Fall-18 Question Solve

1/ (a) Define vector product of two vectors. Discuss the geometrical interpretation of the vector product of two vectors. (See Lecture note → Lecture-14)

(b) Define linear dependence of vectors. Verify whether the set of vectors $\{(1,-1,3), (1,4,5), (2,-3,7)\}$ is linearly dependent or independent.

⇒ (See Lecture note for difinition → Lecture-12)

Set a linear combination of the given vectors equal to zero by using unknown scalars x, y, z:

$$x(1,-1,3) + y(1,4,5) + z(2,-3,7) = (0,0,0)$$

$$\Rightarrow (x, -x, 3x) + (y, 4y, 5y) + (2z, -3z, 7z) = (0,0,0)$$

$\Rightarrow (x+y+2z, -x+4y-3z, 3x+5y+7z) = (0,0,0)$

Equating the corresponding components and forming a system of linear equation we get,

$$\left.\begin{aligned} x+y+2z &= 0 \\ -x+4y-3z &= 0 \\ 3x+5y+7z &= 0 \end{aligned}\right\}$$

Reduce the system into echelon form by the elementary transformations. ~~At first interchange~~

Apply $L_2' = L_2 + L_1$ and $L_3' = L_3 - 3L_1$ and get,

$$\left.\begin{aligned} x+y+2z &= 0 \\ 0+5y-z &= 0 \\ 0+2y+z &= 0 \end{aligned}\right\}$$

Now, Apply $L_3' = L_3 - \frac{2}{5}L_2$ and get,

$$\left.\begin{aligned} x+y+2z &= 0 \\ 0+5y-z &= 0 \\ 0+0+\frac{7}{5}z &= 0 \end{aligned}\right\}$$

The system is in echelon form and has exactly three equations in three unknowns. Hence the system has a zero solution i.e. $x=0$, $y=0$, $z=0$. So the given vectors are linearly independent.

(c) Find the directional derivative of divergence of $\vec{u}$ at the point $(1,2,2)$ in the direction of the outer normal of the sphere $x^2+y^2+z^2=9$ for $\vec{u} = x^4\hat{i}+y^4\hat{j}+z^4\hat{k}$.

$\Rightarrow$ Given, $\vec{u} = x^4\hat{i}+y^4\hat{j}+z^4\hat{k}$

$$\therefore \text{div}\,\vec{u} = \nabla\cdot\vec{u}$$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(x^4\hat{i}+y^4\hat{j}+z^4\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(x^4)+\frac{\partial}{\partial y}(y^4)+\frac{\partial}{\partial z}(z^4)$$

$$= 4x^3+4y^3+4z^3$$

Now, directional derivative of $\text{div}\,\vec{u} = \nabla\,\text{div}\,\vec{u}$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(4x^3+4y^3+4z^3\right)$$

$$= \hat{i}\cdot 12x^2+\hat{j}\cdot 12y^2+\hat{k}\cdot 12z^2$$

Now, directional derivative of $\text{div}\,\vec{u}$ at the point $(1,2,2) = \hat{i}\cdot 12\cdot(1)^2+\hat{j}\cdot 12\cdot(2)^2+\hat{k}\cdot 12\cdot(2)^2$

$$= 12\hat{i}+48\hat{j}+48\hat{k}$$

$$= 12(\hat{i}+4\hat{j}+4\hat{k})$$

Again, we know normal to the sphere $x^2+y^2+z^2=9$ is $= \nabla(x^2+y^2+z^2-9)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-9)$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

Now, normal to the sphere $x^2+y^2+z^2=9$ at the point $(1,2,2) = 2\cdot 1\,\hat{i} + 2\cdot 2\hat{j} + 2\cdot 2\hat{k}$

$$= 2\hat{i} + 4\hat{j} + 4\hat{k}$$

$\therefore$ unit normal vector $= \dfrac{2\hat{i}+4\hat{j}+4\hat{k}}{\sqrt{4+16+16}}$

$$= \frac{2(\hat{i}+2\hat{j}+2\hat{k})}{6}$$

$$= \frac{1}{3}(\hat{i}+2\hat{j}+2\hat{k})$$

So, directional derivative of div $\vec{u}$ along the normal

$$= 12(\hat{i}+4\hat{j}+4\hat{k})\cdot\frac{1}{3}(\hat{i}+2\hat{j}+2\hat{k})$$

$$= 4(1+8+8) = 68 \text{. (Ans)}$$

2(a) Test whether the vector field $\vec{F} = (z^2+2x+3y)\hat{i} + (3x+2y+z)\hat{j} + (y+2xz)\hat{k}$ is irrotational or irrotational. If irrotational then obtain a scalar ϕ function such that grad $\phi = \vec{F}$.

$\Rightarrow$ (See Lecture note $\rightarrow$ Lecture-23)

(b) Evaluate $\int_c \vec{F}\cdot\vec{dr}$ where c is the curve in the xy-plane, $y = x^3$ from $(1,1)$ to $(2,8)$ if $\vec{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$.

$\Rightarrow$ Here, $\int_c \vec{F}\cdot\vec{dr} = \int_c \left[(5xy-6x^2)\hat{i} + (2y-4x)\hat{j}\right]\cdot(dx\hat{i}+dy\hat{j})$

$$= \int_c (5xy-6x^2)\,dx + (2y-4x)\,dy$$

$$= \int_{x=1}^{2} (5x\cdot x^3 - 6x^2)\,dx + (2x^3-4x)\cdot 3x^2 dx$$

Given, $y = x^3$

$\therefore dy = 3x^2dx$

$$= \int_{x=1}^{2} (5x^4 - 6x^2)\,dx + (6x^5 - 12x^3)\,dx$$

$$= \int_{x=1}^{2} (6x^5 + 5x^4 - 12x^3 - 6x^2)\,dx$$

$$= \left[6\cdot\frac{x^6}{6} + 5\cdot\frac{x^5}{5} - 12\cdot\frac{x^4}{4} - 6\cdot\frac{x^3}{3}\right]_1^2$$

$$= \left[x^6 + x^5 - 3x^4 - 2x^3\right]_1^2$$

$$= \left\{(2)^6 + (2)^5 - 3\cdot(2)^4 - 2\cdot(2)^3\right\} - \left\{1^6 + 1^5 - 3\cdot1^4 - 2\cdot1^3\right\}$$

$$= 64 + 32 - 48 - 16 - 1 - 1 + 3 + 2$$

$$= 35.$$ (Ans)

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(c) Find the total work done in moving a particle in a force field given by $\vec{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = t^2+1$, $y = 2t^2$, $z = t^3$ from $t=1$ to $t=2$.

$\Rightarrow$ work done $= \int_C \vec{F}\cdot\vec{dr}$

$$= \int_C (3xy\hat{i} - 5z\hat{j} + 10x\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_C 3xy\,dx - 5z\,dy + 10x\,dz \quad \ldots \text{(i)}$$

Now, the path c given,

$x = t^2+1$ $\quad$ $y = 2t^2$ $\quad$ $z = t^3$

$\Rightarrow \frac{dx}{dt} = 2t$ $\quad$ $\Rightarrow \frac{dy}{dt} = 4t$ $\quad$ $\Rightarrow \frac{dz}{dt} = 3t^2$

$\therefore dx = 2t\,dt$ $\quad$ $\Rightarrow dy = 4t\,dt$ $\quad$ $\therefore dz = 3t^2\,dt$

The equation (i) becomes,

$$\int_{t=1}^{2} \left[3(t^2+1).2t^2.2t\,dt - 5t^3.4t\,dt + 10(t^2+1)3t^2\,dt\right]$$

$$= \int_{t=1}^{2} (12t^5 + 12t^3)\,dt - 20t^4\,dt + (30t^4 + 30t^2)\,dt$$

$$= \int_{t=1}^{2} (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt$$

$$= \left[12\frac{t^6}{6} + 10\frac{t^5}{5} + 12\frac{t^4}{4} + 30\frac{t^3}{3}\right]_1^2$$

$$= \left[2t^6 + 2t^5 + 3t^4 + 10t^3\right]_1^2$$

$$= \{2.2^6 + 2.2^5 + 3.2^4 + 10.2^3\} - \{2.1^6 + 2.1^5 + 3.1^4 + 10.1^3\}$$

$$= 128 + 64 + 48 + 80 - 2 - 2 - 3 - 10$$

$$= 303 .$$ (Ans)

3/ ⓐ Use Green's theorem to evaluate $\int_c (x^2y\,dx + x^2dy)$, where c is the boundary described counter clockwise of the triangle with vertices $(0,0), (1,0)$ and $(1,1)$.

⇒ (See Lecture Note, Lecture-25)

ⓑ Apply Divergence theorem to evaluate $\iint_S \vec{F}.\vec{ds}$, where $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ and S is the surface bounded by the region $x^2+y^2=4$, $z=0$ and $z=3$.

⇒ (See Lecture Note, Lecture-26)

ⓒ Verify Stoke's theorem for $\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and c is its boundary.

⇒ (See Lecture Note, Lecture-27)

4/ ⓐ Define inverse of a matrix. Find the inverse of the following matrix using row canonical form.

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -6 & 0 & 1 & -2 \\ 8 & 1 & -2 & 1 \end{bmatrix}$$

⇒ (See Lecture Note, Lecture-9, Definition Lecture-5)

ⓑ Define echelon and reduced echelon matrix. Find the echelon form and row reduced echelon form of the following matrix: $\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix}$ ⇒ (See Lecture Note, Lecture-6)

5] ⓐ Define rank of a matrix. Determine the rank of the following matrix:

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

⇒ (For definition see Lecture Note, Lecture-7)

Let us reduce the given matrix to echelon form by the elementary row transformations or operations:

At first ~~apply~~ Interchange R_2 and R_1 and get the equivalent matrix,

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Now apply $R_3' = R_3 - 3R_1$ and $R_4' = R_4 - R_1$,

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

Now apply. $R_3' = R_3 - R_2$ and $R_4' = R_4 - R_2$,

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$; which is in row echelon form.

Since there are two non-zero rows, so the rank of the given matrix is 2.

(b) Examine whether the matrix $\frac{1}{6}\begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & -5 & 1 & 1 \\ 3 & 1 & 1 & -5 \\ 3 & 1 & -5 & 1 \end{bmatrix}$ is orthogonal or not.

⇒ (See Lecture Note, Lecture-8)

(c) Find the eigenvalues and the associated eigenvectors of the following matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

⇒ The characteristic matrix of the given matrix is:

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 2 \end{bmatrix}$$

Now the determinant of $(\lambda I - A)$ is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 2) - 6$$

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Now, the characteristic equation of A is,

$$(\lambda-1)(\lambda-2)-6=0$$

$$\Rightarrow \lambda^2-2\lambda-\lambda+2-6=0$$

$$\Rightarrow \lambda^2-3\lambda-4=0$$

$$\Rightarrow \lambda^2-4\lambda+\lambda-4=0$$

$$\Rightarrow \lambda(\lambda-4)+1(\lambda-4)=0$$

$$\Rightarrow (\lambda-4)(\lambda+1)=0$$

$$\therefore \lambda=-1,\ 4$$

Hence, the eigen values of A are $\lambda_1=-1$, $\lambda_2=4$

Now by definition $X=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector of A corresponding to λ if and only if X is a non-trivial solution of $(\lambda I-A)X=0$, that is

$$\begin{bmatrix} \lambda-1 & -2 \\ -3 & \lambda-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \ldots \text{(i)}$$

for $\lambda=-1$, equation (i) becomes,

$$\begin{bmatrix} -2 & -2 \\ -3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} -2x_1-2x_2&=0 \\ -3x_1-3x_2&=0 \end{aligned}\right\}$$

Now apply $L_1'=-\frac{L_1}{2}$ and $L_2'=-\frac{L_2}{3}$ and get.

$$\left.\begin{aligned} x_1+x_2&=0 \\ x_1+x_2&=0 \end{aligned}\right\} \Rightarrow x_1+x_2=0$$

The system is in echelon form and consistent. Since there are more unknowns than equation in the echelon form, so the system has an infinite number of solutions and has a free variable which is x_2.

Let $x_2 = a$ (where a is an arbitrary real number)

$\therefore x_1 = -a$. Therefore, the eigen vectors of A associated to $\lambda = -1$ are of the form $X = \begin{bmatrix} -a \\ a \end{bmatrix}$.

In particular, let $a = 1$, then $X = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to $\lambda = -1$.

For $\lambda = 4$ equation (i) becomes,

$$\begin{bmatrix} 3 & -2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} 3x_1 - 2x_2 &= 0 \\ -3x_1 + 2x_2 &= 0 \end{aligned}\right\}$$

Now apply $L_1' = -L_1$ and get,

$$\left.\begin{aligned} -3x_1 + 2x_2 &= 0 \\ -3x_1 + 2x_2 &= 0 \end{aligned}\right\} \Rightarrow -3x_1 + 2x_2 = 0$$

The system is in echelon form and consistent. Since there are more unknowns than equation in the echelon form, so the system has an infinite number of solutions and has a free variable which is x_2.

Let $x_2 = b$ (where b is an arbitratry real number)

$\therefore x_1 = \frac{2}{3}b$. Therefore; the eigen vector of A associated to $\lambda = 4$ are of the form $X = \begin{bmatrix} \frac{2}{3}b \\ b \end{bmatrix}$

In particular, let $b = 1$, then $X = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$ is an eigenvector of A associated to $\lambda = 4$.

6| ⓐ Define Fourier series. Find the Fourier series of $f(x) = x + x^2$ for $-\pi \leq x \leq \pi$.

$\Rightarrow$ (For definition See Lecture Note. Lecture - 28)

Let, $f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x + \ldots$ (i)

Here, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\,dx = \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi}$

$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} + \frac{\pi^3}{3}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^3}{3}\right)\right]$

$\cancel{\frac{1}{\pi}\ 0} = \frac{1}{\pi}\cdot 2\frac{\pi^3}{3} = \frac{2\pi^2}{3}$.

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\cos nx\,dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx$

$$= \frac{1}{\pi}\left[x.\frac{\sin nx}{n} - 1.\left(-\frac{\cos nx}{n^2}\right)+0\right]_{-\pi}^{\pi} + \frac{1}{\pi}\left[x^2.\frac{\sin nx}{n}\right.$$

$$\left. - 2x.\left(\frac{-\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right) - 0\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[x.\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{-\pi}^{\pi} + \frac{1}{\pi}\left[x^2.\frac{\sin nx}{n} + 2x\frac{\cos nx}{n^2}\right.$$

$$\left. - 2\frac{\sin nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\pi.\frac{\sin n\pi}{n} + \frac{\cos n\pi}{n^2} - \left(-\pi\frac{\sin n(-\pi)}{n} + \frac{\cos n(-\pi)}{n^2}\right)\right]$$

$$+ \frac{1}{\pi}\left[\pi^2\frac{\sin n\pi}{n} + 2\pi\frac{\cos n\pi}{n^2} - 2\frac{\sin n\pi}{n^3}\right.$$

$$\left. - \left(\pi^2\frac{\sin n(-\pi)}{n} - 2\pi\frac{\cos n(-\pi)}{n^2} - 2\frac{\sin n(-\pi)}{n^3}\right)\right]$$

$$= \frac{1}{\pi} * 0 + \frac{1}{\pi}.\cancel{\pi\ \sin}\ 2.\ 2\pi.\frac{\cos n\pi}{n^2}$$

$$= 4.\frac{(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(-\frac{\cos nx}{n}\right) - (2x+1)\left(\frac{-\sin nx}{n^2}\right) + 2\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(\pi+\pi^2)\frac{\cos n\pi}{n} + 2\frac{\cos n\pi}{n^3} + (-\pi+\pi^2)\frac{\cos n\pi}{n}\right.$$

$$\left. - 2\frac{\cos n\pi}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}\cos n\pi\right]$$

$$= -\frac{2}{n}(-1)^n.$$

Substituting the values of a_0, a_n, b_n in (i) we get,

$$x + x^2 = \frac{\pi^2}{3} + 4\left[-\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right]$$
$$-2\left[-\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \cdots\right]$$

(Ans)

(b) Define Fourier series in complex form. Find the complex form of the Fourier series of the following function: $f(x) = \begin{cases} 0 & \text{when } -\pi \le x < 0 \\ 1 & \text{when } 0 \le x \le \pi \end{cases}$

⇒ (See Lecture Note, Lecture-35).

(c) Find the Fourier transform of the following function

$$f(x) = \begin{cases} 1 - x^2 & \text{when } |x| \le 1 \\ 0 & \text{when } |x| > 1 \end{cases}$$

⇒ (See Lecture Note, Lecture-31).

7) (a) Find the Fourier cosine transform of $f(x) = e^{-x^2}$.

⇒ The Fourier cosine transform of $f(x)$ is given by

$$F(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

Putting the value of $f(x)$ we get,

$$F(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} \cos sx \, dx$$

$$\Rightarrow I = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} \cos sx \, dx \quad \cdots (i)$$

Differentiating both sides w.r. to s,

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} . x(-\sin sx) \, dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty x e^{-x^2} \sin sx \, dx$$

$$= -\sqrt{\frac{2}{\pi}} \left[\left| \sin sx \int x e^{-x^2} dx \right|_0^\infty - \int_0^\infty \left\{ \frac{d}{dx} \sin sx \int x e^{-x^2} dx \right\} dx \right]$$

Solving $\int x e^{-x^2} dx$

Let, $-x^2 = t$

$\Rightarrow -2x\,dx = dt$

$\Rightarrow -x\,dx = \frac{dt}{2}$

$$= \int e^t \left(-\frac{dt}{2}\right)$$

$$= -\frac{1}{2} \int e^t dt = -\frac{1}{2} e^t$$

$$= -\frac{1}{2} e^{-x^2} = -\frac{e^{-x^2}}{2}$$

$$\therefore \frac{dI}{ds} = -\sqrt{\frac{2}{\pi}} \left[\left| -\sin sx \cdot \frac{e^{-x^2}}{2} \right|_0^\infty - \int_0^\infty s \cos sx \left(-\frac{e^{-x^2}}{2}\right) dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left| \sin sx \cdot \frac{e^{-x^2}}{2} \right|_0^\infty - \frac{s}{2} \int_0^\infty \cos sx \cdot e^{-x^2} dx \right]$$

$$\Rightarrow \frac{dI}{ds} = \sqrt{\frac{2}{\pi}}\left[(0-0)\right] - \frac{s}{2} I$$

$$\Rightarrow \frac{dI}{ds} = -\frac{s}{2} I$$

$$\Rightarrow \frac{dI}{I} = -\frac{s}{2}\, ds$$

Integrating both sides,

$$\int \frac{dI}{I} = -\frac{1}{2}\int s\, ds$$

$$\Rightarrow \log I = \frac{-1}{2}\cdot\frac{s^2}{2} + c$$

$$\Rightarrow \log I = -\frac{s^2}{4} + c$$

$$\Rightarrow I = e^{-\frac{s^2}{4}+c} = e^{-\frac{s^2}{4}}\cdot e^{c}$$

$$\Rightarrow I = e^{-\frac{s^2}{4}}\cdot A \quad , \text{ where } A = e^{c}$$

Put $s=0$, $\quad I = A = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\, dx$

Let $x^2 = t \Rightarrow x = \sqrt{t}$

$$\Rightarrow 2x\,dx = dt$$

$$\Rightarrow dx = \frac{dt}{2x} = \frac{dt}{2\sqrt{t}}$$

$$\therefore I = \cancel{\int_0^\infty e^{-t}}\ \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-t}\cdot\frac{1}{2\sqrt{t}}\, dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t}\cdot t^{-\frac{1}{2}}\, dt \cdot \sqrt{\frac{2}{\pi}}$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-t}\cdot t^{\left(\frac{1}{2}-1\right)}\, dt = \frac{1}{\sqrt{2\pi}}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{\sqrt{2\pi}}$$

$$= \frac{1}{\sqrt{2}}$$

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$$\left[\because \Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1}\, dx \quad (\text{Gamma function}).\right]$$

$$\therefore I = \frac{1}{\sqrt{2}} e^{-\frac{s^2}{4}}$$

$\therefore$ Fourier cosine transform of e^{-x^2} is $\frac{1}{\sqrt{2}} e^{-\frac{s^2}{4}}$.

(Ans)

ⓑ Find the Fourier sine integral for $f(x) = e^{-\beta x}$, hence evaluate $\int_0^\infty \frac{s \sin sx}{\beta^2 + s^2}\, ds$.

$\Rightarrow$ (See Lecture Note, Lecture-30)

ⓒ (See Lecture-36, Boundary value problem example)

Spring-18 Question Solve

1) (a) Define scalar product of three vectors. Write geometrical interpretation of the scalar product of three vectors.

$\Rightarrow$ (See Lecture Note, Lecture-14)

(b) Define linear dependence and linear independence of vectors. Examine whether the set of vectors $\{(1,-2,1), (2,1,-1), (7,-4,1)\}$ is linearly dependent or independent.

$\Rightarrow$ (See Lecture Note for definition $\rightarrow$ Lecture-12)

Set a linear combination of the given vectors equal to zero by using unknown scalars x, y, z:

$$\cancel{x(2,1,-1)+y(7,-4,1)+z(}$$

$$x(1,-2,1)+y(2,1,-1)+z(7,-4,1)=(0,0,0)$$

$$\Rightarrow (x,-2x,x)+(2y,y,-y)+(7z,-4z,z)=(0,0,0)$$

$$\Rightarrow (x+2y+7z, -2x+y-4z, x-y+z)=(0,0,0)$$

Equating the corresponding components and forming a system of linear equation we get,

$$\left.\begin{aligned} x+2y+7z&=0\\ -2x+y-4z&=0\\ x-y+z&=0 \end{aligned}\right\}$$

Reduce the system into echelon form by the elementary transformations.

Apply $L_2' = L_2 + 2L_1$ and $L_3' = L_3 - L_1$ and get,

$$x + 2y + 7z = 0$$
$$0 + 5y + 10z = 0$$
$$0 - 3y - 6z = 0$$

Now apply $L_3' = L_3 + \frac{3}{5}L_2$ and get,

$$x + 2y + 7z = 0$$
$$0 + 5y + 10z = 0$$
$$0 + 0 + 0 = 0$$

The system is in echelon form and has only 2 equations in 3 unknowns. Hence the system has a non-zero solution. Thus the given vectors are linearly dependent.

~~(e) Find the directional derivative of divergence of $\vec{u}$ at the point (1,2,2) in the direction of the outer normal of the sphere $x^2+y^2+z^2=9$ for $\vec{u} = x^4\hat{i}+$~~

(c) Find the directional derivative of $f(x,y,z) = x^2y^2z^2$ at the point $(1,1,-1)$ in the direction of the tangent to the curve $x = e^t$, $y = 2\sin t + 1$, $z = t - \cos t$ at $t=0$.

$\Rightarrow$ Let $\phi = x^2y^2z^2$

$\therefore$ Directional Derivative of $\phi = \nabla\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y^2z^2)$$

$$= 2xy^2z^2\hat{i} + 2x^2yz^2\hat{j} + 2x^2y^2z\hat{k}$$

$\therefore$ Directional Derivative of ϕ at $(1,1,-1)$

$$= 2\cdot1\cdot1^2\cdot(-1)^2\hat{i} + 2\cdot1^2\cdot1\cdot(-1)^2\hat{j} + 2\cdot1^2\cdot1^2\cdot(-1)\hat{k}$$

$$= 2\hat{i} + 2\hat{j} - 2\hat{k}$$

$\therefore \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$= e^t\hat{i} + (2\sin t + 1)\hat{j} + (t - \cos t)\hat{k}$$

$\therefore$ Tangent vector, $\vec{T} = \dfrac{d\vec{r}}{dt}$

$$= e^t\hat{i} + 2\cos t\,\hat{j} + (1 + \sin t)\hat{k}$$

$\therefore$ Tangent (at $t=0$) $= e^0\hat{i} + 2\cdot\cos 0\cdot\hat{j} + (1+\sin 0)\cdot\hat{k}$

$$= \hat{i} + 2\hat{j} + \hat{k}$$

$\therefore$ Required directional derivative along tangent

$$= (2\hat{i} + 2\hat{j} - 2\hat{k})\cdot\frac{(\hat{i}+2\hat{j}+\hat{k})}{\sqrt{1+4+1}}$$

$$= \frac{2+4-2}{\sqrt{6}} = \frac{4}{\sqrt{6}} \quad \text{(Ans.)}$$

2/ (a) A fluid motion is given by the following function:
$\bar{v} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}$.
Examine whether the motion is rotational or irrotational.
Also obtain a scalar function ϕ such that $\text{grad}\,\phi = \bar{v}$.

$\Rightarrow$

$$\text{curl}\,\bar{v} = \nabla \times \bar{v}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\sin z - \sin x & x\sin z + 2yz & xy\cos z + y^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(xy\cos z + y^2) - \frac{\partial}{\partial z}(x\sin z + 2yz)\right] - \vec{j}\left[\frac{\partial}{\partial x}(xy\cos z + y^2) - \frac{\partial}{\partial z}(y\sin z - \sin x)\right] + \hat{k}\left[\frac{\partial}{\partial x}(x\sin z + 2yz) - \frac{\partial}{\partial y}(y\sin z - \sin x)\right]$$

$$= \hat{i}(x\cos z + 2y - x\cos z - 2y) - \hat{j}(y\cos z - y\cos z) + \hat{k}(\sin z - \sin z)$$

$$= 0\hat{i} - 0\hat{j} + 0\hat{k} = \vec{0}$$

Since $\text{curl}\,\bar{v} = \vec{0}$, so $\bar{v}$ is irrotational.

Again, given $\bar{v} = \text{grad}\,\phi = \nabla\phi$, where ϕ is a scalar function.

Now, the total differential, $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

$$\Rightarrow d\phi = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot\left(\vec{i}dx + \vec{j}dy + \vec{k}dz\right)$$

$$\Rightarrow d\phi = \nabla\phi \cdot d\vec{r} \quad [\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}]$$

$$\Rightarrow d\phi = \bar{v} \cdot d\vec{r}$$

$$\Rightarrow d\phi = \left[(y\sin z - \sin x)\vec{i} + (x\sin z + 2yz)\vec{j} + (xy\cos z + y^2)\hat{k}\right] \cdot (dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k})$$

$$\Rightarrow d\phi = (y\sin z - \sin x)\,dx + (x\sin z + 2yz)\,dy + (xy\cos z + y^2)\,dz$$

$\Rightarrow d\phi = y\sin z\, dx - \sin x\, dx + x\sin z\, dy + 2yz\, dy$
$+ xy\cos z\, dz + y^2 dz$

$\Rightarrow d\phi = (y\sin z\, dx + x\sin z\, dy) + (y^2 dz + 2yz\, dy)$
$+ xy\cos z\, dz - \sin x\, dx$

$\Rightarrow d\phi = \sin z\,(x\,dy + y\,dx) + (y^2 dz + \cancel{y\,2z\,dz}\; z \cdot 2y\,dy)$
$+ xy\cos z\, dz - \sin x\, dx$

$\Rightarrow d\phi = \sin z\, d(xy) + xy\cos z\, dz + (y^2 dz + z \cdot d(y^2))$
$- \sin x\, dx$

$\Rightarrow d\phi = \sin z\, d(xy) + xy\, d(\sin z) + d(y^2 z) - \sin x\, dx$

$\Rightarrow d\phi = d(xy\sin z) + d(y^2 z) - \sin x\, dx$

Now integrating both sides we get,

$\int d\phi = \int d(xy\sin z) + \int d(y^2 z) - \int \sin x\, dx$

$\Rightarrow \phi = xy\sin z + y^2 z + \cos x + c$, which is the required scalar function. (Ans)

2) (b) If $\vec{F} = 3xy\,\hat{i} - y^2\,\hat{j}$, evaluate $\int_c \vec{F} \cdot d\vec{r}$ where c is the curve in the xy-plane, $y = 2x^2$ from $(0,0)$ to $(1,2)$.

$\Rightarrow \int_c \vec{F} \cdot d\vec{r}$

$= \int_c (3xy\,\hat{i} - y^2\,\hat{j}) \cdot (dx\,\hat{i} + dy\,\hat{j})$

$= \int_c (3xy\, dx - y^2 dy)$

$= \int_{x=0}^{1} (3x \cdot 2x^2\, dx - 4x^4 \cdot 4x\, dx)$

$y = 2x^2$
$\therefore dy = 4x\, dx$

$$= \int_{x=0}^{1} \left(6x^3\,dx - 16x^5\,dx\right)$$

$$= \left[6 \cdot \frac{x^4}{4} - 16 \cdot \frac{x^6}{6}\right]_0^1$$

$$= \frac{3}{2}(1)^4 - \frac{8}{3}(1)^6 - 0$$

$$= \frac{3}{2} - \frac{8}{3} = \frac{9-16}{6} = \frac{-7}{6} \quad \text{(Ans)}$$

(c) Find the work done in moving a particle once around a circle C in the xy-plane, if the circle has centre at the origin and radius 3 and if the force field is given by $\vec{F} = (2x-y+z)\hat{i} + (x+y-z^2)\hat{j} + (3x-2y+4z)\hat{k}$.

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_C \left[(2x-y+z)\hat{i} + (x+y-z^2)\hat{j} + (3x-2y+4z)\hat{k}\right] \cdot (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \int_C (2x-y+z)\,dx + (x+y-z^2)\,dy + (3x-2y+4z)\,dz$$

~~Given, $x^2+y^2=3^2$~~

~~$\Rightarrow y^2 = 3^2 - x^2$~~

~~$\therefore y = \pm\sqrt{9-x^2}$~~

~~$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_C (2x-$~~

The parametric equations of given path $x^2+y^2=3^2$ are

$x = 3\cos\theta$, $y = 3\sin\theta$, where θ varies from 0 to 2π.

As in the xy-plane, $z=0$.

$\therefore dx = -3\sin\theta\,d\theta$, $dy = 3\cos\theta\,d\theta$, $dz=0$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (2\cdot 3\cos\theta - 3\sin\theta + 0)(-3\sin\theta\,d\theta) + (3\cos\theta + 3\sin\theta - 0)\, 3\cos\theta\,d\theta + (9\cos\theta - 6\sin\theta + 0)\cdot 0$$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (6\cos\theta - 3\sin\theta)\,\cancel{d}.(-3\sin\theta\, d\theta) + (3\cos\theta + 3\sin\theta)\, 3\cos\theta\, d\theta$$

$$= \int_0^{2\pi} \left(-18\cos\theta\sin\theta\, \cancel{d\theta} + 9\sin^2\theta + 9\cos^2\theta + 9\sin\theta\cos\theta\right) d\theta$$

$$= \int_0^{2\pi} (9 - 9\sin\theta\cos\theta)\, d\theta$$

$$= 9\int_0^{2\pi} (1 - \sin\theta\cos\theta)\, d\theta$$

$$= 9\left[\theta - \frac{\sin^2\theta}{2}\right]_0^{2\pi} = 9\left(2\pi - \frac{\sin^2 2\pi}{2}\right) = 9.\ 2\pi$$

$$= 18\pi. \quad \text{(Ans)}$$

3] (a) (See Lecture Note, Lecture-25)
(b) (See Lecture Note, Lecture-26)
(c) (See Lecture Note, lecture-27)

4] (a) (See Lecture Note, lecture-9)
(b) (See Lecture Note ~~lec~~, lecture-6)

5] (a) (See lecture Note, lecture-7)
(b) (See Lecture Note, lecture-11)

6] (a) (See lecture Note, lecture-29)
(b) (See lecture Note, lecture-29)

7] (a) (See lecture Note, lecture-31)
(b) (See lecture Note, lecture-33)
(c) (See Lecture Note, lecture-32)