Problem Set 04: Predicates, Quantifiers, and Nested Quantifiers

CS/MATH 113 Discrete Mathematics

Spring 2024

The problems below are grouped by the section of the book that they draw from. Many of them are similar to the worked examples in the section. Please consult the section for help if needed. You can also approach the course staff during their consultation hours are listed on Canvas.

Discussing the problems with your peers is encouraged and does not violate academic honesty unless the submitted solutions are highly similar.

1.4 Predicates and Quantifiers

1. Express the negation of each of these statements in terms of quantifiers without using the negation symbol.
   (a) $\forall x(-2 < x < 3)$

   Solution: $\exists x((x \leq -2) \vee (x \geq 3))$

   (b) $\forall x(0 \leq x < 5)$

   Solution: $\exists x((x < 0) \vee (x \geq 5))$

   (c) $\exists x(-4 \leq x \leq 1)$

   Solution: $\forall x((-4 > x) \vee (x > 1))$

   (d) $\exists x(-5 < x < -1)$

   Solution: $\forall x((x \leq -5) \vee (x \geq -1))$

2. Determine whether the following pairs of statements are logically equivalent.
   (a) $\forall x(P(x) \implies Q(x))$ and $\forall x P(x) \implies \forall x Q(x)$

   Solution: Since the predicate P and Q donot take the same truth value for every x, the two quantified statements are not equivalent.

(b) $\forall x(P(x) \iff Q(x))$ and $\forall x P(x) \iff \forall x Q(x)$

> **Solution:** Since the two predicates take the same truth value for every x, the two quantified statements are equivalent.

3. Show that the following pairs of statements are not logically equivalent.

(a) $\forall x P(x) \lor \forall x Q(x)$ and $\forall x(P(x) \lor Q(x))$

> **Solution:** Since both the predicates donot take the same truth value for every x, the statements are not equivalent.

(b) $\exists x P(x) \land \exists x Q(x)$ and $\exists x(P(x) \land Q(x))$

> **Solution:** For the first statment, P and Q can be satisfied by two different values of x. Whereas, the second one insists on a single x sastisfying both predicates. Therefore, they are not equivalent.

1.5 Nested Quantifiers

4. A discrete mathematics class contains 1 CND freshman, 3 CND sophomores, 15 CS sophomores, 2 CND juniors, 2 CS juniors, and 1 CS senior. Express each of these statements in terms of quantifiers and then determine its truth value.

(a) There is a student in the class who is a junior.

> **Solution:**
> j(x): x is a junior
> D(x): x is in discrete maths class
> $\exists x(D(x) \implies j(x))$
> True

(b) Every student in the class is a CS major.

> **Solution:**
> CS(x): x is a CS major
> D(x): x is in discrete maths class
> $\forall x(D(x) \implies CS(x))$
> False

(c) There is a student in the class who is neither a CND major nor a junior.

> **Solution:**
> j(x): x is a junior
> CND(x): x is a CND major

D(x): x is in discrete maths class
$\exists x(D(x) \implies \neg(CND(x) \vee j(x)))$
True

(d) Every student in the class is either a sophomore or a CS major.

Solution:
D(x): x is in discrete maths class
CS(x): x is a CS major
soph(x): x is a sophomore
$\forall x(D(x) \implies (CS(x) \vee Soph(x)))$
False

(e) There is a major such that there is a student in the class in every year of study with that major.

Solution:
Maj(m): m major Y(p, q, r, s): p is a freshman, q is a sophomore, r is a junior, s in senior $\exists m(Maj(m) \implies \exists p \exists q \exists r \exists s P(p, q, r, s))$
False

5. Use predicates, quantifiers, logical connectives, and mathematical operators to express the statement that there is a positive integer that is not the sum of three squares.

Solution:
P(x): x is a positve integer
S(x, p, q, r): x is the sum of squares of p, q, r
$\exists x(P(x) \implies \neg(\exists p \exists q \exists r(S(x, p, q, r))))$

6. Determine the truth value of each of these statements if the domain of each variable consists of all real numbers. Where possible, provide an example if the statement is true, or a counterexample if the statement is False.

(a) $\forall x \exists y(x^2 = y)$

Solution:
True
This statement means that for all real numbers x there exists a value y which is the square of x, $2^2 == 4$

(b) $\forall x \exists y(x = y^2)$

Solution:
False
A counter example would be x = -1, there doesn't exist a numbers whose square is equal to -1

(c) $\exists x \forall y (xy = 0)$

Solution:
True
For x =0, the product of x and y will always be zero.

(d) $\exists x \exists y (x + y \neq y + x)$

Solution: False
According to the commutative property of addition, any order of adding elements gives the same answer. For example, x = 1 and y = -2, x+y = -1 and y+x = -1.

(e) $\forall x (x \neq 0 \implies \exists y (xy = 1))$

Solution:
True
For all numerical values except 0 there exists a multiplicative inverse. For example, if x = 3, then multiplying it with 1/3 will give 1.

(f) $\exists x \forall y (y \neq 0 \implies xy = 1)$

Solution:
False
There only exists one multiplicative inverse for any real number, whereas this statement says that for one x all the values of y are its multiplicative inverse.

(g) $\forall x \exists y (x + y = 1)$

Solution: True
if x=0 and y=1, x+y = 1

(h) $\exists x \exists y (x + 2y = 2 \wedge 2x + 4y = 5)$

Solution:
False
Since both of the equations cannot be solved simultaneously, it is false for all the values of x and y.

(i) $\forall x \exists y (x + y = 2 \wedge 2x - y = 1)$

Solution:
False
Lets consider x=0, y would be equals to 2.

$$2(0) - 2 \neq 1$$

(j) $\forall x \forall y \exists z(z = \frac{x+y}{2})$

> **Solution:**
> True
> The statement says that for every value x and y, there exists an average z. consider x = 1 and y = 3, there exist a value z which is equals to 2.

7. Suppose the domain of the propositional function P(x, y) consists of pairs x and y, where x is $1, 2,$ or 3 and y is $1, 2,$ or 3. Write out these propositions using disjunctions and conjunctions.

(a) $\forall x \forall y P(x, y)$

> **Solution:** $P(1,1) \wedge P(1,2) \wedge P(1,3) \wedge P(2,1) \wedge P(2,2) \wedge P(2,3) \wedge P(3,1) \wedge P(3,2) \wedge P(3,3)$

(b) $\exists x \exists y P(x, y)$

> **Solution:** $P(1,1) \vee P(1,2) \vee P(1,3) \vee P(2,1) \vee P(2,2) \vee P(2,3) \vee P(3,1) \vee P(3,2) \vee P(3,3)$

(c) $\exists x \forall y P(x, y)$

> **Solution:** $(P(1,1) \wedge P(1,2) \wedge P(1,3)) \vee (P(2,1) \wedge P(2,2) \wedge P(2,3)) \vee (P(3,1) \wedge P(3,2) \wedge P(3,3))$

(d) $\forall x \exists y P(x, y)$

> **Solution:** $(P(1,1) \wedge P(2,1) \wedge P(3,1)) \vee (P(1,2) \wedge P(2,2) \wedge P(3,2)) \vee (P(1,3) \wedge P(2,3) \wedge P(3,3))$

8. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.

(a) $\forall x \exists y(x = \frac{1}{y})$

> **Solution:** Lets consider x = 0, then there won't exist any value of y such 0 = 1/y as division by zero is infinity.

(b) $\forall x \exists y(y^2 - x < 100)$

> **Solution:** The quantified statement is true. Therefore, no counterexample exist.

(c) $\forall x \forall y(x^2 \neq y^3)$

> **Solution:** consider x=0 and y=0 then x^2 would be equals to y^3

9. Determine the truth value of the statement $\exists x \forall y (x \leq y^2)$ if the common domain for the variables is as given below. Where possible, provide an example if the statement is true, or a counterexample if the statement is False.

   (a) the positive real numbers.

   > **Solution:** False
   > Consider x = 0.25, then for y = 0.125 we have $y^2 = 1/64$ for which $x \geq y^2$.

   (b) the integers.

   > **Solution:** True
   > consider x = 0, then for every value of y $x \leq y^2$

   (c) the nonzero real numbers.

   > **Solution:** False
   > consider x=1, then for y = 0.5 we have $y^2 = 0.25$ which is not greater than equals to 1. Thus for every value of x there will always exist a y for which $x \geq y^2$

10. Express the quantification $\exists! x P(x)$ using universal and existential quantifiers, and logical operators.

> **Solution:** $\exists x (P(x) \wedge \forall y (P(y) \implies y = x))$