Problem Set 07: Proof Methods

CS/MATH 113 Discrete Mathematics

Spring 2024

For each question below, clearly write the statement to prove or disprove so that its logical structure is evident. Provide formal proofs. See this guide for typesetting proofs in LaTeX.

If your proof is exceeding 10 lines, you are probably on the wrong track.

The following definitions may prove helpful when attempting the problems.

Definition 1 (Prime and composite numbers)**.** A natural number (0, 1, 2, 3, 4, 5, 6, etc.) is called a *prime number* (or a *prime*) if it is greater than 1 and cannot be written as the product of two smaller natural numbers. The numbers greater than 1 that are not prime are called *composite* numbers.

Definition 2 (Even and odd numbers)**.** An *even number* is an integer of the form $x = 2k$ where k is an integer; an *odd number* is an integer of the form $x = 2k + 1$.

Definition 3 (Parity)**.** The *parity of a number* is its property of being even or odd.

Definition 4 (Rational number)**.** A *rational number* can be written as $\frac{p}{q}$ where p and q are integers and $q \neq 0$. A number that is not rational is *irrational.*

1. Show through contraposition: If $x^2 - 6x + 5$ is even, then x is odd.

Solution:

Proof. We will prove the contrapositive: If x is even, then $x^2 - 6x + 5$ is odd.

suppose x is even, which means $x = 2k$ for some integer k.

Then $x^2 - 6x + 5 = (2k)^2 - 6(2k) + 5 = 4k^2 - 12k + 4 + 1$.

We can factor out 2 from the first three terms and get $2(2k^2 - 6k + 2) + 1$.

Since $2k^2 - 6k + 2$ is an integer, we can write $2(2k^2 - 6k + 2) + 1 = 2m + 1$ for some integer m.

Now we can see that $2m + 1$ is odd

Therefore, $x^2 - 6x + 5$ is odd.

We have proved the contrapositive, so we can conclude that the original statement is true. □

2. Provide a counterexample to disprove: If n is an integer and n^2 is divisible by 4, then n is divisible by 4. Explain why it is a counterexample.

Solution:

Proof. Let $n = 2$. Then $n^2 = 4$, which is divisible by 4. However, $n = 2$ is not divisible by 4, since 4 does not divide 2 evenly. Therefore, $n = 2$ is a counterexample to the statement. $\square$

3. Prove using contradiction that $\sqrt{2}$ is irrational.

Solution:

Proof. Suppose $\sqrt{2}$ is rational, and let $\sqrt{2} = \frac{a}{b}$, where a and b are positive integers with no common factor. Then we have

$$2b^2 = a^2.$$

This implies that a^2 is even, and hence a is even. Therefore, we can write $a = 2k$ for some positive integer k. Substituting this into the equation, we get

$$2b^2 = (2k)^2,$$

which simplifies to

$$b^2 = 2k^2.$$

This implies that b^2 is even, and hence b is even. But this contradicts the assumption that a and b have no common factor, since they both have 2 as a factor. Therefore, our initial assumption that $\sqrt{2}$ is rational must be false, and hence $\sqrt{2}$ is irrational. $\square$

For each of the following problems, clearly mention the proof method that you employ.

4. Prove that for $n \in \mathbb{Z}$, n is odd if and only if $5n + 6$ is odd.

Solution:

Proof. We will prove this using proof by equivalency.
odd(x): x is odd
even(x): x is even
First we prove that $odd(n) \implies odd(5n + 6)$ using direct proof
Suppose n is odd. Then $n = 2k + 1$ for some integer k. Then

$$5n + 6 = 5(2k + 1) + 6 = 10k + 11 = 2(5k + 5) + 1 = 2m + 1$$

We can conclude that $5n + 6$ is odd.
We now prove the contraposition of $odd(5n + 6) \implies odd(n)$ which is $even(n) \implies even(5n + 6)$
Suppose $even(n)$ is True. Then $n = 2t$ for some integer t. Then

$$5n + 6 = 5(2t) + 6 = 10k + 6 = 2(5t + 3) = 2s$$

We conclude that $5n + 6$ is even.
Since, both of the implications are proved, the biconditional is also proved. $\square$

5. Prove or disprove: The sum of a rational and an irrational number is a rational number.

Solution:

Proof. Let r be a rational number and i be an irrational number. Then suppose their sum $r + i$ is rational. Then we can write $\frac{p}{q} + i = \frac{a}{b}$, where a, b, p and q are integers and $b \neq 0$, $q \neq 0$. Then we have

$$i = \frac{aq - pb}{bq}.$$

Since $bq \neq 0$, i is a rational number thus contradicting with our assumption that i is irrational.

Therefore, we have disproved the statement by contradiction. $\square$

6. Prove or disprove that for $(x^2 - y^2) \mod 4 \neq 2$ where x and y are integers.
Hint: a) Consider the different cases of parities of x nd y. b) Use the method of *proof by cases* and apply a proof *without loss of generality* described in Section 1.8.2 in the book.

Solution:

Proof. There are four possible cases for the parities of x and y:

- Case 1: $x = 2k$ and $y = 2l$ for some integers k and l.
- Case 2: $x = 2k + 1$ and $y = 2l$ for some integers k and l.
- Case 3: $x = 2k$ and $y = 2l + 1$ for some integers k and l.
- Case 4: $x = 2k + 1$ and $y = 2l + 1$ for some integers k and l.

We will now prove each case one by one

- Case 1: $x^2 - y^2 = (2k)^2 - (2l)^2 = 4(k^2 - l^2)$. This means that $(x^2 - y^2) \mod 4 = 0$.
- Case 2: $x^2 - y^2 = (2k+1)^2 - (2l)^2 = 4(k^2 + k - l^2) + 1$. This means that $(x^2 - y^2) \mod 4 = 1$.
- Case 3: $x^2 - y^2 = (2k)^2 - (2l+1)^2 = 4(k^2 - l^2 - l) - 1$. This means that $(x^2 - y^2) \mod 4 = 3$.
- Case 4: $x^2 - y^2 = (2k+1)^2 - (2l+1)^2 = 4(k^2 + k - l^2 - l)$. This means that $(x^2 - y^2) \mod 4 = 0$.

Since we have covered all possible cases, we have proved that $(x^2 - y^2) \mod 4 \neq 2$ for all integers x and y $\square$

7. Prove or disprove that $2^n + 1$ is prime for every $n \in \mathbb{Z}^+$.

Solution:

Proof. This can be disproved using a counterexample.
Consider n = 3, for which $2^n + 1 = 9$
Since 9 can be written as a product of two 3, it is not a prime number.
Therefore we have disproved the statement. $\square$