether higher productivity firms choose higher or lower specialization is a priori ambiguous. Formally, we have that $\frac{\partial s(z)}{\partial z} > 0 \Leftrightarrow \frac{-c'(z)}{\gamma(z)} < \lambda\phi(s(z))\,(p(s(z)) - c(z))$. Hence, the relative shape of the productivity distribution and marginal cost function governs the cross-sectional behavior of the specialization function, which can exhibit non-monotonicity (see Appendix A.1 for further details).

We conclude by characterizing the choice of surplus posted by intermediate goods producers. To this end, we introduce one last piece of notation: $\gamma^\star(z, z_h) \equiv \lambda\phi(s(z))\exp\{-\lambda\hat{\phi}(z, z_h)\}\gamma(z)$. For a downstream firm, this is the likelihood that a compatible supplier with productivity z is the best option available.[9] We refer to $\gamma^\star(z, \bar{z})$ as *weighted productivity density*. Throughout, we use the notation $\mathbb{E}[\cdot]$ to denote expectations with respect to $\gamma^\star(z, \bar{z})$.

Lemma 2 (Offered Surplus)**.** *The equilibrium offered surplus by an intermediate producer with productivity z equals the expected match surplus obtained by downstream firms when trading*

[8] The SOC reads $wq''(s(z)) > \theta\lambda\exp\{-\lambda\hat{\phi}(z, \bar{z})\}f^{N-1}\left[A''(s(z))\phi(s(z)) + \phi''(s(z))(A(s(z)) - c(z) - x(z))\right.$ $\left.-2(-\phi'(s(z)))A'(s(z))\right]$.

[9] Formally, $\lambda\phi(s(z))\gamma(z)$ is the probability that the downstream firm meets a compatible supplier with productivity z, while $\exp\{-\lambda\hat{\phi}(z, z_h)\}$ is the probability it does not meet any compatible supplier with productivity between z and z_h. Since the measure of final producers is normalized to 1, $\gamma^\star(z, \bar{z})$ lies in the unit interval. Note that $f \equiv 1 - \exp\{-\lambda\bar{\phi}\} = \int_{\underline{z}}^{\bar{z}} \gamma^\star(z, \bar{z})dz$. Since $f < 1$, $\gamma^\star(z.\bar{z})$ does not integrate to 1 over the support of the productivity distribution. The reason is that with density $1 - f$ search is unsuccessful, so that a final producer does not trade with any intermediate producer. Allowing for unsuccessful search as a distinct state with zero productivity makes $\gamma^\star(z, \bar{z})$ a legitimate density function.

with suppliers less productive than z. Formally, solving the differential equation (2), the offered surplus is:

$$x(z) = \int_{\underline{z}}^{z} [A(s(\tilde{z})) - c(\tilde{z})]\gamma^{\star}(\tilde{z}, z)d\tilde{z} = \mathbb{E}_{\tilde{z},\, \tilde{z}<z}[A(s(\tilde{z})) - c(\tilde{z})]. \tag{4}$$

The intuition behind this result is instructive. Recall that intermediate producers post surplus offers to final producers and that such surplus is increasing in productivity. First, a firm with productivity z will necessarily lose any auction against firms with productivity $z' > z$. Next, since intermediate producers have the power to set the offered surplus unilaterally, they will offer the surplus that makes final producers exactly indifferent between accepting and rejecting it.[10] The final producers' outside option in case of no trade with the firm of productivity z is the expected surplus offered by the best compatible firm with productivity $z'' < z$. Lemma 2 formalizes this intuition by noting that the outside option is given by $\mathbb{E}_{\tilde{z},\, \tilde{z}<z}[A(s(\tilde{z})) - c(\tilde{z})]$ as the expected surplus of the second best compatible firm after z. Accordingly, firms with productivity $\underline{z}$ will offer no surplus, $x(\underline{z}) = 0$, while firms with productivity $\bar{z}$ will offer the expected surplus over the entire z distribution, $x(\bar{z}) = \mathbb{E}[A - c]$. Firms strictly in between offer a surplus level in $(0, \mathbb{E}[A - c])$.

To sum up, we have so far established two results in partial equilibrium: i) specialization and offered surplus are strategic complements, and ii) optimal offered surplus equals the outside option of final producers, which is the expected surplus from trading with lower-productivity input providers. In the rest of the section, we study a general equilibrium version of this economy to establish the efficiency properties of equilibrium specialization.

Equilibrium. We assume the representative household gains linear utility from consuming the final good and has convex disutility from labor supply: $\max_{C,\ell}\ C + \psi \log(1-\ell) \quad \text{s.t.} \quad C = w\ell + \overline{\Pi}$, where $\overline{\Pi}$ denotes total profits (both from intermediate and final producers). The labor supply curve is given by $\ell = 1 - \frac{\psi}{w}$. Labor market clearing implies that workers are employed to generate more specialized intermediate products: $1 - \frac{\psi}{w} = Nm\bar{q}$, where $\bar{q} = \int q(s(z))d\Gamma(z)$. Product market clearing requires: $C = y$. Finally, the value of output is given by:

$$y = \underbrace{f^N}_{\text{resilience}} \qquad \underbrace{N\mathbb{E}[A - c]/f.}_{\text{expected match surplus}\,\mid\,\text{active}} \tag{5}$$

The characterization of output in eq. (5) highlights a number of important features. First, the complexity of the production process has ambiguous effects on output: on the one hand, more complex production processes generate larger output, conditional on successfully sourcing all key inputs. On the other hand, the probability that all inputs are sourced decreases. The intuition underlying this first observation is the same as in Levine (2012) and other theories of

[10] This is the core idea behind the well-known *Diamond Paradox* (Diamond, 1971). Still, in our model, final producers are able to extract positive surplus in equilibrium due to their ability to locate multiple intermediate producers simultaneously (Burdett and Judd, 1983).

critical failures, such as Kremer (1993). A similar argument goes through for the endogenous specialization decisions. If firms specialize more, the expected surplus in each transaction increases. On the other hand, the trading probability decreases, reducing the measure of active final producers.

Social planner problem. We study the efficiency properties of equilibrium specialization decisions by comparing market allocation with that dictated by a benevolent social planner. The social planner is subject to the same product market frictions as market participants, so our normative criterion is constrained efficiency. The social planner solves:

$$\max_{s_i(z)} \mathcal{W} = C + \psi \log(1-\ell) \quad \text{s.t.} \quad C = y \text{ and } \ell = Nm\bar{q}.$$

The constrained efficient specialization $\mathcal{S}_i(z)$ of an intermediate producer of productivity z producing input i is given by:

$$\begin{aligned}\mathcal{S}_i(z) = \underset{\hat{s}_i(z)}{\operatorname{argmax}}\ & f^{N-1} \int \lambda\phi(\hat{s}_i(z)) \exp\left\{-\lambda\hat{\phi}(z,\bar{z})\right\} [A(\hat{s}_i(z)) - c(z)]\gamma(z)dz \\ & + f(\hat{s}_i(z)) \sum_{j\neq i}^{N-1} f_j^{N-2}\mathbb{E}[A_j - c] + \psi \log\left(1 - m\bar{q}(\hat{s}_i(z)) - m\sum_{j\neq i}^{N-1} \bar{q}_j\right).\end{aligned}$$

The first line represents the expected match surplus for the input of interest. The second line represents the expected match surplus of the other inputs than the one of interest, along with the disutility of labor supply. Importantly, the expected match surplus of the other inputs is only present in the planner's objective function, while it is not internalized by intermediate producers in equilibrium. Since all input markets are symmetric, the first-order condition for the efficient specialization function is given by:

$$\begin{aligned}\theta\lambda\mathcal{P}(z)\Bigg[A'(\mathcal{S}(z)) - \left(\frac{-\phi'(\mathcal{S}(z))}{\phi(\mathcal{S}(z))}\right)\Bigg([A(\mathcal{S}(z)) - c(z)] - \int_{\underline{z}}^{z} [A(\mathcal{S}(\tilde{z})) - c(\tilde{z})]\gamma^\star(\tilde{z},\bar{z})d\tilde{z} \\ + (N-1)e^{-\lambda\hat{\phi}(\underline{z},z)}\mathbb{E}[A-c]/f\Bigg)\Bigg] = \frac{\psi}{1-Nm\bar{q}}\, q'(\mathcal{S}(z)). \qquad (6)\end{aligned}$$

The social marginal benefit of specialization on the left-hand side of (6) consists of four terms. The first term, $\theta\lambda\mathcal{P}(z)A'(\mathcal{S}(z))$, represents the conditional-match-surplus-enhancing effect of specialization, *i.e.*, the increase in expected match surplus by marginally increasing specialization while keeping trading probability fixed. The second term, $-\theta\lambda\mathcal{P}(z)\,[-\phi'(\mathcal{S}(z))/\phi(\mathcal{S}(z))]$ $[A(\mathcal{S}(z)) - c(z)]$, reflects the trading-probability-diminishing effect of specialization on intermediate producers of productivity z producing the same input, *i.e.*, the reduction in expected match surplus by marginally increasing specialization while keeping fixed both conditional match surplus and average input suitability between any productivity bounds, *i.e.*, $\hat{\phi}(z,\bar{z})$. The third term, $\theta\lambda\mathcal{P}(z)\,[-\phi'(\mathcal{S}(z))/\phi(\mathcal{S}(z))]\int_{\underline{z}}^{z}[A(\mathcal{S}(\tilde{z})) - c(z)]\gamma^\star(\tilde{z},\bar{z})d\tilde{z}$, reflects the trading-probability-enhancing effect of specialization on intermediate producers of productivity lower than z producing the same input, *i.e.*, the increase in expected match surplus by marginally increasing specialization while keeping conditional match surplus and input suitability of own input fixed. The fourth term, $-\theta\lambda\mathcal{P}(z)(N-1)\left[-\phi'(\mathcal{S}(z))/\phi(\mathcal{S}(z))\right]e^{-\lambda\hat{\phi}(\underline{z},z)}\mathbb{E}[A-c]/f$, reflects the trading-probability-diminishing effect of specialization on intermediate producers producing other (complementary) inputs, *i.e.*, the reduction in expected match surplus by marginally increasing specialization of another input. The social marginal cost on the right-hand side of (6) is given by the marginal increase in employment induced by higher specialization, weighted by the marginal disutility of labor supply.

We are interested in the efficiency properties of equilibrium specialization. Comparing (3) with (6) allows us to establish the following result.

Proposition 1 (Efficiency). *The equilibrium is constrained efficient if and only if production is not complex, i.e. $N = 1$. If the production process is complex, i.e. $N > 1$, the equilibrium features over-specialization.*

We discuss the intuition behind Proposition 1 by highlighting the externalities at play in the equilibrium. We start out from the marginal welfare effect of specialization, which efficient specialization sets to zero in eq. (6). Evaluating it at the equilibrium optimal specialization $s^\star(z)$, we obtain:

$$\left.\frac{\partial\mathcal{W}}{\partial s(z)}\right|_{\mathcal{S}(z)=s^\star(z)} \propto \underbrace{\mathbb{E}_{\tilde{z},\tilde{z}<z}\left[A(s^\star(\tilde{z}))-c(\tilde{z})\right]}_{\text{business-stealing externality}} \underbrace{-\,x(z)}_{\substack{\text{appropriability}\\ \text{externality}}} \underbrace{-\,(N-1)e^{-\lambda\hat{\phi}(\underline{z},z)}\mathbb{E}[A-c]/f}_{\text{network externality}}. \quad (7)$$

The socially optimal level of specialization equates eq. (7) to 0. This expression highlights the three key externalities of our model. The first term captures that a higher level of specialization by a firm with productivity z induces a higher trading probability for all firms with productivity below z. Therefore, this represents a positive externality on other firms producing the same good. The second term, $-x(z)$, captures the inability of intermediate producers to fully appropriate the social benefit of their specialization. Indeed, a part $x(z)$ of match surplus is offered to final producers and, therefore, is not appropriated by intermediate producers. Lastly, higher specialization reduces the likelihood that all N inputs are successfully sourced and, therefore, that the final good is produced. This represents a *network externality* on intermediate producers of all the other $N-1$ goods. This external effect is evaluated at the expected conditional social surplus per input, $\mathbb{E}[A-c]/f$.

To evaluate the efficiency properties of equilibrium specialization, consider first the case where production is not complex, *i.e.*, $N = 1$. Then, the third term vanishes, and the economy is efficient if and only if the offered surplus $x(z)$ is equal to the external effect on lower-productivity firms, that is if the *appropriability externality* exactly offsets the *business-stealing externality*. As we discussed in Lemma 2, in equilibrium, firm z offers a surplus $x(z)$ equal to the outside option of final producers, which coincides with the expected social surplus of firms with lower productivity. Hence, when production is not complex, intermediate producers offer a surplus that exactly offsets their external effect on firms with lower productivity. Therefore, the equilibrium, through the price-posting mechanism, attains the efficient level of specialization.[11]

When production is complex ($N > 1$), the first two terms still cancel due to the price-posting mechanism, but the last term does not vanish anymore. The marginal welfare effect, evaluated at equilibrium specialization, is negative as firms do not internalize their negative network externality on intermediate producers of other goods. As the marginal welfare effect is negative, the planner would impose a lower level of specialization and, therefore, the equilibrium features *over-specialization*. Importantly, this effect represents a pure externality as it arises only in no-trade scenarios. Input provider j specialization harms other providers $-j$ whenever the final good producer cannot source the input. As a consequence, there is no price that allows j to internalize this negative effect as no trade occurs. We further clarify the economic forces behind the network externality in the following 2-firm example.

Example: Network Externality. Assume two input providers produce the only two key inputs for final production ($N = 2$). Suppose further that final producers only meet one intermediate producer per input, which implies that the latter do not have to compete and can extract the whole surplus.[12] Denote as $f(s_i)$ the probability that firm i trades with the final producer when its specialization is s_i. The probability that both inputs are sourced is, therefore, $f(s_1)f(s_2)$. Higher specialization reduces trading probability ($f' < 0$) but raises match surplus y_i. Social surplus reads $\mathcal{W} = f(s_1)f(s_2)[y(s_1) + y(s_2)]$. Efficient specialization of firm 1 solves

$$\underbrace{f(\mathcal{S}_1)y'(\mathcal{S}_1)}_{\text{social MB}} = \underbrace{(-f'(\mathcal{S}_1))[y(\mathcal{S}_1) + y(\mathcal{S}_2)]}_{\text{social MC}}.$$

Expected private surplus of firm 1 reads $\pi_1 = f(s_1^\star)f(s_2^\star)y(s_1^\star)$. Equilibrium specialization solves

$$\underbrace{f(s_1^\star)y'(s_1^\star)}_{\text{private MB}} = \underbrace{(-f'(s_1^\star))y(s_1^\star)}_{\text{private MC}}.$$

Evaluating the efficiency condition at the privately optimal specialization, we note that we are left with the term $f'(s_1^\star)y(s_2^\star) < 0$, which captures the negative externality of firm 1 on the trading probability of firm 2. It follows that the social marginal cost is higher than the private one, and firm 1 over-specializes. By symmetry, the same holds for firm 2, as well.

Comparative Statics. In what follows, we provide a series of comparative static results to highlight how equilibrium and efficient specialization choices move with the complexity of production and the intensity of search frictions.

Proposition 2 (Comparative statics: equilibrium specialization)**.** *Consider the partial equilibrium effect of a change in search efficiency λ. The following statements hold:*

[11] Readers familiar with CES monopolistic competition models will have noted the same coincidence of offsetting externalities. We discuss this parallel in detail at the end of this section.

[12] Since in this example each firm is the only producer of its good, there is no positive externality on other firms providing the same input with lower productivity. As a consequence, the fact that the intermediate producer offers $x = 0$ surplus is not an inefficiency per se.

a) there is a subset of firms that increase their optimal level of specialization: $\exists!\ z^\star \in [\underline{z}, \bar{z}]$ *such that* $\frac{ds^\star(z)}{d\lambda} < 0 \quad \forall z < z^\star$ *and* $\frac{ds^\star(z)}{d\lambda} > 0 \quad \forall z > z^\star$*;*

b) if $\lambda\bar{\phi} < 1$*, then all firms increase their optimal level of specialization:* $\frac{ds^\star(z)}{d\lambda} > 0 \quad \forall z \in [\underline{z}, \bar{z}]$.

Consider the partial equilibrium effect of a change in complexity N*. Then all firms decrease their specialization:* $\frac{ds^\star(z)}{dN} < 0 \quad \forall z \in [\underline{z}, \bar{z}]$.

Proposition 2 provides two important comparative statics regarding the partial equilibrium response of specialization, *i.e.*, the change in firm-level specialization by keeping average compatibility probabilities, $\hat{\phi}(z, \bar{z})\ \forall z$, and the wage rate, w, fixed. First, equilibrium specialization is increasing in λ throughout the entire productivity distribution if $\lambda\bar{\phi} < 1$. As firms meet a larger number of counterparties, they are happy to increase specialization and trade-off higher surplus with lower compatibility, as in Kiyotaki and Wright (1993) and Menzio (2023). Second, higher complexity N induces firms to choose a lower level of specialization. Intuitively, for given input finding probability f, more complex production reduces the chances that a final producer is operational. Hence, the marginal benefit of specialization is lower.

We can characterize similar comparative statics on the efficient level of specialization. We now carry out comparative statics exercises on the socially optimal level of specialization, implicitly defined by (6). Our goal is to compare the first-order response of efficient specialization to changes in λ and N with their equilibrium analogs analyzed in Proposition 2. Notice that λ directly affects only the marginal benefit of specialization, that is, the left-hand side of eq. (6). Hence, depending on how the marginal benefit moves with λ, we can pin down the direction of the efficient specialization response. Differentiating the marginal benefit of specialization with respect to λ yields:

$$\frac{d\mathrm{MB}_{\mathcal{S}(z)}}{d\lambda} = \frac{d\mathrm{MB}_{s^\star(z)}}{d\lambda}\bigg|_{s^\star(z)=\mathcal{S}(z)} - \beta(z)(N-1) \tag{8}$$

where $\beta(z) \equiv \theta f^{N-1} e^{-\lambda\bar{\phi}}(-\phi'(s(z)))\Big[(1-\lambda\bar{\phi})\mathbb{E}[A-c]/f + \lambda\int(1-\lambda\hat{\phi}(z,\bar{z}))[A(s(z))-c(z)]\gamma^\star(z,\bar{z})/f\,dz\Big]$ and $\frac{d\mathrm{MB}_{s^\star(z)}}{d\lambda}\big|_{s^\star(z)=\mathcal{S}(z)}$ is the private marginal benefit of specialization evaluated at the efficient specialization. If $\lambda\bar{\phi} < 1$, the private marginal benefit of specialization is increasing in λ (both directly and indirectly through an increase in the offered surplus). On the other hand, the network externality exacerbates as λ gets higher, which pushes down the marginal benefit of efficient specialization. Hence, whether or not specialization should increase as search frictions decline is a priori ambiguous and crucially depends on the complexity of the production process, which governs the magnitude of the network externality. On the contrary, higher complexity both reduces the marginal benefit of specialization and increases its marginal cost. As a result, efficient specialization is decreasing in complexity. We summarize our results in the next proposition.

Proposition 3 (Comparative statics: Efficient specialization)**.** *Consider the first-order effect of a change in search efficiency* λ*. Efficient specialization of firms with productivity* z *is increasing* *in search efficiency if and only if*

$$B(z) > \beta(z)(N-1),$$

where $B(z) = \left.\frac{dMB_{s^\star(z)}}{d\lambda}\right|_{s^\star(z)=\mathcal{S}(z)}$.

Consider the first-order effect of a change in complexity N. Efficient specialization is decreasing in complexity, i.e., $\frac{ds^\star(z)}{dN} < 0 \quad \forall z \in [\underline{z}, \bar{z}]$.

We conclude the discussion of the static model by studying how efficiency changes as we vary the degree of complexity N and the intensity of search frictions λ.

In Figure 1, we use a quantitative version of our model to compare the equilibrium and efficient allocation as we vary N and λ. In Appendix D, we specify the functional form assumptions and the parameter values behind this figure.

(a) Specialization (b) Resilience (c) Output (d) Welfare

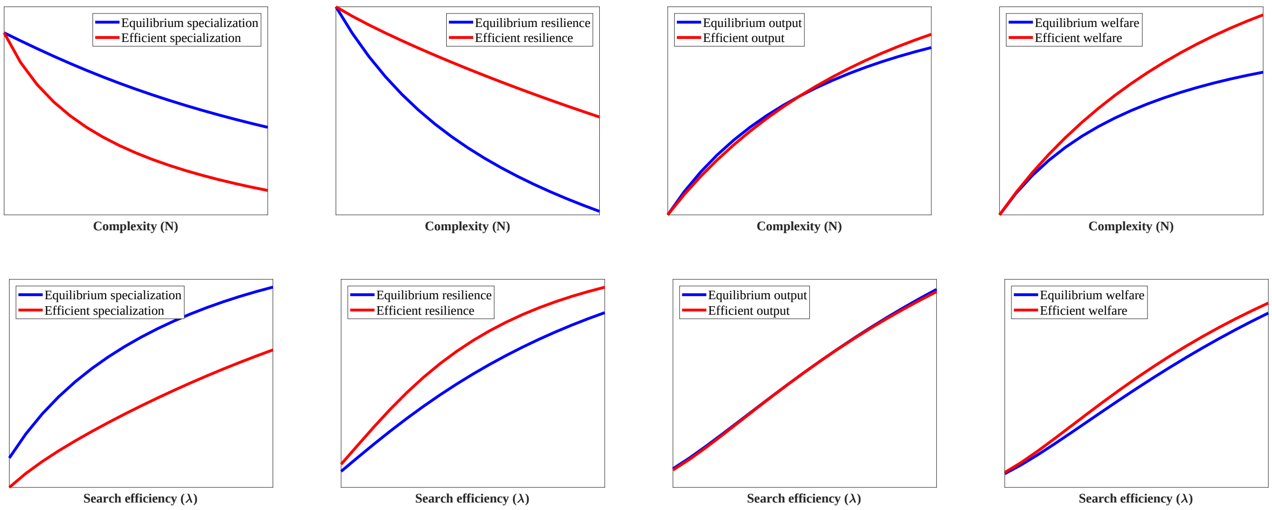

Figure 1: Comparative statics: static model

Note: From left to right, the panels show equilibrium and efficient specialization, resilience, output and welfare as we vary complexity N and search efficiency λ. Note that due to linear utility in consumption, the welfare differences can be read as income-equivalent variations. Details of the quantification are reported in Appendix D.

The leftmost panel of Figure 1 shows the equilibrium and efficient levels of specialization. As highlighted by Proposition 1, whenever production is not complex ($N = 1$), the equilibrium is efficient. Whenever $N > 1$, the economy features over-specialization. As N grows, this inefficiency becomes more severe. As a consequence, resilience is inefficiently low and worsens. Intuitively, as complexity rises, the strength of the network externality increases, inducing larger wedges between the equilibrium and planner allocation. As search frictions decline ($\lambda \uparrow$), firms increase their specialization choices at a faster pace than the social planner. As a consequence, the economy becomes progressively more inefficiently over-specialized. Interestingly, equilibrium output is relatively close to its efficient counterpart, suggesting that most of the welfare loss is driven by inefficiently high use of labor to increase specialization.

We conclude that economies featuring more complex production and, surprisingly, lower search frictions tend to be more inefficient. This is driven by the endogenous choice of firms over-specializing.

Relation to CES - Monopolistic Competition Models. In order to clarify the efficiency properties of the equilibrium, we find it instructive to establish a parallel between our price posting model with endogenous specialization and standard models of monopolistic competition with CES preferences. For the sake of the argument, we consider the stripped-down version of our model where production is not complex, *i.e.*, for $N = 1$.

We think of the mass of varieties in models of monopolistic competition as the counterpart of the average probability of input suitability $\bar{\phi}$ in our framework, which defines the mass of suitable varieties for final producers. For given search efficiency λ, the mass of suitable varieties moves in response to specialization decisions. In monopolistic competition, the mass of varieties moves in response to firm entry, being each variety suitable to the production process of any final producer. In both models, product design decisions are subject to an appropriability externality and a business-stealing externality. First, as long as $x(z) > 0$, intermediate producers do not appropriate the entire match surplus created by their specialization in our model. This pushes towards excess specialization, that is, to create too niche products. Similarly, firms cannot appropriate the additional surplus their entry induces through love-of-variety in monopolistic competition models. This pushes entry to be lower than optimal. Second, in our model, individual specialization decisions affect the trading probability of other firms through competition among the suitable input providers contacted by the same final producer. In this sense, higher specialization is *business-giving*, since it makes it more likely that lower productivity firms will find a match. This effect pushes firms to specialize less than optimal, that is, to create broader products in terms of potential market size, because they do not internalize the positive effect of higher own specialization on the trading probability of other (lower-productivity) firms. Similarly, in monopolistic competition models, firms do not internalize the reduction in profits their entry entails for incumbent firms through a reduction in the CES price index. Exactly as the coincidence of the elasticity of substitution with the inverse elasticity of demand with CES preferences[13], our price posting mechanism makes the equilibrium offered surplus $x(z)$ perfectly balance appropriability and business-stealing externalities. [14]

Relation to Bargaining Models. Our model features important similarities, along with key differences, with models of bargaining over the returns to an investment. We start by interpreting our price posting model through the lens of a bargaining model with endogenous bargaining

[13]This result is more general than CES preferences and generally holds whenever the elasticity of demand coincides with the preference-for-varieties, be it constant or variable, as shown in Zhelobodko et al. (2012) and Parenti et al. (2017).

[14]Albeit present in Menzio (2023), we are the first to highlight the existence of these two sources of externality from endogenous specialization decisions and the way in which the price posting mechanism allows exactly balancing them. We also extend the efficiency result derived in Menzio (2023) to an environment of heterogeneous firms. This result is reminiscent of the relationship between the efficiency results in Dhingra and Morrow (2019) compared to those in Dixit and Stiglitz (1977) in the context of monopolistic competition models with CES preferences.

power. Then, we highlight the importance of the relative timing of investment and surplus splitting to drive the efficiency properties of equilibrium specialization.

To best understand the comparison with bargaining models, it is instructive to normalize the offered surplus by match surplus to identify the offered *share* of total surplus:

$$\xi(z) \equiv \frac{x(z)}{A(s(z)) - c(z)} = \frac{\mathbb{E}_{\tilde{z},\, \tilde{z}<z}[A(s(\tilde{z})) - c(\tilde{z})]}{A(s(z)) - c(z)}. \tag{9}$$

The offered surplus share equals the ratio between the expected match surplus from trading with lower-productivity firms and the match surplus from trading with the firm itself. This is conceptually related to the (exogenous) bargaining power weight in models where firms bargain ex-post over the match surplus. Unlike in those models, our ex-ante price posting mechanism allows endogenizing the relative bargaining power of trading counterparties in an intuitive way. Final producer appropriate a larger share of the surplus – or enjoy higher bargaining power – the higher the number of intermediate producers they can access (higher λ) and the less differentiated inputs are (lower $\mathrm{Var}[s(z)]$). Symmetrically, intermediate producers' *market power* – that is, the ability to appropriate a positive share of the surplus – stems from concentration in final producers' suppliers and product differentiation. Hence, our model not only encompasses rich sources of market power but also allows for endogenous product differentiation via specialization decisions.[15] In sum, the surplus-splitting rule implied by our price posting model is akin to an endogenous bargaining power weight with appealing properties.

We now turn to the role of the relative timing of investment in specialization and surplus splitting to determine equilibrium specialization, as well as its efficiency properties. Bargaining models typically involve an ex-ante investment made by one of the two parties and ex-post bargaining over the returns of such an investment. Since the investing party cannot fully appropriate the returns to its investment, the equilibrium typically features inefficiently low investment due to an appropriability externality, or *hold-up problem*. The key assumption for under-investment to arise is that the investment cost is sunk at the bargaining stage. Importantly, this is not the case in our price posting model. Our intermediate producers invest in specialization at the same time in which they post a transaction price (offered surplus). Rather than bringing about under-investment, the appropriability externality pushes towards over-investment in specialization, as the social cost of lower trading probability is underestimated. Moreover, the competitive force induced by simultaneous search by final producers allows the appropriability externality to be perfectly offset by the resulting business-stealing externality in equilibrium. Hence, our price posting model provides an efficient benchmark to bargaining models featuring hold-up problems.

We provide further details on the relation between our framework and bargaining models in Appendix B.4, in which we lay out a model with sunk specialization choices and ex-post bargaining.

[15] Costly specialization decisions imply that markups larger than unity need to exist in equilibrium for intermediate producers to (at least) break even with their product design costs, exactly as in growth models of expanding variety with R&D investment.

Resilience We conclude the treatment of the static economy with an observation: even though the model is static, the probability that a final producer is operational, or equilibrium *sourcing capacity* $f^N = [1 - \exp\{-\lambda\bar{\phi}\}]^N$, provides an intuitive notion of supply chain *resilience*. In a dynamic setting, its reciprocal, $1/f^N$, equals the average time it takes a final producer to restore production upon losing contact with their suppliers. Therefore, our model of endogenous specialization with complex production provides an appealing way to think about input substitutability and time-to-build in supply chains. Our results suggest that resilience may be lower than efficient due to over-specialized products.

In the next section, we extend our model to a dynamic setting in which firms can face disruptions and ask how this interplays with specialization choices.

3 Dynamic Model

In this section, we extend the static model to a dynamic framework. We assume that final producers face a disruption each period with probability δ, where $\delta \in (0, 1)$. Upon facing a disruption, final producers lose contact with all their input providers.[16] We refer to the average time a final producer does not face any disruption, *i.e.* $1/\delta$, as *robustness*. Intermediate producers never face disruptions and search for new customers in the market in each period. The dynamic model nests the static model as $\delta \to 1$.

Throughout, we restrict our attention to a stationary equilibrium where flows are balanced. Balance of flows dictates that demand for an intermediate producer with specialization s offering surplus x, $\mathcal{D}(s, x)$, *i.e.*, the expected number of final producers sourcing from it, is given by the trading probability, $\mathcal{P}(s, x)$, rescaled by $\theta\lambda/\delta$:

$$\mathcal{D}(s, x) \equiv \frac{\theta\lambda}{\delta}\mathcal{P}(s, x), \tag{10}$$

where the numerator represents the inflow of new customers each period and the denominator the outflow rate of existing customers. In equilibrium, the product market tightness equals the ratio between searching final producers $\mu(f) \equiv \frac{\delta}{\delta+(1-\delta)f^N}$ and intermediate producers m:

$$\theta = \frac{\mu(f)}{m}. \tag{11}$$

The timing convention is as follows: at the end of each period, a share δ of attached final producers faces a disruption and loses all its input providers. At the start of the next period, new final producers replace them. These, together with already existing unattached final producers, search for suppliers.Throughout, we maintain the assumption that m is fixed. In Appendix B.3, we endogenize the mass of intermediate goods producers via a free entry condition.

[16]The model is isomorphic to one where final producers exit the market each period with probability δ and exiting final producers get replaced by an equal measure of new entrants.

Equilibrium. As in the static model, intermediate producers maximize expected operating profits net of product design costs:

$$V(s,x;z) = \max_{s,x} = \mathcal{D}(s,x)\left(A(s) - x - c(z)\right) - wq(s).$$

Notice that the only difference with respect to the static model is that operating profits get discounted at rate δ rather than 1. The equilibrium is characterized by four conditions: the two FOCs pinning down the offered surplus and specialization and the market clearing conditions for labor and goods markets, which we report the full equilibrium in Appendix A. Only the specialization condition differs from the static model:

$$\frac{\theta\lambda\mathcal{P}(z)}{\delta}\left[A'(s(z)) - \left(\frac{-\phi'(s(z))}{\phi(s(z))}\right)\left(A(s(z)) - c(z) - x(z)\right)\right] = wq'(s(z)). \tag{12}$$

Output in this economy can be decomposed into four terms:

$$y = \underbrace{\mu(f)}_{\text{market size}} \quad \underbrace{f^N}_{\text{resilience}} \quad \underbrace{\frac{1}{\delta}}_{\text{robustness}} \quad \underbrace{N\int [A(s(z)) - c(z)]\frac{\gamma^\star(z,\bar{z})}{f}dz}_{\text{expected match surplus|active}}.$$

First, output depends on market size (of investment in specialization) positively. Market size increases if the mass of searching final producers μ increases. Second, output is increasing in the economy's resilience, *i.e.* the probability that a searching final producer finds all the key inputs it needs. Higher resilience can be attained either through a reduction in search frictions ($\lambda \uparrow$), a reduction in aggregate specialization ($\bar{\phi} \uparrow$), or a reduction in complexity ($N \downarrow$). Third, output is increasing in the economy's robustness, *i.e.* the average duration of a match between final producers and intermediate producers. Higher robustness obtains through a reduction in δ. Finally, output is increasing in the expected match surplus conditional on production.

Social planner problem The social planner solves:

$$\max_{s_i(z)} C + \psi\log(1-\ell) \quad \text{s.t.} \quad C = y \text{ and } \ell = Nm\bar{q}.$$

Efficient specialization of intermediate producers with productivity z producing input i is given by:

$$\begin{aligned} s_i(z) = \operatorname*{argmax}_{\hat{s}_i(z)} \; & \frac{\delta}{\delta + (1-\delta)f^{N-1}f(\hat{s}_i(z))}\left(\frac{f^{N-1}}{\delta}\int \lambda\phi(\hat{s}_i(z))e^{-\lambda\hat{\phi}(z,\bar{z})}[A(\hat{s}_i(z)) - c_i(z)]\gamma(z)dz\right. \\ & \left. + \sum_{j\neq i}^{N-1}\frac{f^{N-2}}{\delta}f(\hat{s}_i(z))\mathbb{E}[A_j - c_j]\right) + \psi\log\left(1 - m\left(\sum_{j\neq i}^{N-1}\bar{q}_j + \bar{q}(\hat{s}_i(z))\right)\right). \end{aligned}$$

The first line represents the expected match surplus for input i. The second line represents the expected match surplus of the other inputs $j \neq i$, along with the disutility of labor supply.

By symmetry, efficient specialization of intermediate producers with productivity z producing any input is given by:

$$\theta\lambda\mathcal{P}(z)\Bigg[A'(s(z)) - \left(\frac{-\phi'(s(z))}{\phi(s(z))}\right)\Bigg([A(s(z)) - c(z)] - \int_{\underline{z}}^{z}[A(s(\tilde{z})) - c(\tilde{z})]\gamma^{\star}(\tilde{z},\bar{z})d\tilde{z} \tag{13}$$
$$+ (N-1)e^{-\lambda\hat{\phi}(\underline{z},z)}\mathbb{E}[A-c]/f - N\frac{(1-\delta)f^N}{\delta+(1-\delta)f^N}e^{-\lambda\hat{\phi}(\underline{z},z)}\mathbb{E}[A-c]/f\Bigg)\Bigg] = \frac{\psi}{1-Nm\bar{q}}\,q'(s(z)).$$

Comparing (13) with (6), we observe that a further inefficiency arises in a dynamic setting: intermediate producers do not internalize the effect of their specialization on the equilibrium share of searching final producers. This force pushes the private marginal cost of specialization to exceed the social one. Rearranging (13) allows us to establish the following proposition.

Proposition 4 (Efficiency of the Dynamic Economy). *The economy is efficient if and only if $N = \frac{1}{\mu}$, where $\mu \equiv \frac{\delta}{\delta+(1-\delta)f^N}$. If $N > \frac{1}{\mu}$ then the economy features over-specialization. Furthermore, if production is not complex ($N = 1$) or is infinitely robust ($\delta \to 0$), the economy necessarily features under-specialization.*

We can follow the same steps used in the static setting to derive the marginal welfare effect of equilibrium specialization. We separate such a marginal welfare effect into four terms. The first three are the externalities discussed in the previous section: the business-stealing, appropriability, and network externalities. A fourth effect arises in the dynamic model: intermediate producers choosing their specialization affect the market tightness by varying the probability that inputs are compatible. This effect, akin to the standard search externality in models of endogenous search effort (Pissarides, 2000), is not internalized by intermediate producers.

$$\left.\frac{\partial\mathcal{W}}{\partial s(z)}\right|_{\mathcal{S}(z)=s^{\star}(z)} \propto \underbrace{\mathbb{E}_{\tilde{z},\tilde{z}<z}\left[A(s(\tilde{z})) - c(\tilde{z})\right]}_{\text{business-stealing externality}} \underbrace{-\ x(z)}_{\substack{\text{appropriability}\\ \text{externality}}} \underbrace{-\ (N-1)e^{-\lambda\hat{\phi}(\underline{z},z)}\mathbb{E}[A-c]/f}_{\text{network externality}}$$
$$\underbrace{+\ N\frac{(1-\delta)f^N}{\delta+(1-\delta)f^N}e^{-\lambda\hat{\phi}(\underline{z},z)}\mathbb{E}[A-c]/f}_{\text{search externality}} \tag{14}$$

With this marginal welfare effect decomposition, we can turn to the discussion of Proposition 4. We know from Lemma 2 that the first two terms in (14) cancel out in equilibrium thanks to our price-posting mechanism. As before, consider the case of non-complex production by setting $N = 1$, we can eliminate the network externality, so that we are left with the new source of inefficiency alone. This effect is positive, so the planner would like to induce further specialization. This new external effect implies that the socially optimal level of specialization is higher, all else equal. This also implies that, for $N > 1$, the equilibrium is "less inefficient" than in the static case, as the two network and search externalities partially offset each other.

Proposition 4 further notes that there is a knife-edge case where $N = \frac{1}{\mu}$ in which these two externalities exactly cancel out and the equilibrium is constrained efficient. Furthermore, the critical value for N such that the equilibrium features *over-specialization* is higher than in the static model, since the new externality increases the socially optimal level of specialization.

Indeed, at the static critical value $N = 1$, the economy is *under-specialized* since the network externality vanishes, but the search externality survives.

Figure 2 shows the behavior of equilibrium and efficient specialization, resilience, output, and welfare as we vary complexity N and search efficiency λ. Differently from the static model, Proposition 4 says that there is a knife edge case in which the economy is constrained efficient. As discussed in Proposition 4, the equilibrium can feature both *over-* and *under-specialization*. The leftmost panel shows that as production becomes more complex, the network externality dominates, and the economy becomes increasingly more over-specialized. At low levels of complexity, as shown in Proposition 4, the economy features under-specialization. As established in Proposition 4, the equilibrium under-invests in specialization whenever the production process is not complex (N=1). As search frictions decline ($\lambda \downarrow$), the economy moves towards under-specialization as the equilibrium specialization grows less than the efficient one. As a consequence, the economy displays too much resilience and produces less than efficient.

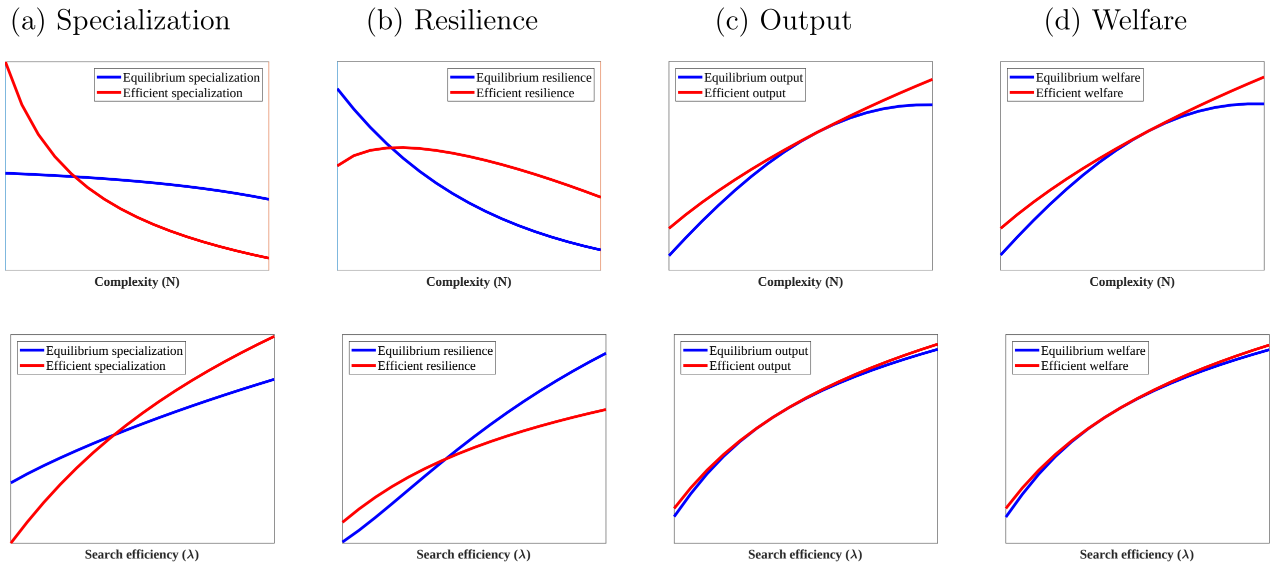

Figure 2: Comparative statics: dynamic model
Note: From left to right, the panels show equilibrium and efficient specialization, resilience, output and welfare as we vary complexity N and search efficiency λ. Note that due to linear utility in consumption, the welfare differences can be read as income-equivalent variations. Details of the quantification are reported in Appendix D.

4 Endogenizing Complexity and Robustness

So far, we have studied economies in which the production process of final producers was taken as given. In this section, we consider final producers' choices on the complexity of their production process as well as prudential investment in redundancies to reduce the likelihood of disruptions.

4.1 Endogenous Complexity

In this section, we consider the problem of firms choosing the complexity of their production process. In particular, firms optimize over the number of key inputs in the production function.

This approach is similar to the core idea of Oberfield (2018), Boehm and Oberfield (2020), Kopytov et al. (2021) and Kim and Shin (2023).

Static model. We allow final goods producers to choose the complexity of the production process to maximize expected profits:

$$N^{\star} = \operatorname*{argmax}_{N \geq 1} f^N N \frac{\mathbb{E}[x]}{f},$$

where f is the probability of finding an input on the market, and $\mathbb{E}[x]/f$ is the expected offered surplus per input conditional on matching. Intuitively increasing complexity induces two opposite effects: on the one hand, adding inputs increases the value of output directly; on the other hand, it reduces the probability that the final good is produced altogether. The optimal level of complexity solves:

$$N^{\star} = \frac{1}{\ln(1/f)}. \tag{15}$$

Hence, complexity is increasing in the input finding probability f. Efficient complexity solves:

$$\mathcal{N} = \operatorname*{argmax}_{N \geq 1} \; f^N N \frac{\mathbb{E}[A-c]}{f} + \psi \log\left(1 - N m \bar{q}\right).$$

Relative to the problem of individual firms, the social planner problem features two differences. First, additional complexity is evaluated at the expected social surplus $\mathbb{E}[A-c]$ instead of the expected private surplus $\mathbb{E}[x]$. Second, the planner accounts for the additional specialization costs associated with additional inputs, taking the form of higher disutility from labor. Hence, efficient complexity is given by:

$$\mathcal{N} = \frac{1}{\ln(1/f)} \left(1 - \frac{w\ell}{y}\right). \tag{16}$$

where y is given by (5) and the labor share $w\ell/y$ represents the ratio between aggregate product design costs and value added. Comparing (15) with (16) allows us to establish the following proposition:

Proposition 5 (Efficiency of the Endogenous Complexity Economy)**.** *The equilibrium is inefficient and features under-resilience, i.e.,* $f^{N^{\star}} < f^{\mathcal{N}}$*.*

Intuitively, final producers do not internalize the marginal product design cost induced by their complexity choice, thereby generating a *reverse hold-up problem*. Choosing a higher level of complexity forces intermediate good producers to invest more in product design. These costs are sunk at the time of price posting As a result, equilibrium sourcing capacity – our notion of “static resilience" – is lower than efficient.

Dynamic model. Equilibrium complexity is the same as in the static model and given by (15). Efficient complexity solves:

$$N^{\star} = \underset{N \geq 1}{\operatorname{argmax}} \; \frac{\delta}{f^N + (1 - f^N)\delta} \left(\frac{f^{N-1}}{\delta} N \mathbb{E}[A - c] \right) + \psi \log\left(1 - mN\bar{q}\right).$$

Efficient complexity is given by:

$$\frac{1}{\mu(f, N) \ln(1/f)} \left(1 - \frac{w\ell}{y}\right). \tag{17}$$

Comparing (17) with (16), we observe that a further inefficiency arises in a dynamic setting. Following the same approach adopted for specialization, the marginal welfare effects of equilibrium complexity can be decomposed into two terms:

$$\left.\frac{\partial \mathcal{W}}{\partial N}\right|_{\mathcal{N}=N^{\star}} \propto \underbrace{-\,w\ell/y}_{\substack{\text{appropriability} \\ \text{externality}}} \quad \underbrace{+\,(1-\mu)}_{\text{search externality}}. \tag{18}$$

First, an appropriability externality induces equilibrium complexity to be higher than efficient. The reason is that final producers fail to internalize the product design costs of each additional input, which are borne by intermediate producers before meeting. The surplus share of such product design costs is represented by the labor share. Second, final producers do not internalize the effect of their complexity choice on the equilibrium share of searching final producers. This search externality pushes the private marginal cost of complexity to exceed the social one. Rearranging (17) allows us to establish the following result:

Proposition 6 (Efficiency of the Dynamic Endogenous Complexity Economy)**.** *The equilibrium is efficient if and only if:*

$$\begin{cases} N = \frac{1}{\mu} \\ LS = 1 - \mu \end{cases}$$

where $\mu = \frac{\delta}{\delta + (1-\delta) f^N}$ *and* $LS = \frac{w\ell}{y}$*. The equilibrium features under-resilience if and only if* $LS^{\star} > 1 - \mu^{\star}$*.*

In general, the equilibrium features both inefficient complexity and specialization. In Figure 3, we study how specialization, complexity, resilience, output, and welfare compare in the equilibrium and in the social planner allocation as we vary search efficiency λ. As the efficiency of the search process increases ($\lambda \uparrow$), the economy features too much specialization and too little complexity. Gains in search efficiency improve welfare but increase the distance between the socially optimal and equilibrium allocation.

(a) Specialization (b) Complexity (c) Resilience (d) Output (e) Welfare

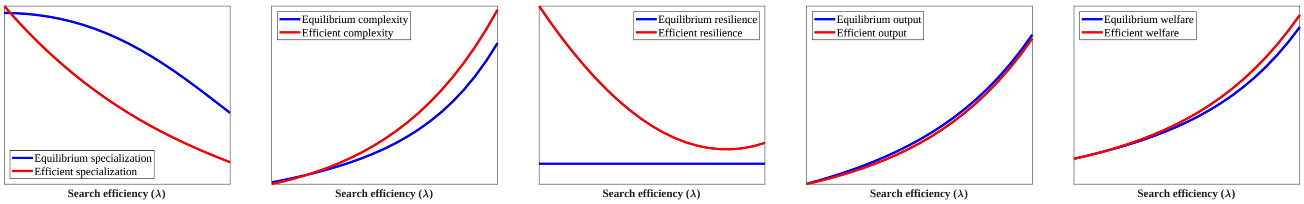

Figure 3: Comparative statics: endogenous complexity model
Note: From left to right, the panels show equilibrium and efficient specialization, complexity, resilience, output and welfare as we vary search efficiency λ. Note that due to linear utility in consumption, the welfare differences can be read as income-equivalent variations. Details of the quantification are reported in Appendix D.

4.2 Endogenous Robustness

So far, we have treated the disruption probability of final producers as an exogenous parameter. In this section, we allow final producers to make prudential investments in robustness aimed at reducing their disruption probability, *e.g.*, through redundancies. Investment in robustness comes at per-period cost $\kappa(r)$ per supply relationship, with $\kappa' > 0$, $\kappa'' < 0$. Specifically, we posit that $\delta = \delta(r)$, with $\delta' < 0$, $\delta'' > 0$. For analytical tractability, we assume that final producers choose their robustness policy before matching with their input providers.[17] We think of robustness policies as the design and implementation of prudential strategies to reduce the exposure to exogenous shocks. Formally, final producers choose their robustness to maximize expected profits:

$$r = \underset{\tilde{r}}{\operatorname{argmax}}\, f^N N \frac{\mathbb{E}[x]/f - \kappa(\tilde{r})}{\delta(\tilde{r})}, \tag{19}$$

where $\mathbb{E}[x]/f$ is the expected offered surplus per input conditional on matching. Equilibrium robustness solves:[18]

$$-\delta'(r)\frac{\mathbb{E}[x]/f - \kappa(r)}{\delta(r)} = \kappa'(r). \tag{20}$$

The privately optimal robustness trades off a lower disruption probability, $-\delta'$, evaluated at the expected present discounted value of each supplier relationship, $\frac{\mathbb{E}[x]/f-\kappa}{\delta}$, against higher private marginal costs, κ'. Importantly, note that firms discount the future with their endogenous disruption probability, δ. Under the assumption that final producers decide their robustness before matching, the expected offered surplus is constant across final producers, and so is their optimal robustness. Due to the presence of robustness costs, final producers need to break even with their share of the match surplus. As a result, the "reservation offered surplus", x_0, of final

[17] More specifically, we assume that final producers commit to a robustness policy r at the search stage for the entire life of the firm before a disruption occurs. Implementing such a robustness policy requires paying a per-period cost $\kappa(r)$ per supplier relationship. If final producers could optimize their robustness policy once matched, intermediate producers would internalize the robustness response of downstream partners when choosing their offered surplus.

[18] The second-order condition holds whenever $\kappa''(r) + 2\frac{(-\delta'(r))}{\delta(r)}\kappa'(r) > \frac{2\delta'^2(r)-\delta''(r)\delta(r)}{\delta(r)^2}\left(\mathbb{E}[x]/f - \kappa(r)\right)$.

producers is positive and pinned down by the following expected break-even condition:

$$x_0 + (N-1)\mathbb{E}[x]/f = N\kappa(r). \tag{21}$$

Final producers need to source N key inputs. Whenever they meet with a potential supplier, they know that production will ultimately entail a $N\kappa(r)$ robustness cost. They are, therefore, willing to accept any surplus such that operating profits exceed that cost, where $(N-1)\mathbb{E}[x]/f$ represents the expected surplus they can obtain from the other $N-1$ suppliers. The reservation offered surplus acts as a boundary condition for the differential equation governing the equilibrium offered surplus (2). Hence, the new equilibrium offered surplus reads:

$$x(z) = x_0 + \mathbb{E}_{\tilde{z},\tilde{z}<z}[A(s(\tilde{z})) - c(\tilde{z})]. \tag{22}$$

Eq. (21) and (22) jointly imply that the equilibrium reservation offered surplus is given by $x_0 = \kappa(r) - \frac{N-1}{N}\mathbb{E}[\tilde{x}(z)]/f$, where $\tilde{x}(z) = \mathbb{E}_{\tilde{z},\tilde{z}<z}[A(s(\tilde{z})) - c(\tilde{z})]$. Note that $x_0 < \kappa(r)$, suggesting that final producers are willing to take on negative net surplus from a single relation since they can cross-subsidize it with the other $N-1$ relationships. Up to the different formulation of the equilibrium offered surplus function, equilibrium specialization is still determined by (12). Interestingly, since $\delta(r)$ discounts the marginal benefit of specialization in (12), equilibrium specialization is increasing in robustness.

Social planner problem. Next, we study the planner problem to evaluate the efficiency properties of equilibrium specialization and robustness decisions. The efficient specialization condition has the same formulation as in the model with exogenous robustness, up to the definition of total match surplus as net of robustness costs. However, the efficiency properties of equilibrium specialization differ significantly from the dynamic model with exogenous robustness due to the fact that the reservation offered surplus is not specific to each intermediate producer. Proposition 7 identifies the reason why the mere presence of costs downstream makes the equilibrium offered surplus inefficient. Corollary 1 applies this general inefficiency result to the endogenous robustness economy. In Appendix A.3, we discuss this point in detail.

Proposition 7 (Efficiency Properties of Equilibrium Specialization with Costs Downstream)**.** *If final producers sustain other costs than input purchases, then the property of the equilibrium offered surplus of perfectly balancing appropriability and business-stealing externalities breaks.*

Corollary 1 (Special Case - Robustness Costs)**.** *Equilibrium firm-level specialization is always inefficient in the endogenous robustness economy due to the presence of robustness costs downstream.*

Intuitively, a social planner would like the reservation offered surplus to vary across the intermediate producers' productivity distribution for business-stealing and appropriability externalities to offset each other in the presence of robustness costs. Since the equilibrium reservation offered surplus is independent of productivity, misallocation of specialization always arises in the cross-section.

Efficient robustness solves the following social planner problem:

$$\mathcal{R} = \underset{r}{\text{argmax}} \;\; \mu(f,r)\frac{f^N}{\delta(r)}N\Big(\mathbb{E}[A-c]/f - \kappa(r)\Big).$$

Notice that robustness costs are borne by active final producers, whose equilibrium mass is $\mu\frac{f^N}{\delta} = \frac{f^N}{\delta+(1-\delta)f^N}$. Efficient robustness is implicitly defined by:

$$-\delta'(\mathcal{R})\left(1-f^N\right)\mu(f,\mathcal{R})\frac{\mathbb{E}[A-c-\kappa(r)]/f}{\delta(\mathcal{R})} = \kappa'(\mathcal{R}). \tag{23}$$

The socially optimal level of robustness trades off a lower disruption probability, $-\delta'$, evaluated at the expected present discounted social value of each supplier relationship, $\left(1-f^N\right)\mu\,\frac{\mathbb{E}[A-c-\kappa]/f}{\delta}$, against the marginal robustness cost, κ'. Comparing (20) with (23), we observe that equilibrium robustness is generally inefficient. Let the aggregate surplus share accruing to final producers be $\tilde{\xi} \equiv \frac{\mathbb{E}[x-\kappa]}{\mathbb{E}[A-c-\kappa]}$. The marginal welfare effect of equilibrium robustness can be decomposed into two terms:

$$\left.\frac{\partial\mathcal{W}}{\partial r}\right|_{\mathcal{R}=r^\star} \propto \underbrace{1-\tilde{\xi}}_{\substack{\text{appropriability}\\ \text{externality}}} \;\; \underbrace{-\left[1-(1-f^N)\mu\right]}_{\text{search externality}}. \tag{24}$$

First, a standard appropriability externality (hold-up problem) induces equilibrium investment in robustness to be lower than efficient, as the flow private marginal benefit, $\mathbb{E}[x-\kappa]$, falls short of the flow social marginal benefit, $\mathbb{E}[A-c-\kappa]$. Second, firms do not internalize the effect on equilibrium tightness induced by their higher robustness. This search externality is evident when noting that the social planner discounts the marginal surplus at a higher rate, $\delta/[(1-f^N)\mu]$, than private firms do, δ. It follows that the search externality induces equilibrium investment in robustness to be higher than efficient. The nature of the search externality is akin to that arising in models of endogenous match destruction, where firms ignore the effect of their reservation productivity choice on equilibrium tightness. As in those models, a specific surplus sharing rule allows decentralizing the efficient allocation (Pissarides, 2000). Formally, efficient robustness obtains in equilibrium (for given aggregate specialization) if the aggregate surplus share accruing to final producers equals the equilibrium share of unattached final producers, *i.e.*, $\tilde{\xi} = (1-f^N)\mu$. Under this condition, the appropriability externality exactly offsets the search externality. Otherwise, the net effect of the appropriability and search externality is ambiguous.[19]

Proposition 8 (Efficiency of Endogenous Robustness Equilibrium)**.** *The equilibrium can feature both under- and over-robustness. For given aggregate specialization, equilibrium robustness is efficient if and only if*

$$\tilde{\xi} = (1-f^N)\mu \;,$$

[19] Standard models of endogenous match destruction imply that either the equilibrium match destruction rate is efficient or lower than efficient. The reason why, in our model, match destruction can be excessive is the presence of the appropriability externality potentially countervailing the search externality.

where $\mu \equiv \frac{\delta(r)}{\delta(r)+(1-\delta(r))f^N}$ *and* $\tilde{\xi} \equiv \frac{\mathbb{E}[x-\kappa]}{\mathbb{E}[A-c-\kappa]}$. *Since equilibrium specialization is increasing in robustness, over-robustness makes over-specialization more likely.*
The equilibrium can feature both under- and over-specialization. For given robustness, equilibrium firm-level specialization is always inefficient.

In general, the equilibrium features inefficient investment in both robustness and specialization. In Figure 4, we study how specialization, robustness, resilience, output and welfare compare in equilibrium and in the planner allocation as we vary complexity N and search efficiency λ.

(a) Specialization (b) Robustness (c) Resilience (d) Output (e) Welfare

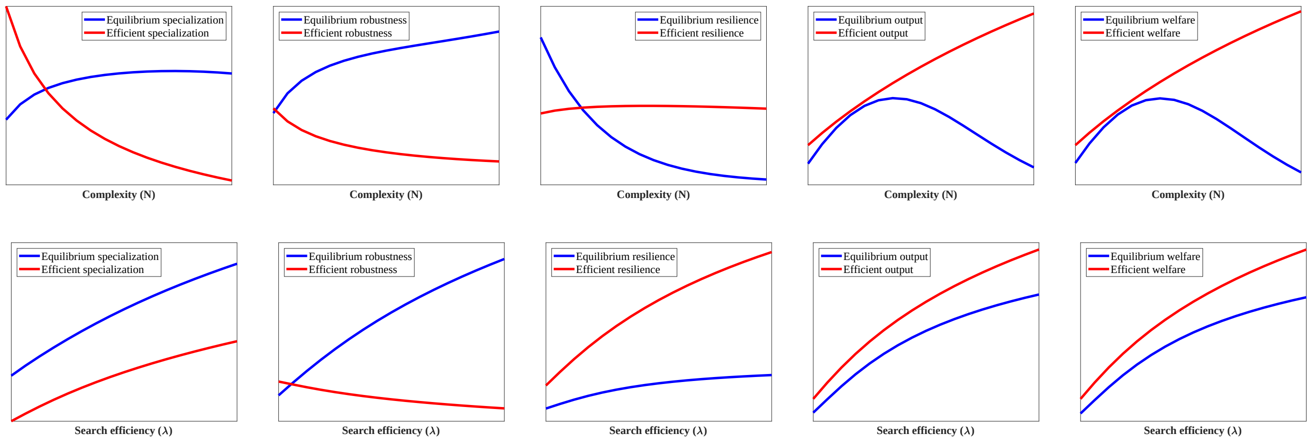

Figure 4: Comparative statics: endogenous robustness model
Note: From left to right, the panels show equilibrium and efficient specialization, robustness, resilience, output and welfare as we vary complexity N and search efficiency λ. Note that due to linear utility in consumption, the welfare differences can be read as income-equivalent variations. Details of the quantification are reported in Appendix D.

As an economy increases its level of production complexity, it tends to over-specialize and over-invest in robustness relative to the constrained efficient allocation. This generates a less-than-efficient level of resilience. Effectively, the social planner and firms in equilibrium choose different points in the trade-off between robustness and resilience. Similarly, as search frictions improve, the economy tends to over-invest in both specialization and robustness compared to the planner.

5 Normative Analysis

In this section, we characterize the optimal policy that decentralizes the constrained efficient allocation in the different economies we have studied in the previous sections.

Static Economy We introduce a targeted tax instrument that allows the planner to redistribute surplus between intermediate and final producers.

From Proposition 1, it follows that intermediate producers offer too high a surplus to final producers. In particular, the efficient surplus equals:

$$\mathcal{X}(z) = x^\star(z) - (N-1)e^{-\lambda\hat{\phi}(\underline{z},z)}\mathbb{E}[A-c]/f. \tag{25}$$

We can write the wedge between the social and private surplus of intermediate producers with productivity z as $\mathcal{X}(z) = x^\star(z) - \tau^\star(z)$ and look for a transaction subsidy schedule that decentralizes the constrained efficient allocation. From (25), we can derive the following result.

Proposition 9 (Optimal Subsidy Schedule in a Static Economy)**.** *The Social Planner can decentralize the constrained efficient allocation via a targeted transaction subsidy schedule $\tau^\star(z)$ such that the price of the transactions is given by $p^\star(z) = p(z) + \tau^\star(z)$, where*

$$\tau^\star(z) = e^{-\lambda\hat{\phi}(\underline{z},z)}T^\star > 0, \tag{26}$$

and $T^\star \equiv (N-1)\,e^{-\lambda\bar{\phi}}\mathbb{E}[A-c]/f > 0$ is a function that only depends on aggregates. The subsidy schedule is decreasing in z and bounded between $[\tau^\star(\bar{z}), \tau^\star(\underline{z})] = [(1-f)^{-1}T^\star, T^\star]$.

Note that, while in equilibrium, it is natural that $x^\star(z) > 0$, $\forall z$, this does not need to be the case in the decentralized constrained efficient allocation. Let $\check{z} : \int_{\underline{z}}^{\check{z}}[A(s(\tilde{z})) - c(z)]\gamma^\star(\tilde{z},\check{z})d\tilde{z} = (N-1)e^{-\lambda\bar{\phi}}\mathbb{E}[A-c]/f$. A social planner would require that intermediate producers with productivity lower than $\check{z}$ offer a negative surplus to final producers for their specialization to be efficient. Notice that this can be sustained in equilibrium as long as final producers make positive total profits in expectation.

Dynamic Model By combining (12) with (13), we can follow the same steps as in the static model to obtain the surplus that decentralizes the constrained efficient allocation:

$$\mathcal{X}(z) = x^\star(z) - (\mu N-1)\,e^{-\lambda\hat{\phi}(\underline{z},z)}\mathbb{E}[A-c]/f. \tag{27}$$

This result allows us to derive the optimal transaction subsidy schedule in the dynamic context.

Proposition 10 (Optimal Subsidy Schedule in a Dynamic Economy)**.** *The Social Planner can decentralize the efficient allocation via a targeted transaction subsidy schedule $\tau_d^\star(z)$ such that the price of the transactions is given by $p^\star(z) = p(z) + \tau_d^\star(z)$, where*

$$\tau_d^\star(z) = e^{-\lambda\hat{\phi}(\underline{z},z)}T_d^\star \lesseqgtr 0, \tag{28}$$

and $T_d^\star \equiv (\mu N-1)\,e^{-\lambda\bar{\phi}}\mathbb{E}[A-c]/f \lesseqgtr 0$ is a function that only depends on aggregates. The subsidy schedule is decreasing in z and bounded between $[\tau_d^\star(\bar{z}), \tau_d^\star(\underline{z})] = [(1-f)^{-1}T_d^\star, T_d^\star]$. Note that $\tau_d^\star(z) > 0$ represents a lump-sum transaction subsidy and $\tau_d^\star(z) < 0$ represents a lump-sum transaction tax.

Proposition 10 shows that in the dynamic economy, the targeting principle is the same as in the static case. However, the tax schedule is shifted since, in general, $T^\star \neq T_d^\star$. Without further restrictions, these cannot be ranked since the two economies feature different levels of specialization in the equilibria. We can characterize a ranking if we consider a special case discussed in the next Corollary.

Corollary 2 (Special Case - Rescaled Cost Function)**.** *Consider the special case in which the cost of specialization $q(s)$ in the dynamic economy is scaled by δ as if the cost can be annuitised. Then,*

a) The equilibrium specialization is the same in the static and dynamic economy;

b) The optimal tax schedule is such that $\tau_d^\star(z) < \tau^\star(z)$, $\forall z$.

The planner can restore efficiency in both economies via a targeted subsidy schedule which rebalances the surplus towards the intermediate producer. When the economy features a dynamic component, the subsidy schedule is shifted downwards as the search externality is already partially undoing over-specialization.

Endogenous Complexity The endogenous complexity economy features two sources of inefficiency, related to endogenous complexity choices by final producers and specialization choices by intermediate producers. As a result, two tax instruments are needed to decentralize the efficient allocation. Proposition 5 shows that the efficiency condition for specialization is the same as in the dynamic model. It follows that the transaction subsidy schedule (28) suffices to induce efficient specialization choices for given complexity. To address inefficient complexity choices for given specialization, we introduce a profit tax schedule to final producers as a function of their complexity choices. Such an instrument should be designed to make appropriability and search externalities in complexity perfectly offset each other. In Proposition 11, we characterize the optimal tax schedule in our endogenous complexity economy.

Proposition 11 (Optimal Tax Schedule in an Endogenous Complexity Economy)**.** *The Social Planner can decentralize the efficient allocation via two instruments:*

1. *A targeted transaction subsidy schedule $\tau_d^\star(z)$ such that the price of the transactions is $p^\star(z) = p(z) + \tau_d^\star(z)$, where $\tau_d^\star(z)$ is given by (28);*

2. *A profit tax or subsidy schedule $\tau_c^\star(N)$ to final producers as a function of their complexity choices:*

$$\tau_c^\star(N) = N^{\frac{1}{\mu}\left(1-\frac{w\ell}{y}\right)-1}, \tag{29}$$

where $\tau_c^\star > 1$ represents a tax and $\tau_c^\star < 1$ represents a subsidy.

Endogenous Robustness As in the endogenous complexity economy, two tax instruments are needed to decentralize the efficient allocation in the endogenous robustness economy. Due to the presence of robustness costs downstream, the transaction subsidy schedule (28) is no longer able to decentralize efficient specialization for given robustness. Plugging the equilibrium offered surplus (22) into the efficient specialization condition, we obtain the new offered surplus that decentralizes the constrained efficient allocation:

$$\mathcal{X}(z) = x^{\star}(z) + \frac{N-1}{N}\mathbb{E}[\tilde{x}]/f - e^{-\lambda\hat{\phi}(\underline{z},z)}\kappa - (\mu N - 1)\, e^{-\lambda\hat{\phi}(\underline{z},z)}\mathbb{E}[A - c - \kappa]/f. \tag{30}$$

This result allows us to derive the optimal transaction subsidy schedule that induces efficient specialization for given robustness in the endogenous robustness economy. Next, we leverage the efficiency condition for robustness derived in Proposition (8) to construct a flat robustness cost tax schedule that induces efficient robustness for given specialization.

Proposition 12 (Optimal Tax Schedule in an Endogenous Robustness Economy)**.** *The Social Planner can decentralize the efficient allocation via two instruments:*

1. *A targeted transaction subsidy schedule $\tau_t^{\star}(z)$ such that the price of the transactions is given by $p^{\star}(z) = p(z) + \tau_t^{\star}(z)$, with*

$$\tau_t^{\star}(z) = -T_0 + e^{\lambda\hat{\phi}(z,\bar{z})}T_t^{\star} \lesseqgtr 0, \tag{31}$$

   where $T_0 = \frac{N-1}{N}\mathbb{E}[\tilde{x}]/f > 0$ and $T_t^{\star} \equiv e^{-\lambda\bar{\phi}}\kappa + (\mu N - 1)\, e^{-\lambda\bar{\phi}}\mathbb{E}[A - c]/f \lesseqgtr 0$ are functions that only depend on aggregates. The subsidy schedule is decreasing in z and bounded between $[\tau_t^{\star}(\bar{z}), \tau_t^{\star}(\underline{z})] = -T_0 + [(1-f)^{-1}T_t^{\star}, T_t^{\star}]$. Note that $\tau_t^{\star}(z) > 0$ represents a lump-sum transaction subsidy and $\tau_t^{\star}(z) < 0$ represents a lump-sum transaction tax;

2. *A flat robustness cost tax or subsidy $\tau_r^{\star}$ to final producers*

$$\tau_r^{\star} = \frac{\tilde{\xi}}{(1 - f^N)\mu}, \tag{32}$$

   where $\tau_r^{\star} > 1$ represents a tax and $\tau_r^{\star} < 1$ represents a subsidy.

6 Conclusions

In this paper, we study a model of frictional supply chain formation where heterogeneous intermediate producers choose the specialization of their goods. Higher specialization increases productivity but reduces the trading probability. Final producers operate a complex production process, meaning that multiple key inputs need to be sourced for final production to take place. This framework delivers a well-defined notion of supply chain resilience, *i.e.*, the time it takes for final producers to return to normal operations following disruptions. In the model, supply chain resilience is negatively affected by the average specialization of goods, the extent of search frictions, and the complexity of the production process.

We study the efficiency properties of equilibrium specialization. We show that endogenous specialization and price posting imply that appropriability and business-stealing externalities perfectly offset each other. Still, intermediate producers fail to internalize the full reduction in social surplus induced by their specialization decisions due to a novel network externality arising from the interplay between frictional markets, endogenous specialization, and complexity in production. The network externality induces over-specialization in competitive equilibrium. In turn, over-specialization implies that supply chains are less resilient than efficient. We characterize how a social planner can decentralize efficient supply chain resilience through a targeted transaction subsidy. The optimal subsidy schedule takes a simple form comprising an aggregate component and a targeting element that depends on the average trading probability of higher-productivity intermediate competitors.

We study multiple extensions of our basic framework. First, we work out a dynamic version of our model. In the dynamic model, a further search externality arises, as intermediate producers do not internalize the effect of their specialization decisions on their meeting probability. Second, we allow final producers to choose the complexity of the production process and to invest in robustness to reduce the probability of facing a disruption. Hold-up problems in production design costs and robustness costs, as well as search externalities, entail that both equilibrium complexity and robustness are generally inefficient. Hence, the social planner needs an additional tax instrument to decentralize the efficient allocation, namely a profit tax on final producers (with endogenous complexity) and a robustness cost subsidy (with endogenous robustness). In all such extensions, if production is sufficiently complex, the network externality induces under-resilience of supply chains in competitive equilibrium.

This paper sheds light on the sources of inefficiencies that are likely to affect supply chain resilience. Our analytical results call for further work to provide quantification to the qualitative mechanisms we have highlighted.

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Appendix

A Additional Derivations and Extensions

A.1 Specialization Function

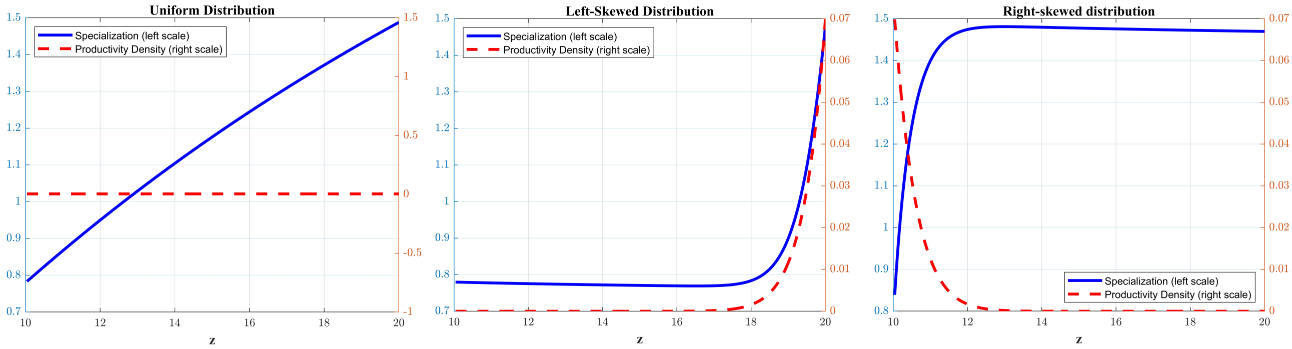

Figure A.1: Specialization function: the role of productivity distribution
Note: The three panels show the cross-sectional shape of the specialization function for different productivity distributions. The left panel considers a uniform distribution $z \sim U[10, 20]$, the middle panel a left-skewed shifted Beta distribution $z \sim \mathcal{B}(1, 18; 10, 20)$, the right panel a right-skewed shifted Beta distribution $z \sim \mathcal{B}(18, 1; 10, 20)$. All the other structural parameters are the same across panels.

Figure A.1 shows how the specialization function is influenced by the shape of the productivity distribution. With a uniform density (right panel), specialization is (approximately) linearly increasing in productivity. A left-skewed productivity distribution (middle panel) entails a nonmonotonic behavior of specialization, which is decreasing in productivity for low productivity levels and exponentially increasing thereafter. Symmetrically, a right-skewed productivity distribution (right panel) gives rise to a mirror-like, nonmonotonic behavior of specialization, which is strongly increasing in productivity for low productivity levels and then gradually decreasing. The qualitative differences across productivity distributions stress the importance of firm heterogeneity as a determinant of specialization choices.

A.2 Dynamic Model

The equilibrium of the dynamic model in Section 3 is given by the following system of equations:

$$x(z) = \int_{\underline{z}}^{z} [A(s(\tilde{z})) - c(\tilde{z})]\gamma^{\star}(\tilde{z}, \bar{z})d\tilde{z}, \tag{33}$$

$$\frac{\theta\lambda\mathcal{P}(z)\}}{\delta}\left[A'(s(z)) - \left(\frac{-\phi'(s(z))}{\phi(s(z))}\right)(A(s(z)) - c(z) - x(z))\right] = wq'(s(z)), \tag{34}$$

$$1 - \frac{\psi}{w} = Nm\bar{q}, \tag{35}$$

$$C = mN\int D(s(z), x(z))[A(s(z)) - c(z)]\gamma(z)dz. \tag{36}$$

The equilibrium conditions represent the first order conditions for the offered surplus and specialization and the labor and goods market clearing conditions, respectively.

A.3 Endogenous Robustness.

The social planner problem reads:

$$\max_{r,\ \hat{s}(z)} \frac{\delta(r)}{\delta(r)+(1-\delta(r))f^{N-1}f(\hat{s}(z))}\frac{f^{N-1}}{\delta(r)}\bigg(\int \lambda\phi(\hat{s}(z))\exp\left\{-\lambda\hat{\phi}(z,\bar{z})\right\}\left[A(\hat{s}(z))-c(z)-\kappa(r)\right]$$
$$\gamma(z)dz+(N-1)\frac{f^{N-1}}{\delta(r)}f(\hat{s}(z))\mathbb{E}\left[A-c-\kappa(r)\right]\bigg)+\psi\log\left(1-m\left((N-1)\bar{q}+\bar{q}(\hat{s}(z))\right)\right).$$

Efficient specialization of intermediate producers with productivity z is given by:

$$\theta\frac{f^{N-1}}{\delta(r)}\left(\frac{\partial\mathbb{E}\left[A-c-\kappa(r)\right]}{\partial s(z)}-(N-1)\frac{-f'(s(z))}{f}\mathbb{E}\left[A-c-\kappa(r)\right]\right)+\theta\frac{f^{N-1}}{\delta(r)}\frac{-f'(s(z))}{f}$$
$$\frac{(1-\delta(r))f^{N}}{\delta+(1-\delta(r))f^{N}}N\mathbb{E}[A-c-\kappa(r)]=\frac{\psi}{1-mN\bar{q}}q'(s(z)). \tag{37}$$

Comparing (37) to (13) we notice that the only difference lies in the definition of total match surplus *net* of robustness costs. Evaluating the planner's first-order condition at the equilibrium specialization yields:

$$\frac{\partial\mathcal{W}}{\partial s(z)}\bigg|_{\mathcal{S}(z)=s^{\star}(z)}\propto\underbrace{\mathbb{E}\left[A(s(\tilde{z}))-c(\tilde{z})-\kappa\right]}_{\text{business stealing externality}}\ \underbrace{-\ x(z)}_{\substack{\text{appropriability}\\ \text{externality}}}\ \underbrace{-\ (N-1)e^{-\lambda\hat{\phi}(\underline{z},z)}\mathbb{E}[A-c-\kappa]/f}_{\text{network externality}}$$
$$\underbrace{+\ \frac{(1-\delta(r))f^{N}}{\delta(r)+(1-\delta(r))f^{N}}e^{-\lambda\hat{\phi}(\underline{z},z)}N\mathbb{E}[A-c-\kappa]/f}_{\text{search externality}}. \tag{38}$$

The only substantive difference with the dynamic model in terms of efficiency properties of equilibrium specialization is that the perfect balance between appropriability and business-stealing externalities breaks down. The reason is that the reservation offered surplus is not targeted to intermediate producers. Formally, the equilibrium offered surplus would keep balancing appropriability and business-stealing externalities if and only if:

$$x_0=(1-e^{-\lambda\hat{\phi}(\underline{z},z)})\kappa\iff\kappa=e^{\lambda\bar{\phi}(\underline{z},z)}\frac{N-1}{N}\mathbb{E}[\tilde{x}(z)]/f\quad\forall z\in[\underline{z},\bar{z}].$$

Since this condition can never hold simultaneously for any productivity level, the net effect of appropriability and business-stealing externalities is ambiguous. This entails that firm-level equilibrium specialization is always inefficient.

B Extensions

In this section, we extend our model along three important dimensions. We start by considering a general production function that allows for arbitrary substitutability or complementarity at the intensive margin among key inputs. This extension brings about a further externality induced by specialization choices associated with complementarity in production. Our second extension focuses on the difference between firm-level disruption and link destruction. So far, we have looked at an economy where final producers lose all their input providers following a disruption. As a direct consequence, final producers need to source all their key inputs anew once a disruption occurs. Here, instead, we study an economy where individual links can break. This extension does not induce any additional inefficiency with respect to the baseline model. Finally, we note that our baseline model features arbitrage opportunities. Specifically, we have not imposed any condition that makes the relative mass of intermediate and final producers adjust to arbitrage away any difference in the expected returns from running either class of firms. In our final extension, therefore, we allow entrepreneurs to enter the market either as intermediate or final producers. This extension brings about a further externality induced by specialization choices related to entry decisions.

B.1 Generalized Production Function

So far, we have looked at an economy in which final producers operate a very specific production function. In particular, we assumed that intermediate goods are i) perfect substitutes at the intensive margin and ii) complements at the extensive margin. While both assumptions matter for the results derived so far, only ii) is crucial for our new network externality to arise. The intuitive reason is that, although goods are complements, an intermediate producer only cares about losing its own share of the surplus if production does not take place. As a consequence, a final producer does not have an instrument to induce bilaterally efficient specialization choices (in a way akin to the inefficiencies induced by limited liabilities). Property ii) is at the heart of this result. If goods are not key, *i.e.*, complementary at the extensive margin, then no network externality arises, and the economy would be statically (constrained) efficient. Note that the same extensive margin complementarity is underlying some of the results in Elliott et al. (2022) and Acemoglu and Tahbaz-Salehi (2023).

We now show that our assumption of perfect substitutability at the intensive margin actually induces fewer inefficiencies than a more general production function would. In particular, suppose that the production function of final producers takes the following generic form:

$$y = \chi(A_1, \ldots, A_N),$$

where χ is a constant-return-to-scale aggregator, which is increasing and concave in each argument. Importantly, we impose $\frac{\partial \chi}{\partial A_i} > 0, \frac{\partial^2 \chi}{\partial A_i^2} < 0 \ \forall i$ and $\chi(A_1, \ldots, A_N) = 0$ if $y_i = 0$ for at least one input i. Hence, we allow for arbitrary substitutability or complementarity at the intensive margin while retaining complementarity at the extensive margin. These assumptions are consistent, for example, with a CES production function with more complementarity than Cobb-Douglas (elasticity of substitution smaller than 1), where all inputs are *key* to the production process in the sense specified above.[20]

Equilibrium. We analyze the equilibrium of the dynamic economy. Intermediate producers maximize expected operating profits net of product design costs:

$$V(s,x;z) = \max_{s,x} = \mathcal{D}(s,x)\left(\tilde{A}(s;\mathbf{s}_{-i}) - x - c(z)\right) - wq(s).$$

where $\mathcal{D}(s,x)$ is given by (10) and $\tilde{A}(s;\mathbf{s}_{-i}) \equiv \frac{\partial\chi}{\partial A_i}A_i$. Notice that the only difference with respect to the baseline dynamic model is that match surplus potentially depends on the specialization vector of all the other input providers $\mathbf{s}_{-i}$. For a generic CES aggregator with substitution parameter $\sigma \in [0,1]$, it holds that $\tilde{A}_i = A_i^{\sigma}\chi^{1-\sigma}$, which nests our baseline model for $\sigma = 1$. Up to the generalized definition of $\tilde{A}$ instead of A, all the equilibrium conditions are the same as in the baseline dynamic model and given by

$$\theta\frac{f^{N-1}}{\delta}\left(\frac{\partial\mathbb{E}[\tilde{A}-c]}{\partial s(z)} - (\mu N - 1)\,\frac{-f'(s(z))}{f}\mathbb{E}[\tilde{A}-c]\right) = \frac{\psi}{1-mN\bar{q}}q'(s(z)) - (N-1)\theta\frac{f^{N-1}}{\delta}\int\frac{\partial\tilde{A}_{-i}}{\partial s(z)}\gamma^{\star}(z)dz.$$

Social planner problem The social planner problem reads:

$$\max_{s_i(z)} C + \psi\log(1-\ell) \quad \text{s.t.} \quad C = y \text{ and } \ell = Nm\bar{q}.$$

Efficient specialization of intermediate producers with productivity z producing input i solves:

$$\begin{aligned}
s_i(z) = \operatorname*{argmax}_{\hat{s}(z)} \; & \frac{\delta}{\delta + (1-\delta)f^{N-1}f(\hat{s}(z))}\left(\frac{f^{N-1}}{\delta}\int\left[\tilde{A}\left(\hat{s}(z),\mathbf{s}_{-i}\right) - c(z)\right]\gamma^{\star}([\hat{s}(z),z],\bar{z})dz\right. \\
& \left. + \sum_{j=1}^{N-1}\frac{f^{N-2}}{\delta}f(\hat{s}(z))\int\left[\tilde{A}\left(s_j(z);\hat{s}(z),\mathbf{s}_{-(i,j)}\right) - c(z)\right]\gamma^{\star}([s_j(z),z],\bar{z})dz\right) \\
& + \psi\log\left(1 - m\left((N-1)\bar{q} + \bar{q}(\hat{s}(z))\right)\right).
\end{aligned}$$

The first line represents the expected match surplus for the input of interest. The second line represents the expected match surplus of the other inputs than the one of interest. The third line represents the disutility of labor supply.

We observe that a further inefficiency arises if inputs are not perfect substitutes at the intensive margin. The reason is that intermediate producers do not internalize the effect of their

[20]These assumptions are also consistent with a CES production function with substitutable inputs, constrained by the requirement of having all inputs to produce. Given our interpretation of the intensive margin of production as input quality, we, therefore, allow for the possibility that conditional on all inputs being sourced, higher quality of one input may more than compensate for the lower quality of another.

specialization on the match surplus generated by other input providers:

$$\left.\frac{\partial \mathcal{W}}{\partial s(z)}\right|_{\mathcal{S}(z)=s^{\star}(z)} \propto \underbrace{\mathbb{E}_{\tilde{z},\tilde{z}<z}\left[\tilde{A}(s(\tilde{z}))-c(\tilde{z})\right]}_{\text{business-stealing externality}} \underbrace{-\,x(z)}_{\substack{\text{appropriability}\\ \text{externality}}} \underbrace{-\,(N-1)e^{-\lambda\hat{\phi}(\underline{z},z)}\mathbb{E}[\tilde{A}-c]/f}_{\text{network externality}} \tag{39}$$
$$\underbrace{+\;N\frac{(1-\delta)f^N}{\delta+(1-\delta)f^N}e^{-\lambda\hat{\phi}(\underline{z},z)}\mathbb{E}[\tilde{A}-c]/f}_{\text{search externality}} \underbrace{+\;\frac{(N-1)e^{\lambda\bar{\phi}(z,\bar{z})}}{\lambda(-\phi'(s(z)))}\frac{\partial \tilde{A}_{-i}}{\partial s(z)}\gamma^{\star}(z)dz}_{\text{production externality}}.$$

The new production externality pushes towards equilibrium under-specialization. Hence, under the more general aggregator χ, the economy features a further inefficiency: individual specialization influences not only the trading probability of other input providers but also their match surplus. We characterize the efficiency properties of the economy with generalized production function in the next Proposition (A.1).

Proposition A.1 (Efficiency of the Dynamic Economy: Generalized production function)**.** *The economy with generalized production function is efficient if and only if:*

$$(\mu N-1)\,\frac{-f'(s(z))}{f}\mathbb{E}[\tilde{A}-c]-(N-1)\int\frac{\partial \tilde{A}_{-i}}{\partial s(z)}\gamma^{\star}(z,\bar{z})dz=0 \quad \forall z\in[\underline{z},\bar{z}].$$

If $N<\frac{1}{\mu}$, then the economy necessarily features under-specialization.

To sum up, when inputs are complements at the extensive margin, equilibrium over-specialization is more likely the more the production process is complex (high N) and the more substitutable input quality is at the intensive margin (the closer σ is to 1 with CES production function).

B.2 Link Destruction

In the model discussed so far link destruction occurs only if the final producer faces a disruption. This is a stark assumption in that, presumably, most link disruption does not require either side of a transaction to leave the market at all. Yet, in our model with extensive margin complementarities, the disappearance of a single key input can completely halt production. [21] In this extension, we consider a variant of our model in which individual links break while both firms involved keep operating. The direct consequence of this assumption is that it is enough for the final producer to replace the broken links to continue production.

The main innovation with respect to the baseline dynamic model is that $\delta\in(0,1]$ now denotes the probability that one of the links between intermediate and final producers gets destroyed. Hence, a disruption happens if the final producer is unable to replace the input

[21] This is reminiscent of the models of critical failures discussed earlier, as well as intimately linked to the role of inventory management in supply chains in shaping an economy's responsiveness to shocks (Alessandria et al., 2011, 2023; Ferrari, 2023; Carreras-Valle, 2023; Carreras-Valle and Ferrari, 2023).

providers who separate each period.[22] The probability that a final producer experiences n link destructions each period is $\binom{N}{n}\delta^n(1-\delta)^{N-n}$, where N always denotes the complexity of the production process. We therefore define as $\rho(f,N)$ the probability that a final producer is operational next period:

$$\rho(f,N) = \sum_{n=0}^{N} \binom{N}{n} \delta^n (1-\delta)^{N-n} f^n. \tag{40}$$

It follows that the probability of being operational conditional on separating with one input provider equals $1-\sum_{n=0}^{N-1}\binom{N-1}{n}\delta^n(1-\delta)^{N-1-n}\left(1-f^{n+1}\right) = f\rho(f,N-1)$. The market tightness in each input market equals $\theta = \frac{\mu(f,N)}{m}$, where the mass of searching final producers is given by:[23]

$$\mu(f,N) \equiv \frac{\delta\left(f^N + 1 - f\rho(f,N-1)\right)}{\delta\left(1-f\rho(f,N-1)\right) + f^N}. \tag{41}$$

Intuitively, the mass of searching final producers takes into account that a share $\nu(f,N) \equiv \frac{f^N}{f^N+1-f\rho(f,N-1)}$ of them is composed of firms that were operating in the last period, and the complementary share is accounted for by inactive firms – the two groups differing by the probability of being active in the current period.[24] Notice that the share of searching final producers nests that of the baseline dynamic model for $\rho(f,N) = f^N$, *i.e.*, if all the searching firms were to source all N inputs at the same time. Intermediate producers' demand, that is, the expected number of final producers sourcing from it, reads:

$$\mathcal{D}(s,x) \equiv \frac{\theta\lambda}{\delta}\mathcal{P}(s,x) = \frac{\theta\lambda\phi(s)\exp\{-\lambda\bar{\phi}(1-G(x))\}\tilde{f}(f,N)}{\delta}, \tag{42}$$

where $\tilde{f}(f,N) \equiv \nu(f,N)\rho(f,N-1) + (1-\nu(f,N))f^{N-1}$.

Equilibrium Intermediate producers maximize expected operating profits net of product design costs:

$$V(s,x;z) = \max_{s,x} \mathcal{D}(s,x)\left(A(s) - x - c(z)\right) - wq(s).$$

Notice that the only difference with respect to the baseline dynamic model is the definition of demand (42) in place of (10). The equilibrium is characterized by the same equations as the baseline dynamic model. The equilibrium specialization function differs from its dynamic model counterpart (12) only in the use of $\tilde{f}(f,N)$ in place of f^{N-1}.

[22]Exactly as in the baseline model, we assume that, following a disruption, a final producer loses contact with all its input providers. Again, this is isomorphic to firms exiting the market if unable to produce and being replaced by new entrants.

[23]The equilibrium share of searching final producers equals $m^s = \delta(1-m^i) + m^i$, where m^i is the equilibrium share of inactive final producers. In turn, the latter is given by $m^i = (1-f\rho(f,N-1))\,\delta(1-m^i) + (1-f^N)m^i$.

[24]Intuitively, firms that were operating in the last period need to source, on average, fewer inputs in the markets than inactive firms, the latter requiring all the N key inputs to be sourced.

Output in this economy is given by:

$$y = \underbrace{\frac{\tilde{f}(f,N)\left(f^N + 1 - f\rho(f,N-1)\right)}{\delta\left(1 - f\rho(f,N-1)\right) + f^N}}_{\text{active firms}} \qquad \underbrace{N\mathbb{E}[A-c]/f}_{\text{expected match surplus|active}} \qquad . \tag{43}$$

Social planner problem The social planner problem reads:

$$\max_{s_i(z)} C + \psi \log(1-\ell) \quad \text{s.t.} \quad C = y \text{ and } \ell = Nm\bar{q}.$$

Efficient specialization of intermediate producers with productivity z producing input i solves:

$$s_i(z) = \underset{\hat{s}(z)}{\text{argmax}} \; \frac{\delta\left(f^{N-1}f(\hat{s}(z)) + 1 - f(\hat{s}(z))\rho(f,N-1)\right)}{\delta\left(1 - f(\hat{s}(z))\rho(f,N-1)\right) + f^{N-1}f(\hat{s}(z))} \left(\frac{\tilde{f}(f,N))}{\delta} \int \left[A(\hat{s}(z)) - c(z)\right] \gamma^\star([\hat{s}(z),\right.$$
$$\left. z], \bar{z})dz + (N-1)\frac{\tilde{f}([f(\hat{s}(z)), f_{-i}], N)}{\delta}\mathbb{E}[A-c]\right) + \psi \log\left(1 - m\left((N-1)\bar{q} + \bar{q}(\hat{s}(z))\right)\right).$$

The first line represents the expected match surplus for the input of interest. The second line represents the expected match surplus of the other inputs than the one of interest, along with the disutility of labor supply. The first order condition is given by:

$$\frac{\theta}{\delta}\left[\tilde{f}(f,N)\frac{\partial \mathbb{E}[A-c]}{\partial s(z)} - \underbrace{(N-1)\left(-\frac{\partial \tilde{f}([f(s(z)), f_{-i}], N)}{\partial s(z)}\bigg|_{\nu=\bar{\nu}}\right)\mathbb{E}[A-c]}_{\text{network externality}}\right] = \frac{\psi}{1 - mN\bar{q}}q'(s(z))$$
$$\underbrace{-\frac{\partial\theta}{\partial s(z)}\frac{\tilde{f}(f,N)}{\delta}N\mathbb{E}[A-c] + \theta\left(-\frac{\partial \tilde{f}(f,N)}{\partial \nu(f,N)}\frac{\partial \nu(f,N)}{\partial s(z)}\right)\frac{N}{\delta}\mathbb{E}[A-c]}_{\text{search externality}}. \tag{44}$$

Comparing (44) with (13), we observe that the model with link destruction features the same sources of externality as the baseline dynamic model. Notice, however, that the search externality is composed of an additional term, catching how much a marginal increase in specialization affects the equilibrium share of searching firms that were operating last period that is, ν. Since higher specialization of input i increases the share of inactive firms, this additional term pushes in the opposite direction with respect to the search externality going through market tightness.[25] Similarly, the network externality gets dampened in magnitude since searching final producers need to source fewer inputs in the market each period.

To characterize the efficiency properties of the equilibrium, it is useful to define as Z and V the equilibrium expressions reflecting the network externality and the search externality as follows:

$$Z(\rho, f) \equiv \tilde{f}(-f')f^{N-1}(1-\delta)[1 - f(\rho - f^{N-1})] - [\delta + f(f^{N-1} - \delta\rho)][f^{N-1}(-f') + f^N(f'\rho - f\rho')](\rho - f^{N-1}),$$

[25] To see this, notice that $\rho'(f,N-1) > \frac{f'(s(z))}{f(s(z))}\rho(f,N-1) \implies f'(s(z))\rho(f,N-1) - f'(s(z))\rho'(f,N-1) > 0$.

$$\mathcal{V}(\rho, f) \equiv [1 - f(\rho - f^{N-1})]^2[\delta + f(f^{N-1} - \delta\rho)][\nu(-\rho' + (1-\nu)(-f')f^{N-1})],$$

where $\rho = \rho(f, N-1)$ to lighten notation.

We characterize the efficiency properties of the economy with link destruction in the next Proposition (A.2).

Proposition A.2 (Efficiency of the Dynamic Economy: Link destruction)**.** *The economy with link destruction is efficient if and only if* $N = 1 + \frac{Z(\rho,f)}{\mathcal{V}(\rho,f)-Z(\rho,f)}$, *where* $Z(\rho, f)$ *and* $\mathcal{V}(\rho, f)$. *If* $N > 1 + \frac{Z(\rho,f)}{\mathcal{V}(\rho,f)-Z(\rho,f)}$, *then the economy with link destruction features over-specialization.*

To sum up, the basic economic forces highlighted in the baseline model go through modeling link destruction rather than firm-level disruptions.

B.3 Free Entry

Until now, we have assumed that the mass of intermediate producers m is fixed, that is, a perfectly inelastic entry margin. We now relax this assumption by positing that m is pinned down by a no-arbitrage condition between operating an intermediate or a final firm.

Equilibrium. The equilibrium is the same as in the baseline dynamic model, up to the addition of the no-arbitrage condition governing the equilibrium mass of intermediate producers. The no-arbitrage condition reads:[26]

$$\frac{f^{N-1}}{\delta}N\mathbb{E}[x] = \theta\frac{f^{N-1}}{\delta}N\mathbb{E}[A - x - c] - w\bar{q}. \tag{45}$$

The no-arbitrage condition (45), the tightness definition (11) and labor market clearing condition (35) jointly pin down the equilibrium mass of intermediate producers as the result of a second-order equation:

$$\begin{aligned} m =& \frac{(N\mathbb{E}[x] + N\bar{q}\mu N\mathbb{E}[A-x-c] + \psi\bar{q}) \pm \sqrt{\Delta}}{2N\bar{q}N\mathbb{E}[x]}, \\ \Delta =& (N\mathbb{E}[x] + N\bar{q}\mu N\mathbb{E}[A-x-c] + \psi\bar{q})^2 - 4N\bar{q}N\mathbb{E}[x]\mu N\mathbb{E}[A-x-c], \end{aligned} \tag{46}$$

Since a higher mass of intermediate producers is associated with lower leisure, the two solutions can be Pareto-ranked according to social welfare. Therefore, we assume that the mass of firms equals the lowest solution to (46). The equilibrium is therefore fully characterized by equations (33)-(36) and (46).

Social planner problem. We consider the problem of a social planner who is constrained by the free entry condition (45). Since the free entry condition pins down m as a function of the surplus sharing rule, we formulate the social planner problem in terms of optimal choice of the

[26] In the formulation of eq. (45) we take a stand on the organization of intermediate production being structured along single firms producing all inputs through distinct product lines.

offered surplus function $x(z)$, taking the equilibrium specialization function as given. The social planner problem reads:

$$\max_{x(z)\leq A(s(x(z)))-c(z)} \qquad \mu\frac{f^{N-1}}{\delta}N\mathbb{E}[A-c]+\psi\log(1-Nm\bar{q})$$

$$\text{s.t.} \quad \mathbb{E}[A-c]=\int[A(s)-c(z)]\gamma^{\star}(z,\bar{z})dz$$

$$m(s,x)=\frac{(N\mathbb{E}[x]+N\bar{q}\mu N\mathbb{E}[A-x-c]+\psi\bar{q})-\sqrt{\Delta}}{2N\bar{q}N\mathbb{E}[x]} \tag{47}$$

$$\frac{\mu\lambda}{m(s,x)}\mathcal{P}(z)\left[A'(s)-\left(\frac{-\phi'(s)}{\phi(s)}\right)(A(s)-c(z)-x(z))\right]=\frac{\psi}{1-Nm\bar{q}}q'(s), \tag{48}$$

where the constraints (47) and (48) follow from solving (45) for m and from the equilibrium specialization for given surplus, respectively. We proceed in three steps. First, we compute the full derivative of specialization with respect to the offered surplus via the implicit function theorem from (48). Second, we compute the derivative of the mass of firms with respect to s and x from (47).

The full derivative of specialization $s(x(z))$ with respect to the offered surplus $x(z)$ reads:

$$\frac{\partial s(x)}{\partial x}=\frac{-\lambda\theta(-\phi'(s))e^{-\lambda\hat{\phi}(z,\bar{z})}\frac{f^{N-1}}{\delta}+\frac{\psi}{(1-Nm\bar{q})^2}\frac{\partial m}{\partial x}\frac{q'(s)}{m}}{\frac{\lambda f^{N-1}}{\delta}\left[\frac{\partial\theta}{\partial s}e^{-\lambda\hat{\phi}(z,\bar{z})}+\theta\lambda(-\phi'(s))\gamma(z)\right][A'(s)\phi(s)+\phi'(s)(A(s)-x(z)-c(z))]\ldots}$$

$$\overline{-\frac{\psi}{(1-Nm\bar{q})^2}\left(\frac{\partial m}{\partial s}\frac{1}{m}+mN\frac{\partial\bar{q}}{\partial s}\right)q'(s)+SOC(z)}. \tag{49}$$

The derivatives of the equilibrium mass of intermediate producers, $m(s(x(z)),x(z))$ with respect to $s(x(z))$ and $x(z)$ read:

$$\frac{\partial m(s,x)}{\partial s}=\frac{\left[\frac{\partial\mathbb{E}[x]}{\partial s}+\frac{\partial\bar{q}}{\partial s}\mu N\mathbb{E}[A-x-c]+N\bar{q}\left(\frac{\partial\mu}{\partial s}N\mathbb{E}[A-x-c]+\mu\frac{\partial\mathbb{E}[A-x-c]}{\partial s}\right)+\frac{\psi}{N}\frac{\partial\bar{q}}{\partial s}-\frac{\Delta^{-\frac{1}{2}}}{2}\frac{\partial\Delta}{\partial s}\right]\ldots}{2(N\bar{q}N\mathbb{E}[x])^2}$$

$$\frac{N\bar{q}N\mathbb{E}[x]-(N\mathbb{E}[x]+N\bar{q}\mu N\mathbb{E}[A-x-c]+\psi\bar{q})\left[\frac{\partial\bar{q}}{\partial s}N\mathbb{E}[x]+N\bar{q}\frac{\partial\mathbb{E}[x]}{\partial s}\right]}{}, \tag{50}$$

$$\frac{\partial m(s,x)}{\partial x}=\frac{\left[\frac{\partial\mathbb{E}[x]}{\partial x}+N\bar{q}\mu\frac{\partial\mathbb{E}[A-x-c]}{\partial x}-\frac{\Delta^{-\frac{1}{2}}}{2}\frac{\partial\Delta}{\partial x}\right]N\bar{q}N\mathbb{E}[x]-(N\mathbb{E}[x]+N\bar{q}\mu N\mathbb{E}[A-x-c]+\psi\bar{q})\,N\bar{q}\frac{\partial\mathbb{E}[x]}{\partial x}}{2(N\bar{q}N\mathbb{E}[x])^2}, \tag{51}$$

where

$$\frac{\partial\Delta(s,x)}{\partial s} = \Big(N\mathbb{E}[x] + N\bar{q}\mu N\mathbb{E}[A-x-c] + \psi\bar{q}\Big)\Big(\tfrac{\partial\mathbb{E}[x]}{\partial s} + \tfrac{\partial\bar{q}}{\partial s}\mu N\mathbb{E}[A-x-c] + N\bar{q}\Big(\tfrac{\partial\mu}{\partial s}N\mathbb{E}[A-x-c]\ldots$$
$$+\mu\tfrac{\partial\mathbb{E}[A-x-c]}{\partial s}\Big) + \tfrac{\psi}{N}\tfrac{\partial\bar{q}}{\partial s}\Big) - 4\Big(\Big[\tfrac{\partial\bar{q}}{\partial s}N\mathbb{E}[x] + N\bar{q}\tfrac{\partial\mathbb{E}[x]}{\partial s}\Big]\mu N\mathbb{E}[A-x-c] + N\bar{q}N\mathbb{E}[x]\Big[\tfrac{\partial\mu}{\partial s}N\ldots$$
$$\mathbb{E}[A-x-c] + \mu\tfrac{\partial\mathbb{E}[A-x-c]}{\partial s}\Big]\Big),$$

$$\frac{\partial\Delta(s,x)}{\partial x} = \Big(N\mathbb{E}[x] + N\bar{q}\mu N\mathbb{E}[A-x-c] + \psi\bar{q}\Big)\Big(\tfrac{\partial\mathbb{E}[x]}{\partial x} + N\bar{q}\mu\tfrac{\partial\mathbb{E}[A-x-c]}{\partial x}\Big) - 4N\bar{q}\mu N\Big(\tfrac{\partial\mathbb{E}[x]}{\partial x}\ldots$$
$$N\mathbb{E}[A-x-c] + N\mathbb{E}[x]\tfrac{\partial\mathbb{E}[A-x-c]}{\partial x}\Big).$$

Finally, we maximize social welfare with respect to $x(z)$, which yields:

$$\theta\frac{f^{N-1}}{\delta}\left(\frac{\partial\mathbb{E}[A-c]}{\partial s(x(z))} - (\mu N-1)\frac{-f'(s(x(z)))}{f}\mathbb{E}[A-c]\right) - \frac{\psi}{1-Nm\bar{q}}\bar{q}' \tag{52}$$
$$= \underbrace{\frac{\psi}{1-Nm\bar{q}}\left(\frac{\partial m(s(x(z)),x(z))}{\partial s(x(z))} + \frac{\partial m(s(x(z)),x(z))/\partial x(z)}{\partial s(x(z))/\partial x(z)}\right)\frac{N\bar{q}}{m}}_{\text{entry externality}}$$

The first line equals the efficient specialization condition of the baseline dynamic model (13).[27] The second line highlights the new source of inefficiency related to the external effect of individual specialization on the equilibrium mass of intermediate producers. Comparing (52) with (13), we observe that the former nests the latter for fixed m. The additional inefficiency term brought about by free entry crucially depends on sign$\left\{\frac{dm}{ds}\right\}$: if it is positive, the entry externality pushes towards equilibrium over-specialization; if negative, towards under-specialization. Either sign is possible in equilibrium. In any case, Proposition (A.3) shows that equilibrium specialization is inefficient whenever it affects entry decisions.

Proposition A.3 (Efficiency of the Dynamic Economy: Free entry)**.** *The economy with free entry is always inefficient.*

B.4 Bargaining.

Our baseline model of specialization posits that intermediate producers post the price of their goods (or, interchangeably, set the offered surplus) ex ante, *i.e.*, before meeting a final producer. In this section, we explore the alternative trading protocol based on ex post bargaining.

To keep this alternative model as close as possible to our static baseline, we assume that final producers can observe productivity z and specialization s of the intermediate producers they meet. When meeting multiple intermediate producers that are suitable for its production process, each final producer picks one intermediate producer to bargain with and loses contact

[27] Since $\left(\frac{\partial s(x(z))}{\partial x(z)}\right)^{-1} = \frac{\partial x(s(z))}{\partial s(z)}$ by the chain rule, maximizing with respect to $x(z)$ by taking $s(x(z))$ as given is indeed isomorphic to maximizing with respect to $s(z)$ by taking $x(z)$ as given.

with the others.[28] As a result of this market design, expected operating profits of intermediate producers with productivity z and specialization s equal:

$$\begin{aligned}\Pi(s;z) &= \left(\sum_{k=0}^{\infty} m_k^d\ \phi(s)\ G\left(X(s,z)\right)^k\right) f^{N-1}(1-\xi)X(s,z)\\ &= \underbrace{\theta\lambda}_{\text{exp \# meetings}} \quad \underbrace{\phi(s)\ \exp\left\{-\lambda\bar{\phi}(1-G(X(s,z)))\right\}}_{\text{matching prob.}|\text{meeting}} \quad \underbrace{f^{N-1}}_{\text{prob. other inputs sourced}} \quad \underbrace{(1-\xi)X(s,z)}_{\text{unit profit}}.\end{aligned}$$

where $G(X(s,z))$ denotes the equilibrium distribution of total match surplus $X(s,z) \equiv A(s) - c(z)$, and $\xi \in [0,1]$ denotes the final producer's bargaining power weight. Anticipating that, in equilibrium, more productive firms produce higher match surplus as well, *i.e.*, $\partial X(s,z)/\partial z > 0$, allows us to denote the offered surplus distribution as $H(z) = G(X(s(z),z)) = \hat{\phi}(\underline{z},z)/\bar{\phi}$. Hence, equilibrium specialization solves:

$$\theta\lambda\mathcal{P}(z)(1-\xi)\left[A'(s(z)) - \left(\frac{-\phi'(s(z))}{\phi(s(z))}\right)(A(s(z))-c(z))\right] = wq'(s(z)). \tag{53}$$

For comparability, we can rewrite the equilibrium specialization condition (3) of the baseline model in terms of shadow bargaining power weight as follows:

$$\theta\lambda\mathcal{P}(z)\left[A'(s(z)) - \left(\frac{-\phi'(s(z))}{\phi(s(z))}\right)(1-\xi(z))\left(A(s(z))-c(z)\right)\right] = wq'(s(z)), \tag{54}$$

where $\xi(z)$ is endogenous and given by (9). Notice that (53) crucially differs from (54) by the fact that the all the term in square bracket is multiplied by the share of the surplus accruing to the intermediate producer, $(1-\xi)$. This reflects the main difference between ex ante price posting and ex post bargaining: investment in specialization gets priced in the former case, whereas it is sunk at the bargaining stage in the latter.

If production is not complex, *i.e.*, $N = 1$, Proposition 1 shows that the offered surplus share (9) is efficient. It follows that the bargaining model with uniform bargaining power weight across the productivity distribution would be efficient if and only if:

$$\begin{aligned}\xi &= \frac{(-\phi'(s(z)))(A(s(z))-c(z))}{(-\phi'(s(z)))(A(s(z))-c(z))-\phi(s(z))A'(s(z))}\xi(z)\\ &= \frac{(-\phi'(s(z)))\mathbb{E}_{\tilde{z},\tilde{z}<z}[A(s(\tilde{z}))-c(\tilde{z})]}{(-\phi'(s(z)))(A(s(z))-c(z))-\phi(s(z))A'(s(z))} \quad \forall z \in [\underline{z},\bar{z}].\end{aligned}$$

Since the left-hand side is constant and the right-hand side is heterogeneous along the productivity distribution, the bargaining model with uniform bargaining power is inefficient.

We now assume that, while negotiating with the highest-productivity intermediate producer, final producers keep the option of bargaining with the other k less productive suitable intermediate producers they are in contact with. Let intermediate producers with lower productivity

[28] This assumption entails that the outside option of both intermediate and final producers in bargaining is zero.

than z be ordered by productivity, with z_1 denoting the highest-productivity one and z_k the lowest-productivity one within such a group. The final producer's outside option in bargaining reads $\omega(z;\{z_1,\ldots,z_k\}) = \zeta\sum_{t=1}^{k}(1-\zeta)^{t-1}[A(s(z_t))-c(z_t)]$. Hence, equilibrium specialization solves:

$$\theta\lambda\phi(s(z))\ \exp\left\{-\lambda\hat{\phi}(z,\bar{z})\right\} f^{N-1}(1-\zeta)\left[A'(s(z))-\left(\frac{-\phi'(s(z))}{\phi(s(z))}\right)(A(s(z))-c(z)-\mathbb{E}[\omega(z)])\right]$$
$$= wq'(s(z)),$$

where the expected outside option of the final producer is given by:

$$\mathbb{E}[\omega(z)] = \sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty} \underbrace{\frac{\lambda^n\exp\{-\lambda\}}{n!}}_{\text{Prob. buyer meets n sellers}} \underbrace{\frac{(n-1)!}{k!(n-k-1)!}(\bar{\phi})^k(1-\bar{\phi})^{n-k-1}}_{\text{Prob. k other sellers are suitable}} \underbrace{\phi(s(z))}_{\substack{\text{Prob. seller is}\\ \text{suitable}}} \underbrace{H(z)^k}_{\substack{\text{Prob. other sellers}\\ \text{less productive}}}$$
$$\mathbb{1}_{\{k>0\}}\zeta\sum_{t=1}^{k}(1-\zeta)^{t-1}\int_{\underline{z}}^{z}\cdots\int_{\underline{z}}^{z_{t-1}}[A(s(z_t))-c(z_t)]\frac{h(z_1)}{H(z)}dz_1\left(\mathbb{1}_{\{t>1\}}\prod_{j=2}^{t}\frac{h(z_j)}{H(z_{j-1})}dz_j\ldots\right.$$
$$\left.+\left(1-\mathbb{1}_{\{t>1\}}\right)\right).$$

The first row equals the probability that the intermediate producer with productivity z is the most productive within the pool of suitable firms its downstream partner is in contact with. The second row equals the expected outside option with respect to the sequence of suitability-probability-weighted productivity densities conditional on the productivity being lower than the preceding level (starting from z), taking the number of other suitable sellers as given. Hence, if production is not complex, the bargaining model with sequential rounds of negotiations is efficient if and only if:

$$\zeta = \frac{(-\phi'(s(z)))[(A(s(z))-c(z))\zeta(z)-\mathbb{E}[\omega(z)]]}{(-\phi'(s(z)))(A(s(z))-c(z)-\mathbb{E}[\omega(z)])-\phi(s(z))A'(s(z))}$$
$$= \frac{(-\phi'(s(z)))[\mathbb{E}_{\tilde{z},\tilde{z}<z}[A(s(\tilde{z}))-c(\tilde{z})]-\mathbb{E}[\omega(z)]]}{(-\phi'(s(z)))(A(s(z))-c(z)-\mathbb{E}[\omega(z)])-\phi(s(z))A'(s(z))}\quad \forall z\in[\underline{z},\bar{z}].$$

Notice that, for $z=\underline{z}$, efficiency when $N=1$ attains if and only if $\zeta=0$, *i.e.*, when intermediate producers have all the bargaining power. Accordingly, final producers would have no bargaining power in negotiations and their outside option would be $\omega(z;\{z_1,\ldots,z_k\})=0\ \forall z$. It follows that also the bargaining model with sequential rounds of negotiations and uniform bargaining power is always inefficient.

The reason why bargaining models are generally inefficient is that matching frictions are endogenous and intertwined with match surplus (Mangin and Julien, 2021). Unlike standard matching models, there is no (iso-elastic) meeting function pinning down the efficient bargaining power weight based on the contribution of the respective market side to aggregate meetings (Hosios, 1990). Since the equilibrium elasticity of demand is heterogeneous across the productivity distribution, so is the efficient bargaining weight. When production is not complex, the price posting protocol adopted in our baseline model allows attaining efficient surplus sharing by the heterogeneous (shadow) bargaining power weight (9). Interestingly, the trading protocol also affects the equilibrium relation between specialization and bargaining power: with ex ante price posting, intermediate producers *decrease* specialization if they appropriate a larger share of the surplus because the opportunity cost of specializing is higher; with ex post price bargaining, intermediate producers *increase* specialization if they appropriate a larger share of the surplus because the hold-up problem constraining their marginal benefit of specialization ameliorates.

C Proofs

Proof of Lemma 1. By the implicit function theorem,

$$\frac{\partial s(z)}{\partial x(z)} = \frac{\theta\lambda \exp\{-\lambda\hat{\phi}(z,\bar{z})\} f^{N-1}(-\phi'(s(z)))}{\theta\lambda \exp\{-\lambda\hat{\phi}(z,\bar{z})\} f^{N-1}\big[\phi(s(z))(-A''(s(z))) + 2(-\phi'(s(z)))A'(s(z))} \cdots \tag{55}$$

$$\overline{-\phi''(s(z))\left(A(s(z) - x(z) - c(z))\right)\big] + wq''(s(z))} > 0.$$

Notice that this expression is always positive due to the SOC holding. ■

Remark 1 (Comparative Statics: market share reallocation from lower search frictions)**.** *Lower search frictions ($\lambda \uparrow$) increases conditional match surplus per input for given specialization function by reallocating market shares towards more productive firms.*

Proof. Consider the direct effect of an increase in λ on conditional match surplus per input:

$$\begin{aligned}
&\frac{d}{d\lambda}\int \frac{\lambda\phi(s(z))\exp\{-\lambda\hat{\phi}(z,\bar{z})\}\gamma(z)(A(s(z)) - c(z))dz}{\int \lambda\phi(s(\tilde{z}))\exp\{-\lambda\hat{\phi}(\tilde{z},\bar{z})\}\gamma(\tilde{z})d\tilde{z}} \\
&= \int \frac{(A(s(z)) - c(z))\gamma^{\star}(z,\bar{z})}{f}\left[-\hat{\phi}(z,\bar{z}) + \int \hat{\phi}(\tilde{z},\bar{z})\frac{\gamma^{\star}(z,\bar{z})}{f}\right] dz \\
&= \frac{1}{f^2}\left(\mathbb{E}[\hat{\phi}(z,\bar{z})]\mathbb{E}[A(s(z)) - c(z)] - f\mathbb{E}[\hat{\phi}(z,\bar{z})\left(A(s(z)) - c(z)\right)]\right) \\
&= \frac{1}{f^2}\Big((1-f)\mathbb{E}[\hat{\phi}(z,\bar{z})]\mathbb{E}[A(s(z)) - c(z)] - f\mathrm{Cov}[\hat{\phi}(z,\bar{z}), (A(s(z)) - c(z))]\Big) \\
&> 0,
\end{aligned}$$

where the last step follows from observing that $\mathrm{Cov}[\hat{\phi}(z,\bar{z}), (A(s(z)) - c(z))] < 0$ as $\partial[\hat{\phi}(z,\bar{z})]/\partial z < 0$ and $\partial[A(s(z)) - c(z)]/\partial z > 0$. ■

Proof of Proposition 1. We start our proof by comparing equations (3) and (6) which implicitly define the private and socially optimal levels of specialization, respectively. First, note that by the labor market clearing condition in the equilibrium, we have that $w = \frac{\psi}{1-Nm\bar{q}}$. Substituting this into (6) we obtain

$$\theta f^{N-1}\lambda\Bigg(\phi(s(z))\ \exp\left\{-\lambda\hat{\phi}(z,\bar{z})\right\}\Bigg[A'(s(z))-\left(\frac{-\phi'(s(z))}{\phi(s(z))}\right)\Bigg([A(s(z))-c(z)]$$
$$-\int_{\underline{z}}^{z}[A(s(\tilde{z}))-c(\tilde{z})]\gamma^{\star}(\tilde{z},\bar{z})d\tilde{z}\Bigg)\Bigg]-(N-1)\frac{(-\phi'(s(z)))\exp\{-\lambda\bar{\phi}\}}{f}\mathbb{E}[A-c]\Bigg)=$$
$$=w\ q'(s(z)).$$

We conjecture that $s^{\star}(z)=s(z)$, where $s(z)$ is the privately optimal specialization, and use eq. (3) to substitute out $wq'(s(z))$:

$$\left.\frac{\partial\mathcal{W}}{\partial s(z)}\right|_{s^{\star}(z)=s(z)}=\theta f^{N-1}\lambda\Bigg(\phi(s(z))\ \exp\left\{-\lambda\hat{\phi}(z,\bar{z})\right\}\Bigg[A'(s(z))-\left(\frac{-\phi'(s(z))}{\phi(s(z))}\right)\Bigg([A(s(z))-c(z)]$$
$$-\int_{\underline{z}}^{z}[A(s(\tilde{z}))-c(\tilde{z})]\gamma^{\star}(\tilde{z},\bar{z})d\tilde{z}\Bigg)\Bigg]-(N-1)\frac{(-\phi'(s(z)))\exp\{-\lambda\bar{\phi}\}}{f}\mathbb{E}[A-c]\Bigg)-$$
$$-\theta\lambda\phi(s(z))\ \exp\left\{-\lambda\hat{\phi}(z,\bar{z})\right\}f^{N-1}\Bigg[A'(s(z))-\left(\frac{-\phi'(s(z))}{\phi(s(z))}\right)(A(s(z))-x(z)-c(z))\Bigg]\lesseqgtr 0.$$

Simplifying we obtain

$$\frac{\partial\mathcal{W}}{\partial s(z)}=-\theta f^{N-1}\lambda\Bigg(\phi'(s(z))\ \exp\left\{+\lambda\hat{\phi}(z,\bar{z})\right\}\left[\int_{\underline{z}}^{z}[A(s(\tilde{z}))-c(\tilde{z})]\gamma^{\star}(\tilde{z},\bar{z})d\tilde{z}\right]$$
$$-(N-1)\frac{(\phi'(s(z)))\exp\{-\lambda\bar{\phi}\}}{f}\mathbb{E}[A-c]\Bigg)+\theta\lambda\phi'(s(z))\exp\left\{-\lambda\hat{\phi}(z,\bar{z})\right\}f^{N-1}x(z)\lesseqgtr 0.$$

Further simplifying, we obtain eq. (7), where we flip the signs since we are dividing by $\phi'<0$

$$\left.\frac{\partial\mathcal{W}}{\partial s(z)}\right|_{\mathcal{S}(z)=s^{\star}(z)}=\int_{\underline{z}}^{z}[A(s(\tilde{z}))-c(\tilde{z})]\,\gamma^{\star}(\tilde{z},z)d\tilde{z}-x(z)-(N-1)e^{-\lambda\hat{\phi}(\underline{z},z)}\mathbb{E}[A-c]/f\lesseqgtr 0.$$

We establish the first statement by noting that when $N=1$, the social marginal value is equal to 0 if and only if

$$x(z)=\int_{\underline{z}}^{z}[A(s(\tilde{z}))-c(\tilde{z})]\,\gamma^{\star}(\tilde{z},z)d\tilde{z},$$

which we know to be true from Lemma 2. Whenever $N>1$, by Lemma 2 we have that

$$\frac{\partial\mathcal{W}}{\partial s(z)}=-(N-1)e^{-\lambda\hat{\phi}(\underline{z},z)}\mathbb{E}[A-c]/f<0.$$

We conclude that when we evaluate the marginal welfare effect of specialization at the privately optimal level, we are left with the network externality term. Since this term is negative, the planner would choose a lower level of specialization. Therefore, whenever production is complex, the equilibrium features *over-specialization.*

■

Proof of Proposition 2. Since only the marginal benefit of specialization depends on λ, the first-order response of equilibrium specialization can be characterized by differentiating the left-hand side of eq. (3) with respect to λ, as follows:[29]

$$\frac{d\text{MB}_{s(z)}}{d\lambda} = \theta\mathcal{P}(z)(1-\lambda\hat{\phi}(z,\bar{z}))\left[A'(s(z)) - \left(\frac{-\phi'(s(z))}{\phi(s(z))}\right)(A(s(z)) - x(z) - c(z))\right]$$
$$+\theta\mathcal{P}(z)\left(\frac{-\phi'(s(z))}{\phi(s(z))}\right)\int_{\underline{z}}^{z}\left(1-\lambda\hat{\phi}(\tilde{z},z)\right)[A(s(\tilde{z})) - c(\tilde{z})]\,\gamma^{\star}(\tilde{z},\bar{z})d\tilde{z}.$$

The first row represents the direct effect of meeting efficiency on specialization, the second row its indirect effect through the response of the offered surplus. The sign of the derivative at z depends upon the signs of the terms $(1-\lambda\hat{\phi}(\tilde{z},\bar{z})),\ \forall\tilde{z}\leq z$. If $\lambda\bar{\phi}<1$, all such terms are positive and so is the total derivative. If $\lambda\bar{\phi}>1$, specialization is decreasing in λ at least at $z=\underline{z}$. We define as $z^{\star}\in[\underline{z}.\bar{z}]$ the productivity level such that $\frac{ds(z)}{d\lambda}<0\ \forall z<z^{\star}$ and $\frac{ds(z)}{d\lambda}>0\ \forall z>z^{\star}$. To formally establish claim a), we rearrange the previous expression by moving $x(z)$ to the second row as follows:

$$\frac{d\text{MB}_{s(z)}}{d\lambda} = \theta\mathcal{P}(z)(1-\lambda\hat{\phi}(z,\bar{z}))\left[A'(s(z)) - \left(\frac{-\phi'(s(z))}{\phi(s(z))}\right)(A(s(z)) - c(z))\right]$$
$$+\theta\mathcal{P}(z)\left(\frac{-\phi'(s(z))}{\phi(s(z))}\right)\int_{\underline{z}}^{z}\left(2-\lambda\hat{\phi}(\tilde{z},\bar{z})\right)[A(s(\tilde{z})) - c(\tilde{z})]\,\gamma^{\star}(\tilde{z},\bar{z})d\tilde{z}$$

Since $x(\underline{z})=0$ and the optimal marginal benefit of specialization is increasing in $s(z)$ from (3), the term in square brackets in the first row is positive for all z (assuming that the lowest-productive firm is the least specialized, as well). The second row is always positive if $\lambda\bar{\phi}<2$, and inherits the cross-sectional properties of the first row. Depending on the shape of the suitability probability function ϕ, $z^{\star}$ could depart from $z:\lambda\hat{\phi}(z,\bar{z})=1$ (which would zero out the first row, *i.e.*, the direct effect of λ being x fixed) in either directions.

The fact that $\frac{ds(x)}{dN}<0\ \forall z\in[\underline{z},\bar{z}]$ follows directly from eq. (3), as the marginal benefit depends on f^N, which is decreasing in N. ■

Proof of Proposition 3. The first claim follows directly from rearranging (8). The second claim follows from noticing that the marginal benefit of efficient specialization in (6) is decreasing in complexity and its marginal cost is increasing in complexity. ■

Proof of Proposition 4. We can follow similar steps as in Proposition 1. We start by using the FOC of the planner problem evaluated at the privately optimal level of specialization and

[29]Notice that $\frac{ds(z)}{d\lambda}=\frac{dMB_{s(z)}/d\lambda}{-SOC(z)}$.

substituting out the marginal cost of specialization:

$$\begin{aligned}\frac{\partial \mathcal{W}}{\partial s(z)} =& -\theta f^{N-1}\lambda\bigg(\phi'(s(z))\ \exp\Big\{-\lambda\hat{\phi}(z,\bar{z})\Big\}\left[\int_{\underline{z}}^{z}[A(s(\tilde{z}))-c(\tilde{z})]\gamma^{\star}(\tilde{z},\bar{z})d\tilde{z}\right] \\ & -(N-1)\frac{(\phi'(s(z)))\exp\{-\lambda\bar{\phi}\}}{f}\mathbb{E}[A-c]\bigg)+\theta\lambda\phi'(s(z))\exp\Big\{-\lambda\hat{\phi}(z,\bar{z})\Big\}f^{N-1}x(z) \\ & -\theta f^{N-1}\lambda N\frac{(1-\delta)f^{N}}{\delta+(1-\delta)f^{N}}\frac{(\phi'(s(z)))\exp\{-\lambda\bar{\phi}\}}{f}\mathbb{E}[A-c]\lesseqgtr 0.\end{aligned}$$

Further simplifying, we obtain eq. (14), where we flip the signs since we are dividing by $\phi' < 0$:

$$\begin{aligned}\frac{\partial \mathcal{W}}{\partial s(z)}\bigg|_{\mathcal{S}(z)=s^{\star}(z)} =& \int_{\underline{z}}^{z}[A(s(\tilde{z}))-c(\tilde{z})]\,\gamma^{\star}(\tilde{z},z)d\tilde{z}-x(z)-(N-1)e^{-\lambda\hat{\phi}(\underline{z},z)}\mathbb{E}[A-c]/f \\ & +\ N\frac{(1-\delta)f^{N}}{\delta+(1-\delta)f^{N}}e^{-\lambda\hat{\phi}(\underline{z},z)}\mathbb{E}[A-c]/f\lesseqgtr 0.\end{aligned}$$

Collecting the network and search externality terms yields:

$$\frac{\partial \mathcal{W}}{\partial s(z)}\bigg|_{\mathcal{S}(z)=s^{\star}(z)} = \int_{\underline{z}}^{z}[A(s(\tilde{z}))-c(\tilde{z})]\,\gamma^{\star}(\tilde{z},z)d\tilde{z}-x(z)-(\mu N-1)e^{-\lambda\hat{\phi}(\underline{z},z)}\mathbb{E}[A-c]/f\lesseqgtr 0.$$

Since $x(z)=\int_{\underline{z}}^{z}[A(s(\tilde{z}))-c(\tilde{z})]\,\gamma^{\star}(\tilde{z},z)d\tilde{z}$, the equilibrium is efficient if and only if $N=\frac{1}{\mu}$. ■

Proof of Proposition 5. Rearranging eq. (15) and (16) yield $f^N = \exp\{-1\}$ and $(f^N)^{\star} = \exp\{-(1-\frac{w\ell}{y})\}$. It follows that, as long as there are some product design costs in the economy, the level of complexity in equilibrium exceeds the efficient one. Note also that if $w\ell = y$ then the efficient level of complexity is $N^{\star} = 0$, which clearly cannot be optimal for the social planner. ■

Proof of Proposition 6. Taking the search externality term in (17) to the left-hand side yields $\theta\frac{f^{N-1}}{\delta}\mathbb{E}[A-c]\Big(1-\ \mu N\ln(1/f)\Big) = \frac{\psi}{1-mN\bar{q}}\bar{q}$. Upon substituting for the labor market clearing and product market clearing conditions (35)-(36), it follows that $\frac{\psi}{1-mN\bar{q}}\bar{q}\frac{\theta f^{N-1}\mathbb{E}[A-c]}{\delta} = \frac{w\ell}{y}$. This first-order condition differs from (16) only by the share of searching final good producers μ multiplying $N\ln(1/f)$. Hence, $(f^N)^{\star} = \exp\Big\{-\frac{1}{\mu}\Big(1-\frac{w\ell}{y}\Big)\Big\}$. Comparing this expression with (15), it follows that complexity is efficient for given specialization if and only if $LS = 1-\mu$. As in the baseline dynamic model, specialization is efficient for given complexity if and only if $N = \frac{1}{\mu}$. If the two efficiency conditions hold simultaneously, the endogenous complexity equilibrium is therefore efficient.

Since $f^N = \exp\{-1\}$ and $f^{N^{\star}} = \exp\{-\frac{1}{\mu^{\star}}(1-LS^{\star})\}$, the equilibrium features under-resilience if and only if $LS^{\star} > 1-\mu^{\star}$. ■

Proof of Proposition 7. Consider the social planner problem of choosing specialization in the presence of any cost K borne by final producers.

$$\max_{\hat{s}(z)} \frac{\delta}{\delta+(1-\delta)f^{N-1}f(\hat{s}(z))}\frac{f^{N-1}}{\delta}\Bigg(\int \lambda\phi(\hat{s}(z))\exp\left\{-\lambda\hat{\phi}(z,\bar{z})\right\}\Big[A(\hat{s}(z))-c(z)-\kappa\Big]$$
$$\gamma(z)dz+(N-1)\frac{f^{N-1}}{\delta}f(\hat{s}(z))\mathbb{E}\left[A-c-\kappa\right]\Bigg)+\psi\log\left(1-m\left((N-1)\bar{q}+\bar{q}(\hat{s}(z))\right)\right),$$

where $\kappa\equiv K/N$. Efficient specialization of intermediate producers with productivity z is given by:

$$\theta\frac{f^{N-1}}{\delta}\left(\frac{\partial\mathbb{E}\left[A-c-\kappa\right]}{\partial s(z)}-(N-1)\frac{-f'(s(z))}{f}\mathbb{E}\left[A-c-\kappa\right]\right)+\theta\frac{f^{N-1}}{\delta}\frac{-f'(s(z))}{f}$$
$$\frac{(1-\delta f^N}{\delta+(1-\delta f^N}N\mathbb{E}[A-c-\kappa]=\frac{\psi}{1-mN\bar{q}}q'(s(z)).$$

Evaluating the planner's first-order condition at the equilibrium specialization yields:

$$\frac{\partial\mathcal{W}}{\partial s(z)}\bigg|_{\mathcal{S}(z)=s^\star(z)}\propto\underbrace{\int_{\underline{z}}^{z}\left[A(s(\tilde{z}))-c(\tilde{z})-\kappa\right]\gamma^\star(\tilde{z},z)d\tilde{z}}_{\text{business stealing externality}}\;-\;\underbrace{x(z)}_{\substack{\text{appropriability}\\\text{externality}}}\;-\;\underbrace{(N-1)e^{-\lambda\hat{\phi}(\underline{z},z)}\mathbb{E}[A-c-\kappa]/f}_{\text{network externality}}$$
$$\underbrace{+\;\frac{(1-\delta f^N}{\delta+(1-\delta f^N}e^{-\lambda\hat{\phi}(\underline{z},z)}N\mathbb{E}[A-c-\kappa]/f,}_{\text{search externality}}$$

where $x(z)=x_0+\mathbb{E}[A(s(\tilde{z}))-c(\tilde{z}),\ \tilde{z}<z]$, with $x_0=\kappa-\frac{N-1}{N}\mathbb{E}[\tilde{x}(z)]/f$ and $\tilde{x}(z)=\mathbb{E}[A(s(\tilde{z}))-c(\tilde{z}),\ \tilde{z}<z]$, is the equilibrium offered surplus. For the equilibrium offered surplus to balance appropriability and business-stealing externalities as in the model without costs downstream, the following needs to hold:

$$x_0=\left(1-e^{-\lambda\hat{\phi}(\underline{z},z)}\right)\kappa\iff\kappa=e^{\lambda\bar{\phi}(\underline{z},z)}\frac{N-1}{N}\mathbb{E}[\tilde{x}(z)]/f\quad\forall z\in[\underline{z},\bar{z}].$$

Since this condition can never hold simultaneously for any productivity level, appropriability and business-stealing externalities are not perfectly balanced by the equilibrium offered surplus when final producers bear other costs than input purchases. ■

Proof of Proposition 8. Comparing (20) with (23) we note that the benefits of higher robustness are evaluated differently by firms and by the planner. Firms obtain a marginal flow benefit $\mathbb{E}[x-\kappa]$ smaller than the social marginal flow benefit $\mathbb{E}[A-c-\kappa]$. This force, in isolation, would induce under-investment in robustness. On the other hand, the planner discounts at the social discount factor $\delta(r)/[(1-f^N)\mu]$, while private firms discount at their destruction rate $\delta(r)$. As a consequence, the planner discounts the marginal surplus more than private firms, which pushes the equilibrium towards investing excessively in robustness. For given aggregate specialization, under-robustness obtains if and only if $\tilde{\zeta}\equiv\frac{\mathbb{E}[x-\kappa]}{\mathbb{E}[A-c-\kappa]}<(1-f^N)\mu$. From this formulation, it is apparent that the lower the share of the match surplus accruing to final producers, the more likely under-robustness is. Equilibrium specialization is efficient if and only if: $N = \frac{1}{\mu}\left(1 + \frac{(1-e^{-\lambda\hat{\phi}(\underline{z},z)})\kappa - x_0}{\mathbb{E}[A-c-\kappa]/f}\right)$ $\forall z \in [\underline{z}, \bar{z}]$. Since this condition never holds for more than one productivity level at a time, equilibrium specialization at the firm level is inefficient. Since the exit probability discounts the marginal benefit of specialization in (12), higher robustness induces higher equilibrium specialization, as well. Inefficiency of firm-level specialization directly follows from 1. ■

Proof of Proposition 9. Given eq. (25), and the definition $x^\star = x(z) + \tau^\star(z)$ we note that $\tau^\star(z) = e^{\lambda\hat{\phi}(z,\bar{z})}\left(-(N-1)\left(\frac{1}{f}-1\right)\mathbb{E}[A-c]\right)$. Using the definition of $T^\star$ we obtain eq. (26. The last statement follows by noting that $e^{\lambda\hat{\phi}(\bar{z},\bar{z})} = 1$ and $e^{\lambda\hat{\phi}(\underline{z},\bar{z})} = e^{\lambda\bar{\phi}} = (1-f)^{-1}$. ■

Proof of Proposition 10. Following the same steps of the proof of Proposition 9 we obtain the untargeted component $T_d^\star$. Similarly, we obtain the bounds by evaluating $e^{\lambda\hat{\phi}(z,\bar{z})}$ at $\underline{z}$ and $\bar{z}$. ■

Proof of Corollary 2. Denote the cost function in the dynamic and static economy as $q_d(s)$ and $q_s(s)$. Suppose that $\frac{q_d(s)}{\delta} = q_s(s)$. Namely, firms in the dynamic economy can annuitize the specialization costs. Then the optimal specialization FOC in eq. (12) is the same as the one in the static economy in eq. (3), hence the optimal level of specialization is the same in the two economies.

Since the level of specialization is the same, so is $\phi(s)$ and all its average $\hat{\phi}$ and $\bar{\phi}$. Comparing the optimal tax schedules then, the two only differ by their untargeted components $T^\star$ and $T_d^\star$. It is immediate that $T_d^\star > T^\star$ and, therefore, that $\tau_d^\star(z) < \tau^\star(z)$, $\forall z$. ■

Proof of Proposition 11. Since the efficiency condition of equilibrium specialization is the same as in the dynamic model, so is the optimal targeted transaction tax schedule $\tau_d^\star(z)$. To derive the optimal profits tax schedule to final producers, we first solve the final producer's profit maximization problem in the presence of a generic tax schedule that depends on complexity:

$$N = \arg\max f^N N \tau_c(N) \frac{\mathbb{E}[x]/f}{\delta}$$

Equilibrium complexity solves: $N = \frac{1}{\ln(1/f)}\left(1 + N\frac{\tau_c'(N)}{\tau_c(N)}\right)$. For equilibrium complexity to be efficient, the following needs to hold:

$$\frac{1}{\mu}\left(1 - \frac{w\ell}{y}\right) = 1 + N\frac{\tau_c'(N)}{\tau_c(N)} \iff \tau_c(N) = N^{\frac{1}{\mu}\left(1-\frac{w\ell}{y}\right)-1},$$

which completes the proof. ■

Proof of Proposition 12. By noting that $-\hat{\phi}(\underline{z},z) = \hat{\phi}(z,\bar{z}) - \bar{\phi}$, the efficient offered surplus (30) directly maps into a transaction tax schedule with intercept $T_0 \equiv \frac{N-1}{N}\mathbb{E}[\tilde{x}]/f$ and firm-specific component $e^{\lambda\hat{\phi}(z,\bar{z})}T_t^\star$, where $T_t^\star \equiv -\left(e^{-\lambda\bar{\phi}}\kappa + (\mu N - 1)\left(\frac{1}{f}-1\right)\mathbb{E}[A-c]\right) < 0$. The proportional robustness cost tax $\tau_r^\star$ follows directly from Prop. (8). ■

Proof of Proposition A.1. The proposition follows directly from eq. (A.1). If inputs are not perfect substitutes, *i.e.*, $\frac{\partial\tilde{A}_j}{\partial s_i(z)} \geq 0\ \forall j = \{1,\ldots,N\}\setminus\{i\}$, the social marginal benefit of specialization increases with respect to the private one, thus making the under-specialization condition of the baseline dynamic model, $N < 1 + \frac{(1-\delta)f^N}{\delta}$, sufficient for equilibrium under-specialization in the model with generalized production function. ■

Proof of Proposition A.2. Taking the dynamic externality term in (44) to the left-hand side allows isolating the wedge between equilibrium and efficient specialization, which is proportional to: $\theta\,(N-1)\left(-\frac{\partial\tilde{f}([f(s(z)),f_{-i}],N)}{\partial s(z)}\bigg|_{\nu=\bar{\nu}}\right) - \frac{\partial\theta}{\partial s(z)}\tilde{f}(f,N)N + \theta\left(-\frac{\partial\tilde{f}(f,N)}{\partial\nu(f,N)}\frac{\partial\nu(f,N)}{\partial s(z)}\right)N$. Working out the relevant derivatives yields:

$$\frac{\partial\tilde{f}([f(s(z)),f_{-i}],N)}{\partial s(z)}\bigg|_{\nu=\bar{\nu}} = -\Big[\bar{\nu}(-\rho'(f,N-1)) + (1-\bar{\nu})(-f'(s(z)))f^{N-1}\Big] < 0$$

$$\frac{\partial\theta}{\partial s(z)} = \theta\frac{(-f'(s(z)))f^{N-1}(1-\delta)}{[1-f(\rho(f,N-1)-f^{N-1})]\,[\delta+f(s(z))(f^{N-1}-\delta\rho(f,N-1))]} > 0$$

$$\frac{\partial\tilde{f}(f,N)}{\partial\nu(f,N)}\frac{\partial\nu(f,N)}{\partial s(z)} = -\left(\rho(f,N-1)-f^{N-1}\right)\frac{f^{N-1}(-f'(s(z))) + f^N\left(f'(s(z))\rho(f,N-1)\ldots\right.}{\left(f^N+1-f\rho(f,N-1)\right)^2}$$

$$\frac{\left.-f(s(z))\rho'(f,N-1)\right)}{} < 0$$

$$\rho'(f,N-1) = (-f'(s(z)))\frac{1}{N-1}\sum_{n=0}^{N-1}\binom{N-1}{n}\delta^n(1-\delta)^{N-1-n}nf^{n-1} < 0.$$

Substituting for such derivatives into the wedge expression, we observe that the wedge equals to 0 if and only if $N = 1 + \frac{Z(\rho,f)}{V(\rho,f)-Z(\rho,f)}$, with $Z(\rho,f)$ and $V(\rho,f)$ being defined in the main text. ■

Proof of Proposition A.3. By rearranging (52), it follows that the equilibrium is efficient if and only if:

$$\theta\frac{f^{N-2}}{\delta}\left(\frac{\delta}{f^N+(1-f^N)\delta}N-1\right)\mathbb{E}[A-c] = -\frac{\psi}{1-Nm\bar{q}}\frac{N\bar{q}}{m(-f'(s(x(z))))}\left(\frac{\partial m(s(x(z)),x(z))}{\partial s(x(z))}+\right.$$
$$\left.\frac{\partial m(s(x(z)),x(z))/\partial x(z)}{\partial s(x(z))/\partial x(z)}\right) \quad \forall z\in[\underline{z},\bar{z}].$$

Since the left-hand side is constant across firms and the right-hand side is firm-specific, the economy with free entry is always inefficient. ■

D Details of the Quantification

In this section, we provide a detailed account of our quantification exercises.

We start with a number of functional form assumptions for $A(s)$, $q(s)$ and $\phi(s)$, as well as the productivity and marginal costs distributions $\gamma(z)$ and $c(z)$. Specifically, we posit iso-elastic match quality and specialization cost functions, $A(s) = a_0\frac{s^{1-\alpha}}{1-\alpha}, \alpha\in(0,1)$ and $q(s) = q_0\frac{s^{1+\eta}}{1+\eta}, \eta\in(0,\infty)$.

We assume match probability of constant specialization semi-elasticity, $\phi(s) = \exp\{-\chi s\}, \chi \in (0, \infty)$. Finally, we consider a uniform productivity distribution, $\gamma(z) = \frac{1}{\bar{z}-\underline{z}}$, and a hyperbolic marginal cost function, $c(z) = \frac{c_0}{z}$. For the endogenous robustness model, we further assume iso-elastic robustness cost function, $\kappa(r) = \kappa_0 \frac{r^{1+\kappa_1}}{1+\kappa_1}, \kappa_1 \in (0, \infty)$, and disruption probability with constant semi-elasticity, $\delta(r) = \delta_0 \exp\{-\delta_1 r\}, \delta_1 \in (0, \infty)$. As apparent, our functional form assumptions are motivated on the ground of parsimony.

Absent any well-established parametrization for the class of models at hand, we proceed by picking parameter values that guarantee equilibrium existence and reasonable equilibrium outcomes. In the light of their special relevance, we carry out comparative statics exercises with respect to search efficiency λ and production complexity N. In what follows we report the values of the background parameters on which the figures in the main text are based.

Table A.1: Parametrization of Figures

Parameter	Description	Value Fig. 1 (Static)	 Fig. 2 (Dynamic)	 Fig. 3 (End. Comp.)	 Fig. 4 (End. Rob.)
Structural parameters					
α	Concavity match quality	1/3	1/3	1/3	1/3
η	Convexity specialization cost	0.5	0.5	0.5	0.5
χ	Semi-elasticity match probability	0.9	0.9	0.9	0.9
$[\underline{z}, \bar{z}]$	Bounds productivity distribution	[10, 20]	[10, 20]	[10, 20]	[10, 20]
ψ	Disutility of labor supply	3	3	3	3
δ_1	Semi-elasticity disruption probability downstream	0	0	0	1
κ_1	Convexity robustness costs	0	0	0	1
Shifters					
q_0	Intercept specialization cost	0.28	0.01	0.40	0.01
a_0	Intercept match quality	3.3	15	3.6	15
c_0	Scalar marginal cost	1.35	3.87	1.9	3.87
m	Measure upstream firms	1	5.5	1.53	5.5
δ_0	Scalar disruption probability downstream	1	0.1	0.6	0.25
κ_0	Scalar robustness costs	0	0	0	20