ANOMALY DETECTION (Outliers)

Date :

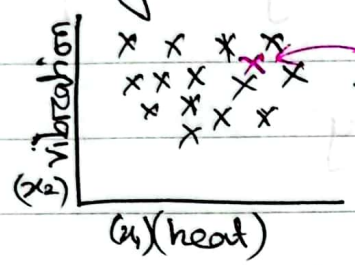
* Machine learning Application
* used for irregular data / unsupervised

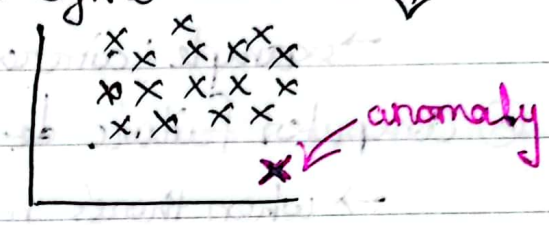
Aircraft engine manufacture example:-

আমার কাছে heat generated + vibration (engine-এর) data আছে

→ m টা engine-এর dataset → নতুন engine এর dataset → (abnormal)

** নতুন engine এর dataset





Anomaly নাকি না বুঝার (mathematically) দুটো model:

1. Univariate Gaussian Model
2. Multivariate Gaussian "

প্রথম একটা model বানাই dataset থেকে → ~~p(x)~~

we access the model by $p(x)$ [probability of (x_{test}) being normal]

ε → threshold probability (value টা - আমাদের দরকার যেটা)

Here if $\boxed{\begin{array}{l} p(x) < \varepsilon \rightarrow \text{anomal} \\ p(x) > \varepsilon \rightarrow \text{normal} \end{array}}$ **

APPLICATIONS:

1. Fraud detection in login/web resource access/credit card:
   - → by detecting normal user behaviour probability
   - → " " normal user's activity model
   - → User's activity associates with
     - * time online length
     - * login frequency.
     - * login location
     - * Credit spending frequency.

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→ unusual user identify by the model

* flag up anything that looks weird
* auto blocking cards/transactions/login attempts.

② Anomal Detection in manufacturing, testing, Quality assurance

→ example : aircraft engine

③ Computer failure ~~detection~~ pre-identify

→ when there's many machines

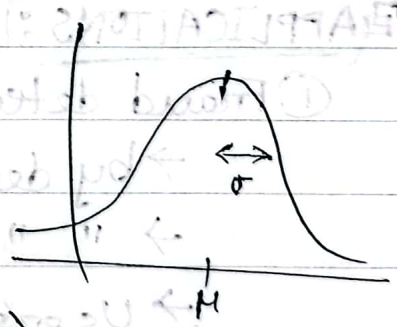
→ things that is accountable : memory use
number of disk accesses
CPU Load, network traffic

→ Defining my own complex feature

anomal behaviour
* CPU network traffic/load
* high network " with low CPU usage (malicious host)
* low " " " high " " (host gone in infinite loop, deadlock)

→ when we see anomalus machine.

GAUSSIAN / NORMAL DISTRIBUTION :



Data set → x

μ → mean / peak

σ → standard deviation (width/spread)

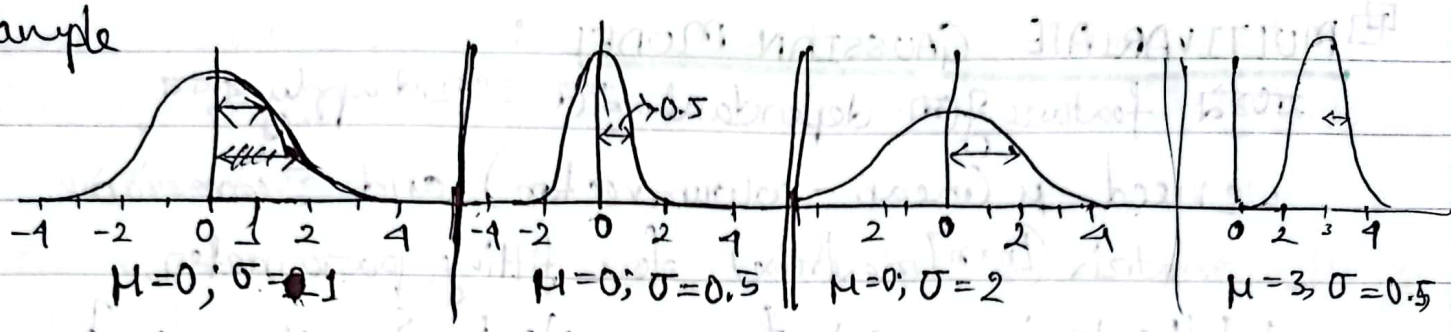
σ^2 → variance

$\sim N(\mu, \sigma^2)$ or $P(x : \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

Area constant ; σ (width) changes

Date :

example



UNIVARIATE GAUSSIAN MODEL :

Informally ① Choose features from dataset (large/small) General properties.

Data set $= \{x1, x2, \dots x(m)\}$ where x_i

② Calculate parameters. ; mean (μ) and standdev (σ)

$\hookrightarrow \mu_j = \frac{1}{m} \sum_{i=1}^{m} x_{j(i)}$

$\hookrightarrow \sigma_j^2 = \frac{1}{m} \sum_{i=1}^{m} (x_{j(i)} - \mu_j)^2$

③ Compute probability $p(x)$ for text point x

$\hookrightarrow p(x) = \prod_{j=1}^{n} p(x_j ; \mu_j, \sigma_j^2)$

$= p(x_1; \mu_1, \sigma_1^2) * p(x_2; \mu_2, \sigma_2^2) * \dots p(x_n; \mu_n, \sigma_n^2)$

$= \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi\sigma_j^2}} e^{\left(-\frac{(x_j - \mu_j)^2}{2\sigma_j^2}\right)}$

Check $p(x) < \varepsilon \rightarrow$ anomaly.

***

Formally :

① Choose features x_i that you think might be indicative of anomalous.

② Fit parameters $\mu_1 \dots \mu_n ; \sigma_1^2 \dots \mu_n^2$; $\mu_j = \frac{1}{m} \sum_{i=1}^{m} x_{j(i)}$; $\sigma_j^2 = \frac{1}{m} \sum_{i=1}^{m} (x_j^{(i)} - \mu_j)^2$

③ compute $p(x) = \prod_{j=1}^{n} p(x_j ; \mu_j, \sigma_j^2) = \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi\sigma_j}} e^{\left(-\frac{(x_j - \mu_j)^2}{2\sigma_j^2}\right)}$

Anomaly if $p(x) < \varepsilon$

CamScanner

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MULTIVARIATE GAUSSIAN MODEL :

যখন feature গুলো dependant হবে তখন apply করব

we need μ (mean - coloumn vector) and Σ covariance matrix ($[n \times n]$ matrix) for fitting parameter.

let the training set of m examples be $\{x_1, x_2 \dots x_m\}$.

so $\mu = \frac{1}{m} \sum_{i=1}^{m} x_i$

$$\Sigma = \frac{1}{m} \sum_{i=1}^{m} (x_i - \mu)(x_i - \mu)^T$$

***

Formal algorithm : ① Fit model; calculate μ, Σ

② We are given with new test example(x)

calculate $p(x)$ by

$$p(x) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{\frac{1}{2}}} e^{\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right)}$$

③ Compare $p(x)$ with ε

if $p(x) < \varepsilon \rightarrow$ anomaly

else normal.

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Example :

** Given heat(x_1) and vibration(x_2) ~~data~~ training data set of aircraft engin → $(x_1, x_2) = (2,2), (3,4), (1,3), (4,4), (5,7), (6,9)$ $(7,8)$ $(10,14)$, $(12,15)$, $(15,19)$

New engine → heat = 10 and vibration = 5. Is the new one faulty?

Threshold probability, $\varepsilon = 0.1$

Ans:

By univariate (but multivariate করা better cause data features গুলো dependent) : Step-1 → x_1, x_2 features.

Step-2

$$\mu 1 = \frac{\Sigma x_1}{m} = \frac{2+3+1+4+5+6+7+10+12+15}{10} = 6.5$$

$$\mu 2 = \frac{\Sigma x_2}{m} = 8.5.$$

$$\sigma_1^2 = \frac{\Sigma (x_1 - \mu)^2}{m} = 18.65. \qquad \sigma_2^2 = \frac{\Sigma (x_2 - \mu)^2}{m} = 29.81$$

Step-3

$$p(10,5) = p(x_1 = 10; \mu_1 = 6.5, \sigma_1^2 = 18.65) * p(x_2 = 5; \mu_2 = 8.5, \sigma_2^2 = 29.81)$$

$$= \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}} * \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}}$$

$$= \frac{1}{4.32 \times 2.5} e^{-\frac{(3.5)^2}{37.3}} * \frac{1}{5.45 \times 2.5} e^{-\frac{(-3.5)^2}{59.62}}$$

$$= 0.067 \times 0.06 = 0.0039.$$

So, $P(10,5) < 0.1$

Thus the aircraft is faulty

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By multivariate :

step-1 calculate μ and Σ covariance

Here features = 2 , $m = 10$

$(m) =$

x_1	x_2
2	2
3	4
1	3
4	4
5	7
6	9
7	8
10	14
12	15
15	19

$\mu = 6.5 \quad 8.5$

$(m-\mu) =$ or, a

x_1	x_2
-4.5	-6.5
-3.5	-4.5
-5.5	-5.5
-2.5	-4.5
-1.5	-1.5
-0.5	0.5
0.5	-0.5
3.5	5.5
5.5	6.5
8.5	10.5

transpos matrix

$a' =$

-4.5	-3.5	-5.5	-2.5	-1.5	-0.5	0.5	3.5	5.5	8.5
-6.5	-4.5	-5.5	-4.5	-1.5	0.5	-0.5	5.5	6.5	10.5

$$a'.a = \begin{bmatrix} 186.5 & 232.5 \\ 232.5 & 298.5 \end{bmatrix}$$

$$\Sigma = \frac{a'.a}{m} = \begin{bmatrix} 18.65 & 23.25 \\ 23.25 & 29.85 \end{bmatrix}$$

step-2

$$|\Sigma| = (18.65 \times 29.85) - (23.25 \times 23.25) = 16.14$$

$$\Sigma^{-1} = \frac{1}{16.14} \begin{bmatrix} 29.85 & -23.25 \\ -23.25 & 18.65 \end{bmatrix} = \begin{bmatrix} 1.85 & -1.44 \\ -1.44 & 1.15 \end{bmatrix}$$

Test data $X = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$ $\quad x - \mu = \begin{bmatrix} 10-6.5 \\ 5-6.5 \end{bmatrix} = \begin{bmatrix} 3.5 \\ -3.5 \end{bmatrix}$

$$(x-\mu)^T = \begin{bmatrix} 3.5 & -3.5 \end{bmatrix}$$

$$(x-\mu)^T * \Sigma^{-1} = \begin{bmatrix} 11.515 & -9.1 \end{bmatrix}$$

$$(x-\mu)^T * \Sigma^{-1} * (x-\mu) = 72.1525$$

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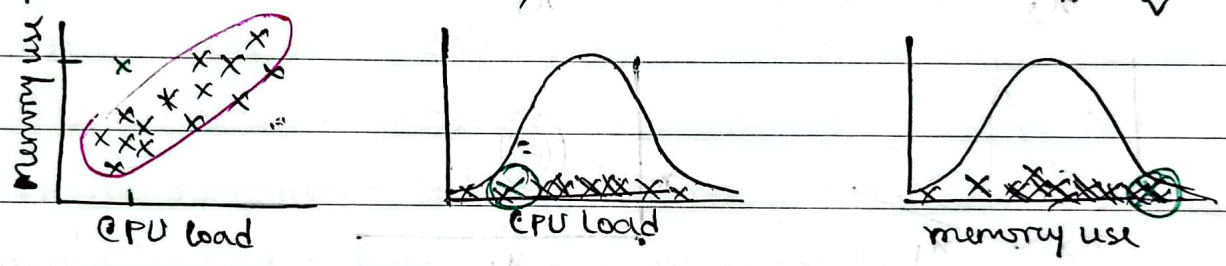
$$P(x) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{\frac{1}{2}}} e^{\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right)}$$

$$= \frac{1}{(6.2832)^{2/2} \sqrt{16.14}} e^{\left(-\frac{1}{2} \times 72.1525\right)} = 8.5143 \times 10^{18}$$

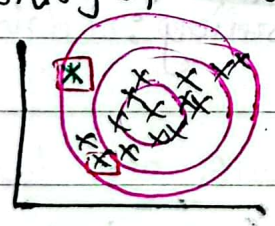
Step-3 $p(x) < 0.1$ so, there's faulty.

*** Why univariate might fail where multivariate works:

→ We use ~~star~~ variance for univariate, So in the graph sometime it may look normal. Example ↓



Our model here make concentric circles around the means. Probability of two red circles are same.



But incase of multivariate we develop $p(x)$ all in one go instead of each feature seperately. We calculate μ, which is an dimensional vector and Σ which is $n \times n$ matrix - the covariance matrix. We use the formula ↓

$$P(x) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} e^{\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right)}$$

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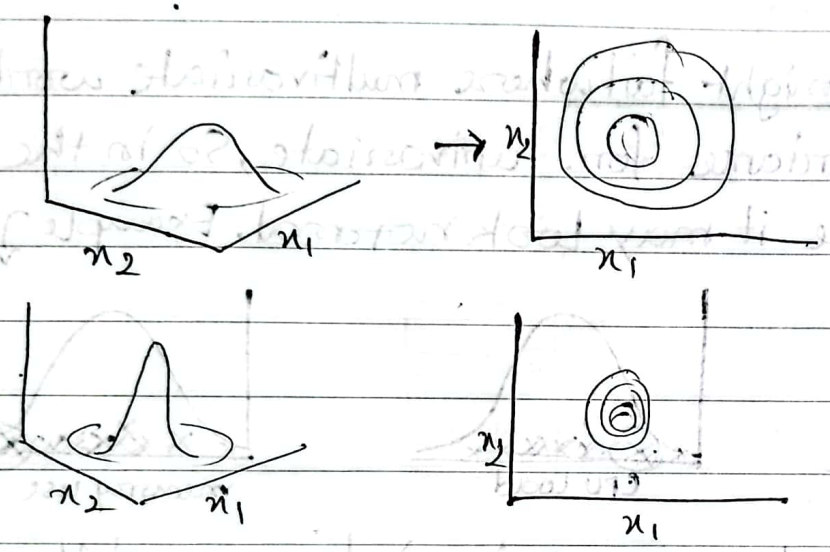
This $p(x)$ gives a 2D model. The width of the bump decreases and height increases and it changes as per data. Therefore, using the values we define the shape of this to better fit the data rather assuming symmetry in every dimension. We can off diagonal values negative to show negative correlation. We can also move the peak.



ANOMALY DETECTION VS Supervised learning : Comparison

→ Where to use ~~Anomaly~~ : Supervised learning :

① Very small no. of positive examples.
- consider using anomaly algo.
- not enough data to learn positive examples

② Very large number of negative example.
- (normal data points / not anomalies)

③ Many types of anomalies
- It's hard for algorithm to detect how anomaly look like but easier to detect what they don't look like

Date :

③ ~~Detecti~~ SPAM email Detection

→ They have similar or common keywords so supervised learning is needed.

④ Reasonably large positive + negative examples, so there's a lot of opportunity for algorithm to learn what anomalies look like

Applications : ① SPAM Detection
② Weather prediction
③ Cancer classification

→ When to use ANOMALY :

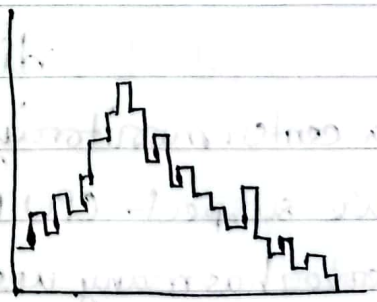
① Fraud detection – sufficient positive examples not available but may use supervised learning if you are a major online retailer – subject to many attacks.

② Manufacturing – huge volumes may make supervised you make assumption about all possible kinds of errors which is unknown.

We can not use Anomaly for non gaussian features.

* NON GAUSSIAN FEATURES :

we need to take out the histogram and check Non gaussian histrogram looks like this →



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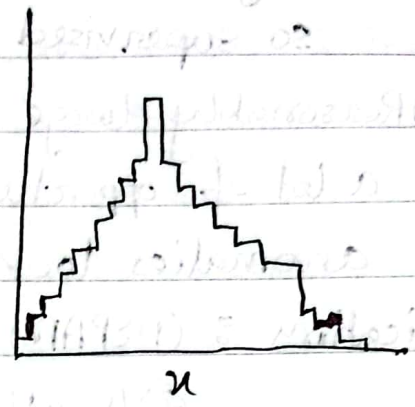
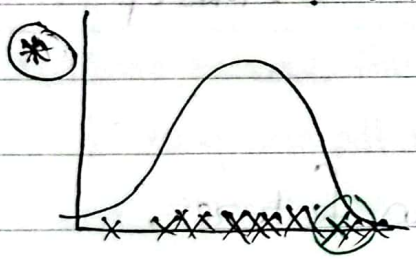
How to make it Gaussian

* play with different transformations.
* might take a log transformation.
* shifted logarithm transformation especially if $x<0$ (negative values) : $\log(x_1+c)$
* play with c to make it look like as gaussian as possible.
* or exponential transformation would do.



FEATURE ENGINEERING from error analysis of anomaly detection systems :

* Good way of (error analysis) coming up with features.
  ↳ ① Run algorithm on CV set., see which ones wrong
  ② Develop features based on trying to understand why algo got those wrong.



p(x) large for normal.
P(x) small for abnormal.

Example x in green is the error.
we need to develop a new feature x_2 to help distinguish the anomalous.

* Data center monitoring example :
  we suspect CPU load and net traffic grow linearly.
  If server has many user — CPU load high, traffic high.
  but → hanged system → infite loop (CPU load grow, network lows)
  BOTNET → CPU load low, network jammed.

Date :

COMPARISON: Univariate vs Multivariate.

Univariate/Original:

① Used more often

② Needs to make extra features manually due to usual combinations.

③ Computationally efficient at no covariance or inverse matrix is used.

④ Cheaper

⑤ Scales better; works good for $n = 100,000$

⑥ Works good for small traing sets; represents optimized but axis-symmtric specialization

Multivariate:

① Used less.

② No need of creating extra features as can capture feature correlation.

③ Less efficient as need to compute inverse matrix

④ Expensive

⑤ Bad scales, bad at $n = 100,000$.

⑥ Needs $m > n$.
- no. of examples > no. of features.
- or else it means it is singular matrix which is non invertible.

Deal with non invertible Matrix:

- when $m < n$. [Examples < features]
  ① We can use univariate [PCA]
  ② Eliminate redundant features (linearly dependent ones) [use sanity check]

Univariate assumptions:

① Gaussian distribution wise assuming each feature.

② Independence assumption.

QUEUING THEORY

Date :

Single Server Exponential Queuing System / M/M/1 (markovian, memoryless)

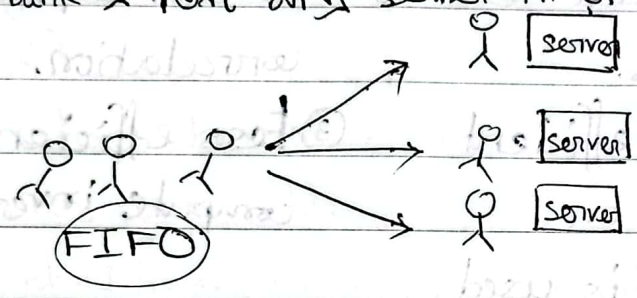
↳ ① Indefinite Queue Capacity (কিছু না বলা থাকলে এটা করব)

② Finite Queue

③ Shoeshine Shop Model

Bank-এ তিনটি মাত্র server থাকায় wait করতে হয় line-এ

System :



customers → data

server → CPU/OS

example

data randomly packet-এ করে চলে আসে modem-এর buffer/queue তে থাকে পরে CPU-কে দেওয়া হয়।

Applications of Queuing theory:

* algorithm design/analysis
* network design
* network performance optimization
* network device design

Infinite Queue Capacity :

(λ) Arrival rate + service rate(μ) (Given)

* Service rate বেশি → system stable.

M/M/1 → Markovian (Discrete time - একই সময়ে দুইটা ঘটনা হবে না)

→ Memoryless (Future-এ যা হবে তা past-এর উপর dependent না)

→ 1 (একটাই server থাকবে; multiple হলে M/M/K হবে)

M/M/K

~~M/M/G/M/M~~

M/G/1 → General (arrival general probability-র উপর dependent)

M/G/K

Date :

math-এ λ, μ দেওয়া থাকবে

বের করব → L: Average no. of person in the system = Lq + served ppl

Lq: " " " " " queue

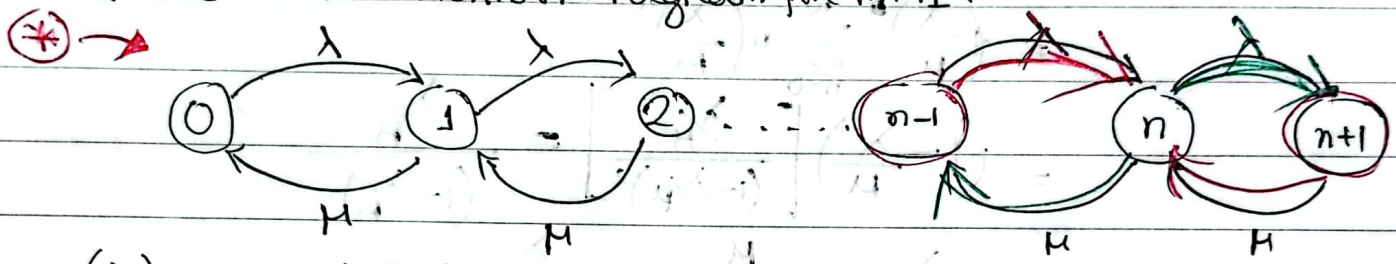
W: time (avg) spent in the sys = Wq + Service time

$= W_q + 1/\mu$

Wq: " " " " " queue

For single server ~~ser~~ sys $\boxed{\text{service rate} = \text{Departure rate}}$

Queue State transition Diagram for M/M/1:



(λ) arrival বাড়া মানে মানুষ বাড়বে

Inflow = Outflow

Inflow = μP_1 = Outflow = λP_0.

λ গুন করব যেখান থেকে বের হবে

μ " " originating node-এর সাথে

inflow outflow

$\lambda P_{n-1} + \mu P_{n+1} = (\lambda+\mu) P_n$

Probability of n person being in the system:

$\mu P_1 = \lambda P_0$ or, $P_1 = \frac{\lambda}{\mu} P_0$.

when $n=0$; $P_1 = \frac{\lambda}{\mu} P_0$

$n=1$; $P_2 = \frac{\lambda}{\mu} P_1 + \left(P_1 - \frac{\lambda}{\mu} P_0\right) = \frac{\lambda}{\mu} P_1 + 0 = \frac{\lambda}{\mu}\frac{\lambda}{\mu} P_0 = \left(\frac{\lambda}{\mu}\right)^2 P_0$

$n=2$; $P_3 = \frac{\lambda}{\mu} P_2 = \left(\frac{\lambda}{\mu}\right)^3 P_0$

$n=4$; $P_4 = \left(\frac{\lambda}{\mu}\right)^4 P_0$

$\boxed{n \rightarrow P_n = \left(\frac{\lambda}{\mu}\right)^n P_0.}$

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Also we know, $\sum_{n=0}^{\infty} P_n = 1 = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n P_0 = \frac{P_0}{(1-\lambda/\mu)}$

So, $P_0 = 1 - \frac{\lambda}{\mu}$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

DERIVATION :

$$L = \sum_{n=0}^{\infty} n P_n = \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

$$= \left(1 - \frac{\lambda}{\mu}\right) \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n$$

$$= \left(1 - \frac{\lambda}{\mu}\right) \left[\frac{\lambda/\mu}{(1-\lambda/\mu)^2}\right] = \frac{\lambda/\mu}{(1-\lambda/\mu)}$$

$$= \frac{\lambda}{\mu} \times \frac{\mu}{(\mu-\lambda)} = \frac{\lambda}{(\mu-\lambda)}$$

$$\boxed{L = \frac{\lambda}{(\mu+\lambda)}}$$

$W = L/\lambda \longrightarrow$ (কারণ 50 customers = 10 min × 5 cust/min
আমি ~~10 min~~ আসছি sys-এ 10 min
হইতে — per min 5জন-এর arrival)

or, $W = \frac{\lambda/(\mu-\lambda)}{\lambda}$

or, $\boxed{W = \frac{1}{(\mu-\lambda)}}$

We know, $W = W_q$ + service time

$= W_q + 1/\text{service rate} = W_q + 1/\mu$

So, $\boxed{W_q = W - 1/\mu = \frac{\lambda}{\mu(\mu-\lambda)}}$

$$L_q = \lambda W_q = \lambda \cdot \frac{\lambda}{\mu(\mu-\lambda)}$$

$$= \frac{\lambda^2}{\mu(\mu-\lambda)}$$

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Exple : per 12 minutes 1 customer arrives, service rate = 8 min per;

→ Here, $\left(\lambda = \frac{1}{12} \quad \mu = \frac{1}{8}\right)$ persons/min.

so, $\cancel{L = \frac{\lambda}{(\mu-\lambda)} = \frac{1/12}{1/8 - 1/12} = \frac{1/12}{4/12} = \frac{1}{4}}$

$$L = \frac{\lambda}{\mu-\lambda} = \frac{1/12}{1/8 - 1/12} = \frac{1/12}{\frac{3-2}{24}} = \frac{24}{12} = 2.$$

$$W = \frac{1}{\mu-\lambda} = \frac{1}{1/8 - 1/12} = \frac{1}{\frac{3-2}{24}} = 24$$

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{1/12}{1/8\left(\frac{3-2}{24}\right)} = \frac{1/12}{\frac{1}{192}} = 16$$

$$L_q = \lambda W_q = \frac{1}{12} \times 16 = \frac{4}{3} \quad \text{(Ans)}$$

** Probability of server being IDLE ?

→ means no customer.

so $\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n P_0 = 1$

or, $\frac{P_0}{\left(1 - \lambda/\mu\right)} = 1$

or, $P_0 = \left(1 - \frac{\lambda}{\mu}\right)$.

** Probability of at least 3 customers ?

→ $P3 + P^4 + P^5 + \dots = 1 - P^0 - P^1 - P^2$

Date :

Shoeshine Shop :

No queue – system occupied দেখলে চলে যাব

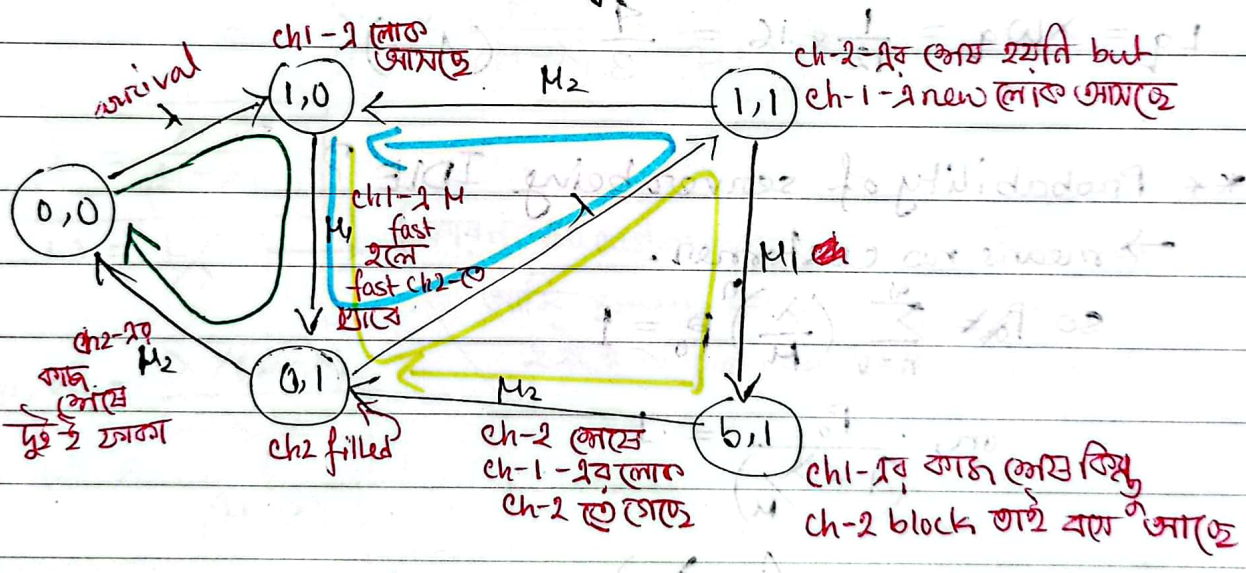
দুইটা chair → 1 2

possibilities	1	2
	0	0
	1	0
	1	1
	1	1
	b	1 (ch1 blocked)

arrival একটাই (λ)

service rate দুইটা – ch1 3 ch2-এর

probability state transition diagram.



State : outflow = inflow

(0,0) $\lambda P_{00} = \mu_2 P_{01}$

(1,0) $\mu_1 P_{10} = \lambda P_{00} + \mu_2 P_{11}$

(0,1) $\mu_2 P_{01} + \lambda P_{01} = \mu_1 P_{10} + \mu_2 P_{b1}$

(1,1) $\mu_1 P_{11} + \mu_2 P_{11} = \lambda P_{01}$

(b,1) $\mu_2 P_{b1} = \mu_1 P_{11}$

$$\boxed{P_{00} + P_{01} + P_{10} + P_{11} + P_{b1} = 1}$$

Date :

Exple: Given, $\lambda = 1$, $\mu_1 = 1$, $\mu_2 = 2$.

So, putting the values, $1 \cdot P_{00} = 2P_{01}$ —— ①

$P_{10} = P_{00} + 2P_{11}$ —— ②

$3P_{01} = P_{10} + 2P_{b1}$ —— ③

$3P_{11} = P_{01}$ —— ④

$2P_{b1} = P_{11}$ —— ⑤

$P_{00} + P_{01} + P_{10} + P_{11} + P_{b1}$ —— ⑥

Here, putting value of P_{10}

$P_{00} + (P_{00} + 2P_{11}) + \frac{1}{3}(P_{10} + 2P_{b1}) + P_{11} + \frac{1}{2}P_{11} = 1$

$\Rightarrow 2P_{00} + 2P_{11} + \frac{1}{3}(P_{00} + 2P_{11}) + \frac{2}{3}\frac{P_{11}}{2} + P_{11} + \frac{1}{2}P_{11} = 1$

$\Rightarrow 2P_{00} + 2P_{11} + \frac{1}{3}P_{00} + \frac{2}{3}P_{11} + \frac{P_{11}}{3} + P_{11} + \frac{1}{2}P_{11} = 1$

$\Rightarrow \frac{7}{3}P_{00} + \frac{9}{3}P_{11} = 1$

$\Rightarrow \frac{7}{3}P_{00} + \frac{9}{3\times 3}P_{01} = 1$

$\Rightarrow \frac{7}{3}P_{00} + \frac{1}{2}P_{00} = 1 \Rightarrow \frac{17}{6}P_{00} = 1$

$P_{00} + (P_{00} + 2P_{11}) + \frac{1}{2}P_{00} + P_{11} + \frac{1}{2}P_{11} = 1$

$\Rightarrow 2P_{00} + 2P_{11} + \frac{1}{2}P_{00} + P_{11} + \frac{1}{2}P_{11} = 1$

$\Rightarrow \frac{5}{2}P_{00} + \frac{7}{2}P_{11} = 1$

$\Rightarrow \frac{5}{2}P_{00} + \frac{7}{6}P_{01} = 1 \Rightarrow \frac{5}{2}P_{00} + \frac{7}{12}P_{00} = 1 \Rightarrow P_{00} = \frac{12}{37}$

So, $P_{01} = \frac{P_{00}}{2} = \frac{6}{37}$ $\quad P_{10} = \frac{12}{37} + \frac{4}{37} = \frac{16}{37}$

$P_{11} = \frac{2}{37}$

$P_{b1} = \frac{1}{37}$

CamScanner

Date :

* Portion of potential customer to enter the system?

মানে প্রথম chair ফাকা না থাকলে কেমনে কি

তাই ch1 ফাকা থাকার probability বলব

$P_{00} + P_{01} = \frac{12}{37} + \frac{6}{37} = \frac{18}{37}$

does not enter বললে

$1 - (P_{00} + P_{01})$

* Mean no. of customers in system? (L)

$L = \text{Average}(x) = E[x] = \sum x P(x)$

$= 0 * P_{00} + 1 * P_{01} + 1 * P_{10} + 2 * P_{11} + 2 * P_{01}$

$= 0 + \frac{6}{37} + \frac{16}{37} + \frac{4}{37} + \frac{2}{37}$

$= \frac{28}{37}$

* average amount of time an entering customer spent? (W)

$W = L/\lambda$ ~~$= \frac{28}{37} \times \frac{1}{1} =$~~

but λ won't work directly cause এর arrival rate not effective as ~~all pe~~ some people not entering

So, $W = \frac{L}{\lambda_a}$ → effective arrival rate.

$\lambda_a = \lambda (P_{00} + P_{01}) = \frac{18}{37}$.

$W = \frac{28}{37} \times \frac{37}{18} = \frac{14}{9}$ (Ans).

Date :

Finite

আমরা জানি $1 = \sum_{n=0}^{\infty} P_n = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n$

যদি ∞ না হয়ে capacity $= N$ বলে দেয়

$$1 = P_0 \sum_{n=0}^{N} \left(\frac{\lambda}{\mu}\right)^n = P_0 \left[\frac{1-(\lambda/\mu)^{N+1}}{1-\lambda/\mu}\right]$$

or, $$P_0 = \left[\frac{1-\lambda/\mu}{1-(\lambda/\mu)^{N+1}}\right]$$

Probability/Fraction → both chair empty : P_{00}

ch1 is empty : $(P_{00}+P_{01})$

ch2 " " : $(P_{00}+P_{10})$.

ch2 is filled : $(P_{01}+P_{11}+P_{b1})$

either or both filled : $(1-P_{00})$ or $(P_{01}+P_{10}+P_{11}+P_{b1})$

both are filled : $P_{11}+P_{b1}$

MARKOV CHAIN → Population problem
→ Gambler's ruin "
→ Random walk "

Date :

For state i there is a fixed probability P_{ij} for next state j.
If X_{n+1} depends on X_n and not on $X_{n-1}, X_{n-2} \cdots$
the process is called Markov Chain.

Transition Probability Matrix →

$$P = \begin{vmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ \vdots & & & \\ P_{i0} & P_{i1} & P_{i2} & \cdots \\ \vdots & & & \end{vmatrix}$$

Example-1

Given, probability of raining today, raining tomorrow $= \alpha$
" " not raining ", but " " $= \beta$.

let there be 2 states → 0 = It rains
1 = not rains.

$$\text{So, } P = \begin{vmatrix} P_{00} & P_{01} \\ \alpha & 1-\alpha \\ P_{10} & P_{11} \\ \beta & 1-\beta \end{vmatrix}$$

P_{00} হচ্ছে বুঝায়, rains today, rains tomorrow
P_{01} " " " " , not tomorrow → $P_{00}+P_{01}=1$

P_{10} " " not rain " , but rains tomorrow
P_{11} " " " " " not " " $P_{10}+P_{11}=1$

Exple-3. Given, C = Cheerful ; S = So So , G = Gloomy.

if Gary's cheerful today then tomorrow he will be C = 0.5, S = 0.4, G = 0.1
" " soso " " tomorrow " " " C = 0.3, S = 0.4, G = 0.3
" " gloomy " " " " " " C = 0.2, S = 0.3, G = 0.5

So, transition Probability Matrix =

	C	S	G
C	0.5	0.4	0.1
S	0.3	0.4	0.3
G	0.2	0.3	0.5

Date :

যদি ques to আসে হতো – He is cheerful, soso, gloomy today then tomorrow. he will be cheerful with 0.5, 0.3, 0.2 ; so so with 0.4, 0.4, 0.3 probability and gloomy with 0.1, 0.3, 0.5 probability – তাহলে colour wise সাজাতাম

Markov chain-এর দুইটা model → ① Random walk
② Gambling

** যদি Markov Chain – আগের দুই বা অধিক state-এর উপর next state depend করে, then এগুলিকে change করব
cause এটা markov chain হয় না but change করে Markov Chain বানাব।

*** let rain ~~p=0.7~~ ✓ ~~rains rained today~~
not rains, ~~p=0.4~~. ✗

Consider three dates:
✓ yesterday = Sunday
✓ today = Monday = yesterday
tomorrow = Tuesday = today.

4 states:
0 : rained today ; rained yesterday
1 : " " ; not " "
2 : not " " ; rained "
3 : not rained today - yesterday.

So, $P_{00} = \underbrace{S\ M}_{0}\ \underbrace{M\ T}_{0} = 0.7$

$P_{01} = \underbrace{SM}_{0}\ \underbrace{MT}_{1} = 0$ (cause M conflict তিনের $\bar{M}$ vs $\bar{M}$)

Date :

~~$R_{10} = \underbrace{S\ m}_{1}\ \underbrace{m\ T}_{0} = 1-0.7 = 0.3$~~.

$P_{02} = \underbrace{S\ m}_{0}\ \underbrace{m\ T}_{2} = 1-0.7 = 0.3$.

$P_{03} = \underbrace{S\ m}_{0}\ \underbrace{m\ T}_{3} = 0$

$P_{10} = \underbrace{S\ m}_{1}\ \underbrace{m\ T}_{0} = 0.5$

$P_{11} = \underbrace{S\ m}_{1}\ \underbrace{m\ T}_{1} = 0$

$P_{12} = \underbrace{S\ m}_{1}\ \underbrace{m\ T}_{2} = 0.5$

$P_{13} = \underbrace{S\ m}_{1}\ \underbrace{m\ T}_{3} = 0$

$P_{20} = S\ m\ \ m\ T = 0$

$P_{21} = S\ m\ \ m\ T = 0.4$

$P_{22} = S\ m\ \ m\ T = 0$

$P_{23} = S\ m\ \ m\ T = 1-0.4 = 0.6$

$P_{30} = S\ m\ \ m\ T = 0$

$P_{31} = S\ m\ \ m\ T = 0.2$

$P_{32} = S\ m\ \ m\ T = 0$

$P_{33} = S\ m\ \ m\ T = 0.8$

given

rained two days, rained tomorrow 0.7

rained today, tomorrow not yesterday 0.5

rain tomorrow, yesterday not today = 0.4

rain tomorrow but not today, yesterday = 0.2

$$P = \begin{vmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{vmatrix}$$

Date :

Chapman-Kolmogorov Equation: use to get n^{th} state probability P^{n}_{ij}.

$$\hookrightarrow P^{n+m}_{ij} = \sum_{k=0}^{\alpha} P^{n}_{ij} P^{m}_{kj}$$

Exple-4 rains ; $\alpha = 0.7$

not rain ; $\beta = 0.4$.

Calculate probability that it will rain 4 days from today that it rained today. (P^4 বের করব)

So, $P = \begin{vmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{vmatrix}$ $P^2 = \begin{vmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{vmatrix} \begin{vmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{vmatrix} = \begin{vmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{vmatrix}$

$P^4 = \begin{vmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{vmatrix} \begin{vmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{vmatrix} = \begin{vmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{vmatrix}$

as rains today + rains 4 days later $= P^4_{00} = 0.5749$.

Exple-5 Transformation-এর matrix use করব

If monday, tuesday rains, what probability for raining on Thursday?

Thursday - tuesday-এর দুইদিন পর, So P^2_{00} বের করা

** যদি বলে Probability that it won't rain on Thursday but rained on monday-tuesday. $\Rightarrow P^2_{02} + P^2_{03}$

** যদি বলে rained thursday ; not monday+tuesday $\rightarrow P^2_{30} + P^2_{31}$

** Probability of of GARY being cheerful after four days, today Gloomy $\rightarrow P^4_{30}$.

** GARY cheerful today - cheerful 4 day later $\rightarrow P^4_{00}$

Date :

** GARY not cheerful 4 days later but today Cyborny.

$\hookrightarrow P^4_{31} + P^4_{32}$

GAMBLER'S RUIN PROBLEM :

Game-এ জিতার probability $= P$ (gains coin)

হারার probability $= 1-P = q$ (gives coin)

Total coin $= N$

So $P_{00} = P_{NN} = 1$

কোনো coin নাই

total coin-এ gambler-কে পাওয়া

$P_{i,i+1} = P = 1 - P_{i,i-1}$

$\Rightarrow P_i = pP_{i+1} + qP_{i-1}$

$\Rightarrow P_{i+1} - P_i = \frac{q}{P}(P_i - P_{i-1})$

so $P_i - P_1 = P_1\left[\left(\frac{q}{P}\right) + \left(\frac{q}{P}\right)^2 + \cdots + \left(\frac{q}{P}\right)^{i-1}\right]$

if $\frac{q}{P} = 1$; then $P_i = P_1[1+1+\cdots+1] = iP_1$ (fair coin toss)

if $\frac{q}{P} \neq 1$, then $P_i = P_1\left[\frac{1-(q/P)^i}{1-(q/P)}\right]$

if $\frac{q}{P} \neq 1$ then $P_N = \left[\frac{1-(q/P)^N}{1-(q/P)}\right]P_1$

$$\Rightarrow P_1 = \left[\frac{1-(q/P)}{1-(q/P)^N}\right]$$

if $q/P = 1$ then $P_N = NP_1$

$$\Rightarrow P_1 = \frac{1}{N}$$

Date :

So,

$$P_N = \begin{cases} \left[\dfrac{1-\left(\frac{q}{p}\right)^N}{1-\left(\frac{q}{p}\right)}\right] P_1, & \text{if } \frac{q}{p} \neq 1 \\ N P_1, & \text{if } \frac{q}{p} = 1 \end{cases}$$

$$P_1 = \begin{cases} \left[\dfrac{1-\left(\frac{q}{p}\right)}{1-\left(\frac{q}{p}\right)^N}\right], & \text{if } P_N \neq \frac{1}{2} \\ \frac{1}{N}, & \text{if } P_N = \frac{1}{2} \end{cases}$$

$$P_i = \begin{cases} \left[\dfrac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N}\right] & ; p \neq \frac{1}{2} \\ \frac{i}{N} & ; p = \frac{1}{2} \end{cases}$$

Example: Given Patty's winning probability = 0.6.

** if Patty ~~coin~~ has 5 coins, Max has 10, what's probability Patty will WIPE OUT MAX.

here, patty; $p = 0.6$; $q = 0.4$.

$N = 5 + 10 = 15$.

as $\frac{q}{p} \neq 1$ so,

$$P_5 = \left[\frac{1-\left(\frac{0.4}{0.6}\right)^5}{1-\left(\frac{0.4}{0.6}\right)^{15}}\right] = 0.87$$

Practice 1:

Given. Barry - Garry play throwing fair dice (2).

If outcome is greater than or equal to 10; then Barry wins.

Barry starts with 80 \$ and Garry with 20 \$

Winner receives 10 \$.

(Probability দেয়া নেই) so → বিজয় condition : 10 or বেশি হলে outcome

$$P = P\{sum = 10\} + P\{sum = 11\} + P\{sum = 12\}.$$

$$= P\{(4,6)(6,4)(5,5)\} + P\{(5,6)(6,5)\} + P\{(6,6)\} = \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{6}{36} = \frac{1}{6}$$

CamScanner

Date :

$q = 1 - \frac{1}{6} = \frac{5}{6}$.

For Barry will wipe garry out :

Again here, 10\$ = 1 coin. for us.

So, Barry has 80\$ or 80/10 = 8 coins

Garry " ~~80~~ 20\$ " 20/10 = 2 coins

$N = 2 + 8 = 10$.

So, $P_8 = \left[\dfrac{1 - \left(\frac{5/6}{1/6}\right)^8}{1 - \left(\frac{5/6}{1/6}\right)^{10}}\right]$ as $q/p \neq 1$

$= 0.0399$.

Practice 2: Barry Garry flipping 2 coins; If at least 1 head comes Barry ~~will~~ wins: Winner receives 20\$. Barry has 40\$ Garry has 120\$, Probability of Barry wipe out Garry?

$p = p\{1 \text{ head}\} + p\{2 \text{ heads}\} = p\{(H,T)(T,H)\} + p\{H,H\}$.

$= 2/4 + 1/4 = 3/4 = 0.75$

$q = 1 - 0.75 = 0.25$.

starting ~~dollar~~ coins $= (40+120) = 160\$$

$N = 160/20 = 8$ coins

Barry has $= 40/20 = 2$ coins

So, $P_2 = \left[\dfrac{1 - \left(\frac{0.25}{0.75}\right)^2}{1 - \left(\frac{0.25}{0.75}\right)^8}\right] = 0.89$.

BAYES THEOREM

Date :

$P(A|B)$

↳ A হওয়ার probability ; B already happend

Throw a die → outcomes : {1, 2, 3, 4, 5, 6}.

$P(2) = 1/6$

$P(2|\text{Even}) = 1/3$

$P(2|\text{Odd}) = 0$

$P(2|\text{outcome} < 3) = 1/2$

$$P(A|B) = \frac{P(B|A) * P(A)}{P(B)}$$

$$P(B) = P(A) * P(B|A) + P(A') * P(B|A')$$

Problems :

** Box contains red balls, ~~blue~~ white balls.

1st Box → red = 60 , white = 40. ; total = 100

2nd " → red = 10 , white = 20 ; total = 30

1 box is selected and a ball is selected ; Probability that red ball selected ; $P(A) = ?$

Sol^n : $P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2)$

$= \left(\frac{1}{2} \times 60/100\right) + \left(\frac{1}{2} \times 10/30\right)$

$= 7/15.$

** probability of that ball being from box 1 ?

$\rightarrow P(B_1|A) = \frac{P(B_1)\,P(A|B_1)}{P(A)} = \frac{\frac{1}{2} \times 60/100}{7/15} = 9/14.$

(for Box 2) $\rightarrow P(B_2|A) = \frac{P(B_2)\,P(A|B_2)}{P(A)} = \frac{\frac{1}{2} \times 10/100}{7/15} = 0.107$

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Date :

** machines producing items:

m1 produces 20% items ; 1% defective

m2 " 30% " ; 2% "

m3 " 50% " ; 3% "

- determine the probability that it was from m2.

Let M_i = items selected by machines.

$P(M_1) = 0.2$

$P(M_2) = 0.3$

$P(M_3) = 0.5$

D_B = defected item.

$P(D|M_1) = 0.01$

$P(D|M_2) = 0.02$

$P(D|M_3) = 0.03$

$$P(M_2|D) = \frac{P(M_2)P(D|M_2)}{P(D)}$$

$$= \frac{P(M_2)\cdot P(D|M_2)}{P(M_1)P(D|M_1)+P(M_2)P(D|M_2)+P(M_3)P(D|M_3)}$$

$$= \frac{0.3 \times 0.02}{(0.2)(0.01)+(0.3)(0.02)+(0.5)(0.03)}$$

$$= 0.26$$

** Murder case :

60% Homicide case — wife got killed + husband murderer.

Blood found matched husband but also 20% population.

Probability of husband being murderer with evidence (blood)?

Here, $P(H) = 0.6$; $P(H^c) = 0.4$

$P(E|H) = 1$ → husband killer + evidance match him.

$P(E|H^c) = 0.2$ → evidance match 20% population.

$$P(H|E) = \frac{P(H)P(E|H)}{P(H)P(E|H)+P(H^c)P(E|H^c)}$$

$$= \frac{0.6 \times 1}{(0.6 \times 1)+(0.4 \times 0.2)} = 0.882$$

Date :

** Test detects .98% of a disease.

rest 2% is no (1% is false positive)

0.5% population has it..

if you test positive, what probability you actually have it?

let, E = test positive

H = I have it.

$P(E|H) = 0.98$

$P(E|H^c) = 0.001$

$P(H) = 0.005$; $P(H^c) = 0.995$

$$P(H|E) = \frac{\cancel{P(E)}\, P(H)\, P(E|H)}{P(H)\{P(E|H) + P(H^c)\, P(E|H^c)}$$

$$= \frac{0.98 \times 0.005}{(0.98)(0.005) + (0.001)(0.995)}$$

$$= 0.33$$

RANDOM VARIABLE

Date :

↳ quantity takes values from random Experiment

* determined by outcomes.
* assinged (fixed) probabilities to the possible values.

Random experiments : dice throw, coin throw.

** Expectation of Random Variable : $E[X] = \sum_{x, P(x)>0} x \cdot p(x)$

random. Var.

probability

Maths : (Sheldon Ross Book)

(2.1) 2 faire dice is thrown.

Probabilities → $P\{x=2\} = P\{1,1\} = \frac{1}{36}$

$P\{x=3\} = P\{(2,1)\ (1,2)\} = \frac{2}{36}$

$P\{x=4\} = P\{(2,2)\ (3,1)\ (1,3)\} = \frac{3}{36}$

$P\{x=5\} = P\{\cancel{(4,3)}\ \cancel{(3}\ (4,1)\ (1,4)\ (3,2)\ (2,3)\} = \frac{4}{36}$

$P\{x=6\} = P\{(1,5)\ (5,1)\ (2,4)\ (4,2)\ (3,3)\} = \frac{5}{36}$

$P\{x=7\} = P\{(1,6)\ (6,1)\ (2,5)\ (5,2)\ (3,4)\ (4,3)\} = \frac{6}{36}$

$P\{x=8\} = P\{\cancel{(1,7)}\ \cancel{(7,1)}\ (2,6)\ (6,2)\ (3,5)\ (5,3)\ (4,4)\} = \frac{5}{36}$

$P\{x=9\} = P\{(3,6)\ (6,3)\ (4,5)\ (5,4)\} = \frac{9}{36}$

$P\{x=10\} = P\{(5,5)\ (4,6)\ (6,4)\} = 3/36$

$P\{x=11\} = P\{(6,5)\ (5,6)\} = \frac{2}{36}$

$P\{x=12\} = P\{(6,6)\} = 1/36$

So, $E[X] = \left(2 \times \frac{1}{36} + 3 \cdot \frac{2}{36} + 4 \cdot \frac{3}{36} + \cdots\cdots + 12 \times \frac{1}{36}\right)$.

$= 7.$

Date :

(2.2) Given, 2 fair coin thrown — $Y = 0, 1, 2$ where Y is no. of Heads.

So $P\{Y=0\} = P\{(T,T)\} = 1/4$

$P\{Y=1\} = P\{(H,T)(T,H)\} = \frac{2}{4}$

$P\{Y=2\} = P\{(H,H)\} = 1/4$

So, $P = 1 = \frac{1}{4} + \frac{2}{4} + \frac{1}{4}$

$E[Y] = (0 \times 1/4) + (1 \times 2/4) + (2 \times 1/4)$

$= 1.0$

RV 2 types → (1) Discrete (countable) — PMF
- → geometric
- → bernoulli
- → Binomial (mostly used)
- → poisson (λ)

(2) Continuous (uncountable) — PDF
- → uniform
- → Exponential ($1/\lambda$)
- → Gaussian.

Discrete:

Probability mass function $p(a)$ of x → $p(a) = P\{x = a\}$

x → random variable; a → given value

Cumulative distribution function F,

$F(a) = \sum p(x_i)$. where $F(a) = P\{x <= a\}$.

⇒ Bernoulli: 1 trial. → $P(1) = P\{x=1\} = p$ (success)

$P(0) = P\{x=0\} = (1-p)$ (failure).

success range : $0 \le p \le 1$.

⇒ Geometric : A coin tossed

N = no. of flips before first head.

heads probability = p ; no heads = 1 − p.

Date :

$P\{N=1\} = P\{H\} = p$

$P\{N=2\} = P\{(T,H)\} = (1-p)(p)$

$P\{N=3\} = P\{(T,T,H)\} = (1-p)(1-p)p = (1-p)^2 p.$

$\vdots$

$P\{N=x\} = P\{\underbrace{\overbrace{T, T \ldots T}^{x}, H}_{x-1}\} = (1-p)^{x-1} \cdot p \quad x \geq 1$

⇒ Bionomical : (multiple trial)

probability of i success → $p(i) = \binom{n}{i} p^i (1-p)^{n-i}$

Maths:

(2.6) Given → 4 fair coin flipped.

Probability of 2H and 2T = ?

$P\{x=2\} = \binom{4}{2} p^2 (1-p)^2$

$= \binom{4}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2$

$= 0.375$ or $3/8$!

** যদি বলতো → 3 heads, 1 tail.

Then; $P\{x=3\} \binom{4}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^1 = 1/4$

↳ cause 3H → {(H,H,H,T)(HHHH)}
so $2/4 = 1/2$

(2.7) 0.1 probability of defective item in a machine

probability of at most 1 defective among 3 :

let, $n=3$, $i = 1, 0$; At most 1 ~~এর মধ্যে~~

$P\{x=0\} + P\{x=1\} = \binom{3}{0}(0.1)^0(1-0.1)^3 + \binom{3}{1}(0.1)^1(1-0.1)^2$

Date :

$= 0.972$.

** At least 1 defected বললে $i = 1, 2, 3$.

** no more than 1 " " $i = 0, 1$

** no less than 1 " " $i = 1, 2, 3$.

** at most 2 defected " $i = 0, 1, 2$

(2.8) Airplane crashing probability $= 1-p$.

success flight needs 50% engine working.

For what value of p a 4 engine or 2 engine plane is preferable?

Sol^n: 4 engine: $\binom{4}{2} p^2 (1-p)^2 + \binom{4}{3} p^3 (1-p)^1 + \binom{4}{4} p^4 (1-p)^0$.

$= 6p^2(1-p)^2 + 4p^3(1-p) + p^4$. [at least 2 works]

at least 1 works

2 engine: $\binom{2}{1} p^1 (1-p)^1 + \binom{2}{2} p^2 (1-p)^0$

$= 2p(1-p) + p^2$

so, 4 engined plane is safer if

$$6p^2(1-p)^2 + 4p^3(1-p) + p^4 \geq 2p(1-p) + p^2$$

$$\Rightarrow 6p^2(1-p)^2 + 4p^3(1-p) \geq 2p(1-p) - p^4 + p^2$$

$$\Rightarrow 6p(1-p)^2 + 4p^2(1-p) \geq \cancel{2(1-p) - p^3}\ 2p(1-p) - p^3 + p$$

$$\Rightarrow 6p(1-2p+p^2) + 4p^2 - 4p^3 \geq 2 - 2p - p^3$$

$$\Rightarrow 6p - 12p^2 + 6p^3 + 4p^2 - 4p^3 \geq 2 - 2p - p^3$$

$$\Rightarrow 8p - 8p^2 + 3p^3 \geq 2$$

$$\Rightarrow 3p^3 - 8p^2 + 7p - 2 \geq 0$$

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$(p-1)^2(3p-2) \geq 0$.

So, $p \geq \frac{2}{3}$

If probability fall than $2/3$ then 2 engines are forfeited.

$\Rightarrow$ Poisson: for i, $P(i) = P\{X=i\} = e^{-\lambda}\frac{\lambda^i}{i!}$; $i = 0, 1 \ldots$

2.10 Typographical errors in a single page; $\lambda = 1$.
probability of at least one error = ?

Sol^n: $P\{x \geq 1\} = P\{1 \text{ error}\} + P\{2 \text{ error}\} + \ldots$

$= 1 - P\{0 \text{ error}\}$

$= 1 - P\{x=0\}$

$= 1 - e^{-1}\frac{1^0}{0!} = 1 - e^{-1} = 0.633$.

2.11 highway accident; $\lambda = 3$; probability of no accident today = ?

Sol^n: $P(x=0) = \frac{e^{-3}\,3^0}{0!} = e^{-3} = 0.05$

*** α-particle example where average 3.2 α is given off.
probability of no more than 2 α particles appearance.

So, $P(x \leq 2) = e^{-3.2}\frac{3.2^0}{0!} + e^{-3.2}\frac{3.2^1}{1!} + \frac{3.2^2}{2!}e^{-3.2}$.

$= e^{-3.2}(9.32) = 0.38$.

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Continous RV: Probability Density Faunction.

⇒ Uniform : range - এর ভেতর হলেই probability

$P\{a \le x \le b\} = \int_a^b f(x)\,dx$

$P\{x \le a\} = \int_{-\infty}^{a} f(x)\,dx$

$P\{x \ge a\} = \int_a^{\infty} f(x)\,dx$

define করতে বললে : Uniformly distributed over the interval (0,1) if its probability density function (PDF) is given by $f(x) = \begin{cases} 1, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$

or, $(\alpha, \beta) \rightarrow$ PDF $\rightarrow f(x) = \begin{cases} \frac{1}{\beta-\alpha}, & \text{if } \alpha<x<\beta \\ 0, & \text{otherwise} \end{cases}$

(2.14) x is uniformly distributed over (0,10)

Steps: ① আগে PDF বের করব $\frac{1}{\beta-\alpha}$

② integration করব.

Here PDF,

$$\frac{1}{10-0} = \frac{1}{10}$$

Now, $P\{x<3\} = \int_0^3 \frac{dx}{10} = \frac{3}{10}$

$P\{x>7\} = \int_7^{10} \frac{dx}{10} = \frac{3}{10}$

$P\{1<x<6\} = \int_1^6 \frac{dx}{10} = \frac{5}{10} = \frac{1}{2}$.

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⇒ Exponential Random Var: life time মাপি ; $\lambda > 0$.

a continuous rv whose PDF for some $\lambda > 0$ is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

Memoryless এটা – কতদিন পাড় হইছে তার উপর Lifetime depend না!

mean দেওয়া থাকলে → $\lambda = \frac{1}{\text{mean}}$.

Example: Math শুরুতে formula derive করব

** Exponentially distributed time of a person is bank with mean 10 mis ; এর মানে $\lambda = \frac{1}{10}$

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} \lambda e^{-\lambda y} dy = \int_{0}^{x} \lambda e^{-\lambda y} dy$$

$$= -\lambda \frac{1}{\lambda} \left[e^{-\lambda y} \right]_0^x$$

$$= -\left(e^{-\lambda x} - e^{0} \right) = 1 - e^{-\lambda x}$$

~~Here,~~ $P(X > x) = 1 - (1 - e^{-\lambda x})$

$$= \cancel{1} - \cancel{1} + e^{-\lambda x}$$

$$= e^{-\lambda x}$$

for more than 15 minutes:

here, $P(X > 15) = e^{-15 \cdot \frac{1}{10}} = 0.23$

More than 15 minute that, she still is in bank for 10 min

$$P(X > 5) = e^{-5 \frac{1}{10}} = 0.604.$$

Date:

Memory less Property:

*** Exponential Distribution: RV x is memoryless. if

$$P\{x > s+t \mid x > t\} = P\{x > s\}. \quad \text{for all } s, t \geq 0$$

let x be the lifetime of some instrument.
It has already survived t time and it lives for at least s more ~~to~~ time. Both period is same that is the instrument does not know that it has been already used for t time.
The condition for this equation is:

$$\frac{P\{x > s+t, x > t\}}{P\{x > t\}} = P\{x > s\}.$$

$$\Rightarrow P\{x > s+t\} = P\{x > s\} \cdot P\{x > t\}$$

So when x is exponentially distributed for $e^{-\lambda(s+t)} = e^{-\lambda s} \cdot e^{-\lambda t}$, it follows that exponentially distributed random variables are memoryless.

*** Geometric Distribution: let x be memoryless for $s, t \geq 0$.
Here, PMF for geometric rv is $f(x) = p(1-p)^x$; $x = 0, 1, 2 \ldots$
probability that X is equal or greater than x
is $P(X \geq x) = (1-p)^x$; $x = 0, 1, 2 \ldots$
The conditional probability of interest, $P(x \geq s+t \mid x \geq t)$

$$= \frac{P(x \geq s+t, x \geq t)}{P(x \geq t)}.$$

$$= \frac{P(x \geq s+t)}{P(x \geq t)} = \frac{(1-p)^{s+t}}{(1-p)^t}$$

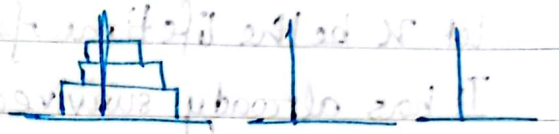
$$= (1-p)^s = P(x \geq s).$$

RECURRENT PROBLEMS

Date :

3 types of recurrent problems : ① Tower of Hanoi
② Pizza Cutting
③ Josephus.

Tower of Hanoi (Derivations):



Single TOH : $T_n = T_{n-1} + 1 + T_{n-1}$

$$= 2T_{n-1} + 1$$
$$= 2(2T_{n-2} + 1) + 1$$
$$= 2^2 T_{n-2} + 2 + 1$$
$$= 2^2 (2T_{n-3} + 1) + 2 + 1$$
$$= 2^3 T_{n-3} + 2^2 + 2 + 1$$
$$\vdots$$
$$= 2^n T_{n-n} + 2^{n-1} + 2^{n-2} \ldots\ldots + 2^n$$
$$= \frac{(2^n - 1)}{(2-1)} = \boxed{2^n - 1} \; **$$

Basis : $T_0 = 2^n - 1 = 2^0 - 1 = 0$.

Hypothesis : $T_n = 2^n - 1$

Induction : $\cancel{T_{n+1} = 2^{n+1} - 1}$

$$\cancel{= 2^n \cdot 2 - 1}$$
$$\cancel{= 2(T_n + 1) - 1}$$
$$\cancel{= 2T_n + 2 - 1 = 2T_n}$$

$$T_{n+1} = 2T_n + 1$$
$$= 2(2^n - 1) + 1$$
$$= 2^{n+1} - 2 + 1$$
$$= 2^{n+1} - 1.$$

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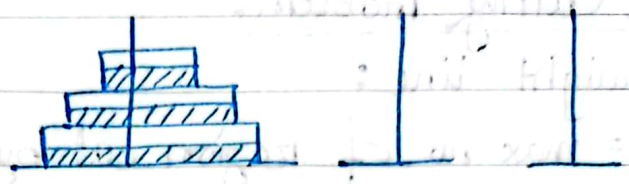
Double TOH :



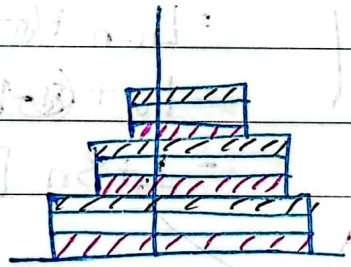
$$T_n = T_{n-1} + 2 + T_{n-1}$$
$$= 2T_{n-1} + 2$$
$$= 2^2(T_{n-1} + 2) + 2$$
$$= 2^3 T_{n-3} + 2^3 + 2^2 + 2$$
$$\vdots$$
$$= 2^n T_{n-n} + 2^n + 2^{n-1} + \cdots + 2$$
$$= 2[2^0 + 2^1 + \cdots + 2^{n-1}]$$
$$= 2\,\frac{(2^{n-1+1} - 1)}{(2-1)} = 2(2^n - 1)$$

Basis: $T_0 = 2(2^0 - 1) = 0$.

Hypothesis: $T_{n-1} = 2(2^{n-1} - 1)$.

Induction: $T_n = 2T_{n-1} + 2$.
$$= 2[2(2^{n-1} - 1)] + 2$$
$$= 2(2^{n-2} - 2) + 2$$
$$= 2^{n-1} - 4 + 2$$
$$= 2(2^n - 1)$$

Triple TOH :



Triple TOH = 1.5 × Double TOH

$$T_n = T_{n-1} + 3 + T_{n-1}$$
$$= 2T_{n-1} + 3$$
$$= 2(2T_{n-2} + 3) + 3$$
$$= 2^2 T_{n-2} + 2.3 + 3.$$
$$= 2^2(2T_{n-3} + 3) + 2.3 + 3$$
$$= 2^3 T_{n-3} + 2^2.3 + 2.3 + 3.$$
$$\vdots$$
$$= 2^n T_{n-n} + 2^{n-1}.3 + \cdots + 2^0.3.$$
$$= 3[2^0 + 2^1 + \cdots + 2^{n+1}]$$
$$= 3\,\frac{(2^{n-1+1} - 1)}{(2-1)}$$
$$= 3(2^n - 1)$$

Base: $T_0 = 3(2^0 - 1) = 0$

Hypothesis: $T_{n-1} = 3.(2^{n-1} - 1)$

Induction: $T_n = 2T_{n-1} + 3$
$$= 2.3(2^{n-1} - 1) + 3$$
$$= 2.3.2^{n-1} - 2.3 + 3$$
$$= 2^n.3 - 2.3 + 3.$$
$$= 2^n.3 - 3. \quad [\because -6+3]$$
$$= 3(2^n - 1).$$

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Date :

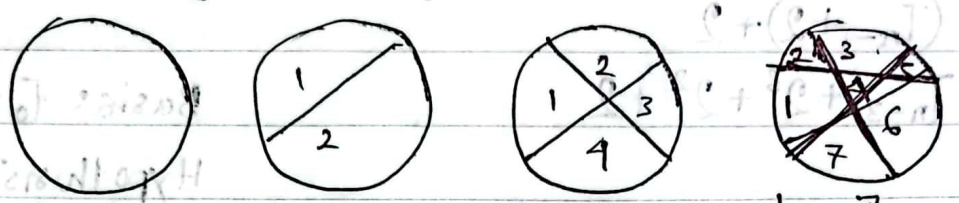
Lines in the plane:

Pizza Cutting Problem:

Straight line :

L_n = max no. of region cut by st. lines.

(*) normally :



$L_0 = 1$ কাটেনি

$L_1 = 2$. $\Rightarrow L_1 = L_0 + 1$

$L_2 = 4$ $L_2 = L_1 + 2$

$L_3 = 7$ $L_3 = L_2 + 3$.

Here, $L_0 = 1$

$L_1 = L_0 + 1$

$L_2 = L_1 + 2$

$L_3 = L_2 + 3$

$\vdots$

$L_n = L_{n-1} + n$.

Open formula:

$L_n = L_{n-1} + n$

$= L_{n-2} + (n-1) + n$

$= L_{n-3} + (n-2) + (n-1) + n$

$= L_{n-n} + (n-n+1) \cdots + (n-1) + n$.

$= L_0 + 1 + \cdots + (n-1) + n$.

$= L_0 + S_n$ $[\because S_n = 1 + 2 + \cdots + (n-1) + n]$

Closed formula

$L_n = L_0 + S_n$

$= 1 + \frac{n(n+1)}{2}$ $[\because L_0 = 1]$

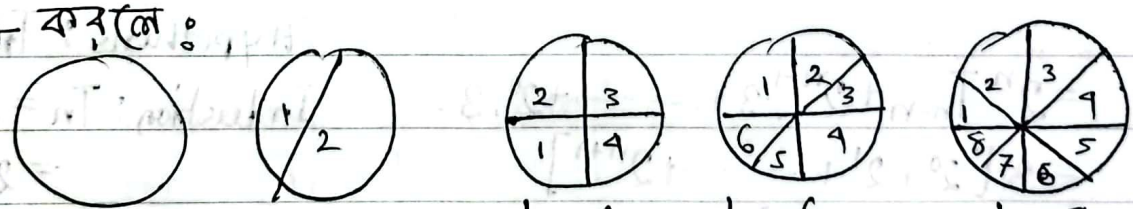
Basis: $L_0 = \frac{0(0+1)}{2} + 1 = 1$

Hypo: $L_{n-1} = \frac{(n-1)n}{2} + 1$

Induction: $L_n = L_{n-1} + n = \frac{(n-1)n}{2} + 1 + n$

$= \frac{n(n+1)}{2} + 1$

(*) Intersect করলে :



$L_0 = 1$

$L_1 = 2$ $L_1 = L_0 + 1$

$L_2 = 4$. $L_2 = L_1 + 2$

$L_3 = 6$ $L_3 = L_2 + 2$

$L_4 = 8$ $L_4 = L_3 + 2$

Here, $L_0 = 1$ $L_3 = L_2 + 2$

$L_1 = L_0 + 1$ $L_4 = L_3 + 2$

$L_2 = L_1 + 2$ $L_5 = L_4 + 2$.

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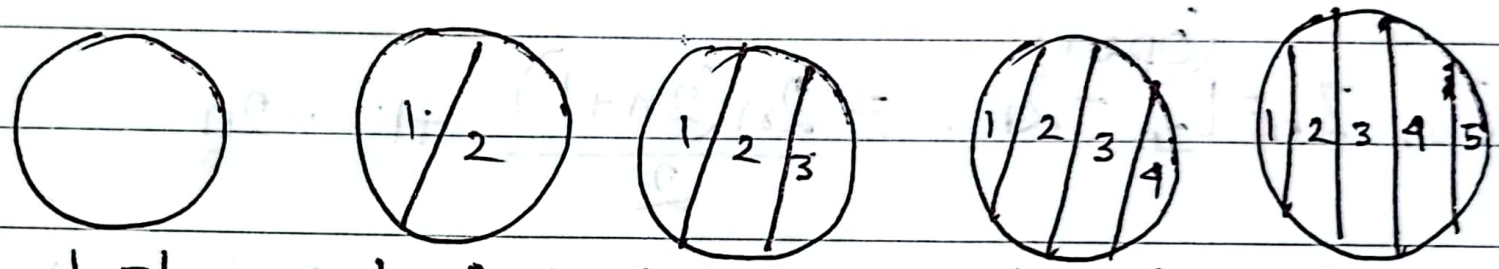
* Parallel :



$L_0 = 1$ $\quad L_1 = 2$ $\quad L_2 = 3$ $\quad L_3 = 4$ $\quad L_4 = 5$.

$L_1 = L_0 + 1$ $\quad L_2 = L_1 + 1$ $\quad L_3 = L_2 + 1$ $\quad L_4 = L_3 + 1$

So, $L_n = L_{n-1} + 1$

$= L_{n-2} + 2$.

$= L_{n-3} + 3$.

$\vdots$

$= L_{n-n} + n$.

$= L_0 + n$. [open]

$= 1 + n$. [closed]

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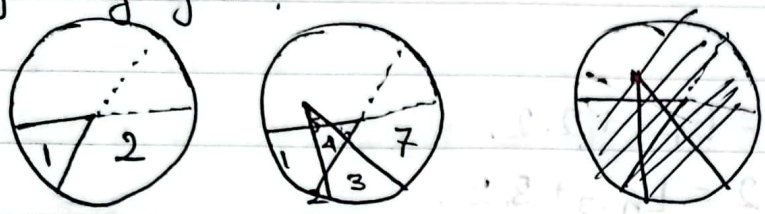
Zig : ZigZag-এর অর্ধেক.



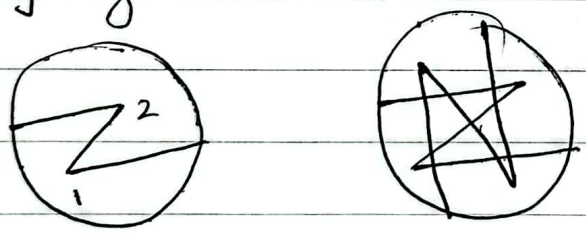
1zig 2zig

Zig	Lines	Line region $\frac{(n+1)n}{2}+1$	Zig region	lost
1	2	4	2	2
2	4	~~10~~ 11	7	4 (2n)
3	6	22	16	6 (2n)

So, $Z_n = \overset{(\text{open})}{L_{2n}} - 2n = \frac{2n(2n+1)}{2} + 1 - 2n$

$= 2n^2 + n + 1 - 2n = 2n^2 - n + 1$ (Closed)

ZigZag



$ZZ_1 = 2$ $ZZ_2 = 12$.

ZZ	Lines	Line region	ZZ region	lost
1	3	7	2	5
2	6	22	12	10 (5n)
3	9	46	31	15 (5n)

So, $ZZ_n = \overset{(\text{open})}{L_{3n}} - 5n = \frac{3n(3n+1)}{2} + 1 + 5n = \frac{9n^2+3n}{2} + 1 + 5n$

$= \frac{9n^2+7n+2}{2}$ (close)

Date :

W Object:

W	lines	line region	W region	lost
1	4	11	2	9
2	8	37	19	18

$9n$., $n=2$

(Open) $W_n = L_{4n} - 9n$.

$$= \frac{4n(4n+1)}{2} + 1 - 9n$$

$$= 8n^2 + 2n + 1 - 9n$$

$$= 8n^2 - 7n + 1.$$ (Closeformula)

Date :

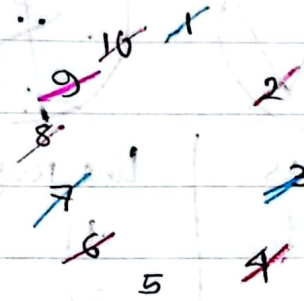
Josephus Problem :

Flavious Josephus circle-এর মানুষ পাশের জনকে মারবে

$J(1) = 1$ survivor

$J(10) = 5$ "

original problem → $J(41) = 19$ বাঁচছিল



2 4 6 8 10 3 7 1 9 5 → survivor

J(2n) : (2 cases)

even $J(2n) = 2J(n) - 1$

odd $J(2n+1) = 2J(n) + 1$

example : $J(100) = 2J(50) - 1$ (even)

$= 4J(25) - 2 - 1$

$= 8J(12) + 4 - 2 - 1$

$= 16J(6) - 8 + 4 - 2 - 1$

$= 32J(3) - 16 - 8 + 4 - 2 - 1$

$= 64J(1) + 32 - 16 - 8 + 4 - 2 - 1$

$= 73$

J(n)

$n = 2^m + l$ → largest, nearest power of 2 to n.

Exple – যদি $n = 100$ হয় then $2^m = 64$ হওয়া উচিত (2^6).

" $n = 41$ " " $2^m = 32$ " " (2^5).

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
J(n)	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

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So, $J(2^m + l) = 2l + 1$

Basis : $J(2^0 + 0) = 2.0 + 1 = 1$

Hypothesis : for values between 1 to m-1

$$J(2^m + l) = 2l + 1$$

$$\Rightarrow J(2^m + \tfrac{l}{2}) = 2\left(\tfrac{l}{2}\right) + 1 = l + 1$$

Induction:

$$J(2n) = 2l + 1$$

$$= 2J(n) - 1$$

here : $2n = 2^m + l$

$$n = 2^{m-1} + \tfrac{l}{2}$$

$$J(2n) = 2J(2^{m-1} + \tfrac{l}{2}) - 1$$

$$= 2[2 \times \tfrac{l}{2} + 1] - 1$$

$$= 2l + 2 - 1 = 2l + 1.$$

$$\cancel{J(2n+1) = 2J(2^{m-1} + \tfrac{l}{2}) + 1}$$

$$\cancel{= 2[2 \times \tfrac{l}{2} + 1] + 1}$$

$$\cancel{=}$$

$$2n + 1 = 2^m + l.$$

$$\Rightarrow 2n = 2^m + l - 1$$

$$\Rightarrow n = 2^{m-1} + \frac{l-1}{2}$$

$$J(2n+1) = 2J(2^{m-1} + \tfrac{l-1}{2}) + 1$$

$$= 2[2\tfrac{l-1}{2} + 1] + 1$$

$$= 2[l - 1 + 1] + 1$$

$$= 2l + 1.$$

Date :

Josephus Binary: $\boxed{J(n) = J(2^m + l) = 2l+1}$

** $J(100) = 2 \times 36 + 1 = 73$

$\boxed{1}100100$ → শুধু MSB বাদে বাকি same

$\boxed{0}100100$

$100100\,1$ → left shif করে 1 যোগ করব
or, left rotation "

** $J(13) = 2 \times 5 + 1 = 11$

1101

0101

1011

$$n = (b_m\ b_{m-1}\ b_{m-2} \ldots\ldots b_2\ b_1\ b_0)_2$$
$$l = (0\ b_{m-1}\ b_{m-2} \ldots\ldots b_2\ b_1\ b_0)_2$$
$$2l = (b_{m-1}\ b_{m-2} \ldots\ldots b_2\ b_1\ b_0\ 0)_2$$
$$2l+1 = (b_{m-1}\ b_{m-2} \ldots\ldots b_2\ b_1\ b_0\ b_m)_2$$

Find values for n for which $n/2$ person survives.

$\Rightarrow J(n) = n/2$

or, $2l+1 = \dfrac{2^m + l}{2}$

~~or, $2l = \dfrac{2^m+l}{2} - 1$~~

~~or, $l = \dfrac{2^m+l}{4} - \dfrac{1}{2} = \dfrac{\ }{2}$~~

or, $4l+1 = 2^m + l$

or, $3l = 2^m - 1$

or, $l = \dfrac{2^m - 1}{3}$

m	l	$n = 2^m + l$	$J(n) = 2l+1$
1	0	2	1
3	2	10	5
5	10	42	21
7	42	170	85

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Practice : *** 3 values for $n/3^{th}$ survivor

$J(n) = n/3$

$\Rightarrow 2l+1 = \frac{2^m + l}{3}$

$\Rightarrow 6l+3 = 2^m + l$

$\Rightarrow 5l = 2^m - 3$

$\Rightarrow l = \frac{2^m - 3}{5}$.

m	l	$n = 2^m + l$	$J(n) = 2l+1$
11	409	2457	819.
3	1	9	3
7	25	153	51

*** values for last person.

$J(n) = n$.

$\Rightarrow 2l+1 = 2^m + l$

$\Rightarrow l = 2^m - 1$

*** values for $n/4^{th}$ survivor

$J(n) = n/4$

$\Rightarrow 2l+1 = \frac{2^m + l}{4}$

$\Rightarrow 8l+4 = 2^m + l$

$\Rightarrow l = \frac{2^m - 4}{7}$

*** values for 25th survivor.

$J(n) = 25$

$\Rightarrow 2l+1 = 25$

$\Rightarrow l = \frac{24}{2} = 12$

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Radixed based Josephus.

Example : Given, $f(1) = 34 \leftarrow \alpha_1$

$f(2) = 54 \leftarrow \alpha_2$

$f(3n+0) = 10 f(n) + 76 \rightarrow \beta_0$

$f(3n+1) = 10 f(n) - 2 \rightarrow \beta_1$

$f(3n+2) = 10 f(n) + 8 \rightarrow \beta_2$

c, d

$\therefore f(19) = ?$

$\therefore f(25) = ?$

এখানে $d = 3$, $c = 10$,

So, for $f(19) \rightarrow$

$$\begin{array}{r|l} 3 & 19 \\ 3 & 6 - 1 \\ 3 & 2 - 0 \\ 3 & 0 - 2 \end{array} \quad (201)_3.$$

So, $f(19) = (201)_3$

$= (\alpha_2 \ \beta_0 \ \beta_1)_{10}$

$= (5 \quad 76 \quad -2)_{10}$

$= (5 \times 10^2 + 76 \times 10^1 + -2 \times 10^0)_{10}$

$= 500 + 760 + -2$

$= 1258$

for $f(25) \rightarrow$

$$\begin{array}{r|l} 3 & 25 \\ 3 & 8 - 1 \\ 3 & 2 - 2 \\ 3 & 0 - 2 \end{array}$$

$f(25) = (221)_3$

$= (\alpha_2 \ \beta_2 \ \beta_1)_3$

$= (5 \quad 8 \quad -2)_{10}$

$= 500 + 80 - 2 = 578$

Date :

In original Josephus problem :

$f(1) = \alpha$

$f(2n+0) = 2f(n) - \beta_0$

$f(2n+1) = 2f(n) + \beta_1$

$\alpha = 1 ; \ \beta_0 = -1 \ ; \ \beta_1 = 1$

** find J(100) using Radix →

$n = (11 \ 00 \ 1 \ 0 \ 0)_2$

$= (\alpha \ \beta_1 \ \beta_0 \ \beta_0 \ \beta_1 \ \beta_0 \ \beta_0)_{10}$

$= (1 \ 1 \ -1 \ -1 \ 1 \ -1 \ -1)_{10}$

$= 64 + 32 - 16 - 8 + 4 - 2 - 1 = 73$

Good, bad persons.

** There are 2n people in a circle, first n are good guys, last n are bad, if we kill every mth person show that bad guys die first.

let, n = 3, so 6 people in circle, 6 1, 5 2, 4 3

bad guys → LCM(4,5,6)

m = LCM(4,5,6).

So, 2 | 4, 5, 6
2, 5, 3.

m = 60

if n = 4. so circle has 8 people

m = LCM(5,6,7,8)

= 840.

but circle-এর শুরুতে bad people থাকলে আর শেষে good guys থাকলে bad guys-এর position বের করব GCD (✓) not LCM (✗)

PERCEPTRON ALGORITHM.

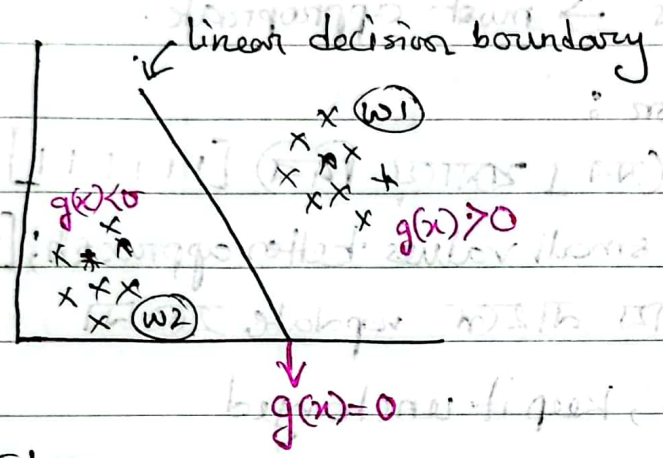
Date :

দুইটা pattern linearly seperable হইলে apply করা যায়

suppose $w1 = (1,1) \cdot (1,-3)\ (4,5)$

$w2 = (2,2.5)\ (0,2)\ (2,3)$

perceptron use করে linear decision boundary বের করব



3 Steps :-

① Training pattern linearly seperable না হলে (Low dimension-এ) then তাকে high dimension-এ নেই

use PHI ϕ - f^n to make 2D set go 6-D.

$$y = \phi(x) = [x_1^2 \ \ x_2^2 \ \ x_1x_2 \ \ x_1 \ \ x_2 \ \ 1]^T$$

last -এ $w_0 = 1$ বসিয়ে নেবে to keep at least 1 weight constant and independent like the term of st. line / hyper plane (i.e. $y = mx + c$)

weight vector $\rightarrow [w^T \ \ w_0] = a^T$

Augmented " $\rightarrow$ $\begin{bmatrix} x \\ 1 \end{bmatrix} y$

We perform $\rightarrow g(y) = a^T y = [w^T \ \ w_0] * \begin{bmatrix} x \\ 1 \end{bmatrix}$

If $g(y) > 0$; $y \in w_1$

else $y \in w_2$

② Class-2 normalize করি negation করে $g(y) < 0$ is replaced by $g(-y) > 0$ to make it simple

Date :

③ Decide learning rate $= \alpha$ in range $(0,1]$:

$\alpha = 0$ হবে না cause তাহলে weight update হবে না

$\alpha > 1$ হবে " " update বেশি বড় হয়ে যাবে (unstable).

$\alpha = 0.5$ best

· 1/no. of training patterns → most appropriate.

Decide initial weight vector:

arbitarily setting. (সব 1 বসায়ে দিব) [1 1 1 1 1 1]

uniform random small values better approach; [-1, 1]

0 বসাব না সব গুলায় না হলে update হবে না

Updating: $g(y) > 0$ is perfect, keep it unchanged

$g(y) < 0$ not okay

↳ so $w(i+1) = w(i) + \alpha \cdot Y_{ij}$ (new weight বানাব next iteration-এর জন্য)

Misclassification শেষ হলে iteration stop করব

Practice:

let, $\alpha = 0.8$

$$w = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad w^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$\omega_1 = (1,1)\ (1,-3)\ (4,5)$ $\quad c = (1,1)(1,3)(4,5)(2,2.5)(0,2)(2,3)$

$\omega_2 = (2,2.5)\ (0,2)\ (2,3)$

Date :

$y = [x_i^2 \quad y_i^2 \quad x_i y_i \quad x_i \quad y_i \quad 1]^T$

points		y_{ij}					
(1,1)	y_{11}	1	1	1	1	1	1
(1,-3)	y_{12}	1	9	-3	1	-3	1
(4,5)	y_{13}	16	25	20	4	5	1
(2, 2.5)	$-y_{21}$	-4	-6.25	-0.5	-2	-2.5	-1
(0,2)	$-y_{22}$	0	-4	0	0	-2	-1
(2,3)	$-y_{23}$	-4	-9	-6	-2	-3	-1

<u>One at a time approach:</u> mismatt $[g(y)<0]$ পাইলেই $w = w + \alpha y^T$ change করব

Iteration 1:

$$w^T y_{11} = [1 \ 1 \ 1 \ 1 \ 1 \ 1] * \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= 6 > 0.$$

$$w^T y_{12} = [1 \ 1 \ 1 \ 1 \ 1 \ 1] * \begin{bmatrix} 1 \\ 9 \\ -3 \\ 1 \\ -3 \\ 1 \end{bmatrix}$$

$$= 1 + 9 - 3 + 1 - 3 + 1 = 6 > 0$$

$$w^T y_{13} = [1 \ 1 \ 1 \ 1 \ 1 \ 1] \begin{bmatrix} 16 \\ 25 \\ 20 \\ 4 \\ 5 \\ 1 \end{bmatrix} = 16 + 25 + 20 + 4 + 5 + 1$$

$$= 71 > 0.$$

Date :

$$w^T y_{21} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} * \begin{bmatrix} -4 \\ -6.25 \\ -5 \\ -2 \\ -2.5 \\ -1 \end{bmatrix}$$

$$= -4 - 6.25 - 5 - 2 - 2.5 - 1$$

$$= -20.75 < 0 \quad [\text{mismatch}]$$

so $$w = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} + 0.8 * \begin{bmatrix} -4 \\ -6.25 \\ -5 \\ -2 \\ -2.5 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2.2 & -4 & -3 & -0.6 & -1 & -0.2 \end{bmatrix}$$

$$w^T y_{22} = \begin{bmatrix} -2.2 & -4 & -3 & -0.6 & -1 & -0.2 \end{bmatrix} * \begin{bmatrix} 0 \\ -4 \\ 0 \\ 0 \\ -2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 + 16 + 0 + 0 + 2 + 0.2 \end{bmatrix}$$

$$= 18.2 > 0$$

$$w^T y_{23} = \begin{bmatrix} -2.2 & -4 & -3 & -0.6 & -1 & -0.2 \end{bmatrix} * \begin{bmatrix} -4 \\ -9 \\ -6 \\ -2 \\ -3 \\ -1 \end{bmatrix} = 8.8 + 36 + 18 + 1.2 + 3 + 0.2$$

$$= 67.2 > 0.$$

Date :

Summary:

y_{ij}	w	$w^T y$
1 1 1 1 1 1	1 1 1 1 1 1	6
1 9 -3 1 -3 1	1 1 1 1 1 1	6
16 25 20 4 5 1	1 1 1 1 1 1	71
-4 -6.25 -5 -2 -2.5 -1	1 1 1 1 1 1	-20.75
0 -4 0 0 -2 -1	-2.2 -4 -3 -0.6 -1 -0.2	18.2
-4 -9 -6 -2 -3 -1	-2.2 -4 -3 -0.6 -1 -0.2	67.2

আবার iteration হবে

<u>Many at a time</u>: (same ভাবে update করতে থাকব)
mismatch পেলেও update!!

$w^T y_{11} = [1\ 1\ 1\ 1\ 1\ 1]$ $6 > 0$

$w^T y_{12} = 6 > 0$

$w^T y_{13} = 71 > 0$

$w^T y_{21} = [1\ 1\ 1\ 1\ 1\ 1] * = -20.75 < 0$

$$w^T y_{22} = [1\ 1\ 1\ 1\ 1\ 1] * \begin{bmatrix} 0 \\ -4 \\ 0 \\ 0 \\ -2 \\ -1 \end{bmatrix} = 0 - 4 + 0 + 0 - 2 - 1 = -7 < 0$$

$$w^T y_{23} = [1\ 1\ 1\ 1\ 1\ 1] * \begin{bmatrix} -4 \\ -9 \\ -6 \\ -2 \\ -3 \\ -1 \end{bmatrix} = -4 - 9 - 6 - 2 - 3 - 1 = -25 < 0$$

Date :

Summary:

y_{ij}	w	$w^T y$
1 1 1 1 1 1	1 1 1 1 1 1	6
1 1 -3 1 -3 1	1 1 1 1 1 1	6
16 25 20 4 5 1	1 1 1 1 1 1	71
-4 -6.25 -5 -2 -2.5 -1	1 1 1 1 1 1	-20.75
0 -4 0 0 -2 -1	1 1 1 1 1 1	-7
-4 -9 -6 -2 -3 -1	1 1 1 1 1 1	-25

$$\sum y_{ij} = y_{21} + y_{22} + y_{23}$$

$$= [-4 \quad -6.25 \quad -5 \quad -2 \quad -2.5 \quad -1] +$$
$$[0 \quad -4 \quad 0 \quad 0 \quad -2 \quad -1] +$$
$$[-4 \quad -9 \quad -6 \quad -2 \quad -3 \quad -1]$$
$$= [-8 \quad -19.25 \quad -11 \quad -4 \quad -7.5 \quad -3]$$

$$w = w + \alpha * \sum y_{ij}$$

$$= [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1] + 0.8 \begin{bmatrix} -8 \\ -19.25 \\ -11 \\ -4 \\ -7.5 \\ -3 \end{bmatrix}$$

$$= [-5.4 \quad -14.4 \quad -7.8 \quad -2.2 \quad -5 \quad -1.4]$$

চলতেই থাকবে ∴

STERN BROCOT TREE

Date :

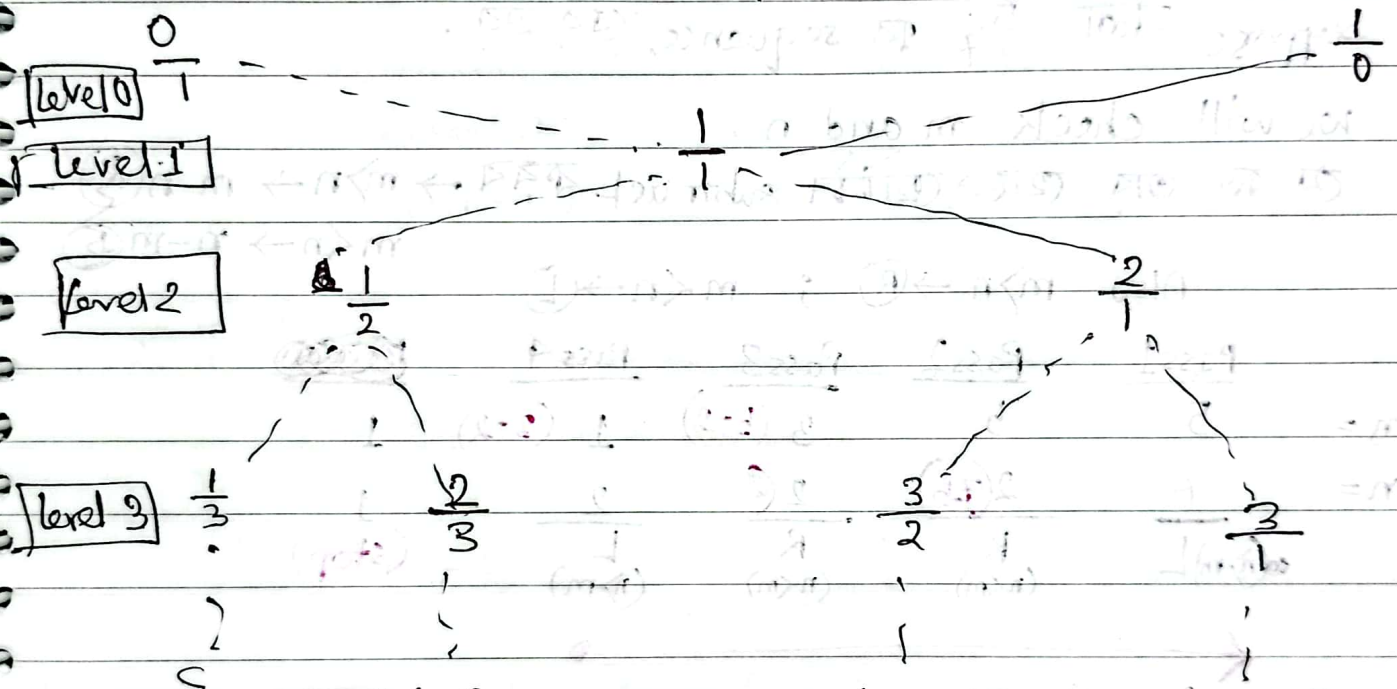
If $\gcd(m,n)=1$ → no common prime factor = relatively prime. $(m \perp n)$

constructing set of nonnegative fractions m/n wit $m \perp n$ is called stern brocot tree.

starts with $\frac{0}{1}$ $\frac{1}{0}$ (if $m=0$ and $n=1$)

which means $\frac{m}{n}$ $\frac{m'}{n'}$ and there root will be $\frac{m+m'}{n+n'}$

always (সবসময়) left upper node -এর সাথে root -কে যোগ করে left child বানাব ; left upper node না থাকলে parent -এর টা নেব



চলতেই থাকবে। But এতো বড় tree থেকে nonnegative number fetch করা possible না। so we have forgulae

*** যদি বলে $f(LRRL) = ?$

memorise → $L = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$R = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

Date :

so for LRRL → multiply the matrices:

$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\times\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\times\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\times\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}\times\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}=\begin{bmatrix}3 & 4\\2 & 3\end{bmatrix}=\frac{2+3}{4+3}=\frac{5}{7}$$

numerator denominator

*** Suppose বলা 5/7 এর sequence বের কর.

we will check m and n.

যে বড় তার থেকে ছোটটা subtract করব → m>n → m−n (R).

m<n → n−m (L)

Also m>n → (R) ; m<n → (L)

	Pass1	Pass2	Pass3	Pass4	
m =	5	5	3 (5−2)	1 (3−2)	1
n =	7	2 (7−5)	2	2	1
	(n>m) L	R (n<m)	R (n<m)	L (n>m)	(stop)

(LRRL)

*** level চাইলে pass-এর সংখ্যা বলব +1 → ~~level 5~~ Here

এখানে → 4+1=5.

~~MLE & MAP Classifier.~~ Date :

Proofs :

① $\boxed{\frac{m}{n} \text{ and } \frac{m'}{n'} \text{ are consecutive fractions.}}$

$\frac{m}{n} \times \frac{m'}{n'}$

$m'n - mn'$

we have $m'n - mn' = 1$

Basis: $\Rightarrow \frac{m}{n} = \frac{0}{1}$ and $\frac{m'}{n'} = \frac{1}{0}$; $m'n - mn' = 1 \cdot 1 - 0 \cdot 0 = 1$

Hypothesis: $m'n - mn' = 1$ is true for any stage in Stern Brocot tree.

Induction: median of $\frac{m}{n}$ and $\frac{m'}{n'}$ is $\frac{m+m'}{n+n'}$

So, $(m+m')n - m(n+n') = \cancel{mn} + nm' - \cancel{mn} - mn' = m'n - mn'$
$= 1$

$m'(n+n') - n'(m+m') = m'n + \cancel{m'n'} - n'm - \cancel{n'm'} = m'n - n'm$
$= 1$ (Proved)

② If $\frac{m}{n} < \frac{m'}{n'}$ and all values are non negative

$\boxed{\text{then } \frac{m}{n} < \frac{m+m'}{n+n'} < \frac{m'}{n'}}$

So, $\frac{m+m'}{n+n'} - \frac{m}{n} = \frac{\cancel{mn} + m'n - \cancel{mn} - mn'}{n(n+n')} = \frac{m'n - mn'}{n(n+n')} = \frac{1}{n(n+n')}$

$\frac{m'}{n'} - \frac{m+m'}{n+n'} = \frac{m'n + \cancel{m'n'} - mn' - \cancel{m'n'}}{n'(n+n')} = \frac{m'n - mn'}{n'(n+n')} = \frac{1}{n'(n+n')}$

as $n > 0$ and $n' > 0$ and so is $\frac{1}{n(n+n')}, \frac{1}{n'(n+n')} > 0$

Thus, $\frac{m}{n} < \frac{m+m'}{n+n'} < \frac{m'}{n'}$.

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Date :

③ Is there any positive fraction a/b with $a \perp b$ omitted from Stern Brocot tree.?

let, $\frac{m}{n} = \frac{0}{1} < \frac{a}{b} < \frac{m'}{n'} = \frac{1}{0}$.

so let $\frac{a}{b} = \frac{m+m'}{n+n'}$ and

Either $\frac{m+m'}{n+n'} = \frac{a}{b}$ we win.

But if $\frac{m+m'}{n+n'} < a/b$ we set $m \leftarrow m+m'$
$n \leftarrow n+n'$

and move to right subtree.

if $\frac{m+m'}{n+n'} > a/b$ then $m' \leftarrow m+m'$
$n' \leftarrow n+n'$

and move to left subtree.

and inequality comes closer to equality to a/b as either m or n or m' or n' increases at each step, so we must win after at most $(a+b)$ steps.

MLE and MAP Classifier
↳ Maximum likelihood Event → (MLE not affected by prior probability)
Maximum-A-Posterior Probability.
(Bayes 3 বলি)

Date: ***

* Easy problem : When shape + parameters given.
  shape → graph
  parameters → (mean(μ), variance(σ^2)).

* not easy problem : Shape দেওয়া but parameters নাই.
  Shape থেকে বের করে নিব. (MLE / MAP).

* not easy + difficult : Shape ও নাই parameter ও নাই (perceptron)
  Just কিছু labeled supervised data আছে

* Most difficult : labeled ও নাই – unsupervised data আছে

Practice:

(1.2) *** Camera detects cats and dogs. Cats and dogs small ears probability → 0.8 and 0.1 respectively. A large ear animal detected

(i) probability of the observed animal being a dog:

Let small ears probability = S.
Large " " = L
Dog is observed = D
Cat " " = C.

Given that it is Dog, probability of Small ears : $P(S|D) = 0.1$
So " " " " " , " " large ears : $P(L|D) = 1 - 0.1 = 0.9$

Same for cats $P(S|C) = 0.8$
$P(L|C) = 0.2$

Here, $P(L|D) > P(L|C) \rightarrow 0.9 > 0.2$ as per MLE (ii)
So it is dog.

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same data for 1.2 – 1.5 maths.

Date :

(1.2) Given, $\boxed{\begin{array}{l} \text{Salmon}: \mu = 5, \ \sigma^2 = 1 \\ \text{Bass}: \mu = 10 \quad \sigma^2 = 4 \end{array}}$ | Data/Length, $l = 7$ inch.

i) MLE : P(Data | Class)

$$P(l=7 \mid \text{Salmon}) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(l-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{(7-5)^2}{2}} = \frac{1}{\sqrt{2\pi}} (0.135)$$

$$P(l=7 \mid \text{Bass}) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{(7-10)^2}{8}} = \frac{1}{\sqrt{2\pi}} \cdot \frac{(0.32)}{2} = \frac{0.162}{\sqrt{2\pi}}$$

So, Bass's probability high.

ii) Bayes : P(Class | Data).

$$P(\text{Salmon} \mid l=7) = \frac{P(\text{Salmon})\, P(l=7 \mid \text{Salmon})}{\cancel{P(l=7)}}$$

$$P(\text{Bass} \mid l=7) = \frac{P(\text{Bass})\, P(l=7 \mid \text{Bass})}{\cancel{P(l=7)}}$$

P(Salmon), P(Bass) দেওয়া নাই, so $\frac{1}{2}$ ধরব

Thus $P(\text{Salm} \mid l=7) = \frac{1}{2} \times \frac{1}{\sqrt{2\pi}} (0.135)$

$P(\text{Bass} \mid l=7) = \frac{1}{2} \times \frac{1}{\sqrt{2\pi}} (0.162)$

~~P(Bass)~~ Bass > Salmon.

iii) a) What if Salmon and Bass equally likely.

using MAP as MLE is not affected by probability prior.

So, when P(Salmon) = P(Bass) = $\frac{1}{2}$ $\boxed{(50-50)\%}$

Date :

So, $P(\text{Salmon} \mid L=7) = P(\text{Sea Bass} \mid L=7)$

$$\Rightarrow \cancel{\frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{(l-15)^2}{1}} \times \frac{1}{2}}{P(l-7)}} = \cancel{\frac{\frac{1}{\sqrt{2\pi}} e^{-(l-10)}}{P}}$$

$$\Rightarrow \frac{\frac{1}{\sqrt{2\pi}}(0.135) \times \cancel{\frac{1}{2}}}{\cancel{P(l=7)}} = \frac{\frac{1}{\sqrt{2\pi}}(0.16) \times \cancel{\frac{1}{2}}}{\cancel{P(l=7)}}$$

$\therefore$ Still Bass higher.

b) if Salmon is 2 likely a Bass

So, $P(\text{Salmon}) = \frac{2}{3}$

$P(\text{Bass}) = \frac{1}{3}$

Thus, $P(\text{Salmon} \mid L=7) = P(\text{Bass} \mid L=7)$

$$\Rightarrow \frac{\cancel{\frac{1}{\sqrt{2\pi}}}(0.135) \times \frac{2}{3}}{\cancel{P(l=7)}} = \frac{\cancel{\frac{1}{\sqrt{2\pi}}}(0.16) \times \frac{1}{3}}{\cancel{P(l=7)}}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}}(0.09) = \frac{1}{\sqrt{2\pi}}(0.053)$$

$\therefore$ Salmon is larger.

(1.3) μ, σ^2 দেওয়া but l দেওয়া নাই।

At decision boundary $P(\text{Salmon}) = P(\text{Bass})$

So,

$\cancel{P(l \mid \text{Salmon}) = P(l \mid \text{Bass})}$

$P(\text{Salmon} \mid L) = P(\text{Bass} \mid L)$

$$\Rightarrow \frac{P(\text{Salmon})P(l \mid \text{Salmon})}{\cancel{P(l)}} = \frac{P(\text{Bass})P(l \mid \text{Bass})}{\cancel{P(l)}}$$

Date :

$$\Rightarrow \left(\frac{1}{2}\right)\cdot\frac{1}{\sqrt{2\pi}}\left(e^{-\frac{(l-5)^2}{2}}\right) = \left(\frac{1}{2}\right)\frac{1}{2\sqrt{2\pi}}\left(e^{-\frac{(l-10)^2}{8}}\right)$$

$$\Rightarrow e^{-\frac{(l-5)^2}{2}} = \frac{e^{-\frac{(l-10)^2}{8}}}{2}$$

$$\Rightarrow -\frac{(l-5)^2}{2} = -\log_e 2 - \frac{(l-10)^2}{8}$$ [∵ log বসায়ে e উভয় দিই]

$$\Rightarrow \frac{8(l-5)^2}{2} = 8\log_e 2 + \frac{8(l-10)^2}{8}$$ [∵ multiply by 8]

$$\Rightarrow 4(l-5)^2 = 8\log_e 2 + (l-10)^2$$

$$\Rightarrow 4l^2 - 40l + 100 = 5.54 + l^2 - 20l + 100$$

$$\Rightarrow 3l^2 - 20l - 5.54 = 0$$ [apply $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$] (or use calculator)

so, $l = 6.9$ inch

(1.5) l বার দেওয়া but বলা Salmon twice than bass.

So. $$\frac{2}{3}\times\left(\frac{1}{2}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{(l-5)^2}{2}} = \frac{1}{3}\times\left(\frac{1}{2}\right)\frac{1}{\sqrt{2\pi}}\cdot\frac{e^{-\frac{(l-10)^2}{8}}}{2}$$

$$\Rightarrow 4e^{-\frac{(l-5)^2}{2}} = e^{-\frac{(l-10)^2}{8}}$$

$$\Rightarrow \log_e 4 - \frac{(l-5)^2}{2} = -\frac{(l-10)^2}{8}$$

$$\Rightarrow 2(1.38) - (l^2 - 10l + 25) = \frac{-l^2 + 20l - 100}{4}$$

$$\Rightarrow 8(2.78 - l^2 + 10l - 25) = -l^2 + 20l - 100$$

$$\Rightarrow -4l^2 + 40l - 88.88 = -l^2 + 20l - 100$$

Date :

$\cancel{\Rightarrow 7l^2 - 60l + 77.76 = 0}$

$\cancel{\Rightarrow l = 6.97}$

$\Rightarrow 3l^2 - 20l - 11.12 = 0$

$\Rightarrow l = 7.182$ inch (Ans)

(2.1) Suppose কোনো μ, σ^2 নাই just 12টা fish-এর data sets দেওয়া (name, length) → (Salmon, 4) (bass, 8) (bass, 12), (Salmon, 5) (sal, 6) (sal, 4) (bass, 10) (bass, 12). (Sal, 5) (sal, 5) (sal, 4) (sal, 7)

For salmon, mean $= (4+5+4+5+5+4+7+6)/8 = 5$.

$$\text{variance} = \frac{(4-5)^2+(5-5)^2+(4-5)^2+(5-5)^2+(5-5)^2+(4-5)^2+(7-5)^2+(6+5)^2}{8}$$

$$= \frac{1+0+1+0+0+1+4+1}{8}$$

$$= 8/8 = 1$$

Bass, mean $= (8+12+10+12)/4 = 10.5$.

$$\text{variance} = (8-10.5)^2+(12-10.5)^2+(10-10.5)^2+(12-10.5)^2/4$$

$$= 2.75.$$

Prior probabilities:

$P(\text{Salmon}) = 8/12 = 0.67$

$P(\text{Bass}) = 4/12 = 0.34$

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BIONOMIAL COEFFICIENT

Date :

$\binom{n}{k}$ ~~n choose~~ choose k elements from n.

$\hookrightarrow {}^nC_k$

Prove $\cancel{\binom{n}{k}} = \dfrac{n!}{k!\,(n-k)!} = \dfrac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$

So, $LHS = \dfrac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$

$= \dfrac{n(n-1)(n-2)\cdots(n-k+1)(n-k)(n-k-1)\cdots 1}{k!\,(n-k)!}$

$= \dfrac{n!}{k!\,(n-k)!}$ (R.H.S)

Rules:

$\binom{r}{k} \rightarrow$ 0 if $k<0$ (-ve)

1 if $k=0$

0 if $k>r$

1 if $k=r$

4 ति proof → 1. Symmetry Identity $\binom{r}{k} = \binom{r}{r-k}$

2. Absorption Identity $\binom{r}{k} = \frac{r}{k} \times \binom{r-1}{k-1}$

3. Addition Identity $\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$

4. Summation Identity. $\sum_{k \le n} \binom{r+k}{k} = \binom{r+n+1}{n}$

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Date :

Symmetry Identity: $\binom{n}{k} = \binom{n}{n-k}$

$\binom{4}{3} = \binom{4}{4-3} = \binom{4}{1}$

তিন — তিনটা

same.

proof: $LHS = \binom{n}{k} = \dfrac{n!}{k!(n-k)!}$

$$= \frac{n!}{(n-(n-k))!\,(n-k)!}$$

$$= \binom{n}{n-k}$$

Symmetry holds or not?

** if $n<0$; n negative.

let $n=-1$

whether $\binom{-1}{k} = \binom{-1}{-1-k}$ holds or not?

** when $k=0$;

$LHS = \binom{-1}{0} = 1$

$RHS = \binom{-1}{-1-0} = \binom{-1}{-1} = 0$ [$\because k<0$]

$LHS \neq RHS$. doesn't hold

** when $k>0$; (+ve).

$$LHS = \binom{-1}{k} = \frac{-1 \cdot -2 \cdot -3 \cdots\cdots (-1-k+1)}{k!}$$

$$= \frac{(-1)^k \cdot -2 \cdot -3 \cdots\cdots -k}{1 \cdot 2 \cdot 3 \cdots\cdots k}$$

$$= (-1)^k = \pm 1$$

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$R.H.S = \binom{-1}{-1-K} = 0$

$\Rightarrow$ then K totally becomes negative!

This $LH.S \neq RHS$. (doesn't hold)

** if $K<0$; (-ve)

$LHS = \binom{-1}{-K} = 0$

$RHS = \binom{-1}{-1-(-K)} = \binom{-1}{-1+K}$

$= \dfrac{-1 \cdot -2 \cdot -3 \cdots\cdots (\cancel{-1}+\cancel{1}-K-1)}{(K-1)!}$

$= \dfrac{-1 \; -2 \; -3 \cdots\cdots -(K-1)}{1 \cdot 2 \cdot 3 \cdots\cdots (K-1)}$

$= (-1)^{K-1} = \pm 1.$

Again $LHS \neq RHS$. (doesn't hold)

Date:

Absorption Identity:

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

$$= \frac{n}{k}\left(\frac{(n-1)!}{(k-1)!\,(n-k)!}\right)$$

$$= \frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

Show that $(n-k)\binom{n}{k} = n\binom{n-1}{k}$

$$L.H.S = (n-k)\binom{n}{k}$$

$$= (n-k)\binom{n}{n-k} \quad [\because \text{using symmetry}]$$

$$= \cancel{(n-k)}\left(\frac{n}{\cancel{n-k}}\right)\binom{n-1}{n-k-1} \quad [\because \text{using absorption}]$$

$$= n\binom{n-1}{n-k-1}$$

$$= n\binom{n-1}{\cancel{n}-1-\cancel{n}+k\cancel{+1}} \quad [\because \text{using symmetry}]$$

$$= n\binom{n-1}{k} = RHS.$$

Addition formula:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

↳

10 ta বল থেকে 3 টা নিব;

So, 9টা থেকে 3টা নেওয়া + 9টা হতে 3-1টা কম নেওয়া

same কথা.

Existence of addition formula Date:

Show that $\binom{r}{k} = \left(\frac{r}{k}\right)\binom{r-1}{k-1}$

$$\Rightarrow k\binom{r}{k} = r\binom{r-1}{k-1} \quad \text{--- (I)}$$

$\Rightarrow$ ~~ekc~~

also we know, $(r-k)\binom{r}{k} = r\binom{r-1}{k} \quad \text{--- (II)}$

now, (I) + (II) :

$$k\binom{r}{k} + (r-k)\binom{r}{k} = r\binom{r-1}{k-1} + r\binom{r-1}{k}$$

$$\Rightarrow \cancel{k\binom{r}{k}} + r\binom{r}{k} - \cancel{k\binom{r}{k}} = r\binom{r-1}{k-1} + r\binom{r-1}{k}$$

$$\Rightarrow r\binom{r}{k} = r\binom{r-1}{k-1} + r\binom{r-1}{k}$$

$$\Rightarrow \binom{r}{k} = \binom{r-1}{k-1} + \binom{r-1}{k} \quad \text{(exists)}$$

Summation formula:

prove : $\sum_{k \le n} \binom{m+k}{k} = \binom{m+n+1}{n}$ for $m+k \geq 0$

or, $k \geq -m$

$$L.H.S = \sum_{-m \le k \le n} \binom{m+k}{k}$$

$$= \sum_{-m \le k \le n} \binom{m+k}{m+\cancel{k}-\cancel{k}} \quad [\because \text{using symmetry}]$$

$$= \sum_{-m \le k \le n} \binom{m+k}{m}$$

Date :

replace k by $k-m$

$$\sum_{-m \le k-m \le n} \binom{\cancel{m}+k-\cancel{m}}{m}$$

$$= \sum_{0 \le k \le m+n} \binom{k}{m}.$$

$$= \binom{m+n+1}{m+1}$$

$$= \binom{m+n+1}{\cancel{m}+n+\cancel{1}-\cancel{m}-\cancel{1}}$$

$$= \binom{m+n+1}{n} = RHS$$ [verified]

Show $\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$

$$L.H.S = \binom{r}{m}\binom{m}{k} = \frac{r!}{\cancel{m!}(r-m)!} \times \frac{\cancel{m!}}{k!(m-k)!}$$

$$= \frac{r!}{k!(r-m)!(m-k)!}$$

$$= \frac{r!}{k!(r-k)!} \times \frac{(r-k)!}{(m-k)!(r-m)!}$$

$$= \binom{r}{k}\binom{r-k}{m-k}$$ (Showed)

$= RHS.$

INTEGER FUNCTIONS

Date :

(↓ FLOOR, CEIL ↑)

FLOOR: $\lfloor x \rfloor$ greatest integer less than or equal to x is n such that

$$\boxed{\begin{array}{l} n \leq x \\ n+1 > x \end{array}}$$

CEIL: $\lceil x \rceil$ The least integer greater than or equal to x is n such that $n \geq x$

$\Rightarrow n-1 < x$.

$\lfloor 2.6 \rfloor = 2$

$\lfloor 3 \rfloor = 3$.

$\lfloor -1.5 \rfloor = -2$

$\lceil 2.6 \rceil = 3$.

$\lceil 3 \rceil = 3$.

$\lceil -1.5 \rceil = -1$

Rules:

① $\lfloor x \rfloor = x$ or $\lceil x \rceil = x \iff x$ is integer

② $\lfloor -x \rfloor = -\lceil x \rceil$

$\lceil -x \rceil = -\lfloor x \rfloor$

③ $\lfloor x \rfloor = n \iff n \leq x < n+1$

④ $\lfloor x \rfloor = n \iff x-1 < n \leq x$

⑤ $\lceil x \rceil = n \iff n-1 < x \leq n$.

⑥ $\lceil x \rceil = n \iff x \leq n < x+1$

⑤ $\lfloor x+n \rfloor = \lfloor x \rfloor + n$, integer n.

⑥ $n \leq x \iff n \leq \lfloor x \rfloor$

⑦ $x \leq n \iff \lceil x \rceil \leq n$.

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Date :

If n is an m-bit integer, then prove that

$$m = \lfloor \log n \rfloor + 1$$

n is an m-bit integer

n is like $\underbrace{1xxxx\cdots x}_{m \text{ bits}}$; which means.

n is at least $\underbrace{1000\cdots 0}_{m \text{ bits}} = 2^{m-1}$

n is at most $\underbrace{1111\cdots 1}_{m \text{ bits}} = \underbrace{1000\cdots 0}_{m+1 \text{ bits}} - 1 = 2^m - 1$

So, $2^{m-1} \leq n \leq 2^m$

$\Rightarrow \log 2^{m-1} \leq \log n \leq \log 2^m$

$\Rightarrow (m-1)\cancel{\log 2} \leq \log n \leq m\cancel{\log 2}$

$\Rightarrow (m-1) \leq \log n \leq m$

$\Rightarrow \lfloor \log n \rfloor = (m-1)$ [$\because n \leq x < n+1 \Leftrightarrow \lfloor x \rfloor = n$]

$\Rightarrow m = \lfloor \log n \rfloor + 1$

Date :

Example : $n = 35$; $m = ?$

$m = \lfloor \log_2 35 \rfloor + 1$

$= 5 + 1$

$= 6$ (Ans.) 6 bits is needs to write 32. and 5 for 31.

Prove/disprove $\rightarrow \lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor$; for $x \geq 0$.

If x is integer, $\lfloor x \rfloor = x$.

$L.H.S = \lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor = R.H.S$.

again if x is not integer,

let, $m = \lfloor \sqrt{\lfloor x \rfloor} \rfloor$

$\Rightarrow m \leq \sqrt{\lfloor x \rfloor} < m+1$ $[\because n = \lfloor x \rfloor \Leftrightarrow n \leq x < n+1]$

$\Rightarrow m^2 \leq \lfloor x \rfloor < (m+1)^2$

$\Rightarrow m^2 \leq x < (m+1)^2$ $[\because n \leq x \Leftrightarrow n \leq \lfloor x \rfloor]$

$\Rightarrow m^2 \leq \sqrt{x} < (m+1)^2$

$\Rightarrow m \leq \sqrt{x} < (m+1)$

$\Rightarrow m = \lfloor \sqrt{x} \rfloor$ $[\because n \leq x < n+1 \Leftrightarrow n = \lfloor x \rfloor]$

$L.H.S = R.H.S$. (Proved)

MOD: $n \% m$.

$n = \lfloor \frac{n}{m} \rfloor \times m + n \bmod m$.

$\Rightarrow n \bmod m = n - m \lfloor \frac{n}{m} \rfloor$; $y \neq 0$.

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Date :

Example : ⊛ $43 \bmod 10 = 43 - 10 \left\lfloor \frac{43}{10} \right\rfloor$

$= 43 - (10 \times 4)$

$= 3.$

⊛ $5 \bmod -3 = 5 - (-3) \left\lfloor \frac{5}{-3} \right\rfloor$

$= 5 - 3 \left\lceil \frac{5}{3} \right\rceil$ $[\because \lfloor -x \rfloor = -\lceil x \rceil]$

$= -1$

⊛ $-5 \bmod 3 =$ (same)

⊛ $-5 \bmod -3 = -5 - (-3) \left\lfloor \frac{5}{-3} \right\rfloor$

$= -5 + 3 \times 1$

$= -2.$

⊛ $x \bmod 0 = x$ (predefined value).

Properties :

$c(x \bmod y) = cx \bmod cy$

$LHS = c(x \bmod y)$

$= c\left(x - y \left\lfloor \frac{x}{y} \right\rfloor\right).$

$= cx - cy \left\lfloor \frac{x}{y} \right\rfloor$

$= cx - cy \left\lfloor \frac{cx}{cy} \right\rfloor$

$= cx \bmod cy. = RHS.$

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Date :

prove or disprove $(x \bmod ny) \bmod y = x \bmod y$.

if y is 0. : $\rightarrow ny = 0$.

$$LH.S = (x \bmod 0) \bmod y$$
$$= x \bmod y. = RHS.$$

if $y \neq 0$,

$$LHS = (x \bmod ny) \bmod y$$
$$= \left(x - ny\left\lfloor \frac{x}{ny} \right\rfloor\right) \bmod y$$
$$= \left(x - ny\left\lfloor \frac{x}{ny} \right\rfloor\right) - y\left\lfloor \frac{x - ny\left\lfloor \frac{x}{ny} \right\rfloor}{y} \right\rfloor$$
$$= x - ny\left\lfloor \frac{x}{ny} \right\rfloor - y\left\lfloor \frac{x}{y} - \frac{ny\left\lfloor \frac{x}{ny} \right\rfloor}{y} \right\rfloor$$
$$= x - \cancel{ny\left\lfloor \frac{x}{ny} \right\rfloor} - y\left\lfloor \frac{x}{y} \right\rfloor + \cancel{ny\left\lfloor \frac{x}{ny} \right\rfloor}$$
$$= x - y\left\lfloor \frac{x}{y} \right\rfloor$$
$$= x \bmod y = RH.S.$$ (Proved).