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ORIGINAL ARTICLE

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Evaluating the quality level of a product with multiple quality characterisitcs

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Abstract Most of the studies of quality systems or product assessment deal with a single quality characteristic to determine the quality loss. From the customer's point of view, however, products are often judged by more than one quality characteristic. For this reason, a multivariate quality loss function is required as an extension to the Taguchi loss function to capture the overall losses caused by bad quality when multiple quality characteristics are present. A numerical example is illustrated showing that using inappropriate univariate loss functions will give an underestimated quality loss or even ignored the loss incurred because of the poor quality.

Keywords Loss function · Multivariate loss function · Product quality · Quality characteristics · Taguchi Method

1 Introduction

In manufacturing processes, conformance to standards is a very critical aspect. It relates to how closely the final product matches its design specification and ultimately to customer satisfaction. Most companies began to evaluate the costs of quality in order to achieve conformance. Traditionally, quality costs can be classified into four categories:

- Appraisal costs: monitoring or inspection costs
- Prevention costs: maintenance of equipment, training operators and improving procedures
- Internal costs: rejected, reworked, scrapped, materials
- External costs: warranties, loss of market share, and sales of return

The main objective of management is to achieve the minimum in the total quality costs as shown in Fig. 1. The curve of the total quality costs is typically flat around the optimum and permits a wide range in the selection of percentage of product conforming without much sacrifice in cost. It is shown that the minimum point (total quality costs) occurs when the costs of prevention and appraisal equal the costs of internal and external failure. In practice, this point is difficult to identify. The main problem is to measure the external costs that are associated with warranties, loss of sale or customer dissatisfaction. To reduce the cost of poor quality or quality losses is the primary focus of quality improvement.

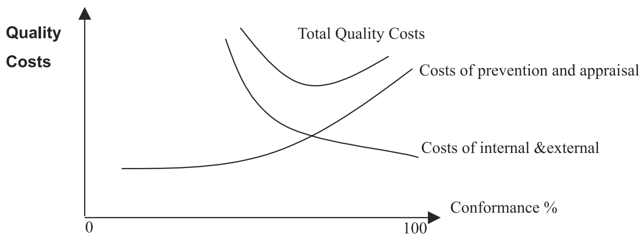

Fig. 1. Costs variation with conformance %

As a result, one of the most critical issues encountered in the area of quality engineering is the selection of a proper quality loss function to relate a key quality characteristic of a product to its quality performance. The loss function can be used to evaluate loss in term of monetary units. The loss function enables managers to quantify unobservable costs. Therefore, it aids manufacturers to evaluate more rigorously quality improvement projects. However, using inappropriate loss functions will lead to inaccurate results that give either underestimate or overestimate of the expected quality costs [1, 2].

2 Costs of quality – univariate case

Most classical studies [3, 4] have considered a step loss function as a quality measurement system. This approach assumes that costs do not depend on the actual value of the quality characteristic as long as it is within the specification. This function, for

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a single quality characteristic, can be written as:

$$L_s(y) = \begin{cases} 0 & LSL \le y \le USL \\ C_r & \text{otherwise} \end{cases} , \tag{1}$$

where C_r is a cost due to non-conformance, which is, herein, called rejection cost. Although this step loss function has been widely used and because of the mathematical convenience, there is a general consensus that the Taguchi loss function may be a better approximation for the measurement of customer dissatisfaction with product quality based on quality characteristic(s). Dr. Taguchi stated that *quality loss is defined as the loss a product costs society from the time the product is released from shipment* [5]. His approach can also evaluate the quality cost for items within the specification limits.

The Taguchi loss function presented in Fig. 2 illustrates the quality loss incurred whenever the quality characteristic differs from the target value. Variations from the target value, which are still within the specification limits, do not cause the company to incur any internal costs (step loss function approach). However, these deviations may cause customer dissatisfaction, in which case the company may incur external quality costs that include repair, warranties or loss of market share (refer to the shaded area). The Taguchi loss function for a single quality characteristic can be written as [6]:

$$L_q = \begin{cases} k(y-\mu)^2 & LSL \le y \le USL \\ C_r & \text{otherwise} \end{cases} . \tag{2}$$

Define k as C_r/Δ^2, where Δ is the tolerance of the specification limits and C_r is the rejection cost. When the output value is equal the target value, the loss is zero. However, when the value of $\mu \le LSL$ or $\mu \ge USL$, the loss is equal to the rejection cost. Therefore, the loss increases quadradically away from the intended mean. To facilitate the analysis, it is assumed that the upper and the lower specification limits are equidistant from the mean, i.e., $\Delta = (USL - \mu) = (\mu - LSL)$.

The expected quality cost (internal and external costs) per unit produced can be obtained by adding the integral of the Taguchi loss function to the values of the specification limits and the integral of the step loss function outside the values of the

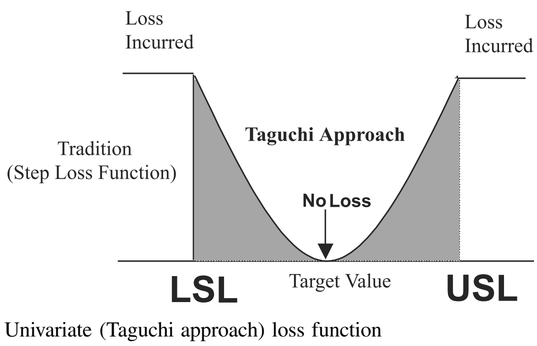

Fig. 2. Univariate (Taguchi approach) loss function

specification limits, i.e.:

$$E[L(y)] = k\int_{LSL}^{USL} (y-\mu)^2 f(y)\,\mathrm{d}y + \int_{USL}^{\infty} C_r f(y)\,\mathrm{d}y + \int_{-\infty}^{LSL} C_r f(y)\,\mathrm{d}y , \tag{3}$$

where μ is the target value and $f(y)$ is the probability density function of y, which is assumed to be normally distributed.

This function estimates the expected quality loss for a given stable manufacturing process. The Taguchi loss function enables managers to quantify unobservable costs. Therefore, it aids the manufacturer to evaluate more rigorously quality improvement projects. The estimated cost savings from using the Taguchi method is a quantifiable recognition caused by the quality improvement proposal. It indicates the reduction in the cost of external failure incurred by the loss warranties, excess inventory, customer dissatisfaction, and eventually loss of market share.

Furthermore, the information obtained from Eq. 3 can also be used for the economic design of $\bar{x}$ control charts [7], in which the expected quality cost is used in order to find the most economic control limits and parameters for the control charts, hence minimising the overall costs. The economic design of EPC also uses the information of off-target (Taguchi's approach), in order to seek an optimal control policy that will minimise the overall costs [8]. Ganeshan et al. [9] applied the Taguchi loss function into the economic production quantity model, in order to find the optimal lot size. As a review of the literature [1–10] shows, most researchers have focused their work on a single quality characteristic, whereas products are often judged by more than one quality characteristic. Therefore, to avoid mistakes in decision-making, a multivariate quality loss function is required as an extension of the step loss and Taguchi loss function to capture the overall losses (costs) caused by bad quality, when multiple quality characteristics are present.

3 Costs of quality – multivariate case

In this section, a mathematical model will be presented that allows the calculation of the loss function when the key quality characteristics of the process or product are more than one.

3.1 Rejection cost

A product is said to be rejected if the quality characteristic is outside the specification limits. However, the costs of nonconforming, rejection cost, may differ in different scenarios. For example, in the case where a product characteristic is less than the lower specification limit, the product will be scrapped with a cost C_{rL}. On the other hand, if the product quality characteristic is greater than the upper specification then the product will be reworked with a cost C_{rU}. Hence, the rejection cost can be written as:

$$L_R(y) = \begin{cases} C_{rL} & y < LSL \\ 0 & LSL \le y \le USL \\ C_{rU} & y > USL \end{cases} . \quad (4)$$

Consider n quality characteristics, $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)^T$. The rejection cost can be defined as $C_r(y_1, y_2, \ldots, y_n) = 0$ when $LSL_i \le y_i \le USL_i$ for all i $(i = 1, 2, \ldots, n)$. This is based on the concept of a single step loss function, that is, there is no quality loss when the product performance falls within the specification region. Considering the two quality characteristics shown in Fig. 3, the rejection region I and II represent the ones where a product characteristic fails to meet one of the specification limits and both of the specification limits, respectively. It is assumed that the rejection cost for a product characteristic falling within region I is lower than that of region II. Therefore, a special scenario is worth considering. This scenario is when equal numbers of product quality characteristics falling out of the specifications incur equal rejection costs. It should be noted that the number of rejection regions is equal to the number of quality characteristics. The overall multivariate rejection cost can be shown as:

$$L_R(y_1, y_2, \ldots, y_n) = \begin{cases} 0 & LSL_i \le y_i \le USL_i \\ C_r^I & (y_1, y_2, \ldots, y_n) \in R_1 \\ C_r^{II} & (y_1, y_2, \ldots, y_n) \in R_2 \\ \vdots & \\ \vdots & \\ C_r^n & (y_1, y_2, \ldots, y_n) \in R_n \end{cases} , \quad (5)$$

where R_j is the set of $(y_1, y_2, \ldots, y_n)$ where j quality characteristics fail to fall within the specifications. For example, if two quality characteristics are considered, $R_1 = \{(y_1, y_2) : (y_1 < LSL_1$ and $LSL_2 \le y_2 \le USL_2)$ or $(y_1 > USL_1$ and $LSL_2 \le y_2 \le USL_2)$ or $(LSL_1 \le y_1 \le USL_1$ and $y_2 < LSL_2)$ or $(LSL_1 \le y_1 \le USL_1$ and $y_2 > USL_2)\}$, and $R_2 = \{(y_1, y_2) : (y_1 < LSL_1$ and $y_2 < USL_2)$ or $(y_1 < USL_1$ and $y_2 > USL_2)$ or $(y_1 > USL_1$ and $y_2 < LSL_2)$ or $(y_1 > USL_1$ and $y_2 > USL_2)\}$.

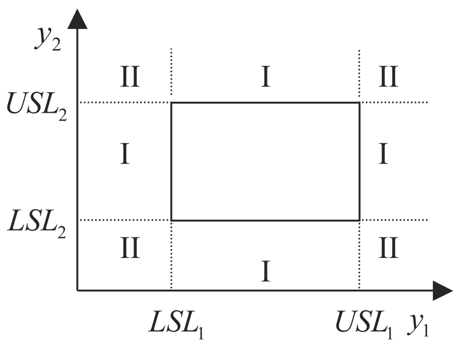

Fig. 3. Rejection regions for two quality characteristics

The expected rejection cost associated with region I can be shown as: $C_r^I \cdot P(R^I)$, where C_r^I and $P(R^I)$ represent the unit rejection cost associated with region I and the probability that the product falls within region I, respectively. Similarly, the expected rejection costs associated with region II can be written as $C_r^{II} \cdot P(R^{II})$. As a result, the expected total rejection cost becomes:

$$\sum_{j=1}^{n} C_r^j P(R^j) , \quad (6)$$

where j represents the type of rejection region for $j = I, II, \ldots, n$.

3.2 Taguchi loss function

Let n quality characteristics and their associated customer-identified targets be donated by $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)^T$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)^T$, respectively. Assume that $L_Q(y)$ is a twice-differentiable function in the neighbourhood of μ gives

$$L_Q(\boldsymbol{y}) = L(\boldsymbol{\mu}) + L'(\boldsymbol{\mu})^T(\boldsymbol{y} - \boldsymbol{\mu}) + 1/2(\boldsymbol{y} - \boldsymbol{\mu})^T L''(\boldsymbol{\mu})^T(\boldsymbol{y} - \boldsymbol{\mu}) + \ldots . \quad (7)$$

Since $L_Q(y)$ has a minimal value at $y = \mu$ and thus $L'(\mu) = 0$. Moreover, we are only interested in the major variable term in Eq. 7. Thus, ignoring the constant terms and the 3rd or higher-order terms, we have $L_Q(y) = 1/2(y - \mu)^T H(\mu)(y - \mu)$, where $H(\mu)$ is a "Hessian matrix" for $L_Q(y)$.

Using the summation notation, we have (see also [11, 12])

$$L_Q(y) = \begin{cases} \sum_{i=1}^{n} \sum_{j=1}^{i} k_{ij}(y_i - \mu_i)(y_j - \mu_j) & LSL_i \le y_i \le USL_i \\ C_r(y_1, y_2, \ldots, y_n) & \text{otherwise} \end{cases} , \quad (8)$$

where

$$k_{ii} = \frac{1}{2}\left(\frac{\partial^2 L(y_1, y_2, \ldots, y_n)}{\partial y_i^2}\right)\Bigg|_{y=\mu} \quad (\forall i)$$

$$k_{ij} = \frac{1}{2}\left(\frac{\partial^2 L(y_1, y_2, \ldots, y_n)}{\partial y_i \partial y_j}\right)\Bigg|_{y=\mu} \quad (\forall i \ne j)$$

k_{ij}'s can be determined by using the regression method [13] or solving a system of simultaneous linear equations.

When the specification region of interest is implemented, the expected loss where n quality characteristics are present can be expressed as

$$E\left[L_Q(y_1, y_2, \ldots, y_n)\right] = \iint \ldots \int L_Q(y_1, y_2, \ldots, y_n) \times f(y_1, y_2, \ldots, y_n)\,\mathrm{d}y_1\,\mathrm{d}y_2 \ldots \mathrm{d}y_n . \quad (9)$$

3.3 The expected total cost

Considering where two quality characteristics are present, the expected quality cost can be shown as:

$$
\begin{aligned}
E\left[L(y_1, y_2)\right] = C_r^I \Bigg[& \int_{-\infty}^{LSL_2} \int_{LSL_1}^{USL_1} f(y_1, y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2 \\
& + \int_{LSL_2}^{USL_2} \int_{-\infty}^{LSL_1} f(y_1, y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2 \\
& + \int_{USL_2}^{\infty} \int_{LSL_1}^{USL_1} f(y_1, y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2 \\
& + \int_{LSL_2}^{USL_2} \int_{USL_1}^{\infty} f(y_1, y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2 \Bigg] \\
& + C_r^{II} \Bigg[\int_{-\infty}^{LSL_2} \int_{-\infty}^{LSL_1} f(y_1, y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2 \\
& + \int_{USL_2}^{\infty} \int_{-\infty}^{LSL_1} f(y_1, y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2 \\
& + \int_{USL_2}^{\infty} \int_{USL_1}^{\infty} f(y_1, y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2 \\
& + \int_{-\infty}^{LSL_2} \int_{USL_1}^{\infty} f(y_1, y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2 \Bigg] \\
& + \int_{LSL_2}^{USL_2} \int_{LSL_1}^{USL_1} L_Q(y_1, y_2) f(y_1, y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2 \,, \quad (10)
\end{aligned}
$$

where $L_Q = k_{11}(y_1 - \mu_1)^2 + k_{12}(y_1 - \mu_1)(y_2 - \mu_2) + k_{22} \cdot (y_2 - \mu_2)^2$ and $f(y_1, y_2)$ is a joint probability density function for y_1 and y_2, which follows a bivariate normal distribution.

4 Numerical example

In this section, a numerical example is illustrated showing that using only a single quality characteristic as a key performance indicator is inappropriate to reflect the actual quality loss incurred by bad quality when the product performance depends on more than one quality characteristic. A C program was written for performing the calculation of Eq. 10.

Consider that there are two quality characteristics of interest, y_1 and y_2, which are statistically independent, and they are assumed to follow a bivariate normal distribution with $\sigma_1 = 0.8$ and $\sigma_2 = 1.5$. The specification region limits are given by $8 \le y_1 \le 12$ and $17 \le y_2 \le 23$, respectively, in which the internal failure costs (rejection costs) of \$20 for product quality characteristic(s) fall into the region I and \$40 for that of region II. The customer identified target values are $\mu_1 = 10$ and $\mu_2 = 20$. Furthermore, it is assumed that the loss coefficients k_{11}, k_{12}, and k_{22} are 5, 2 and 2, respectively. Table 1 summarises the expected quality costs for different approaches.

Table 1. Expected quality costs for different approaches

	Univariate (\$)	Multivariate (\$)
Step loss function		0.912
y_1	0.248	
y_2	1.158	
Taguchi loss function		9.521
y_1	3.162	
y_2	4.768	
Total	9.09	10.68

If the company only uses either one of the quality characteristics as the key quality performance for measuring the quality loss then the expected quality loss, when using the step loss function Eq. 1, would be \$0.248 and \$0.912 for y_1 and y_2, respectively. Further, if Eq. 3 were used for quality loss measurement then the quality loss for y_1 would be \$3.162 and y_2 would be \$4.768. It is shown that the traditional approach (step loss function) always underestimates or even ignores the expected quality loss due to customer dissatisfaction. Using the multivariate expected loss function Eq. 10, the expected quality cost is \$10.68. The loss function and its contour are shown in Figs. 4 and 5, respectively. For this particular example, the multivariate expected quality loss of \$10.68 is greater when compared with the sum of quality losses for single quality characteristic (\$9.09 = \$3.41 + \$5.68). It is found that measuring quality characteristics independently, there is a probability of underestimating the expected quality loss, and hence reduce the costs for quality improvement projects in terms of appraisal and prevention costs. However, if the value of k_{12} is negative then the expected quality loss will be less than the sum of quality losses for single quality characteristic. In this case, the company will overestimate the expected quality loss of the product.

The main factor contributing to different results for these two approaches (univariate vs. multivariate) is that the multi-

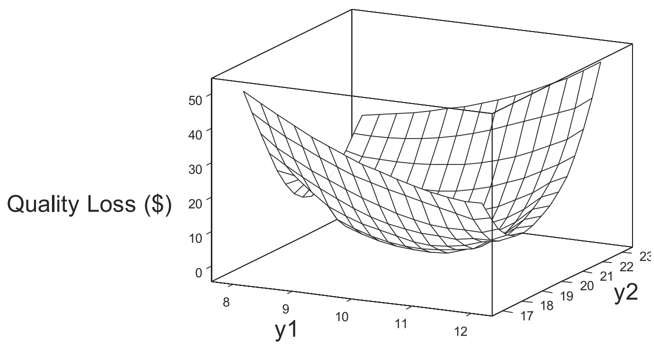

Fig. 4. Multivariate loss function

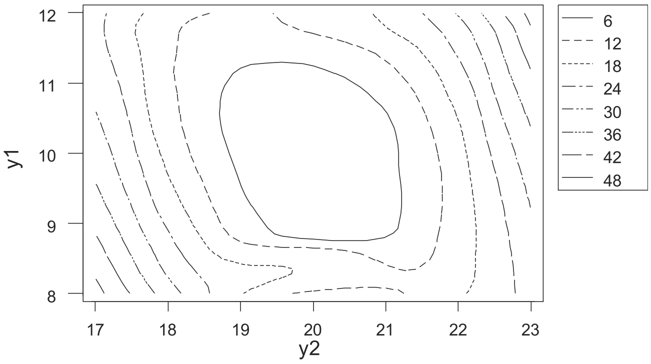

Fig. 5. Contour plot for the multivariate loss function

variate approach has considered the customers perception of the quality characteristics as interdependent, leading to a cost (k_{12}), which accounts for the customer's reactions regarding the product quality based on multi-characteristics, whereas, the univariate approach only judges quality characteristics individually.

5 Conclusion

The loss function can be used to evaluate quality loss in term of monetary units. The information obtained can be used for designing economic control charts and/or EPC control policy. However, using inappropriate loss function will lead to inaccurate results that will underestimate or overestimate the expected quality loss, hence affecting the optimal control policy of EPC, and the control limits and the parameters of the economic control charts. It is shown that the multivariate loss function is a more informative approach to calculate the quality level of a multi-characteristics product.

In this paper, a multivariate quality loss function is presented for evaluating the quality loss incurred by exceeding quality limits of multi-characteristics products or process. This loss function enables the company to quantify unobservable costs. Therefore, it aids the manufacturer in evaluating quality improvement projects more rigorously. In modern manufacturing environments, the Taguchi approach is beneficial when the costs associated with poor quality are to be minimised. However, in order to achieve a stable process as well as minimising costs, one has to combine the Taguchi approach with a quality control scheme.

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