Fourier series

Periodic Function:

If $F(t) = f(t+T) = f(t+2T) = \cdots\cdots$ then $f(t)$ is called the periodic function of period T. As for example:

$$\sin x = \sin(x+2\pi) = \sin(x+4\pi) = \cdots\cdots$$

So, $\sin x$ is a periodic function of period 2π.

Fourier series:

A series of sines and cosines of an angle and its multiples of the form.

$$\frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \cdots\cdots + a_n\cos nx + \cdots\cdots + b_1\sin x + b_2\sin 2x + \cdots\cdots + b_n\sin nx.$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} \sin nx$$ is called the Fourier series, where $a_0, a_1, a_2 \cdots\cdots a_n$, are constant.

Here,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx, \text{ or } \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$a_n = \frac{1}{\pi}\int_0^{2\pi} \cos nx\,dx = a_n \boxed{= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx.}$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\,\cancel{\cos nx}\,\sin nx\,dx,$ where $n = 1, 2, 3 \cdots$

Some useful integrals:

(i) $\int_0^{2\pi} \sin nx\, dx = 0$

(ii) $\int_0^{2\pi} \cos nx\, dx = 0$

(iii) $\int_0^{2\pi} \sin^2 nx\, dx = \pi$

(iv) $\int_0^{2\pi} \cos^2 nx\, dx = \pi$

(v) $\int_0^{2\pi} \sin nx \cdot \sin mx\, dx = 0$

(vi) $\int_0^{2\pi} \cos nx \cdot \cos mx\, dx = 0$

(vii) $\int_0^{2\pi} \sin nx \cdot \cos mx\, dx = 0$

(viii) $\int_0^{2\pi} \cos nx \cdot \sin mx = 0$

(ix) ~~sin~~ $\sin n\pi = 0$, $\cos n\pi = (-1)^n$, where $n \in I$

(x) $[u, v]_1 = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

where,

$$[u,v]_1 = \int uv\, dx,\ v_1 = \int v\, dx,\ v_2 = \int v_1\, dx$$

and so on.

$$u' = \frac{du}{dx},\ u'' = \frac{d^2u}{dx^2} \text{ and so on.}$$

$$[u,v] = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$

$$[u,v] = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$

Ex:1 Find the fourier series of $f(x) = x$, $0 < 2\pi$ and sketch its graph from $x = -4\pi$ to $x = 4\pi$.

Soln: Let,

$$F(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x + \cdots \quad (1)$$

Here,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = 1/\pi \int_0^{2\pi} x\,dx$$

$$= 1/\pi \left[\frac{x^2}{2}\right]_0^{2\pi}$$

$$= 1/2\pi\,(4\pi^2 - 0) = 2\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= 1/\pi \int_0^{2\pi} x \cos nx\,dx \qquad [u,v] = uv_1 - u'v_2 + u''v_3 - \cdots$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\left(-\frac{\cos nx}{n^2}\right) + 0\right]_0^{2\pi}$$

$$= 1/\pi \left[x\,\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{2\pi}$$

$$= 1/\pi \left[\left\{2\pi \cdot \frac{\sin n \cdot 2\pi}{} + \frac{\cos n\,2\pi}{n^2}\right\} - \left(0 + \frac{1}{n^2}\right)\right]$$

$$= 1/\pi \left[\left(0 + \frac{1}{n^2}\right) - \left(0 + \frac{1}{n^2}\right)\right] = 0$$

$u,v = uv_1 - u'v_2 + u''v_3 - \cdots$

Now, $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx \, dx$

$= \frac{1}{\pi}\int_0^{2\pi} x \sin nx \, dx$

$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1\left(-\frac{\sin nx}{n^2}\right) + 0\right]_0^{2\pi}$

$= \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{2\pi}$

$= \frac{1}{\pi}\left[\left\{-\frac{2\pi\cos 2n\pi}{n} + \frac{\sin 2\pi n}{n^2}\right\} - (-0+0)\right]$

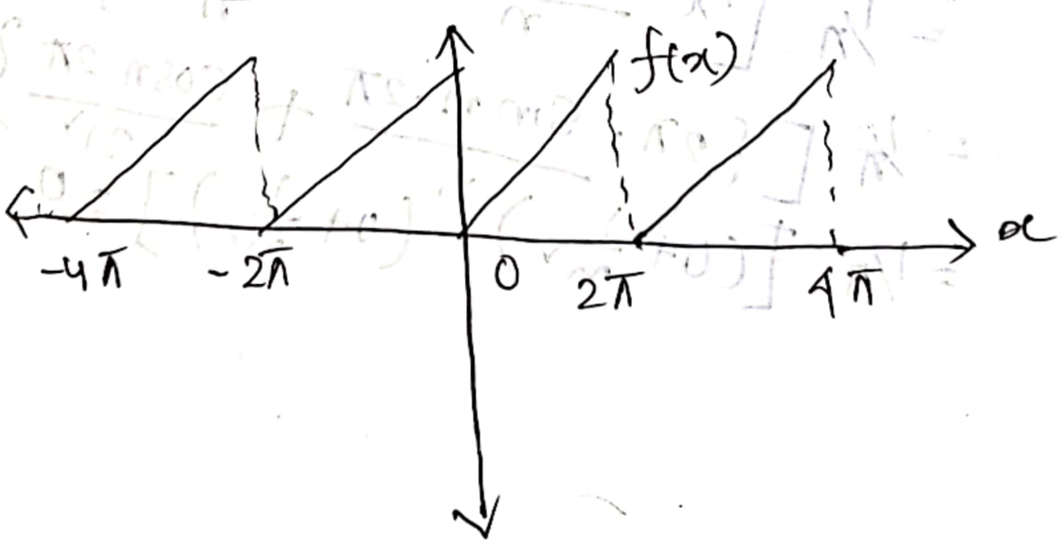
$= \frac{1}{\pi}\left(\frac{-2\pi\cos 2n\pi + \sin 2\pi n^2}{n}\right)$

$= \frac{1}{\pi}\left(-\frac{2\pi}{n}\right) = -\frac{2}{n}$

from (1),

$x = \frac{2\pi}{2} - \frac{2}{1}\sin x - \frac{2}{2}\sin 2x - \frac{2}{3}\sin 3x - \cdots$

$= \pi - 2\left[\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \cdots\right]$



Ex: find the fourier series of the following function:

$$f(x) = \begin{cases} -1 & \text{for } -\pi < x < -\pi/2 \\ 0 & \text{for } -\pi/2 < x < \pi/2 \\ 1 & \text{for } \pi/2 < x < \pi \end{cases}$$

$\Rightarrow$ we know,

let $f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x + \cdots$

Here,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{-\pi/2} (-1)\,dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} (0)\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} (1)\,dx$$

$$= -\frac{1}{\pi}[x]_{-\pi}^{-\pi/2} + 0 + \frac{1}{\pi}[x]_{\pi/2}^{\pi}$$

$$= -\frac{1}{\pi}\left[-\pi/2 + \pi\right] + 0 + \frac{1}{\pi}\left[\pi - \pi/2\right]$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\cos nx\,dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(0)\cos nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} 1\cos nx\,dx$$

$$= -\frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{-\pi/2} + 0 + \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi/2}^{\pi}$$

$$= -\frac{1}{\pi}\left[-\frac{1}{n}\sin\frac{n\pi}{2} + \frac{1}{n}\sin n\pi\right] + \frac{1}{\pi}\left[\frac{1}{n}\sin n\pi - \frac{1}{n}\sin\frac{n\pi}{2}\right]$$

$$= -\frac{1}{\pi}\left[-\frac{1}{n}\sin\frac{n\pi}{2}\right] + \frac{1}{\pi}\left[-\frac{1}{n}\sin\frac{n\pi}{2}\right]$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\sin nx\,dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(0)\sin nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi}(1)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{-\pi}^{-\pi/2} + 0 + \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{\pi n}\left[\cos n\pi/2 - \cos n\pi\right] - \frac{1}{\pi n}\left[\cos n\pi - \cos n\pi/2\right]$$

$$= \frac{1}{\pi n}\left[\cos n\pi/2 - \cos n\pi\right] + \frac{1}{\pi n}\left[\cos n\pi/2 - \cos n\pi\right]$$

* let $f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x + \cdots$

$$= \frac{2}{n\pi}\left[\cos n\pi/2 - \cos n\pi\right]$$

$$\therefore b_1 = \frac{2}{1\pi}\left[\cos \pi/2 - \cos\pi\right] = 2/\pi\left[0+1\right] = 2/\pi$$

$$\therefore b_2 = \frac{2}{2\cdot\pi}\left[\cos\pi - \cos 2\pi\right] = 1/\pi\left[-1-1\right] = -2/\pi.$$

Ex: find the fourier series for the periodic function:

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

$\Rightarrow$ let $f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x + \cdots$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} 0\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} (x)\,f(x)$$

$$= 0 + 1/\pi\left[\frac{x^2}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\pi^2 - (-\pi)^2\right]$$

$$= \frac{1}{2\pi}\left[\pi^2 - \pi^2\right] = 0$$

$$uv = uv_1 - u'v_2 + u''v_3 - u'''v_4$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(0)\times\cos nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x\times\cos nx\,dx$$

$$= 0 + \frac{1}{\pi}\left[x\left[\frac{\sin nx}{n}\right] - 1\left[-\frac{\cos nx}{n^2}\right] + 0\right]_{-\pi}^{\pi}$$

$$= 0 + \frac{1}{\pi}\left[\cancel{\frac{\pi}{n}(\sin n\pi)} + \cancel{\frac{1}{n^2}\left[\cos n\pi\right]} - \cancel{\frac{\pi}{n}(\sin n\pi)} - \cancel{\frac{1}{n^2}\left[\cos n\pi\right]}\right]$$

$$= 0$$

Even odd function :

Even

$f(x) = f(-x)$

$\cos x, x^2$

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$b_n = 0$

Odd

$f(x) = -f(-x)$

$\sin x, x, x^3$

$a_0 = 0$

$a_n = 0$

$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$.

Fall 2021

Ex: Find the fourier series expansion of the periodic function $f(x) = x^2$, $-\pi \leq x \leq \pi$. Hence find the sum of the series.

$\Rightarrow$ Given,

$f(x) = x^2$; $-\pi \leq x \leq \pi$, which is a even function.

So, $b_n = 0$

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi} \times \frac{\pi^3}{3}$

$= \frac{2\pi^2}{3}$

$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$

$= 2/\pi\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right) + 0\right]_0^{\pi}$

$= \frac{2}{\pi}\left[\left\{x^2\left(\frac{\sin n\pi}{n}\right) + 2\pi\,\frac{\cos n\pi}{n^2} - \frac{2\sin n\pi}{n^3}\right\} - 0\right]$

$= \frac{2}{\pi}\left[0 + \frac{2\pi}{n^2}(-1)^n - 0\right] = \frac{4}{n^2}(-1)^n$

$b_n = 0$

let,

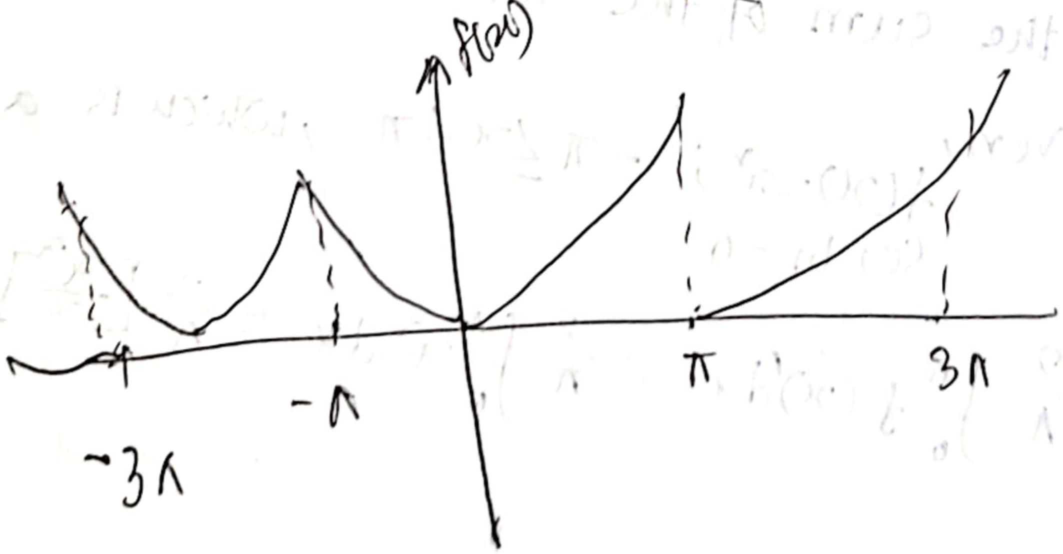
$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + b_1\sin x + b_2\sin 2x + \cdots$

$= \frac{2\pi^2/3}{2} - \frac{4}{1^2}\cos x + \frac{4}{2^2}\cos 2x + \cdots$

Ans

$= \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\cdots\right)$

Ans:



Spring 2021

~~Ex:~~ Find the fourier series expansion of the periodic function $f(x) = x^3$ for $-\pi < x < \pi$.

Let,

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x + \cdots$$

Now,

$a_0 = 0$

$a_n = 0$

$$\frac{2}{\pi}\left[\pi^3 \frac{(-\cos n\pi)}{n} + 6\pi \frac{\cos n\pi}{n^3}\right]$$

$$b_n = \frac{2}{\pi}\int_0^\pi f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^\pi x^3 \sin nx \, dx$$

$$= \frac{2}{\pi}\left[x^3\left(\frac{-\cos nx}{n}\right) - 3x^2\left(\frac{-\sin nx}{n^2}\right) + 6x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right) + 0\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\pi^3 \frac{(-\cos n\pi)}{n} + 6\pi \frac{\cos n\pi}{n^3}\right].$$

$$= 2(-1)^n \left[-\pi^2/n + 6/n^3\right]$$

$$f(x) = 2\left[-\left(-\frac{\pi^2}{1} + \frac{6}{1}\right)\sin x + \frac{1}{2}\left(-\frac{\pi^2}{2} + \frac{6}{8}\right)\sin 2x + \cdots\right]$$

Ans

Fourier series in complex form:

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{\frac{in\pi x}{l}} + \sum_{n=1}^{\infty} c_{-n} e^{-\frac{in\pi x}{l}}$$ where,

$$c_0 = \frac{a_0}{2} = \frac{1}{2l}\int_0^{2l} f(x)\,dx$$

$$c_n = \frac{1}{2l}\int_0^{2l} f(x)\, e^{-\frac{in\pi x}{l}}\,dx$$

$$c_{-n} = \frac{1}{2l}\int_0^{2l} f(x)\, e^{\frac{in\pi x}{l}}$$

Example: Obtain the complex form of the fourier series of the function $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ 1, & 0 \le x \le \pi \end{cases}$

Soln: Here,

Period $= \pi - (-\pi)$

$= \pi + \pi$

$= 2\pi$

$\Rightarrow 2l = 2\pi$

$\Rightarrow l = \pi$ $\therefore l = \pi$

For the given function, the fourier series in complex form;

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{inx} + \sum_{n=1}^{\infty} c_{-n} e^{-inx} \quad \text{---(1)}$$

Where,

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\cdot dx + \int_{0}^{\pi} 1\,dx\right]$$

$$= 1/2\pi\ \left[0 + [x]_0^\pi\right]$$

$$= 1/2\pi\ [\pi - 0] = 1/2 .$$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\cdot e^{-inx}dx + \int_{0}^{\pi} 1\cdot e^{-inx}\,dx\right]$$

$$= \frac{1}{2\pi}\int_{0}^{\pi} 1\cdot e^{-inx}\,dx$$

$$= 1/2\pi\left[\frac{e^{-inx}}{-in}\right]_0^\pi$$

$$= -\frac{1}{2\pi i n}\left[e^{-in\pi} - e^{0}\right]$$

$$= -\frac{1}{2\pi i n}\left[\cos n\pi - i\sin n\pi - 1\right]$$

$$= -\frac{1}{2\pi i n}\left[(-1)^n - 1\right]$$

$e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

$e^{in\pi} = \cos n\pi + i\sin n\pi$

$e^{-in\pi} = \cos n\pi - i\sin n\pi$

$\therefore \sin n\pi = 0$

$\therefore \cos n\pi = (-1)^n$

$$= \begin{cases} 0, \text{ when } n \text{ is even} \\ \frac{1}{in\pi}, \text{ '' } n \text{ is odd} \end{cases}$$

Now,

$$C_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{inx}\, dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0 \cdot e^{inx} dx + \int_{0}^{\pi} 1 \cdot e^{inx} dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{0}^{\pi} e^{inx} dx\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{inx}}{in}\right]_{0}^{\pi}$$

$$= \frac{1}{2\pi in}\left[e^{in\pi} - e^{0}\right]$$

$$= \frac{1}{2\pi in}\left[\cos n\pi + i\sin n\pi - 1\right]$$

$$= \frac{1}{2\pi in}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0, \text{ when } n \text{ is even} \\ -\frac{1}{in\pi}, \text{ '' } n \text{ is odd} \end{cases}$$

$$f(x) = \frac{1}{2} + \left[\frac{1}{i\pi} e^{ix} + \frac{1}{3i\pi} e^{3ix} + \frac{1}{5i\pi} e^{5ix} + \cdots \right] + \left[-\frac{1}{i\pi} e^{-ix} + \frac{(-1)}{3i\pi} e^{-3ix} + \cdots \right]$$

Ans:

Fourier Transforms

If $f(x)$ is a function,

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

The inverse fourier transform of a function $F(s)$,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s)\, e^{-isx}\, ds$$

The fourier sine transform of a function $f(x)$,

$$f(s) = \sqrt{2/\pi} \int_0^{\infty} f(x) \cdot \sin sx\, dx$$

The inverse " " " of a " $f(s)$,

$$f(x) = \sqrt{2/\pi} \int_0^{\infty} f(x) \cdot \sin sx\, ds.$$

The fourier cosine transform of a function $f(x)$,

$$f(s) = \sqrt{2/\pi} \int_0^{\infty} f(x) \cos sx\, dx$$

The inverse " " " " " " "

$$f(x) = \sqrt{2/\pi} \int_0^{\infty} f(x) \cdot \cos sx\, ds.$$

$x<a$
$-x<a$
$\Rightarrow x>-a$

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Ex: Find the fourier transform of the function $f(x) = \begin{cases} 1 & \text{for } |x|<a \\ 0 & \text{for } |x|>a \end{cases}$

$x<a$
$-x<a$
$\Rightarrow x>-a$

Sol: The fourier transform of function $f(x)$,

$$f(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{isx}\, dx$$

Substituting the value of $f(x)$, we get,

$$f(s) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 1 \cdot e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is} \right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{is} \left[e^{ias} - e^{-ias} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2}{s} \cdot \frac{e^{ias} - e^{-ias}}{2i}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2 \sin as}{s} = \sqrt{2/\pi}\, \frac{\sin as}{s}$$

Ans

Ex: Find the fourier transform of the following function: $f(x) = \begin{cases} 1-x^2 & \text{if } |x| \le 1 \\ 0 & \text{if } |x| > 1 \end{cases}$

$x \le 1$
$-x \le 1$
$x \ge -1$

$\Rightarrow$ We know,

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx \quad \text{---(1)}$$

Substituting the value of $f(x)$ in (1), we get;

$$f(s) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[(1-x^2) \cdot \frac{e^{isx}}{is} - (-2x) \cdot \frac{e^{isx}}{(is)^2} + (-2) \frac{e^{isx}}{(is)^3} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left\{ 0 + (-2) \frac{e^{is}}{s^2} - (-2) \frac{e^{is}}{is^3} \right\} - \left\{ 0 + (2) \frac{e^{-is}}{s^2} - (-2) \frac{e^{-is}}{is^3} \right\} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[-2 \frac{e^{is}}{s^2} + 2 \frac{e^{is}}{is^3} - 2 \frac{e^{-is}}{s^2} - 2 \frac{e^{-is}}{is^3} \right]$$

(Ans:)

Spring 2021

Ex: Determine the fourier transform of
$$f(x) = \begin{cases} x & \text{when } |x| \le a \\ 0 & \text{when } |x| > a \end{cases}$$

$\boxed{x \le a}$

$-x \le a$

$\boxed{\Rightarrow x \ge -a}$

$\Rightarrow$ We know,
$$f(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x\, e^{isx}\, dx$$
$$= \frac{1}{\sqrt{2\pi}} \left[x \cdot \frac{e^{isx}}{is} - 1 \times \frac{e^{isx}}{(is)^2} + 0 \right]_{-a}^{a}$$
$$= \frac{1}{\sqrt{2\pi}} \left[\left\{ a \cdot \frac{e^{isa}}{is} + 1 \times \frac{e^{isa}}{s^2} \right\} - \left\{ (-a) \cdot \frac{e^{-isa}}{is} + 1 \times \frac{e^{-isa}}{s^2} \right\} \right]$$
$$= \frac{1}{\sqrt{2\pi}} \left[\frac{ae^{isa}}{is} + \frac{e^{isa}}{s^2} + \frac{ae^{-isa}}{is} - \frac{e^{-isa}}{s^2} \right]$$

(Ans:)

Fourier Sine Transform

Ex: Find the fourier sine transform of the following function:

$$f(x) = \begin{cases} \sin x & \text{when } 0<x<a \\ 0 & \text{when } x>a \end{cases}$$

⇒ We know,

$$f(s) = \sqrt{2/\pi} \int_0^\infty f(x) \cdot \sin sx \, dx$$

$$= \sqrt{2/\pi} \int_0^a f(x) \sin sx \, dx + \sqrt{2/\pi} \int_a^\infty f(x) \sin sx \, dx$$

$$= \sqrt{2/\pi} \left[\int_0^a \sin x \cdot \sin sx \, dx + \int_a^\infty 0 \cdot \sin sx \, dx \right]$$

$$= \sqrt{2/\pi} \int_0^a \sin x \cdot \sin sx \, dx + 0$$

SC

CS

CC

$s+s$

$s-s$

$c+c$

$$= \sqrt{2/\pi} \times \frac{1}{2} \int_0^a 2 \sin x \cdot \sin sx \, dx$$

$$= \sqrt{2/\pi} \times \frac{1}{2} \int_0^a \left[\cos(x - sx) - \cos(x + sx) \right] dx$$

$$= \sqrt{2/\pi} \times \frac{1}{2} \int_0^a \left[\cos(1-s)x - \cos(1+s)x \right] dx$$

$$= \sqrt{2/\pi} \times \frac{1}{2} \left[\frac{x \sin(1-s)}{(1-s)} - \frac{x \sin(1+s)}{(1+s)} \right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left\{ \frac{a \cdot \sin(1-s)}{(1-s)} - \frac{a \sin(1+s)}{(1+s)} \right\} - 0 \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(1-s)a}{(1-s)} - \frac{\sin(1+s)a}{(1+s)} \right]$$

(Ans)

Ex: find the fourier sine transform of $\frac{e^{-ax}}{x}$

$\Rightarrow$ Here,

$$f(x) = \frac{e^{-ax}}{x}$$

we know,

$$f(s) = \sqrt{2/\pi} \int_0^\infty f(x) \cdot \sin sx \, dx$$

$$= \sqrt{2/\pi} \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx$$

Then,

$$I = \sqrt{2/\pi} \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx \quad \text{---- (1)}$$

Differentiating (1) with respect to "s" we get,

$$\frac{dI}{ds} = \sqrt{2/\pi} \int_0^\infty \frac{e^{-ax}}{\cancel{x}} \cos sx \cdot \cancel{x} \, dx$$

$$= \sqrt{2/\pi} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{2/\pi}\left[\frac{e^{-ax}}{a^2+s^2}(s\sin sx - a\cos sx)\right]_0^\infty$$

$$\Rightarrow \frac{dI}{ds} = \sqrt{2/\pi}\cdot\left[\frac{a}{a^2+s^2}\right]$$

$\left[\because e^{-ax}\cos bx = \frac{e^{-ax}}{a^2+b^2}(b\sin bx - a\cos bx)\right]$

$$\Rightarrow dI = \sqrt{2/\pi}\cdot\frac{a}{a^2+s^2}\,ds$$

$$\Rightarrow \int dI = \sqrt{2/\pi}\; a\int\frac{ds}{a^2+s^2}$$

$$\Rightarrow I = \sqrt{2/\pi}\cdot a\cdot\frac{1}{a}\tan^{-1}(s/a) + A$$

$\left[\because \int\frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c\right]$

We can get,

$A = 0$

$$\therefore I = \sqrt{2/\pi}\,\tan^{-1}(s/a)$$

Ans.

Fourier cosine transform

* Find the fourier cosine transform of $f(x) = e^{-2x} + 4e^{-3x}$.

$\Rightarrow$ We know,

$$f(s) = \sqrt{2/\pi} \int_0^\infty f(x) \cos sx \, dx$$

Now,

$$f(s) = \sqrt{2/\pi} \int_0^\infty \left(e^{-2x} + 4e^{-3x}\right) \cos sx \, dx$$

$$= \sqrt{2/\pi} \int_0^\infty e^{-2x} \cos sx \, dx + \sqrt{2/\pi} \int_0^\infty 4e^{-3x} \cos sx \, dx$$

$$= \sqrt{2/\pi} \left[\frac{e^{-2x}}{2^2+s^2} (s \sin sx - 2\cos sx) \right]_0^\infty +$$

$$4\sqrt{2/\pi} \left[\frac{e^{-3x}}{9+s^2} (s \sin sx - 3\cos sx) \right]_0^\infty$$

$$= \sqrt{2/\pi} \cdot \frac{2}{2^2+s^2} + 4\sqrt{2/\pi} \cdot \frac{3}{9+s}$$

(Ans)

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Ex: Find the fourier cosine transform of $\frac{e^{-ax}}{x}$

$\Rightarrow$ Here,

$$f(x) = \frac{e^{-ax}}{x}$$

We know,

$$f(s) = \sqrt{2/\pi} \int_0^\infty f(x) \cdot \cos . sx \, dx$$

$$= \sqrt{2/\pi} \int_0^\infty \frac{e^{-ax}}{x} \cdot \cos . sx \, dx$$

Then,

$$I = \sqrt{2/\pi} \int_0^\infty \frac{e^{-ax}}{x} \cos . sx \, dx \quad \text{---} (1)$$

Differentiating (1) with the respect to "s",

$$\frac{dI}{ds} = \sqrt{2/\pi} - \int_0^\infty \frac{e^{-ax}}{\cancel{x}} \sin . sx . \cancel{x} \, dx$$

$$\Rightarrow \frac{dI}{ds} = \sqrt{2/\pi} - \int_0^\infty e^{-ax} \sin . sx \, dx$$

$$\Rightarrow \frac{dI}{ds} = \sqrt{2/\pi} \left[\frac{e^{-ax}}{a^2+s^2} (-a \sin sx - s \cos sx) \right]_0^\infty$$

$$\Rightarrow \frac{dI}{ds} = \sqrt{2/\pi} \left[\frac{-s}{a^2+s^2} \right]$$

$$\Rightarrow dI = \sqrt{2/\pi} \times \frac{-s}{a^2+s^2} \, ds$$

$\Rightarrow$

$$\Rightarrow \int dI = \sqrt{2/\pi} \times (\bar{\omega}S) \int \frac{1}{a^2+s^2}\, ds$$

$$\Rightarrow I = \sqrt{2/\pi}\, (\bar{\omega}S) \cdot \frac{1}{a} \tan^{-1}(s/a) + A$$

we get,

$A = 0$

$$\therefore I = \sqrt{2/\pi} \left(\frac{\bar{\omega}S}{a}\right) \tan^{-1}(s/a)$$

Ans.

Inverse fourier

Ex: Find the inverse fourier cosine transform of $F(S) = e^{-s}$

$\Rightarrow$ We know,

$$f(S) = \sqrt{2/\pi}\int_0^\infty f(x) \cdot \cos sx \cdot ds$$

$$= \sqrt{2/\pi}\int_0^\infty e^{-s} \cos sx \, ds$$

$$= \sqrt{2/\pi}\left[\frac{e^{-s}}{1+x^2}(x \sin sx - 1 \cdot \cos sx)\right]_0^\infty$$

$$= \sqrt{2/\pi} \times \frac{1}{1+x^2}$$

Ans.

Ex: 2 Find the inverse fourier sine transform of $F(S) = \frac{e^{-as}}{s}$.

$\Rightarrow$ Here, $f(s) = \frac{e^{-as}}{s}$

We know,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(s) \cdot \sin . sx \cdot ds$$

$$= \sqrt{2/\pi} \int_0^\infty \frac{e^{-as}}{s} \sin \cdot sx \cdot ds$$

$$I = \sqrt{2/\pi} \int_0^\infty \frac{e^{-as}}{s} \sin \cdot sx \cdot ds \quad —(1)$$

Now,

$$\frac{dI}{dx} = \sqrt{2/\pi} \int_0^\infty \frac{e^{-as}}{\cancel{s}} \cos \cdot sx \cdot \cancel{s}\, ds$$

$$\Rightarrow \frac{dI}{dx} = \sqrt{2/\pi} \int_0^\infty e^{-as} \cos sx \, ds$$

$$= \sqrt{2/\pi} \left[\frac{e^{-as}}{a^2+x^2} (x \sin xs - a \cos xs) \right]_0^\infty$$

$$= \sqrt{2/\pi} \frac{a}{a^2+x^2}$$

$$\Rightarrow dI = \sqrt{2/\pi} \frac{a}{a^2+x^2} dx$$

$$\Rightarrow \int dI = \sqrt{2/\pi} \times a \int \frac{1}{a^2+x^2}\, dx$$

$$\Rightarrow I = \sqrt{2/\pi} \times \cancel{a^2} \frac{1}{\cancel{a}} \tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\Rightarrow I = \sqrt{2/\pi} \times \tan^{-1}(x/a) + c \text{ ----(2)}$$

Now,

$c = 0$

Then,

$$I = \sqrt{2/\pi} \times \tan^{-1}(x/a)$$

Ans:

$b = x$

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* Find the inverse fourier sine transform of $F(s) = e^{-3s}$

$\Rightarrow$ Here, $F(s) = e^{-3s}$

We know,

$$F(x) = \sqrt{2/\pi} \int_0^\infty f(s) \cdot \sin \cdot sx \, ds$$

$$\Rightarrow f(x) = \sqrt{2/\pi} \int_0^\infty e^{-3s} \cdot \sin \cdot sx \, ds$$

$$\cancel{\Rightarrow f(x) = \sqrt{2/\pi} \left[\frac{e^{-3s}}{3^2+x^2} (-3 \sin xs - x \cos xs) \right]_0^\infty}$$

$$\cancel{\Rightarrow f(x) = \sqrt{2/\pi} \times \left(\frac{-x}{3^2+x^2} \right)}$$

$$\cancel{\therefore f(x) = \sqrt{2/\pi} \times \left(-x/3^2+x^2 \right)}$$

$$\Rightarrow f(x) = \sqrt{2/\pi} \left[\frac{e^{-3s}}{9+x^2} (-3 \sin xs - x \cos xs) \right]_0^\infty$$

$$\rightarrow f(x) = \sqrt{2/\pi} \left(\frac{x}{9+x^2} \right)$$