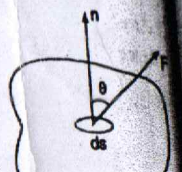
...l of $\bar{F}$ along the normal

...re n is the unit normal vector to an element ds and

$$\hat{n} = \frac{\text{grad} f}{|\text{grad} f|}$$

$$ds = \frac{dx\,dy}{(\hat{n}\cdot k)}$$



Surface integral of F over S $= \Sigma \bar{F}\cdot\hat{n} = \iint_S (\bar{F}\cdot\hat{n})\,ds$

Note. (1) Flux $= \iint_S (\bar{F}\cdot\hat{n})\,ds$ where $\bar{F}$ represents the velocity of a liquid.

If $\iint_S (\bar{F}\cdot\hat{n})\,ds = 0$, then $\bar{F}$ is said to be a *solenoidal* vector point function.

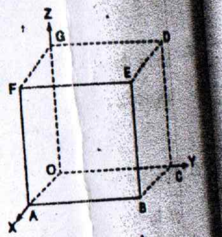
Example 94. *Show that* $\iint_S \bar{F}\cdot\hat{n}\,ds = \frac{3}{2}$, *where*

$\vec{F} = 4xzi - y^2j + yzk$ *and S is the surface of the cube bounded by the planes,*

$x = 0, x = 1, y = 0, y = 1, z = 0, z = 1.$

Solution.

S. No.	Surface	Outward Normal	ds	Eq. of surface
1	OABC	$-k$	$dx\,dy$	$z = 0$
2	DEFG	k	$dx\,dy$	$z = 1$
3	OAFG	$-j$	$dx\,dz$	$y = 0$
4	BCDE	j	$dx\,dz$	$y = 1$
5	ABEF	i	$dy\,dz$	$x = 1$
6	OCDG	$-i$	$dy\,dz$	$x = 0$



$$\iint_S \bar{F}\cdot\hat{n}\,ds = \iint_{OABC} \bar{F}\cdot\hat{n}\,ds + \iint_{DEFG} \bar{F}\cdot\hat{n}\,ds + \iint_{OAFG} \bar{F}\cdot\hat{n}\,ds + \iint_{BCDE} \bar{F}\cdot\hat{n}\,ds + \iint_{ABEF} \bar{F}\cdot\hat{n}\,ds + \iint_{OCDG} \bar{F}\cdot\hat{n}\,ds \quad ...(1)$$

$$\iint_{OABC} \bar{F}\cdot n\,ds = \iint_{OABC} (4xzi - y^2j + yzk)\cdot(-k)\,dx\,dy = \int_0^1\int_0^1 -yz\,dx\,dy = 0 \quad (\text{as } z = 0)$$

$$\iint_{DEFG} (4xzi - y^2j + yzk)\cdot k\,dx\,dy = \iint_{DEFG} yz\,dx\,xy = \int_0^1\int_0^1 y(1)\,dx\,dy$$

$$= \int_0^1 dx\left[\frac{y^2}{2}\right]_0^1 = [x]_0^1\,\frac{1}{2} = \frac{1}{2}$$

$$\iint_{OAFG} (4xzi - y^2j + yzk)\cdot(-j)\,dx\,dz = \iint_{OAFG} y^2\,dx\,dz = 0 \quad (\text{as } y = 0)$$

$$\iint_{BCDE} (4xzi - y^2j + yzk)\cdot j\,dx\,dz = \iint_{BCDE} (-y^2)\,dx\,dz$$

$$= -\int_0^1 dx\int_0^1 dz = -(x)_0^1\,(z)_0^1 = -1 \quad (\text{as } y = 1)$$

$$\iint_{ABEF} (4xzi - y^2j + yzk)\cdot i\,dy\,dz = \iint 4xz\,dy\,dz = \int_0^1\int_0^1 4(1)\,z\,dy\,dz$$

$$= 4(y)_0^1\left(\frac{z^2}{2}\right)_0^1 = 4(1)\left(\frac{1}{2}\right) = 2$$

$$\iint_{OCDG} (4xzi - y^2j + yzk)(-i)\,dy\,dz = \int_0^1\int_0^1 -4xz\,dy\,dz = 0 \quad (\text{as } x = 0)$$

On putting these values in (1), we get

$$\iint_S F\cdot\hat{n}\,ds = 0 + \frac{1}{2} + 0 - 1 + 2 + 0 = \frac{3}{2}$$

Proved

VOLUME INTEGRAL

Let $\bar{F}$ be a vector point function and volume V enclosed by a closed surface.

The volume integral $= \iiint_V \bar{F}\,dv$

Example 95. *If* $\bar{F} = 2zi - xj + yk$, *evaluate* $\iiint_V \bar{F}\,dv$ *where v is the region bounded* ...*surfaces*

$x = 0,\ y = 0,\ x = 2,\ y = 4,\ z = x^2,\ z = 2.$ (A.M.I.E.T.E., Winter 1995)

Solution. $\iiint \bar{F}\,dv = \iiint (2zi - xj + yk)\,dx\,dy\,dz$

$$= \int_0^2 dx\int_0^4 dy\int_{x^2}^2 (2zi - xj + yk)\,dz$$

$$= \int_0^2 dx\int_0^4 dy\,[z^2 i - xzj + yzk]_{x^2}^2$$

$$= \int_0^2 dx\int_0^4 dy\,[4i - 2xj + 2yk - x^4i + x^3j - x^2yk]$$

$$= \int_0^2 dx\left[4yi - 2xyj + y^2k - x^4yi + x^3yj - \frac{x^2y^2}{2}k\right]_0^4$$

$$= \int_0^2 (16i - 8xj + 16k - 4x^4i + 4x^3j - 8x^2k)\,dx$$

$$= \left[16x - 4x^2j + 16xk - \frac{4x^5}{5}i + x^4j - \frac{8x^3}{3}k\right]_0^2$$

$$= 32i - 16j + 32k - \frac{128}{5}i + 16j - \frac{64}{3}k$$

$$= \frac{32i}{5} + \frac{32k}{3} = \frac{32}{15}(3i + 5k)$$

Ans.

GREEN'S THEOREM (For a plane)

Statement. If $\phi(x, y), \psi(x, y), \frac{\partial\phi}{\partial y}$ and $\frac{\partial\psi}{\partial x}$ be continuous functions over a region R ...ded by simple closed curve C in $x - y$ plane, then

$$\oint_C (\phi\,dx + \psi\,dy) = \iint_R \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dx\,dy$$

and

$$-\oint_c F_3\, dz = \iint_S \left(\frac{\partial F_3}{\partial y} \cos\alpha - \frac{\partial F_3}{\partial x} \cos\beta \right) ds = \quad \ldots(7)$$

On adding (5), (6) and (7) we get

$$\oint_c (F_1\, dx + F_2\, dy + F_3\, dz)$$

$$= \iint_S \left(\frac{\partial F_1}{\partial z} \cos\beta - \frac{\partial F_1}{\partial y} \cos\gamma + \frac{\partial F_2}{\partial x} \cos\gamma - \frac{\partial F_2}{\partial z} \cos\alpha + \frac{\partial F_3}{\partial y} \cos\alpha - \frac{\partial F_3}{\partial x} \cos\beta \right) ds$$

Proved

Example 99. *Using Stoke's theorem or otherwise, evaluate*

$$\int_c [(2x - y)\, dx - yz^2\, dy - y^2 z\, dz]$$

where c is the circle $x^2 + y^2 = 1$, corresponding to the surface of sphere of unit radius.

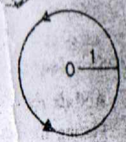
Solution. $\int_c [(2x - y)\, dx - yz^2\, dy - y^2 z\, dz]$

$$= \int_c [(2x - y)\, i - yz^2 j - y^2 z\, k] \cdot (i\, dx + j\, dy + k\, dz)$$

By Stoke's theorem $\oint \overline{F} \cdot dr = \iint_S \text{curl}\, \overline{F} \cdot \hat{\eta}\, ds \quad \ldots(1)$

$$\text{curl}\, \overline{F} = \overline{\nabla} \times \overline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= (-2yz + 2yz)\, i - (0 - 0)\, j + (0 + 1)\, k = k$$



Putting the value of curl $\overline{F}$ in (1) we get

$$= \iint k \cdot \hat{\eta}\, ds$$

$$= \iint k . \hat{\eta} \frac{dx\, dy}{\hat{\eta} . k} \qquad \left[\because ds = \frac{dx\, dy}{(\hat{\eta} \cdot k)} \right]$$

$$= \iint dx\, dy$$

$$= \pi$$

Example 100. *Apply Stoke's theorem to find the value of*

$$\int_c (y\, dx + z\, dy + x\, dz)$$

where c is the curve of intersection of $x^2 + y^2 + z^2 = a^2$ and $x + z = a$. (Nagpur, Summer 2001)

Solution. $\int_c (y\, dx + z\, dy + x\, dz)$

$$= \int_c (yi + zj + xk) . (i\, dx + j\, dy + k\, dz)$$

$$= \int (yi + zj + xk) . d\overline{r}$$

$$= \iint \text{curl}\, (yi + zj + xk) \cdot \overline{n}\, ds \quad \text{(By Stoke's Theorem)}$$

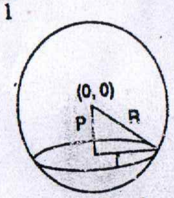
$$= \iint_S \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \times (yi + zj + xk) \cdot \hat{n}\, ds \quad \ldots(1)$$

$$= \iint_S -(i + j + k) \cdot \hat{n}\, ds$$

...e S is the circle formed by the intersection of $x^2 + y^2 + z^2 = a^2$ and $x + z = a$.

$$\hat{\eta} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right)(x + z - a)}{|\nabla\phi|} = \frac{i + k}{\sqrt{1 + 1}}$$

$$\therefore \quad \hat{n} = \frac{i}{\sqrt{2}} + \frac{k}{\sqrt{2}}$$



Putting the value of $\hat{n}$ in (1) we have

...en Integral

$$= \iint_S -(i + j + k) \cdot \left(\frac{i}{\sqrt{2}} + \frac{k}{\sqrt{2}} \right) ds$$

$$= \iint_S -\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) ds \qquad r^2 = R^2 - p^2 = a^2 - \frac{a^2}{2} = \frac{a^2}{2}$$

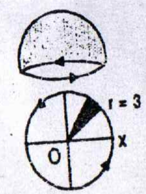
$$= \frac{-2}{\sqrt{2}} \iint_S ds = \frac{-2}{\sqrt{2}} \pi \left(\frac{a}{\sqrt{2}} \right)^2 = -\frac{\pi a^2}{\sqrt{2}}$$

Ans.

Example 101. *Use Stoke's theorem to evaluate $\int_c \vec{v} \cdot \vec{dr}$ where $\vec{v} = y^2 i + xyj + xzk$, and the bounding curve of the hemisphere $x^2 + y^2 + z^2 = 9$, $z > 0$, oriented in the positive direction.*

Solution. By Stokes theorem

$$\int_c \vec{v} \cdot \vec{dr} = \iint_S (\text{curl}\, \vec{v}) \cdot \hat{\eta}\, ds = \iint_S (\nabla \times \vec{v}) \cdot \hat{\eta}\, ds$$



$$\nabla \times \vec{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & xy & xz \end{vmatrix} = (0 - 0)\, i - (z - 0)\, j + (y - 2y)\, k$$

$$= -zj - yk$$

$$\hat{\eta} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right)(x^2 + y^2 + z^2 - 9)}{|\nabla\phi|}$$

$$= \frac{2xi + 2yj + 2zk}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{xi + yj + zk}{\sqrt{x^2 + y^2 + z^2}} = \frac{xi + yj + zk}{3}$$

$$(\nabla \times \vec{v}) \cdot \hat{\eta} = (-zj - yk) \cdot \frac{xi + yj + zk}{3} = \frac{-yz - yz}{3} = \frac{-2yz}{3}$$

$$\hat{\eta} \cdot k\, ds = dx\, dy \text{ or } \frac{xi + yj + zk}{3} . k\, ds = dx\, dy \text{ or } \frac{z}{3} ds = dx\, dy$$

$$ds = \frac{3}{z} dx\, dy$$

$$\therefore \quad \iint_S (\nabla \times \vec{v}) \cdot \hat{\eta}\, ds = \iint \left(\frac{-2yz}{3} \right) \left(\frac{3}{z} dx\, dy \right) = -\iint 2y\, dx\, dy$$

$$= -2\int\int r\sin\theta\, r\, d\theta\, dr = -2\int_0^{2\pi} \sin\theta\, d\theta \int_0^3 r^2\, dr$$

$$= -2(-\cos\theta)_0^{2\pi}\left(\frac{r^3}{3}\right)_0^3 = -2(-1+1)\,9 = 0$$

Ans.

Example 102. *Directly or by Stoke's theorem, evaluate* $\iint_s \text{curl}\, \vec{v}\cdot\hat{n}\, ds$, $\vec{v} = iy + jz + kx$, *s is the surface of the paraboloid* $z = 1 - x^2 - y^2$, $z \geq 0$ *and* $\hat{n}$ *is the unit normal to s.*

(A.M.I.E.T.E. Summer 1997)

Solution. $\overline{\nabla}\times\vec{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -i - j - k$

Obviously $\hat{\eta} = -k$.

Therefore $(\nabla\times\overline{V}).\ \hat{\eta} = (-i-j-k).-k = +1$

Hence $\iint_S (\nabla\times\overline{V}).\ \hat{\eta}\, ds = \iint_S (1)\, dx\, dy = \iint_S dx\, dy$

$$= \pi\,(1)^2 = \pi$$ **Ans.** (Area of circle $= \pi r^2$)

Example 103. *Use the Stoke's theorem to evaluate*

$$\int_C [(x+2y)\, dx + (x-z)\, dy + (y-z)\, dz]$$

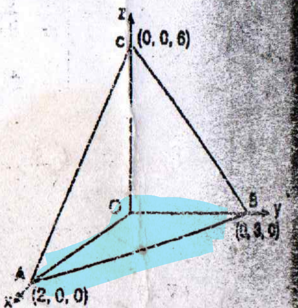
where c is the boundary of the triangle with vertices (2, 0, 0), (0, 3, 0) *and* (0, 0,6) *oriented in the anti-clockwise direction.*

(A.M.I.E.T.E. Summer 1997)

Solution. $\int_C \overline{F}.\ \overline{dr} = \int_C [(x+2y)\ dx + (x-z)\ dy + (y-z)\ dz)]$

$$= \int_C [(x+2y)\, i + (x-z)\, j + (y-z)\, k].\ [\, i\, dx + j dy + k dz]$$

$\therefore$ $\overline{F} = (x+2y)\, i + (x-z)\, j + (y-z)\, k$



$$\overline{\nabla}\times\overline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y & x-z & y-z \end{vmatrix}$$

$$= (1+1)\, i - (0-0)\, j + (1-2)\, k = 2i - k$$

S is the surface of the plane $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$,

$\hat{\eta}$ is the normal to the plane ABC.

Normal Vector $= \nabla\phi = [i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}]\ \left[\frac{x}{2} + \frac{y}{3} + \frac{z}{6} - 1\right]$

$$= \frac{i}{2} + \frac{j}{3} + \frac{k}{6} = \frac{1}{6}(3i + 2j + k)$$

$$\hat{\eta} = \frac{\frac{1}{6}(3i+2j+k)}{\frac{1}{6}\sqrt{9+4+1}} = \frac{1}{\sqrt{14}}(3i+2j+k)$$

$$(\overline{\nabla}\times\overline{F}).\ \hat{\eta} = (2i-k).\ \frac{1}{\sqrt{14}}(3i+2j+k) = \frac{1}{\sqrt{14}}(6-1) = \frac{5}{\sqrt{14}}$$

$$\int \overline{F}.\ \overline{dr} = \iint_S (\text{curl}\,\overline{F}).\ \hat{\eta}\, ds$$

$$= \iint_s \frac{5}{\sqrt{14}} ds = \frac{5}{\sqrt{14}}\iint_R \frac{dx\, dy}{k.\frac{1}{\sqrt{14}}(3i+2j+k)}$$

$$= 5\iint_R dx\, dy$$

where R is the projection of S on the $x-y$ plane i.e. triangle OAB.

$= 5$. Area of triangle $OAB = \frac{5}{2}(2\times 3) = 15$ **Ans.**

Example 104. *Apply Stoke's theorem to calculate*

$$\int_c 4y\, dx + 2z\, dy + 6y\, dz$$

where c is the curve of intersection of

$x^2 + y^2 + z^2 = 6z$ *and* $z = x + 3$.

Solution. $\int_c \overline{F}\cdot\overline{dr} = \int_c 4y\, dx + 2z\, dy + 6y\, dz$

$$= \int_c (4yi + 2zj + 6yk)\cdot(idx + jdy + kdz)$$

$\therefore$ $\overline{F} = 4yi + 2zj + 6yk$

$$\overline{\nabla}\times\overline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4y & 2z & 6y \end{vmatrix} = (6-2)\, i - (0-0)\, j + (0-4)\, k = 4i - 4k$$

S is the surface of the circle $x^2 + y^2 + z^2 = 6z$, $z = x + 3$. $\hat{\eta}$ is normal to the plane $x - z + 3 = 0$

$$\hat{\eta} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot(x-z+3)}{|\nabla\phi|} = \frac{i-k}{\sqrt{1+1}} = \frac{i-k}{\sqrt{2}}$$

$$(\nabla\times F)\cdot\hat{\eta} = (4i-4k)\cdot\frac{i-k}{\sqrt{2}} = \frac{4+4}{\sqrt{2}} = 4\sqrt{2}$$

$$\int_c \overline{F}\cdot\overline{dr} = \iint_S (\text{curl}\, F)\cdot\hat{\eta}\, ds$$

$$= \iint_S 4\sqrt{2}\,(dx\, dz) = 4\sqrt{2}\ \text{(area of circle)}$$

Centre of the sphere $x^2+y^2+(z-3)^2 = 9$, $(0, 0, 3)$ lies on the plane $z = x+3$. It means that the given circle is a great circle of sphere, where radius of the circle is equal to the radius of the sphere.

Radius of circle $= 3$, Area $= \pi(3)^2 = 9\pi$

$$\iint_S (\nabla\times F)\cdot\hat{\eta}\, ds = 4\sqrt{2}\,(9\pi) = 36\sqrt{2}\,\pi$$ **Ans.**

Example 105. *Verify Stoke's theorem for*

$$\overline{F} = (x^2+y-4)\,i + 3xyj + (2xz+z^2)\,k$$

over the surface of hemisphere $x^2+y^2+z^2 = 16$ above the xoy-plane.

Solution. $\int_c \overline{F}\cdot\overline{dr}$ where c is the boundary circle $x^2+y^2 = 16$

(bounding the hemispherical surface)

$$= \int_c [(x^2+y-4)\,i + 3xyj + (2xz+z^2)\,k]\cdot(idx+jdy)$$

$$= \int_c (x^2+y-4)\,dx + 3xy\,dy$$

Putting $x = 4\cos\theta$, $y = 4\sin\theta$, $dx = -4\sin\theta\, d\theta$, $dy = 4\cos\theta\, d\theta$

$$= \int_0^{2\pi} [(16\cos^2\theta + 4\sin\theta - 4)(-4\sin\theta\, d\theta) + (192\sin\theta\cos^2\theta\, d\theta)]$$

$$= 16\int_0^{2\pi} [-4\cos^2\theta\sin\theta - \sin^2\theta + \sin\theta + 12\sin\theta\cos^2\theta]\, d\theta$$

$$= 16\int_0^{2\pi} (\cancel{8\sin\theta\cos^2\theta}\ \sin^2\theta + \sin\theta)\, d\theta$$

$$= -16\int_0^{2\pi} \sin^2\theta\, d\theta \qquad \text{(other integrals of odd powers of sin and cosine = 0)}$$

$$= -16\times 4\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta = -64\left(\frac{1}{2}\,\frac{\pi}{2}\right) = -16\pi$$

To evaluate surface integral

$$\nabla\times\overline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix}$$

$$= (0-0)\,i - (2z-0)\,j + (3y-1)\,k = -2zj + (3y-1)\,k$$

$$\hat{\eta} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-16)}{|\nabla\phi|}$$

$$= \frac{2xi+2yj+2zk}{\sqrt{4x^2+4y^2+4z^2}} = \frac{xi+yj+zk}{\sqrt{x^2+y^2+z^2}} = \frac{xi+yj+zk}{4}$$

$$(\nabla\times\overline{F})\cdot\hat{\eta} = [-2zj + (3y-1)\,k]\cdot\frac{xi+yj+zk}{4} = \frac{-2yz+(3y-1)\,z}{4}$$

$$\hat{\eta}\cdot ds = dx\,dy \quad\text{or}\quad \frac{xi+yj+zk}{4}\cdot k\, ds = dx\,dy \ \text{or}\ \frac{z}{4}\, ds = dx\,dy$$

$$\therefore \quad ds = \frac{4}{z}\, dx\, dy$$

$$\iint (\nabla\times F)\cdot\hat{\eta}\, ds = \iint \frac{-2yz+(3y-1)\,z}{4}\left(\frac{4}{z}\, dx\, dy\right)$$

$$= \iint [-2y + (3y-1)]\, dx\, dy = \iint (y-1)\, dx\, dy$$

$$= \int_{-4}^{4} dx \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (y-1)\, dy \quad [\because x^2+y^2 = 16,\ y = \sqrt{16-x^2}\,]$$

$$= \int_{-4}^{4} dx \left[\frac{y^2}{2} - y\right]_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}$$

$$= \int_{-4}^{4} dx \left[\frac{16-x^2}{2} - \sqrt{16-x^2} - \frac{16-x^2}{2} - \sqrt{16-x^2}\right]$$

$$= -2\int_{-4}^{4} \sqrt{16-x^2}\, dx = -2\left[\frac{x}{4}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_{-4}^{4}$$

$$= -2\left[8\left(\frac{\pi}{2}\right) + 8\left(\frac{\pi}{2}\right)\right] = -16\pi$$

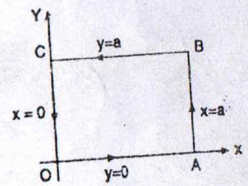
The line integral is equal to the surface integral hence Stokes theorem is verified.

Proved

Example 106. *Verify Stoke's Theorem for the function*

$$F = x^2 i - xyj$$

integrated round the square in the plane $z = 0$ and bounded by the lines $x = 0$, $y = 0$, $x = a$, $y = a$.



Solution.

$$\nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -xy & 0 \end{vmatrix}$$

$$= (0-0)\,i - (0-0)\,j + (-y-1)\,k = -yk$$

($\hat{\eta}\perp$ to xy plane *i.e.* k)

$$\iint_S (\nabla\times F)\cdot\hat{\eta}\, ds = \iint_S (-yk)\cdot k\, dx\, dy$$

$$= \int_0^a dx \int_0^a -y\,dy = \int_0^a dx \left[-\frac{y^2}{2}\right]_0^a = -\frac{a^2}{2}(x)_0^a = -\frac{a^3}{2} \qquad \text{...(1)}$$

To obtain line integral

$$\int_C \overline{F}\cdot\overline{dr} = \int_C (x^2 i - xyj)\cdot(i\,dx + j\,dy) = \int_C (x^2\,dx - xy\,dy)$$

where c is the path $OABCO$ as shown in the figure. ...(2)

Also $\int_C F\cdot dr = \int_{OABCO} F\cdot dr = \int_{OA} F\cdot dr + \int_{AB} F\cdot dr + \int_{BC} F\cdot dr + \int_{CO} F\cdot dr$

...g OA, $y = 0$, $dy = 0$

$$\int_{OA} \overline{F}\cdot d\overline{r} = \int_{OA} (x^2dx - xydy) = \int_0^a x^2dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}.$$

(*ii*) Along $AB\,x = a$, $dx = 0$

$$\int_{AB} \overline{F}\cdot d\overline{r} = \int_{AB} (x^2\,dx - x\,y\,d\,y) = \int_a^0 -a\,y\,d\,y = -a\left[\frac{y^2}{2}\right]_0^a = -\frac{a^3}{2}$$

(*iii*) Along $BC\,y = a$, $dy = 0$

$$\int_{BC} \overline{F}\cdot d\overline{r} = \int_{BC} (x^2dx - xy\,dy) = \int_a^0 x^2dx = \left[\frac{x^3}{3}\right]_a^0 = -\frac{a^3}{3}$$

(*iv*) Along $CO\,x = 0$, $dx = 0$

$$\int_{CO} \overline{F}\cdot d\overline{r} = \int_{CO} (x^2dx - xy\,dy) = 0$$

Line	Eq.		Lower Limit	Upper Limit
OA	$y = 0$	$dy = 0$	$x = 0$	$x = a$
AB	$x = a$	$dx = 0$	$y = 0$	$y = a$
BC	$y = a$	$dy = 0$	$x = a$	$x = 0$
CO	$x = 0$	$dx = 0$	$y = a$	$y = 0$

Putting the values of these integrals in (2), we have

$$\int_C \overline{F}\cdot d\overline{r} = \frac{a^3}{3} - \frac{a^3}{2} - \frac{a^3}{3} + 0 = -\frac{a^3}{2} \quad \ldots(3)$$

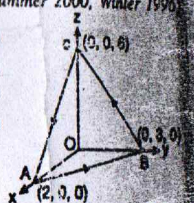
From (1) and (3), $\iint_S (\nabla\times\overline{F})\cdot\hat{\eta}\,ds = \int_C \overline{F}\cdot d\overline{r}$

Hence Stoke's theorem is verified.

Example 107. *Verify Stoke's theorem for* $\overline{F} = (x+y)\,i + (2x - z)\,j + (y+z)\,k$ *for the surface of a triangular lamina with vertices (2, 0, 0), (0, 3, 0) and (0, 0, 6).*

(*A.M.I.E.T.E., Summer 2000, Winter 1996*)

Solution. Here the path of integration c consists of the straight lines AB, BC, CA where the co-ordinates of A, B, C are (2, 0, 0), (0, 3, 0) and (0, 0, 6) respectively. Let s be the plane surface of triangle ABC bounded by c. Let $\hat{\eta}$ be unit normal vector to surface S. Then by Stokes theorem, we must have



$$\oint_c F\cdot dr = \iint_s \text{curl } F\cdot\hat{\eta}\,ds \ldots (1)$$

L.H.S. of (1) =

$$\int_{ABC} \overline{F}\cdot d\overline{r} = \int_{AB} \overline{F}\cdot d\overline{r} + \int_{BC} \overline{F}\cdot d\overline{r} + \int_{CA} F\cdot dr$$

Line	Eq. of line		Lower Limit	Upper Limit
AB	$\frac{x}{2} + \frac{y}{3} = 1$, $z = 0$	$dy = -\frac{3}{2}dx$	At A $x = 2$	At B $x = 0$
BC	$\frac{y}{3} + \frac{z}{6} = 1$, $x = 0$	$dz = -2dy$	At B $y = 3$	At C $y = 0$
CA	$\frac{x}{2} + \frac{z}{6} = 1$, $y = 0$	$dz = -3dx$	At C $x = 0$	At A $x = 2$

Along line AB, $z = 0$, equation of AB is $\frac{x}{2} + \frac{y}{3} = 1$

or $y = \frac{3}{2}(2 - x)$, $dy = -\frac{3}{2}dx$

At A, $x = 2$, At B, $x = 0$, $\overline{r} = xi + yj$

$$\int_{AB} \overline{F}\cdot d\overline{r} = \int_{AB} [(x+y)\,i + 2xj + yk]\cdot(idx + jdy)$$

$$= \int_{AB} (x+y)\,dx + 2xdy = \int_{AB}\left(x + 3 - \frac{3x}{2}\right)dx + 2x\left(-\frac{3}{2}dx\right)$$

$$= \int_2^0 \left(-\frac{7x}{2} + 3\right)dx = \left(-\frac{7x^2}{4} + 3x\right)_2^0 = (7-6) = +1$$

Along line BC, $x = 0$, Eq. of BC is $\frac{y}{3} + \frac{z}{6} = 1$ or $z = 6 - 2y$, $dz = -2dy$

At B, $y = 3$, At C, $y = 0$, $r = yj + zk$

$$\int_{BC} F\cdot dr = \int_{BC} [yi - zj + (y+z)\,k]\cdot(jdy + kdz) = \int_{BC} -zdy + (y+z)\,dz$$

$$= \int_3^0 (-6 + 2y)\,dy + (y + 6 - 2y)(-2dy) = \int_3^0 (4y - 18)\,dy = \left(2y^2 - 18y\right)_3^0 = 36$$

Along line CA, $y = 0$, Eq. of CA, $\frac{x}{2} + \frac{z}{6} = 1$ or $z = 6 - 3x$, $dz = -3dx$

At C, $x = 0$, at A, $x = 2$, $r = xi + zk$

$$\int_{CA} \overline{F}\cdot d\overline{r} = \int_{CA} [xi + (2x - z)\,j + zk]\cdot[dxi + dzk] = \int_{CA} xdx + zdz$$

$$= \int_0^2 xdx + (6 - 3x)(-3dx) = \int_0^2 (10x - 18)\,dx = \left[5x^2 - 18x\right]_0^2 = -16$$

$$\text{L.H.S. of (1)} = \int_{ABC} \overline{F}\cdot d\overline{r} = \int_{AB} \overline{F}\cdot d\overline{r} + \int_{BC} \overline{F}\cdot d\overline{r} + \int_{CA} \overline{F}\cdot d\overline{r} = 1 + 36 - 16 = 21 \quad \ldots (2)$$

$$\text{Curl } \overline{F} = \overline{\nabla}\times\overline{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\times[(x+y)\,i + (2x - z)\,j + (y+z)\,k]$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = (1+1)\,i - (0-0)\,j + (2-1)\,k = 2i + k$$

Equation of the plane of ABC is $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$

Normal to the plane ABC is

$$\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left(\frac{x}{2} + \frac{y}{3} + \frac{z}{6} - 1\right) = \frac{i}{2} + \frac{j}{3} + \frac{k}{6}$$

$$\text{Unit Normal Vector} = \frac{\frac{i}{2} + \frac{j}{3} + \frac{k}{6}}{\sqrt{\frac{1}{4} + \frac{1}{9} + \frac{1}{36}}} \quad ; \quad \hat{\eta} = \frac{1}{\sqrt{14}}(3i + 2j + k)$$

$$\text{R.H.S. of (1)} = \iint_s \text{curl } \overline{F}\cdot\hat{\eta}\,ds = \iint_s (2i + k)\cdot\frac{1}{\sqrt{14}}(3i + 2j + k)\,\frac{dx\,dy}{\frac{1}{\sqrt{14}}(3i + 2j + k)\cdot k}$$

$$= \iint_s \frac{(6+1)}{\sqrt{14}}\,\frac{dx\,dy}{\frac{1}{\sqrt{14}}} = 7\iint dx\,dy = 7 \text{ Area of } \Delta OAB$$

$$= 7\left(\frac{1}{2}\times 2\times 3\right) = 21 \quad \ldots (3)$$

With the help of (2) and (3), we find (1) is true and so Stoke's theorem is verified.

Example 108. *Verify Stoke's theorem for*

$$\overline{F} = (y - z + 2)\,i + (yz + 4)\,j - (xz)\,k$$

... is its boundary traversed in the clockwise direction. ... paraboloid $2z = x^2 + y^2$

Ans. -20π (A.M.I.E.T.E., Winter 1997)

...aluate $\int_C F \cdot dR$ where $F = yi + xz^3j - zy^3k$, C is the circle

$$x^2 + y^2 = 4,\ z = 1.5 \quad \textit{(Delhi 1992)}$$

Ans. $\frac{19}{2}\pi$

6. If S is the surface of the sphere $x^2+y^2+z^2 = 9$, prove that $\int_s \text{curl } F \cdot ds = 0$

7. Verify Stoke's theorem for the vector field $\overline{F} = (2y+z)\,i + (x-z)\,j + (y-x)\,k$ over the portion of the plane $x+y+z = 1$ cut off by the co-ordinate planes.

8. Verify Stoke's theorem for $\overline{F} = -y^3i + x^3j$ and the closed curve C is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Ans. $\frac{3\pi}{4}ab(a^2+b^2)$

9. Verify Stoke's theorem for $\overline{F} = (x^2+y^2)\,i - 2xyj$ taken round the rectangle bounded by the lines $x = 0, x = a, y = 0, y = b$.

(Mysore 1997S)

10. Evaluate $\int_c \overline{F}\,dr$ by Stoke's theorem for $\overline{F} = yzi + zxj + xyk$ and C is the curve of intersection of $x^2+y^2 = 1$ and $y = z^2$.

Ans. 0

11. If $\vec{F} = (y^2+z^2-x^2)\,i + (z^2+x^2-y^2)\,j + (x^2+y^2-z^2)\,k$, evaluate $\int \text{curl } \vec{F}.\,\hat{n}\,ds$ integrated over the portion of the surface. $x^2+y^2-2ax+az = 0$, above the plane $z = 0$ and verify Stoke's theorem; where $\hat{n}$ is unit vector normal to the surface.

(A.M.I.E.T.E., Winter 2002) Ans. $4\pi a^3$

5.45 GAUSS'S THEOREM OR DIVERGENCE THEOREM

(Relation between surface integral and volume integral)

Statement. The surface integral of the normal component of a vector function F taken around a closed surface S is equal to the integral of the divergence of F taken over the volume V enclosed by the surface S.

Mathematically $\iint_S F \cdot \hat{n}\,ds = \iiint_V \text{div}\,\overline{F}\,dv$

Proof. Let $\overline{F} = F_1i + F_2j + F_3k$.

Putting the value of F, n in the statement of the divergence theorem we have

$$\iint_S (F_1i+F_2j+F_3k)\,\hat{n}\,ds = \iiint_V \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(F_1i+F_2j+F_3k)\,dx\,dy\,dz$$

$$= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz \quad \text{...(1)}$$

We require to prove (1).

Let us first evaluate $\iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz$.

$$\iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \iint_R \left[\int_{z=f_1(x,y)}^{z=f_2(x,y)} \frac{\partial F_3}{\partial z}\,dz\right] dx\,dy = \iint_R \left[F_3(x,y,z)\right]_{z=f_1(x,y)}^{z=f_2(x,y)} dx\,dy$$

$$= \iint_R [F_3(x,y,f_2) - F_3(x,y,f_1)]\,dx\,dy \quad \text{...(2)}$$

For the upper part of the surface i.e. S_2, we have

...

Again for the lower part of the surface i.e. S_1 we have,

$$dx\,dy = -\cos r_1\,ds_1 = \hat{n}_1 \cdot k\,ds_1$$

$$\iint_R F_3(x,y,f_2)\,dx\,dy = \iint_{S_2} F_3\hat{n}_2 \cdot k\,ds_2$$

$$\iint_R F_3(x,y,f_1)\,dx\,dy = -\iint_{S_1} F_3\hat{n}_1 \cdot k\,ds_1$$

Putting these values in (2) we have

$$\iiint_V \frac{\partial F_3}{\partial z}\,dv = \iint_{S_2} F_3\hat{n}_2 \cdot k\,ds_2 + \iint_{S_1} F_3\hat{n}_1 \cdot k \cdot ds_1$$

$$= \iint_S F_3\hat{n} \cdot k\,ds \quad \text{...(3)}$$

Similarly, it can be shown that

$$\iiint_V \frac{\partial F_2}{\partial y}\,dv = \iint_S F_2\hat{n} \cdot j\,ds \quad \text{...(4)}$$

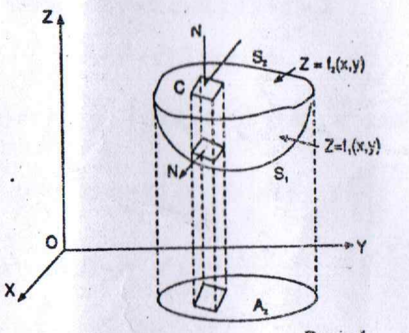
$$\iiint_V \frac{\partial F_1}{\partial x}\,dv = \iint_S F_1\hat{n} \cdot i\,ds \quad \text{...(5)}$$

Adding (3), (4) & (5) we have

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dv$$

$$= \iint_S (F_1i + F_2j + F_3k) \cdot \hat{n} \cdot ds \quad \text{...(4)}$$

$$\iiint_V (\nabla \cdot \overline{F})\,dv = \iint_S \overline{F} \cdot \hat{n}\,ds$$



Proved

Example 109. *Use Divergence theorem to evaluate*

$$\int_S \vec{A} \cdot d\vec{S}, \text{ where } A = x^3i + y^3j + z^3k$$

and S is the surface of the sphere $x^2+y^2+z^2 = a^2$.

Solution. $\iint_S \vec{A} \cdot d\vec{S} = \iiint \text{div}\,A\,dV$

$$= \iiint \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot (x^3i + y^3j + z^3k)\,dV$$

$$= \iiint (3x^2+3y^2+3z^2)\,dV = 3\iiint (x^2+y^2+z^2)\,dV$$

On putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, we get

$$= 3\iiint r^2(r^2\sin\theta\,dr\,d\theta\,d\phi) = 3 \times 8\int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{2}} \sin\theta\,d\theta \int_0^a r^4\,dr$$

$$= 24(\phi)_0^{\frac{\pi}{2}}(-\cos\theta)_0^{\frac{\pi}{2}}\left(\frac{r^5}{5}\right)_0^a = 24\left(\frac{\pi}{2}\right)(-0+1)\left(\frac{a^5}{5}\right) = \frac{12\pi a^5}{5} \quad \text{Ans.}$$

Example 110. *Use Divergence theorem to evaluate* $\iint_S F \cdot dS$ *where* $F = 4xi - 2y^2j + z^2k$ *and S is the surface bounding the region* $x^2+y^2 = 4$, $z = 0$ *and* $z = 3$.

(A.M.I.E.T.E., Summer 2003, 2001, 1997, Kerala 1995)

Solution. By Divergence theorem,

$$\iint_S F\cdot dS = \iiint_V div\, F dV$$

$$= \iiint_V \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot(4\,xi - 2\,y^2j + z^2k)\,dV$$

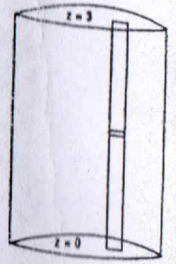
$$= \iiint_V (4-4\,y+2\,z)\,dx\,dy\,dz$$

$$= \iint dx\,dy \int_0^3 (4-4\,y+2\,z)\,dz$$

$$= \iint dx\,dy\,(4z-4\,y\,z+z^2)_0^3$$

$$= \iint (12-12\,y+9)\;dx\,dy$$

$$= \iint (21-12\,y)\;dx\,dy$$



Let us put $x = r\cos\theta$, $y = r\sin\theta$

$$= \iint (21-12\,r\sin\theta)\;r\,d\theta\,dr$$

$$= \int_0^{2\pi} d\theta \int_0^2 (21\,r - 12\,r^2\sin\theta)\,dr$$

$$= \int_0^{2\pi} d\theta\left(\frac{21\,r^2}{2} - 4\,r^3\sin\theta\right)_0^2$$

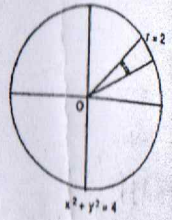
$$= \int_0^{2\pi} d\theta\;(42-32\sin\theta)$$

$$= (42\,\theta + 32\cos\theta)_0^{2\pi}$$

$$= 84\,\pi + 32 - 32$$

$$= 84\,\pi$$

Ans.



Example 111. *Apply the Divergence theorem to compute* $\iint_s u\cdot n\,ds$, *where s is the surface of the cylinder* $x^2+y^2 = a^2$ *bounded by the planes* $z = 0$, $z = b$ *and where* $u = ix - jy + kz$.

(Nagpur, Winter 2000)

Solution. By Gauss's Divergence theorem

$$\iint u\cdot n ds = \iiint_V (\nabla\cdot u)\,dv$$

$$= \iiint_V \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot(ix - jy + kz)\,dv$$

$$= \iiint_V \left(\frac{\partial x}{\partial x} - \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right)dv$$

$$= \iiint_V (1-1+1)\,dv$$

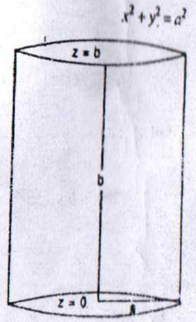
$$= \iiint_V dv$$

$$= \iiint_V dx\,dy\,dz$$

= Volume of the cylinder

$$= \pi\,a^2\,b$$

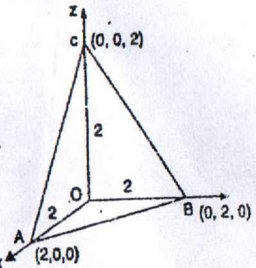
Ans.



...mple 112. *Evaluate surface integral* $\iint F\cdot n\,ds$ *where* $F = (x^2+y^2+z^2)(\hat{i}+\hat{j}+\hat{k})$, *...urface of the tetrahedron* $x = 0$, $y = 0$, $z = 0$, $x+y+z = 2$ *and n is the unit normal ...ward direction to the closed surface S.*

...tion. By Divergence theorem

$$\iint_S \overline{F}\cdot\hat{\eta}\,ds = \iiint_V div\,F\cdot dv$$



...is the surface of tetrahedron $x = 0, y = 0, z = 0, x+y+z = 2$

$$\iiint_V \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot(x^2+y^2+z^2)(i+j+k)\,dv$$

$$\iiint_V (2x+2\,y+2\,z)\,dv = 2\iiint_V (x+y+z)\,dx\,dy\,dz$$

$$2\int_0^2 dx\int_0^{2-x} dy\int_0^{2-x-y}(x+y+z)\,dz = 2\int_0^2 dx\int_0^{2-x} dy\left(xz+yz+\frac{z^2}{2}\right)_0^{2-x-y}$$

$$2\int_0^2 dx\int_0^{2-x} dy\left[2x - x^2 - xy + 2y - xy - y^2 + \frac{(2-x-y)^2}{2}\right]$$

$$2\int_0^2 dx\left[2xy - x^2y - x\,y^2 + y^2 - \frac{y^3}{3} - \frac{(2-x-y)^3}{6}\right]_0^{2-x}$$

$$2\int_0^2 dx\left[2x(2-x) - x^2(2-x) - x(2-x)^2 + (2-x)^2 - \frac{(2-x)^3}{3} + \frac{(2-x)^3}{6}\right]$$

$$2\int_0^2\left(4x - 2x^2 - 2x^2 + x^3 - 4x + 4x^2 - x^3 + (2-x)^2 - \frac{(2-x)^3}{3} + \frac{(2-x)^3}{6}\right]$$

$$2\left[2x^2 - \frac{4\,x^3}{3} + \frac{x^4}{4} - 2x^2 + \frac{4\,x^3}{3} - \frac{x^4}{4} - \frac{(2-x)^3}{3} + \frac{(2-x)^4}{12} - \frac{(2-x)^4}{24}\right]_0^2$$

$$2\left[-\frac{(2-x)^3}{3} + \frac{(2-x)^4}{12} - \frac{(2-x)^4}{24}\right]_0^2 = 2\left[\frac{8}{3} - \frac{16}{12} + \frac{16}{24}\right] = 4$$

Ans.

...xample 113. *Use the Divergence theorem to evaluate*

$$\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$$

...S is the portion of the plane $x + 2\,y + 3\,z = 6$ *which lies in the first Octant.* (Kerala 1995)

(U.P. I Semester Winter 2003;

Solution. $\iint_S (f_1\,dy\,dz + f_2\,dx\,dz + f_3\,dx\,dy)$

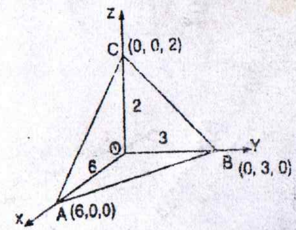
$$= \iiint_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right)dx\,dy\,dz$$

...S is a closed surface bounding a volume V.

$$\therefore \iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$$

$$= \iiint_V \left[\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right]dx\,dy\,dz$$

$$= \iiint_V (1+1+1)\,dx\,dy\,dz$$



$$\iint_{OCE} \overline{F}\cdot\hat{\eta}\, ds = \iint_{OCE} (2x^2yi - y^2j + 4xz^2k)\,(-i)\, ds = \iint_{OCE} -2x^2y\, ds = 0$$

because in OCE (y–z) plane, $x = 0$

$$\iint_{ABD} \overline{F}\cdot\hat{\eta}\, ds = \iint_{ABD} (2x^2yi - y^2j + 4xz^2k)\,(i)\, ds = \iint_{ABD} 2x^2y\, ds$$

$$= \iint 2x^2y\, dy\, dz = \int_0^3 dz \int_0^{\sqrt{9-z^2}} 2(2)^2\, y\, dy$$

$$= 8\int_0^3 dz \left[\frac{y^2}{2}\right]_0^{\sqrt{9-z^2}} = 4\int_0^3 dz\,(9-z^2)$$

$$= 4\left[9z - \frac{z^3}{3}\right]_0^3 = 4\,[27-9] = 72$$

$$\therefore \iint_S F\cdot\hat{\eta}\, ds = 108 + 0 + 0 + 0 + 72 = 180 \qquad \ldots(2)$$

From (1) and (2)

$$\iiint_V \nabla\cdot F\, dV = \iint_S F\cdot\hat{\eta}\, ds$$

Hence the theorem is verified.

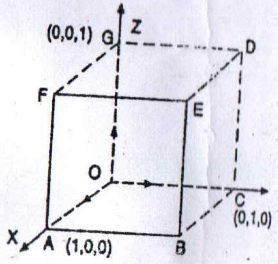
Apply

Example 116. *Verify Divergence Theorem, given that* $\overline{F} = 4xzi - y^2j + yzk$ *and S is the surface of the cube bounded by the planes* $x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0,\ z = 1$.

Solution. $\overline{\nabla}\cdot\overline{F} = \left(i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z}\right)\cdot(4xzi - y^2j + yzk)$

$$= 4z - 2y + y$$

$$= 4z - y$$



Volume Integral $= \iiint \nabla\cdot F\, dv$

$$= \iiint (4z - y)\, dx\, dy\, dz$$

$$= \int_0^1 dx \int_0^1 dy \int_0^1 (4z - y)\, dz$$

$$= \int_0^1 dx \int_0^1 dy\,(2z^2 - yz)_0^1 = \int_0^1 dx \int_0^1 dy\,(2-y)$$

$$= \int_0^1 dx \left(2y - \frac{y^2}{2}\right)_0^1 = \int_0^1 dx\left(2 - \frac{1}{2}\right) = \frac{3}{2}\int_0^1 dx = \frac{3}{2}(x)_0^1 = \frac{3}{2} \qquad \ldots(1)$$

To evaluate $\iint_S F\cdot\hat{\eta}\, ds$ where S consists of six plane surfaces.

Over the face $OABC$, $z = 0,\ dz = 0,\ \hat{\eta} = -k,\ ds = dx\, dy$

$$\iint F\cdot\hat{\eta}\, ds = \int_0^1\int_0^1 (-y^2j)\cdot(-k)\, dx\, dy = 0$$

Over the face $BCDE$, $y = 1,\ dy = 0,\ \hat{\eta} = j,\ ds = dx\, dz$

$$\iint F\cdot\hat{\eta}\, ds = \int_0^1\int_0^1 (4xzi - j + zk)\cdot(j)\, dx\, dz = \int_0^1\int_0^1 -dx\,dz$$

$$= -\int_0^1 dx \int_0^1 dz = -(x)_0^1(z)_0^1 = -(1)(1) = -1$$

Over the face $DEFG$, $z = 1,\ dz = 0,\ \hat{\eta} = k,\ ds = dx\, dy$

$$\iint F\cdot\hat{\eta}\, ds = \int_0^1\int_0^1 [4x(1) - y^2j + y(1)k]\cdot(k)\, dx\, dy$$

$$= \int_0^1\int_0^1 y\, dx\, dy = \int_0^1 dx \int_0^1 y\, dy = (x)_0^1\left(\frac{y^2}{2}\right)_0^1 = \frac{1}{2}$$

Over the face $OCDG$, $x = 0,\ dx = 0,\ \hat{\eta} = -i,\ ds = dy\, dz$

$$\iint F\cdot\hat{\eta}\, ds = \int_0^1\int_0^1 (oi - y^2j + yzk)\cdot(-i)\, dy\, dz = 0$$

Over the face $AOGF$, $y = 0,\ dy = 0,\ \hat{\eta} = -j,\ ds = dx\, dz$

$$\iint F\cdot\hat{\eta}\, ds = \int_0^1\int_0^1 (4xzi)\cdot(-j)\, dx\, dz = 0$$

Over the face $ABEF$, $x = 1,\ dx = 0,\ \hat{\eta} = i,\ ds = dy\, dz$

$$\iint F\cdot\hat{\eta}\, ds = \int_0^1\int_0^1 [(4zi - y^2j + yzk)\cdot(i)]\, dy\, dz = \int_0^1\int_0^1 4z\, dy\, dz$$

$$= \int_0^1 dy \int_0^1 4z\, dz = \int_0^1 dy\,(2z^2)_0^1 = 2\int_0^1 dy$$

$$= 2(y)_0^1 = 2$$

On adding we see that over the whole surface

$$\iint F\cdot\hat{\eta}\, ds = \left(0 - 1 + \frac{1}{2} + 0 + 0 + 2\right) = \frac{3}{2} \qquad \ldots(2)$$

From (1) and (2)

$$\iiint_V \nabla\cdot F\, dv = \iint_S \overline{F}\cdot\hat{\eta}\, ds \qquad \textbf{Verified.}$$

Exercise 5.22

1. Use Divergence theorem to evaluate

$$\iint_S (y^2z^2i + z^2x^2j + x^2y^2k)\cdot \overline{ds}$$

where S is the upper part of the sphere $x^2 + y^2 + z^2 = 9$ above x–y plane. **Ans.** $\dfrac{243\pi}{8}$

2. Evaluate $\int_S (\nabla\times\overline{F})\cdot ds$, where S is the surface of the paraboloid $x^2 + y^2 + z = 4$ above the xy-plane and $\overline{F} = (x^2 + y - 4)\,i + 3xyj + (2xz + z^2)\,k$. **Ans.** -4π

3. Evaluate $\iint_S [xz^2\,dy\,dz + (x^2y - z^3)\,dz\,dx + (2xy + y^2z)\,dx\,dy]$ where S is the surface enclosing a region bounded by hemisphere $x^2 + y^2 + z^2 = 4$ above X-Y plane. (*A.M.I.E.T.E., Summer 1998*)

4. Verify Gauss divergence theorem for $\overrightarrow{F} = x^2\,i + y^2\,\hat{j} + z^2\,\hat{k}$ on the surface S of the cuboid formed by the planes $x = 0,\ x = a,\ y = 0,\ y = b,\ z = 0$ and $z = c$. (*A.M.I.E.T.E., Dec. 2004*)

For Q-3

vectors

Ex: 94, 95, 99, 102, 103, 104, 105
106, 110, 111, 116

Matrix

pdf ⇒ given to CR