Proof. We know that

$$\cos u(t-x)=\cos(ut-ux)$$

or
$$\cos u(t-x)=\cos ut\cos ux+\sin ut\sin ux$$

Then equation (5) of article 14.3, can be written as

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\,(\cos ut\cos ux+\sin ut\sin ux)\,du\,dt$$

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos ut\,\cos ux\,du\,dt+\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\sin ut\ \text{si}\ldots$$

Case 1. When $f(t)$ is odd.

$\therefore$ $f(t)\cos ut$ is odd hence $\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos ut\cos ux\,du\,dt=0$

[For odd function $\int_{-a}^{a} f(\ldots$

For even function $\int_{-a}^{a} f(x)\,dx=\ldots$

From (6) we have

$$\therefore\quad f(x)=\frac{2}{\pi}\int_0^{\infty}\sin ux\,du\int_0^{\infty} f(t)\sin ut\,dt$$

The relation (7) is called **Fourier sine integral.**

Case 2. When $f(t)$ is even.

$\therefore$ $f(t)\sin ut$ is odd. $\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\sin ut\sin ux\,du\,dt=0$

$\therefore$ $f(t)\cos ut$ is even.

From (6) we have

$$f(x)=\frac{2}{\pi}\int_0^{\infty}\cos ux\,du\int_0^{\infty} f(t)\cos ut\,dt$$

The relation (8) is known as Fourier cosine integral.

FOURIER'S COMPLEX INTEGRAL

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\,du\int_{-\infty}^{\infty} f(t)e^{-iut}\,dt$$

Proof. We know that $\int_{-a}^{a} f(x)\,dx=0$ if $f(x)$ is odd function.

$$\therefore\quad \int_{-\infty}^{\infty}\sin u(t-x)\,du=0$$ [since $\sin u(t-x)$ is od...

Obviously we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt\int_{-\infty}^{\infty}\sin u(t-x)\,du=0$$

$$\frac{i}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt\int_{-\infty}^{\infty}\sin u(t-x)\,du=0$$

Multiplying by i)

On adding (5) and (9) we have

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos u(t-x)\,du\,dt+\frac{i}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt\int_{-\infty}^{\infty}\sin u(t\ldots$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt\int_{-\infty}^{\infty}[\cos u(t-x)+i\sin u(t-x)]\,du$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt\int_{-\infty}^{\infty} e^{iu(t-x)}\,du$$

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\,du\int_{-\infty}^{\infty} f(t)e^{iut}\,dt \qquad \ldots($$

...(10) is called Fourier's Complex Integral.

...ple 1. *Express the function*

$$f(x)=\begin{cases}1 & \text{when } |x|\le 1\\ 0 & \text{when } |x|>1\end{cases}$$

...r integral. Hence evaluate

$$\int_0^{\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda$$

...n. The Fourier Integral for $f(x)$ is

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\,dt\,d\lambda$$

$$=\frac{1}{\pi}\int_0^{\infty}\int_{-1}^{1}\cos\lambda(t-x)\,dt\,d\lambda \qquad (\text{since } f(t)=1)$$

$$=\frac{1}{\pi}\int_0^{\infty}\left[\frac{\sin\lambda(t-x)}{\lambda}\right]_{-1}^{1}d\lambda$$

$$=\frac{1}{\pi}\int_0^{\infty}\frac{\sin\lambda(1-x)+\sin\lambda(1+x)}{\lambda}\,d\lambda$$ By sin C + sin ...

$$f(x)=\frac{2}{\pi}\int_0^{\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda$$

$$\int_0^{\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda=\frac{\pi}{2}f(x)$$

$$\int_0^{\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda=\begin{cases}\frac{\pi}{2} & \text{for } |x|<1\\ 0 & \text{for } |x|>1\end{cases}$$

...$|x|=1$, which is a point of discontinuity of $f(x)$, value of integral $=\frac{\pi/2 \ldots}{2}$

...mple 2. *Find the Fourier sine integral for*

$$f(x)=e^{-\beta x}$$

...ce show that $\frac{\pi}{2}e^{-\beta x}=\int_0^{\infty}\frac{\lambda\sin\lambda x}{\beta^2+\lambda^2}\,d\lambda$

...tion. The Fourier sine transform of $f(x)$ is

$$f(x)=\frac{2}{\pi}\int_0^{\infty}\sin\lambda x\,d\lambda\int_0^{\infty} f(t)\sin\lambda t\,dt$$

Putting the value of $f(x)$ in (1) we get

$$e^{-\beta x}=\frac{2}{\pi}\int_0^\infty \sin\lambda x\,d\lambda\int_0^\infty e^{-\beta t}\sin\lambda t\,dt$$

$$=\frac{2}{\pi}\int_0^\infty \sin\lambda x\,d\lambda\left[\frac{e^{-\beta t}}{(\beta^2+\lambda^2)}(-\beta\sin\lambda t-\lambda\cos\lambda t)_0^\infty\right]$$

$$=\frac{2}{\pi}\int_0^\infty \sin\lambda x\,d\lambda\left[0+\frac{\lambda}{\beta^2+\lambda^2}\right]$$

$$e^{-\beta x}=\frac{2}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{\beta^2+\lambda^2}d\lambda \quad \text{or} \quad \frac{\pi}{2}e^{-\beta x}=\int_0^\infty \frac{\lambda\sin\lambda x}{\beta^2+\lambda^2}d\lambda.$$

Example 3. *Using Fourier cosine integral representation of an appropriate function, th*

$$\int_0^\infty \frac{\cos wx}{k^2+w^2}dw=\frac{\pi e^{-kx}}{2k},$$ $x >$

Solution. We know that Fourier integral is

$$f(x)=\frac{2}{\pi}\int_0^\infty \cos ux\,du\int_0^\infty f(t)\cos ut\,dt$$

Putting the value of $f(t)$ and replacing u by w we get

$$e^{-kx}=\frac{2}{\pi}\int_0^\infty \cos wx\,dw\int_0^\infty e^{-kt}\cos wt\,dt$$

$$=\frac{2}{\pi}\int_0^\infty \cos wx\,dw\left[\frac{e^{-kt}}{k^2+w^2}\{-k\cos wt+w\sin wt\}_0^\infty\right]$$

$$=\frac{2}{\pi}\int_0^\infty \cos wx\,dw\left[0+\frac{k}{k^2+w^2}\right]=\frac{2k}{\pi}\int_0^\infty \frac{\cos wx\,dw}{k^2+w^2}$$

or $$\int_0^\infty \frac{\cos wx}{k^2+w^2}=\frac{\pi e^{-kx}}{2k}$$

14.6 FOURIER TRANSFORMS

We have done in Article 14.5 that

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}du\int_{-\infty}^{\infty} f(t)e^{iut}dt$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}ds\left[\int_{-\infty}^{\infty} f(t)e^{ist}dt\right] \qquad (u=s)$$

Putting $\int_{-\infty}^{\infty} f(t)e^{ist}dt=F(s)$ in (1) we get

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}.F(s)ds$$

In (2) $F(s)$ is called the Fourier transform of $f(x)$.

In (3) $f(x)$ is called the inverse Fourier transform of $F(s)$.

For reasons of symmetry, we multiply both $F(x)$ and $F(s)$ by $\sqrt{\frac{1}{2\pi}}$ instead of havi

r $\frac{1}{2\pi}$ in only one function. Thus, we obtain the definition of Fourier transforms as

$$\boxed{\begin{aligned} F(s)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist}dt \\ f(x)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds \end{aligned}}$$

RIER SINE AND COSINE TRANSFORMS

m equation (7) of Article 14.4 we know that

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin sx\,ds\int_0^\infty f(t)\sin st\,dt \qquad (s=u)$$

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \sin sx\,ds\,F(s) \qquad \text{... (1)}$$

$$F(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin st\,dt \qquad \text{... (2)}$$

equation (2), $F(s)$ is called **Fourier sine transform** of $f(x)$.
equation (1), $f(x)$ is called the **Inverse Fourier sine transform** of $F(s)$
m equation (8) of Article 14.4, we have

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \cos sx\,F(s)\,dx \qquad \text{... (3)}$$

$$F(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos st\,dt \qquad \text{... (4}$$

equation (4), $F(s)$ is called **Fourier cosine transform** of $F(x)$.
equation (3), $f(x)$ is called the **inverse Fourier cosine transform** of $F(s)$.

mple 4. *Find the Fourier transform of*

$$f(x)=\begin{cases} 1 & \text{for } |x|<a \\ 0 & \text{for } |x|>a \end{cases}$$

ution. The Fourier transform of a function $f(x)$ is given by

$$F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x).e^{isx}dx$$

bstituting the value of $f(x)$, we get

$$F(s)=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a} 1\,e^{isx}dx=\left[\frac{e^{isx}}{is}\right]_{-a}^{a}=\frac{1}{\sqrt{2\pi}}\frac{1}{(is)}\left[e^{ias}-e^{-ias}\right]$$

$$=\frac{1}{\sqrt{2\pi}}\frac{2}{s}.\frac{e^{ias}-e^{-ias}}{2i}=\frac{1}{\sqrt{2\pi}}\frac{2\sin sa}{s}=\sqrt{\frac{2}{\pi}}\frac{\sin sa}{s}$$

mple 5. *Find the Fourier transform of*

$$f(x)=\begin{cases} 1-x^2 & \text{if } |x|\le 1, \\ 0 & \text{if } |x|>1. \end{cases}$$

ution.

$$f(x)=\begin{cases} 1-x^2 & -1<x<1 \\ 0 & |x|>1 \end{cases}$$

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The Fourier transform of a function $f(x)$ is given by

$$F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x).e^{isx}.dx$$

Substituting the values of $f(x)$ in (1), we get

$$F(s)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)e^{isx}dx$$

Integrating by parts, we get $\left[[uv]_1=uv_1-u'v_2+u''u_3 ...\right]$

$$F(s)=\frac{1}{\sqrt{2\pi}}\left[(1-x^2)\frac{e^{isx}}{is}-(-2x)\frac{e^{isx}}{(is)^2}+(-2)\frac{e^{isx}}{(is)^3}\right]_{-1}^{1}$$

$$=\frac{1}{\sqrt{2\pi}}\left[-2\frac{e^{is}}{s^2}+2\frac{e^{is}}{is^3}-2\frac{e^{is}}{s^2}-\frac{e^{-is}}{is^3}\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[-\frac{2}{s^2}(e^{is}+e^{-is})+\frac{2}{is^3}(e^{is}-e^{-is})\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[-\frac{2}{s^2}(2\cos s)+\frac{2}{is^3}(2i\sin s)\right]$$

$$=\frac{1}{\sqrt{2\pi}}\frac{4}{s^3}[-s\cos s+\sin s]$$

Example 6. *Find the Fourier sine and cosine transforms of* $f(x)=e^{-ax}$.

Solution. The Fourier sine transform of $f(x)$ is given by

$$F(s)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\,dx$$

Putting the value of $f(x)$ we get

$$F(s)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\sin sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\frac{e^{-ax}}{a^2+s^2}[-a\sin sx-s\cos sx]_0^{\infty}$$

$$=-\sqrt{\frac{2}{\pi}}\frac{e^{-ax}}{a^2+s^2}[a\sin sx+s\cos sx]_0^{\infty}$$

$$=\sqrt{\frac{2}{\pi}}\left[-0+\frac{1}{a^2+s^2}\times s\right]=\sqrt{\frac{2}{\pi}}\frac{s}{a^2+s^2}$$

The Fourier cosine transform is

$$F(s)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}\{-a\cos sx+s\sin sx\}\right]_0^{\infty}=\sqrt{\frac{2}{\pi}}\left[0+\frac{a}{a^2+s^2}\right]$$

$$=\sqrt{\frac{2}{\pi}}\frac{a}{a^2+s^2}$$ **Ans.**

7. *Find Fourier sine transform of* $\frac{1}{x}$.

$$F_s\left(\frac{1}{x}\right)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\sin sx}{x}dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\sin\theta}{\frac{\theta}{s}}\frac{d\theta}{s}$$

Putting $s x=\theta$ so that $s\,dx=d\theta$

$$=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\sin\theta}{\theta}d\theta=\sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2}\right)=\sqrt{\frac{\pi}{2}}$$ **Ans.**

8. *Find the Fourier cosine transform of*
$f(x)=e^{-2x}+4e^{-3x}$

The Fourier cosine transform of $f(x)$ is given by

$$F(s)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\,dx$$

the value of $f(x)$ we get

$$F(s)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}(e^{-2x}+4e^{-3x})\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^{\infty}e^{-2x}\cos sx\,dx+\sqrt{\frac{2}{\pi}}\int_0^{\infty}4e^{-3x}\cos sx\,dx$$

$$\left(\because \int e^{-ax}\cdot\cos bx\,dx=\frac{e^{-ax}}{a^2+b^2}\{b\sin bx-a\cos b$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{2}{s^2+4}+4\cdot\frac{3}{s^2+9}\right]=2\sqrt{\frac{2}{\pi}}\left[\frac{1}{s^2+4}+\frac{6}{s^2+9}\right]$$

9. *Find the Fourier sine transform of*

$$f(x)=\frac{e^{-ax}}{x}.$$

The sine transform of the function $f(x)$ is given by

$$F(s)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\ \sin sx\,dx$$

the value of $f(x)$, we get

$$F(s)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{e^{-ax}}{x}\sin sx\,dx$$

Differentiating both sides w.r.t. 's' we get

$$\frac{d}{ds}[F(s)]=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{e^{-ax}}{x}(x\cos sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos ax\,dx = \sqrt{\frac{2}{\pi}}\,\frac{a}{s^2+a^2}$$

Integrating w.r.t. 's' we get

$$F(s) = \sqrt{\frac{2}{\pi}}\int \frac{a}{s^2+a^2}\,ds = \sqrt{\frac{2}{\pi}}\tan^{-1}\frac{s}{a}+c$$

For $s = 0$, $\quad F(s) = 0$

Putting these values in the above equation we get

$$0 = 0 + c \quad \text{or} \quad c = 0 \qquad \therefore \quad F(s) = \sqrt{\frac{2}{\pi}}\tan^{-1}\frac{s}{a}$$

Example 10. *Find the Fourier sine and cosine transforms of*

$$f(x) = 1, \quad 0 < x < a$$
$$= 0, \quad x > a$$

Solution. Fourier sine Transform

$$F(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx = \sqrt{\frac{2}{\pi}}\int_0^a 1\sin sx\,dx$$

$$F(s) = \sqrt{\frac{2}{\pi}}\left[-\frac{\cos sx}{s}\right]_0^a = \sqrt{\frac{2}{\pi}}\left[-\frac{\cos as}{s}+\frac{1}{s}\right]$$

Fourier cosine transform

$$F(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx = \sqrt{\frac{2}{\pi}}\int_0^a 1\cos sx\,dx = \sqrt{\frac{2}{\pi}}\left[\frac{\sin sx}{s}\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\frac{\sin as}{s}$$

Example 11. *Find the Fourier cosine transform of*

$$f(x) = x \quad \text{for} \quad 0 < x < \frac{1}{2}$$
$$= 1 - x \quad \text{for} \quad \frac{1}{2} < x < 1$$
$$= 0 \quad \text{for} \quad x > 1.$$

Write the inverse transform.

Solution. Fourier cosine transform

$$F(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{1/2} x\cos sx\,dx + \sqrt{\frac{2}{\pi}}\int_{1/2}^1 (1-x)\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[x\frac{\sin sx}{s} - \left(\frac{-\cos sx}{s^2}\right)\right]_0^{1/2} + \sqrt{\frac{2}{\pi}}\left[(1-x)\frac{\sin sx}{s} - (-1)\ldots\right.$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{1}{2}\frac{\sin s/2}{s} + \frac{\cos s/2}{s^2} - \frac{1}{s^2}\right] + \sqrt{\frac{2}{\pi}}\left[-\frac{\cos s}{s^2} - \frac{1}{2}\frac{\sin s/2}{s}\ldots\right.$$

$$= \sqrt{\frac{2}{\pi}}\left[-\frac{\cos s}{s^2} + \frac{2\cos s/2}{s^2} - \frac{1}{s^2}\right]$$

Example 12. *Find the Fourier transform of the function*

$$f(x) = \begin{cases} 1+\dfrac{x}{a} & (-a < x < 0) \\ 1-\dfrac{x}{a} & 0 < x < a \\ 0 & \text{otherwise} \end{cases}$$

(*U.P., III Semester, Summer 2002*)

Solution. Fourier transform of $f(x)$ is given by

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cdot e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^0\left(1+\frac{x}{a}\right)e^{isx}\,dx + \frac{1}{\sqrt{2\pi}}\int_0^a\left(1-\frac{x}{a}\right)e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left(1+\frac{x}{a}\right)\times\frac{e^{isx}}{is} - \left(\frac{1}{a}\right)\frac{e^{isx}}{-s^2}\right]_{-a}^0 + \frac{1}{\sqrt{2\pi}}\left[\left(1-\frac{x}{a}\right)\frac{e^{isx}}{is} - \left(-\frac{1}{a}\right)\frac{e^{isx}}{-s^2}\right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{is} + \frac{1}{a}\frac{1}{s^2} + \frac{1}{a}\frac{e^{-isa}}{-s^2}\right] + \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a}\cdot\frac{e^{isa}}{-s^2} - \frac{1}{is} + \frac{1}{a}\frac{1}{s^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2}{as^2} + \frac{1}{-as^2}\left(e^{isa}+e^{isa}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2}{as^2} - \frac{2}{as^2}\cos sa\right] = \frac{1}{\sqrt{2\pi}}\,\frac{2}{as^2}\,[1-\cos as]$$

$$= \frac{2}{\sqrt{2\pi}\,as^2}\,2\sin^2\frac{as}{2} = \frac{2\sqrt{2}\,\sin^2\frac{as}{2}}{\sqrt{\pi}\,as^2}$$

Ans.

Example 13. *Find Fourier sine and cosine transform of (a)* x^{n-1}. *(b)* $\dfrac{1}{\sqrt{x}}$.

Solution. (*a*) $F_s(x^{n-1}) = \sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^\infty \sin sx\cdot x^{n-1}\,dx =$

$$F_c(x^{n-1}) = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos sx\cdot x^{n-1}\,dx$$

$$F_c(x^{n-1}) + F_s(x^{n-1}) = \frac{2}{\sqrt{\pi}}\int_0^\infty (\cos sx + i\sin sx)\,x^{n-1}\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{isx}x^{n-1}\,dx \qquad \left[\text{Putting } isx = -t,\quad x = -\frac{t}{is},\quad dx = -\frac{dt}{is}\right.$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-t}\left(-\frac{t}{is}\right)^{n-1}\left(-\frac{dt}{is}\right)$$

$\ldots \frac{1}{3^4} + \frac{1}{5^4} + \ldots = \frac{\pi^4}{96} \quad \ldots \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots = \frac{\pi^4}{90}$

FOURIER SERIES IN COMPLEX FORM

Fourier series of a function $f(x)$ of period $2l$ is

$$f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{l} + a_2 \cos\frac{2\pi x}{l} + \ldots\ldots + a_n \cos\frac{n\pi x}{l} + \ldots\ldots$$
$$+ b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + \ldots\ldots + b_n \sin\frac{n\pi x}{l} + \ldots\ldots \quad \ldots (1)$$

We know that $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

Putting the values of $\cos x$ and $\sin x$ in (1), we get

$$f(x) = \frac{a_0}{2} + a_1 \frac{e^{\frac{i\pi x}{l}} + e^{\frac{-i\pi x}{l}}}{2} + a_2 \frac{e^{\frac{2i\pi x}{l}} + e^{\frac{-2i\pi x}{l}}}{2} + \ldots + b_1 \frac{e^{\frac{i\pi x}{l}} - e^{\frac{-i\pi x}{l}}}{2i} + b_2 \frac{e^{\frac{2i\pi x}{l}} - e^{\frac{-2i\pi x}{l}}}{2i} + \ldots\ldots$$

$$= \frac{a_0}{2} + (a_1 - ib_1) e^{\frac{i\pi x}{l}} + (a_2 - ib_2) e^{\frac{2i\pi x}{l}} + \ldots + (a_1 + ib_1) e^{\frac{-i\pi x}{l}} + (a_2 + ib_2) e^{\frac{-2i\pi x}{l}} + \ldots\ldots$$

$$= c_0 + c_1 e^{\frac{i\pi x}{l}} + c_2 e^{\frac{2i\pi x}{l}} + \ldots\ldots + c_{-1} e^{\frac{-i\pi x}{l}} + c_{-2} e^{\frac{2i\pi x}{l}} + \ldots\ldots$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{\frac{in\pi x}{l}} + \sum_{n=1}^{\infty} c_{-n} e^{\frac{-in\pi x}{l}}$$

where $c_n = \frac{1}{2}(a_n - ib_n)$, $c_{-n} = \frac{1}{2}(a_n + ib_n)$

$$c_0 = \frac{a_0}{2} = \frac{1}{2}\frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$c_n = \frac{1}{2}\left[\frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx - \frac{i}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx\right] \Rightarrow c_n = \frac{1}{2}\frac{1}{l}\int_0^{2l} f(x)\left\{\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right\}dx$$

$$c_n = \frac{1}{2l}\int_0^{2l} f(x)\, e^{\frac{-in\pi x}{l}}\,dx$$

$$c_{-n} = \frac{1}{2l}\int_0^{2l} f(x)\, e^{\frac{in\pi x}{l}}\,dx$$

***Example 23.* Obtain the complex form of the Fourier series of the function**

$$f(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ 1 & 0 \le x \le \pi \end{cases}$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\cdot e^{-inx}\,dx + \int_{0}^{\pi} 1\cdot e^{-inx}\,dx\right] = \frac{1}{2\pi}\int_{0}^{\pi} e^{-inx}\,dx = \frac{1}{2\pi}\left[\frac{e^{-inx}}{-in}\right.$$

$$= -\frac{1}{2n\pi i}\left[e^{-in\pi} - 1\right] = -\frac{1}{2n\pi i}[\cos n\pi - 1] = -\frac{1}{2n\pi i}\left[(-1)^n - 1\right]$$

$$= \begin{cases} \dfrac{1}{in\pi}, & n \text{ is odd} \\ 0 \quad, & n \text{ is even} \end{cases}$$

$$f(x) = \frac{1}{2} + \frac{1}{i\pi}\left[\frac{e^{ix}}{1} + \frac{e^{3ix}}{3} + \frac{e^{5ix}}{5} + \ldots\ldots\right] + \frac{1}{i\pi}\left[\frac{e^{-ix}}{-1} + \frac{e^{-3ix}}{-3} + \frac{e^{-5ix}}{-5} + \ldots\right.$$

$$= \frac{1}{2} - \frac{1}{i\pi}\left[\left(e^{ix} - e^{-ix}\right) + \frac{1}{3}\left(e^{3ix} - e^{-3ix}\right) + \frac{1}{5}\left(e^{5ix} - e^{-5ix}\right) + \ldots\ldots\right]$$

Exercise 12.6

Find the complex form of the Fourier series of

1. $f(x) = e^{-x}, \; -1 \le x \le 1.$ **Ans.** $\sum_{n=-\infty}^{\infty} \frac{(-1)^n (1 - in\pi)}{1 + n^2\pi^2}$

2. $f(x) = e^{ax}, \; -l < x < l$ **Ans.** $\frac{2}{\pi} - \frac{2}{\pi}\left[\frac{e^{2it} + e^{-2it}}{1\cdot 3} + \frac{e^{4it} + e^{-4it}}{3\cdot 5} + \ldots\right.$

3. $f(x) = \cos ax, \; -\pi < x < \pi$ **Ans.** $\frac{a}{\pi}\sin a\pi \ldots$

Properties of Eigenvectors:

i) The eigenvector v of a matrix A is not unique.

ii) If $\lambda_1, \lambda_2, \dots, \lambda_n$ are distinct eigenvalues of $n \times n$ matrix then corresponding eigenvectors $v_1, v_2, \dots, v_n$ form a linearly independent set.

iii) If two or more eigenvalues are equal it may or may not be possible to get linearly independent eigenvectors corresponding to equal roots.

Note: → Two eigenvectors v_1 and v_2 are called orthogonal vectors if $v_1^T v_2 = 0$.

→ Eigenvectors of a symmetric matrix correspondi to different eigenvalues are orthogonal.

** Normalized form of vectors: To find normal form of $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, we divide each element by $\sqrt{a^2+b^2}$

For example, normalized form of $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ is $\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$

Note: "Eigen" is a German word meaning "Proper" or "own".

Eigenvectors are also called proper vectors, characteristic vectors or latent vectors and eigenvalues are called proper values, characteristic values or latent roots.

⊛⊛ To <u>find the eigenvalue of A</u>, we write

$Av = \lambda v$ as $Av = \lambda I v \Rightarrow (A - \lambda I)v = 0$.

where $A - \lambda I$ is called characteristic matrix of A

⊛ <u>Some important properties of eigenvalues:</u>

1. Any square matrix A and its transpose A^T have same eigenvalues.
2. The product of the eigenvalues of a matrix A is equal to the determinant of A.
3. If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of A, then the eigenvalues of
   i) KA are $K\lambda_1, K\lambda_2, \cdots, K\lambda_n$
   ii) A^m are $\lambda_1^m, \lambda_2^m, \cdots, \lambda_n^m$
   iii) A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \cdots, \frac{1}{\lambda_n}$
4. The sum of the eigenvalues of a matrix is equal to the trace (sum of all diagonal elements) of the matrix.

(c) Characteristic Equation:

The eqn $|A-\lambda I|=0$ is called the characteristic eqn of the matrix A.

e.g. $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$.

(d) Characteristic Roots or Eigenvalues:

The roots of characteristic eqn $|A-\lambda I|=0$ are called characteristic roots of A.

e.g. $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$\Rightarrow (\lambda-1)(\lambda-1)(\lambda-5)=0$

$\Rightarrow \lambda = 1, 1, 5$

$\therefore$ Characteristic roots are 1, 1, 5.

(*) Defn of Eigenvalues and Eigenvectors:

If A is an $n \times n$ matrix, then a non-zero vector v in R^n is called an eigenvector of A if Av is a scalar multiple of v, that is, $Av = \lambda v$ for some scalar λ. The scalar λ is called an eigenvalue of A and v is said to be an eigenvector of A corresponding to λ.

$\rightarrow$ The set of all vectors is called the eigenspace of λ.

③

$\therefore$ Let $f(x) = e^{-x}$ so that $F_c(s) = \frac{1}{1+s^2}$ → show

By Parsaval's identity for cosine transform $F_c(s)$

$\int f(x)\cos sx\, dx$

$=$

$$\frac{2}{\pi}\int_0^{\infty}[F_c(s)]^2 ds = \int_0^{\infty}[f(x)]^2 dx$$

$$\Rightarrow \frac{2}{\pi}\int_0^{\infty}\left[\frac{1}{1+s^2}\right]^2 ds = \int_0^{\infty}[e^{-x}]^2 dx$$

$$= \int_0^{\infty} e^{-2x} dx = \left[\frac{e^{-2x}}{-2}\right]_0^{\infty}$$

$$\Rightarrow \int_0^{\infty}\left[\frac{1}{1+s^2}\right]^2 ds = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

$$\therefore \int_0^{\infty}\frac{1}{(1+x^2)^2}dx = \frac{\pi}{4}.$$ [Proved]

Do. ex-18, 19

* ⓧ Prob. Find the Fourier transform of

$$f(x) = \begin{cases} 1-|x| & \text{if } |x|<1 \\ 0 & \text{if } |x|>1 \end{cases}$$

and hence find the value $\int_0^{\infty}\frac{\sin t}{t}dt$.

Solⁿ: $$F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(1-|x|)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(1-|x|)(\cos sx + i\sin sx)\,dx$$

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Additional Sheet

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② ~~Find~~ the Fourier sine transform of $f(x) = \frac{e^{-ax}}{x}$.

Solⁿ: The sine transform of the function $f(x)$ is given by

$$F(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

Substituting the value of $f(x)$, we get

$$F(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx = I \text{ (say)}$$

Then $I = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \cdot \sin sx \, dx \quad \cdots\cdots (1)$

Differentiating ① w.r.t. s we get

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \cdot x \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \cancel{\sqrt{\frac{2}{\pi}} \mathcal{L}\{\cos sx\}} \qquad \left[\because \cancel{\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt} \right.$$

$$\left. \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}, \; s>0 \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$$

Now integrating,

$$I = \int \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2} \, ds$$

$$= \sqrt{\frac{2}{\pi}} \cdot a \int \frac{ds}{a^2+s^2} = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a}\right) + c$$

For $s=0$, this gives ; $I = 0 + c \Rightarrow 0 = 0 + c \Rightarrow c = 0$

$I = c$

$$\therefore I = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a}\right).$$

(Ans.)

Find the Fourier cosine transform of e^{-x}, $x \geq 0$. and verify the inverse cosine transform of e^{-x}.

Sol^n: By def^n of Fourier cosine transform of $f(x)$ for $0 < x < \infty$, we have

$$f_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(x) \cos sx \, dx \quad \ldots\ldots (1)$$

Now, $f_c(s) = \int_0^\infty e^{-x} \cos sx \, dx$ since $F(x) = e^{-x}$

$$= \left[\frac{e^{-x} \sin sx}{s}\right]_0^\infty + \int_0^\infty \frac{e^{-x} \sin sx}{s} dx$$

$$= 0 + \frac{1}{s}\left\{\left[\frac{-e^{-x} \cos sx}{s}\right]_0^\infty - \int_0^\infty \frac{e^{-x} \cos sx}{s} dx\right]\right\}$$

$$= 0 + \frac{1}{s^2} - \frac{1}{s^2} f_c(s)$$

$$\Rightarrow f_c(s) + \frac{1}{s^2} f_c(s) = \frac{1}{s^2}$$

$$\Rightarrow \left\{\frac{s^2+1}{s^2}\right\} f_c(s) = \frac{1}{s^2}$$

$$\therefore f_c(n) = \sqrt{\frac{2}{\pi}} \frac{1}{s^2+1} = \sqrt{\frac{2}{\pi}} \frac{1}{s^2+1}$$

Hence the Fourier cosine transform of e^{-x} is $\sqrt{\frac{2}{\pi}} \frac{1}{s^2+1}$.

def^n of the inverse Fourier cosine transform,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_c(s) \cos sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{1}{1+s^2} \cos sx \, ds$$

$$e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{\cos sx}{1+s^2} ds \quad \cdots\cdots (1)$$

~~From the Fourier integral of an even function, we have,~~

$$f(x) = \frac{2}{\pi} \int_0^\infty du \int_0^\infty f(x) \cos s\cancel{t} \cos sx \, dx$$

$$\Rightarrow f(x) = \frac{2}{\pi} \int_0^\infty ds \int_0^\infty f(x) \cos st \cos sx \, dx \quad \cdots (2)$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \cos sx \, ds \cdot \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos st \, dx$$

Taking $f(x) = e^{-x}$ in (2) we get

$$e^{-x} = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos sx \, ds \cdot \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-t} \cos s \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+s^2} \cos sx \, dn$$

$$= \frac{2}{\pi} \int_0^\infty \frac{1}{1+s^2} \cos sx \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\cos sx}{1+n^2} ds$$

$$\therefore \int_0^\infty \frac{\cos sx}{1+s^2} ds = \frac{\pi}{2} e^{-x}$$

Combining (1) & (3)

$f(x) = \frac{2}{\pi} \cdot \frac{\pi}{2} e^{-x} = e^{-x}$ which is the reqd. inverse F. Cosine transf. of $F_c(s) = \frac{1}{1+s^2}$

$$= \frac{32}{n^2\pi^2}(\cos n\pi - 1)$$ which is the finite Fou[rier]

cosine transform of $F(x) = 2x$.

If $n=0$, $f_c(n) = f_c(0) = \int_0^4 2x \cdot \cos 0 \, dx$

$$= \int_0^4 2x\,dx = 2\left[\frac{x^2}{2}\right]_0^4$$

$$= 2\left[\frac{16}{2} - 0\right] = 16$$

Prob. Find the Fourier transform of $f(x)$ defined by

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

and hence evaluate (i) $\int_{-\infty}^{\infty} \frac{\sin ua \cos ux}{u}\,du$ (ii) $\int_0^{\infty} \frac{\sin u}{u}\,du$.

Solⁿ: By defⁿ of Fourier transform

$$F(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{iut}\,dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{iut}\,dt$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{iut}}{iu}\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{iu}\left[e^{iua} - e^{-iua}\right]$$

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$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2}{u} \cdot \frac{e^{iua} - e^{-iua}}{2i}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2}{u} \operatorname{Sin} ua$$

$$= \sqrt{\frac{2}{\pi}} \frac{\operatorname{Sin} ua}{u}$$

By the def^n of inverse Fourier transform, we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du \qquad \therefore \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du = \begin{cases} 1, & |x| \\ 0, & |x \end{cases}$$

....

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\operatorname{Sin} ua}{u} \cdot e^{-iux}\, du$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Sin} ua}{u} \left[\operatorname{Cos} ux - i \operatorname{Sin} ux \right] du$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Sin} ua \operatorname{Cos} ux}{u}\, du - \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Sin} ua \operatorname{Sin} ux}{u}\, du \quad \ldots (2)$$

The integrand in the 2nd integral of (2) on R.H.S. is odd so this integral is zero.

ince combining ① and ② we have,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos ux \sin ua}{u}\, du = \begin{cases} 1, & |x|<a \\ 0, & |x|>a \end{cases}$$

or, $$\int_{-\infty}^{\infty} \frac{\sin ua \cos ux}{u}\, du = \begin{cases} \pi, & |x|<a \\ 0, & |x|>a \end{cases}$$

(ii) If $x=0$ and $a=1$ then from the above r

we have,

$$\int^{\infty}$$