(i) Inverse of $\begin{bmatrix} 4 & 3 \\ -7 & 1 \end{bmatrix}$ is

(i) $\begin{bmatrix} \frac{1}{4} & \frac{1}{3} \\ \frac{-1}{7} & 1 \end{bmatrix}$ (ii) $\frac{1}{25}\begin{bmatrix} 4 & 3 \\ -7 & 1 \end{bmatrix}$ (iii) $\frac{1}{25}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (iv) $\frac{1}{25}\begin{bmatrix} 1 & -3 \\ 7 & 4 \end{bmatrix}$

(j) In the matrix equation $\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ the values of x and y are

(i) $x = 3, y = -1$ (ii) $x = 2, y = 5$ (iii) $x = 1, y = -1$ (iv) $x = -1, y$ ...

(k) The rank of $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is equal to

(i) 4 (ii) 3 (iii) 5 (iv) 1

Ans. (a) (iv), (b) (iii), (c) (ii), (d) (ii), (e) (iii), (f) (i), (g) (ii), (h) (iv), (i) (iv), (j) (iii), ...

21. An $n \times n$ homogeneous system of equations $AX = 0$ is given. The rank of A is $r < n$. The...

(a) $n - r$ independent solutions (b) r independent solutions

(c) no solution (d) $n - 2r$ independent solutions.

4.41 LINEAR DEPENDENCE AND INDEPENDENCE OF VECTORS

Vectors (matrices) $X_1, X_2, \ldots X_n$ are said to be dependent if

(1) all the vectors (row or column matrices) are of the same order.

(2) n scalars $\lambda_1, \lambda_2, \ldots \lambda_n$ (not all zero) exist such that

$$\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + \ldots\ldots + \lambda_n X_n = 0$$

Otherwise they are linearly independent.

Example 53. *Examine the following vectors for linear dependence and find the relation if it exists.*

$$X_1 = (1, 2, 4),\ X_2 = (2, -1, 3),\ X_3 = (0, 1, 2),\ X_4 = (-3, 7, 2)$$

Solution. Consider the matrix equation

$$\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + \lambda_4 X_4 = 0$$

$$\Rightarrow \lambda_1 (1, 2, 4) + \lambda_2 (2, -1, 3) + \lambda_3 (0, 1, 2) + \lambda_4 (-3, 7, 2) = 0$$

$$\lambda_1 + 2\lambda_2 + 0\lambda_3 - 3\lambda_4 = 0$$
$$2\lambda_1 - \lambda_2 + \lambda_3 + 7\lambda_4 = 0$$
$$4\lambda_1 + 3\lambda_2 + 2\lambda_3 + 2\lambda_4 = 0$$

This is the homogeneous system

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad A\lambda = 0$$

$R_2 - 2R_1,\ R_3 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 + 2\lambda_2 - 3\lambda_4 = 0$$
$$-5\lambda_2 + \lambda_3 + 13\lambda_4 = 0$$
$$\lambda_3 + \lambda_4 = 0$$

$\lambda_4 = t$,
$$\lambda_3 + t = 0,\ \lambda_3 = -t$$
$$-5\lambda_2 - t + 13t = 0,\ \lambda_2 = \frac{12t}{5}$$
$$\lambda_1 + \frac{24t}{5} - 3t = 0 \text{ or } \lambda_1 = \frac{-9t}{5}$$

Hence the given vectors are linearly dependent.

Substituting the values of λ in (1), we get

$$-\frac{9tX_1}{5} + \frac{12t}{5}X_2 - tX_3 + tX_4 = 0 \quad \text{or} \quad -\frac{9X_1}{5} + \frac{12X_2}{5} - X_3 + X_4 = 0$$

$$9X_1 - 12X_2 + 5X_3 - 5X_4 = 0$$ **Ans.**

Example 54. *Define linear dependence and independence of vectors.*

Examine for linear dependence [1, 0, 2, 1], [3, 1, 2, 1], [4, 6, 2, –4], [–6, 0, –3, –4] *and find the relation between them, if possible.*

Solution. Consider the matrix equation

$$\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + \lambda_4 X_4 = 0 \quad \ldots(1)$$

$$\lambda_1 (1, 0, 2, 1) + \lambda_2 (3, 1, 2, 1) + \lambda_3 (4, 6, 2, -4) + \lambda_4 (-6, 0, -3, -4) = 0$$

$$\lambda_1 + 3\lambda_2 + 4\lambda_3 - 6\lambda_4 = 0$$
$$0\lambda_1 + \lambda_2 + 6\lambda_3 + 0\lambda_4 = 0$$
$$2\lambda_1 + 2\lambda_2 + 2\lambda_3 - 3\lambda_4 = 0$$
$$\lambda_1 + \lambda_2 - 4\lambda_3 - 4\lambda_4 = 0$$

$$\begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 2 & 2 & 2 & -3 \\ 1 & 1 & -4 & -4 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 - 2R_1,\quad R_4 - R_1$

$$\begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 0 & -4 & -6 & 9 \\ 0 & -2 & -8 & 2 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 + 4R_2,\quad R_4 + 2R_2$

$$\begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 18 & 9 \\ 0 & 0 & 4 & 2 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_4 - \frac{2}{9} R_3$$

$$\begin{bmatrix} 1 & 3 & 4 & -6 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 18 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 + 3\lambda_2 + 4\lambda_3 - 6\lambda_4 = 0$$

$$\lambda_2 + 6\lambda_3 = 0$$

$$18\lambda_3 + 9\lambda_4 = 0$$

Let $\lambda_4 = t$,

$$18\lambda_3 + 9t = 0 \text{ or } \lambda_3 = \frac{-t}{2}$$

$$\lambda_2 - 3t = 0 \text{ or } \lambda_2 = 3t$$

$$\lambda_1 + 9t - 2t - 6t = 0$$

$$\lambda_1 = -t$$

Substituting the values of λ_1, λ_2, λ_3 and λ_4 in (1), we get

$$-tX_1 + 3tX_2 - \frac{t}{2}X_3 + tX_4 = 0 \text{ or } \boxed{2X_1 - 6X_2 + X_3 - 2X_4 = 0}$$

Example 55. *Is the system of vectors*

$$X_1 = (2, 2, 1)^T,\ X_2 = (1, 3, 1)^T,\ X_3 = (1, 2, 2)^T$$

linearly dependent.

Solution. Here $X_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, $X_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ (*T* stands for transposition)

Consider the matrix equation

$$\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 = 0$$

$$\lambda_1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2\lambda_1 + \lambda_2 + \lambda_3 = 0$$

$$2\lambda_1 + 3\lambda_2 + 2\lambda_3 = 0$$

$$\lambda_1 + \lambda_2 + 2\lambda_3 = 0$$

which is the homogeneous equation.

$$R_1 \longleftrightarrow R_3$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 - 2R_1,\ R_3 - 2R_1 \qquad R_3 + R_2$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & -1 & -3 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & -5 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 + \lambda_2 + 2\lambda_3 = 0$$

$$\lambda_2 - 2\lambda_3 = 0$$

$$-5\lambda_3 = 0 \Rightarrow \lambda_3 = 0$$

$$\lambda_2 = 0 \text{ and } \lambda_1 = 0$$

...us non-zero values of λ_1, λ_2, λ_3 do not exist which can satisfy (1). Hence by definition, ...n system of vectors is not linearly dependent. **Ans.**

Exercise 4.18

...the following system of vectors for linear dependence. If dependent, find the relation between them.

... $(1, -1, 1)$, $X_2 = (2, 1, 1)$, $X_3 = (3, 0, 2)$. **Ans.** Dependent, $X_1 + X_2 - X_3 = 0$

... $(1, 2, 3)$, $X_2 = (2, -2, 6)$. **Ans.** Independent

... $(3, 1, -4)$, $X_2 = (2, 2, -3)$, $X_3 = (0, -4, 1)$. **Ans.** Dependent, $2X_1 - 3X_2 - X_3 = 0$

... $V_1 = (1, -1, 0)$, $V_2 = (0, 1, -1)$, $V_3 = (0, 0, 1)$ be elements of R^3. The set of vectors $\{V_1, V_2, V_3\}$ is ...linearly independent (*b*) linearly dependent (*c*) null (*d*) none of these (*AMIETE Dec. 2005*) **Ans.** (*b*)

... $(1, 1, 1, 3)$, $X_2 = (1, 2, 3, 4)$, $X_3 = (2, 3, 4, 7)$. **Ans.** Dependent, $X_1 + X_2 - X_3 = 0$

... $(1, 1, -1, 1)$, $X_2 = (1, -1, 2, -1)$, $X_3 = (3, 1, 0, 1)$. **Ans.** Dependent, $2X_1 + X_2 - X_3 = 0$

... $(1, -1, 2, 0)$, $X_2 = (2, 1, 1, 1)$, $X_3 = (3, -1, 2, -1)$, $X_4 = (3, 0, 3, 1)$.

Ans. Dependent, $X_1 + X_2 - X_4 = 0$

... $(1, 1, 0, 1)$, $V_2 = (1, 1, 1, 1)$, $V_3 = (4, 4, 1, 1)$ and $V_4 = (1, 0, 0, 1)$

(*A.M.I.E.T.E., June 2006*) **Ans.** Linearly independent

...the value of α so that the vectors $(1, 2, 9, 8)$, $(2, 0, 0, \alpha)$, $(\alpha, 0, 0, 8)$, $(0, 0, 1, 0)$ are linearly ...endent. (*A.M.I.E.T.E., Summer 2005*) **Ans.** $\alpha = \pm 4$.

...RTITIONING OF MATRICES

...matrix. A matrix obtained by deleting some of the rows and columns of a matrix ...to be sub matrix.

...example, $A = \begin{bmatrix} 4 & 1 & 0 \\ 5 & 2 & 1 \\ 6 & 3 & 4 \end{bmatrix}$, then $\begin{bmatrix} 4 & 1 \\ 5 & 2 \end{bmatrix}, \begin{bmatrix} 5 & 2 \\ 6 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ are the sub matrices.

...itioning: A matrix may be subdivided into sub matrices by drawing lines parallel to ...and columns. These sub matrices may be considered as the elements of the original ...

...example,

$$A = \left[\begin{array}{cc:ccc} 2 & 1 & 0 & 4 & 1 \\ 1 & 0 & 2 & 3 & 4 \\ \hdashline 4 & 5 & 1 & 6 & 5 \end{array}\right]$$

$$A_{11} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix},\quad A_{12} = \begin{bmatrix} 0 & 4 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

$$A_{21} = [4 \quad 5],\quad A_{22} = [1 \quad 6 \quad 5]$$

...en we may write $\quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

...the matrix is partitioned. The dotted lines divide the matrix into sub-matrices. ...A_{21}, A_{22} are the sub-matrices but behave like elements of the original matrix A. The ...can be partitioned in several ways.