...ample 37. *Determine the rank of a matrix*

$$\begin{bmatrix} 1 & 4 & 5 \\ 2 & 5 & 8 \\ 3 & 7 & 22 \end{bmatrix}$$

...lution.
$$\begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 22 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & -2 & -2 \\ 0 & -5 & 7 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & -5 & 7 \end{bmatrix} -\frac{1}{2}R_2 \sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 12 \end{bmatrix} R_3 + 5R_2$$

Rank = Number of non-zero rows = 3 **Ans.**

...ple 38. *Find the rank of the matrix*

$$\begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$$

...ution.
$$\begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & -2 \\ 0 & -2 & 14 & -4 \\ 0 & -2 & 14 & -4 \end{bmatrix} \begin{matrix} R_2 + 2R_1 \\ R_3 + 3R_1 \\ R_4 + 5R_1 \end{matrix}$$

$$\sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 2R_2 \\ R_4 - 2R_2 \end{matrix}$$

...e the 4th order and 3rd order minors are zero. But a minor of second order

$$\begin{vmatrix} 3 & -2 \\ 7 & -2 \end{vmatrix} = -6 + 14 = 8 \neq 0$$

...k = Number of non-zero rows = 2 **Ans.**

...ample 39. *Find the rank of the matrix*

$$\begin{bmatrix} 3 & -4 & -1 & 2 \\ 1 & 7 & 3 & 1 \\ 5 & -2 & 5 & 4 \\ 9 & -3 & 7 & 7 \end{bmatrix}$$

...lution.
$$\begin{bmatrix} 1 & 7 & 3 & 1 \\ 3 & -4 & -1 & 2 \\ 5 & -2 & 5 & 4 \\ 9 & -3 & 7 & 7 \end{bmatrix} R_1 \leftrightarrow R_2$$

$R_2 - 3R_1, R_3 - 5R_1, R_4 - 9R_1,\quad R_3 - \frac{37}{25}R_2,\ R_4 - \frac{66}{25}R_2 \quad R_4 - \frac{4}{3}R_3$

$R_3 - \frac{37}{25} R_2, \; R_4 - \frac{66}{25} R_2, R$

$$\begin{bmatrix} 1 & 7 & 3 & 1 \\ 0 & -25 & -10 & -1 \\ 0 & -37 & -10 & -1 \\ 0 & -66 & -20 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 7 & 3 & 1 \\ 0 & -25 & -10 & -1 \\ 0 & 0 & \frac{24}{5} & \frac{12}{25} \\ 0 & 0 & \frac{32}{5} & \frac{16}{25} \end{bmatrix} \sim \begin{bmatrix} 1 & 7 & 3 & 1 \\ 0 & -25 & -10 & -1 \\ 0 & 0 & \frac{24}{5} & \frac{12}{25} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank = 3 **Ans.**

Example 40. *Find the rank of*

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Solution.

$$\xrightarrow{R_1 \leftrightarrow R_2} \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \xrightarrow{R_2 - 2R_1,\; R_3 - 3R_1,\; R_4 - 6R_1} \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$$\xrightarrow{R_3 - \frac{4}{5}R_2,\; R_4 - \frac{9}{5}R_2} \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & \frac{33}{5} & \frac{22}{5} \\ 0 & 0 & \frac{33}{5} & \frac{22}{5} \end{bmatrix} \xrightarrow{R_4 - R_3} \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & \frac{33}{5} & \frac{22}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = Number of non-zero rows = 3 **Ans.**

Exercise 4.16

Find the rank of the following matrices:

1. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$ **Ans.** 2

2. $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix}$ **Ans.** 3

4. $\begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ **Ans.** 2

6. $\begin{bmatrix} 2 & 4 & 3 & -2 \\ -3 & -2 & -1 & 4 \\ 6 & -1 & 7 & 2 \end{bmatrix}$

7. $\begin{bmatrix} 3 & 4 & 1 & 1 \\ -2 & 4 & 3 & 6 \\ -1 & -2 & 6 & 4 \\ 1 & -1 & 2 & -3 \end{bmatrix}$ **Ans.** 4

8. $\begin{bmatrix} 9 & 3 & 1 & 0 \\ 3 & 0 & 1 & -6 \\ 1 & 1 & 1 & 1 \\ 0 & -6 & 1 & 9 \end{bmatrix}$ (A.M.I.E.T.E., Dec.

$\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 2 & 3 \end{bmatrix}$ **Ans.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$ (A.M.I.E.T.E., Sum. 2002) **Ans.** 2

$\begin{bmatrix} 5 & 7 & 12 \\ 2 & 3 & 5 \\ 6 & 9 & 15 \end{bmatrix}$ **Ans.** $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$, Rank = 2

12. $\begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$ **Ans.** $\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$, Rank = 3

$\begin{bmatrix} -3 & -1 \\ 1 & 1 \\ 0 & 2 \\ -2 & 0 \end{bmatrix}$ **Ans.** 2

...are two square matrices of order n such that $AB = 0$, then rank of

...ast one of A, B is less than n. (b) both A and B is less than n

...of A, B is less than n (a) at least one of A, B is zero. (A.M.I.E.T.E., Dec. 2004)

...k of the diagonal matrix $\begin{bmatrix} -1 & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 0 \\ & & & & & 4 \end{bmatrix}$ is

(b) 2, (c) 3, (d) 4 **Ans.** 3 *i.e.*, (c)

...k of the matrix $\begin{bmatrix} 2 & -4 & 6 \\ -1 & 2 & -3 \\ 3 & -6 & 9 \end{bmatrix}$ is

(b) 2, (c) 0, (d) 1 **Ans.** 1, (d)

...a non-zero column vector, $(n \times 1)$, then the rank of matrix AA^T is

(b) 1. (c) $n - 1$. (d) n. **Ans.** (a)

...k of matrix $\begin{bmatrix} \mu & -1 & 0 \\ 0 & \mu & -1 \\ -1 & 0 & \mu \end{bmatrix}$ is 2, for μ equals

...y row number (b) 3 (c) 1 (d) 2 **Ans.** (c)

...ue of μ for which the rank of the matrix $A = \begin{bmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & -1 & 0 \\ 0 & 0 & \mu & -1 \\ -6 & 11 & -6 & 1 \end{bmatrix}$

(b) 1 (c) 4 (d) -1 (A.M.I.E.T.E. 2005) **Ans.** (a)

...OLUTION OF SIMULTANEOUS EQUATIONS

...matrix of the coefficients of x, y, z is reduced into Echelon form by elementary row ...tions. At the end of the row transformation the value of z is calculated from the last ...nd value of y and the value of x are calculated by the backward substitution.

...mple 41. *Solve the following equations*

$$x - y + 2z = 3, \quad x + 2y + 3z = 5, \quad 3x - 4y - 5z = -13$$

...tion. In the matrix form, the equations are written in the following form.

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 3 \\ 3 & -4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -13 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \\ 0 & -1 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -22 \end{bmatrix} \begin{matrix} R_2 - R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \\ & & -32 \end{bmatrix} \begin{bmatrix} x \\ y \\ \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -64 \end{bmatrix} R_3 + \frac{1}{3} R_2$$

$$x-y+2z=3$$
$$3y+z=2$$
$$\frac{-32}{3}z=\frac{-64}{3} \quad \text{or} \quad z=2$$

Putting the value of z in (2) we get

$$3y+2=2 \quad \text{or} \quad y=0$$

Putting the value of y, z in (1) we get

$$x-0+4=3 \quad \text{or} \quad x=-1$$
$$x=-1, y=0, z=2$$ **Ans.**

Example 42. Find all the solutions of the system of equations

$$x_1+2x_2-x_3=1,\ 3x_1-2x_2+2x_3=2,\ 7x_1-2x_2+3x_3=5$$

Solution. $\begin{bmatrix} 1 & 2 & -1 \\ 3 & -2 & 2 \\ 7 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

$$\underset{R_2-3R_1,\ R_3-7R_1}{\begin{bmatrix} 1 & 2 & -1 \\ 0 & -8 & 5 \\ 0 & -16 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}},\quad \underset{R_3-2R_2}{\begin{bmatrix} 1 & 2 & -1 \\ 0 & -8 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}$$

$$x_1+2x_2-x_3=1$$
$$-8x_2+5x_3=-1$$

Let $\quad x_3=k$

Putting $x_3=k$ in (2) we get

$$-8x_2+5k=-1 \quad \text{or} \quad x_2=\frac{1}{8}(5k+1)$$

Substituting the values of x_3, x_2 in (1) we get

$$x_1+\frac{1}{4}(5k+1)-k=1$$

$\therefore \quad x_1=1+k-\frac{5k}{4}-\frac{1}{4}=-\frac{k}{4}+\frac{3}{4}$

$\therefore \quad x_1=-\frac{k}{4}+\frac{3}{4},\ x_2=\frac{5k}{8}+\frac{1}{8},\ x_3=k$

The equations have infinite solutions

Q. 1. Solve the following system of equations.

$x_1+2x_2-x_3=3,\ 3x_1-x_2+2x_3=1,\ 2x_1-2x_2+3x_3=2,\ x_1-x_2+$

(A.M.I.E.T.E.

Solution. Try your self.

Example 43. Express the following system of equations in matrix form by the elimination method (Gauss Jordan method).

$$2x_1+x_2+2x_3+x_4=6,\ 6x_1-6x_2+6x_3+12x_4=36$$
$$4x_1+3x_2+3x_3-3x_4=-1,\ 2x_1+2x_2-x_3+x_4=10$$

Solution. The equations are expressed in matrix form as

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 6 \\ 36 \\ -1 \\ 10 \end{bmatrix} \text{ or } \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & -9 & 0 & 9 \\ 0 & 1 & -1 & -5 \\ 0 & 1 & -3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 6 \\ 18 \\ -13 \\ 4 \end{bmatrix}\begin{matrix} R_2-3R_1 \\ R_3-2R_1 \\ R_4-R_1 \end{matrix}$$

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & -5 \\ 0 & 1 & -3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 6 \\ -2 \\ -13 \\ 4 \end{bmatrix} R_2 \to \frac{R_2}{-9}$$

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & -3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 6 \\ -2 \\ -11 \\ 6 \end{bmatrix}\begin{matrix} R_3-R_2 \\ \\ R_4-R_2 \end{matrix}$$

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 13 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 6 \\ -2 \\ -11 \\ 39 \end{bmatrix} R_4-3R_3$$

$$2x_1+x_2+2x_3+x_4=6 \quad \text{...(1)}$$
$$x_2-x_4=-2 \quad \text{...(2)}$$
$$-x_3-4x_4=-11 \quad \text{...(3)}$$
$$13x_4=39 \quad \text{or} \quad x_4=3$$

Putting the value of x_4 in (3) we get

$$-x_3-12=-11 \quad \text{or} \quad x_3=-1$$

Putting the value of x_4 in (2) we get

$$x_2-3=-2 \quad \text{or} \quad x_2=1$$

Substituting the values of x_4, x_3, x_2 in (1) we get

$$2x_1+1-2+3=6 \text{ or } 2x_1=4 \quad \text{or} \quad x_1=2$$
$$x_1=2, x_2=1, x_3=-1, x_4=3$$ **Ans.**

Example 44. *Find the general solution of the system of equations:*

$$3x_1+2x_3+2x_4=0$$
$$-x_1+7x_2+4x_3+9x_4=0$$
$$7x_1-7x_2-5x_4=0$$

Solution. The system of equations in the matrix form is expressed as

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ -1 & 7 & 4 & 9 \\ 7 & -7 & 0 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 7 & 4 & 9 \\ 3 & 0 & 2 & 2 \\ 7 & -7 & 0 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} -1 & 7 & 4 & 9 \\ 0 & 21 & 14 & 29 \\ 0 & 42 & 28 & 58 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} R_2 + 3R_1 \\ R_3 + 7R_1 \end{matrix}$$

$$\begin{bmatrix} -1 & 7 & 4 & 9 \\ 0 & 21 & 14 & 29 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 - 2R_2$$

$$-x_1 + 7x_2 + 4x_3 + 9x_4 = 0$$

$$21x_2 + 14x_3 + 29x_4 = 0$$

Let $\quad x_4 = a,\ x_3 = b$

From (2) $\quad 21x_2 + 14b + 29a = 0 \quad \text{or} \quad x_2 = -\dfrac{2b}{3} - \dfrac{29a}{21}$

From (1), $-x_1 + 7\left(-\dfrac{2b}{3} - \dfrac{29a}{21}\right) + 4b + 9a = 0$

$$x_1 = -\frac{2a}{3} - \frac{2b}{3}$$

$$x_1 = -\frac{2}{3}(a+b), \quad x_2 = -\frac{1}{21}(29a + 14b),$$

$$x_3 = b,\ x_4 = a \quad \textbf{Ans.}$$

4.38 TYPES OF LINEAR EQUATIONS

(1) Consistent. A system of equations is said to be *consistent*, if they have one o[r more] solution *i.e.*

Unique solution	Infinite solutions
$x + 2y = 4$	$x + 2y = 4$
$3x + 2y = 2$	$3x + 6y = 12$

(2) Inconsistent. If a system of equation has no solution, it is said to be inconsis[tent]

$$x + 2y = 4$$

$$3x + 6y = 5 \qquad \textit{(A.M.I.E.T.E., De}$$

4.39 CONSISTENCY OF A SYSTEM OF LINEAR EQUATIONS

$$a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n = b_2$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots a_{mn}x_n = b_m$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}$$

$$AX = B$$

$$C = [A, B] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

[is] the *augmented* matrix.

[(1)] **Consistent equations.** If Rank A = Rank C

[(i)] *Unique solution:* Rank A = Rank $C = n$ $\qquad$ where n = number of unknowns.

[(ii)] *Infinite solution:* Rank A = Rank $C = r$, $r < n$ $\quad \neq n$

[(2) Inco]nsistent equations. If Rank $A \neq$ Rank C.

[Exa]mple 45. *Show that the equations*

$$2x + 6y = -11, \quad 6x + 20y - 6z = -3, \quad 6y - 18z = -1$$

[are not c]onsistent.

[Sol]ution. Augmented matrix $C = [A, B]$

$$\begin{bmatrix} 2 & 6 & 0 & \cdot & -11 \\ 6 & 20 & -6 & \cdot & -3 \\ 0 & 6 & -18 & \cdot & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & 6 & 0 & \cdot & -11 \\ 0 & 2 & -6 & \cdot & 30 \\ 0 & 6 & -18 & \cdot & -1 \end{bmatrix} R_2 - 3R_1 \sim \begin{bmatrix} 2 & 6 & 0 & -11 \\ 0 & 2 & -6 & 30 \\ 0 & 0 & -0 & -91 \end{bmatrix} R_3 - 3R_2$$

[Th]e rank of C is 3 and the rank of A is 2.

[Ra]nk of $A \neq$ Rank of C.

[Th]e equations are not consistent. **Ans.**

[Ex]ample 46. *Test for consistency and solve:*

$$5x + 3y + 7z = 4, \quad 3x + 26y + 2z = 9, \quad 7x + 2y + 10z = 5$$

[So]lution. The augmented matrix $C = [A, B]$

$$\begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{5} & \frac{7}{5} & : & \frac{4}{5} \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix} \frac{1}{5}R_1$$

$$\begin{bmatrix} 1 & \frac{3}{5} & \frac{7}{5} & : & \frac{4}{5} \\ 0 & \frac{121}{5} & -\frac{11}{5} & : & \frac{33}{5} \\ 0 & -\frac{11}{5} & \frac{1}{5} & : & -\frac{3}{5} \end{bmatrix} \begin{matrix} R_2 - 3R_1 \\ R_3 - 7R_1 \end{matrix} \sim \begin{bmatrix} 1 & \frac{3}{5} & \frac{7}{5} & : & \frac{4}{5} \\ 0 & \frac{121}{5} & -\frac{11}{5} & : & \frac{33}{5} \\ 0 & 0 & 0 & : & 0 \end{bmatrix} R_3 + \frac{1}{11}R_2$$

[Ra]nk of $A = 2$ = Rank of C

[H]ence the equations are consistent. But the rank is less than 3 *i.e.* number of unknowns.

[So] solutions are infinite.

$$\begin{bmatrix} 1 & \frac{3}{5} & \frac{7}{5} \\ 0 & \frac{121}{5} & -\frac{11}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{33}{5} \\ 0 \end{bmatrix}$$

$$x + \frac{3}{5}y + \frac{7}{5}z = \frac{4}{5}$$

$$\frac{121}{5}y - \frac{11z}{5} = \frac{33}{5} \text{ or } 11y - z = 3$$

Let $z = k$ then $\quad 11y - k = 3$ or $y = \frac{3}{11} + \frac{k}{11}$

$$x + \frac{3}{5}\left[\frac{3}{11} + \frac{k}{11}\right] + \frac{7}{5}k = \frac{4}{5} \quad \text{or} \quad x = -\frac{16}{11}k + \frac{7}{11}$$

Example 47. Discuss the consistency of the following system
$2x + 3y + 4z = 11, \quad x + 5y + 7z = 15, \quad 3x + 11y + 13z = 25.$
If found consistent, solve it. (A.M.I.E.T.E.,

Solution. The augmented matrix $C = [A, B]$

$$\begin{bmatrix} 2 & 3 & 4 & 11 \\ 1 & 5 & 7 & 15 \\ 3 & 11 & 13 & 25 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 7 & 15 \\ 2 & 3 & 4 & 11 \\ 3 & 11 & 13 & 25 \end{bmatrix} R_1 \leftrightarrow R_2$$

$R_2 - 2R_1, R_3 - 3R_1 \qquad -\frac{1}{7}R_2, -\frac{1}{4}R_3$

$$\sim \begin{bmatrix} 1 & 5 & 7 & 15 \\ 0 & -7 & -10 & -19 \\ 0 & -4 & -8 & -20 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 7 & 15 \\ 0 & 1 & \frac{10}{7} & \frac{19}{7} \\ 0 & 1 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 7 & 15 \\ 0 & & \frac{10}{7} & \frac{19}{7} \\ 0 & 0 & \frac{4}{7} & \frac{16}{7} \end{bmatrix} R_3$$

Rank of $C = 3$ = Rank of A.

Hence the system of equations is consistent with unique solution.

$x + 5y + 7z = 15 \quad \ldots (1) \qquad y + \frac{10z}{7} = \frac{19}{7} \quad \ldots (2) \qquad \frac{4z}{7} = \frac{16}{7} \Rightarrow z$

From (2) $\quad y + \frac{10 \times 4}{7} = \frac{19}{7} \Rightarrow y = -3$

From (1) $\quad x + 5(-3) + 7(4) = 15 \Rightarrow x = 2$

$x = 2, \; y = -3, \; z = 4$

Example 48. Determine for what values of λ and μ the following equations have (i) no solution; (ii) a unique solution; (iii) infinite number of solutions.

$x + y + z = 6, \quad x + 2y + 3z = 10, \quad x + 2y + \lambda z = \mu \qquad$ (A.M.I.E.T.E., Su

Solution.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$$AX = B$$

$$C = (A, B) = \begin{bmatrix} 1 & 1 & 1 & \cdot & 6 \\ 1 & 2 & 3 & \cdot & 10 \\ 1 & 2 & \lambda & \cdot & \mu \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & \cdot & 6 \\ 0 & 1 & 2 & \cdot & 4 \\ 0 & 1 & \lambda - 1 & \cdot & \mu - 6 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \cdot & 6 \\ 0 & 1 & 2 & \cdot & 4 \\ 0 & 0 & \lambda - 3 & \cdot & \mu - 10 \end{bmatrix} R_3 - R_2$$

There is no solution if $R(A) \neq R(C)$

i.e. $\quad \lambda - 3 = 0$ or $\lambda = 3$ and $\mu - 10 \neq 0$ or $\mu \neq 10$

There is a unique solution if $R(A) = R(C) = 3$

i.e. $\quad \lambda - 3 \neq 0 \quad$ or $\quad \lambda \neq 3$, μ may have any value. → ?

There are infinite solutions if $R(A) = R(C) = 2$

$\lambda - 3 = 0$ or $\lambda = 3$ and $\mu - 10 = 0$ or $\mu = 10$ **Ans.**

Example 49. *Find for what values of k the set of equations*
$2x - 3y + 6z - 5t = 3, \quad y - 4z + t = 1, \quad 4x - 5y + 8z - 9t = k$
has (i) no solution (ii) infinite number of solutions. *(A.M.I.E.T.E., Dec. 2006, Summer 2004)*

Solution. The augmented matrix $C = [A, B]$ *(R.G.P.V. Bhopal Feb. 2006)*

$$\begin{bmatrix} 2 & -3 & 6 & -5 & \cdot & 3 \\ 0 & 1 & -4 & 1 & \cdot & 1 \\ 4 & -5 & 8 & -9 & \cdot & k \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & 6 & -5 & \cdot & 3 \\ 0 & 1 & -4 & 1 & \cdot & 1 \\ 0 & 1 & -4 & 1 & \cdot & k-6 \end{bmatrix} R_3 - 2R_1$$

$$R_3 - R_2$$

$$\begin{bmatrix} 2 & -3 & 6 & -5 & \cdot & 3 \\ 0 & 1 & -4 & 1 & \cdot & 1 \\ 0 & 0 & 0 & 0 & \cdot & k-7 \end{bmatrix}$$

There is no solution if $\rho(A) \neq \rho(C)$

$k - 7 \neq 0 \quad$ or $\quad k \neq 7, \; \rho(A) = 2$ and $\rho(C) = 3.$

There are infinite solutions if $\rho(A) = \rho(C) = 2$

$$k - 7 = 0, \; k = 7 \quad \textbf{Ans.}$$

$$\begin{bmatrix} 2 & -3 & 6 & -5 \\ 0 & 1 & -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$2x - 3y + 6z - 5t = 3 \qquad \ldots(1)$$

$$y - 4z + t = 1 \qquad \ldots(2)$$

Let $t = k_1$ and $z = k_2$

From (2) $\qquad y - 4k_2 + k_1 = 1 \quad$ or $\quad y = 1 + 4k_2 - k_1$

From (1), $2x-3-12k_2+3k_1+6k_2-5k_1 = 3$

or $\quad 2x = 6+6k_2+2k_1$

or $\quad x = 3+3k_2+k_1$

$y = 1+4k_2-k_1$

$z = k_2, \quad t = k_1$

4.40 HOMOGENEOUS EQUATIONS

Let $A_{m \times n}$ be the coefficient matrix

Homogeneous Equations
AX = 0

|A| = 0
Consistent with infinitely many solutions

|A| ≠ 0
Consistent with unique trivial solution, x = 0, y = 0, z = 0

Example 50. Determine the value of λ so that the equations

$$2x+y+2z = 0,\ x+y+3z = 0,\ 4x+3y+\lambda z = 0$$ have a non-ze

Solution.

$$C = \begin{bmatrix} 2 & 1 & 2 & 0 \\ 1 & 1 & 3 & 0 \\ 4 & 3 & \lambda & 0 \end{bmatrix} \sim \overset{R_1 \leftrightarrow R_2}{\begin{bmatrix} 1 & 1 & 3 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 3 & \lambda & 0 \end{bmatrix}} \sim \overset{R_2-2R_1,\ R_3-4R_1}{\begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & -1 & \lambda-12 & 0 \end{bmatrix}} \sim \overset{R_3-}{\begin{bmatrix} 1 & \\ 0 & - \\ 0 & \end{bmatrix}}$$

For infinite solutions R (C) = R (A) = 2 < Number of unknowns

$$\therefore \quad \lambda-8 = 0, \lambda = 8$$

Example 51. Find the values of k such that the system of equations $x+ky+3z = 0,\ 4x+3y+kz = 0,\ 2x+y+2z = 0$ has non-trivial solution.

Solution. The set of equations is written in the form of matrices

$$\begin{bmatrix} 1 & k & 3 \\ 4 & 3 & k \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = B,\ C = (A, B) = \begin{bmatrix} 1 & k & 3 & 0 \\ 4 & 3 & k & 0 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

On interchanging first and third rows we have

$$\sim \begin{bmatrix} 2 & 1 & 2 & 0 \\ 4 & 3 & k & 0 \\ 1 & k & 3 & 0 \end{bmatrix} \sim \overset{R_2-2R_1,\ R_3-\frac{1}{2}R_1}{\begin{bmatrix} 2 & 1 & 2 & 0 \\ 0 & 1 & k-4 & 0 \\ 0 & k-\frac{1}{2} & 2 & 0 \end{bmatrix}} \sim \overset{R_3-\left(k-\frac{1}{2}\right)R_2}{\begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & k-4 \\ 0 & 0 & 2-\left(k-\frac{1}{2}\right)(k- \end{bmatrix}}$$