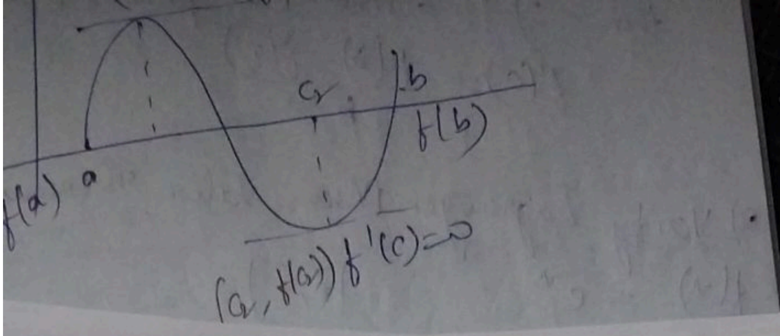


Mathematics – I Assignment

1Q) Find the Taylor's series for $f(x) = e^x$ about $x = -1$.

Sol: Taylor's theorem about $x = a$

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots \infty.$$

Here $a = -1$

$f(x) = e^x$	$f(a) = \frac{1}{e}$
$f(a) = f(-1) = e^{-1} = \frac{1}{e}$	$f'(-1) = \frac{1}{e}$
$f'(x) = e^x$	$f''(-1) = \frac{1}{e}$
$f''(x) = e^x$	$f'''(-1) = \frac{1}{e}$
$f'''(x) = e^x$	

$$\therefore f(x) = f(-1) + (x+1) f'(-1) + \frac{(x+1)^2}{2!} f''(-1) + \frac{(x+1)^3}{3!} f'''(-1) + \cdots \infty.$$

$$f(x) = \frac{1}{e} + \frac{(x+1)}{1!}\left(\frac{1}{e}\right) + \frac{(x+1)^2}{2!}\left(\frac{1}{e}\right) + \frac{(x+1)^3}{3!}\left(\frac{1}{e}\right) + \cdots \infty.$$

$$\therefore f(x) = e^x = \frac{1}{e}\left[1 + (x+1) + \frac{(x+1)^2}{2!} + \frac{(x+1)^3}{3!} + \frac{(x+1)^4}{4!} + \cdots \infty\right]$$

2Q) Discuss the Rolle Theorem for $f(x) = \tan x$, $0 \le x \le 2\pi$

Sol: Theorem – If $f : [a,b] \to R \ni$

① f is continuous in $[a,b]$

② f is differentiable in (a,b).

③ $f(a) = f(b)$

Then $\exists$ atleast one point $c \in (a,b) \ni f'(c) = 0$.

Here $f(x) = \tan x$ where $x \in [0, 2\pi]$

$\because f\left(\frac{\pi}{2}\right) = \tan\frac{\pi}{2} = \infty =$ undefined

and $\because \frac{\pi}{2}$ lies b/w $[0, 2\pi]$

$\therefore$ The function $f(x) = \tan x$ in $[0, 2\pi]$ is discontinuous

$\therefore$ It doesn't satisfies rolls theorem

3Q) State and proove Langranges Mean Value Theorem.

Sol: Statement – Let $f(x)$ be the function such that

(i) f is continuous in $[a,b]$

(ii) f is differentiable in (a,b).

then $\exists$ at least one value $c \in (a,b) \ni$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Proof –

Let $g(x) = f(x) + Ax$ in $[a,b]$ where 'A' is constant to be determined and satisfie all conditions of Rolls Theorem.

$$\therefore g(a) = g(b)$$

$$f(a) + Aa = f(b) + Ab.$$

$$A = \frac{f(b)-f(a)}{-(b-a)}. \quad \text{--- (1)}$$

Substitute 'A' in $g(x) = f(x) + Ax$

then $g(x)$ satisfies all conditions of Rolls theorem. Hence $c \in (a,b) \ni g'(c) = 0$.

$$f'(c) + A = 0.$$

$$f'(c) = -A \quad [\because \text{substituting (1).}]$$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Hence Proved.

4Q) Find the eqn of circle of curvature of the curve $y = e^x$ at $(0,1)$.

Sol.

$$y = e^x$$

$$\frac{dy}{dx} = y_1 = e^x \Rightarrow y_1(0,1) = e^0 = 1$$

$$\frac{d^2y}{dx^2} = y_2 = e^x \Rightarrow y_2(0,1) = e^0 = 1$$

$$\rho = \text{Radius} = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+e^{2x})^{3/2}}{e^x}$$

$$= \frac{(1+1)^{3/2}}{1} = 2^{3/2} = 2^1 \cdot 2^{1/2} = 2\sqrt{2}$$

$$\therefore \rho = 2\sqrt{2}$$

$$X = x - \frac{y_1}{y_2}(1+y_1^2)$$

$$= 0 - \frac{1}{1}(1+1)$$

$$\boxed{X = -2}$$

$$Y = y + \frac{1}{y_2}[1+y_1^2]$$

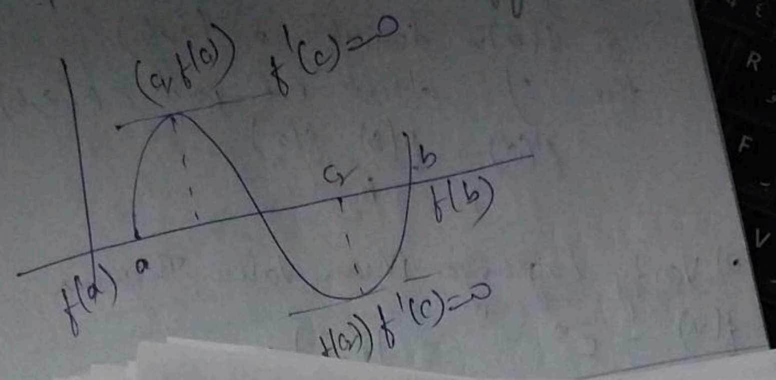
$$Y = 1 + \frac{1}{1}[1+1]$$

$$\boxed{Y = 3}$$

$\therefore$ Equation of circle of curvature is

$$(x-X)^2 + (y-Y)^2 = \rho^2$$

$$(x+2)^2 + (y-3)^2 = 8.$$



5Q) Verify Cauchy's mean value theorem for $f(x) = e^{-x}$ and $g(x) = e^{x}$ in $[a, b]$.

Sol.

(1) Since, e^{x} and e^{-x} are exponential terms. ∴ They are continuous everywhere especially in $[a, b]$

(2) ∴ They are also derivable every where especially in (a, b).

∴ It satisfies Cauchy's Theorem

∴ ∃ exists atleast one 'c' $\in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

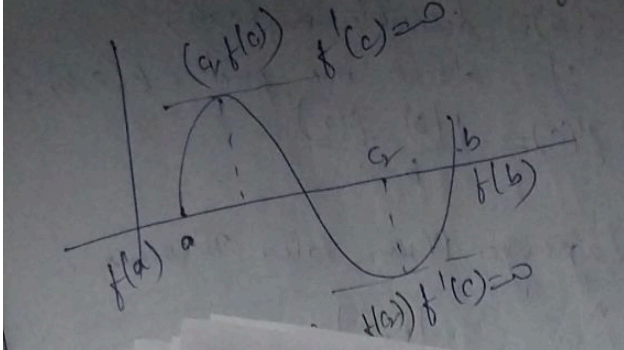
$$\frac{-e^{-c}}{e^{c}} = \frac{e^{-b} - e^{-a}}{e^{b} - e^{a}}$$

$$+\frac{1}{e^{2c}} = \frac{\frac{\cancel{e^{a} - e^{b}}}{e^{a+b}}}{\cancel{e^{b} - e^{a}}}$$

$$\frac{1}{e^{2c}} = \frac{1}{e^{a+b}}$$

$$\therefore 2c = a + b$$

$$\therefore c = a+b/2$$



6Q) Obtain the fourth degree Taylors polynomial to $f(x) = e^{2x}$ about $x=0$.

Sol:

$f(x) = e^{2x}$ $\quad$ $x=a$, $a=0$. $\quad$ $f(0)=1$

$f'(x) = 2e^{2x}$ $\quad$ $f'(0)=2$

$f''(x) = 4e^{2x}$ $\quad$ $f''(0)=4$

$f'''(x) = 8e^{2x}$ $\quad$ $f'''(0)=8$

$f''''(x) = 16e^{2x}$ $\quad$ $f''''(0)=16$.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f''''(0) + \dots \infty.$$

Here the 4th degree polynomial is

$$\frac{x^4}{4!} \times f''''(0) = \frac{x^4}{\cancel{4}\times 3\times \cancel{2}\times 1} \times \cancel{16}\,\cancel{4}\,2$$

$$= \frac{2x^4}{3}.$$

$\therefore$ 4th degree polynomial is

$$\frac{2x^4}{3}.$$

7Q) Obtain Taylors eqn at the point $x=0$. to the function $f(x)=e^x$.

(OR)

Obtain Taylors polynomial approx. of degree 'n' to function $f(x)=e^x$ ~~at~~ about point ~~$f(x)=e^x$~~ $x=0$.

Sol:

$f(x)=e^x$ $\quad x=a$ $\quad a=0$.

$f'(x)=e^x$

$f''(x)=e^x$

$\vdots$

$f^n(x)=e^x$

$f(0)=1$

$f'(0)=1$

$f''(0)=1$

$\vdots$

$f^n(x)=e^0=1$.

$$f(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}\cdots\frac{x^n}{n!}$$

$$\therefore e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\cdots\frac{x^n}{n!}+\cdots\infty$$

$\therefore$ Taylors Polynomial approx. of degree 'n' is $\frac{x^n}{n!}$.

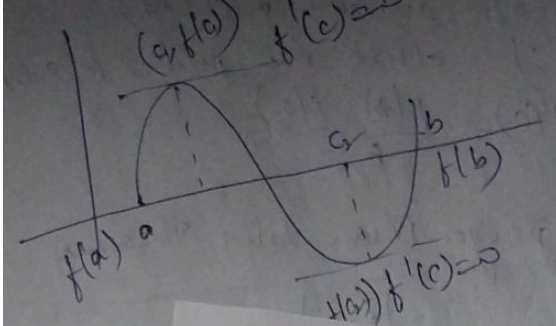
8Q) Using Langranges Mean value theorem, show that $1+x<e^x<1+xe^x$

Sol:

$1+x<e^x<1+xe^x$

$x<e^x-1<xe^x$

$x<e^x-e^0<xe^x$



dividing by x

$$1 < \frac{e^x - e^0}{x} < e^x$$

$$e^0 < \frac{e^x - e^0}{x} < e^x$$

$\therefore$ Let $f(y) = e^y$ [$\because$ e^y is exponential it is continuous and differentiable]

① $f(y)$ is continuous in $[0, x]$

② $f(y)$ is differentiable in $(0, x)$

$\therefore$ It satisfies LMVT $\therefore c \in (0, x)$

$$\therefore f'(c) = \frac{f(x) - f(0)}{x - 0}$$

$$\cancel{e^y} \; e^c = \frac{e^x - e^0}{x - 0}$$

$$\therefore \boxed{e^c = \frac{e^x - 1}{x}} \text{ — ①}$$

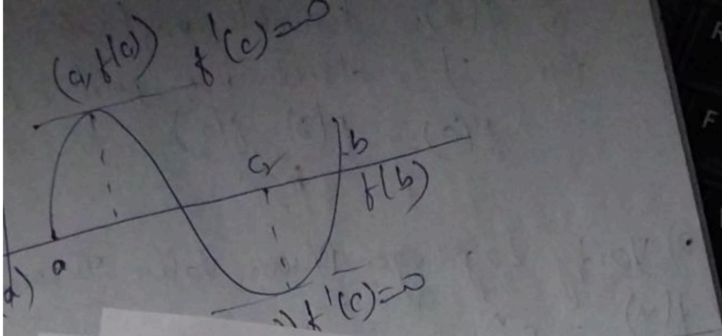
we know that $0 < c < x$ (through LMVT)

$$\therefore e^0 < e^c < e^x$$

$$1 < \frac{e^x - 1}{x} < e^x$$

$$\boxed{1 + x < e^x < xe^x + 1}$$

(Hence Proved)



9Q) Verify Rolle Theorem to the function
$f(x) = (x+2)^3 (x-3)^4$ in $(-2,3)$.

Sol. $\because$ $f(x)$ is is a polynomial it is continuous and differentiable every where

① $f(x)$ is continuous in $[-2,3]$

② $f(x)$ is differentiable in $(-2,3)$

③ $f(-2) = 0$ $\quad \therefore f(-2) = f(3)$
$f(3) = 0$

$\therefore$ It satisfies Rolle Theorem

$\therefore$ $\exists$ atleast one 'c' $\in (-2,3) \ni f'(c) = 0$.

$$f'(c) = 3(c+2)^2 (c-3)^4 + 4(c-3)^3 (c+2)^3 = 0.$$

$$(c+2)^2 (c-3)^3 \left[3(c-3) + 4(c+2)\right] = 0.$$

$$3c - 9 + 4c + 8 = 0.$$

$$7c - 1 = 0.$$

$$\therefore \boxed{c = 1/7}$$

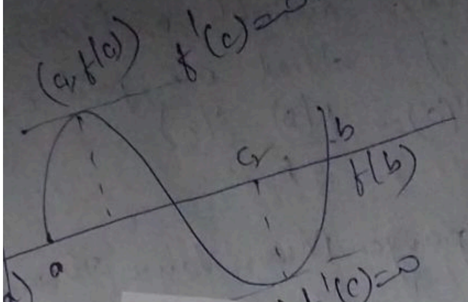
10Q) Find Taylor Series expansion of the function
$f(x) = \sin x$ about the pt $x = \pi/4$.

Sol: $f(x) = \sin x$ | $f(\pi/4) = 1/\sqrt{2}$

$f'(x) = \cos x$ | $f'(\pi/4) = 1/\sqrt{2}$

$f''(x) = -\sin x$ | $f''(\pi/4) = -1/\sqrt{2}$

Sol ① $\log x$,



$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots \infty$$

$$\sin x = \frac{1}{\sqrt{2}} + \left(x - \pi/4\right)\frac{1}{\sqrt{2}} - \frac{(x-\pi/4)^2}{2!}\frac{1}{\sqrt{2}} - \frac{(x-\pi/4)^3}{3!}\frac{1}{\sqrt{2}} + \cdots \infty$$

$\Rightarrow$ Therefore expansion is

$$\sin x = \frac{1}{\sqrt{2}}\left[1 + (x-\pi/4) - \frac{(x-\pi/4)^2}{2!} - \frac{(x-\pi/4)^3}{3!} + \frac{(x-\pi/4)^4}{4!} + \cdots \infty\right]$$

11Q) Show that evolute of cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ is another cycloid.

Sol:

$x = a\theta - a\sin\theta$ | $y = a - a\cos\theta$

$\frac{dx}{d\theta} = a - a\cos\theta$ | $\frac{dy}{d\theta} = a\sin\theta$

$$y_1 = \frac{dy/d\theta}{dx/d\theta} = \frac{\cancel{a}\sin\theta}{\cancel{a}(1-\cos\theta)} = \frac{\cancel{2}\sin\theta/2\cos\theta/2}{\cancel{2}\sin^2\theta/2} = \cot\theta/2$$

$$\boxed{y_1 = \cot\theta/2}$$

$$y_2 = -\operatorname{cosec}^2\theta/2\left(\frac{1}{2}\right)\frac{d\theta}{dx} = \frac{-\operatorname{cosec}^2\theta/2}{2\left(\frac{dx}{d\theta}\right)}$$

$$= \frac{-\operatorname{cosec}^2\theta/2}{2a(1-\cos\theta)} = \frac{-\operatorname{cosec}^2\theta/2}{2a(2\sin^2\theta/2)} = -\frac{\operatorname{cosec}^4\theta/2}{4a}$$

$$\boxed{y_2 = \frac{-\operatorname{Cosec}^4 \theta/2}{4a}}$$

$$X = x - \frac{y_1}{y_2}\left[1 + y_1^2\right]$$

$$X = x - \frac{\cot \theta/2}{\left(\frac{-\operatorname{Cosec}^4\theta/2}{4a}\right)} \cancel{\left[1 + \cot^2\theta/2\right]} \quad \left[\because \left[\operatorname{Cosec}^2\theta - \cot^2\theta = 1\right]\right]$$

$$= x + 4a \frac{\cos\theta/2}{\sin\theta/2} \sin^2\theta/2$$

$$X = x + 2a\sin\theta$$

$$X = \cancel{a - a\cos\theta}\ a\theta - a\sin\theta + 2a\sin\theta$$

$$\cancel{X = a(1 - \cos\theta + 2\sin\theta)} \quad \boxed{X = a(\theta + \sin\theta)}$$

$$Y = y + \frac{1}{y_2}\left[1 + y_1^2\right]$$

$$= y + \frac{1}{\frac{-\operatorname{Cosec}^4\theta/2}{4a}} \cancel{\left[1 + \cot^2\theta/2\right]}$$

$$= y - 4a\sin^2\theta/2 = y - (2a)(2\sin^2\theta/2)$$

$$= a - a\cos\theta - 2a(1 - \cos\theta)$$

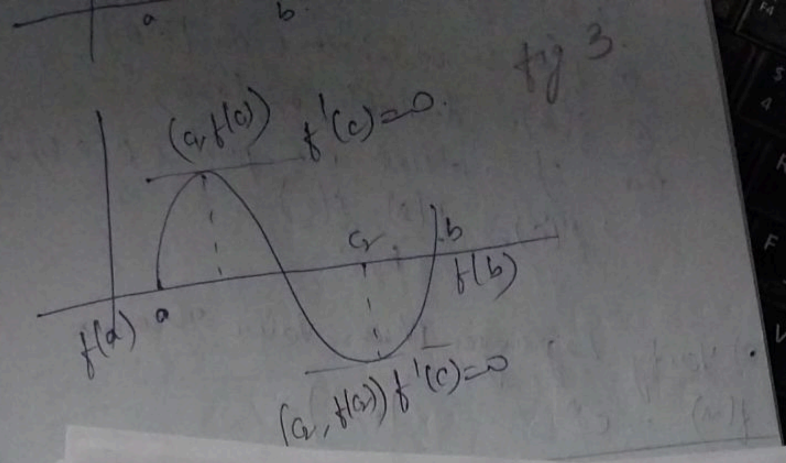
$$= a - a\cos\theta - 2a + 2a\cos\theta$$

$$= a - a\cos\theta$$

$$= a\cos\theta - a$$

$$\boxed{Y = a(\cos\theta - 1)}$$

$\therefore$ Evolute of cycloid is a cycloid.



12Q) Using LMVT show that $|\cos b - \cos a| \le |b-a|$.

Sol: Let $f(x) = \cos x$ in (a, b)

$\because$ $f(x)$ is a trignometric function it is continuous and differentiable

(1) $f(x)$ is continuous in $[a, b]$

(2) $f(x)$ is differentiable in (a, b)

$\therefore$ It satisfies LMVT.

$$\therefore \quad f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$-\sin c = \frac{\cos b - \cos a}{b-a}$$

Applying Mod

$$|\sin x| = \left|\frac{\cos b - \cos a}{b-a}\right|$$

we know that $|\sin x| \le 1$

$$\therefore \quad \left|\frac{\cos b - \cos a}{b-a}\right| \le 1$$

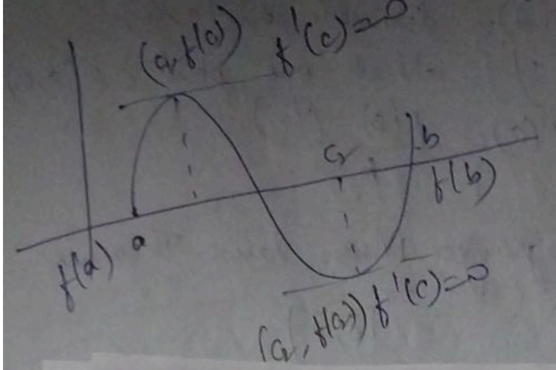
$$\therefore \quad |\cos b - \cos a| \le |b-a|$$

(Hence Prooved)

13Q) Find 'c' of Cauchys mean theorem for $f(x) = \log x$ and $g(x) = 1/x$ in the interval $[1, e]$.

Sol: (1) $\log x$, $1/x$ are continuous in $[1, e]$

(2) $\log x$, $1/x$ are differentiable in $(1, e)$



$\therefore$ It satisfies Cauchys Theorem

$$\therefore \quad \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\boxed{\begin{aligned} f'(x) &= 1/x \\ g'(x) &= \frac{-1}{x^2} \end{aligned}}$$

$$\boxed{\begin{aligned} b &= e \\ a &= 1 \end{aligned}}$$

$$\frac{\frac{1}{\cancel{x}c}}{\frac{-1}{c^2}} = \frac{\log e - \log 1}{\frac{1}{e} - \frac{1}{1}}$$

$$-c = \left(\frac{1-0}{1-e}\right)e$$

$$-c = \frac{e}{1-e} \Rightarrow \therefore \boxed{c = \frac{e}{e-1}}$$

14Q) Find radius of curvature at any point on the cardoid $r = a(1-\cos\theta)$

Sol.

$$r = a(1-\cos\theta)$$

$$r_1 = \frac{dr}{d\theta} = +a\sin\theta$$

$$r_2 = \frac{d^2r}{d\theta^2} = +a\cos\theta$$

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

$$\therefore \quad \rho = \frac{(a^2+a^2\cos^2\theta - 2a^2\cos\theta + a^2\sin^2\theta)^{3/2}}{a^2+a^2\cos^2\theta - 2a^2\cos\theta + 2a^2\sin^2\theta \cancel{+a^2\cos\theta} - (+a\cos\theta)(a-a\cos\theta)}$$

$$\rho = \frac{(2a^2 - 2a^2\cos\theta)^{3/2}}{a^2 + a^2\cos^2\theta - 2a^2\cos\theta + 2a^2\sin^2\theta - a^2\cos\theta + a^2\cos^2\theta}$$

$$= \frac{(2a^2)^{3/2}(1-\cos\theta)^{3/2}}{3a^2 - 3a^2\cos\theta}$$

$$= \frac{2\sqrt{2}\,a^{\cancel{3}}(1-\cos\theta)^{3/2}}{3\cancel{a^2}(1-\cos\theta)}$$

$$= \frac{2\sqrt{2}\,a\sqrt{1-\cos\theta}}{3} = \frac{2\sqrt{2}\,a}{3}\sqrt{2\sin^2\theta/2}$$

$$= \frac{4a\sin\theta/2}{3}$$

$$\therefore \boxed{\rho = \frac{4a}{3}\sin\theta/2}$$

15Q) Find the envelope of the family of lines $y = mx + \sqrt{1+m^2}$ where 'm' is the parameter

Sol:

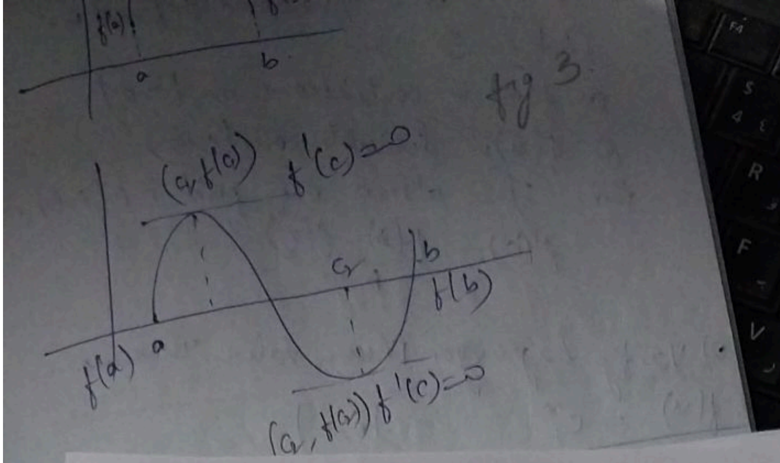
$$y = mx + \sqrt{1+m^2}$$

$$y - mx = \sqrt{1+m^2}$$

SOBS

$$y^2 + m^2x^2 - 2mxy = 1 + m^2$$

$$m^2(1-x^2) + (2xy)m + (1-y^2) = 0$$



It is of the form $A\alpha^2 + B\alpha + C = 0$

To find envelope of a quadratic in terms of 'α' as parameter:

$$B^2 - 4AC = 0.$$

$B = 2xy$; $A = (1-x^2)$; $C = 1-y^2$

$$4x^2y^2 - 4(1-x^2)(1-y^2) = 0.$$

$$4x^2y^2 - 4\left[1 - y^2 - x^2 + x^2y^2\right] = 0$$

$$4x^2y^2 - 4 + 4y^2 + 4x^2 - \cancel{4x^2y^2} = 0$$

$$\cancel{4x^2y^2} - 4 + 4y^2 + 4x^2$$

$\boxed{x^2+y^2=1}$ → is the envelope

18Q) Discuss the applicability of LMVT for $f(x) = 2x+3$ in $[2,4]$.

Sol: $f(x)$ is a polynomial

(i) $f(x)$ is continuous in $[2,4]$

(ii) $f(x)$ is differentiable in $(2,4)$

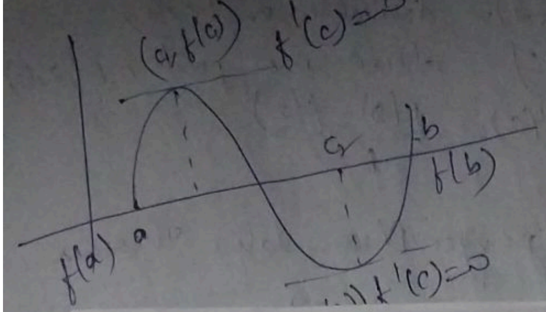
∴ Satisfies LMVT

∴ $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ $\quad [\because f'(x) = 2]$

$$2 = \frac{11-7}{2} = 2$$

∴ $2 = 2$

∴ LMVT satisfies always.



17Q) Find radius of curvature of the curve $x = 2\cos t$; $y = 2\sin t$ at any t.

Sol:

$$\frac{dx}{dt} = -2\sin t$$

$$\frac{dy}{dt} = 2\cos t$$

$$\boxed{y_1 = -\cot t}$$

$$y_2 = -(-\operatorname{cosec}^2 t)(1)\frac{dt}{dx}$$

$$= \frac{\operatorname{cosec}^2 t}{\left(\frac{dx}{dt}\right)} = \frac{\operatorname{cosec}^2 t}{\cancel{(2\cos t)}\,(-2\sin t)}$$

$$\boxed{y_2 = \frac{-\operatorname{cosec}^3 t}{2}}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+\cot^2 t)^{3/2}}{-\frac{\operatorname{cosec}^3 t}{2}} = \frac{(\operatorname{cosec}^2 t)^{3/2}}{-\frac{\operatorname{cosec}^3 t}{2}}$$

$$\rho = \frac{-2\cancel{\operatorname{cosec}^3 t}}{\cancel{\operatorname{cosec}^3 t}} = -2.$$

$\therefore \rho = -2$ but we know radius can't be negative $\therefore$ $\boxed{\rho = 2}$

18Q) State and Prove Rolles Theorem.

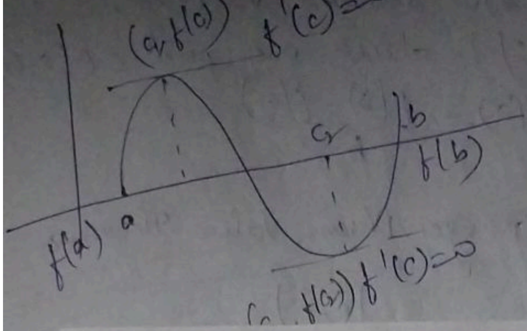
Sol: Statement – Let $f(x)$ be a function that satisfy

(i) It is continuous in $[a,b]$.

(ii) It is differentiable in (a,b).

(iii) $f(a) = f(b)$

then $\exists$ atleast one point c in (a,b) $\ni$ $f'(c) = 0$.



19Q) Find Taylor Series expansion of $f(x) = x^3 + 3x^2 + 2x + 3$ about $x = 1$.

Sol.

$\boxed{a = 1}$

$f(x) = x^3 + 3x^2 + 2x + 3$	$f(1) = 9$
$f'(x) = 3x^2 + 6x + 2$	$f'(1) = 11$
$f''(x) = 6x + 6$	$f''(1) = 12$
$f'''(x) = 6.$	$f'''(1) = 6$
$f''''(x) = 0.$	$f''''(1) = 0$

$$f(x) = 9 + \frac{(x-1)}{1!}(4) + \frac{(x-1)^2}{2!}(12) + \frac{(x-1)^3}{3!}(6)$$

$\therefore$ The above statement is taylor series expansion of $f(x)$ about $x = 1$.

20Q) Find the envelope of family of lines $y = ax + a^2$

Sol.

$y = ax + a^2$

$a^2 + a(x) + (-y) = 0$

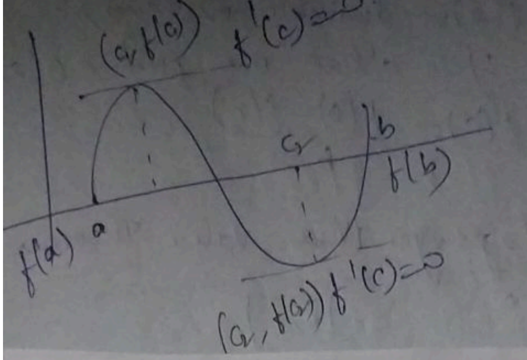
$A\alpha^2 + B\alpha + C = 0$, $[\because$ Here $\alpha = a]$

$\Rightarrow$ Envelope is $B^2 - 4AC$

$x^2 - 4(1)(-y) = 0.$

$x^2 + 4y = 0.$

$\therefore$ Envelope is $\boxed{x^2 + 4y = 0}$



21Q) Find the center of circle of curvature of the curve $y = e^x$ at $(0,1)$.

Sol:

$y_1 = e^x = 1;\ y_2 = e^x = 1.$

$X = x - \frac{y_1}{y_2}[1+y_1^2]$

$X = 0 - \left(\frac{1}{1}\right)[1+1] = -2$

$\boxed{X = -2}$

$Y = y + \frac{1}{y_2}[1+y_1^2]$

$Y = 1 + \frac{1}{1}[1+1] = 3$

$\boxed{Y = 3}$ $\therefore$ Centre is $(-2,3)$.

22Q) Find the radius of curvature for the curve $y^2 = x^3+8$ at $(-2,0)$.

Sol:

$\cancel{2}y^2 = x^3+8$

$\cancel{2y^2 y}$ $2yy_1 = 3x^2$

$\frac{dy}{dx} = y_1 = \frac{3x^2}{2y} = \frac{3(-2)^2}{2(0)} = \infty.$

$\boxed{\frac{dx}{dy} = 0}$

$\cancel{y_2 = \frac{3}{2}\left(\frac{2xy - y_1x^2}{y^2}\right) = \frac{6xy}{2y^2} - \frac{3}{2}\left(\frac{3x^2}{2y}\right)\frac{x^2}{y^2}}$

$$\boxed{\frac{dx}{dy} = \frac{2y}{3x^2} = 0}$$

$$\frac{d^2x}{dy^2} = \left(\frac{2}{3}\right)\left[\frac{x^2(1) - y(2x)\overset{(0)}{\left(\frac{dx}{dy}\right)}}{x^4}\right]$$

$$= \frac{2}{3}\left[\frac{x^2 - 0}{x^4}\right] = \frac{2}{3}\left(\frac{\cancel{x^2}}{x^{\cancel{4}2}}\right)$$

$$\frac{d^2x}{dy^2} = \frac{2}{3x^2} = \frac{\cancel{2}}{3(\cancel{4})_2} = \frac{1}{6}$$

$$\therefore \rho = \frac{\left(1 + \left(\frac{dx}{dy}\right)^2\right)^{3/2}}{\cancel{\frac{d^2x}{dy^2}}\ \frac{d^2x}{dy^2}} = \frac{(1+0)^{3/2}}{\frac{1}{6}} = 6$$

$$\therefore \boxed{\rho = 6}$$

23Q) Find the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\cancel{\frac{2x}{a^2} + \frac{2y\,y_1}{b^2} = 0}$$

$$\cancel{\boxed{y_1 = -\frac{xb^2}{a^2y}}}$$

Let $x = a\cos\theta$, $y = b\sin\theta$

$$\frac{dx}{d\theta} = -a\sin\theta \; ; \; \frac{dy}{d\theta} = b\cos\theta$$

$$y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$$

$$y_2 = \frac{d^2y}{dx^2} = -\frac{b}{a}(-\csc^2\theta)\frac{d\theta}{dx} = -\frac{b}{a^2}\csc^3\theta$$

$$X = x - \frac{y_1}{y_2}[1+y_1^2] \; ; \; Y = y + \frac{1}{y_2}[1+y_1^2]$$

$$X = \left(\frac{a^2-b^2}{a}\right)\cos^3\theta \; ; \; Y = -\frac{(a^2-b^2)}{b}\sin^3\theta$$

$(aX)^{2/3} = (a^2-b^2)^{2/3}\cos^2\theta$ and

$(bY)^{2/3} = (a^2-b^2)^{2/3}\sin^2\theta$

$$(aX)^{2/3} + (bY)^{2/3} = (a^2-b^2)^{2/3}.$$

Locus of (X, Y) is

$(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}$ is a

required evolute

24Q) Find the envelope of the family of curves $\frac{a^2}{x}\cos\theta - \frac{b^2}{y}\sin\theta = c$ where θ being the parameter.

Sol: given $\frac{a^2}{x}\cos\theta - \frac{b^2}{y}\sin\theta = c$ —— (1)

d.w.r.to θ

$-\frac{a^2}{x}\sin\theta - \frac{b^2}{y}\cos\theta = 0$. —— (2)

Squaring and ~~substracting~~ Adding (1) and (2)

$(1)^2 + (2)^2$

$$\frac{a^4}{x^2}\cos^2\theta + \frac{b^4}{y^2}\sin^2\theta - \cancel{\frac{2a^2b^2}{xy}\cos\theta\sin\theta} = c^2$$

$$\frac{a^4}{x^2}\sin^2\theta + \frac{b^4}{y^2}\cos^2\theta + \cancel{\frac{2a^2b^2}{xy}\cos\theta\sin\theta} = 0$$

(+) + + + +

$$\frac{a^4}{x^2} + \frac{b^4}{y^2} + 0 = c^2$$

$\therefore$ Envelope is

$$\boxed{\frac{a^4}{x^2} + \frac{b^4}{y^2} = c^2}$$

25Q) Find the envelope of the family of curves $y = mx + m^2$, where 'm' is parameter.

Sol: $m^2 + mx = y$

$m^2 + mx - y = 0 \rightarrow$ to find envelope of a quadratic

$B^2 - 4AC = 0$

$x^2 - 4(1)(-y) = 0$

$\boxed{x^2 + 4y = 0}$ is the envelope

26Q) State and Prove Cauchey's Mean Value theorem

Sol: Statement - If $f : [a,b] \rightarrow R$ and $g : [a,b] \rightarrow R$ $\ni$

(i) f, g are continuous in $[a,b]$

(ii) f, g are differentiable in (a,b).

(iii) $g'(x) \neq 0 \; \forall \; x \in (a,b)$

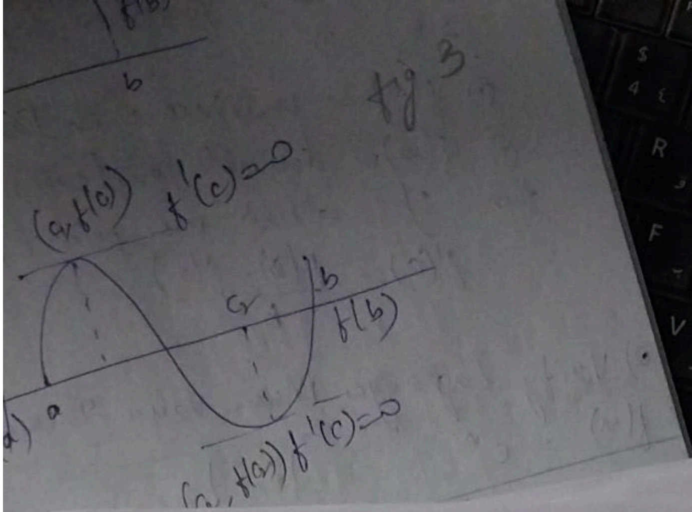
then there exists atleast one value $c \in (a,b) \ni$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof - Define $h : [a,b] \rightarrow R \ni h(x) = f(x) + Ag(x)$ where $A \in R$ and satisfies all conditions of Rolls theorem. $h(a) = h(b) \Rightarrow f(a) + Ag(a) = f(b) + Ag(b)$

$$A = -\left[\frac{f(b) - f(a)}{g(b) - g(a)}\right] \longrightarrow (1)$$

where $g(b) - g(a) \neq 0$



By Rolls Theorem,

we have

$h'(c) = 0 \Rightarrow f'(c) + Ag'(c) = 0$

$A = \dfrac{-f'(c)}{g'(c)}$ — (2)

from (1) and (2) we get

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

(Hence Proved)

27Q) Find evolute of the curve $xy = 1$.

Sol. $xy = 1 \Rightarrow y = 1/x \Rightarrow \boxed{y_1 = -\frac{1}{x^2}}$

$\Rightarrow \boxed{y_2 = \frac{2}{x^3}}$

$\cancel{xy_1 + y = 0}$ $\cancel{y_2}$

$\boxed{\cancel{y_1 = -y/x}}$

$$X = x - \frac{y_1}{y_2}\left[1 + y_1^2\right]$$

$$X = x - \frac{\left(\frac{-1}{x^2}\right)}{\left(\frac{2}{x^3}\right)}\left[1 + \frac{1}{x^4}\right]$$

$$= x + \frac{x}{2}\,\frac{\left[x^4 + 1\right]}{x^4}$$

$$= x + \frac{x}{2} + \frac{1}{2x^3}$$

$$X = \frac{3x}{2} + \frac{1}{2x^3}$$

$$\boxed{X = \frac{3x}{2} + \frac{1}{2x^3}}$$

$$Y = y + \frac{1}{y_2}\left[1 + y_1^2\right]$$

$$Y = y + \frac{1}{\left(\frac{2}{x^3}\right)}\left[1 + \frac{1}{x^4}\right]$$

$$= y + \frac{x^3}{2}\frac{\left[x^4+1\right]}{x^4}$$

$$= y + \frac{x^3}{2} + \frac{1}{2x} \qquad \left[\because \frac{1}{x} = y\right]$$

$$\boxed{Y = \frac{3y}{2} + \frac{1}{2y^3}}$$

$$X+Y = \frac{3(x+y)}{2} + \frac{x^3+y^3}{2x^3y^3}$$

$$= \frac{1}{2}\left(3x + 3y + \frac{x^3+y^3}{x^3y^3}\right) \qquad \left[\because xy = 1\right]$$

$$= \frac{1}{2}\left[3x^2(xy) + 3y(xy) + x^3+y^3\right]$$

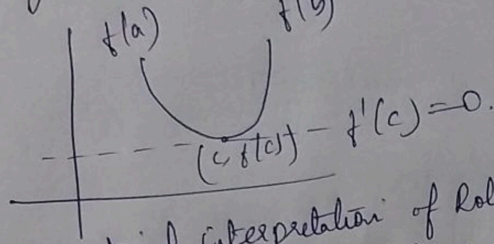
$$X+Y = \frac{1}{2}(x+y)^3 \Rightarrow (X+Y)^{2/3} = \frac{(x+y)^2}{2^{2/3}}$$

$$X-Y = -\frac{(x-y)^3}{2} \Rightarrow (X-Y)^{2/3} = \frac{(x-y)^2}{2^{2/3}}$$

$$\boxed{(X+Y)^{2/3} - (X-Y)^{2/3} = 4^{2/3}} \rightarrow \text{is an evolute}$$

28Q) Write geometrical representation of Roll's theorem.

S.f.



The geometrical interpretation of Roll's theorem is that if $f(x)$ is a continuous function in $[a,b]$ and differentiable function in (a,b) then there is a point $c \in (a,b)$ where the tangent to curve $f(x)$ is horizontal or we can say it is parallel to the x-axis.

29Q) Find the equation of circle of curvature of the curve $y = \sin x$ at $\left(\frac{\pi}{2}, 1\right)$.

S.f.

$y = \sin x$

$y_1 = \cos x = 0$

$y_2 = -\sin x = -1$

$\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2} = \dfrac{(1+0)^{3/2}}{-1} = -1$

$\therefore$ radius $= \rho = 1$

$X = x - \dfrac{y_1}{y_2}\left[1+y_1^2\right]$; $Y = y + \dfrac{1}{y_2}\left[1+y_1^2\right]$

$Y = 1 + \dfrac{1}{(-1)}[1] = 0$

$\boxed{X = \pi/2}$ $\boxed{Y = 0}$

$\therefore$ Equation of circle of curvature is
$$(x-\pi/2)^2 + y^2 = 1$$

30Q) Verify Roll's mean value theorem for $f(x) = \frac{\sin x}{e^x}$ on $[0,\pi]$.

Sol: $\sin x$ is a trigo func $\therefore$ Continuous everywhere

e^x is an exponential function $\therefore$ Continuous everywhere

① $\therefore f(x)$ is continuous in $[0,\pi]$

② $f(x)$ is differentiable in $(0,\pi)$

③ $f(0) = 0$ $\quad \therefore f(0) = f(\pi)$
$f(\pi) = 0$

$\therefore$ It satisfies Roll's Theorem

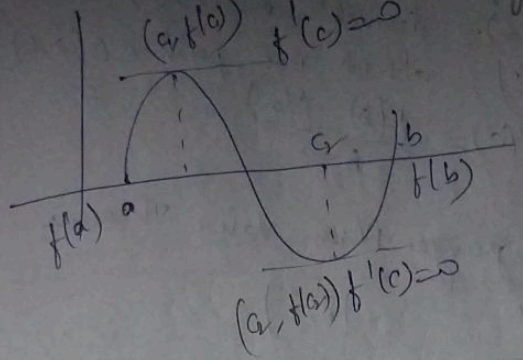
$\therefore \exists$ atleast one point 'c' $\ni f'(c) = 0$.

$$f(x) = \frac{\sin x}{e^x} \Rightarrow f'(x) = \frac{e^x \cos x - e^x \sin x}{e^{2x}} \cancel{=0}.$$

~~cos x = sin~~

$\therefore f'(c) = 0$

$$e^c \cos c - e^c \sin c = 0$$
$$\cos c = \sin c$$
$$\therefore \boxed{c = \pi/4.}$$



31Q) Find the envelope of the family of straight line $x\cos\alpha + y\sin\alpha = a$ where 'α' is the parameter.

Sol:

$$x\cos\alpha + y\sin\alpha = a \quad \text{--- (1)}$$

d.w.r.to α

$$-x\sin\alpha + y\cos\alpha = 0 \quad \text{--- (2)}$$

$(1)^2 + (2)^2$

$$x^2\cos^2\alpha + y^2\sin^2\alpha + \cancel{2xy\cos\alpha\sin\alpha} = a^2$$

$$x^2\sin^2\alpha + y^2\cos^2\alpha - \cancel{2xy\cos\alpha\sin\alpha} = 0$$

$$x^2 + y^2 = a^2$$

$\therefore \boxed{x^2+y^2=a^2} \to$ is an envelope.

32Q) Find evolute of the curve $x = a\cos^3 t$; $y = a\sin^3 t$

Sol:

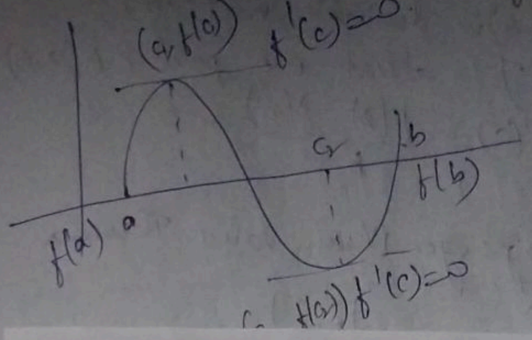
$$\frac{dx}{dt} = 3a\cos^2 t(-\sin t)$$

$$\frac{dy}{dt} = 3a\sin^2 t\cos t$$

$$\frac{dy}{dx} = y_1 = -\frac{\cancel{3a}\sin^{\cancel{2}} t\,\cancel{\cos t}}{\cancel{3a}\cos^{\cancel{2}} t\,\cancel{\sin t}} = -\tan t$$

$$y_2 = -\sec^2 t\,\frac{dt}{dx} = \frac{+\sec^2 t}{+3a\cos^2 t\sin t}$$

$$\boxed{y_2 = \frac{\sec^4 t}{3a\sin t}}$$



$$X = x - \frac{y_1}{y_2}\left[1+y_1^2\right]$$

$$= x - \frac{(-\tan t)}{\frac{\sec^4 t}{3a\sin t}}\left[\cancel{1+\tan^2 t}\right]$$

$$= x + 3a\sin^2 t\cos t$$

$$\boxed{X = a\cos^3 t + 3a\sin^2 t\cos t}$$

$$Y = y + \frac{1}{y_2}\left[1+y_1^2\right]$$

$$\boxed{Y = a\sin^3 t + 3a\cos^2 t\sin t}$$

$$(X+Y)^{2/3} + (X-Y)^{2/3} = a^{2/3}\left[(\cos t+\sin t)^2 + (\cos t-\sin t)^2\right]$$

$$= 2a^{2/3}$$

$\therefore$ the locus of (X, Y) is $(x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3}$

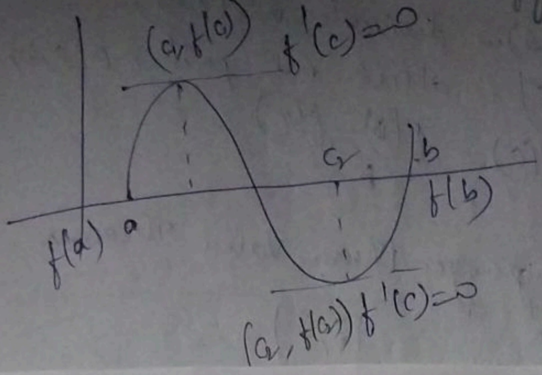
33Q) Find the circle of curvature of the curve $xy=9$ at the point $(1,9)$.

Sol.

$$xy = 9$$

$$y = \frac{9}{x}$$

$$\boxed{\begin{aligned} y_1 &= \frac{-9}{x^2} = -9 \\ y_2 &= \frac{18}{x^3} = 18 \end{aligned}}$$



$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+81)^{3/2}}{18} = \frac{(82)^{3/2}}{18}$$

$2 \,|\, 82$
$\;\;\;\, 41$

$$\boxed{\rho = \frac{(82)^{3/2}}{18}}$$

$$X = x - \frac{y_1}{y_2}\left[1+y_1^2\right]$$

$$= 1 - \frac{(-\cancel{9})}{\cancel{18}\,2}\left[1+81\right]$$

$$= 1 + \frac{1}{\cancel{2}}\left[\cancel{82}\right]41$$

$$\boxed{X = 42}$$

$$Y = y + \frac{1}{y_2}\left[1+y_1^2\right]$$

$$= 9 + \frac{1}{\cancel{18}\,9}\left[\cancel{82}\right]41$$

$$\boxed{Y = \frac{81+41}{9} = \frac{122}{9}}$$

$\therefore$ circle of curvature is

$$(x-42)^2 + \left(y - \frac{122}{9}\right)^2 = \frac{(82)^3}{(18)^2}$$