Ex: 70 : Find the directional derivative of div($\bar{u}$) at (1,2,2) in the direction of the outer normal to the sphere $x^2+y^2+z^2=9$ for $\bar{u}=x^4\hat{i}+y^4\hat{j}+z^4\hat{k}$

Sol^n: div $\bar{u}$ $=\nabla\cdot\bar{u}$

$=(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z})\cdot(x^4\hat{i}+y^4\hat{j}+z^4\hat{k})$

$=4x^3+4y^3+4z^3$

Outer normal of the sphere $=\nabla(x^2+y^2+z^2-9)$

$$=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-9)=2x\ldots$$

Outer normal of the sphere at (1, 2, 2) $=2i+4j+4k$

Directional derivative $=\nabla(4x^3+4y^3+4z^3)$

$$=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)(4x^3+4y^3+4z^3)=12x\ldots$$

Directional derivative at (1, 2, 2) $=12i+48j+48k$

Directional derivative along the outer normal $=(12i+48j+48k)\cdot\frac{2i+4j\ldots}{\sqrt{4+16\ldots}}$

$$=\frac{24+192+192}{6}=68$$

Example 71. *Find the directional derivative of the* divergence of *$f(x,y,z)=xyi+xy^2j+z^2k$ at the point (2, 1, 2) in the direction of the outer... sphere, $x^2+y^2+z^2=9$.*

Solution. $f(x,y,z)=xyi+xy^2j+z^2k$

Divergence of $f(x,y,z)=\nabla\cdot f$ $=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot(xyi+xy^2j+z^2k)=\ldots$

Directional derivative of divergence of $(y+2xy+2z)$

$$=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)(y+2xy+2z)$$

$$=2yi+(1+2x)j+2k$$

Directional derivative at the point (2, 1, 2) $=2i+5j+2k$

Normal to the sphere $=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-9)=2\ldots$

Normal at the point (2, 1, 2) $=4i+2j+4k$

Directional derivative along normal at (2, 1, 2) $=(2i+5j+2k)\cdot\frac{(4i+2\ldots}{\sqrt{16+\ldots}}$

$$=\frac{1}{6}(8+10+8)=\frac{13}{3}$$

Question. If $r=xi+yj+zk$, and $r=|r|$ find div$\left(\frac{r}{r^2}\right)$

5.38 CURL

The curl of a vector point function F is defined as below

curl $\bar{F}=\nabla\times\bar{F}$ $\quad(\bar{F}=F_1i+F_2j+F_3k)$

$$=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)\times(F_1i+F_2j+F_3k)$$

$$=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$=i\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)-j\left(\frac{\partial F_3}{\partial x}-\frac{\partial F_1}{\partial z}\right)+k\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$$

...is a vector quantity.

...SICAL MEANING OF CURL

...w $V=\omega\times\bar{r}$, where ω is the angular velocity, V is the linear velocity and $\bar{r}$ is the ...tor of a point on the rotating body.

curl $V=\nabla\times V$ $\quad\left[\begin{array}{l}\omega=\omega_1i+\omega_2j+\omega_3k\\ r=xi+yj+zk\end{array}\right]$

$$=\nabla\times(\omega\times\bar{r})$$

$$=\nabla\times[(\omega_1i+\omega_2j+\omega_3k)\times(xi+yj+zk)]$$

$$=\nabla\times\begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$=\nabla\times[(\omega_2z-\omega_3y)i-(\omega_1z-\omega_3x)j+(\omega_1y-\omega_2x)k]$$

$$=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times[(\omega_2z-\omega_3y)i-(\omega_1z-\omega_3x)j+(\omega_1y-\omega_2x)k]$$

$$=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2z-\omega_3y & \omega_3x-\omega_1z & \omega_1y-\omega_2x \end{vmatrix}$$

$$=(\omega_1+\omega_1)i-(-\omega_2-\omega_2)j+(\omega_3+\omega_3)k$$

$$=2(\omega_1i+\omega_2j+\omega_3k)=2\omega$$

...2ω which shows that curl of a vector field is connected with rotational properties ...field and justifies the name *rotation* used for curl.

...$F=0$, the field F is termed *irrotational*.

...le 72. *Calculate the curl of the vector $xyzi+3x^2yj+(xz^2-y^2z)k$.*

...on. curl $\bar{f}=\nabla\times\bar{f}=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times f$

$$f=xyzi+3x^2yj+(xz^2-y^2z)k$$

out of worked problems only tick marked will be in Q-02 exercises

Ex: 5.20
Ex: 5.19
Tick marked problems

Ex: 68 - Find the value of n for which the vector $r^n\bar{r}$ is solenoidal.

see H.K Dass worked out

$$\text{Curl } f = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$$= -2yzi - (z^2 - xy)\,j + (6xy - xz)\,k$$

$$= -2yzi + (xy - z^2)\,j + (6xy - xz)\,k$$

Example 73. *A vector $\bar{r}$ is defined by $\bar{r} = ix + jy + kz$. If $|\bar{r}| = r$ then show t $r^n \bar{r}$ is irrotational.*

Solution. $\text{Curl } \bar{F} = \nabla \times \bar{F} = \nabla \times r^n \bar{r}$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times (x^2+y^2+z^2)^{n/2}(xi + y$$

$$= \begin{vmatrix} i & j \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ (x^2+y^2+z^2)^{n/2}x & (x^2+y^2+z^2)^{n/2}y \end{vmatrix}$$

$$= \left[\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}(2yz) - \frac{n}{2}(x^2+y^2+z^2)\right.$$

$$-\left[\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}(2xz) - \frac{n}{2}(x^2+y^2+z\right.$$

$$+\left[\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}(2xy) - \frac{n}{2}(x^2+y^2\right.$$

$$= 0$$

Hence $r^n \bar{r}$ is irrotational.

Example 74. *Show that gradient field describing a motion is irrotational.*

Solution. Let a field be $f(x, y, z)$.

$$\text{Gradient } f(x,y,z) = \nabla f.$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$\text{Curl of Gradient } f(x,y,z) = \nabla \times \left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}\right)$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times \left(i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}\right)$$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix}$$

$$= i\left[\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right] - j\left[\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right] + k\left[\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right]$$

$$= 0$$

rl of Gradient field is zero, its motion is irrotational. **Proved.**

ple 75. *Show that $\vec{V}(x,y,z) = 2xyz\,i + (x^2z + 2y)\,j + x^2y\,k$ is irrotational and find tion $u(x,y,z)$ such that $\vec{V} = \text{grad}(u)$.*

n. $V(x,y,z) = 2xyz\,i + (x^2z + 2y)\,j + x^2y\,k$.

$$\text{curl } V = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times [2xyzi + (x^2z + 2y)j + x^2yk]$$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xyz & x^2z + 2y & x^2y \end{vmatrix}$$

$$= (x^2 - x^2)\,i - (2xy - 2xy)\,j + (2xz - 2xz)\,k = 0$$

$V(x, y, z)$ is irrotational.

corresponding scalar function u, consider the following relations given

$$v = \text{grad}(u) \quad \text{or} \quad v = \nabla(u) \qquad \ldots(1)$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

$$= \left(i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} + k\frac{\partial u}{\partial z}\right)\cdot(i\,dx + j\,dy + k\,dz)$$

$$= \nabla u \cdot \vec{dr} = \vec{v}\cdot\vec{dr} \qquad \text{From (1)}$$

$$= [2xyz\,i + (x^2z + 2y)\,j + x^2y\,k]\cdot(i\,dx + j\,dy + k\,dz)$$

$$= 2xyz\,dx + (x^2z + 2y)\,dy + x^2y\,dz$$

$$= y\,(2xz\,dx + x^2\,dz) + (x^2z)\,dy + 2y\,dy$$

$$= [y\,d(x^2z) + (x^2z)\,dy] + 2y\,dy = d(x^2yz) + 2y\,dy$$

ating we get $\quad u = x^2yz + y^2$ **Ans.**

ple 76. *Show that $\vec{F} = (y^2 + 2xz^2)\,i + (2xy - z)\,j + (2x^2z - y + 2z)\,\hat{k}$ is irrotational find its scalar potential.* (A.M.I.E.T.E., Dec. 2004)

tion. curl $\vec{F}$

$$= \nabla \times \vec{F} = \left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right] \times [(y^2 + 2xz^2)\,\hat{i} + (2xy - z)\,\hat{j} + (2x^2z - y + 2z)\,\hat{k}.]$$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 + 2xz^2 & 2xy - z & 2x^2z - y + 2z \end{vmatrix}$$

$$= (-1+1)\,i - (4xz - 4xz)\,j + (2y - 2y)\,k$$

$$= 0$$

irrotational.

$$F = \text{grad } u \text{ or } F = \nabla(u)$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

$$= \left(i\frac{\partial u}{\partial x} + j\frac{\partial u}{\partial y} + k\frac{\partial u}{\partial z}\right).(idx + jdy + kdz)$$

$$= \nabla u \,.\, dr$$

$$= \overline{F}\,.\,dr$$

$$= [(y^2 + 2xz^2)\hat{i} + (2xy - z)\hat{j} + (2x^2z - y + 2z)\overline{k}]\,.\,(idx + jdy + kdz)$$

$$= (y^2 + 2xz^2)dx + (2xy - z)\,dy + (2x^2z - y + 2z)\,dz$$

$$= (y^2\,dx + 2xy\,dy) - (zdy + ydz) + (2xz^2\,dx + 2x^2z\,dz) + 2z\,dz$$

$$u = \int (y^2dx + 2xy\,dy) - (zdy + y\,dz) + (2xz^2\,dx + 2x^2z\,dz) + 2z\,dz$$

$$= xy^2 - yz + x^2z^2 + z^2 + C$$

Example 77. *A fluid motion is given by*

$$\overline{v} = (y\sin z - \sin x)\,i + (x\sin z + 2yz)\,j + (xy\cos z + y^2)\,k$$

is the motion irrotational? If so, find the velocity potential.

Solution. Curl $\overline{v} = \overline{\nabla} \times \overline{v}$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times (y\sin z - \sin x)\,i + (x\sin z + 2yz)\,j + (\ldots$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\sin z - \sin x & x\sin z + 2yz & xy\cos z + y^2 \end{vmatrix}$$

$$= (x\cos z + 2y - x\cos z - 2y)\,i - [y\cos z - y\cos z]\,j + (\sin z - \ldots$$

$$= 0$$

Hence, the motion is irrotational.

so $\quad v = \nabla\phi$ where ϕ is called velocity potential.

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \left(i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}\right)\cdot(idx + jdy + kdz)$$

$$= \nabla\phi\cdot\overline{dr} = \overline{v}\cdot\overline{dr}$$

$$= [(y\sin z - \sin x)\,i + (x\sin z + 2yz)\,j + (xy\cos z + y^2)\,k]\cdot(idx + \ldots$$

$$= (y\sin z - \sin x)\,dx + (x\sin z + 2yz)\,dy + (xy\cos z + y^2)\,dz$$

$$= (y\sin z\,dx + xdy\sin z + xy\cos z\,dz) - \sin x\,dx + (2yz\,dy + y^2dz)$$

$$= d(xy\sin z) + d(\cos x) + d(y^2z)$$

$$\phi = \int d(xy\sin z) + \int d(\cos x) + \int d(y^2z)$$

$$\phi = xy\sin z + \cos x + y^2z + c$$

Velocity potential $= xy\sin z + \cos x + y^2z + c.$ **Ans.**

Example 78. Show that the vector field represented by

$$(z^2 + 2x + 3y)\,\vec{i} + (3x + 2y + z)\,\vec{j} + (y + 2zx)\,\vec{k}$$

is irrotational but not solenoidal. Also obtain a scalar ϕ function such that grad $\phi = \vec{F}$

(A.M.I.E.T.E., Winter 2003)

Solution. Curl $\vec{F} = \vec{\Delta} \times \overline{F}$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 + 2x + 3y & 3x + 2y + z & y + 2zx \end{vmatrix} = i\,(1 - 1) - j\,(2z - 2z) + k\,(3 - 3) = 0$$

$\vec{F}$ is irrotational.

$$\text{div } \vec{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left[(z^2 + 2x + 3y)i + (3x + 2y + z)\,j + (y + 2zx)\,k\right]$$

$= (2 + 2 + 2x)$ which is not equal to zero.

$\vec{F}$ is not solenoidal.

$\vec{F} = \nabla\phi$ where ϕ is a scalar function

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz. \qquad \text{(Total differentiation)}$$

$$= \left(i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}\right).(i\,dx + j\,dy + k\,dz) = \nabla\phi.\,d\,\overline{r} = \overline{F}\,.\,\overline{dr} \qquad [F = \overline{\nabla}\,\phi].$$

$$= \left[(z^2 + 2x + 3y)i + (3x + 2y + z)\,j + (y + 2zx)k\right].\,[i\,dx + j\,dy + k\,dz]$$

$$= (z^2 + 2x + 3y)\,dx + (3x + 2y + z)\,dy + (y + 2zx)\,dz$$

$$= z^2dx + 2xdx + 3ydx + 3xdy + 2ydy + zdy + ydz + 2zxdz$$

$$\phi = \int(z^2dx + 2zxdz) + 3\int(ydx + xdy) + \int(zdy + y\,dz) + \int 2xdx + \int 2ydy$$

$$= xz^2 + 3xy + yz + x^2 + y^2 + c$$ **Ans.**

Example 79. *Let $\vec{V}(x, y, z)$ be a differentiable vector function and $\phi(x, y, z)$ be a scalar function. Give an expression for div $(\phi\vec{V})$ in terms of ϕ, $\vec{V}$, div $\vec{V}$ and $\nabla\phi$.*

Solution. Let $\vec{V} = V_1i + V_2j + V_3k$

$$\text{div}(\phi\vec{V}) = \nabla\cdot(\phi\vec{V}) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)[\phi V_1i + \phi V_2j + \phi V_3k]$$

$$= \frac{\partial}{\partial x}(\phi V_1) + \frac{\partial}{\partial y}(\phi V_2) + \frac{\partial}{\partial z}(\phi V_3) = \left(\phi\frac{\partial V_1}{\partial x} + \frac{\partial\phi}{\partial x}V_1\right) + \left(\phi\frac{\partial V_2}{\partial y} + \frac{\partial\phi}{\partial y}V_2\right) + \left(\phi\frac{\partial V_3}{\partial z} + \frac{\partial\phi}{\partial z}V_3\right)$$

$$= \phi\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right) + \left(\frac{\partial\phi}{\partial x}V_1 + \frac{\partial\phi}{\partial y}V_2 + \frac{\partial\phi}{\partial z}V_3\right)$$

$$= \phi\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).(V_1i + V_2j + V_3k) + \left(i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}\right).(V_1i + V_2j + V_3k)$$

$$\text{div}(\phi\vec{V}) = \phi\,(\text{div}\,\vec{V}) + (\text{grad}\,\phi)\,.\,\vec{V}$$ **Ans.**

Example 80. *Prove that* curl $(\phi \overline{F}) = (\text{gard } \phi) \times \overline{F}$, if $\overline{F}$ is irrotatio… is a scalar function.

Solution. $\overline{F} = F_1 i + F_2 j + F_3 k$

$$\nabla \times (\phi \overline{F}) = \nabla \times (\phi F_1 i + \phi F_2 j + \phi F_3 k)$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\times(\phi F_1 i + \phi F_2 j + \phi F_3 k) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi F_1 & \phi F_2 & \phi F_3 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(\phi F_3) - \frac{\partial}{\partial z}(\phi F_2)\right] i - \left[\frac{\partial}{\partial x}(\phi F_3) - \frac{\partial}{\partial z}(\phi F_1)\right] j + \left[\frac{\partial}{\partial x}(\phi F_2) \ldots\right.$$

$$= \left[\phi\frac{\partial F_3}{\partial y} + F_3\frac{\partial \phi}{\partial y} - \phi\frac{\partial F_2}{\partial z} - F_2\frac{\partial \phi}{\partial z}\right] i - \left[\phi\frac{\partial F_3}{\partial x} + F_3\frac{\partial \phi}{\partial x} - \phi\frac{\partial F_1}{\partial z} \ldots\right.$$

$$+\left[\phi\frac{\partial F_2}{\partial x} + F_2\frac{\partial \phi}{\partial x} - \phi \ldots\right.$$

$$= \phi\left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) i + \left(\phi\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) k\right]$$

$$+\left[\left(F_3\frac{\partial \phi}{\partial y} - F_2\frac{\partial \phi}{\partial z}\right) i + \left(F_1\frac{\partial \phi}{\partial z} - F_3\frac{\partial \phi}{\partial x}\right) j + (F_1 \ldots\right.$$

$$= \phi\left[\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\times(F_1 i + F_2 j + F_3 k)\right] + \begin{vmatrix} i & j & k \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \phi(\nabla \times \overline{F}) + (\nabla\phi)\times(F_1 i + F_2 j + F_3 k) = \phi(\nabla\times\overline{F}) + (\nabla\phi)\times \ldots$$

$$= (\nabla\phi)\times\overline{F} + \phi(\nabla\times\overline{F}) = (\text{grad}\,\phi)\times\overline{F} \qquad (\nabla\times\overline{F} = 0)$$

Example 81. *Suppose that $\vec{U}, \vec{V}$ and f are continuously differentiable f… Then div $(\vec{U}\times\vec{V}) = \vec{V}.\,\text{curl}\,\vec{U} - \vec{U}.\,\text{curl}\,\vec{V}$*

Solution. Let $\vec{U} = u_1 i + u_2 j + u_3 k$, $\vec{V} = v_1 i + v_2 j + v_3 k$

$$\vec{U}\times\vec{V} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= (u_2 v_3 - u_3 v_2) i - (u_1 v_3 - u_3 v_1) j + (u_1 v_2 \ldots$$

$$\text{div}\,(\overline{U}\times\overline{V}) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)[(u_2 v_3 - u_3 v_2) i - (u_1 v_3 - u_3 v_1) j + \ldots$$

$$= \frac{\partial}{\partial x}(u_2 v_3 - u_3 v_2) + \frac{\partial}{\partial y}(-u_1 v_3 + u_3 v_1) + \frac{\partial}{\partial z}(u_1 v_2 - u_2 v_1)$$

$$= u_2\frac{\partial v_3}{\partial x} + v_3\frac{\partial u_2}{\partial x} - u_3\frac{\partial v_2}{\partial x} - v_2\frac{\partial u_3}{\partial x} - u_1\frac{\partial v_3}{\partial y} - v_3\frac{\partial u_1}{\partial y}$$

$$+ u_3\,\partial v_1 + v_1\,\partial u_3 + u_1\,\partial v_2 + v_2\,\partial v_1 \ldots$$

$$= v_1\left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right) + v_2\left(-\frac{\partial u_3}{\partial x} + \frac{\partial u_1}{\partial z}\right) + v_3\left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)$$

$$+ u_1\left(-\frac{\partial v_3}{\partial y} + \frac{\partial v_2}{\partial z}\right) + u_2\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + u_3\left(\frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial x}\right)$$

$$= (v_1 i + v_2 j + v_3 k)\left[i\left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right) + j\left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\right) + k\left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)\right]$$

$$-(u_1 i + u_2 j + u_3 k)\left[i\left(-\frac{\partial v_2}{\partial z} + \frac{\partial v_3}{\partial y}\right) + j\left(-\frac{\partial v_3}{\partial x} + \frac{\partial v_1}{\partial z}\right) + k\left(-\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}\right)\right]$$

$$\ldots(\nabla\times\overline{U}) - \overline{U}\cdot(\nabla\times V) = \overline{V}\cdot\text{curl}\,\overline{U} - \overline{U}\cdot\text{curl}\,\overline{V}$$ **Proved**

…tice :

…$(\overline{A}\times\overline{r})$ if curl $\overline{A} = 0$.

… *Prove that,*

$$\nabla\times(\overline{F}\times\overline{G}) = \overline{F}(\nabla\cdot\overline{G}) - \overline{G}(\nabla\cdot\overline{F}) + (\overline{G}\cdot\nabla)\overline{F} - (\overline{F}\cdot\nabla)\overline{G}$$

$$\nabla\times(\overline{F}\times\overline{G}) = \Sigma\, i\times\frac{\partial}{\partial x}(\overline{F}\times\overline{G})$$

$$= \Sigma\, i\times\left(\frac{\partial F}{\partial x}\times\overline{G} + \overline{F}\times\frac{\partial G}{\partial x}\right) = \Sigma\, i\times\left(\frac{\partial F}{\partial x}\times\overline{G}\right) + \Sigma\, i\times\left(\overline{F}\times\frac{\partial G}{\partial x}\right)$$

$$= \Sigma\left[(i\cdot\overline{G})\frac{\partial F}{\partial x} - \left(i\cdot\frac{\partial F}{\partial x}\right)\overline{G}\right] + \Sigma\left[\left(i\cdot\frac{\partial G}{\partial x}\right)\overline{F} - (i\cdot\overline{F})\frac{\partial G}{\partial x}\right]$$

$$= \Sigma(\overline{G}\cdot i)\frac{\partial F}{\partial x} - \overline{G}\,\Sigma\left(i\cdot\frac{\partial F}{\partial x}\right) + \overline{F}\,\Sigma\left(i\cdot\frac{\partial G}{\partial x}\right) - \Sigma(\overline{F}\cdot i)\frac{\partial G}{\partial x}$$

$$= \overline{F}\left(\Sigma\, i\frac{\partial G}{\partial x}\right) - \overline{G}\,\Sigma\left(i\cdot\frac{\partial F}{\partial x}\right) + \Sigma(\overline{G}\cdot i)\frac{\partial F}{\partial x} - \Sigma(\overline{F}\cdot i)\frac{\partial G}{\partial x}$$

$$= \overline{F}\,\nabla\overline{G} - \overline{G}\,\nabla\cdot\overline{F} + (\overline{G}\cdot\nabla)\overline{F} - (\overline{F}\cdot\nabla)\overline{G}$$ **Proved**

…actice :

$$\ldots = (\overline{G}\cdot\nabla)\overline{F} + (\overline{F}\cdot\nabla)\overline{G} + \overline{G}\times(\nabla\times\overline{F}) + \overline{F}\times(\nabla\times\overline{G})$$

…3. *Prove that, for every field $\overline{V}$; div curl $\overline{V} = 0$.*

Let $\overline{V} = V_1 i + V_2 j + V_3 k$

$$\text{div}\,(\text{curl}\,\vec{V}) = \nabla\cdot(\nabla\times\vec{V})$$

$$= \nabla\cdot\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left[i\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) - j\left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right) + k\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$$= \frac{\partial^2 V_3}{\partial x \partial y} - \frac{\partial^2 V_2}{\partial x \partial z} - \frac{\partial^2 V_3}{\partial y \partial x} + \frac{\partial^2 V_1}{\partial y \partial z} + \frac{\partial^2 V_2}{\partial z \partial x} - \frac{\partial^2 V_1}{\partial z \partial y}$$

$$= \left(\frac{\partial^2 V_1}{\partial y \partial z} - \frac{\partial^2 V_1}{\partial z \partial y}\right) + \left(\frac{\partial^2 V_2}{\partial z \partial x} - \frac{\partial^2 V_2}{\partial x \partial z}\right) + \left(\frac{\partial^2 V_3}{\partial x \partial y} - \frac{\partial^2 V_3}{\partial y \partial x}\right)$$

$$= 0$$

Similar Question for Practice :

Show that div $(\nabla \phi_1 \times \nabla \phi_2) = 0$.

Example 84. *$\overline{r_1}$ and $\overline{r_2}$ are vectors joining the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ to a variable point $P(x, y, z)$, show that curl $(\overline{r_1} \times \overline{r_2}) = 2(\overline{r_2} - \overline{r_1})$.*

Solution. $\overline{r_1} = \overrightarrow{PP_1} = (x_1 - x)\,i + (y_1 - y)\,j + (z_1 - z)\,k$

$$\overline{r_2} = \overrightarrow{PP_2} = (x_2 - x)\,i + (y_2 - y)\,j + (z_2 - z)\,k$$

$$\overline{r_1} \times \overline{r_2} = [(x_1 - x)\,i + (y_1 - y)\,j + (z_1 - z)\,k] \times [(x_2 - x)\,i + (y_2 - y)\,j + (z_2 - z)\,k]$$

$$= \begin{vmatrix} i & j & k \\ x_1 - x & y_1 - y & z_1 - z \\ x_2 - x & y_2 - y & z_2 - z \end{vmatrix} = [(y_1 - y)(z_2 - z) - (y_2 - y)(z_1 - z)]\,i - [(x_1 - x)(z_2 - z)$$

$$- (x_2 - x)(z_1 - z)]\,j + [(x_1 - x)(y_2 - y) - (x_2 - x)(y_1 - y)]\,k$$

$$= (y_1 z_2 - y_1 z - y z_2 - y_2 z_1 + y_2 z + y z_1)\,i - (x_1 z_2 - x_1 z - x z_2 - x_2 z_1$$

$$+ x_2 z + x z_1)\,j + (x_1 y_2 - x_1 y - x y_2 - x_2 y_1 + x_2 y + x y_1)\,k$$

$$\text{curl}\,(\overline{r_1} \times \overline{r_2}) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times (\overline{r_1} \times \overline{r_2})$$

$$\text{curl}\,(\overline{r_1} \times \overline{r_2}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \begin{matrix} (y_1 z_2 - y_1 z - y z_2 \\ - y_2 z_1 + y_2 z + y z_1) \end{matrix} & \begin{matrix} (-x_1 z_2 + x_1 z + x z_2 \\ + x_2 z_1 - x_2 z - x z_1) \end{matrix} & \begin{matrix} (x_1 y_2 - x_1 y - x y_2 \\ - x_2 y_1 + x_2 y + x y_1) \end{matrix} \end{vmatrix}$$

$$= (-x_1 + x_2 - x_1 + x_2)\,i - (-y_2 + y_1 + y_1 - y_2)\,j + (-z_1 + z_2 - z_1 + z_2)\,k$$

$$= 2(x_2 - x_1)\,i + 2(y_2 - y_1)\,j + 2(z_2 - z_1)\,k$$

$$= 2[(x_2 i + y_2 j + z_2 k) - (x_1 i + y_1 j + z_1 k)] = 2(\overline{r_2} - \overline{r_1})$$

Example 85. *$\overline{a}$ is a constant vector, show that $\overline{a} \times (\nabla \times \overline{r}) = \nabla(\overline{a} \cdot \overline{r}) - (\overline{a} \cdot \nabla)\overline{r}$*

Solution. $\overline{a} = a_1 i + a_2 j + a_3 k$, $\overline{r} = r_1 i + r_2 j + r_3 k$

$$\nabla \times \overline{r} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r_1 & r_2 & r_3 \end{vmatrix} = i\left(\frac{\partial r_3}{\partial y} - \frac{\partial r_2}{\partial z}\right) - \left(\frac{\partial r_3}{\partial x} - \frac{\partial r_1}{\partial z}\right)j + k\left(\frac{\partial r_2}{\partial x} - \frac{\partial r_1}{\partial y}\right)$$

$$\overline{a} \times (\nabla \times \overline{r}) = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ \frac{\partial r_3}{\partial y} - \frac{\partial r_2}{\partial z} & -\frac{\partial r_3}{\partial x} + \frac{\partial r_1}{\partial z} & \frac{\partial r_2}{\partial x} - \frac{\partial r_1}{\partial y} \end{vmatrix}$$

$$= \left[a_2\left(\frac{\partial r_2}{\partial x} - a_2\frac{\partial r_1}{\partial y}\right) - \left(-a_3\frac{\partial r_3}{\partial x} + a_3\frac{\partial r_1}{\partial z}\right)\right] i - \left[a_1\frac{\partial r_2}{\partial x} - a_1\frac{\partial r_1}{\partial y} - a_3\frac{\partial r_3}{\partial y} + a_3\frac{\partial r_2}{\partial z}\right] j$$

$$+ \left[-a_1\frac{\partial r_3}{\partial x} + a_1\frac{\partial r_1}{\partial z} - a_2\frac{\partial r_3}{\partial y} + a_2\frac{\partial r_2}{\partial z}\right] k$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(a_1 r_1 + a_2 r_2 + a_3 r_3) - \left[a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right](r_1 i + r_2 j + r_3 k)$$

$$= \nabla(\overline{a} \cdot \overline{r}) - (\overline{a} \cdot \nabla)\,\overline{r}$$ **Proved.**

By the formula of vector triple product

$$\overline{a} \times (\nabla \times \overline{r}) = (\overline{a} \cdot \overline{r})\nabla - (a \cdot \nabla)\,\overline{r} = \nabla(\overline{a} \cdot \overline{r}) - (\overline{a} \cdot \nabla)\,\overline{r}$$ **Proved.**

Example 86. *If $\overrightarrow{A} = \nabla \times (\overrightarrow{\phi i})$ where $\nabla^2 \phi = 0$, show that*

$$\overrightarrow{A} \cdot \nabla \times \overrightarrow{A} = \frac{\partial \phi}{\partial z}\frac{\partial^2 \phi}{\partial y \partial x} - \frac{\partial \phi}{\partial y} \cdot \frac{\partial^2 \phi}{\partial z \partial x}$$

Solution. $\overrightarrow{A} = \nabla \times (\phi i) = \nabla \phi \times i$

$$\nabla \times \overrightarrow{A} = \nabla \times (\nabla \phi \times i) \qquad [\because \overline{a} \times (\overline{b} \times c) = (\overline{a} \cdot c)\,\overline{b} - (\overline{a} \cdot \overline{b})\,c]$$

$$= (i \cdot \nabla)\nabla\phi - (\nabla \cdot \nabla \phi)\,i = \left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right] \cdot i\,(\nabla \phi) - (\nabla^2 \phi)\,i$$

$$= \frac{\partial}{\partial x}\left(i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}\right) - \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right) i$$

$$= i\frac{\partial^2 \phi}{\partial x^2} + j\frac{\partial^2 \phi}{\partial x \partial y} + k\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial x^2}\,i - \frac{\partial^2 \phi}{\partial y^2}\,i - \frac{\partial^2 \phi}{\partial z^2}\,i = -\frac{\partial^2 \phi}{\partial y^2}\,i - \frac{\partial^2 \phi}{\partial z^2}\,i + j\frac{\partial^2 \phi}{\partial x \partial y} + k\frac{\partial^2 \phi}{\partial x \partial z}$$

$$\overrightarrow{A} \cdot (\nabla \times \overrightarrow{A}) = (\nabla \phi \times i)\left[-\frac{\partial^2 \phi}{\partial y^2}\,i - \frac{\partial^2 \phi}{\partial z^2}\,i + \frac{\partial^2 \phi}{\partial x \partial y}\,j + \frac{\partial^2 \phi}{\partial x \partial z}\,k\right]$$

$$= \left[\left(i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}\right) \times i\right]\left[-\frac{\partial^2 \phi}{\partial y^2}\,i - \frac{\partial^2 \phi}{\partial z^2}\,i + j\frac{\partial^2 \phi}{\partial x \partial y} + k\frac{\partial^2 \phi}{\partial x \partial z}\right]$$

$$= \left(-k\frac{\partial \phi}{\partial y} + j\frac{\partial \phi}{\partial z}\right)\left(-\frac{\partial^2 \phi}{\partial y^2}\,i - \frac{\partial^2 \phi}{\partial x \partial y}\,j - \frac{\partial^2 \phi}{\partial z^2}\,i - \frac{\partial^2 \phi}{\partial x \partial z}\,k\right) = -\frac{\partial \phi}{\partial y}\frac{\partial^2 \phi}{\partial x \partial z} + \frac{\partial \phi}{\partial z}\frac{\partial^2 \phi}{\partial x \partial y}$$ **Proved.**

Example 87. *If r is the distance of a point (x, y, z) from the origin, prove that curl* $\left(K \times \text{grad}\frac{1}{r}\right)$ + grad $\left(k \cdot \text{grad}\frac{1}{r}\right) = 0$ *where k is the unit vector in the direction OZ.*

(U.P.I. Semester, Winter 2004)

Solution. $r^2 = (x - 0)^2 + (y - 0)^2 + (z - 0)^2 = x^2 + y^2 + z^2 \Rightarrow \frac{1}{r} = (x^2 + y^2 + z^2)^{-1/2}$

$$\text{grad}\,\frac{1}{r} = \nabla\frac{1}{r}$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \times (x^2+y^2+z^2)^{-1/2} \quad = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x\ldots$$

$$= -(x^2+y^2+z^2)^{-3/2}(xi+yj+zk),$$

$$\text{curl}\left(k \times \text{grad}\frac{1}{r}\right) = \nabla \times \left(k \times \text{grad}\frac{1}{r}\right) = -\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \times \left[(x^2+y^2+z^2\ldots \right.$$

$$\text{curl}\left(k \times \text{grad}\frac{1}{r}\right) = \nabla \times \left(k \times \text{grad}\frac{1}{r}\right) = -\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \times \left[(x^2+y^2\ldots \right.$$

$$= -\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{(x^2+y^2+z^2)^{3/2}} & \frac{-x}{(x^2+y^2+z^2)^{3/2}} & 0 \end{vmatrix}$$

$$= -\frac{3}{2}\,\frac{(-x)(2z)}{(x^2+z^2+y^2)^{5/2}}\, i + \frac{(-3/2)\,y\,(2z)}{(x^2+y^2+z^2)^{5/2}}\, j + \left[\frac{(-3/2)(-x)(2x)}{(x^2+y^2+z^2)^{5/2}} \ldots \right.$$

$$-\frac{(-3/2)(y)(2y)}{(x^2+y^2+z^2)^{5/2}} \ldots$$

$$= \frac{-3xz}{(x^2+y^2+z^2)^{5/2}}\, i - \frac{3yz}{(x^2+y^2+z^2)^{5/2}}\, j + \frac{(3x^2-x^2-y^2-z^2+3y^2 \ldots}{(x^2+y^2+z^2)^{5/2}}$$

$$= \frac{-3xzi - 3yzj + (x^2+y^2-2z^2)\,k}{(x^2+y^2+z^2)^{5/2}}$$

$$k \cdot \text{grad}\frac{1}{r} = k \cdot \left[-(x^2+y^2+z^2)^{-3/2}(xi+yj+zk) \right] = \ldots$$

$$\text{grad}\left(k \cdot \text{grad}\frac{1}{r}\right) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \frac{-z}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{i(-z)(-\frac{3}{2})(2x)}{(x^2+y^2+z^2)^{5/2}} + \frac{j(-z)(-3/2)(2y)}{(x^2+y^2+z^2)^{5/2}} + \left[\frac{(-z)(-3/2)(2z)}{(x^2+y^2+z^2)^{5/2}} \ldots \right.$$

$$= \frac{3xzi + 3yzj + (3z^2-x^2-y^2-z^2)\,k}{(x^2+y^2+z^2)^{5/2}} = \frac{3xzi + 3yzj - (x^2+y^2-2z^2)\,k}{(x^2+y^2+z^2)^{5/2}}$$

Adding (1) and (2) we get $\text{curl}\left(k \times \text{grad}\frac{1}{r}\right) + \text{grad}\left(k \cdot \text{grad}\frac{1}{r}\right) = 0$

Example 88. (*a*) *By taking* $\vec{F} = P\nabla q$, *where P and q are scalars, prove*

$$\nabla \cdot F = P\nabla^2 q + \nabla P \cdot \nabla q.$$

Solution.

$$\vec{F} = P\nabla q$$

$$= P\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) q = P\left(i\frac{\partial q}{\partial x} + j\frac{\partial q}{\partial y} + \ldots \right)$$

$$\nabla \cdot \vec{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \cdot P\left(i\frac{\partial q}{\partial x} + j\frac{\partial q}{\partial y} + k\frac{\partial q}{\partial z} \right)$$

...ating by product rule

$$\ldots \frac{\partial P}{\partial y} + k\frac{\partial P}{\partial z} \Big) \cdot \left(i\frac{\partial q}{\partial x} + j\frac{\partial q}{\partial y} + k\frac{\partial q}{\partial z} \right) + P\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \cdot \left(i\frac{\partial q}{\partial x} + j\frac{\partial q}{\partial y} + k\frac{\partial q}{\partial z} \right)$$

$$\ldots + P\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 q \;=\; \nabla P \cdot \nabla q + P\nabla^2 q \qquad \textbf{Proved}$$

...88. (*b*) *Establish the relation* (*A.M.I.E.T.E., Dec. 2004*)

Curl Curl $\vec{f} = \vec{\nabla}\,\text{div}\,\vec{f} - \nabla^2 \vec{f}$ (*U.P., I Semester, Winter 2002*)

Let $\vec{f} = f_1 i + f_2 j + f_3 k$, then by definition,

$$\ldots \vec{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) i + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) j + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) k$$

Curl curl $\vec{f}$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} & \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} & \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) - \frac{\partial}{\partial z}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \right] i + \text{two similar terms}$$

$$= \left[\frac{\partial^2 f_2}{\partial y \partial x} - \frac{\partial^2 f_1}{\partial y^2} - \frac{\partial^2 f_1}{\partial z^2} + \frac{\partial^2 f_3}{\partial z \partial x} \right] i + \text{two similar terms}$$

$$= \left[\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_2}{\partial y \partial x} + \frac{\partial^2 f_3}{\partial z \partial x} - \left(\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2} \right) \right] i + \text{two similar terms}$$

$$= \left[\frac{\partial}{\partial x}\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f_1 \right] i + \text{two similar terms}$$

$$= \frac{\partial}{\partial x}(\text{div}\vec{f} - \nabla^2 f_1)\, i + \frac{\partial}{\partial y}(\text{div}\vec{f} - \nabla^2 f_2)\, j + \frac{\partial}{\partial z}(\text{div}\vec{f} - \nabla^2 f_3)\, k$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \text{div}\vec{f} - \nabla^2 \left[f_1 i + f_2 j + f_3 k \right]$$

$$= \text{grad div}\,\vec{f} - \nabla^2 \vec{f} \qquad \textbf{Proved.}$$

Exercise 5.18

... $= xi + yj + zk$ and $r = |r|$, show that (*i*) div $\left(\frac{\vec{r}}{|\vec{r}|^3}\right) = 0$, (*ii*) div (grad r^n) $= n(n+1)\,r^{n-2}$,

... div $(r\phi) = 3\phi + r$ grad ϕ.

... the divergence and curl of the vector field $V = (x^2-y^2)\,i + 2xyj + (y^2-xy)\,k$.

Ans. Divergence $\vec{V} = 4x$, Curl $\vec{V} = (2y-x)\,i + yj + 4yk$

... Show that the vector $V = (x+3y)\,i + (y-3z)\,j + (x-2z)\,k$ is solenoidal.

... Show that the vector field $\vec{F} = 2x(y^2+z^3)\,i + 2x^2yj + 3x^2z^2k$ is conservative. Find its scalar potential and the work down by it in moving a particle from $(-1, 2, 1)$ to $(2, 3, 4)$. (*AMIETE Dec. 2005*)

... that for any vector field V, div (curl V) = 0.

5. Show that the vector field given by $\vec{A} = 3x^2 y\, i + (x^3 - 2yz^2)\, j + (3z^2 - 2y^2 z)\, k$ is irrot[ational and] solenoidal. Also find a scalar function ϕ such that grad $\phi = \vec{A}$

(A.M.I.E.T.E., Winter 2002) **Ans.** $\phi = x^3 y$ ...

6. Prove that:

(i) $\nabla \cdot (\phi A) = \nabla\phi \cdot A + \phi(\nabla \cdot A)$ (ii) $\nabla(A \cdot B) = (A \cdot \nabla) B + (B \cdot \nabla) A + A \times (\nabla \times B) ...$

(iii) $\nabla \times (A \times B) = (B \cdot \nabla) A - B(\nabla \cdot A) - (A \cdot \nabla) B + A(\nabla \cdot B)$

(iv) If $\bar{a}$ is a constant vector and $\vec{r} = x\bar{i} + y\bar{j} + z\bar{k}$. Show that curl $(\bar{a} \times \bar{r}) = 2\bar{a}$

(A.M.I.E.T.E. ...)

7. If $\vec{V} = \dfrac{xi + yj + zk}{\sqrt{x^2 + y^2 + z^2}}$, find the value of div $\bar{V}$ and curl $\bar{V}$. (U.P., I Semester ...)

Ans. div $\bar{V} = \dfrac{2}{\sqrt{x^2 + y^2 + z^2}}$ ...

8. A fluid motion is given by $V = (y + z)\, i + (z + x)\, j + (x + y)\, k$.

(i) Is this motion irrotational? If so, find the velocity potential.

Ans. Yes, velocity potential = ...

(ii) Is the motion possible for an incompressible fluid?

9. Express the vector field $zi + 2xj + yk$ in cylindrical coordinates. **Ans.** $iz + j\,2r\cos$ ...

10. Express the vector field $xi + 2yj + yzk$ in spherical polar coordinates.

Ans. $r \sin\theta \cos\phi\, i + 2r \sin\theta \sin\phi\, j + r \sin\theta$ ...

11. If ρ, ϕ, z are cylindrical coordinates, show that grad $(\log \rho)$ and grad ϕ are solenoidal ...

12. Prove that the cylindrical coordinates system is orthogonal.

13. Obtain the expression for $\nabla^2 f$ in spherical coordinates from their coorresponding expression ... curvilinear coordinates.

14. (a) $\nabla \cdot (\phi \bar{F}) = (\nabla\phi) \cdot \bar{F} + \phi(\nabla \cdot \bar{F})$ (b) div grad $\dfrac{x}{r^3} = 0$ (A.M.I.E.T.E., S...)

15. (a) $\nabla \times (\phi \bar{F}) = (\nabla\phi) \times \bar{F} + \phi(\nabla \times \bar{F})$ (b) $\nabla \times \dfrac{(\bar{A} \times \bar{R})}{r^n} = \dfrac{(2-n)\bar{A}}{r^n} + \dfrac{n(\bar{A} \cdot \bar{R})}{r^{n+2}}$ ...

16. $\nabla \cdot (\bar{F} \times \bar{G}) = \bar{G} \cdot (\nabla \times \bar{F}) - \bar{F} \cdot (\nabla \times \bar{G})$

17. $\nabla \times (\bar{F} \times \bar{G}) = (\bar{G} \cdot \nabla)\bar{F} - \bar{G}(\nabla \cdot \bar{F}) - (\bar{F} \cdot \nabla)\bar{G} + \bar{F}(\nabla \cdot \bar{G})$

18. $\nabla(\bar{F} \cdot \bar{G}) = (\bar{G} \cdot \nabla)\bar{F} + (\bar{F} \cdot \nabla)\bar{G} + \bar{G} \times (\nabla \times \bar{F}) + \bar{F} \times (\nabla \times \bar{G})$

19. $\nabla \cdot (\nabla \phi) = \nabla^2 \phi$ 20. $\nabla \times (\nabla \phi) = 0$ 21. $\nabla \cdot (\nabla \times \bar{F})$ ...

22. div $(f \nabla g)$ − div $(g \nabla f) = f\nabla^2 g - g\nabla^2 f$ 23. $\nabla \times (\nabla \times \bar{F}) = \nabla(\nabla \cdot \bar{F}) - \nabla^2 \bar{F}$

24. Div (grad ϕ_1 × grad ϕ_2) = 0

25. If $\phi \bar{F} = \nabla \psi$ where ϕ, ψ are scalar functions of x, y, z and a vector field, show that $\bar{F} \cdot$ curl ...

26. Evaluate div $(\bar{A} \times \bar{r})$ if curl $\bar{A} = 0$. (b) Prove that curl $(\bar{a} \times \bar{r}) = 2a$

27. Find curl F where F = grad $(x^3 + y^3 + z^3 - 3xyz)$.

28. Find out values of a, b, c for which $\bar{v} = (x + y + az)\, i + (bx + 3y - z)\, j + (3x + cy + z)\, k$ is ...

(Nagpur, Winter 2000) **Ans.** $a = 3, b =$ ...

29. Choose the correct answer.

(i) The parametric representation of a surface is $\vec{r} = au \cosh vi + bu \sinh vj + u^2 k$. Th[e equation] of the surface in certesian form is

(a) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = z$, (b) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = z$, (c) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = u^2$, (d) ...

(ii) The unit vector tangent to the curve $x = t, y = t^2, z = t^3$ at the point $(-1, 1, -1)$ is

(a) $(1/\sqrt{14})(i + 2j + 3k)$ (b) $(1/\sqrt{14})(i - 2j + 3k)$ (c) $(1/\sqrt{3})(i + j + k)$ (d) $(1/\sqrt{3})(i - j + k)$.

(iii) If $\bar{A}$ is constant vector and $\bar{R} = xi + yj + zk$, then grad $(\bar{A} \cdot \bar{R})$ is

(a) $\bar{A}$ (b) $2\bar{A}$ (c) $\bar{R}$ (d) $2\bar{R}$. (A.M.I.E.T.E., Summer 2002)

(iv) The value of λ for which the vector field $\vec{v} = (x + 3y)\, i + (y - 2z)\, j + (x + \lambda z)\, k$ is solenoidal is

(a) 0, (b) 2, (c) −2, (d) 1. (Madras, 2006)

(v) The value of λ so that the vector $\vec{u} = (x + 3y)\vec{i} + (y - 2z)\vec{j} + (x + \lambda z)\vec{k}$ is a solenoidal vector, is

(a) (−2) (b) 3 (c) 1 (d) None of these.

(vi) If $r^2 = x^2 + y^2 + z^2$ then $\vec{\nabla} \cdot \left(\dfrac{\vec{r}}{r}\right)$ is equal to

(a) 0 (b) $3r$ (c) r^2 (d) $\dfrac{2}{r}$.

(vii) Show that the vector field $\bar{f} = \dfrac{\bar{r}}{r^3}$ is irrotational as well as solemoidal. Find the scalar potential.

(A.M.I.E.T.E. Summer 2004, Winter 2001) **Ans.** $-\dfrac{1}{\sqrt{r}} + c$

(viii) The direction in which the directional derivative of $f(x, y) = \dfrac{(x^2 - y^2)}{xy}$ at (1, 1) equals to zero is given by the ray at one of the angle with positive direction of x-axis.

(a) 60° (b) 45° (c) 135° (d) None of these.

(ix) The circulation of F round the curve c, where $\bar{F} = (2x + y^2)\, i + (3y - 4x)\, j$ and c is the curve $y = x^2$ from (0, 0) to (1, 1) and the curve $y^2 = x$ from (1, 1) to (0, 0) is (A.M.I.E.T.E., Summer 2001)

(a) $\dfrac{49}{30}$ (b) $-\dfrac{49}{30}$ (c) $\dfrac{51}{30}$ (d) $-\dfrac{51}{30}$

(x) The vector defined by $\vec{v} = e^x \sin yi + e^x \cos yj$, is

(a) rotational ; (b) irrotational ; (c) solenoidal ; (d) rotational in part of space.

(xi) Let $f(x, y, z) = c$ represent the equation of a surface. The unit normal to this surface is

(a) grad (f)/ | grad (f) | ; (b) grad (f) ;

(c) div [grad (f)] ; (d) Curl [grad (f)].

(xii) Find out values of $a, b\, c$ for which $\bar{V} = (x + y + az)\, i + (bx + 3y - z)\, j + (3x + cy + z)\, k$ is irrotational (A.M.I.E.T.E. Winter 2000)

(xiii) If $f = \tan^{-1}\left(\dfrac{y}{x}\right)$ then div (grad f) is equal to

(a) 1 (b) −1 (c) 0 (d) 2

(xiv) The value of curl (grad f), where $f = 2x^2 - 3y^2 + 4z^2$ is

(a) $4x - 6y + 8z$ (b) $4xi - 6yj + 8zk$ (c) 0 (d) 3

(xv) The curl of the gradient of a scalar function U is (A.M.I.E.T.E., Dec. 2005)

(a) 1 (b) $\nabla^2 U$ (c) ∇U (d) 0 **Ans.** ( )

(xvi) The value of the integral $\int_C y\, ds$, where C is the curve $y = 2\sqrt{x}$ from $x = 3$ to $x = 24$, is

(A.M.I.E.T.E., Dec. 2005)

(a) 156 (b) 153 (c) 150 (d) 158

Ans. (i) (b), (ii) (a), (iii) (a), (iv) (c), (v) (a), (vi) (d), (vii) $-\dfrac{1}{\sqrt{r}} + c$, (viii) (b), (ix) (c), (x) (b), (c)

(xi) (a), (xii) $[a = 3, b = 1, c = -1]$ (xiii) (c) (xiv) (c), (xv) (d)

5.40 INTEGRATION OF VECTORS

LINE INTEGRAL

Let $\overline{F}(x, y, z)$ be a vector function and a curve AB.

Line integral of a vector function $\overline{F}$ along the curve AB is defined as integral of the component of $\overline{F}$ along the tangent to the curve AB.

Component of $\overrightarrow{F}$ along a tangent PT at P

$$= \text{Dot product of } \overline{F} \text{ and unit vector along } PT$$

$$= \overline{F} \cdot \frac{\overrightarrow{dr}}{ds} \qquad \left(\frac{\overrightarrow{dr}}{ds} \text{ is a unit vector along tangent } PT\right)$$

$$\text{Line integral} = \Sigma\, \overline{F} \cdot \frac{\overrightarrow{dr}}{ds} \text{ from } A \text{ to } B \text{ along the curve}$$

$$\therefore \text{Line integral} = \int_c \left(F \cdot \frac{\overline{dr}}{ds}\right) ds = \int_c \overline{F} \cdot \overline{dr}$$

Note (1) Work. If $\overline{F}$ represents the variable force acting on a particle along arc

the total work done $= \int_A^B \overline{F} \cdot \overline{dr}$

(2) Circulation. If $\overline{F}$ represents the velocity of a liquid then $\oint_c \overline{F} \cdot \overline{dr}$ is c

circulation of F round the curve c.

If $\oint \overline{F} \cdot \overline{dr} = 0$, then, the filed F is called conservative, i.e. no work is don

energy is conserved.

(3) When the path of integration is a closed curve then notation of integratio

place of $\int$.

Example 89. *If a force $\overline{F} = 2x^2yi + 3xyj$ displaces a particle in the xy-plan (0, 0) to (1, 4) along a curve $y = 4x^2$. Find the work done.*

Solution. Work done

$$= \int_c \overline{F} \cdot \overline{dr}$$

$$= \int_c (2x^2 y\, i + 3xy\, j) \cdot (dx\, i + dy\, j)$$

$$= \int_c (2x^2y\, dx + 3xy\, dy)$$

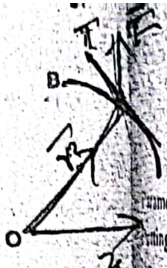
Putting the value of y and dy we get $\begin{pmatrix} y = 4x^2 \\ dy = 8x\, dx \end{pmatrix}$

$$= \int_0^1 \cdot [\,2x^2 (4x^2)\, dx + 3x (4x^2)\, 8x\, dx\,]$$

$$= 104 \int_0^1 x^4 dx = 104 \left(\frac{x^5}{5}\right)_0^1 = \frac{104}{5}$$

Example 90. *Compute $\int_c F \cdot dr$, where $F = \dfrac{iy - jx}{x^2 + y^2}$ and c is the circle $x^2 + y^2$ … ersed counter clockwise.*

Solution. $\overline{r} = ix + jy + kz, \quad d\overline{r} = i\,dx + j\,dy + k\,dz$



$$\int_c F \cdot dr = \int_c \frac{iy - jx}{x^2 + y^2} \cdot (idx + jdy + kdz)$$

$$= \int_c \frac{ydx - xdy}{x^2 + y^2} = \int_c (ydx - xdy) \qquad [\because x^2 + y^2 = 1]$$

Parametric equation of the circle are $x = 1 \cos\theta$, $y = 1 \sin\theta$.

Putting $x = \cos\theta$, $y = \sin\theta$, $dx = -\sin\theta\, d\theta$, $dy = \cos\theta\, d\theta$

$$= \int_0^{2\pi} \sin\theta\,(-\sin\theta\, d\theta) - \cos\theta\,(\cos\theta\, d\theta)$$

$$= -\int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\, d\theta = -\int_0^{2\pi} d\theta$$

$$= -(\theta)_0^{2\pi} = -2\pi$$

Ans.

Example 91. *If $F = \nabla\phi$ show that the work done in moving a particle in the force field $A(x_1, y_1, z_1)$ to $B(x_2, y_2, z_2)$ is independent of the path joining the two points.*

Solution

$$\text{Work done} = \int_A^B F \cdot dr = \int_A^B \nabla\phi \cdot dr$$

$$= \int_A^B \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\phi \cdot (i\, dx + j\, dy + k\, dz)$$

$$= \int_A^B \left[i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}\right] \cdot (i\, dx + j\, dy + kdz)$$

$$= \int_A^B \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz\right)$$

$$= \int_A^B d\phi = [\phi]_A^B = \phi(B) - \phi(A)$$

$$= \phi(x_2, y_2, z_2) - \phi(x_1, y_1, z_1)$$

Hence the work done depends only on the end points A and B not on the path from A

Note : If $\int_A^B \overline{F} \cdot \overline{dr}$ is to be proved to be independent of path, then $\overline{F} = \nabla\phi$

here F is called **conservative** (irrotational) vector field and ϕ is called the **scalar potential.**

and $\nabla \times \overline{F} = \nabla \times \nabla\phi = 0$

Example 92. *Show that the integral*

$\int_{(1,2)}^{(3,4)} (xy^2 + y^3)\, dx + (x^2 y + 3xy^2)\, dy$ is independent of the path joining the points (1, 2) and (3, 4). Hence, evaluate the integral.

Solution. $\int_{(1,2)}^{(3,4)} (xy^2 + y^3)\, dx + (x^2 y + 3xy^2)\, dy$

$$= \int_{(1,2)}^{(3,4)} [(xy^2 + y^3)\, i + (x^2y + 3xy^2)\, j] \cdot (idx + jdy)$$

$$= \int_{(1,2)}^{(3,4)} \overline{F} \cdot d\overline{r}$$

$$\nabla \times \overline{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy^2 + y^3 & x^2y + 3xy^2 & 0 \end{vmatrix}$$

$$= (0-0)\, i + (0-0)\, j + (2xy + 3y^2 - 2xy - 3y^2)\, k$$

$$= 0$$

If $\overline{\nabla} \times \overline{F} = 0$ then $\overline{F} = \overline{\nabla} \phi$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$$

$$= \left(\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j\right) \cdot (idx + jdy) = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y}\right) \phi \cdot (idx + jdy)$$

$$= \overline{\nabla} \phi \cdot \overline{d}r = \overline{F} \cdot \overline{d}r$$

$$= (xy^2 + y^3)\, dx + (x^2y + 3xy^2)\, dy$$

$$= [(xdx)\, y^2 + x^2\, (ydy)] + [y^3\, dx + x\, (3y^2\, dy)]$$

$$= \frac{1}{2} d\,(x^2 y^2) + d\,(xy^3)$$

$$\phi = \int \left[\frac{1}{2} d\,(x^2 y^2) + d\,(x y^3)\right]$$

$$= \frac{1}{2} x^2 y^2 + xy^3$$

$$[\phi]_{(1,2)}^{(3,4)} = \left[\frac{1}{2} x^2 y^2 + xy^3\right]_{(1,2)}^{(3,4)} = \left[\frac{1}{2} (9)\,(16) + (3)\,(64)\right] - \left[\frac{1}{2} (1)\,(4) + (1)\,(8)\right]$$

$$= 72 + 192 - 2 - 8 = 254$$

Example 93. *Determine whether the line integral*

$\int_c (2xyz^2\, dx + (x^2 z^2 + z \cos yz)\, dy + (2x^2 yz + y \cos yz)\, dz$ is independent of the integration ? If so, then evaluate it from $(1,0,1)$ to $\left(0, \frac{\pi}{2}, 1\right)$.

Solution. $\int_c (2xy\, z^2\, dx + (x^2 z^2 + z \cos y\, z)\, dy + (2x^2\, yz + y \cos yz)\, dz$

$$= \int_c (2xy\, z^2\, i + (x^2 z^2 + z \cos y\, z)\, j + (2x^2\, yz + y \cos yz)\, k\,] \cdot (\, idx + jdy + kdz)$$

$$= \int_c F \cdot dr$$

This integral is independent of path of integration if

$$\overline{F} = \nabla \phi \quad \Rightarrow \quad \nabla \times \overline{F} = 0$$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x\, yz^2 & x^2 z^2 + z \cos y z & 2x^2\, y\, z + y \cos yz \end{vmatrix}$$

$= (x \cos yz - yz \sin yz - 2x^2 z - \cos yz + yz \sin yz)\, i - (4xyz - 4x\, yz)\, j + (2xz^2 - 2xz^2)\, k = 0$

...the line integral is independent of path.

$$d\,\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \qquad \text{(Total differentiation)}$$

$$= \left(i \frac{\partial \phi}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial \phi}{\partial z}\right) \cdot (idx + j\, dy + kdz) = \nabla \phi \cdot dr = F \cdot dr$$

$$= [\,(2xyz^2 i + (x^2 z^2 + z \cos y\, z)\, j + (2x^2 yz + y \cos yz)\, k\,] \cdot (idx + j\, dy + kdz)$$

$$= 2xyz^2\, dx + (x^2 z^2 + z \cos yz)\, dy + (2x^2 yz + y \cos yz)\, dz$$

$$= [(2x\, dx)\, yz^2 + x^2\, (dy)\, z^2 + x^2 y\, (2z\, dz)] + [(\cos yz\, dy)\, z + (\cos yz\, dz)\, y]$$

$$= d\,(x^2 y z^2) + d\,(\sin yz)$$

$$\phi = \int d\,(x^2 y z^2) + \int d\,(\sin yz) = x^2 y z^2 + \sin yz$$

$$[\phi]_A^B = \phi\,(B) - \phi\,(A) = [x^2 y z^2 + \sin yz\,]_{(0, \frac{\pi}{2}, 1)} - [x^2 y z^2 + \sin yz\,]_{(1, 0, 1)}$$

$$= \left[0 + \sin\left(\frac{\pi}{2} \times 1\right)\right] - [0 + 0] = 1 \qquad \textbf{Ans.}$$

Exercise 5.19

1. ...the work done by a force $yi + xj$ which displace a particle from origin to a point $(i + j)$. **Ans.** 1

2. ...vector field is given by $\overline{F} = (x^2 - y^2 + x)\,\hat{i} - (2xy + y)\,\hat{j}$. show that the field is irrotational and find ...scalar potential. Hence evaluate line integral $\int \overline{F} \cdot d\overline{r}$ from (1, 2) to (2, 1). **Ans.** $-\frac{35}{6}$
(*A.M.I.E.T.E., Summer 2003*)

3. ...ow that $V = (2xy + z^3)\, i + x^2\, j + 3xz^2\, k$, is a conservative field. Find its scalar potential ϕ such that ...grad ϕ. Find the work done by the force V in moving a particle from (1, −2, 1) to (3, 1, 4).
Ans. $x^2 y + xz^3$, 202

4. ...w that the line integral $\int_c (2xy + 3)\, dx + (x^2 - 4z)\, dy - 4y\, dz$ ...re c is any path joining (0, 0, 0) to (1, −1, 3) does not depend on the path c and evaluate the line integral.
Ans. 14

5. ...d the work done in moving a particle once round the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, $z = 0$ under the field ...force given by $F = (2x - y + z)\, i + (x + y - z^2)\, j + (3x - 2y + 4z)\, k$, is the field of force conservative?
(*A.M.I.E.T.E., Winter 2000*) **Ans.** 40π

6. ...d the work done by a force $\overrightarrow{F} = \sin y\, \hat{i} + x\, (1 + \cos y)\, \hat{j} + z\, \hat{k}$ by moving a particle once around the ...circle $x^2 + y^2 = a^2$.
(*A.M.I.E.T.E., Dec. 2004*)

7. Show that the vector field $\overrightarrow{F} = (yze^{xyz} - 4x)\, \hat{i} + (xze^{xyz} + z)\, \hat{j} + (xye^{xyz} + y)\, \hat{k}$ is conservative. Hence evaluate the line integral $\int (yze^{xyz} - 4x)\, dx + (xze^{xyz} + z)\, dy + (xye^{xyz} + y)\, dz$ along a path joining the points (0, 0, 0) to (1, 1, 1)
(*A.M.I.E.T.E., Dec. 2004*)

8. A force field $\overline{F}$ is said to be conservative if
(*a*) Curl $\overline{F} = 0$ (*b*) grad $\overline{F} = 0$ (*c*) Div $\overline{F} = 0$ (*d*) Curl (grad $\overline{F}$) = 0 (*A.M.I.E.T.E., Dec. 2006*) **Ans.** (*d*)

... SURFACE INTEGRAL

Let F be a vector function and S be the given surface.

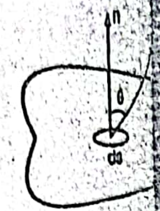
Surface integral of a vector function $\overline{F}$ over the surface S is defined as the integral of the ...ponents of $\overline{F}$ along the normal to the surface.

Component of $\overline{F}$ along the normal

$= \overline{F} \cdot n$ where n is the unit normal vector to an element ds and

$$\hat{n} = \frac{\text{grad} f}{|\text{grad} f|}$$

$$ds = \frac{dx\,dy}{(\hat{n} \cdot k)}$$



Surface integral of F over S $= \Sigma \overline{F} \cdot \hat{n} = \iint_S (\overline{F} \cdot \hat{n})\, ds$

Note. (1) Flux $= \iint_S (\overline{F} \cdot \hat{n})\, ds$ where $\overline{F}$ represents the velocity of a liquid.

If $\iint_S (\overline{F} \cdot \hat{n})\, ds = 0$, then $\overline{F}$ is said to be a *solenoidal* vector point function.

Example 94. *Show that* $\iint_S \overline{F} \cdot \hat{n}\, ds = \frac{3}{2}$, *where*

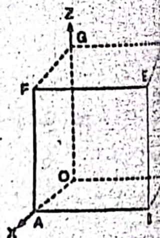
$\overrightarrow{F} = 4xz\,i - y^2 j + yz\,k$ *and S is the surface of the cube bounded by the planes,*

$x = 0, x = 1, y = 0, y = 1, z = 0, z = 1.$

Solution.

S. No.	Surface	Outward Normal	ds	Eq. of surface
1	OABC	$-k$	$dx\,dy$	$z=0$
2	DEFG	k	$dx\,dy$	$z=1$
3	OAFG	$-j$	$dx\,dz$	$y=0$
4	BCDE	j	$dx\,dz$	$y=1$
5	ABEF	i	$dy\,dz$	$x=1$
6	OCDG	$-i$	$dy\,dz$	$x=0$

$$\iint_S \overline{F} \cdot \hat{n}\, ds = \iint_{OABC} \overline{F} \cdot \hat{n}\, ds + \iint_{DEFG} \overline{F} \cdot \hat{n}\, ds + \iint_{OAFG} \overline{F} \cdot \hat{n}\, ds + \iint_{BCDE} \overline{F} \cdot \hat{n}\, ds + \iint_{ABEF} \overline{F} \cdot \hat{n}\, ds + \iint_{OCDG} \overline{F} \cdot \hat{n}\, ds \quad ...(1)$$



$$\iint_{OABC} \overline{F} \cdot n\, ds = \iint_{OABC} (4xz\,i - y^2 j + yz\,k) \cdot (-k)\, dx\, dy = \int_0^1\int_0^1 -yz\, dx\, dy = 0 \quad \text{(as}$$

$$\iint_{DEFG} (4xzi - y^2 j + yz\,k) \cdot k\, dx\, dy = \iint_{DEFG} yz\, dx\, xy = \int_0^1\int_0^1 y(1)\, dx\, dy$$

$$= \int_0^1 dx \left[\frac{y^2}{2}\right]_0^1 = [x]_0^1 \frac{1}{2} = \frac{1}{2}$$

$$\iint_{OAFG} (4\,xz\,i - y^2 j + yz\,k) \cdot (-j)\, dx\, dz = \iint_{OAFG} y^2\, dx\, dz = 0$$

$$\iint_{BCDE} (4\,xz\,i - y^2 j + yz\,k) \cdot j\, dx\, dz = \iint_{BCDE} (-y^2)\, dx\, dz$$

$$= -\int_0^1 dx \int_0^1 dz = -(x)_0^1\,(z)_0^1 = -1 \quad \text{(as } y = 1)$$

$$\iint_{ABEF} (4\,xzi - y^2 j + yzk) \cdot i\, dy\, dz = \iint 4\,xz\, dy\, dz = \int_0^1\int_0^1 4\,(1)\,z\, dy\, dz$$

$$= 4\,(y)_0^1 \left(\frac{z^2}{2}\right)_0^1 = 4\,(1)\left(\frac{1}{2}\right) = 2$$

$$\iint_{OCDG} (4\,xz\,i - y^2 j + yz\,k)\,(-i)\, dy\, dz = \int_0^1\int_0^1 -4\,xz\, dy\, dz = 0 \quad \text{(as } x = 0)$$

On putting these values in (1), we get

$$\iint_S F \cdot \hat{n}\, ds = 0 + \frac{1}{2} + 0 - 1 + 2 + 0 = \frac{3}{2}$$

Proved

VOLUME INTEGRAL

Let $\overline{F}$ be a vector point function and volume V enclosed by a closed surface.

The volume integral $= \iiint_V \overline{F}\, dv$

Example 95. *If* $\overline{F} = 2z\,i - x\,j + y\,k$, *evaluate* $\iiint_V \overline{F}\, dv$ *where v is the region bounded by the surfaces*

$x = 0,\ y = 0,\ x = 2,\ y = 4,\ z = x^2,\ z = 2.$

Solution. $\iiint \overline{F}\, dv = \iiint (2\,zi - xj + yk)\, dx\, dy\, dz$

$$= \int_0^2 dx \int_0^4 dy \int_{x^2}^2 (2\,zi - xj + yk)\, dz$$

$$= \int_0^2 dx \int_0^4 dy\, [z^2 i - xzj + yzk]_{x^2}^2$$

$$= \int_0^2 dx \int_0^4 dy\, [4\,i - 2\,xj + 2\,yk - x^4 i + x^3 j - x^2 yk]$$

$$= \int_0^2 dx \left[4\,yi - 2\,xyj + y^2 k - x^4 yi + x^3 yj - \frac{x^2 y^2}{2} k\right]_0^4$$

$$= \int_0^2 (16\,i - 8\,xj + 16\,k - 4\,x^4 i + 4\,x^3 j - 8\,x^2 k)\, dx$$

$$= \left[16\,xi - 4\,x^2 j + 16\,xk - \frac{4\,x^5}{5} i + x^4 j - \frac{8\,x^3}{3} k\right]_0^2$$

$$= 32\,i - 16\,j + 32\,k - \frac{128}{5} i + 16\,j - \frac{64}{3} k$$

$$= \frac{32\,i}{5} + \frac{32\,k}{3} = \frac{32}{15}(3\,i + 5\,k)$$

Ans.

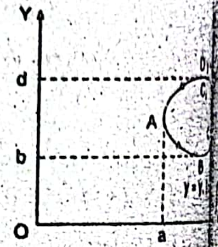
GREEN'S THEOREM (For a plane)

Statement. If $\phi(x, y)$, $\psi(x, y)$, $\frac{\partial \phi}{\partial y}$ and $\frac{\partial \psi}{\partial x}$ be continuous functions over a region R bounded by simple closed curve C in $x - y$ plane, then

$$\oint_C (\phi\, dx + \psi\, dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$$

Proof. Let the curve C be divided into two curves C_1 (ABC) and C_2 (C…

Let the equation of the curve C_1 (ABC) be,
$y = y_1(x)$ and equation of the curve C_2 (CDA) be
$y = y_2(x)$.



Let us see the value of

$$\iint_R \frac{\partial \phi}{\partial y}\,dx\,dy = \int_{x=a}^{x=c}\left[\int_{y=y_1(x)}^{y=y_2(x)}\frac{\partial \phi}{\partial y}\,dy\right]dx$$

$$= \int_a^c \left[\phi(x,y)\right]_{y=y_1(x)}^{y=y_2(x)}\,dx$$

$$= \int_a^c \left[\phi(x,y_2)-\phi(x,y_1)\right]dx = -\int_c^a \phi(x,y_2)\,dx-\int_a^c \phi(x, …$$

$$= -\left[\int_c^a \phi(x,y_2)\,dx+\int_a^c \phi(x,y_1)\,dx\right]$$

$$= -\left[\int_{C_2} \phi(x,y)\,dx+\int_{C_1} \phi(x,y)\,dx\right] = -\oint_C \phi(x,y)\,dx$$

Thus $\oint_C \phi\,dx = -\iint_R \frac{\partial \phi}{\partial y}\,dx\,dy$

Similarly, it can be shown that

$$\oint_C \psi\,dy = \iint \frac{\partial \psi}{\partial x}\,dx\,dy$$

On adding (1) and (2), we get

$$\oint (\phi\,dx+\psi\,dy) = \iint_R \left(\frac{\partial \psi}{\partial x}-\frac{\partial \phi}{\partial y}\right)dx\,dy$$

Note. Green's theorem in vector form

$$\int_C \bar{F}\cdot d\bar{r} = \iint_R (\nabla\times\bar{F})\cdot k\,dR$$

where $\bar{F} = \phi i+\psi j$, $\bar{r} = xi+yj$, k is a unit vector along z-axis and $dR = dx\,dy$.

Example 96. *A vector field $\bar{F}$ is given by $\bar{F} = \sin y\,i+x(1+\cos y)\,j$*

Evaluate the line integral $\int_C \bar{F}\cdot d\bar{r}$ where C is the circular path given by …

(A.M.I.E.T.E. …)

Solution. $\bar{F} = \sin y\,i+x(1+\cos y)\,j$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C [\sin y\,i+x(1+\cos y)\,j]\cdot(i\,dx+j\,dy) = \int_C \sin y\,dx+ …$$

On applying Green's theorem

$$\oint_c (\phi\,dx+\psi\,dy) = \iint_s \left(\frac{\partial \psi}{\partial x}-\frac{\partial \phi}{\partial y}\right)dx\,dy$$

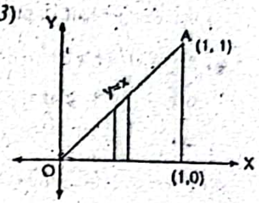
$$= \iint_s [(1+\cos y)-\cos y]\,dx\,dy$$

where s is the circular plane surface of radius a.

$$= \iint_s dx\,dy = \text{Area of circle} = \pi a^2.$$

Example 97. *Using Green's theorem evaluate $\int_c (x^2 y\,dx+x^2\,dy)$ where c is the boundary described counter clockwise of the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$.* (U.P. I semester Winter 2003)

Solution. By Green's theorem



$$\int_c (\phi\,dx+\psi\,dy) = \iint_R \left(\frac{\partial \psi}{\partial x}-\frac{\partial \phi}{\partial y}\right)dx\,dy$$

$$\int_c (x^2 y\,dx+x^2\,dy) = \iint_R (2x-x^2)\,dx\,dy$$

$$= \int_0^1 (2x-x^2)\,dx\int_0^x dy = \int_0^1 (2x-x^2)\,dx\,[y]_0^x$$

$$= \int_0^1 (2x-x^2)(x)\,dx = \int_0^1 (2x^2-x^3)\,dx = \left(\frac{2x^3}{3}-\frac{x^4}{4}\right)_0^1$$

$$= \left(\frac{2}{3}-\frac{1}{4}\right) = \frac{5}{12}$$ **Ans.**

Example 98. *Use Green's theorem to evaluate*

$$\int_c (x^2+xy)\,dx+(x^2+y^2)\,dy$$

where c is the square formed by the lines $y = \pm 1$, $x = \pm 1$.

Solution. $\int_c (x^2+xy)\,dx+(x^2+y^2)\,dy$

By Green's theorem $\oint_c (\phi\,dx+\psi\,dy) = \iint \left(\frac{\partial \psi}{\partial x}-\frac{\partial \phi}{\partial y}\right)dx\,dy$

$$= \int_{-1}^{1}\int_{-1}^{1}\left[\frac{\partial}{\partial x}(x^2+y^2)-\frac{\partial}{\partial y}(x^2+xy)\right]dx\,dy$$

$$= \int_{-1}^{1}\int_{-1}^{1}(2x-x)\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1} x\,dx\,dy$$

$$= \int_{-1}^{1} x\,dx\int_{-1}^{1} dy = \int_{-1}^{1} x\,dx\,(y)_{-1}^{1} = \int_{-1}^{1} x\,dx\,(1+1)$$

$$= \int_{-1}^{1} 2x\,dx = (x^2)_{-1}^{1} = 1-1 = 0.$$ **Ans.**

Exercise 5.20

… $[(3x^2-6yz)\,dx+(2y+3xz)\,dy+(1-4xyz^2)\,dz]$ from $(0,0,0)$ to $(1,1,1)$ along the path c … straight line from $(0,0,0)$ to $(0,0,1)$ then to $(0,1,1)$ and then to $(1,1,1)$.

… theorem in plane for

$$\int_c (x^2+2xy)\,dx+(y^2+x^3y)\,dy$$

… with the vertices $P(0,0)$, $Q(1,0)$, $R(1,1)$ and $S(0,1)$.

Ans. $-\frac{1}{2}$

… theorem for $\int_c (x^2-2xy)\,dx+(x^2y+3)\,dy$ around the boundary c of the region $y^2 = 8x$

4. Use Green's theorem in a plane to evaluate the integral $\int_c [(2x^2 - y^2)\,dx + (x^2 + y^2)\,d$

where c is the boundary in the x-y plane of the area enclosed by the x-axis and the semi

in the upper half xy-plane.

5. Apply Green's theorem to evaluate $\oint_c [(y - \sin x)\,dx + \cos x\,dy]$

where c is the plane triangle enclosed by the lines

$$y = 0,\ x = \frac{\pi}{2} \text{ and } y = \frac{2x}{\pi}.$$

6. Either directly or by Green's theorem, evaluate the line integral $\int_c e^{-x}(\cos y\,dx - \sin y$

where c is the rectangle with vertices $(0, 0)$, $(\pi, 0)$, $\left(\pi, \frac{\pi}{2}\right)$ and $\left(0, \frac{\pi}{2}\right)$.

7. Verify the Green's theorem to evaluate the line integral (a) $\int_c (2y^2\,dx + 3x\,dy)$ (b)

where c is the boundary of the closed region bounded by $y = x$ and $y = x^2$.

(a) **Ans.** $\frac{7}{30}$ (b) (*A.M.I.E.T.E., Winter*

8. Evaluate $\int\int_s \overline{F}\cdot ds$ where $\overline{F} = xy\,i - x^2 j + (x + z)\,k$ and S is region of the plane

first octant (*A.M.I.E.T.E., 2004, A.M.I.E.T.E., Winter*

9. The line integral $\int_c x^2\,dx + y^2\,dy$, where C is the boundary of the region $x^2 + y^2 \le$

(a) 0. (b) a (c) πa^2 (d) $\frac{1}{2}\pi a^2$ (A.M.I

5.44. STOKE'S THEOREM (Relation between Line Integral and Surface In

Statement. Surface integral of the component of curl $\overline{F}$ along the norm S, taken over the surface S bounded by curve C is equal to the line integ point function $\overline{F}$ taken along the closed curve C.

Mathematically $\oint_c \overline{F}\cdot d\overline{r} = \int\int_S \text{curl}\,\overline{F}\cdot\hat{\eta}\,ds$

where $\hat{\eta} = \cos\alpha\,i + \cos\beta\,j + \cos\gamma\,k$ is a unit external normal to any surface ds

Proof. Let

$$\overline{r} = xi + yj + zk$$
$$d\overline{r} = i\,dx + j\,dy + k\,dz$$
$$F = F_1 i + F_2 j + F_3 k$$

On putting the values of $\overline{F}.d\overline{r}$ in the statement of the theorem

$$\oint_c (F_1 i + F_2 j + F_3 k)\cdot(i\,dx + j\,dy + k\,dz)$$

$$= \int\int_S \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\times(F_1 i + F_2 j + F_3 k).(\cos\alpha\,i + \cos$$

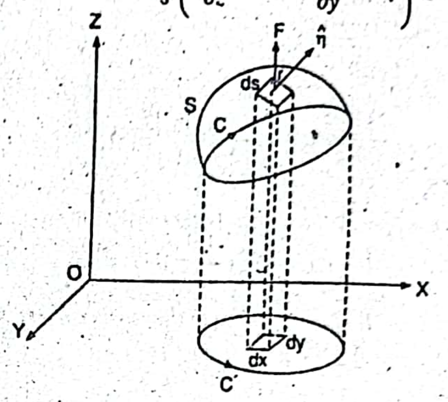
$$\oint (F_1\,dx + F_2\,dy + F_3\,dz) = \int\int_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)i + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)j + \left(\right.\right.$$

$$(i\cos\alpha +$$

$$= \int\int_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos$$

first prove

$$\oint_c F_1\,dx = \int\int_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right)ds \quad ...(2)$$



quation of the surface S be $z = g(x, y)$. The projection of the surface on x-y R.

$$\oint_c F_1(x, y, z)\,dx = \oint_c F_1[x, y, g(x, y)]\,dx$$

$$= -\int\int_R \frac{\partial}{\partial y}F_1(x, y, g)\,dx\,dy \quad \text{[By Green's Theorem]}$$

$$= -\int\int_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial g}{\partial y}\right)dx\,dy \quad ...(3)$$

ion cosines of the normal to the surface $z = g(x, y)$ are given by

$$\frac{\cos\alpha}{-\frac{\partial g}{\partial x}} = \frac{\cos\beta}{-\frac{\partial g}{\partial y}} = \frac{\cos\gamma}{1}$$

= projection of ds on the x-y plane $= ds\cos\gamma$

values of ds in R.H.S. of (2)

$$\beta - \frac{\partial F_1}{\partial y}\cos\gamma\Big)ds = \int\int_R \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right)\frac{dx\,dy}{\cos\gamma}$$

$$= \int\int_R \left(\frac{\partial F_1}{\partial z}\frac{\cos\beta}{\cos\gamma} - \frac{\partial F_1}{\partial y}\right)dx\,dy$$

$$= \int\int_R \left[\frac{\partial F_1}{\partial z}\left(-\frac{\partial g}{\partial y}\right) - \frac{\partial F_1}{\partial y}\right]dx\,dy$$

$$= -\int\int_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial g}{\partial y}\right)dx\,dy \quad ...(4)$$

d (4), we get

$$\oint_c F_1\,dx = \int\int_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right)ds \quad ...(5)$$

$$\oint_c F_2\,dy = \int\int_S \left(\frac{\partial F_2}{\partial x}\cos\gamma - \frac{\partial F_2}{\partial z}\cos\right.$$