Assignment 1
Communication Theory

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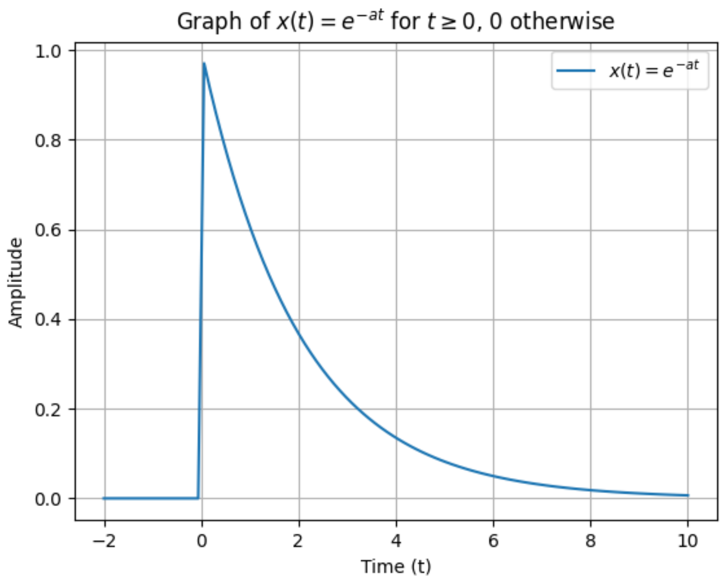
Question 1 :

Consider the signal $x(t)$ defined as $x(t) = e^{-at}$ for $t \geq 0$ and 0 otherwise. What is the bandwidth required to transmit 95% of the signal?

Solution :



Starting with the Parseval's theorem:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int |X(w)|^2 dw$$

$$\text{Energy(E)} = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{0}^{\infty} e^{-2at} dt = \left.\frac{e^{-2at}}{-2a}\right|_0^{\infty} = \frac{1}{2a}$$

Now Let band width $= B$

$$\Rightarrow \quad \frac{1}{2\pi} \int_{-B}^{B} |X(w)|^2 d\omega = \frac{95}{100} \times \frac{1}{2a}$$

Now $X(w) = \frac{1}{a+jw}$.

$$\Rightarrow \frac{1}{2\pi} \int_{-B}^{B} \frac{1.dw}{a^2 + w^2} = \frac{0.95}{2a}$$

$$\frac{1}{2\pi} \cdot \frac{1}{a} \tan^{-1}\left(\frac{w}{a}\right)\Big|_{-B}^{B} = \frac{0.95}{2a}$$

$$= \tan^{-1}\left(\frac{B}{a}\right) - \tan^{-1}\left(-\frac{B}{a}\right) = \frac{0.95\pi}{a} \cdot a$$

$$\Rightarrow 2\tan^{-1}\left(\frac{B}{a}\right) = 0.95\pi.$$

$$\Rightarrow \tan^{-1}\left(\frac{B}{a}\right) = 0.475 \times 180^\circ$$

$$\Rightarrow \tan(85.5^\circ) = \frac{B}{a.}$$

$$\boxed{\Rightarrow B = 12.70a \quad radians}$$

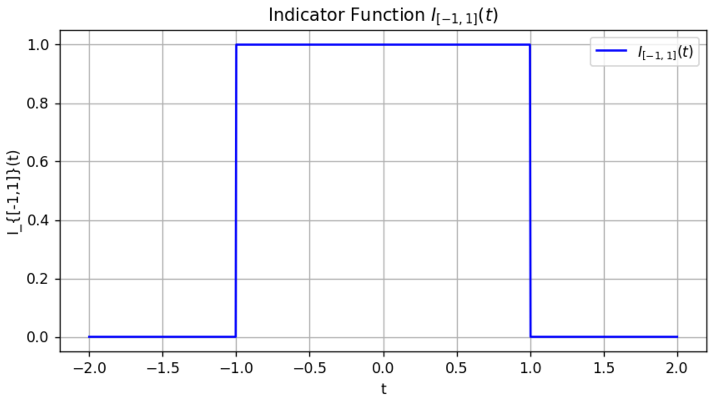
Therefore bandwidth required to transmit 95% of the signal is 12.70a rad or $\frac{12.7a}{2\pi} = \boxed{2.022aHz}$

Question 2 :

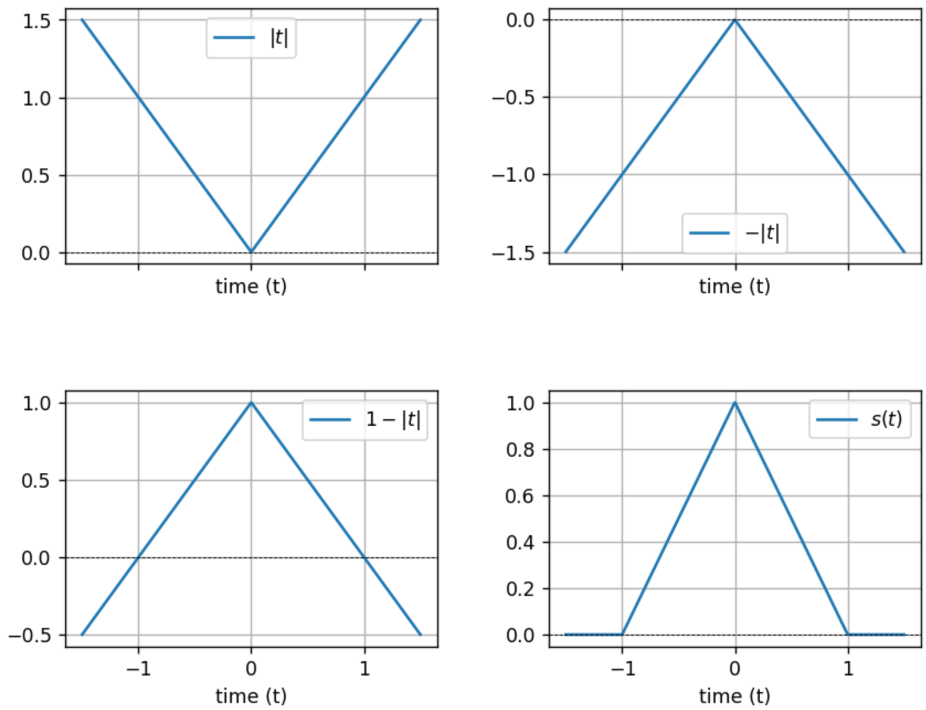
Consider the tent signal $s(t) = (1 - |t|)I_{[-1,1]}(t)$.
(a) Find and sketch the Fourier transform $S(f)$.
(b) Compute the 99% energy containment bandwidth in KHz, assuming that the unit of time is milliseconds.

Solution :

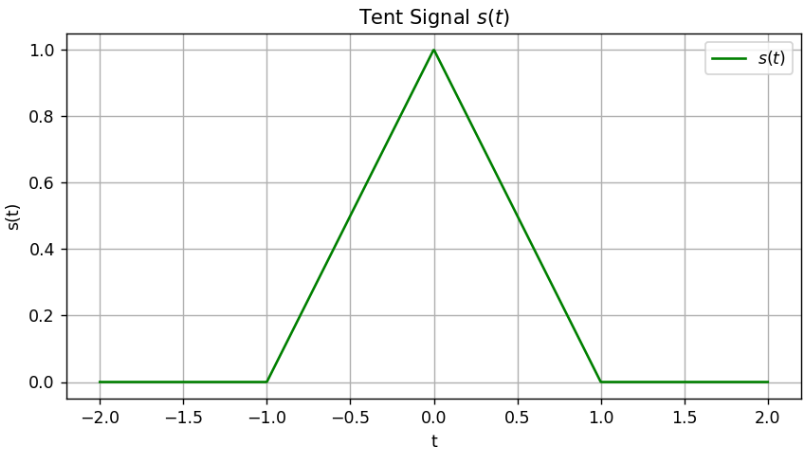
(a) Below is the plot of $I_{[-1,1]}(t)$:



In below figure , I have shown step by step plots how to sketch s(t):



Below is the sketch of our final $s(t)$:



Now therefore :

$$s(t) = \begin{cases} 1+t & \text{if } -1 \le t < 0 \\ 1-t & \text{if } 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Now finding Fourier transform of $s(t)$:

$$\int_{-1}^{0} (1+t)e^{-j\omega t}dt + \int_{0}^{1} (1-t)e^{-j\omega t}dt$$

Now for a instance let's focus on first term:
Let $u = -t$ in the first integral. When $t = -1$, $u = 1$, and when $t = 0$, $u = 0$. Also, $dt = -du$. Substituting these into the integral:

$$\begin{aligned} \int_{-1}^{0} (1+t)e^{-j\omega t}dt &= \int_{1}^{0} (1-u)e^{j\omega u}(-du) \quad \text{[Substitute } u = -t \text{ and } dt = -du] \\ &= \int_{0}^{1} (1-u)e^{j\omega u}du \quad \text{[Reversing the limits of integration]} \\ &= \int_{0}^{1} (1-t)e^{j\omega t}dt \quad \text{[Renaming the dummy variable back to } t]. \end{aligned}$$

So, $\int_{-1}^{0}(1+t)e^{-j\omega t}dt$ is indeed equal to $\int_{0}^{1}(1-t)e^{j\omega t}dt$, and the transition is justified through the change of variable $u = -t$.

$$\Rightarrow \int_0^1 (1-t)e^{j\omega t}dt + \int_0^1 (1-t)e^{-j\omega t}dt.$$

$$\Rightarrow \int_0^1 (1-t)\left(e^{j\omega t} + e^{-j\omega t}\right)dt.$$

$$\Rightarrow \int_0^1 e^{j\omega t} + e^{-j\omega t} - \int_0^1 t\left(e^{j\omega t} + e^{-j\omega t)}dt.$$

Now using trigonometric identities:-

$$\Rightarrow S(f) = \int_0^1 2\cos\omega t dt - \int_0^1 2t\cos\omega t dt$$

$$= 2\frac{\sin\omega t}{\omega}\bigg|_0^1 - 2\left[t\frac{\sin\omega t}{\omega} - \int_0^1 \frac{\sin\omega t}{\omega}\right]_0^1$$

$$= 2\frac{\sin\omega t}{\omega}\bigg|_0^1 - 2\left[\frac{t\sin\omega t}{\omega t} + \frac{\cos\omega t}{\omega^2}\right]_0^1$$

$$= \frac{2\sin\omega}{\omega} - 2\left[\frac{\sin\omega}{\omega} + \frac{\cos\omega}{\omega^2} - \frac{1}{\omega^2}\right]$$

$$= 2\left[\frac{1}{\omega^2} - \frac{\cos\omega}{\omega^2}\right]$$

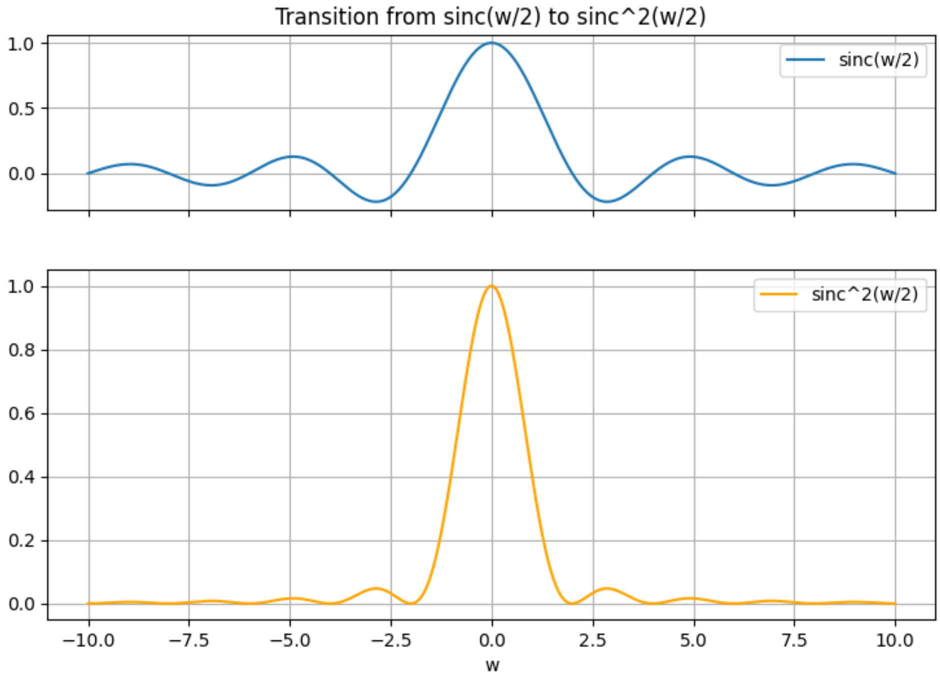
$$= \frac{2}{\omega^2}\left(2\sin^2\frac{\omega}{2}\right) \Rightarrow \frac{4\sin^2 - \left(\frac{\omega}{2}\right)}{\left(\frac{\omega}{2}\right)^2} \times \left(\frac{1}{2}\right)^2$$

$$= \frac{\sin^2\left(\frac{w}{2}\right)}{\left(\frac{w}{2}\right)^2} = \boxed{\mathrm{sinc}^2\left(\frac{w}{2}\right)}$$

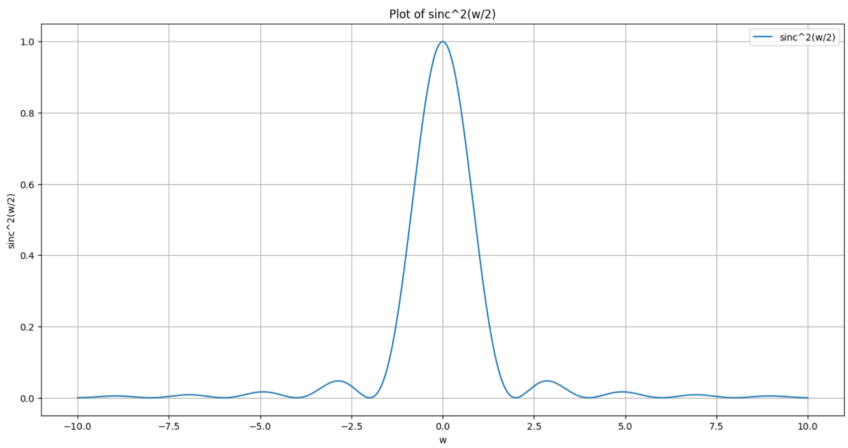
Therefore :

$$\mathcal{F}\{s(t)\} \rightarrow \mathrm{sinc}^2\left(\frac{w}{2}\right)$$

Below is the sketch of sinc^2 :



Drawing $\text{sinc}^2\left(\frac{w}{2}\right)$ graph by taking help to $\text{sinc}\left(\frac{w}{2}\right)$



Seperate plot of $\text{sinc}^2\left(\frac{w}{2}\right)$

(b) Starting with the Parseval's theorem:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int |X(w)|^2 dw$$

As $\int_{-1}^{0} (1+t)dt = \int_{0}^{1} (1-t)dt$ which is proved in part 'a'.So:

$$\text{Energy(E)} = \int_{-\infty}^{\infty} |x(t)|^2 dt = 2\int_{0}^{1} (1-t)^2 dt$$

Let's expand the square and integrate each term:

$$= \int_{0}^{1} (1 - 2t + t^2) dt$$

Now integrate each term:

$$= \int_{0}^{1} 1dt - \int_{0}^{1} 2tdt + \int_{0}^{1} t^2 dt$$

$$= [t]_0^1 - 2\left[\frac{t^2}{2}\right]_0^1 + \frac{1}{3}\left[t^3\right]_0^1$$

$$= 1 - 2\left(\frac{1}{2}\right) + \frac{1}{3}$$

$$= 1 - 1 + \frac{1}{3} = \frac{1}{3}$$

$$\rightarrow 2\int_{0}^{1} (1-t)^2 dt = 2 \times \frac{1}{3} = \frac{2}{3}$$

Now according to our question :

$$\frac{99}{100} \times \frac{2}{3} = \frac{1}{2\pi} \int_{-B}^{B} |X(w)|^2 d\omega, \quad \text{where 'B' is band width.}$$

Now calculating RHS part:

$$\frac{1}{2\pi} \int_{-B}^{B} |X(w)|^2 d\omega = \frac{1}{2\pi} \int_{-B}^{B} \operatorname{sinc}^4\left(\frac{w}{2}\right) d\omega$$

$$= \frac{1}{2\pi} \int_{-B}^{B} \left(\frac{4}{w^2}\right) \sin^4\left(\frac{w}{2}\right) d\omega$$

After calculation in matlab we get B=0.58KHz.

Question 3 :

Let $x(t)$ and $y(t)$ be two periodic signals with period T_0, and let x_n and y_n denote the Fourier series coefficients of these two signals.
(a) Show that

$$\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)y^*(t)dt = \sum_{n=-\infty}^{\infty} x_n y_n^*$$

This relation is known as Parseval's relation for the Fourier series. Show that the Rayleigh's relation for periodic signals is a special case of this relation. Rayleigh's Relation is shown below. Rayleigh's Relation:

$$\sum_{n=-\infty}^{\infty} |x_n|^2$$

(b) Show that for all periodic physical signals that have finite power, the coefficients of the Fourier series expansion x_n tend to zero as $n \to \infty$.
(c) Use Parseval's relation in part (a) to prove the following identity.

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots + \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$

Hint: Find Fourier series expansion of $f(x) = x^2, x \in [-\pi, \pi]$ and then use Parseval's identity.

Solution :

(a) As given $x(t)$ and $y(t)$ are periodic with T_0 and x_n, y_n are their fourier series coefficients. Then we know that :

$$x(t) = \sum_{n=-\infty}^{\infty} x_n \cdot e^{j\frac{2\pi}{T_0}\cdot nt}$$

$$y(t) = \sum_{n=-\infty}^{\infty} y_n \cdot e^{j\frac{2\pi}{T_0}\cdot nt}$$

Now, let's consider the left-hand side (LHS) of the given equality:

$$\text{LHS} = \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)y^*(t)\, dt$$

We want to express this in terms of the Fourier series coefficients. To do this, we can use the expressions for x_n and y_n:

$$
\begin{aligned}
\text{LHS} &= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)y^*(t)\,dt \\
&= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} \left(\sum_{n=-\infty}^{\infty} x_n e^{j\frac{2\pi n}{T_0}t}\right)\left(\sum_{n=-\infty}^{\infty} y_n^* e^{-j\frac{2\pi n}{T_0}t}\right) dt \\
&= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} \left(x_1 e^{j\frac{2\pi 1}{T_0}t} + x_2 e^{j\frac{2\pi 2}{T_0}t}.....\right)\left(y_1^* e^{-j\frac{2\pi 1}{T_0}t} + y_2^* e^{-j\frac{2\pi 2}{T_0}t}.....\right) dt
\end{aligned}
$$

Now by using the orthogonality property of complex exponentials when we open two $\sum$ and multiply exponentials with different n value, it will become zero as they are orthagonal and finally only we get sum of those whose product involved same indices.So therefore we can write above step as :

$$
\begin{aligned}
&= \frac{1}{T_0}\sum_{n=-\infty}^{\infty} x_n y_n^* \int_{\alpha}^{\alpha+T_0} e^{j\frac{2\pi(n-n)}{T_0}t}\,dt \\
&= \frac{1}{T_0}\sum_{n=-\infty}^{\infty} x_n y_n^*(T_0) = \boxed{\sum_{n=-\infty}^{\infty} x_n y_n^*}
\end{aligned}
$$

In General :

$$
\begin{aligned}
\text{LHS} &= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)y^*(t)\,dt \\
&= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} \left(\sum_{m=-\infty}^{\infty} x_m e^{j\frac{2\pi m}{T_0}t}\right)\left(\sum_{k=-\infty}^{\infty} y_k^* e^{-j\frac{2\pi k}{T_0}t}\right) dt \\
&= \frac{1}{T_0}\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x_m y_k^* \int_{\alpha}^{\alpha+T_0} e^{j\frac{2\pi(m-k)}{T_0}t}\,dt
\end{aligned}
$$

Now, let's use the orthogonality property of complex exponentials. If n is an integer, then:

$$
\int_{\alpha}^{\alpha+T_0} e^{j\frac{2\pi n}{T_0}t}\,dt = \begin{cases} T_0, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}
$$

Applying this property to the integral, we get:

$$
\text{LHS} = \frac{1}{T_0}\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x_m y_k^* T_0 \delta_{m,k}
$$

Now, the only terms that survive in the sum are those where $m = k$, so we can simplify the expression:

$$\text{LHS} = \sum_{n=-\infty}^{\infty} x_n y_n^*$$

This is the right-hand side (RHS) of the given equality, and thus we have shown that:

$$\frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} x(t) y^*(t)\, dt = \sum_{n=-\infty}^{\infty} x_n y_n^*$$

This completes the proof for part a.

Now if we take the special case where $x(t) = y(t)$, the expression becomes:

$$\frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} x(t) y^*(t)\, dt = \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} |x(t)|^2\, dt$$

Rewriting this expression in terms of the Fourier series coefficients x_n:

$$\begin{aligned}
\frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} |x(t)|^2\, dt &= \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} \left(\sum_{n=-\infty}^{\infty} x_n e^{j \frac{2\pi n}{T_0} t} \right) \left(\sum_{k=-\infty}^{\infty} x_n^* e^{-j \frac{2\pi n}{T_0} t} \right) dt \\
&= \frac{1}{T_0} \sum_{n=-\infty}^{\infty} x_n x_n^* \int_{\alpha}^{\alpha+T_0} e^{j \frac{2\pi (n-n)}{T_0} t}\, dt \\
&= \frac{1}{T_0} \sum_{n=-\infty}^{\infty} |x_n|^2 . T_0 = \boxed{\sum_{n=-\infty}^{\infty} |x_n|^2}
\end{aligned}$$

Therefore, in the case where $x(t) = y(t)$, we have:

$$\frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} x(t) y^*(t)\, dt = \sum_{n=-\infty}^{\infty} |x_n|^2$$

This is Rayleigh's relation, expressing the total power of a signal in terms of its Fourier coefficients.

. . . .

(b) A finite power periodic signal will have the following property:

$$P_x = \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} |x(t)|^2 \, dt < \infty$$

This represents the power averaged over a period (T) of the signal, ensuring that it is finite. For a periodic signal $x(t)$ with finite power, Parseval's theorem states:

$$\frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} |x(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |x_n|^2$$

Now, since P is finite, the sum $\sum_{n=-\infty}^{\infty} |X_n|^2$ must converge. The convergence of the sum $\sum_{n=-\infty}^{\infty} |X_n|^2$ means that the terms $|X_n|^2$ must approach zero as $n \to \infty$. This is a necessary condition for the sum to converge while remaining finite.

Therefore, it can be concluded that the individual Fourier coefficients X_n tend to zero as $n \to \infty$.

Let's prove it by taking an example:
Let $x(t) = \sin(t)$ which is periodic and has a finite avg power of $\frac{1}{2}$ watts (I have calculated it in notes). Now :

$$c_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt.$$

$$x(t) = \sin(t) = \frac{e^{jt} - e^{-jt}}{2j}$$

$$\Rightarrow c_n = \frac{1}{T} \int_0^T \left(\frac{e^{jt} - e^{-jt}}{2j} \right) e^{-jn\omega_0 t} dt$$

$$\Rightarrow c_n = \frac{1}{2jT} \int_0^T e^{jt(1-n\omega_0)} - e^{-jt(1+n\omega_0)} dt.$$

Now, as $n \longrightarrow \infty$

$$\Rightarrow \lim_{n\to\infty} C_n = \frac{1}{2jT} \int_0^T e^{-\infty} - e^{-\infty} dt$$

$$\Rightarrow \lim_{n\to\infty} c_n = \frac{1}{2jT} \int_0^T 0 dt = 0$$

$$\Rightarrow \lim_{n\to\infty} c_n = 0$$

Hence proved.

. . . .

(c) Proving the Identity:
Let's consider the function $f(x) = |x|, x \in [-\pi, \pi]$

$$f(-x) = |-x| = f(x)$$

$\Rightarrow f(x)$ is continous and even function on $[-\pi, \pi]$

$$\text{Therefore} \Rightarrow b_n = 0$$

Now, Fourier Series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\text{As } b_n = 0 \Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi} \int_0^{\pi} xdx \quad \{ \text{ as } f(x) \text{ is even } \}$$

$$\Rightarrow a_0 = \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi} \Rightarrow \frac{2}{\pi} \left(\frac{\pi^2}{2} \right) = \pi$$

$$\boxed{\Rightarrow a_0 = \pi}$$

Now

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)dx \Rightarrow \frac{2}{\pi} \int_0^{\pi} x \cos(nx)dx$$

$$= \frac{2}{\pi} \left[x \frac{\sin(nx)}{n} - \int \frac{\sin(nx)}{n} dx \right]$$

$$= \frac{2}{\pi} \left[\frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\cos(\pi n)}{n^2} - \frac{1}{n^2} \right] = \frac{2}{\pi} \left(\frac{\cos(n\pi) - 1}{n^2} \right)$$

Now when $n = \text{even} \Rightarrow \cos(n\pi) = 1 \Rightarrow a_n = 0$
when $n = \text{odd} \Rightarrow \cos(n\pi) = -1 \Rightarrow a_n = -4/n^2\pi$

$$\text{Therefore} \quad a_n = \begin{cases} -\dfrac{4}{n^2\pi}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

Therefore the fourier series is:

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$$

Now according to parseval's Identity: -

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2dx = \frac{a_0^2}{4} + \sum_{n=1}^{\infty}\left(\frac{a_n^2+b_n^2}{2}\right)$$

we have calculated that $a_0 = \pi, b_n = 0$ and.

$$a_n = 0(\text{ for } n = \text{ even }), a_n = \frac{-4}{n^2\pi}\text{ (for } n = \text{ odd }\Big)$$

Now,

$$\begin{aligned}
&\text{Now, } \frac{1}{2\pi}\int_{-\pi}^{\pi}|x|^2dx = \frac{\pi^2}{4} + \sum_{n=1}^{\infty}\frac{16}{2}\frac{1}{(2n-1)^4\pi^2}\\
&= \frac{1}{\pi}\int_0^{\pi}x^2dx = \frac{\pi^2}{4} + 8\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4\pi^2}\\
&\frac{1}{\pi}\frac{\pi^3}{3} - \frac{\pi^2}{4} = 8\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4\pi^2}\\
&\frac{\pi^2}{12} = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} \Rightarrow \frac{\pi^4}{96} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}
\end{aligned}$$

Therefore on expanding we get :

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots + \frac{1}{(2n+1)^4}$$

This completes the proof.

Question 4 :

Determine the Fourier transform of each of the following signals :
(a) $\operatorname{sinc}^3 t$
(b) t sinc(t)
(c) $te^{-\alpha t}\cos(\beta t)$

Solution :

(b)Given, to find F.T of $f(t) = t\sin c(t) \Rightarrow t \cdot \frac{\sin(t)}{t} = \sin(t)$

$$\begin{aligned}
\mathcal{F}\{t\,\operatorname{sinc}(t)\} &= \int_{-\infty}^{\infty} \sin(t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} \left(\frac{e^{jt} - e^{-jt}}{2j}\right) e^{-j\omega t}dt \\
&= \frac{1}{2j}\left[\int_{-\infty}^{\infty} e^{jt}e^{-j\omega t} - \int_{-\infty}^{\infty} e^{-jt}e^{-j\omega t}dt\right] \\
&= \frac{1}{2j}\left[\mathcal{F}\{e^{jt}\} - \mathcal{F}\{e^{-jt}\}\right]
\end{aligned}$$

We know that the Fourier Transform of the exponential function is:

$$\mathcal{F}\{e^{jt}\} = 2\pi\delta(\omega - 1) \quad \text{and} \quad \mathcal{F}\{e^{-jt}\} = 2\pi\delta(\omega + 1)$$

$$\begin{aligned}
\text{Therefore} \quad \mathcal{F}\{t\,\operatorname{sinc}(t)\} &= \frac{\pi}{j}[\delta(\omega - 1) - \delta(\omega + 1)] \\
&\Rightarrow -j\pi[\delta(\omega - 1) - \delta(\omega + 1)]
\end{aligned}$$

$$\Rightarrow \quad t\,\operatorname{sinc}(t) \overset{\mathcal{F}}{\longleftrightarrow} -j\pi[\delta(\omega - 1) - \delta(\omega + 1)]$$

Question 8 :

Consider a passband signal of the form

$$u_p(t) = a(t)\cos(200\pi t)$$

where $a(t) = \text{sinc}(2t)$ and the unit of time is in microseconds.
(a) What is the frequency band occupied by $u_p(t)$?
(b) The signal $u_p(t)\cos(199\pi t)$ is passed through a lowpass filter to obtain an output $b(t)$. Give an explicit expression for $b(t)$, and sketch $B(f)$ (if $B(f)$ is complex-valued, sketch its real and imaginary parts separately).
(c) The signal $u_p(t)\sin(199\pi)t$ is passed through a lowpass filter to obtain an output $c(t)$. Give an explicit expression for $c(t)$, and sketch $C(f)$ (if $C(f)$ is complex-valued, sketch its real and imaginary parts separately).
(d) Can you reconstruct $a(t)$ from simple real-valued operations performed on $b(t)$ and $c(t)$? If so, sketch a block diagram for the operations required. If not, say why not.

Solution :

(a) Given, a passband signal

$$u_p(t) = a(t)\cos(200\pi t)$$
$$\Rightarrow u_p(t) = \text{sinc}(2t)\cos(200\pi t)$$

$$FT\{u_p(t)\} = FT\{\text{sinc}(2t)\} * FT\{\cos(200\pi t)\}$$

From the duality property:

$$x(t) \xleftrightarrow{FT} x(\omega)$$
$$x(t) \xleftrightarrow{FT} 2\pi x(-\omega)$$
$$x(at) \xleftrightarrow{FT} \frac{2\pi}{a} x\left(\frac{-\omega}{|a|}\right)$$

Now $I_{[-T,T]}(t) \xleftrightarrow{FT} 2T\,\text{sinc}(\omega T)$

$$\Rightarrow 2T\,\text{sinc}(tT) \xleftrightarrow{FT} 2\pi I_{[-T,T]}(\omega)$$

putting $T = 2$,

$$\Rightarrow 4\,\text{sinc}(2t) \xleftrightarrow{FT} 2\pi I_{[-2,2]}(\omega)$$
$$\boxed{\Rightarrow \text{sinc}(2t) \xleftrightarrow{FT} \frac{\pi}{2} I_{[-2,2]}(\omega)}$$

Now finding $FT(\cos(200\pi t))$:

$$F(\cos(200\pi t)) = \int_{-\infty}^{\infty} \left(\frac{e^{200j\pi} + e^{-200j\pi t}}{2} \right) e^{-j\omega t} dt$$
$$= \frac{1}{2} \left[\int_{-\infty}^{\infty} e^{200j\pi t} e^{-j\omega t} dt + \int_{-\infty}^{\infty} e^{-200j\pi t} e^{j\omega t} dt \right]$$

We know that the Fourier transform of an exponential function from duality is:

$$F\{e^{jt}\} = 2\pi\delta(\omega - 1) \text{ and } F\{e^{-jt}\} = 2\pi\delta(\omega + 1)$$
$$\Rightarrow F \cdot T(\cos(200\pi t)) = \frac{2\pi}{2}[\delta(\omega - 200\pi) + \delta(\omega + 200\pi)]$$
$$= \pi[\delta(\omega - 200\pi) + \delta(\omega + 200\pi)]$$

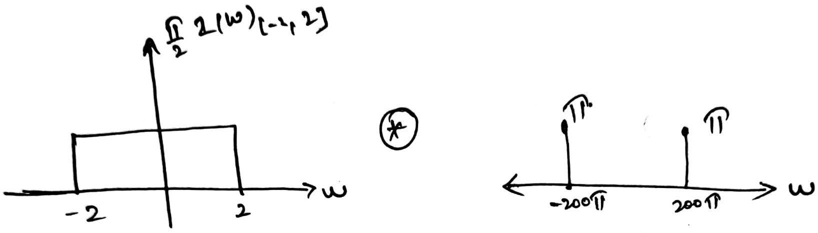
In terms of f we can it as :

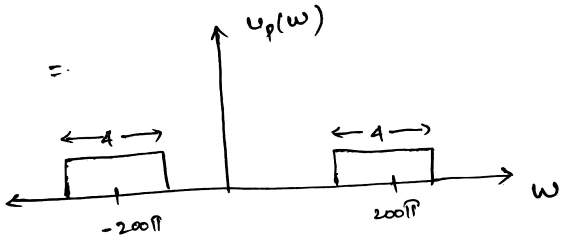
$$= \frac{1}{2}[\delta(f - 100) + \delta(f + 100)]$$

$$\boxed{\Rightarrow \cos(200\pi t) \xleftrightarrow{FT} \pi[\delta(\omega - 200\pi) + \delta(\omega + 200\pi)]}$$

Now,

Therefore $\quad F \cdot T\{u_p(t)\} = \dfrac{\pi}{2} I_{[-2,2]}(\omega) * \pi[\delta(\omega - 200\pi) + \delta(\omega + 200\pi)]$





Hence,

$$\begin{cases} \frac{\pi^2}{2}, & -200\pi - 2 \leq \omega \leq -200\pi + 2 \\ \frac{\pi^2}{2}, & 200\pi - 2 \leq \omega \leq 200\pi + 2 \\ 0, & \text{otherwise} \end{cases}$$

Therefore bandwidth is 4 rad or $\frac{2}{\pi}$Hz.

. . . .

(b)