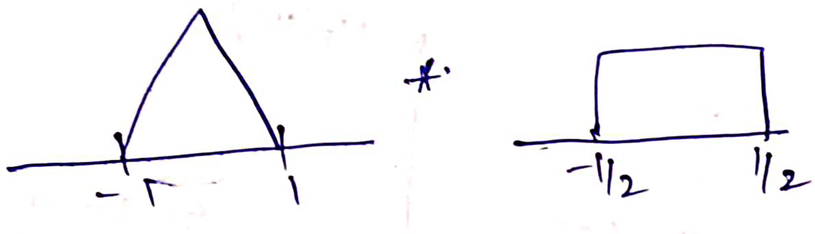
(4a) FT of $x(t) = \frac{\sin^3(t)}{t^3}$

we know that :

$$\frac{\sin(t)}{t} \xleftrightarrow{FT} rect(f)$$

$$sinc^2(t) \longleftrightarrow tri(f)$$

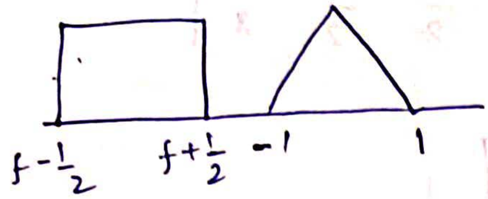
$\therefore$ 

$= sinc^3\left(\frac{\omega}{2}\right)$

$\therefore$ On applying duality and convoluting rect and tri we get F.T $(sinc^3(t))$.

4a

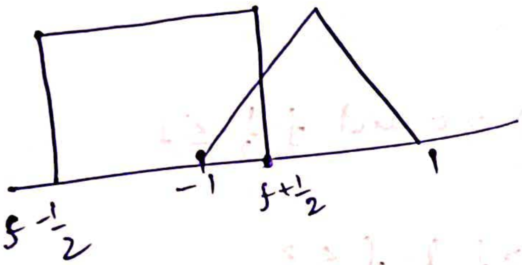
Case 1 :-



if $f+\frac{1}{2} \le -1$ $\Rightarrow X(f)=0$

$f \le -\frac{3}{2}$

Case 2 :-



if $-1 \le f+\frac{1}{2} \le 0$

$\Rightarrow -\frac{3}{2} \le f \le -\frac{1}{2}$

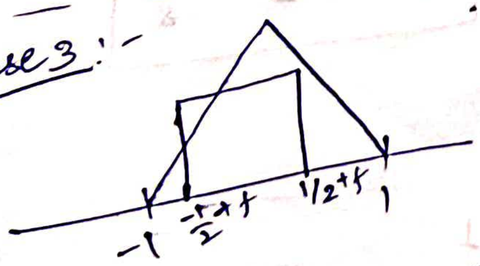
$$X(f) = \int_{-1}^{f+1/2} (1+\tau)\, d\tau$$

$$= \tau + \frac{\tau^2}{2} \Big|_{-1}^{f+\frac{1}{2}}$$

$$= f + \frac{1}{2} + \frac{f^2}{2} + \frac{1}{8} + \frac{f}{2} + 1 - \frac{1}{2}$$

$$= \boxed{\frac{f^2}{2} + \frac{3f}{2} + \frac{9}{8}}$$

Case 3 :-



if $f+\frac{1}{2} \ge 0$ & $f-\frac{1}{2} \ge -1$

$\Rightarrow \frac{1}{2} \ge f \ge -\frac{1}{2}$.

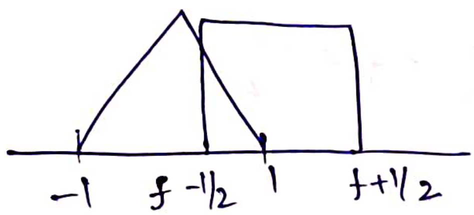
$$X(f) = \int_{f-1/2}^{0} (1+\tau)\, d\tau + \int_{0}^{f+1/2} (1-\tau)\, d\tau$$

$$= \tau + \frac{\tau^2}{2} \Big|_{f-1/2}^{0} + \tau - \frac{\tau^2}{2} \Big|_{0}^{f+1/2}$$

$$= -\left[f - \frac{1}{2} + \frac{f^2}{2} - \frac{f}{2} + \frac{1}{8}\right] + \left[f + \frac{1}{2} - \frac{f^2}{2} - \frac{f}{2} - \frac{1}{8}\right]$$

$$= -f^2 + 1 - \frac{1}{4} \Rightarrow \boxed{\frac{3}{4} - f^2}$$

Case 4 :-



if $f - \frac{1}{2} \geq 0$ and $f - \frac{1}{2} \leq 1$

$= f \geq \frac{1}{2}$ & $f \leq \frac{3}{2}$

$\Rightarrow \frac{1}{2} \leq f \leq \frac{3}{2}$

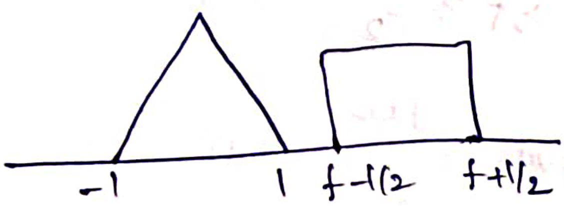
$$X(f) = \int_{f-1/2}^{1} (1-\tau)\, d\tau$$

$$= \left[\tau - \frac{\tau^2}{2}\right]_{f-1/2}^{1} = \left(1 - \frac{1}{2}\right) - \left(f - \frac{1}{2} - (f - 1/2)^2\right)$$

$$= \frac{1}{2} - f + \frac{1}{2} + \frac{f^2}{2} - \frac{f}{2} + \frac{1}{8}$$

$$= \boxed{\frac{f^2}{2} - \frac{3f}{2} + \frac{9}{8}}$$

Case5 :-



if $f - \frac{1}{2} \geq 1$

$\Rightarrow \boxed{f \geq \frac{3}{2}}$ $\Rightarrow \boxed{X(f) = 0}$

$$\therefore X(f) = \begin{cases} \frac{f^2}{2} + \frac{3f}{2} + \frac{9}{8} & -\frac{3}{2} \le f \le -\frac{1}{2} \\ \frac{3}{4} - f^2 & -\frac{1}{2} \le f \le \frac{1}{2} \\ \frac{f^2}{2} - \frac{3f}{2} + \frac{9}{8} & \frac{1}{2} \le f \le \frac{3}{2} \\ 0 & \text{, otherwise.} \end{cases}$$

4 c $t e^{-\alpha t} \cos(\beta t)$

$$x(t) \xrightarrow{F.T} X(\omega)$$

$$t x(t) \xleftrightarrow{F.T} j \frac{dX(\omega)}{d\omega}$$

$$e^{-\alpha t} \xrightarrow{F.T} \int_{-\infty}^{\infty} e^{-\alpha t} e^{-j2\pi f t} dt$$

$$\int_{0}^{\infty} e^{-\alpha t} e^{-j2\pi f t} dt$$ (assuming Right handed signal)

$$= \frac{-1}{\alpha + j2\pi f} [0 - 1]$$

$$= \frac{1}{\alpha + j2\pi f}$$ $\Rightarrow$ as $\omega = 2\pi f$

$$\Rightarrow \frac{1}{\alpha + j\omega}$$

$$e^{-\alpha t} \xrightarrow{F.T} \frac{1}{\alpha + j\omega}$$

$$e^{-\alpha t} \xrightarrow{F.T} j \frac{d}{d\omega} \frac{1}{\alpha + j\omega}$$

$$= \frac{-j}{(\alpha + j\omega)^2} (j) = \frac{1}{(\alpha + j\omega)^2}$$

$$t e^{-\alpha t} \xleftrightarrow{F\cdot T} \frac{1}{(\alpha+j\omega)^2}$$

$$t e^{-\alpha t} \cos\beta t = t e^{-\alpha t}\left[\frac{e^{j\beta t}+e^{-j\beta t}}{2}\right]$$

$$t e^{-\alpha t} \cos\beta t \xrightarrow{F\cdot T} \frac{1}{(\alpha+j\omega)^2} \circledast \left[\delta(\omega-\beta)+\delta(\omega+\beta)\right]$$

$$= \frac{\pi}{(\alpha+j\omega)^2} \circledast \left[\delta(\omega-\beta)+\delta(\omega+\beta)\right]$$

$$= \pi\left[\frac{1}{(\alpha+j(\omega-\beta))^2}+\frac{1}{(\alpha+j(\omega+\beta))^2}\right]$$

$$\Rightarrow t e^{-\alpha t} \cos\beta t \xleftrightarrow{F\cdot T} \frac{\pi}{(\alpha+j(\omega-\beta))^2}+\frac{\pi}{(\alpha+j(\omega-\beta))^2}$$

Q6)) Given,

$$u_p(t) = \text{sinc}(2t)\cos(100\pi t), \quad v_p(t) = \text{sinc}(t)\sin\left(101\pi t + \frac{\pi}{4}\right)$$

$$u(t) = \left(u_p(t) + j\tilde{u}_p(t)\right) e^{-j2\pi f_c t}$$

$$u(t) = \text{sinc}(2t)\left[\cos(100\pi t) + j\sin(100\pi t)\right] e^{-100\pi j t}$$

$$= \text{sinc}(2t)\, e^{100\pi j t - 100\pi j t}$$

$$\boxed{u(t) = \text{sinc}(2t)}$$

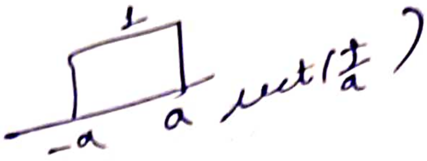
$$v_p(t) = \left[v_p(t) + j v_p(t)\right] e^{-j2\pi f_c t}$$ where $\tilde{v}_p(t) = \text{sinc}(t)\left\{-\cos\left(101\pi t + \frac{\pi}{4}\right)\right.$

$$v(t) = \text{sinc}(t)\, e^{-j(\pi t - \frac{\pi}{4})}\, e^{j100\pi t}\, e^{-j(100)\pi t}$$

$$\Rightarrow \boxed{v(t) = \text{sinc}(t)\, e^{-j(\pi t - \frac{\pi}{4})}}$$

(b)

Bandwidth =

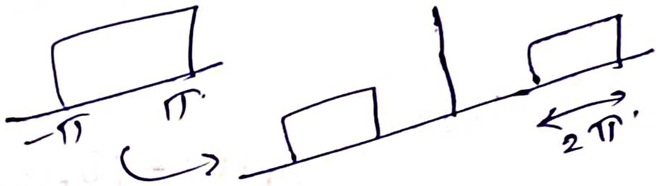


$\mathrm{sinc}(at) \xleftrightarrow{FT}$ $a\ \mathrm{rect}(\frac{t}{a})$

$u_p(t) = \mathrm{sinc}\, 2t \cos 100\pi t$

$= \frac{\sin 2\pi t}{\sin \pi t} \cdot \cos(100\pi t)$

$\boxed{\text{Bandwidth} = 2\pi}$



~~If we consider negative frequency also then Bandwidth = 2+2 = 4Hz~~

as $\omega = 2\pi f$

$\Rightarrow \boxed{1\,Hz = f_{bandwidth}}$

$V_p(t) = \mathrm{sinc}(t) \sin(101\pi t + \frac{\pi}{4})$

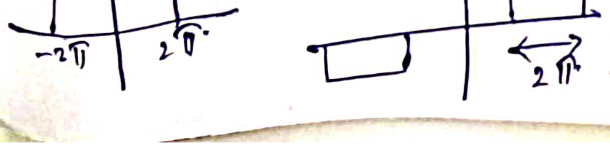
$= \mathrm{sinc}(t)\left\{\frac{1}{\sqrt{2}}(\sin(101\pi t) + \cos(101\pi t))\right\}$

we know: $\cos(2\pi f_0 t) \xleftrightarrow{FT} \frac{1}{2}\{\delta(f-f_0) + \delta(f+f_0)\}$

$\sin(2\pi f_0 t) \xleftrightarrow{F.T} \frac{j}{2}\{\delta(f-f_0) - \delta(f+f_0)\}$

If we look at $\mathrm{sinc}(t) \sin 101\pi t$

bandwidth $= 4\pi$



$\boxed{f_{bandwidth} = 2\,Hz}$

(c) Inner product:-

$$\langle u_p, v_p \rangle = \int u_p(t) \cdot v_p(t)\, dt$$

$$= \text{sinc}(t)\, \text{sinc}(2t)\, e^{-j(\pi t - \pi/4)}$$

(d) convolution:-

$$y_p(f) = U_p(f) \cdot V_p(f) \quad \text{(convolution property)}$$

$$y_p(f) = \frac{1}{4}\left[\text{rect}\left(\frac{f-100\pi}{4\pi}\right) + \text{rect}\left(\frac{f+100\pi}{4\pi}\right)\right]$$

$$y_p(f) = \frac{1}{\sqrt{2}}\,\frac{1}{2}\left[\text{rect}\left(\frac{f-101\pi}{2\pi}\right) + \text{rect}\left(\frac{f+101\pi}{2\pi}\right)\right]$$

$$+ \frac{1}{\sqrt{2}}\,\frac{1}{2j}\left[\text{rect}\left[\frac{f-101\pi}{2\pi}\right] - \text{rect}\left[\frac{f+101\pi}{2\pi}\right]\right]$$

$$y_p(f) = \frac{1}{4\sqrt{2}}.\ \frac{1}{2}.\left[\text{rect}\left(\frac{f-101\pi}{2\pi}\right) + \text{rect}\left(\frac{f+101\pi}{2\pi}\right)\right]$$

$$+ \frac{1}{4}\,\frac{1}{2\sqrt{2}j}\left[\text{rect}\left(\frac{f-101\pi}{2\pi}\right) - \text{rect}\left(\frac{f+101\pi}{2\pi}\right)\right]$$

After taking Inverse fourier transform we will get

$$\boxed{y_p(t) = \frac{1}{4\sqrt{2}}\ \text{sinc}(\pi t)\left[\cos(101\pi t) + \sin(101\pi t)\right]}$$

(7) $u(t) = \mathbb{1}_{[-1,1]}(t)\cos(100\pi t)$

$h(t) = \mathbb{1}_{[0,3]}(t)\sin(100\pi t)$

$y(t) = h(t) * u(t)$

$$= \begin{cases} \int_0^{t+1} \sin(100\pi t)\cos(100\pi t)\, d\tau, & t \in [-1,1] \\ \int_{t-1}^{t+1} \sin(100\pi t)\cos(100\pi t)\, d\tau, & t \in [1,2] \\ \int_{t-1}^{3} \sin(100\pi t)\cos(100\pi \tau)\, d\tau, & t \in [2,4] \end{cases}$$

Now Integrating :-

$$\int \sin(100\pi t)\cos(100\pi(t-\tau))\, d\tau$$

$$= \int \sin(100\pi \tau)\cos(100\pi\tau - 100\pi t)\, d\tau$$

$$= \int \frac{\sin(200\pi\tau - 100\pi t)}{2} + \frac{\sin(100\pi + \tau)}{2}\, d\tau$$

$$= -\cos(100\pi\tau - 100\pi t) + \frac{\sin(100\pi t)\tau}{2} + C$$

$$\Rightarrow \int_0^{t+1} \sin(100\pi t)\cos(100\pi(t-\tau))\, d\tau, \quad t \in [-1,1]$$

$$= \left[\frac{-\cos(100\pi(\tau - t))}{400\pi} + \frac{\sin(100\pi t)\tau}{2} \right]_0^{t+1}$$

$$= \frac{-\cos(100\pi)}{400\pi} + \frac{\sin(100\pi t)(t+1)}{2} + \frac{\cos(0)}{400\pi} = 0$$

$$= \left(\frac{t+1}{2}\right) \sin(100\pi t) \quad \forall\ t \in [-1,1]$$

$$\int_{t-1}^{t+1} \sin(100\pi t) \cos(100\pi (t-\tau)) d\tau, \quad t \in [1,2]$$

$$= \left[-\frac{\cos(100\pi(\tau - t))}{400\pi} + \frac{\sin(100\pi t)}{2} \tau \right]_{t-1}^{t+1}$$

$$= \sin(100\pi t), \quad t \in [1,2]$$

$$\int_{t-1}^{3} \sin(100\pi t) \cos(100\pi (t-\tau)) d\tau, \quad t \in [2,4]$$

$$= \left[-\frac{\cos(100\pi(\tau - t))}{400\pi} + \frac{\sin(100\pi + \tau)}{2} \tau \right]_{t-1}^{3}$$

$$= -\frac{\cos(100\pi(3-t))}{400\pi} + \frac{3}{2}\sin(100\pi t) + \frac{\cos(-100\pi)}{400\pi}$$

$$- \frac{\sin(100\pi t)}{2}(t-1)$$

$$= 2\sin(100\pi t) - \frac{t}{2}\sin(100\pi t)$$

$$= \left(2 - \frac{t}{2}\right)\sin(100\pi t)$$

$$\Rightarrow \quad y(t) = \begin{cases} \frac{t+1}{2}\sin(100\pi t), & t \in [-1,1] \\ \sin(100\pi t), & t \in [1,2] \\ \left(2 - \frac{t}{2}\right)\sin(100\pi t), & t \in [2,4] \\ 0, & \text{othelwise.} \end{cases}$$

we can also do as:

$$\int_a^b \frac{1}{2}\sin(200\pi t)\, d\tau$$

$$\frac{1}{2}\left[-\frac{\cos(200\pi t)}{200\pi}\right]_a^b$$

$$= \frac{1}{400\pi}\left[-\cos(200\pi t)\right]_a^b$$

$$y(t) = \begin{cases} \frac{1}{400\pi}\left[1-\cos(200\pi(t+1))\right] & t\in[-1,1] \\ \frac{1}{400\pi}\left[\cos(200\pi(t-1))-\cos(200\pi(t+1))\right] & t\in[1,2] \\ \frac{1}{400\pi}\left[\cos(200\pi(t-1))-1\right] & t\in[2,4] \\ 0 & \text{, otherwise.} \end{cases}$$

(5) (a) $\int_0^{\infty} e^{-\alpha t} \operatorname{sinc}^2(t)\, dt.$

$$x(t) = e^{-\alpha|t|} \operatorname{sinc}^2(t)$$

symmetric about y axis

$$\therefore \int_{-\infty}^{\infty} e^{-\alpha|t|} \operatorname{sinc}^2(t)\, dt = 2\int_0^{\infty} e^{-\alpha t} \operatorname{sinc}^2(t)\, dt.$$

consider $x_1(t) = e^{-\alpha|t|}$

$$= \int_{-\infty}^{0} e^{\alpha t} e^{-j2\pi f t} dt + \int_0^{\infty} e^{-\alpha t} e^{-2\pi j f t}.$$

$$= \frac{e^{(\alpha - j2\pi f)}}{\alpha - j2\pi f}\Bigg|_{-\infty}^{0} + \int_{\infty}^{0} \frac{e^{-(\alpha + j2\pi f)t}}{(\alpha + j2\pi f)}$$

$$X_1(f) = \frac{1}{\alpha - j2\pi f} + \frac{1}{\alpha + j2\pi f} - \frac{2\alpha}{\alpha^2 + 4\pi^2 j^2}$$

Consider $x_2(t) = \operatorname{sinc}^2(t)$

$$X_2(f) = \pi - \pi^2 |f| \quad , |f| < \frac{1}{\pi}$$

& 0 , otherwise

as $x(t) = x_1(t) \cdot x_2(t)$

$$X(f) = X_1(f) * X_2(f)$$

$$X(f) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(f-\tau)\, d\tau$$

$$X(0) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(-\tau)\, d\tau$$

$$X(0) = \int_{-\infty}^{\infty} \frac{2\alpha}{\alpha^2 + 4\pi^2\tau^2} \left(\pi - \pi^2 |\tau|\right) d\tau$$

$$= \int_{-\frac{1}{\pi}}^{\frac{1}{\pi}} \frac{2\alpha}{\alpha^2 + 4\pi^2\tau^2} \left(\pi - \pi^2 |\tau|\right) d\tau$$

$$= \int_{-\frac{1}{\pi}}^{0} \frac{\frac{2\pi}{\alpha}}{\alpha^2 + \frac{4\pi^2\tau^2}{\alpha^2}} + \frac{\pi^2\tau(2\alpha)}{\alpha^2 + 4\pi^2\tau^2}\, d\tau$$

$$= 2\tan^{-1}\left(\frac{2}{\alpha}\right) + a + b.$$

where

$$a = \int_{-\frac{1}{\pi}}^{0} \frac{2\alpha\pi^2\tau}{\alpha^2 + 4\pi^2\tau^2}\, d\tau \qquad \left(\tau^2 = u \quad 2\tau\, d\tau = du\right)$$

$$= \int_{\frac{1}{\pi^2}}^{0} \frac{\alpha\pi^2\, du}{\alpha^2 + 4\pi^2 u} = \frac{\alpha\pi^2}{4\pi^2} \Big|_{1/\pi^2}^{0} \ln\left(4\pi^2 u + \alpha^2\right)$$

$$= \frac{\alpha}{4} \ln\left(\frac{\alpha^2}{4+\alpha^2}\right)$$

$$b = -\int_0^{1/\pi} \frac{\alpha\pi^2 du}{\alpha^2+4\pi^2 u} = \frac{\alpha\pi^2}{4\pi^2}\Big|_{1/\pi^2}^{0} \ln(4\pi^2 u+\alpha^2)$$

$$= \frac{\alpha}{4}\ln\left(\frac{\alpha^2}{4+\alpha^2}\right)$$

$$\therefore X(0) = 2\tan^{-1}\left(\frac{2}{\alpha}\right) + \frac{\alpha}{2}\ln\left(\frac{\alpha^2}{4+\alpha^2}\right)$$

we found $X(0) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$. when $f=0$

$$X(0) = \int_{-\infty}^{\infty} x(t)\, dt.$$

as done earlier:-

$$\int_0^{\infty} e^{-\alpha t}\,\mathrm{sinc}^2(t)\,dt = \frac{\alpha}{2}\int_{-\infty}^{\infty} e^{-\alpha|t|}\,\mathrm{sinc}^2(t)\,dt$$

$$= \frac{1}{2}X(0)$$

$$\therefore \boxed{\text{Ans} = \tan^{-1}\left(\frac{2}{\alpha}\right) + \frac{\alpha}{4}\ln\left|\frac{\alpha^2}{4+\alpha^2}\right|}$$

(b) $\int_0^{\infty} e^{-\alpha t}\cos(\beta t)\,dt = ?$

$$\frac{1}{2}\int_{-\infty}^{\infty} x(t)\,dt = \int_0^{\infty} e^{-\alpha t}\cos(\beta t)\,dt$$

$$X(0) = \int_{-\infty}^{\infty} x(t)\,dt$$

$$X(t) = X_1(f) * X_2(f)$$

where $X_2(t) = e^{-\alpha|t|}$

from 5(a) we know

$$X_1(f) = \frac{2\alpha}{\alpha^2 + 4\pi^2 j}$$

$$X_2(f) = x_2(t) = \cos(\beta t)$$

$$= \cos\left(2\pi\left(\frac{\beta}{2\pi}\right)t\right)$$

$$= \frac{1}{2}\left[\delta\left(f - \frac{\beta}{2\pi}\right) + \delta\left(f + \frac{\beta}{2\pi}\right)\right)$$

Now $g(t) * \delta(t - t_0) = g(t - t_0)$

$$\therefore X_1(f) * X_2(f)$$

$$X(f) = \frac{1}{2}\left[X_1\left(f - \frac{\beta}{2\pi}\right) + X_1\left(f + \frac{\beta}{2\pi}\right)\right]$$

$$X(f) = \frac{1}{2}\left[\frac{2\alpha}{\alpha^2 + 4\pi^2\left(f - \frac{\beta}{2\pi}\right)^2} + \frac{2\alpha}{\alpha^2 + 4\pi^2\left(f + \frac{\beta}{2\pi}\right)^2}\right.$$

$$\boxed{\Rightarrow X(0) = \frac{2\alpha}{\alpha^2 + \beta^2}}$$

$$\therefore \int_{-\infty}^{\infty} x(t)\,dt = X(0) = \int_0^{\infty} e^{-\alpha t}\cos(\beta t)\,dt = \frac{X(0)}{2}$$

$$\boxed{= \int_0^{\infty} e^{-\alpha t}\cos\beta t\,dt = \frac{\alpha}{\alpha^2 + \beta^2}}$$

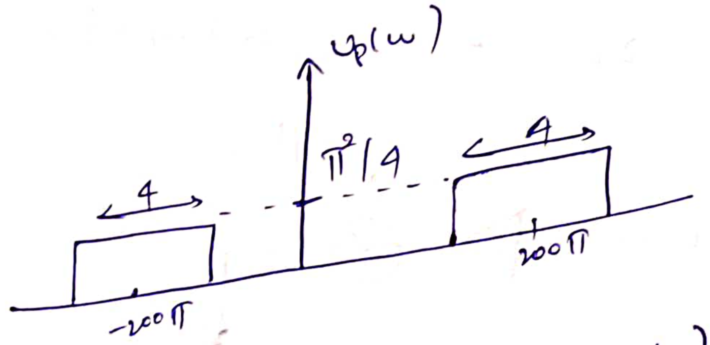
8(b) Let $x(t) = v_p(t)\cos(199\pi t)$

$$x(t) \longrightarrow \boxed{LPF} \longrightarrow b(t)$$

$$x(t) \xrightarrow{F.T} X(\omega)$$

$$v_p(t) \xrightarrow{F.T} V_p(\omega)$$

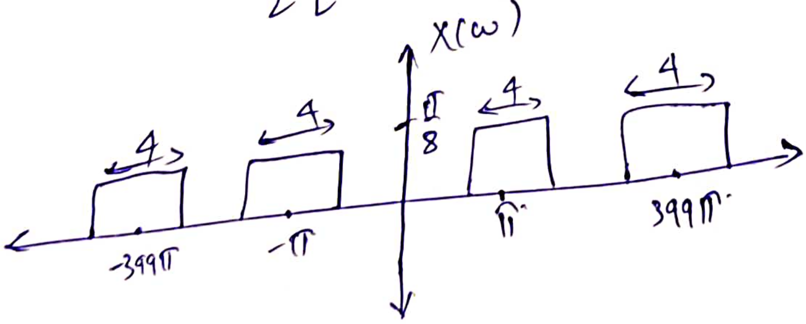
$$V_p(\omega) = \frac{\pi}{4} \mathbb{I}_{[-200\pi-2,\, -200\pi+2]} + \frac{\pi}{4} \mathbb{I}_{[200\pi-2,\, 200\pi+2]}$$



$$X(\omega) = V_p(\omega) * FT(\cos(199\pi t))$$

$$= V_p(\omega) * \frac{1}{2}[\delta(\omega - 199\pi) + \delta(\omega + 199\pi)]$$

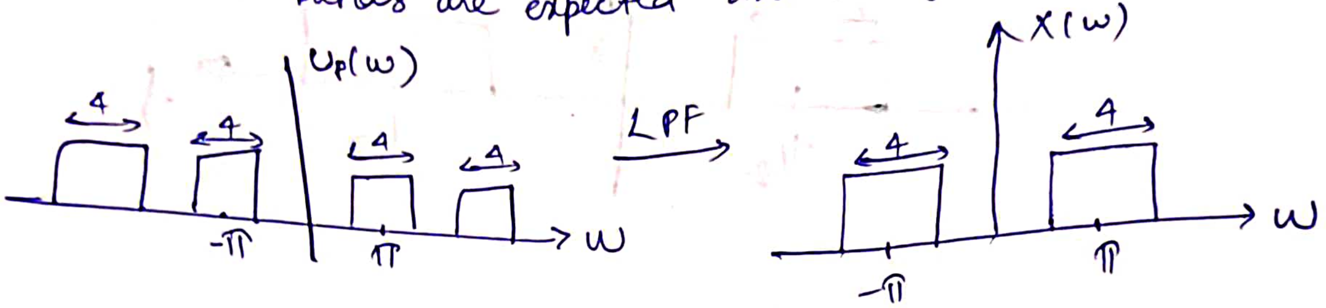
$$= \frac{1}{2}[V_p(\omega - 199\pi) + V_p(\omega + 199\pi)]$$

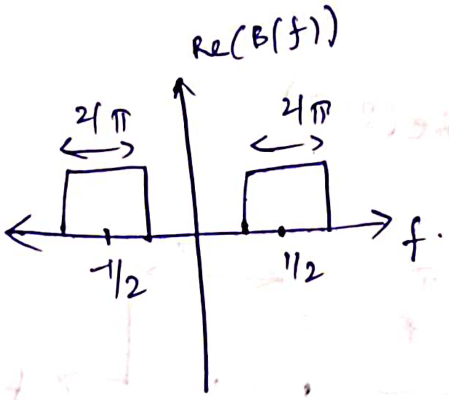


$$= \frac{\pi}{8}\left[\mathbb{I}_{[-2-399\pi,\, -399\pi+2]} + \mathbb{I}_{[-\pi-2,\, -\pi+2]} + \right.$$

$$\left. \mathbb{I}_{[399\pi-2,\, 399\pi+2]} + \mathbb{I}_{[\pi-2,\, \pi+2]}\right]$$

when $b(t)$ is passed the LPF. so the frequency domain their two bands are expected about angular freq π & $-\pi$





(8 c) Let $y(t) = U_p(t)\sin(199\pi t)$

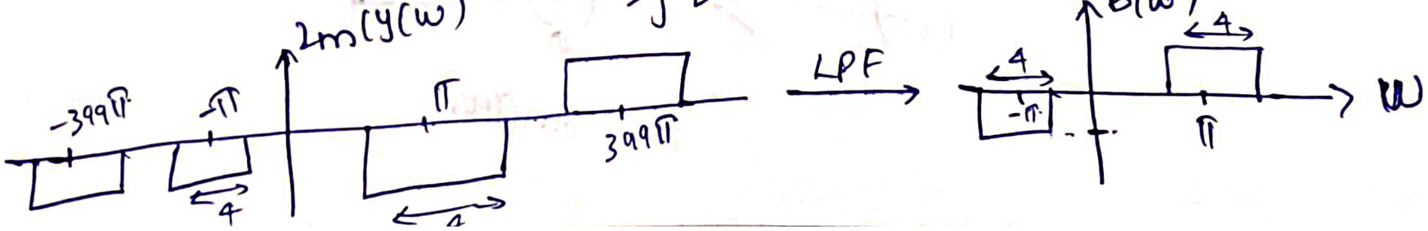
$$y(t) \xrightarrow{F.T} Y(\omega)$$

$$u_p(t) \xrightarrow{F.T} U_p(\omega)$$

$$Y(\omega) = U_p(\omega) * F.T(\sin(199\pi t))$$

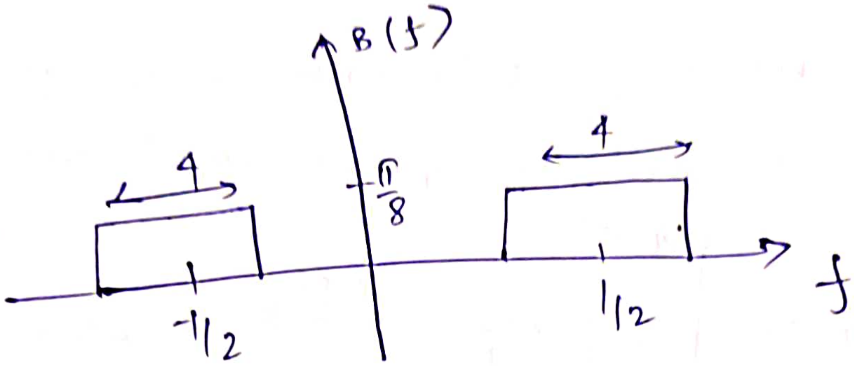
$$= V_p(\omega) * \frac{1}{2j}\left[\delta(\omega - 199\pi) - \delta(\omega + 199\pi)\right]$$

$$= \frac{1}{2j}\left[V_p(\omega - 199\pi) - U_p(\omega + 199\pi)\right]$$

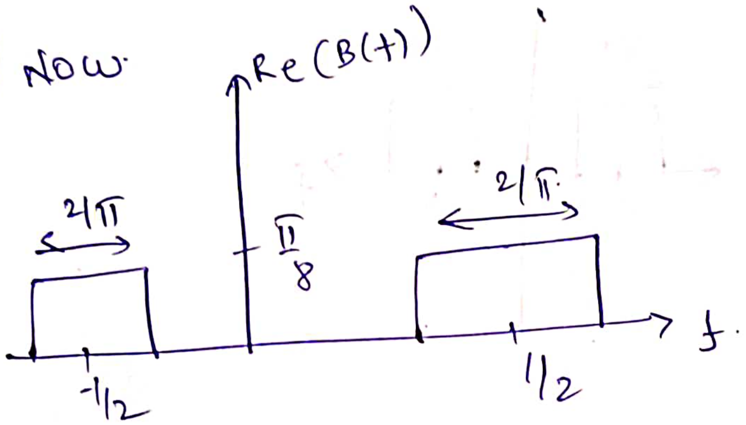


$$Re(B(f)) = 0$$

$$Im(B(f)) = \frac{-\pi}{8} I_{[-2-\pi,\, -\pi+2]} - \frac{\pi}{8} I_{[\pi-2,\, \pi+2]}$$



(f) Now



complex envelope $= u_c(t) + j u_s(t) = 2(b(t) - j c(t)) = u(t)$

$\Rightarrow$ Real valued $\Rightarrow$

$$a(t) = 2b(t)\cos(\pi t) - 2c(t)\sin(\pi t)$$