Discrete Mathematics
CSE1203

* Book: | Lecture-1 | Date- 3/01/2019

1. Rosen, K. H., Discrete Mathematics and its Applications, McGraw-Hill, 6th Edition, 2007. [7th Edition, 2011, 8th Edi 2018] [Tata McGraw-Hill, 6th Edition, 2008]

2. Johnsonbaugh, R., Discrete Mathematics, Pearson, 6th Edition, 2005.

* Chapter 1. Logic and Proofs [Mathematical Logic / Formal Logic]

Topic 1.1. Propositional Logic [Propositional calculus]

A] Proposition:

Definition:

- Declarative sentence/statement, either true or false, but not both
- Assigned one of the two distinct values, True an False or T and F.
- Denoted by p, q, r, ...

Examples:

* The integer 20 is a prime
* Dhaka is the capital of Bangladesh.

Not Examples:

* What is your name?
* Solve the problem.

B] Negation of a proposition:

* Definition:

* Let p be a proposition
* Negation of p is denoted by $\neg p$, and read 'not p'.
* $\neg p$ is True if p is false, and $\neg p$ is False when p is True.
* 'Not' ($\neg$) is a unary operator.

Examples:

p: '9 is divisible by 3'; $\neg p$: 'It is not the case that 9 is divisible by 3' or '9 is not divisible by 3'.

* Truth Table:

p	$\neg p$
F	T
T	F

* $\neg(\neg p)$ is logically equivalent to ($\equiv$) to p.

Proof using a Truth table:

p	$\neg p$	$\neg(\neg p)$
F	T	F
T	F	T

$\therefore \neg p(\neg p) \equiv p$

c) Compound propositions involving common Binary Operators or connectives.

i) conjunction

Definition:

* p, q - propositions
* conjunction of p and q is denoted by $p \wedge q$ and read 'p and q'.
* Proposition which is True only when both p and q are True.

Examples:

* p : 'The boy prepares his lessons regularly'.
* q : 'The boy helps his parents everyday'.
* $p \wedge q$: 'The boy prepares his lessons regularly and [/although / but] helps his parents everyday.'

Truth table:

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

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ii) Disjunction:

Definition:

* p, q - propositions
* Disjunction of p and q is denoted by $p \vee q$ and read 'p or q'.
* Proposition which is False only when both p and q are False.

Examples:

* p: 'You can take physics this semester.'
* q: 'You can take chemistry this semester.'
* $p \vee q$: 'You can take Physics or chemistry [any one or both of Physics and chemistry] this semester.'

Truth table:

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

iii) Exclusive disjunction:

Definition:

* p, q - propositions
* Exclusive Disjunction of p and q is denoted by $p \oplus q$ and read 'p or exclusive q'.
* Proposition which is True only when exactly one of p and q, ~~are~~ is True.

Examples:

* p: 'You can take tea'.
* q: 'You can take coffee'.
* $p \oplus q$: 'You can take tea or coffee, but not both.'

Truth table:

p	q	$p \oplus q$
F	F	F
F	T	T
T	F	T
T	T	F

* Prove using truth tables that $p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$

p	q	$p \oplus q$	$p \vee q$	$p \wedge q$	$\neg(p \wedge q)$	$(p \vee q) \wedge \neg(p \wedge q)$
F	F	F	F	F	T	F
F	T	T	T	F	T	T
T	F	T	T	F	T	T
T	T	F	T	T	F	F

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* Lecture-2 | Date- 6/01/2019

D) Conditional Statements or Implications:

Definition:

* p, q - propositions

* Compound proposition, denoted by $p \to q$, and most commonly read 'p implies q', which is False only when p is True, but q is False.

Examples:

If you fall ill, then you miss the examination.

p: 'You fall ill.'

q: 'You miss the examination.'

$p \to q$, [p implies q]

* p - the antecedent or the premise of the conditional statement

q - the conclusion or the consequence of the conditional statement.

* Other common expressions.

① If p then q. ② q, whenever p.

③ q, if p. ④ q follows from p.

* Truth table:

p	q	$p \to q$	$\neg p$	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$	$\neg p \vee q$
F	F	T	T	T	F	T	T
F	T	T	T	F	F	T	T
T	F	F	F	T	T	F	F
T	T	T	F	F	F	T	T

Complete the table and be assured that $p \to q \equiv \neg(p \wedge \neg q)$,

and $p \to q \equiv \neg p \vee q$.

* If I am elected, then I do all the good things.

E) <u>Biconditional Statements or Bi-implications:</u>

Definition:

* p, q - propositions
* compound proposition, denoted by $p \leftrightarrow q$, and most commonly read 'p if and only if q', which is <u>True only when</u> p and q have the <u>same truth values</u>

<u>Examples:</u>

The angles of a triangle are equal to each other <u>if and only if</u> the triangle is equilateral.

p: 'The angles of a triangle are equal to each other.'

q: 'The triangle is equilateral.'

$p \leftrightarrow q$ [p if and only if q]

Other common expressions:

p iff q.

q, whenever p, and p, whenever q.

Truth Table:

p	q	$p \leftrightarrow q$	$\neg p$	$\neg q$	$p \to q$	$q \to p$	$\neg p \vee q$	$\neg q \vee p$	$(p \to q) \wedge (q \to p)$	$(\neg p \vee q) \wedge (\neg q \vee p)$
F	F	T	T	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T	F	F	F
T	F	F	F	T	F	T	F	T	F	F
T	T	T	F	F	T	T	T	T	T	T

complete the table and be assured that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

and $p \leftrightarrow q \equiv (\neg p \vee q) \wedge (\neg q \vee p)$

* H.W.

p	q	$p \leftrightarrow q$	$\neg p$	$\neg q$	$p \to q$	$q \to p$	$\neg p \vee q$	$\neg q \vee p$	$(p \to q) \wedge (q \to p)$	$(\neg p \vee q) \wedge (\neg q \vee p)$
T	T	T	F	F	T	T	T	T	T	T
T	F	F	F	T	F	T	F	T	F	F
F	T	F	T	F	T	F	T	F	F	F
F	F	T	T	T	T	T	T	T	T	T

F] Special Propositional conventions: [Lecture-3] Date- 8/1/2019

③ Converse, Inverse and contrapositive of a Conditional statement

p	q	$p \to q$	$q \to p$	$\neg p$	$\neg q$	$\neg p \to \neg q$	$\neg q \to \neg p$
F	F	T	T	T	T	T	T
F	T	T	F	T	F	F	T
T	F	F	T	F	T	T	F
T	T	T	T	F	F	T	T

complete the table and be assured.

- contrapositive is equivalent to the conditional ($\neg q \to \neg p \equiv p \to q$)
- converse ($q \to p$) is not
- inverse ($\neg p \to \neg q$) is also not

But the converse and inverse of a conditional are equivalent to each other ($q \to p \equiv \neg p \to \neg q$)

* Exercise on ["You are ill" → "You miss the examination"].
p q

$p \to q$: If you are ill, then you miss the examination.

contrapositive ($\neg q \to \neg p$): "If you ~~are~~ do not miss the exam, then you are not ill."

converse ($q \to p$): If You miss the examination, then you are ill.

inverse ($\neg p \to \neg q$): If you are not ill, then you do not miss the exam.

② <u>Tautology:</u> compound proposition, the truth value of which is always True.

For proof: $p \vee \neg p$, $p \leftrightarrow \neg(\neg p)$, $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

[$p \vee \neg p$]

p	$\neg p$	$\neg(\neg p)$	$p \leftrightarrow \neg(\neg p)$
F	T	F	T
T	F	T	T

[$p \leftrightarrow \neg(\neg p)$]

*

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$
F	F	F	T	T	T	T	T
F	T	T	F	T	F	~~T~~ F	T
T	F	T	F	F	T	~~T~~ F	T
T	T	T	F	F	F	F	T

③ <u>Contradiction:</u> compound proposition that is always False.

For proof: $p \wedge \neg p$, $p \leftrightarrow \neg p$, $(p \to q) \leftrightarrow (p \wedge \neg q)$

p	$\neg p$	$p \wedge \neg p$
F	T	F
T	F	F

$p \wedge \neg p$

p	$\neg p$	$p \leftrightarrow \neg p$
F	T	F
T	F	F

$p \leftrightarrow \neg p$

p	q	$p \to q$	$\neg q$	$p \wedge \neg q$	$(p \to q) \leftrightarrow (p \wedge \neg q)$
F	F	T	T	F	F
F	T	T	F	F	F
T	F	F	T	T	F
T	T	T	F	F	F

④ Contingency:

Neither tautology nor a contradiction.

⑤ Order of precedence of logical operators:

$\neg$; $\wedge$, $\vee$; $\to$, $\leftrightarrow$

⑥ p is logically equivalent to q if $p \leftrightarrow q$ is tautology.

Lecture-4 (10-1-19)

* G| Common Propositional Equivalences:[Important Laws of Propositional Logic]

① Double negation law: $\neg(\neg p) \equiv p$

② Idempotent laws: $p \wedge p \equiv p$, $p \vee p \equiv p$

③ Commutative laws: $p \wedge q \equiv q \wedge p$, $p \vee q \equiv q \vee p$

④ Associative laws: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$, $(p \vee q) \vee r \equiv p \vee (q \vee r)$

⑤ Distributive laws: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

⑥ De Morgan's laws: $\neg(p \wedge q) \equiv \neg p \vee \neg q$, $\neg(p \vee q) \equiv \neg p \wedge \neg q$

⑦ Identity laws: $p \wedge T \equiv p$, $p \vee F \equiv p$

⑧ Domination laws: $p \wedge F \equiv F$, $p \vee T \equiv T$

⑨ Complement laws/ Negation laws: $p \wedge \neg p \equiv F$, $p \vee \neg p \equiv T$

⑩ absorption laws: $p \wedge (p \vee q) \equiv p$, $p \vee (p \wedge q) \equiv p$

* Other useful propositional equivalences:

*(11) $p \to q \equiv \neg(p \wedge \neg q) \equiv \neg p \vee q$ [Elimination of conditionals]

*(12) $p \to q \equiv \neg q \to \neg p$ [contrapositive]

(13) $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$

(14) $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$

(15) $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$

(16) $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

*(17) $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$ [Elimination of Biconditionals]

(18) $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

(19) $p \leftrightarrow q \equiv q \leftrightarrow p \equiv \neg p \leftrightarrow \neg q \equiv \neg q \leftrightarrow \neg p$

* Exercise:

(1) Verify the above equivalences using truth tables.

$\leftrightarrow$ corresponding Tautology

* Associative Laws: $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

p	q	r	$q \wedge r$	$p \wedge (q \wedge r)$	$p \wedge q$	$(p \wedge q) \wedge r$	$p \wedge (q \wedge r) \leftrightarrow (p \wedge q) \wedge r$
F	F	F	F	F	F	F	T
F	F	T	F	F	F	F	T
F	T	F	F	F	F	F	T
F	T	T	T	~~T~~ F	F	F	T
T	F	F	F	~~T~~ F	F	F	T
T	F	T	F	~~T~~ F	F	F	T
T	T	F	F	~~T~~ F	T	F	T
T	T	T	T	T	T	T	T

2) Verify using only propositional equivalences:

(i) $(p \wedge \neg q) \vee (q \wedge p) \equiv p$

(ii) $\neg(\neg p \vee q) \wedge (\neg p \vee (q \wedge \neg p)) \equiv F$

(iii) $(\neg p \wedge (\neg q \vee \neg p)) \vee p \equiv T$

(iv) $(\neg p \vee \neg q) \wedge q \wedge \neg(\neg p) \equiv q \wedge ((\neg q \vee p) \wedge \neg q)$

(v) $(q \wedge r) \vee \neg p \leftrightarrow (\neg p \vee q) \wedge (\neg p \vee r) \equiv T$

Ans:

(iv) $\Rightarrow$ L.H.S. $= (\neg p \vee \neg q) \wedge q \wedge \neg(\neg p)$

$= (\neg p \vee \neg q) \wedge q \wedge p$ [Double Negation]

$= (\neg p \vee \neg q) \wedge p \wedge q$ [commutative law]

$= \neg(p \wedge q) \wedge p \wedge q$ [De Morgan]

$= \neg(p \wedge q) \wedge (p \wedge q)$ [commutative law]

$= \neg x \wedge x$ Taking $x = p \wedge q$

$= F$ [complement law]

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Lecture-5 (13-1-19)

Topic 1.2: Predicate Logic / Predicate calculus

A] Predicates:

Definition:

- declarative sentence or statement with one or more variables
- turns into a proposition after assigning values to the variables
- set of possible values of variables: Universe of discourse (uod) or Domain
- assigning values: binding the variables

Examples:: [Domain/uod: set of integers]

- $x-5>8$: $P_1(x)$, say; 1-place predicate; $P_1(2)$ = False.
- $x+y=4$: $P_2(x,y)$, say; 2-place predicate; $P_2(1,3)$ = True.
- $x+y=z$; $P_3(x,y,z)$, say; 3-place predicate; $P_3(1,2,3)$ = True.

✓ $P_3(2,2,5)$ returns the value of the propositional function P_3 at $(2,2,5)$, which is False.

✓ Two possible outcome of a propositional function: True, False.

✓ Two possible predicates in the domain 'set of all people': Father(x,y): 'x is the father of y.' Male(x): 'x is a male.'

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B) Quantified Predicates:

- When the values of the variables in a predicate are specified in some particular way, then the predicate is said to be quantified.
- Fully quantified predicates are propositions.
- Two quantifiers are widely used: Universal Quantifier, Existential Quantifier

* Universal Quantifier:

- Denoted by $\forall$, and read 'for all' / 'for each'.
- Examples: [Domain/uod: set of integers]

$P_4(x): x^2 \geq 0$ $\qquad$ $P_5(x): x+2 > 10$

$\forall x P_4(x)$: True $\qquad$ $\forall x P_5(x)$: False

(x)

- $\forall x P(x) \equiv P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \ldots$; when the elements of the uod can be listed as $x_1, x_2, x_3 \ldots$
- A counterexample is sufficient to show 'falsehood' of a universally quantified predicate, that is, to disprove it.

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* Existential Quantifier:

- Denoted by '∃', and read 'for some', meaning 'there is/exists at least one'

i.e. $\exists x\, P_4(x) = \text{True}$ $\qquad \exists x\, P_5(x) = \text{True}$

- $\exists x\, P(x) \equiv P(x_1) \vee P(x_2) \vee P(x_3) \vee \ldots$, when the elements of the uod can be listed as $x_1, x_2, x_3, \ldots$

- To prove or to show the 'truth' of an existentially -∃ quantified predicate one example is sufficient

* Other quantifier like 'There is exactly one' (uniqueness quantifier), 'There is exactly two', etc. are not commonly used.

Lecture - 6 (15-1-19)

C] Negation of Quantified Predicates:

* $\neg \forall x P(x) \equiv \exists x \neg P(x)$

$[\neg(P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \ldots) \equiv \neg P(x_1) \vee \neg P(x_2) \vee \neg P(x_3) \vee \ldots]$

* $\neg \exists x P(x) \equiv \forall x \neg P(x)$

$[\neg(P(x_1) \vee P(x_2) \vee P(x_3) \vee \ldots) \equiv \neg P(x_1) \wedge \neg P(x_2) \wedge \neg P(x_3) \wedge \ldots]$

* $\neg \forall x \neg P(x) \equiv \exists x P(x)$

* $\neg \exists x \neg P(x) \equiv \forall x P(x)$

D] Expressing Natural Language Sentences with Quantified predicates and Logical connectives

[UoD : Z]

1. There are some odd integers that are not primes.

$\exists x\,(odd(x) \wedge \neg Prime(x))$.

2. The only even prime is two.

$\forall x\,((Even(x) \wedge Prime(x)) \rightarrow (x = 2))$

3. Not for all cases primes are odd.

$\neg \forall x\,(Prime(x) \rightarrow odd(x)) \equiv \exists x \neg(Prime(x) \rightarrow odd(x)) \equiv$
$\exists x \neg(\neg Prime(x) \vee odd(x)) \equiv \exists x\,(Prime(x) \wedge \neg odd(x))$.

④ The sum of two negative numbers is always negative.

$$\forall x\, \forall y\, (((x<0) \wedge (y<0)) \rightarrow ((x+y)<0))$$

[uod: All People]

⑤ Every guest gets at least one gift.

$$\forall x\, (\text{Guest}(x) \rightarrow \exists y\, (\text{Gift}(y) \wedge \text{Gets}(x,y)))$$

লাল স স লাল

() লাল – Scope of x, () স – scope of y; $\exists$ - nested, within the scope of x.

⑥ If x is the father of y and y is the father of z, then x is the grandfather of z.

$$\forall x\, \forall y\, \forall z\, ((\text{Father}(x,y) \wedge \text{Father}(y,z)) \rightarrow \text{Grandfather}(x,z))$$

⑦ Nobody likes everybody.

$$\neg \exists x\, \forall y\, \text{Likes}(x,y) \equiv \forall x\, \neg \forall y\, \text{Likes}(x,y) \equiv$$

$$\forall x\, \exists y\, \neg \text{Likes}(x,y).$$

* Exercise from text book [Section 1.3, Domain: All people]

7. $C(x)$: 'x is a comedian.' $F(x)$: 'x is funny.'

a) $\forall x\, (C(x) \rightarrow F(x))$: Every comedian is funny.

b) $\forall x\, (C(x) \wedge F(x))$: Everyone is a comedian, and is also funny.

c) $\exists x\,(C(x) \to F(x))$: Some people, if they are comedians, are funny.

d) $\exists x\,(C(x) \wedge F(x))$: Some people are comedians, and are also funny.

25. a) No one is perfect. $\Rightarrow$ $\neg \exists x\, \text{Perfect}(x) \equiv \forall x\, \neg \text{Perfect}(x)$

b) Not everyone is perfect. $\Rightarrow$ $\neg \forall x\, \text{Perfect}(x) \equiv \exists x\, \neg \text{Perfect}(x)$

c) All your friends are perfect. $\Rightarrow$ $\forall x\,(\text{YourFriend}(x) \to \text{Perfect}(x))$

d) At least one of your friends is Perfect. $\exists x\,(\text{YourFriend}(x) \wedge \text{Perfect}(x))$

e) Everyone is your friend and is perfect. $\forall x\,(\text{YourFriend}(x) \wedge \text{Perfect}(x))$

f) Everyone is your friend or someone is not perfect.

f) Not everyone is your friend

$\neg \forall x\, \text{YourFriend}(x) \vee \exists y\, \neg \text{Perfect}(y)$.

* There are paradoxes:

Is the declaration 'I am a liar' a proposition?

= (1st quiz में आएगा)

Lecture-7 (17-1-19) / (24-1-19 → बृहस्पतिवार) - 1st Quiz

Topic 1.3 Proofs

A] Basic concepts:

* Theorem: Proposition that can be proved to be true.

* Proof: Argument, that is, sequence of propositions that demonstrates the 'truth' of a theorem.

* A proof may include:

  (i) Axioms or Postulates:
  Definitions/ Primary assumptions/
  Facts known to be True apriori / Explicit facts;

  (ii) Theorems proved earlier;

  (iii) Hypotheses of the theorem, that is, propositions that are assumed to be true;

  (iv) Proposition inferred (derived) from given propositions.

* Inference or Reasoning: Process of constructing a proof using some rules.

* Lemma: Simple theorem used to prove complicated ones.

* Fallacy: Invalid (incorrect) reasoning, commonly due to improper use of implication.

* Corollary: Proposition derived directly from a theorem.

* Conjecture: Proposition with unknown truth value.

Examples:

(i) Goldbach's conjecture: Every even integer n, $n > 2$, can be shown as the sum of two primes.

(ii) Twin prime conjecture: There are infinitely many twin primes (pair of primes that differ by 2), like $(3,5), (5,7), \ldots$

B] Rules of Inference of Propositional Logic:

* $P_1, P_2, P_3, \ldots, P_n \vdash q$ is an inference rule iff $P_1 \wedge P_2 \wedge P_3 \wedge \ldots \wedge P_n \rightarrow q$ is a tautology.

[$\vdash$: infers / proves / derives]

* common inference rules:

1] Addition: $p \vdash p \vee q$

2] Simplification: $p \wedge q \vdash p$

3] Conjunction: $p, q \vdash p \wedge q$

4] Modus pones: $p, p \rightarrow q \vdash q$

5] Modus tollens: $\neg q, p \rightarrow q \vdash \neg p$

6) Hypothetical syllogism: $p \to q,\ q \to r \vdash p \to r$

7) Disjunctive syllogism: $p \vee q,\ \neg p \vdash q$;

$p \vee q,\ \neg q \vdash p$;

8) Resolution: $p \vee q,\ \neg p \vee r \vdash q \vee r$

* Exercise:

1) Prove the tautologies corresponding to the rules of inference described.

(5) $\neg q,\ p \to q \vdash \neg p$

$\Rightarrow (\neg q \wedge (p \to q)) \to \neg p$

p	q	$\neg q$	$p \to q$	$\neg q \wedge (p \to q)$	$\neg p$	$(\neg q \wedge (p \to q)) \to \neg p$
F	F	T	T	T	T	T
F	T	F	T	F	T	T
T	F	T	F	F	F	T
T	T	F	T	F	F	T

2) Derive a fact (proposition) from a given knowledge base (KB, set of facts):

i) r from $\{p \to s,\ q \vee p,\ s \to r,\ \neg q\}$; ii) q from $\{q \leftrightarrow p, p\}$

iii) $p \wedge t$ from $\{r \wedge p,\ s \vee t,\ \neg s\}$; iv) $s \vee u$ from $\{p \to q,\ t \wedge \neg r,\ q \to r,\ p \vee s\}$

(i) KB:

To be derived: r

1. $p \rightarrow s$
2. $q \vee p$
3. ~~$p \rightarrow r$~~ $s \rightarrow r$
4. $\neg q$

From, 1, 3:

5. $p \rightarrow r$ [Hypothetical syllogism]

From, 2, 4:

6. p [Disjunctive syllogism]

From 5, 6:

7. r [Modus pones].

(Derived)

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Lecture-8 (20-1-19)

~~D) Most common combination of inference Rules of propo~~

c) Inference Rules Involving Quantified Predicates:

① Universal Instantiation:

$\forall x\, P(x) \vdash P(s)$, where s is any specified element of the domain.

['Everyone dies.' ⊢ 'Kasim dies']

② Universal Generalization:

$P(a) \vdash \forall x\, P(x)$, where a is an arbitrary element of the domain.

['A girl student gets special stipend.' ⊢ 'Every girl student gets special stipend.']

③ Existential Instantiation:

$\exists x\, P(x) \vdash P(c)$, where c is a certain specified element of the domain.

['Some mammals can fly.' ⊢ 'The bat is a mammal that can fly.']

④ **Existential Generalization:**

$P(c) \vdash \exists x\, P(x)$, where c is a certain specified element of the domain.

['The whale is a mammal that lives in water.'
$\vdash$ 'Some mammals live in water.']

D] Most common combination of inference Rules of Propositional and Predicate Logic:

i) **Universal Modus Ponens:**

$\forall x\,(P(x) \rightarrow Q(x))$
$P(s)$, where s is any specified element of the domain

$\therefore Q(s)$

ii) **Universal Modus tollens:**

$\forall x\,(P(x) \rightarrow Q(x))$
$\neg Q(s)$, where s is any specified element of the domain

$\therefore \neg P(s)$

* **An illustrative example:**

'Every CSE student takes CSE1203. Karim is a CSE student.'

Prove: 'Karim takes CSE1203.'

$\Rightarrow \quad \forall x\,(\text{CSE student}(x) \rightarrow \text{takes}(x, \text{CSE}1203))$

$\text{CSE Student}(\text{karim})$

$\therefore \text{Takes}(\text{karim}, \text{CSE}1203)$

E] Important Deductive Methods of Proving Theorems:

Domain: Z; $n, k \in Z$

1) Direct Proof:

- To prove that $p \rightarrow q$ is True, assume that p is True, and then show that q is True.
- Example: If $\underbrace{n \text{ is even}}_{p}$ then $\underbrace{n^2 \text{ is even}}_{q}$. $[p \rightarrow q]$

Say, $n = 2k$; p is True,

Then, $n^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2$

$\therefore n^2$ is even, $\therefore$ q is True.

2) Indirect or Contrapositive Proof:

- To prove that $p \rightarrow q$ is True, assume that $\neg q$ is True, and then show that $\neg p$ is True.
- Example: If $\underbrace{3n+2 \text{ is odd}}_{p}$ then $\underbrace{n \text{ is odd}}_{q}$. $[p \rightarrow q]$

Say, $\neg q$ is True.

That is, n is even.

Say, $n = 2k$

So, we get -

$$3n+2 = 3 \times 2k + 2 = 6k + 2$$

$$= 2(3k+1), \text{ even, not odd}$$

That is $\neg p$ is True.

3) Proof by contradiction:

- To prove that p is True, assume that $\neg p$ is True, and then derive an absurdity/contradiction.

- Example: $2n+1$ is odd. [P]

Say $2n+1$ is not odd, that is it is even.

$2n+1 = 2k$; $k \in Z$

$\Rightarrow k = n + \frac{1}{2}$, $k \notin Z$

$\therefore (k \in Z) \wedge (k \notin Z)$

contradiction.

4) Counter Example method for universally quantified predicates:

• Example: Proposed: 'Every bird can fly'.

counterex: 'An ostrich, although is a bird, can't fly.'

$\forall x \, (Bird(x) \rightarrow canFly(x))$

$\Rightarrow \equiv Bird(ostrich) \wedge \neg canFly(ostrich)$

$\Rightarrow \equiv \neg (\forall x \, (Bird(x) \rightarrow canFly(x)))$

$\equiv \neg (\forall x (\neg Bird(x) \vee canFly(x)))$

$\equiv \exists x \neg (\neg Bird(x) \vee canFly(x))$

$\equiv \exists x \, (Bird(x) \wedge \neg canFly(x))$

5) One Example method for an Existentially Quantified Predicate:

• Example: Proposed: 'Some mammals can fly.'

cited: 'A bat is a mammal and it can fly.'

6] Proving logical Equivalences, tautologies and contradiction using truth tables.

7] Simplification using laws of logical equivalences

8] Deriving from a knowledgebase using inference rules.

Lecture - 9 (22-1-19)

F] Non Deductive Proof Techniques:

[Example based]

* Mathematical Induction:

Example: $1+2+3+\ldots+n = n(n+1)/2$

- Let $P(n)$ denote $1+2+3+4+\ldots+n = n(n+1)/2$
- Idea: $P(1) =$ True ; $P(2) =$ True, ... ; so, $P(n) =$ True

Formal inductive proof in three steps:

- Basis: $P(1)$ holds; Proved True.

$P(1)$: $1 = 1(1+1)/2$

$1 = 1$, True

- Hypothesis: Say, $P(k) =$ True, for $k \geq 1$

$P(k)$: $1+2+3+\ldots+k = k(k+1)/2$

• Induction Step:

If $P(K) = \text{True}$, then $P(K+1) = \text{True}$, that is, $P(K) \rightarrow P(K+1)$

Proof of the step:

$$P(K+1) : 1+2+3+ \cdots + (K+1) = \frac{(K+1)(K+2)}{2}$$

$$\begin{aligned} L.H.S. &= 1+2+3+ \cdots + (K+1) \\ &= 1+2+3+ \cdots + K + (K+1) \\ &= K(K+1)/2 + (K+1) \quad \text{[According to the hypothesis]} \\ &= \frac{K(K+1) + 2(K+1)}{2} \\ &= \frac{(K+1)(K+2)}{2} = R.H.S. \end{aligned}$$

[We have $P(1)$, $P(K) \rightarrow P(K+1)$ shown True; So, we can take $P(1+1) = \text{True}$, $P(2+1) = \text{True}$, . . .]

* Empirical Induction:

Show that 'About 90% of the Bangladesh population takes interest in politics.'

We can take help of statistical data analysis to prove or disprove it.

* Analogy:

[Induction based on similarity among examples]

Prove that "Electric current flow at a junction can be computed with the help of water flow at a crossing of water channels;

** Non deductive, especially empirical and analogical procedures are not always very accurate, but very useful and common in natural world.

Lecture-10 (27-1-19)

Chapter 2. Set Theory

Topic 2.1. Basic concepts

A) Sets and set notations:

* Sets:

- Unordered collection of objects (distinct objects, if not said otherwise);
- These objects are elements or members of the set;
- A set is said to contain its elements;
- An object either belongs to ($\in$) or does not belong to ($\notin$) the set;

• Assumed, $S \notin S$, although a set may contain other sets.

• Russell's paradox: $S = \{x \mid x \notin x\}$ does not exist.

* Set notations to describe sets:

(i) Listing: $S_1 = \{1, 2, 3, 4, \ldots, 9\}$

(ii) Set-builder: $S_1 = \{x \mid x$ is a positive integer and $x < 10\}$

(| = such that)

* Important sets of number:

• $N = \{0, 1, 2, 3, \ldots\}$ – Natural number set

• $Z = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$ – set of integers

• $Q = \{p/q \mid p, q \in Z \wedge q \neq 0\}$ – set of Rational numbers

• $[N, Z, Z^+, Q, Q^+, R, R^+]$

• $Z^+ = \{1, 2, 3, \ldots\}$; R – set of real numbers.

B] Important concepts related to sets:

a) Set relations: Subset, $\subseteq$, $A \subseteq A$; Proper subset, $\subset$; Superset, $N \subset Z \subset Q \subset R$; $A \subset B$ if $\forall x (x \in A \rightarrow x \in B) \wedge \exists y (y \in B \wedge y \notin A)$; Equal sets (if $A \subseteq B \wedge B \subseteq A$)

b) Empty set: $\{\}$, $\emptyset$, $\{x \mid x \in Z \wedge x < 5 \wedge x > 10\}$;

$\{\emptyset\}$, $\{\{\}\}$, $\{a\}$ or $\{\{a, b\}\}$ - singleton, not empty; $\emptyset \subseteq A$, for any set A.

c) Universal set: A universal set (U).

d) Complement of a set, A', $\{x \mid x \in U \wedge x \notin A\}$;

$(A')' = A$, $\emptyset' = U$, $\emptyset = U'$.

e) Finite and infinite sets: Finite sets, cardinal numbers, $|A|$; Infinite sets may only be compared to each other, may be countable or uncountable.

(number of numbers)

7) Power set: $P(A)$; $|P(A)| = 2^n$, where $|A| = n$,

$P(\emptyset) = \{\emptyset\}$; $P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$; counting

$P(A)$ using bit strings; $P(N)$ or $P(Z)$ is uncountable

Power set of any infinite set is uncountable.

$A = \{a, b, c\}$

$P(A) = \{ \emptyset,$ — 000

$\{a\},$ — 100

$\{b\},$ — 010

$\{c\},$ — 001

$\{a, b\}$ — 110

$\{a, c\}$ — 101

$\{b, c\}$ — 011

$\{a, b, c\}$ — 111

$\}$

8) Cartesian product: Ordered pairs / n-tuples, $\times$,

$A \times \emptyset = \emptyset$; $A \times B \neq B \times A$, except for special cases like $\emptyset$ and $A = B$,

$|A \times B| = |A| \times |B|$ when $A \neq B$, $A \times B \times C$ — A, B cities and C — airlines providing service, $A_1 \times A_2 \times A_3 \times \ldots A_n$ — contains ordered n-tuples.

Lecture-11 (23-1-19)

h) **Concatenation of sets:**

$AB = \{ab \mid a \in A \text{ and } b \in B\}$.

$A\emptyset = \emptyset = \emptyset A$, $AB \neq BA$ except for special cases like $\emptyset$ and $A = B$.

Topic 2.2. Set Operations

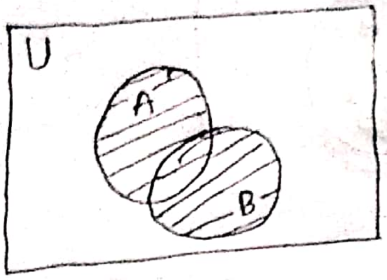
1) **Set Union:**

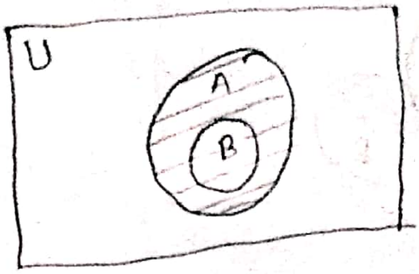
$\cup$, $A \cup B = \{x \mid x \in A \vee x \in B\}$, membership table, Venn diagrams, $A \cup B = B \cup A$; $A \cup \emptyset = \emptyset \cup A = A$; $A \cup U = U \cup A = U$; $A \cup A' = U$; $A \cup A = A$.

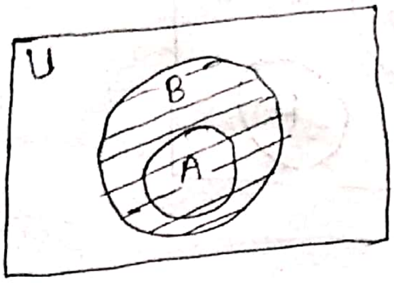
A	B	A ∪ B
0	0	0
0	1	1
1	0	1
1	1	1

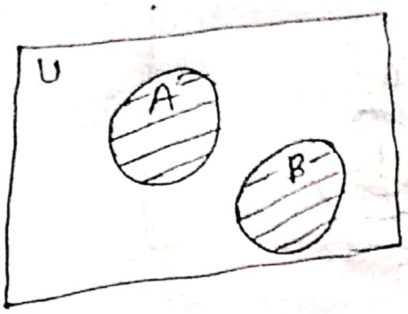
0 - does not belongs to

1 - belongs to









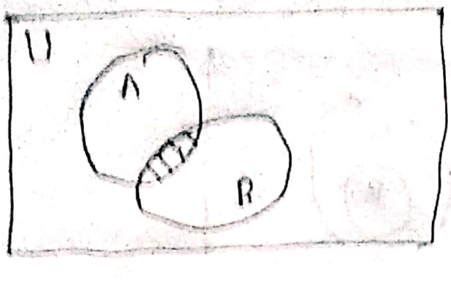
2) Set intersection:

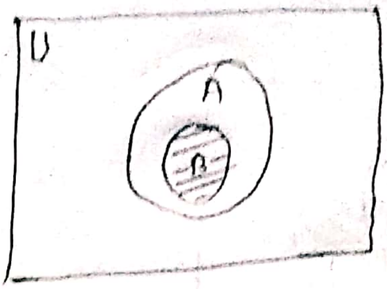
$\cap$, $A \cap B = \{x \mid x \in A \wedge x \in B\}$, membership table, Venn diagrams, $A \cap B = B \cap A$, $A \cap \emptyset = \emptyset \cap A = \emptyset$, $A \cap U = U \cap A = A$, $A \cap A' = \emptyset$, $A \cap A = A$, disjoint sets: $A \cap B = \emptyset$, $|A \cup B| = |A| + |B| - |A \cap B|$.

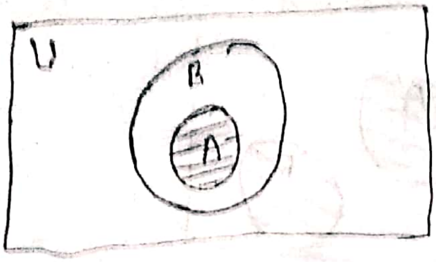
A	B	$A \cap B$
0	0	0
0	1	0
1	0	0
1	1	1

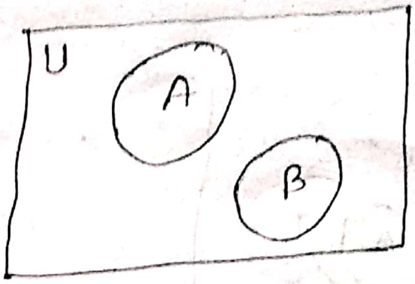
0 - does not belongs to.

1 - belongs to









3) Set Difference:

→ Not a basic operation, $A - B = \{x \mid x \in A \wedge x \notin B\}$

$= A \cap B'$, membership table, Venn diagrams,

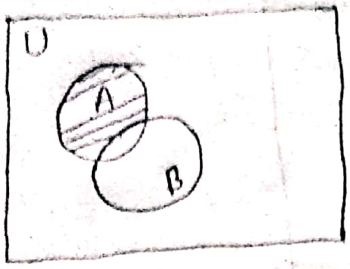
$A - B \neq B - A$ except for special cases like $\emptyset$ and

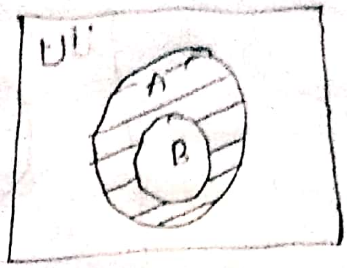
$A = B$, $A - \emptyset = A$, $\emptyset - A = \emptyset$, $A - U = \emptyset$, $U - A = A'$,

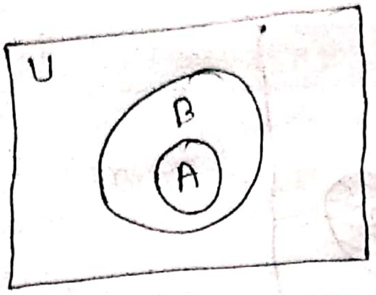
$A - A' = A$, $A - A = \emptyset$

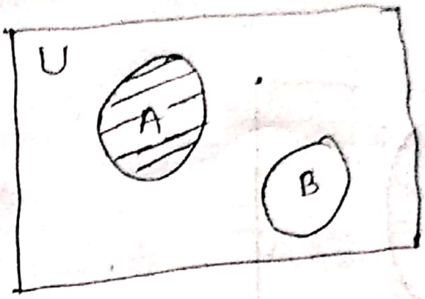
A	B	A − B
0	0	0
0	1	0
1	0	1
1	1	0

0 – does not belongs to
1 – belongs to.









4] Symmetric Set Difference:

$\oplus$, Not a basic operation, $A \oplus B = \{x \mid x \in A \vee$ (exclusive) $x \in B\}$

$= (A \cup B) - (A \cap B) = (A - B) \cup (B - A) = (A \cup B) \cap (A \cap B)'$

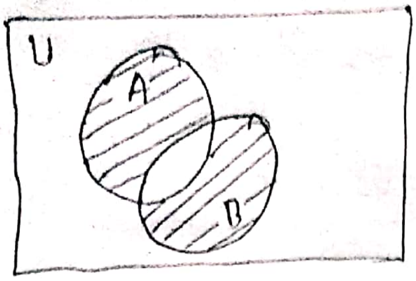
$= (A \cup B) \cap (A' \cup B')$, membership table, venn diagrams, $A \oplus B = B \oplus A$, $A \oplus \emptyset = \emptyset \oplus A = A$,

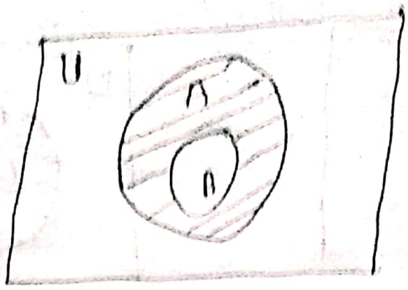
$A \oplus U = U \oplus A = A'$, $A \oplus A' = U$, $A \oplus A = \emptyset$.

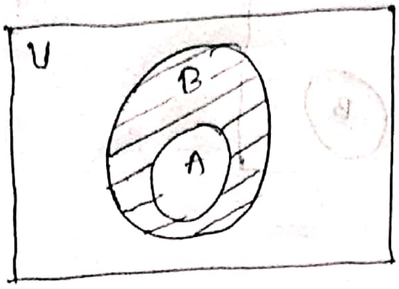
A	B	$A \oplus B$
0	0	0
0	1	1
1	0	1
1	1	0

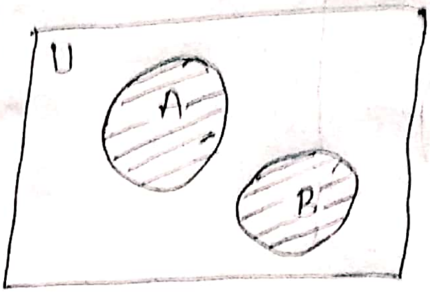
0 - does not belongs to

1 - belongs to.









* Generalized union and intersection:

- $A_1 \cup A_2 \cup A_3 \cup \dots A_n = \bigcup_{i=1}^{n} A_i$, contains distinct elements of All A_i .
- $A_1 \cap A_2 \cap A_3 \dots A_n = \bigcap_{i=1}^{n} A_i$, contains elements common in All A_i .

Lecture-12 (31-1-13)

Topic 2.3. Laws of Set operations [Set Identities]

* Say, A, B and C are subsets of a specific U.

1) **Complementation law:** $(A')' = A$

2) **Idempotent laws:** $A \cup A = A$, $A \cap A = A$

3) **Commutative laws:** $A \cup B = B \cup A$, $A \cap B = B \cap A$

4) **Associative laws:**

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

5) **Distributive laws:**

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

6) **De Morgan's laws:**

$$(A \cup B)' = A' \cap B', \quad (A \cap B)' = A' \cup B'$$

7) **Identity laws:** $A \cup \emptyset = A$, $A \cap U = A$

8) **Domination laws:** $A \cup U = U$, $A \cap \emptyset = \emptyset$

9) **Complement/Negation laws:** $A \cup A' = U$, $A \cap A' = \emptyset$

10) **absorption laws:** $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$

10] Prove by membership table:

$$A \cup (A \cap B) = A$$

A	B	$A \cap B$	$A \cup (A \cap B)$
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

* Identities are used to simplify and transform expressions, and also to show equivalences of expressions involving set operations.

* Exercise:

- Verify the laws using membership tables.
- Prove the following equalities using only definitions and set identities:

i] $(A \oplus B) \cup (A \cap B) = A \cup B$

ii] $(A \cap A) \cup (B \cap (B' \cup A))' = U$

iii] $((A \cap B) \cup (B \cap C' \cap A)) \cup (A \cap B') = A$

iv] $((B' \cup A') \cap (A' \cup B)) \cup (A' \cap B) = A'$

v] $((A\cup B)\cap A')' \cup B = U$

vi] $A' \cap (A \oplus B) = A' \cap B$

$\Rightarrow$

(i) $(A \oplus B) \cup (A \cap B) = A \cup B$

L.H.S. $= (A \oplus B) \cup (A \cap B)$

$= ((A\cup B) - (A\cap B)) \cup (A \cap B)$ [Definition]

$= ((A\cup B) \cap (A\cap B)') \cup (A \cap B)$ [Definition] [$\because$ $x - y = x \cap y'$]

$= (A\cap B) \cup ((A\cup B) \cap (A\cap B)')$ [commutative law]

$= ((A\cap B) \cup (A\cup B)) \cap ((A\cap B) \cup (A\cap B)')$ [Distributive law]

$= ((A\cap B) \cup (A \cup B)) \cap U$ [complement law]

$= \underbrace{(A\cap B)}_{x} \cup (\underset{y}{A} \cup \underset{z}{B})$ [Identity law]

$= ((A\cap B) \cup A) \cup B$ [Associative law]

$= (A \cup (A \cap B)) \cup B$ [commutative law]

$= A \cup B$ [Absorption law]

$=$ R.H.S.

* Quiz-2 (14-2-19) syllabus upto today.

Lecture - 13 (3-2-19)

Chapter 3. Relations and Functions

Topic 3.1. Relations:

A) Basic Concepts:

* A Binary Relation or Relation, R from a set A to a set B is a subset of $A \times B$.

• Example: $A_1 = \{1, 2, 3, 5\}$, $B_1 = \{4, 6, 9, 15\}$

$R_1 = \{(a, b) \mid a \in A_1 \wedge b \in B_1 \wedge$ 'a is the square root of b'$\}$

$= \{(2, 4), (3, 9)\}$

$\therefore R_1$ stands for 'is the square root of'.

$R_1 \subseteq A_1 \times B_1$

we write, $2 R_1 4$ and $3 R_1 9$.

↓

'3' is the square root of '9'.

* R from A to B may also be $\emptyset$ & equal to $A \times B$ (Empty relation & Universal relation).

* Relation R on a set A is a subset of A X A.

• Example 1. R_2 standing for 'is greater than', defined on N as follows:

$$R_2 = \{ (x,y) \mid x,y \in N \wedge x>y \}$$

$$= \{(1,0),(2,0),(3,0),\ldots,(2,1),(3,1),(4,1),\ldots,\ldots,\}$$

$1 R_2 0$ $3 R_2 0$ $2 R_2 1$ $4 R_2 1$

• Example 2. R_3: 'is a brother of' on the set of all people

* n-ary relation, R on sets $A_1, A_2, A_3 \ldots, A_n$:

$$R \subseteq A_1 \times A_2 \times A_3 \ldots \times A_n.$$

• Examples:

Ternary (3-ary), R_4 on $A = \{1, 2, 3, \ldots, 12\}$, defined as follows:

$$R_4 = \{ (x,y,z) \mid x,y,z \in A \wedge x^2 + 5y = z \}$$

$$= \{(1,1,6),(1,2,11),(2,1,9)\}$$

Ternary $R_5 = \{(\text{Karim, CSE, 1st year}), (\text{Rahim, EEE, 2nd year}), \ldots\}$

[Database application]

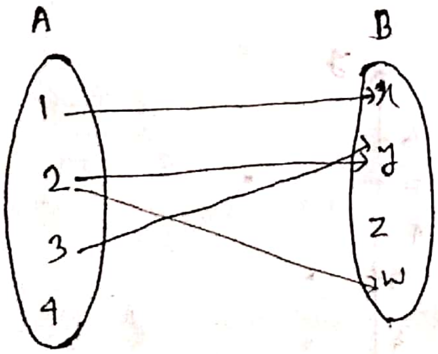
B) Representation of binary relations:

- Set representation:

Example: $A = \{1, 2, 3, 4\}$, $B = \{x, y, z, w\}$,

$R = \{(1, x), (2, y), (2, w), (3, y)\}$

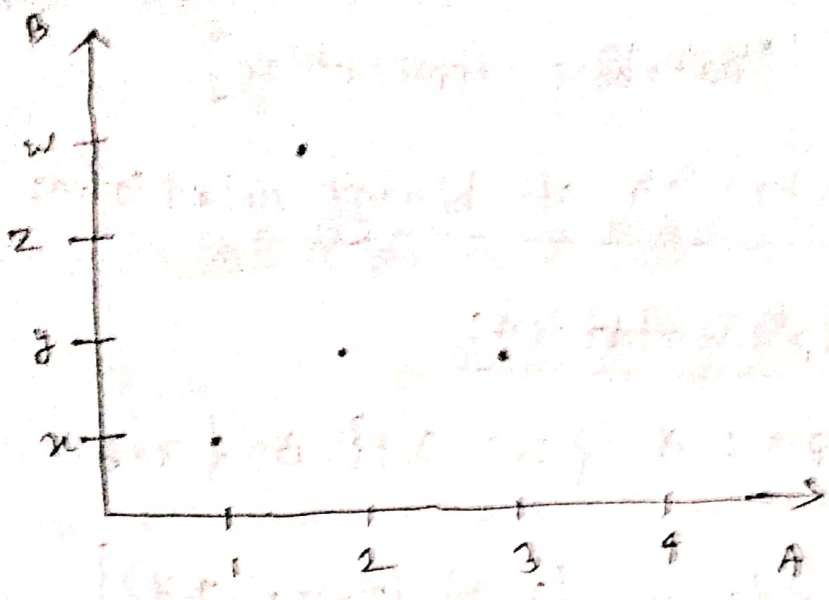
- Arrow diagrams:



- Tabular/matrix:

	x	y	z	w
1	1	0	0	0
2	0	1	0	1
3	0	1	0	0
4	0	0	0	0

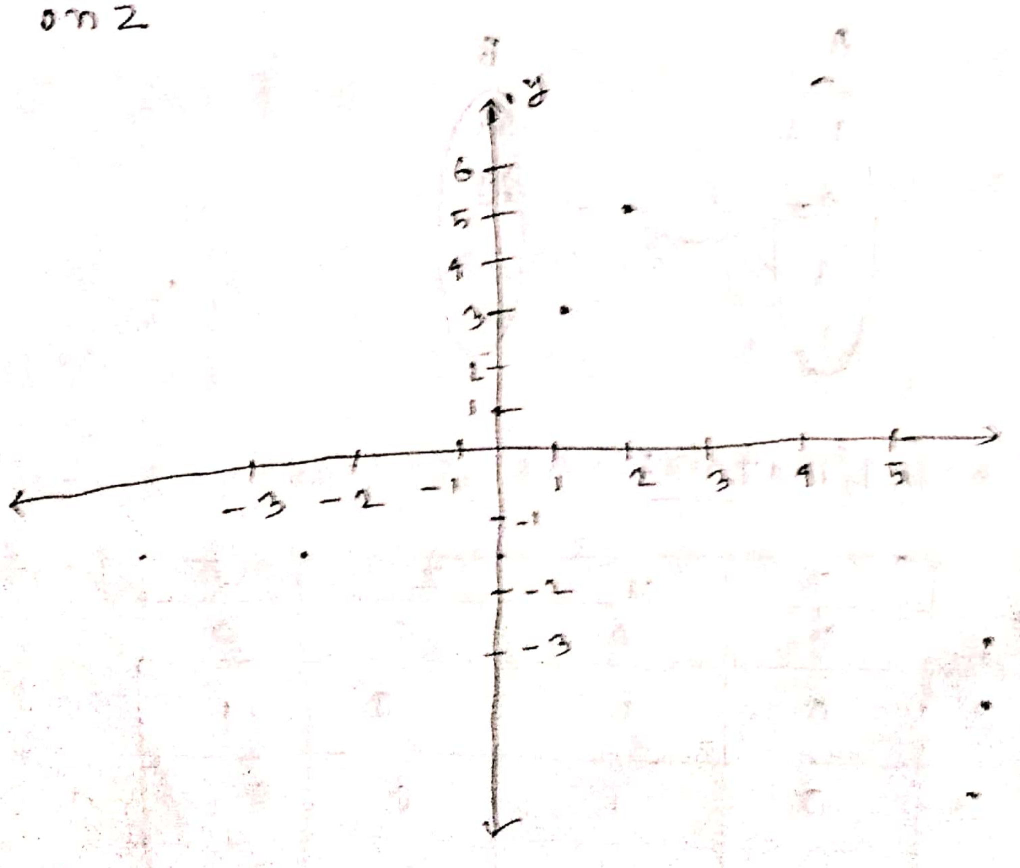
• Coordinate/Graph representation:



*

$y = 2x + 1$

on 2



c) Important Properties of Binary Relations:

Say, R is a binary relation on a set A,

that is, $R \subseteq A \times A$.

i) Reflexivity:

- R is reflexive if $\forall a \in A$,

  aRa, that is, $(a, a) \in R$.

- Examples:

  Reflexive: $\leq, \geq, =$ on Z; $\subseteq$ on $P(z)$.

  Not Reflexive: $<, >$, on Z; $\subset$ on $P(z)$.

ii) Symmetry:

- R is symmetric if $\forall a, b \in A$

  $(a, b) \in R \rightarrow (b, a) \in R$, that is,

  whenever aRb, then bRa.

- Examples: $=$ in general algebra; $\equiv$ in proposition-al calculus;

  'is a brother of' on the set of all male humans

- Not examples: 'is a brother of' on the set of all humans; $\leq, <$ on Z.

Lecture-14 (5-2-19)

iii) Antisymmetry:

- R is antisymmetric if for every distinct $a, b \in A$,

$(a,b) \in R \rightarrow (b,a) \notin R$, that is,

whenever aRb, then $b \not R a$.

- Examples: $<, \leq$ on Z; $\subset, \subseteq$ on $P(z)$.
- Not examples: 'is a brother of' on the set of all humans.

- Not symmetric is not necessarily antisymmetric.

iv) ~~Transivity~~:

iv) Transitivity:

- R is Transitive if $\forall a, b, c \in A$

$((a,b) \in R \wedge (b,c) \in R) \rightarrow (a,c) \in R$ that is,

whenever (aRb) and (bRc), then aRc.

- Examples: $\leq, <$ on Z; $\subset, \subseteq$ on $P(z)$;

'is a brother of' on the set of all male human.

- Not example: 'is the mother of' on the set of all humans.

* Other properties that are not commonly used:
  Irreflexive, Asymmetric, Both symmetric and antisymmetric.

D] Equivalence Relations and classes:

* A binary relation R on a set A is said to be an equivalence relation if R is reflexive, symmetric, and transitive.

• **Examples:**

= in general algebra; ≡ in propositional calculus;

$\equiv_m$ on N for $m \in Z^+$ (congruence modulo m on N for any positive integer m)

[$x \equiv_m y$, reads x "congruent modulo m" y, which means x and y leave the same remainder after integer division by m,
Application: Hashing function]

* $R \subseteq A \times A$

=, R

$a = a$ → Reflexive.

$a = b \rightarrow b = a$ → Symmetric

$a = b \wedge b = c \rightarrow a = c$ → Transitive.

*

$p \equiv p$

$p \equiv q \rightarrow q \equiv p$

$(p \equiv q) \wedge (q \equiv r) \rightarrow (p \equiv r)$

• verify $\equiv_3$ on N for reflexivity, symmetry and transitivity.

$6 \equiv_3 6$ Reflexive

$(7 \equiv_3 13) \rightarrow (13 \equiv_3 7)$ symmetric

$(7 \equiv_3 13) \wedge (13 \equiv_3 25) \rightarrow (7 \equiv_3 25)$ transitive.

• R on **R** such that $(a, b) \in R \leftrightarrow (a-b) \in Z$

a	b	a−b
1.4	1.4	0
1.4	−2.6	4
−2.6	1.4	−4
−2.6	5.4	−8
5.4	3.4	2
−2.6	3.4	−6

1.4 R 1.4 → Reflexive

1.4 R −2.6
−2.6 R 1.4 → symmetric

−2.6 R 5.4
5.4 R 3.4 → −2.6 R 3.4 → transitive

Lecture-15 (7-2-19)

- R on the set of English words such that ~~[illegible]~~ $(x,y) \in R \leftrightarrow |x| = |y|$, meaning x and y are of equal length.

$x \; R \; y$

$\overset{a}{\text{Men}} \; R \; \overset{b}{\text{Men}}$ → Reflexive

$\overset{a}{\text{Men}} \; R \; \overset{b}{\text{hen}}$

$\overset{b}{\text{hen}} \; R \; \overset{a}{\text{Men}}$ → Symmetric

self R help

help R Best → Transitive

self R Best

* $\Sigma = \{0, 1\}$ → alphabet
* $|\varepsilon| = 0$

- $S = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, \ldots\} = \Sigma^*$, $L = \{1, 10, 100, 1000, 10000, 100000, \ldots\} = L(10^*)$ → a regular language

I_L on S such that

$x I_L y$ iff $\forall z \in S$,

$xz \in L \rightarrow yz \in L$, and

$xz \notin L \rightarrow yz \notin L$.

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x	y
10 (a)	1000 (b)
1000 (b)	10 (a)

$a \, I_L \, b$
$b \, I_L \, a$ → symmetric

$\underset{a}{10} \; I_L \; \underset{a}{10}$ → Reflexive

$\underset{a}{100} \; I_L \; \underset{b}{1000000}$

$\underset{b}{1000000} \; I_L \; \underset{c}{10}$ → I_L is transitive.

$\underset{a}{100} \; I_L \; \underset{c}{10}$

* Equivalence classes:

- An equivalence relation R, on a set, A partitions A into equivalence classes.
- Example:

  $\equiv_3$ on N

  Partition: $\{\{0, 3, 6, 9, 12, \dots\}, \{1, 4, 7, 10, \dots\}, \{2, 5, 8, 11, \dots\}\}$

  Classes: $C_1 = \{0, 3, 6, 9, 12, \dots\}$, $C_2 = \{1, 4, 7, 10, \dots\}$,

  $C_3 = \{2, 5, 8, 11, \dots\}$

- If R is an equivalence relation on A, then there are $A_1, A_2, A_3, \ldots$ such that

$$A_i \subseteq A,$$

$$\forall a, b \in A_i \ (a,b) \in R, \text{ that is, } aRb,$$

$$A_i \cap A_j = \emptyset, \text{ for } i \neq j,$$

$$\text{and } A_1 \cup A_2 \cup A_3 \cup \ldots = A.$$

Here, A_i is said to be an equivalence class, $\{A_1, A_2, A_3, \ldots\}$ - a partition of A, for $e \in A_i$, e is called a representative of the class A_i, and representatives of different classes are not equivalent to each other.

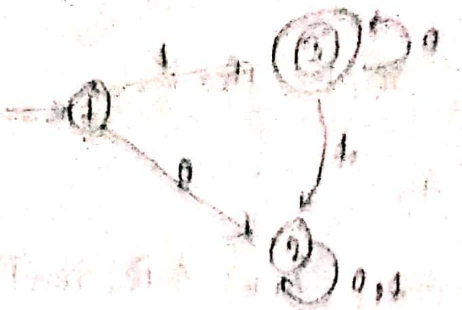
* Exercise:

1. Find the equivalence classes related to I_L discussed above.

2. Consider the relation $R_1 = \{(a,a), (a,c), (a,f), (b,b), (b,e), (c,a), (c,c), (c,f), (e,b), (e,e), (f,a), (f,c), (f,f)\}$ on $\{a, b, c, e, f\}$.

Demonstrate that R_1 is an equivalence relation, and also find the equivalence classes that it yields.

①→



DFA

Deterministic

Finite Automaton

Classes:

① $\{\varepsilon\}$

② $\{0, 00, 01, \dots\}$

③ $\{1, 10, 100, 1000, \dots\}$

Lecture-16 (12-2-19)

Topic - 3.2. Functions

A| Basic Features of functions/ Mappings/ Transformations:

* A function f from a set A to a set B associates exactly one element from B to each and every element of A.

In other words, f is a binary relation from A to B in the following way:

$f = \{(a, b) \mid a \in A \wedge b \in B \wedge \forall a \in A$ there is exactly one $b \in B\}$

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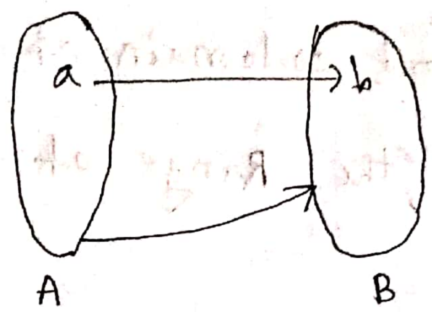
• Explanation:



• f is a function from A to B.

• we write:

$f: A \rightarrow B$

[f maps A to B]

$f(a) = b$

[a: argument of f
b: value of f at a]

we call,

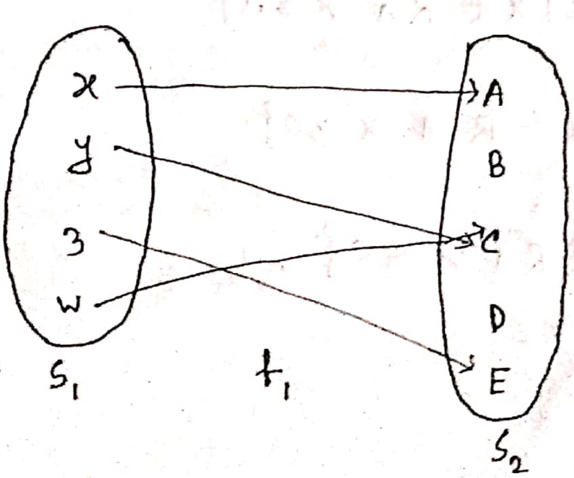
b: the <u>Image</u> of a; a: a <u>Preimage</u> of b;

A: the <u>Domain</u> of f; B: the <u>codomain</u> of f;

Set of all images: Range of f.

• Examples:

1)



S_1: Students;

S_2: Grades obtained in CSE 1203;

$f_1: S_1 \rightarrow S_2$;

$f_1(x) = A$, $f_1(y) = C$, ...;

S_1 : Domain of f_1;

S_2 : codomain of f_1;

$\{A, C, E\}$: Range of f_1;

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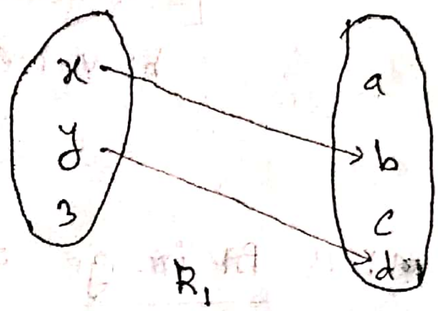
2) f_2 is a function on Z, that is, $f_2 : Z \to Z$, such that $f_2(x) = x^2$.

Z: the Domain and codomain of f_2

$\{0, 1, 4, 9, 16, \dots\}$: the Range of f_2

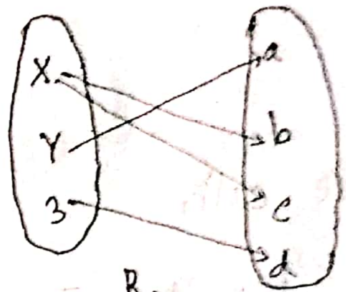
- **Not examples:**

1)



R_1

2)



R_2

3) $y = f(x)$ is defined as $y^2 = x$.

In fact, two functions: $f_1(x) = \sqrt{x}$, $f_2(x) = -\sqrt{x}$.

Domain for both $= \{x \mid x \in R \wedge x \geq 0\}$

Range of $f_1 = \{x \mid x \in R \wedge x \geq 0\}$

Range of $f_2 = \{x \mid x \in R \wedge x \leq 0\}$.

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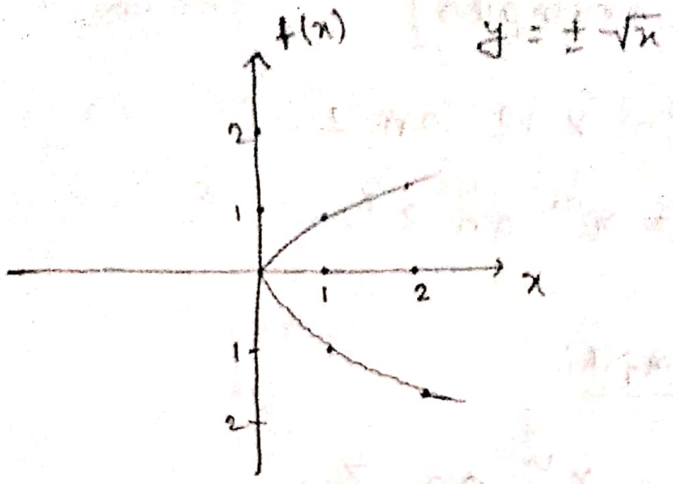


B] Fundamental classes of Functions:

✓ Say, $f : A \to B$.

I. <u>one-to-one function / Injective function / Injection</u>

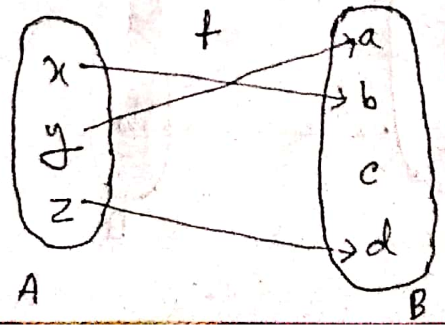
- f is an injection <u>iff</u>

$$\forall x, y \in A \, (x \neq y \to f(x) \neq f(y)).$$

Equivalently, f is an injection <u>iff</u> <u>no two elements</u> of the domain have the <u>same image</u>.

- <u>Examples:</u>

<u>1]</u> [Idealized]



An element of B can be the image of at most one element of A.

• [concrete examples]

2. $f(x) = x+1$ on $\mathbb{Z}$.

3. $f(x) = x^2$ on $\mathbb{Z}^+$.

• Not example:

$$f(x) = x^2 \text{ on } \mathbb{Z}.$$

Lecture-17 (17-2-19)

11. Onto function / surjective function / surjection:

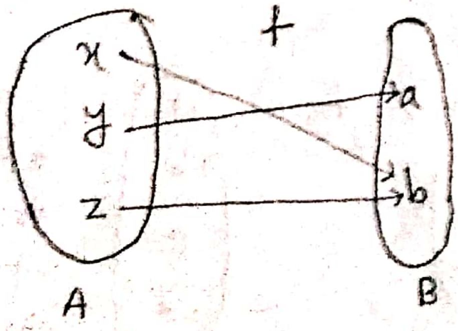
• f is a surjection iff

$$\forall y \in B\,(\exists x \in A\,(f(x) = y)).$$

Equivalently, f is a surjection iff each element of B is the image of one or more elements of A.

• Examples:

1. (Idealized)

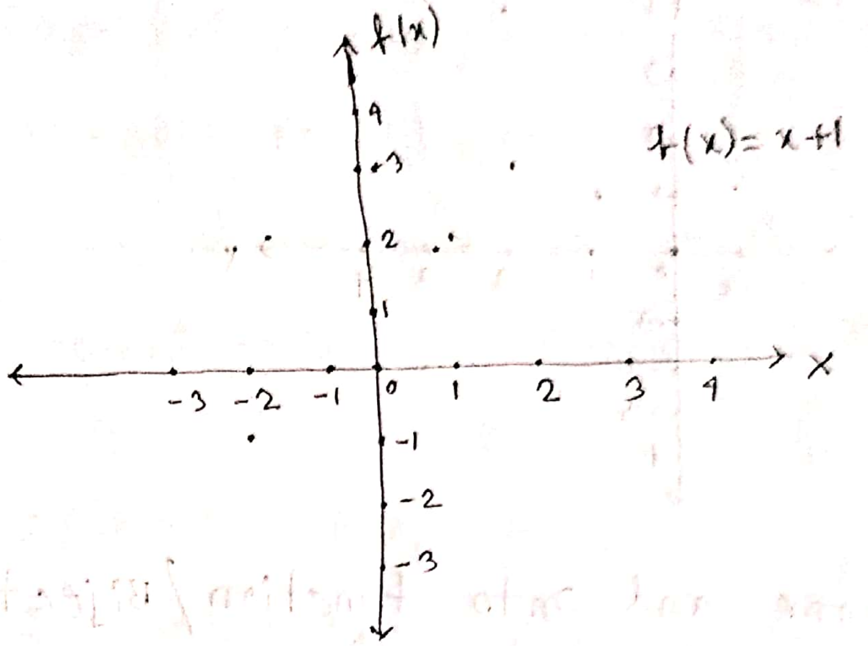


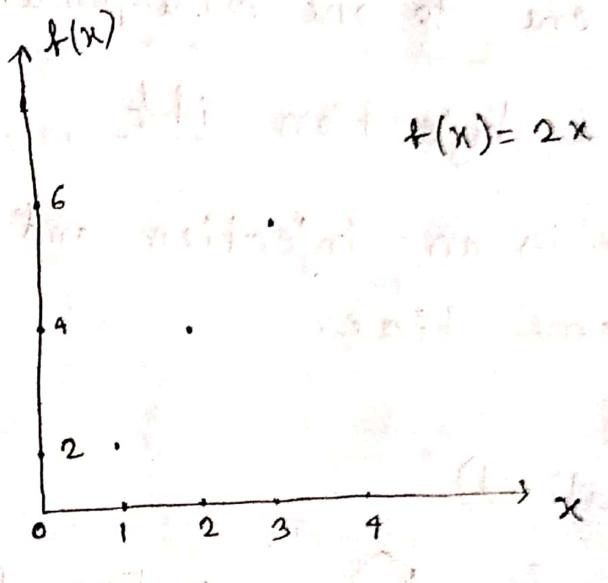
codomain and range of f are equal.

• [concrete examples]

2. $f(x) = x+1$ on $\mathbb{Z}$.

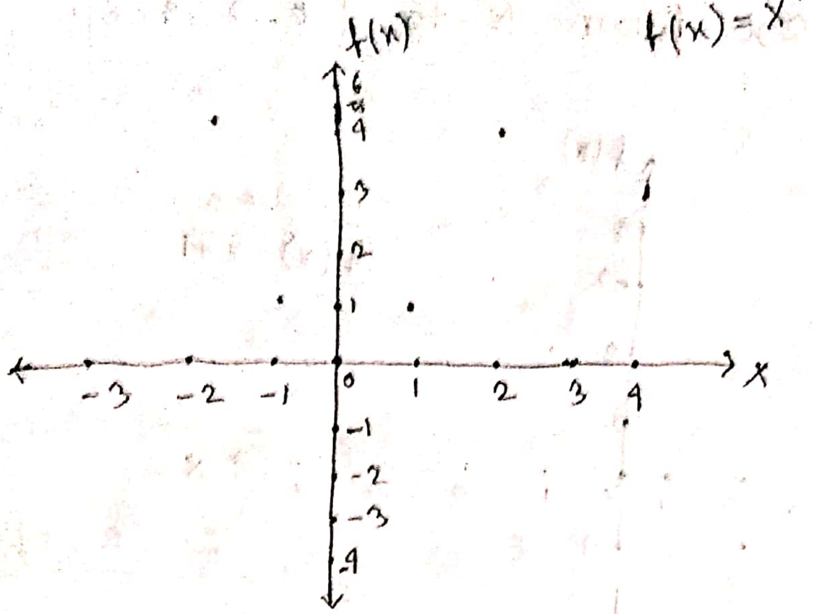
3. $f(x) = 2x$ from $\mathbb{N}$ to $\{0, 2, 4, 6, \dots\}$





• **Not example:**

$f(x) = x^2$ on $\mathbb{Z}$



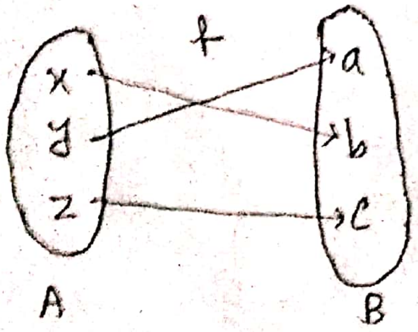
III. One-to-one and onto function / Bijective function / Bijection / one-to-one correspondence:

• f is a bijection iff,

it is an injection and surjection at the same time.

• **Examples:**

1. (Idealized)



For finite A and B,

$|A| = |B|$.

[concrete examples]

2. $f(x) = x+1$ on $\mathbb{Z}$.

3. $f(x) = 2x$ from $\mathbb{N}$ to $\{0, 2, 4, 6, \dots\}$

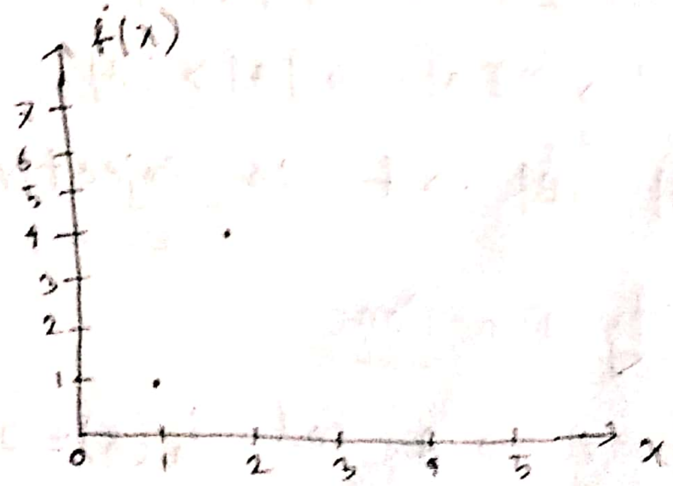
* Some important facts about functions and their fundamental classes:

i] Not all relations are functions.

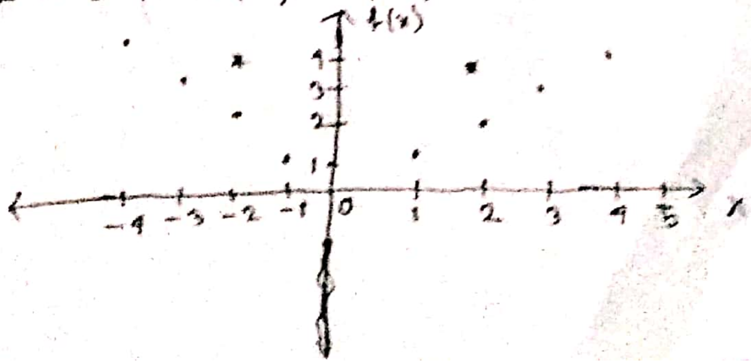
ii] All combination of class properties are possible:

ⓐ $\neg I \wedge \neg S$: $f(x) = x^2$ on $\mathbb{Z}$

ⓑ $I \wedge \neg S$: $f(x) = x^2$ on $\mathbb{N}$.

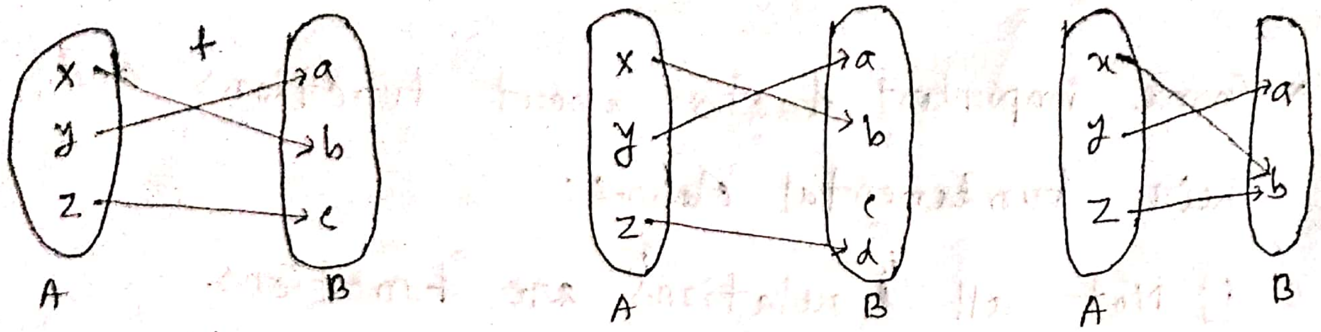


ⓒ $\neg I \wedge S$: $f(x) = |x|$ from $\mathbb{Z}$ to $\mathbb{N}$.



(d) $I \wedge S$: $f(x) = x + 1$ on $\mathbb{Z}$.

*



$I \wedge S$ $I \wedge \neg S$ $\neg I \wedge S$

* for finite A and B,

a. f is $I \wedge S \rightarrow |A| = |B|$.

b. f is $I \wedge \neg S \rightarrow |A| < |B|$.

c. f is $\neg I \wedge S \rightarrow |A| > |B|$.

d. $|A| = |B| \rightarrow f$ is injective, iff f is surjective

* Identity Function:

$i_A : A \rightarrow A$ and $i_A(x) = x$.

It is a bijection.

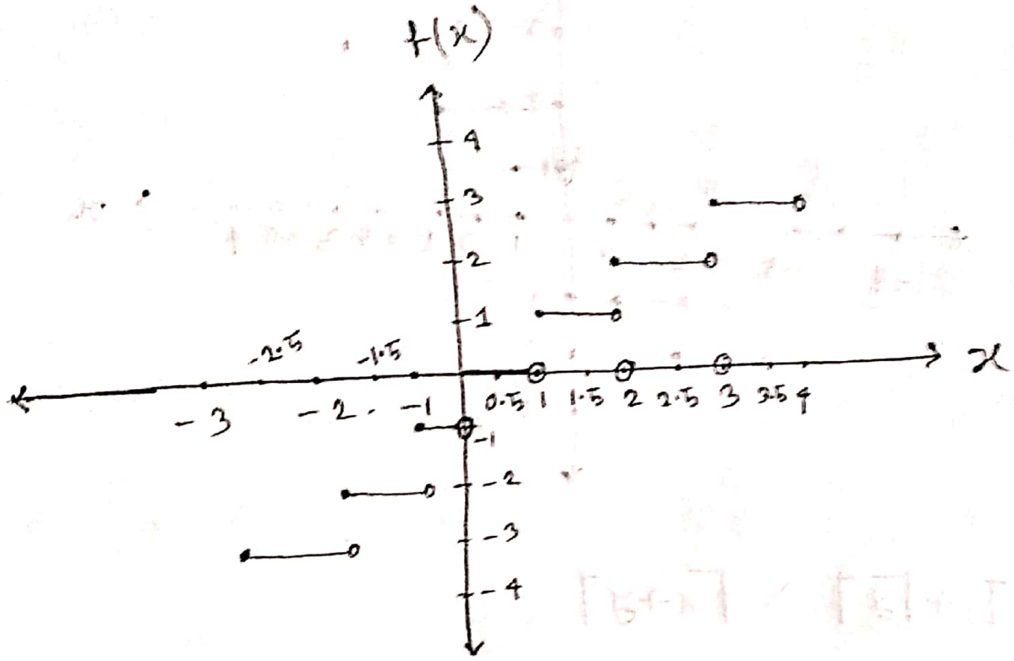
Lecture - 18 (19-2-19)

c) Some Important Discrete Functions:

1) Floor function from R to Z:

$y = f(x) = \lfloor x \rfloor = n,$

where $n \leq x < n+1$, for $x \in R$ and $n \in Z$.



- $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x+y \rfloor$
- . . .

let,

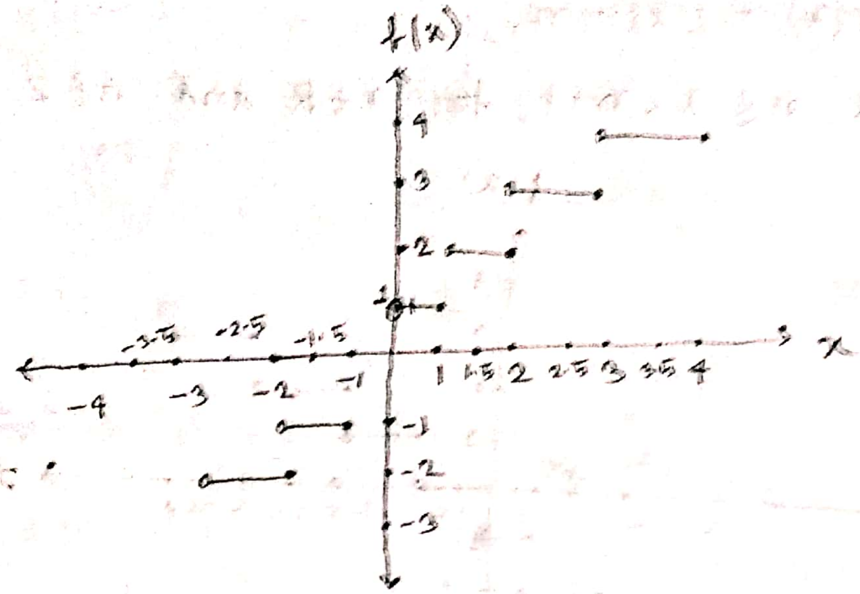
	x	y	$x+y$
	1.3	2.5	3.8
floor	1	2	3
	1.8	2.7	4.5
floor	1	2	4

- $\neg I \wedge S$

2) Ceiling function from R to Z:

$y = f(x) = \lceil x \rceil = n+1,$

where $n < x \leq n+1$, for $x \in R$ and $n \in Z$.



• $\lceil x \rceil + \lceil y \rceil \geq \lceil x+y \rceil$

• . . .

	x	y	x+y
	1.3	2.5	3.8
ceil	2	3	4
	1.8	2.7	4.5
ceil	2	3	5

• ¬ I ∧ S .

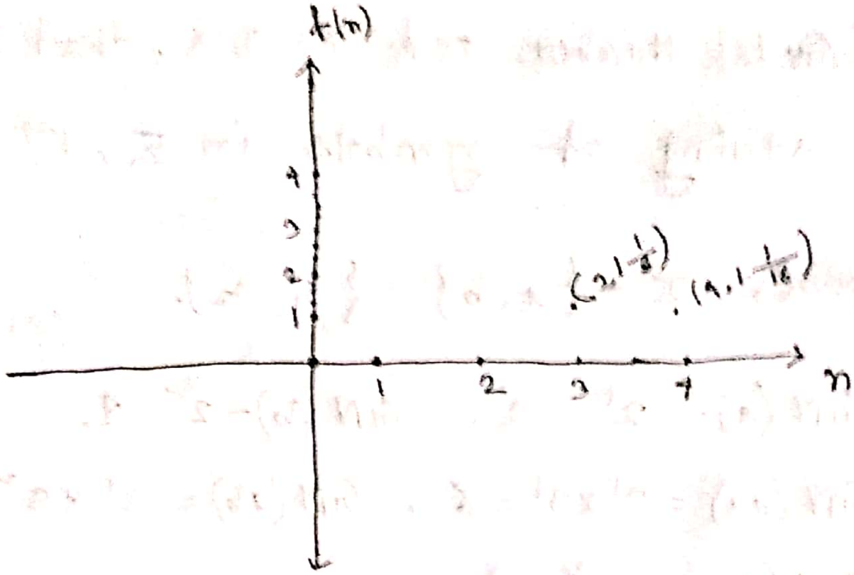
3) Discrete numeric functions:

Function of the type, $f: N \to R$

Example:

$f(n) = 0$, for $0 \le n \le 2$,

$f(n) = 1 + 2^{-n}$, for $n \ge 3$



• $\neg I \wedge \neg S$

4) Functions with strings as arguments:

i) Say, $\Sigma = \{a, b\}$. [Σ - alphabet]

$\Sigma^* = \{\varepsilon, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$

$L: \Sigma^* \to N$, such that $L(x) = n$, where n is the number of symbols in x, that is, $|x|$.

$L(\varepsilon) = 0$, $L(a) = 1$, $L(b) = 1$, $L(aa) = 2$, $L(ab) = 2$, . . .

$\neg I \wedge S$

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ii) $GN: \Sigma^* \to N$, such that

$GN(x) = 2^{i1} \times 3^{i2} \times 5^{i3} \times \ldots \times (n^{th} \text{ prime})^{in}$

where $i1, i2, i3, \ldots, in \in \{1, 2, 3, \ldots, m\}$,

for $|\Sigma| = m$, and say, $\Sigma = \{s_1, s_2, \ldots, s_m\}$.

[GN – Gödel Number, and $x \in \Sigma^*$, that is, x is a string of symbols in Σ, 1st prime = 2]

Suppose, $\Sigma = \{a, b\} = \{s_1, s_2\}$.

$GN(a) = 2^1 = 2$, $GN(b) = 2^2 = 4$,

$GN(aa) = 2^1 \times 3^1 = 6$, $GN(ab) = 2^1 \times 3^2 = 18$,

$GN(ba) = 2^2 \times 3^1 = 12$, . . .

$GN(\varepsilon)$ is taken 1.

$\therefore$ I $\wedge$ 7S.

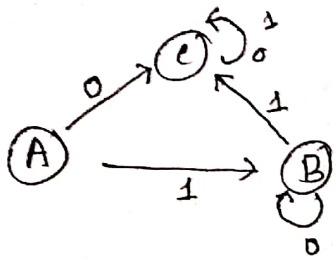
5] Transition function of automata (machines):

Consider the DFA for $L(10^*)$ shown earlier:

States, $S = \{A, B, C\}$, $\Sigma = \{0, 1\}$

$\delta : S \times \Sigma \rightarrow S$,

$\delta(A,0) = C$, $\delta(A,1) = B$, $\delta(B,0) = B$, $\delta(B,1) = C$, ...



$\cdot \neg I \wedge \neg S.$

A, 0	C
A, 1	B
B, 0	B
B, 1	C
C, 0	C
C, 1	C

Lecture – 19 (3-3-19)

D] Composition and Inverse of functions:

- Say, f and g are functions such that

  $g : A \to B$, $f : C \to D$,

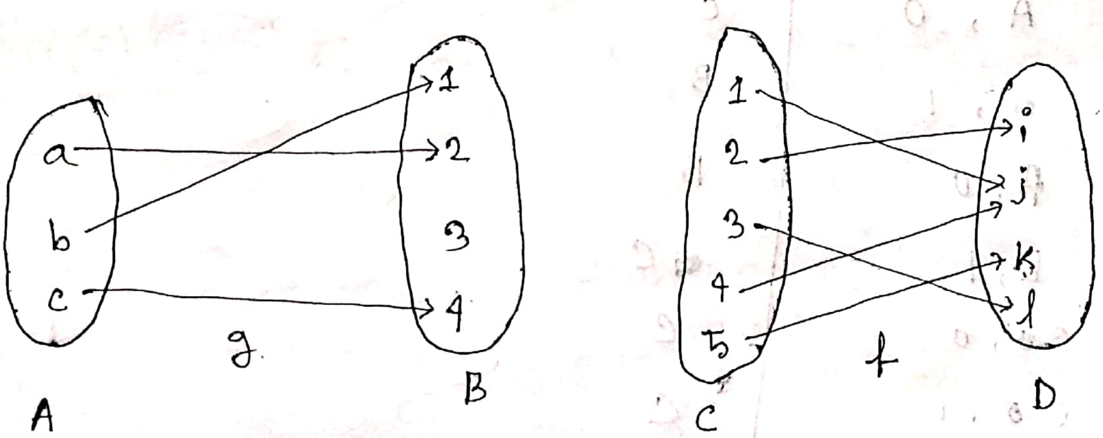
  and the range of g is a subset of the domain of f.

  [Special case : $B = C$]

Composition of f and g is a function from A to D, denoted by $f \circ g$, such that

$$f \circ g(x) = f(g(x)), \text{ for } \forall x \in A.$$

Abstract example:



$f \circ g(a) = i$, $f \circ g(b) = j$, $f \circ g(c) = j$; $f \circ g : \neg I \wedge \neg S$

Concrete example:

$f_1(x) = 2x+1$ on R, $f_2(x) = x-2$ on R.

- $f_1 \circ f_2(x) = f_1(f_2(x)) = f_1(x-2) = 2(x-2)+1$
$= 2x-3$

- $f_2 \circ f_1(x) = f_2(f_1(x)) = f_2(2x+1) = 2x+1-2$
$= 2x-1$

- Say, $f: X \to Y$, and f is a bijection.

The inverse of f, denoted by f^{-1}, is a function such that

$f^{-1}: Y \to X$, and

$f^{-1}(f(x)) = x$, for all $x \in X$, and $f(f^{-1}(y)) = y$, for all $y \in Y$.

* A function is invertible iff it is bijective.

Example:

$f(x) = 2x$ from N to $\{0, 2, 4, 6, \ldots\}$

$f^{-1}(x) = x/2$.

[one-to-one correspondence]

3rd Quiz এর পর্যন্ত

Chapter 4. Counting and countability

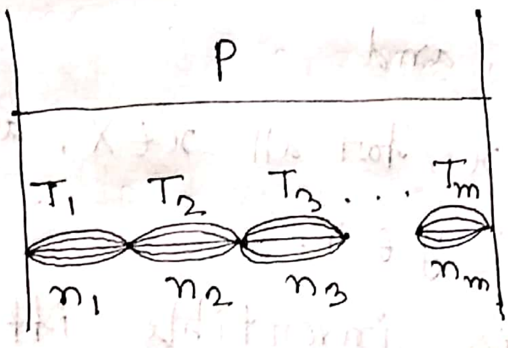
Topic 4.1. Basic counting Principles:

A] The Product Rule:

- Say, a process P can be broken down to sequence of m tasks:

$$T_1, T_2, T_3, \ldots, T_m,$$

and say, T_i can be done in n_i different ways after up to T_{i-1} is done.



- There are $n_1 \times n_2 \times n_3 \times \ldots \times n_m$ different ways to execute the process P.

Examples:

1] How many 5-length bit strings are possible?

0/1	0/1	0/1	0/1	0/1
2	2	2	2	2

$\therefore 2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32.$

2] How many 4-digit binary numbers are possible?

0 0 0 0

0 0 0 1

0 0 1 0

⋮ ⋮ ⋮ ⋮

1 1 1 1

$\therefore 2^4 = 16$

3] How many 3-digit numbers (decimal) are possible?

0 0 0

0 0 1

0 0 2

0 0 9

0 1 0

0 1 1

0 1 2

⋮

9 9 9

$\therefore 10 \times 10 \times 10 = 10^3 = 1000$.

4] How many car license numbers are possible that have 2 capital letters at the beginning, followed by a hyphen, and then by 3 digits?

A – Z	A – Z	–	0 – 9	0 – 9	0 – 9
26	26	1	10	10	10

$\therefore 26^2 \times 1 \times 10^3$.

5] Observe the code fragment below:

$k := 0$

for $i_1 = 1$ to n_1

for $i_2 = 1$ to n_2

⋮

for $i_m = 1$ to n_m

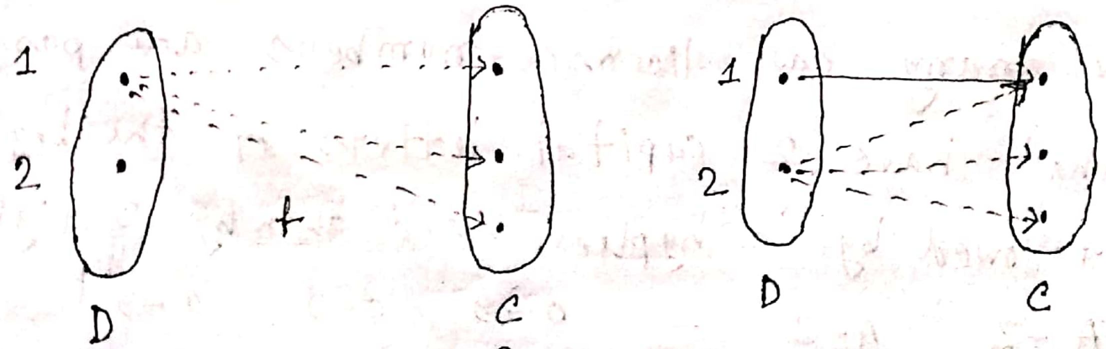
$k := k+1$

Value of k after execution of the program fragment:

$n_1 \times n_2 \times n_3 \times \ldots \times n_m$.

6] $f : D \to C$, $|D| = m$, $|C| = n$.

what would be the number of different possible functions of this kind?



$\frac{3}{1}$ $\frac{3}{2}$

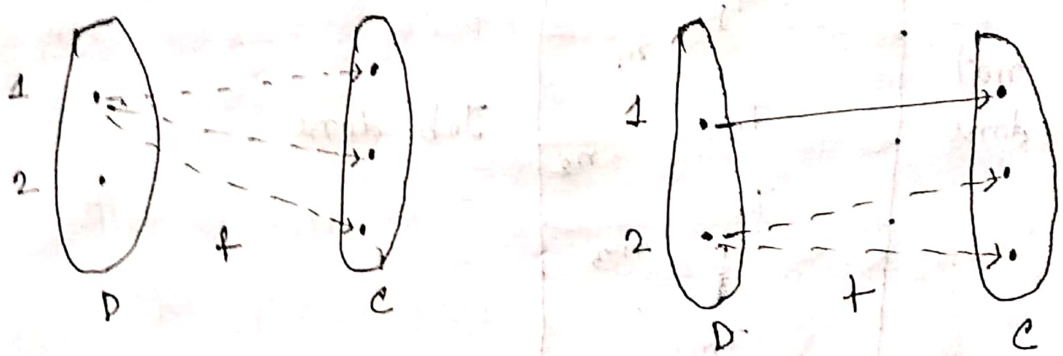
$\therefore 3^2$

$\therefore\ n^m \qquad |C|^{|D|}$

n = cardinal number of range

m = cardinal number of domain

7) Suppose, $f: D \to C$, $|D| = m$, $|C| = n$, f is an I n I S.



$\frac{3}{1} \qquad \frac{2}{2} \qquad 3 \times 2$

In general: $n \times (n-1) \times (n-2) \times \cdots \times (n-(m-1))$.

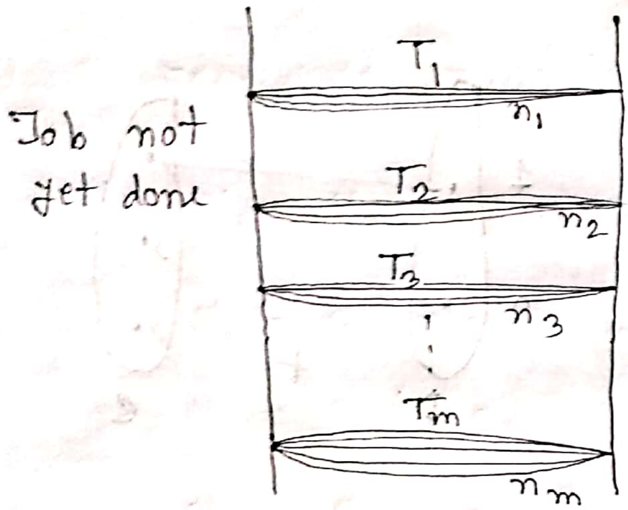
* If f is an I n I S, that is, f is a bijection, then $|D| = |C| = n$, and the number is

$n \times (n-1) \times (n-2) \times \cdots \times 1 = n!$.

Lecture- 20 (5-3-19)

B] The Sum Rule:

* Suppose, one of the tasks, from among m different tasks, $T_1, T_2, T_3, \ldots, T_m$, is required to be carried out to have a job done.



And say, there are $n_1, n_2, n_3, \ldots, n_m$ different ways to carryout the tasks respectively.

- Then there are $n_1 + n_2 + n_3 + \ldots + n_m$ different ways to get the job done.

□ Examples:

1] Program fragment for a job:

```
for i=1 to n
    if condition = True
        then do Task1 [x:=2i^2+1]
```

else do $Task_2$ $[x_i = 4i + 2]$

∴ Possible ways of getting the job done : $n + n = 2n$.

2) Either a teacher or a student may represent the department in a committee.

No. of teachers = m,

No. of students = n, and nobody is both a teacher and a student.

∴ There are $m + n$ different ways of representing the department.

c) Combination of Product and Sum Rules :

Example 1 : How many identifiers are possible under the following conditions:

i) At most 3 letters or digits are used;

ii) 1st symbol is a lowercase letter;

iii) There are 50 reserved words (keywords), each of 3 lowercase letters each.

$N_1 = 26$ [of length 1]

$$\frac{a-z}{26}$$

$N_2 = 26 \times (26 + 26 + 10)$

$$\frac{a-z,\ A-Z,\ 0-9}{26+26+10}$$

$N_3 = 26 \times (26+26+10) \times (26+26+10) - 50.$

Total $= N_1 + N_2 + N_3$

Example 2:

How many passwords of length 6 to 8 are possible, if

i) only lowercase letters and digits are used, and

ii) a password must contain at least one digit

$N = N_6 + N_7 + N_8$, where

$N_6 = 36^6 - 26^6$

$N_7 = 36^7 - 26^7$

$N_8 = 36^8 - 26^8$

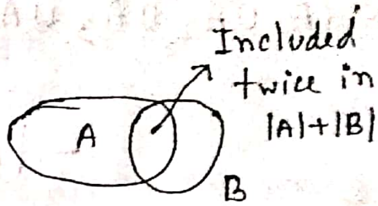
$\underline{a-z, 0-9}$, $\underline{a-z, 0-9}$, ..., less

$\underline{a-z}$, $\underline{a-z}$, ...

D] Principle of Inclusion and Exclusion:

* Simple form:

$$|A \cup B| = |A| + |B| - |A \cap B|$$



* Example:

How many integers from 1 to 20 are divisible by 2 or 3?

Let, A contain those divisible by 2, and
B contain those divisible by 3.

So, $A \cup B$: divisible by 2 or 3,
$A \cap B$: divisible by 2 and 3.

$$\therefore |A \cup B| = |A| + |B| - |A \cap B|$$

$$= \lfloor 20/2 \rfloor + \lfloor 20/3 \rfloor - \lfloor 20/6 \rfloor$$

$$= 10 + 6 - 3$$

$$= 13$$

▷ General form of the principle of Inclusion and exclusion:

$$|A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i \le j \le n} |A_i \cap A_j| + \sum_{1 \le i \le j \le k \le n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n|$$

- $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

- $|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4| - |A_1 \cap A_2| - |A_1 \cap A_3|$

$$- |A_1 \cap A_4| - |A_2 \cap A_3| - \cdots - |A_3 \cap A_4|$$

$$+ |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|$$

$$- |A_1 \cap A_2 \cap A_3 \cap A_4|.$$

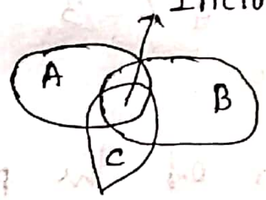
Lecture – 21 (7–3–19)

Example:

How many binary strings of length 5 begin with 0 or end with 1 or have 1 at the central places?

A:	0	0/1	0/1	0/1	0/1	$\lvert A\rvert = 1\times2\times2\times2\times2 = 16$
B:	0/1	0/1	0/1	0/1	1	$\lvert B\rvert = 2\times2\times2\times2\times1 = 16$
C:	0/1	1	1	1	0/1	$\lvert C\rvert = 2\times1\times1\times1\times2 = 4$

Included thrice in $|A| + |B| + |C|$



$A \cap B$:	0	0/1	0/1	0/1	1	$\lvert A\cap B\rvert = 1\times2\times2\times2\times1 = 8$
$A \cap C$:	0	1	1	1	0/1	$\lvert A\cap B\rvert = 1\times1\times1\times1\times2 = 2$
$B \cap C$:	0/1	1	1	1	1	$\lvert A\cap B\rvert = 2\times1\times1\times1\times1 = 2$
$A \cap B \cap C$:	0	1	1	1	1	$\lvert A\cap B\cap C\rvert = 1\times1\times1\times1\times1 = 1$

$\therefore |A \cup B \cup C| = 16 + 16 + 9 - 8 - 2 - 2 + 1 = 37 - 12 = 25.$

* Exercise:

Solve the following problems and name the major counting principles involved.

1) Department: 20 teachers; 250 male students; 150 female students

i) 1 teacher and 3 male students and 2 female students ~~two~~ form a delegation; How many possible delegations? [Hint: combination + Product rule]

$\Rightarrow N_1 = {}^{20}C_1 = 20$

$N_2 = {}^{250}C_3 =$

$N_3 = {}^{150}C_2 =$

$\therefore N = N_1 \times N_2 \times N_3 =$

ii) 1 teacher or 3 male students or 2 female students form a delegation; How many possible delegations? [Hint: combination + Sum rule]

$\Rightarrow N = N_1 + N_2 + N_3$

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2) How many integers between 21 and 200 (including 21 and 200) are divisible by 3 or 5 or 9?

Ans.(84)

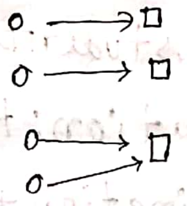
[Hint: X: 1-200, Y: 1-20; X-Y to be found.]

⇒

E] The Pigeonhole Principle:

* Simple form:

If there are k boxes to contain more than k objects, then at least one box must contain more than one objects.



3 boxes (holes)

4 objects (pigeons)

* Problems:

1) What is the smallest class size, for which it is obvious that at least two students have the same grade in a course, if the set of possible grades, G = {A+, A, A-, B+, B, B-, C+, C, D, F}?

Boxes: Grades, $|G| = 10$

objects: Students

10 boxes; so, needed $10+1$, that is, 11 objects.

2) What is the size of the smallest group of people, for whom at least two must have the same birthday according to the English calender?

$\Rightarrow$ Boxes: $365+1$ days,

objects: People

366 boxes; so needed $366+1$, that is, 367 objects

3) Say, $f: D \to C$, and $|D| > |C|$, for finite D & C.

Can f be an injection?

$\Rightarrow$ Boxes: $|C|$, objects: $|D|$

As $|D| > |C|$, at least one of C is the image of two or more of D.

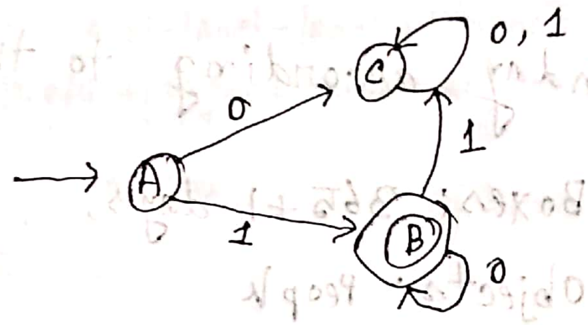
So, f can't be an injection.

4) How many single shoes at least are to be examined from a mess of 100 pairs to find a proper pair for sure?

$\Rightarrow$ Boxes: 100, so objects required: 101.

5) If there is a string of length $\geq$ the number of states of a DFA and if it is accepted by the DFA, then there must be a cycle, and the language is infinite.

$\Rightarrow$



$L = \{1, 10, \underline{100}, 1000, \dots\}$

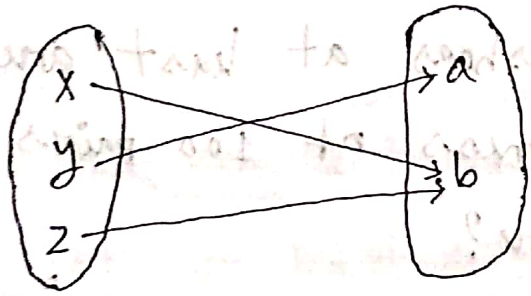
(এইক্ষেত্রে loop পাওয়া যাবে)

Lecture-22 (10-3-19)

* Generalized Pigeonhole principle:

m boxes — At least one box contain ~~at least~~

n objects at least $\lceil n/m \rceil$ objects



$\lceil 3/2 \rceil = \lceil 1.5 \rceil = 2$

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Problems:

1) Grades: A, B, C, D, E

class size: 66

what is the minimum size of the largest group obtaining the same grade?

$\Rightarrow$

A	1	2	13
B	1	2	13
C	1	2	13
D	1	2	13
E	1	2	13

Boxes: 5

objects: 66

$\lceil 66/5 \rceil = \lceil 13.2 \rceil = 14$

2) Sample size of a survey: 100 persons.

what is the minimum size of the largest group having birthday in same month?

$\Rightarrow$ Boxes: 12

objects: 100

$\lceil 100/12 \rceil = \lceil 8.33 \rceil = 9$.

3) How many people are to be interviewed to find for sure at least 9 with birthday in same month?

$\Rightarrow$ Boxes: 12 object $= n$

$\lceil n/12 \rceil = 9$.

$n = 108$

$= 107$

$= 106$

$= 105$

$= 104$

$= 103$

$= 102$

$= 101$

$= 100$

$= 99$

$= 98$

$= 97$

$n_{min} = 97$

$n_{min} = 12 \times (9-1) + 1$

$= 97$

$n_{max} = 12 \times 9 = 108$

$\therefore$ Answer: Any number from 97 to 108.

Topic-4.1 Recurrence Relation

✓ Say $a_i, a_{i+1}, a_{i+2}, \ldots$, for $a_n, i \in N$ is a sequence.

- Recurrence Relation: An equation that defines a term, a_n, using one or more preceding terms, given one or more term values as initial conditions.
- The Solution: The sequence itself.

✓ Examples:

1) Rabbits and Fibonacci numbers:

- Each pair gets matured at the age of 2 moths, and gives birth to 1 pair every next month.

- Number of pairs in the ~~nth~~ nth month,

f_n:

$$f_n = f_{n-1} + f_{n-2} \quad , \text{for } n \in N, \ n \geq 2,$$
$$f_0 = 0, \ f_1 = 1.$$

$f_0 = 0$
$f_1 = 1$
$f_2 = 1$
$f_3 = 2$
$f_4 = 3$

2) Tower of Hanoi:

Say, H_n = Number of moves required to transfer n discs from A to C with the help of C, if required.

$$H_n = 2H_{n-1} + 1 \ , \text{ for } n \in N, \ n \geq 2, \ H_1 = 1.$$

$H_2 = 2H_1 + 1 = 3$

$H_3 = 7$

$H_4 = 15$

$H_5 = 31$

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3) Find the solution, up to a_9, of the recurrence relation,

$$a_n = a_{n-1} + 2a_{n-3}, \text{ for } n \in N, n \geq 3,$$

$$a_0 = 1, \; a_1 = 2, \; a_2 = 0$$

$\Rightarrow$

$a_3 = a_2 + 2a_0 = 0 + 2\times1 = 2$

$a_4 = a_3 + 2a_1 = 2 + 2\times2 = 6$

$a_5 = a_4 + 2a_2 = 6 + 2\times0 = 6$

$a_6 = a_5 + 2a_3 = 6 + 2\times2 = 10$

$a_7 = a_6 + 2a_4 = 10 + 2\times6 = 22$

$a_8 = a_7 + 2a_5 = 22 + 2\times6 = 34$

$a_9 = a_8 + 2a_6 = 34 + 2\times10 = 54.$ (Ans.)

4) Compound Interest:

$$P_n = P_{n-1} + 0.11\, P_{n-1}$$

$$= 1.11\, P_{n-1}$$

$$= (1.11)^n \times P_0$$

Suppose, P_n - amount after n years with 11% interest

$P_0 = 10000$, for $n \in N$, $n \geq 1$

Exercise:

Find the solution up to a_6 of the recurrence relation:

i) $a_n = a_{n-1} + (-1)^n a_{n-2}$, for $n \in N$, $n \geq 2$, $a_0 = 0$, $a_1 =$

ii) $a_n = n a_{n-1} + a_{n-2}^2$, for $n \in N$, $a_0 = -1$, $a_1 = 0$

$\Rightarrow$ i) $a_2 = a_1 + (-1)^2 \cdot a_0 = 1 + 0 = 1$

$a_3 = a_2 + (-1)^3 \cdot a_1 = 1 - 1 = 0$

$a_4 = a_3 + (-1)^4 \cdot a_2 = 0 + 1 = 1$

$a_5 = a_4 + (-1)^5 \cdot a_3 = 1 - 0 = 1$

$a_6 = a_5 + (-1)^6 a_4 = 1 + 1 = 2$ (Ans.)

ii) $a_2 = 2a_1 + a_0^2 = 2 \times 0 + (-1)^2 = 1$

$a_3 = 3a_2 + a_1^2 = 3 \times 1 + 0 = 3$

$a_4 = 4a_3 + a_2^2 = 4 \times 3 + (1)^2 = 13$

$a_5 = 5a_4 + a_3^2 = 5 \times 13 + (3)^2 = 74$

$a_6 = 6a_5 + a_4^2 = 6 \times 74 + (13)^2 = 444 + 169 = 613.$ (Ans.)

Lecture- 23 (12-3-19)

Topic- 4.3. Countable and Uncountable sets

✓ A finite set is always said to be countable, because the elements can be indexed or numbered in the way they are described.

✓ An infinite set is not always countable. It is countable, if there is a bijection to it from N or vice-versa, that means, if its elements can be indexed.

✓ Examples of countable Infinite sets:

1) $\{1, 3, 5, 7, \ldots\} = O \rightarrow$ (English O) 13

$\{0, 1, 2, 3, \ldots\} = N$

$f: N \rightarrow O$

$f(n) = 2n+1$

$\therefore$ I $\wedge$ S.

$f^{-1} = ? \Rightarrow f^{-1}: O \rightarrow N$

$\therefore f^{-1}(x) = \frac{x-1}{2}$

2) $\{0, 2, 4, 6, \ldots\} = E$

$\{0, 1, 2, 3, \ldots\} = N$

$f: N \to E$

$f(n) = 2n$

$\therefore$ I $\wedge$ S

$f^{-1} = ? \Rightarrow f^{-1}: E \to N$

$\therefore f^{-1}(x) = \frac{x}{2}$

3) $\{0, 1, -1, 2, -2, \ldots\} = Z$

$\{0, 1, 2, 3, \ldots\} = N$

$f: N \to Z$

$f(n) = -n/2$, for $n \in E$

$= (n+1)/2$, for $n \in O$

$\therefore$ I $\wedge$ S

$f^{-1} = ? \Rightarrow f^{-1}: Z \to N$

$\therefore f^{-1}(x) = -\frac{2x}{\cancel{2}}$, for ~~n ∈ E~~ $x \in Z^-$

$= 2x - 1$, for $x \in Z^+$

4) $\Sigma = \{a, b\}$

$\Sigma^* = \{\varepsilon, a, b, aa, ab, ba, bb, aaa, \ldots\}$

$N = \{0, 1, 2, 3, 4, 5, 6, 7, \ldots\}$

$f: N \to \Sigma^*$, $f(n) = ?$ [Try using GN]

✓ Examples of uncountable sets:

1) $P(N) = 2^N = \ldots$, can't be indexed.

- Power set of an infinite set is uncountable.
- $P(\Sigma^*)$, that is, 2^{Σ^*}, for a nonempty Σ, is uncountable.
- Languages are uncountable, but computers (Turing Machines) are countable.

2) $\{ x \mid x \in R \wedge x > 1.0001 \wedge x < 1.0002 \}$

✓ Can we index prime numbers?

- Primality check of positive integer n:

  Divisibility by 2, 3, 5, 7, ... m (prime), such that $m \leq sqrt(n)$.

  Not efficient; This is a field of active research in Cryptography, Network Security, etc.

- $P_1 \times P_2 \times P_3 \times P_4 \times \cdots \times P_m + 1$ is always a prime, where $P_1 = 2$, $P_2 = 3$, . . .

  [otherwise, $P_1 \times P_2 \times P_3 \times P_4 \times \cdots \times P_m$ and 1 have common factors other than 1 !!!]

- A new prime can be found using a new known sequence, $P_1, P_2, P_3, P_4, \ldots, P_m$. So, there are infinitely many primes!

- Not all primes can be written as $P_1 \times P_2 \times P_3 \times P_4 \times \cdots \times P_m + 1$.

  For examples: 5, 11, 13, 17, 19, 23, 29, . . .

  * There is always a scope of discovering a new prime !!!

* No general formula yet for primes!!!

* So, the question remains open!!!

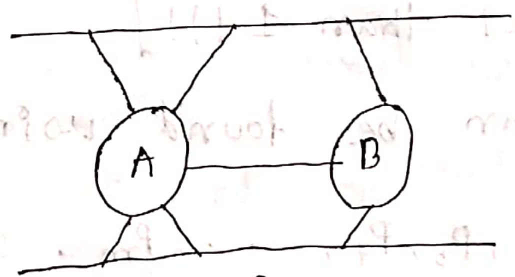
Lecture- 24 (19-3-19)

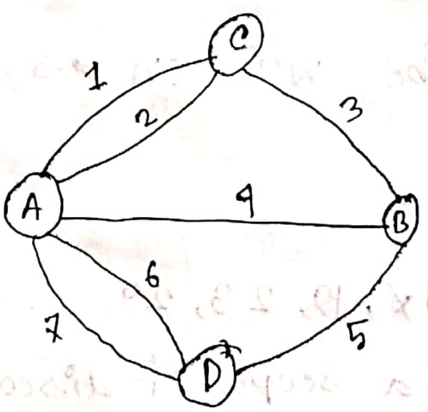
Chapter-5. Graph Theory

Topic 5.1. Introduction

✓ Graphs - very widely used data structure for problem representation and decision making.

✓ königsburg 7- bridges problem, Euler and Birth of a new science; Graph Theory.





Vertex / vertices / Nodes

$\{A, B, C, D\} = V$

Edge / Edges

$\{1, 2, 3, 4, 5, 6, 7\} = E$

(V, E)

Degree of $A = 5$, $D = 3$

✓ Modeling computer networks, Transport routes, chemical structures, . . . , object relationships; All NP-Complete problems.

* Basic Concepts:

1. Graph/ Undirected Graph:

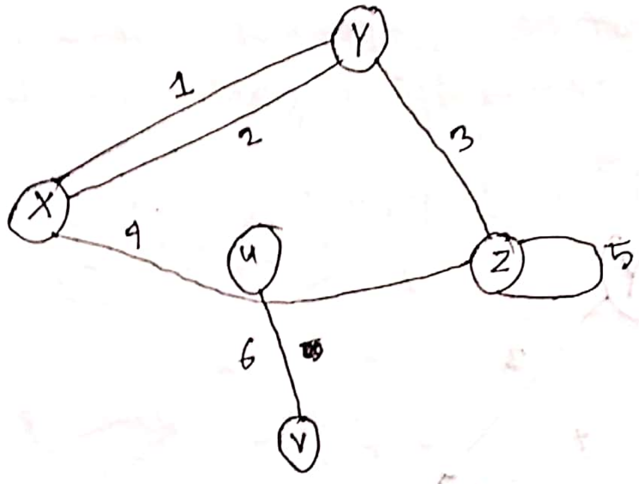


Fig: [G_1]

$G_1 = (V_1, E_1)$,

$V_1 = \{x, y, z, u, v\}$,

$E_1 = \{1, 2, 3, 4, 5, 6\}$

5 → a loop

1,2 → Parallel/multiple edges

$(\{u,v\}, \{6\})$ → an isolated subgraph

$(\{x,y,z\},\{1,2,3,4,5\})$ → another " "

• Set of vertices must be nonempty.

• If no loops, no multiple edges and no isolated subgraph, then it is called a simple graph.

• An edge has two ends.

• Edges meet only at vertices.

2) Directed Graph:

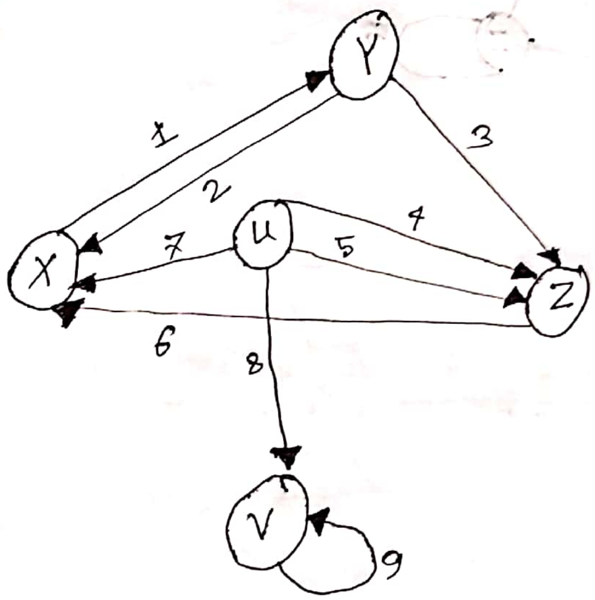


Fig-[G_2]

$G_2 = (V_2, E_2)$,

$V_2 = \{x, y, z, u, v\}$,

$E_2 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

9 → a loop

4, 5 → Parallel / multiple edges

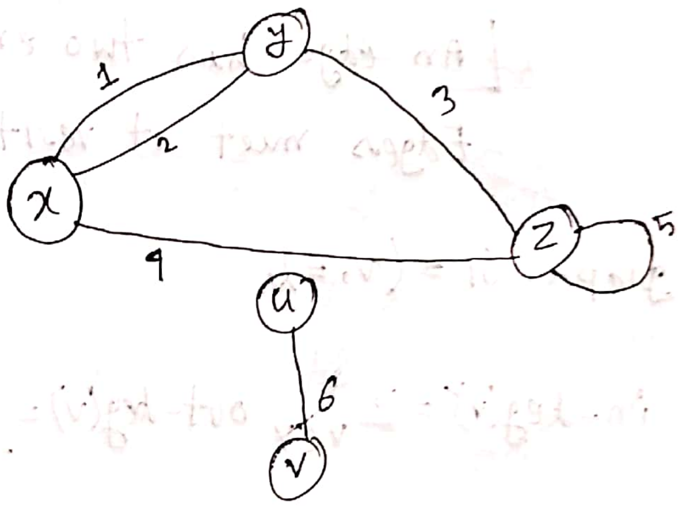
• Edges are 'directed' or 'pointed' to' vertices

• An edge <u>originates</u> at a vertex in one end and <u>terminates</u> at a vertex in another end.

• If no loops, no multiple edges and no isolated subgraph, then it is called a <u>simple directed graph</u>.

3] <u>Adjacent vertices/neighbors:</u>

[v is adjacent to u in G_2, not vice-versa]



[G_1]

[G_2]

4] Degree of a vertex, $deg(x) = 3$ in G_1

[Number of edges incident with]

5] In-degree (in-deg) and out-degree (out-deg)

6] Disconnected graph – with isolated subgraph $\rightarrow [G_1]$

7] <u>The Handshaking Theorem:</u>

✓ For an undirected graph, $G = (V, E)$,

$$\cancel{\sum_{v} t}$$

$$\sum_{v \in V} deg(v) = 2|E| = 2e$$

[An edge has two ends; Edges meet at vertices.]

✓ For a directed graph, $G = (V, E)$,

$$\sum_{v \in V} \text{in-deg}(v) = \sum_{v \in V} \text{out-deg}(v) = |E| = e$$

and, $$\sum_{v \in V} deg(v) = \sum_{v \in V} (\text{in-deg}(v) + \text{out-deg}(v)) = 2|E| = 2e$$

Lecture-25 (24-3-19)

Topic 5.2. Special Types of Graphs

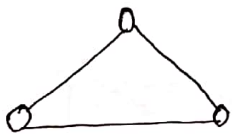
1.| Complete Graphs :

Exactly one edge connect any two vertices.

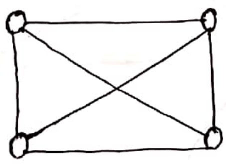
$K_2, K_3, K_4, K_5, \ldots$

K_2 :

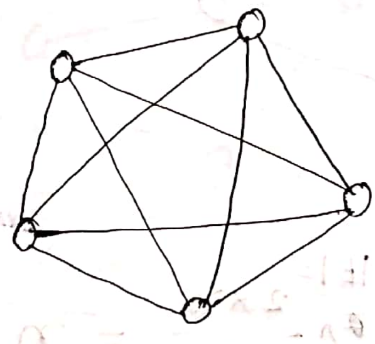
K_3 :



K_4 :



K_5 :



⋮

$G = (V, E)$

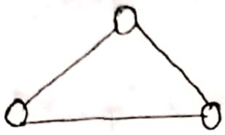
No. of edges of K_n: $|E| = (n-1)n/2$

2] Cycle Graphs/cycles:

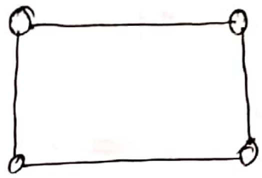
only one simple cycle or circuit.

$C_3, C_4, C_5, \ldots$

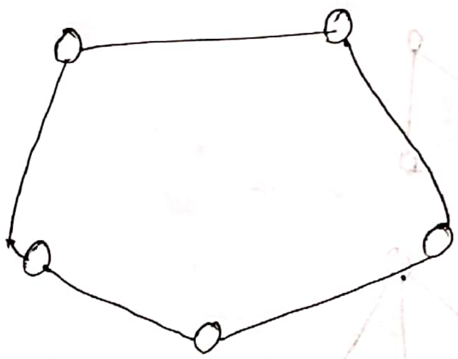
C_3 :



C_4 :



C_5 :



$\vdots$

No. of edges of C_n: $|E| = \frac{2 \times n}{2} = n$.

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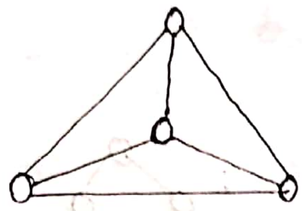
3) Wheels :

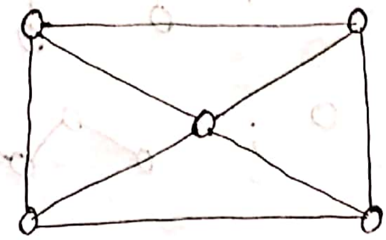
cycle with one additional vertex that is adjacent to all other.

$W_3, W_4, W_5, \ldots$

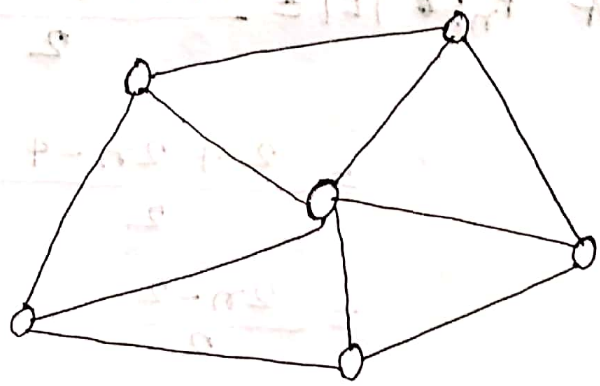
W_3 :



W_4 :



W_5 :



⋮

No. of edges of W_n : $|E| = \frac{3n+n}{2} = 2n$.

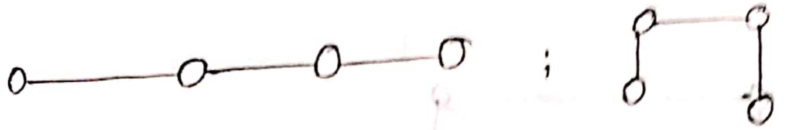
4) Paths/ Path Graphs:

Incomplete simple connected circuit; single path

P_2 , P_3 , P_4 , P_5 , . . .

P_2 : 

P_3 : 

P_4 : 

P_5 : 

No. of edges of P_n: $|E| = \frac{1+1+(n-2)\times 2}{2}$

$$= \frac{2+2n-4}{2}$$

$$= \frac{2n-2}{2}$$

$$= n-1 .$$

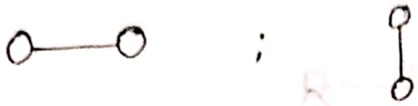
5] Trees:

$T_1, T_2, T_3, T_{4,[a]}, T_{4,[b]}, T_{5,[a]}, T_{5,[b]}, T_{5,[c]}, \ldots$

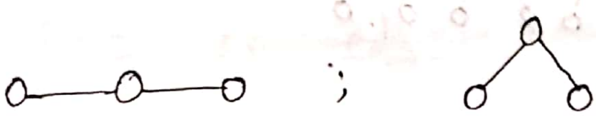
$T_{4[a]}$,

T_1 :

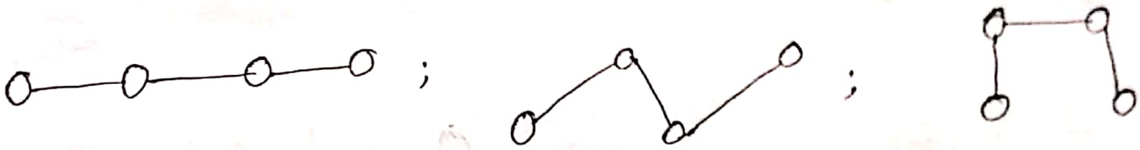
T_2 :



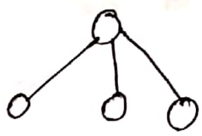
T_3 :



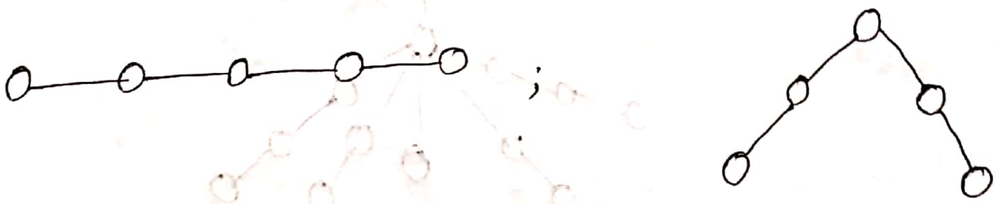
$T_{4,1}$:



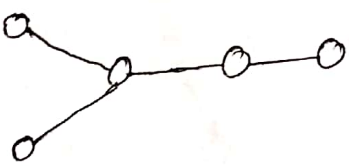
$T_{4,2}$:



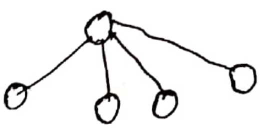
$T_{5,1}$:



$T_{5,2}$:



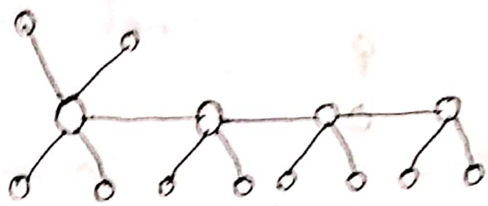
$T_{5,3}$:



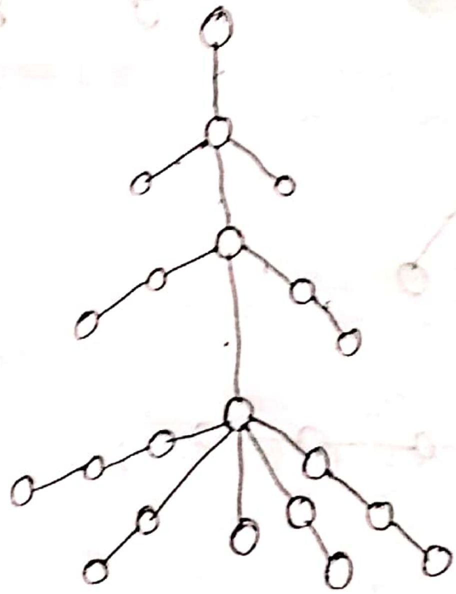
Caterpillar, christmas Tree, . . .

✓ A tree is a simple graph with no circuits /cycles.

Caterpillar:



Christmas Tree:



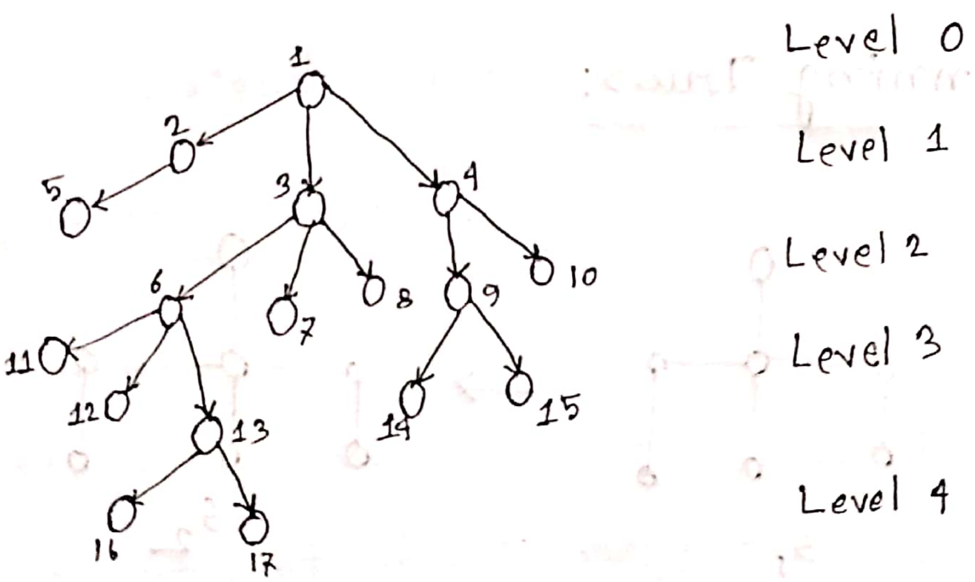
* Quiz - 4 (7-4-19, রবিবার)

Lecture – 26 (28-3-19)

✓ **Properties of trees:**

- T_n has $n-1$ edges;
- There exists a unique path between any two vertices;
- Deleting an edge results in a disconnected graph, which is a 'Forest' with exactly two trees.
- A new edge between any two vertices results in a graph with one circuit.

* **Rooted Trees:**



Parent-child: 1-2 , 1-3, 1-4, . .

Ancestor of 16: 13, 6, 3, 1

Root : 1 ; Taken to have no ancestors;

Descendent of 4: 9, 10, 14, 15

Descendent of 1: 2 - 17

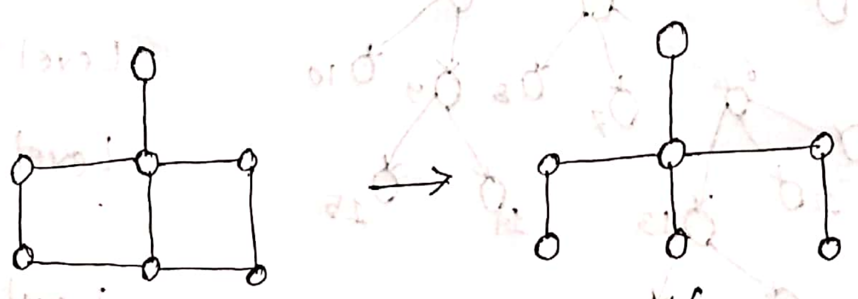
Siblings: 2, 3, 4; 6, 7, 8; 9, 10; . . .

- Leaves; Internal vertices; Branching factor: 3; maximum no of children.

  Binary trees; ordered (left to right) children or subtrees.

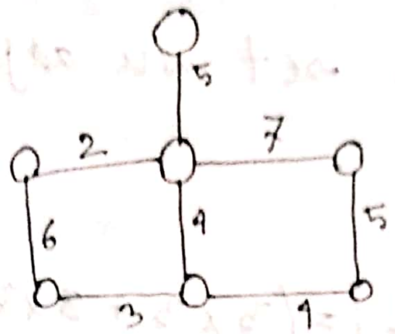
(BT: Branching factor: 2)

* Spanning Trees:

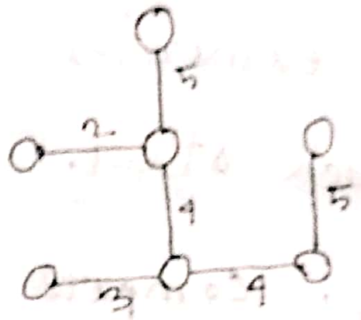


S_1 S_2

- $V(S_2) = V(S_1)$
  $E(S_2) \subseteq E(S_1)$ and S_2 is a tree;
- S_2 is a spanning tree of graph S_1.



S_3



S_4

- S_3 : weighted graph
- S_4 : Possibly a minimum weight spanning tree of graph S_3

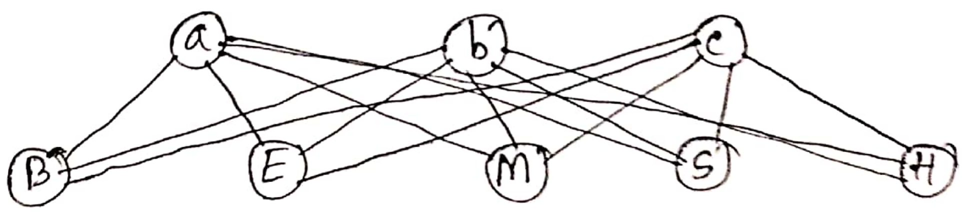
- Various algorithms are there to obtain a minimum-weight spanning tree.

6) <u>Bipartite Graphs:</u>

- Example 1:

Say, a, b, c - students; B, E, M, S, H - courses.

Each Student takes every course.



- A partition of V contains 2 sets, where no two elements of the same set are adjacent to each other.

- $K_{3,5}$: complete bipartite ; $|E| = (5\times3 + 3\times5)/2$
$= 3\times5$.

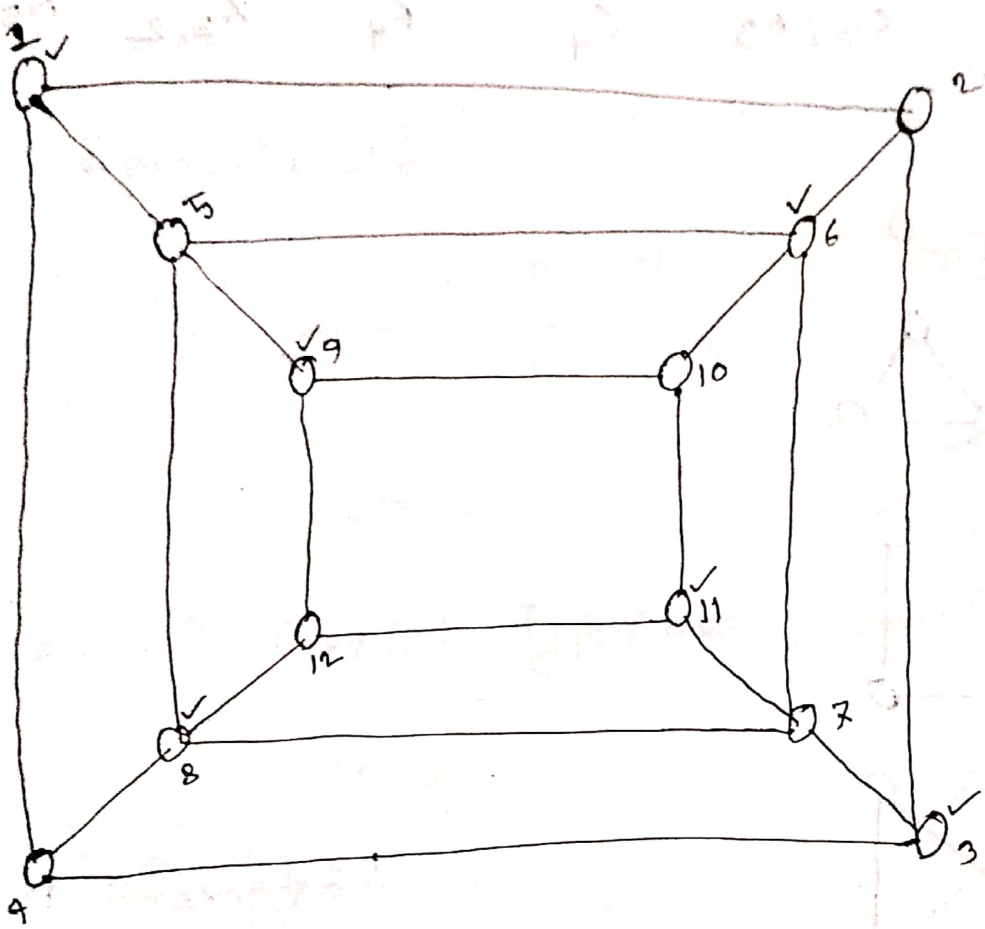
- To determine whether a given graph is bipartite or not, coloring alternate vertex with the same color may be applied.

Lecture-27 (31-3-19)

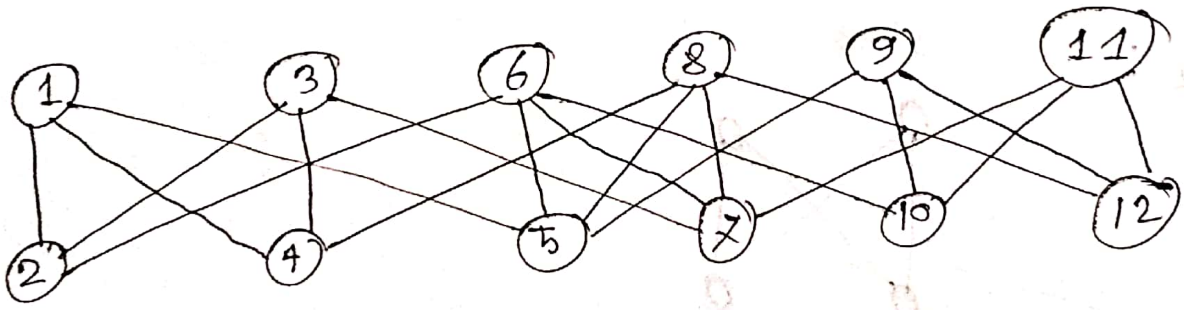
- Example-2:

3 concentric rectangles, having the corners of one connected to the corresponding corners of inner rectangles.

*After coloring, it is found that a partition of V contains 2 sets, where no two elements of the same set are adjacent to each other.



✓ → Red colour



Bipartite, but not complete :

$V = \{1, 2, 3, \ldots, 12\}$

$V_1 = \{1, 3, 6, 8, 9, 11\}$

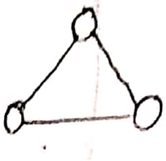
$V_2 = \{2, 4, 5, 7, 10, 12\}$

7| Regular graphs:

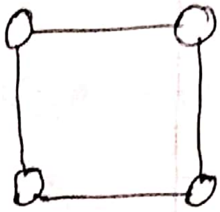
P_2/T_2 $\quad C_3/K_3$ $\quad C_4$ $\quad K_4$ $\quad K_{2,2}$ $\quad K_{3,3}$ ...

P_2/T_2 :

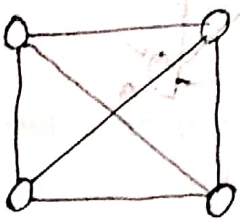
C_3/K_3 :



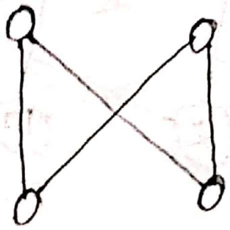
C_4 :



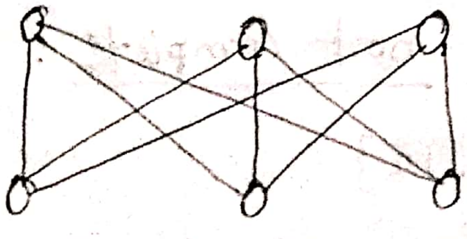
K_4 :



$K_{2,2}$:



$K_{3,3}$:



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All the vertices are of the same degree;

Degree $= m$, vertices $= n$: m-regular with n vertices,

$R_{m,n}$; $|E| = m \times n/2$;

m or n must be even!

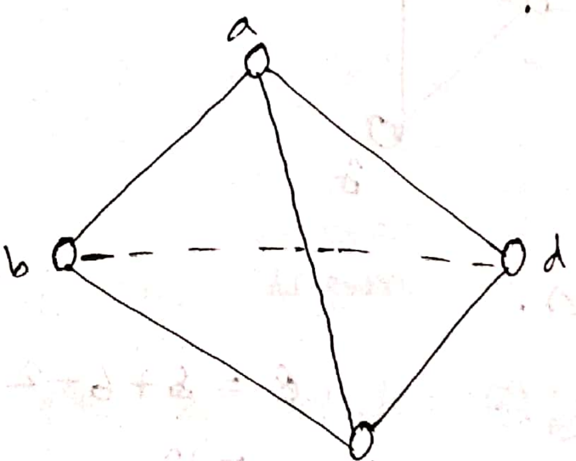
($R_{5,7}$, $R_{3,7}$ not possible)

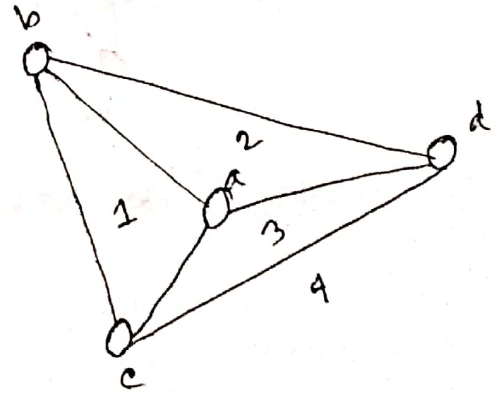
Topic 5.2. Special Types of Graphs (continued)

8) Planar graphs:

Examples:

- Tetrahedron - 4 faces, including an infinite face.





4 faces

4 - infinite face.

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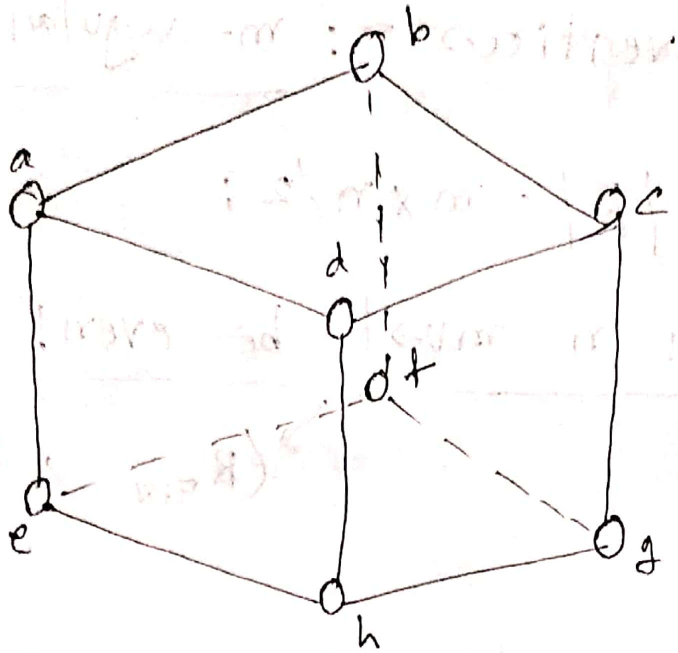
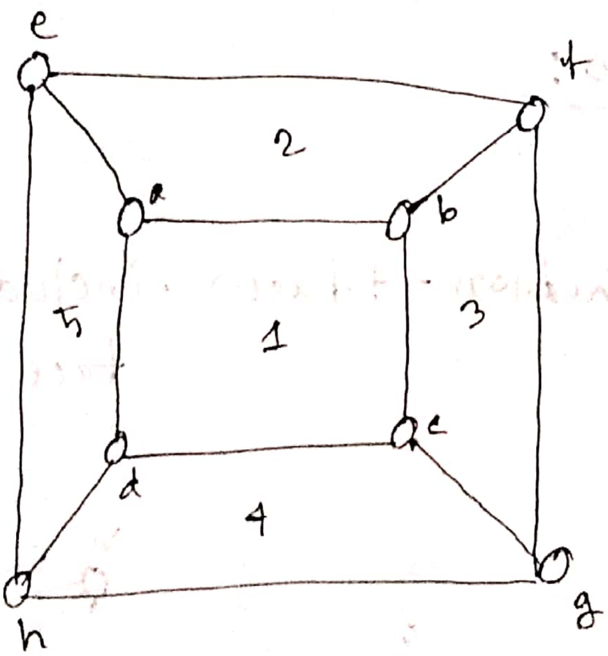
• Cube - 6 faces.





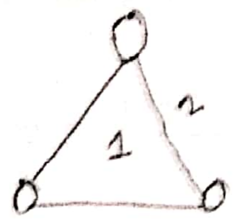
6 (→ infinite face)

6 faces.

$e = 8 + 6 - 2$

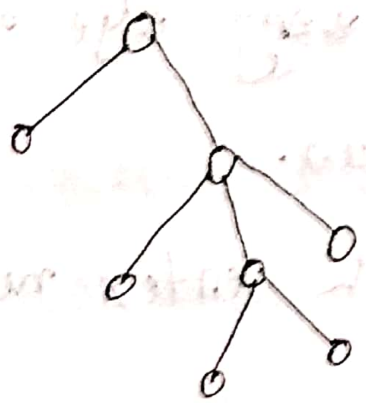
$= 12$

- C_3 - 2 faces



2 faces.

- Any tree - 1 face



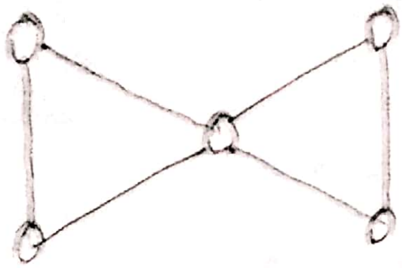
1 face.

$e = 8 + 1 - 2 = 7 \longrightarrow$ or, $(n-1)$

✓ Can be drawn on a 2-D plane with crossing at a vertex only.

✓ Euler's formula for planar graphs: $e = v + f - 2$.

9] Eulerian Graphs:

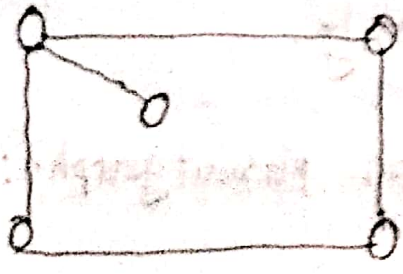


✓ Eulerian circuit/cycle: Round tour/trail/

walk that takes every edge once, and visits all the vertices.

Degree of each vertex needs to be even.

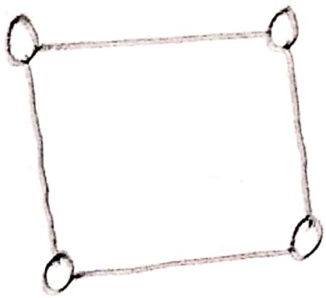
✓ Semi-Eulerian: The circuit is not complete, although each edge taken.

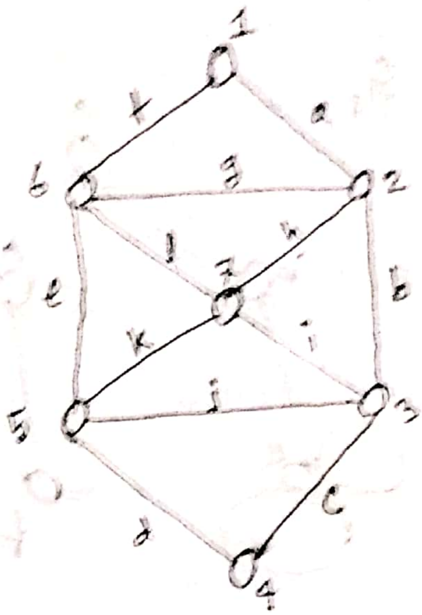


10| Hamiltonian Graphs:

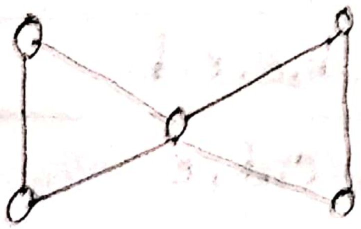
Hamiltonian circuit/cycle:

✓ Round Tour that takes each vertex once.





Both E & H.



E, but not H.

H: 1-a-2-h-7-i-3-c-4-d-5-e-6-f

E: 1-a-

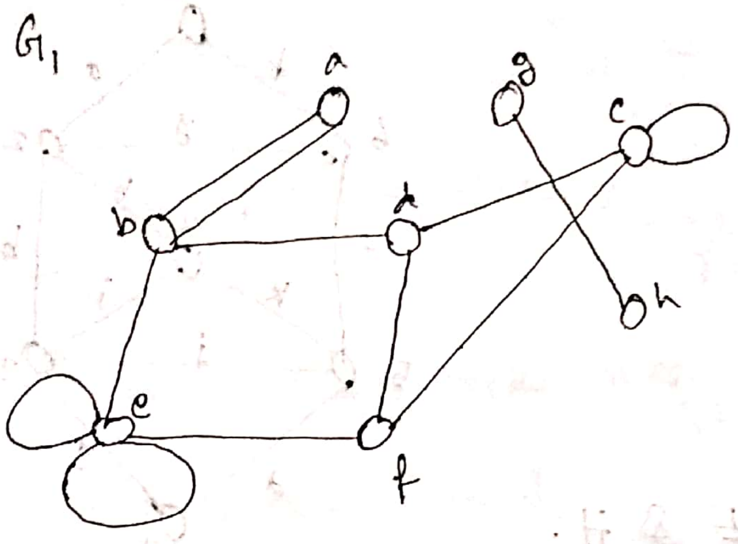
✓ **Semi-Hamiltonian:** cycle not complete, although each vertex visited.

But Eulerian.

Lecture - 28 (2-4-19)

Topic 5.3 Representation of Graphs in Computers

1| Representation using Adjacency Lists :



vertex	Adjacent vertices	vertex	Adjacent vertices
a	b, b	e	b, e, e, f
b	a, a, d, e	f	c, d, e
c	c, d, f	g	h
d	b, c, f	h	g

- Multiple edges between a & b
- Loops : 3
- $|E| = (21+3)/2 = 12$ (Adjacent vertices + 3 loops)
- Isolated subgraph involving g & h.

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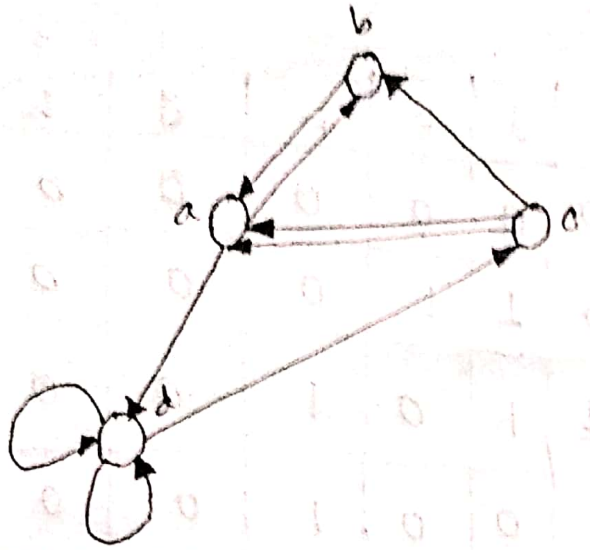
G_2



vertex	Adjacent vertices
a	b, d
b	a
c	a, a, b
d	c, d, d

- Σ in-deg $= 3+2+1+3 = 9$
- Σ out-deg $= 2+1+3+3 = 9$
- $|E| = 9$.

✓ Decisions from the lists: Loops, multiple edges, In-degree, out-degree, Number of edges, path/Tour/cycle between vertices, Isolated subgraph, etc.

✓ Appropriate algorithms are required.

2] Representation using Adjacency Matrices:

[G_1]

	a	b	c	d	e	f	g	h	
a	0	2	0	0	0	0	0	0	2
b	2	0	0	1	1	0	0	0	4
c	0	0	1	1	0	1	0	0	3-1 4
d	0	1	1	0	0	1	0	0	3
e	0	1	0	0	2	1	0	0	4-1 5
f	0	0	1	1	1	0	0	0	3
g	0	0	0	0	0	0	0	1	1
h	0	0	0	0	0	0	1	0	1
	2	4	3	3	4	3	1	1	

* diagonal entry (\) গুলোকে 2 দিয়ে গুণ করে যোগ করতে হবে (degree বের করার সময়)

✓ Diagonal entries: Number of loops.

✓ Simple graph: completely Boolean and symmetric.

✓ Other decisions based on sum of the entries in a row/column with special attention to diagonal entries.

[G2]

	a	b	c	d	out-degree
a	0	1	0	1	2
b	1	0	0	0	1
c	2	1	0	0	3
d	0	0	1	2	3
in-degree	3	2	1	3	

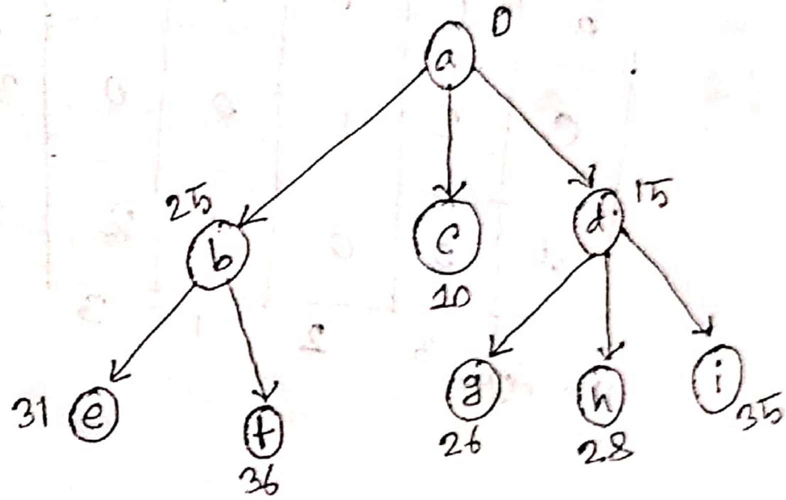
✓ Not symmetric

✓ Sum of the entries in a row : out-degree

✓ Sum of the entries in a column : in-degree.

Lecture-29 (9-4-19)

3| Representation of Rooted Trees :



- Node : State
- Characteristics: Evaluation function value

Possible node structure:

Name	Parent	Characteristics
a	Nil	0
b	a	25
c	a	10

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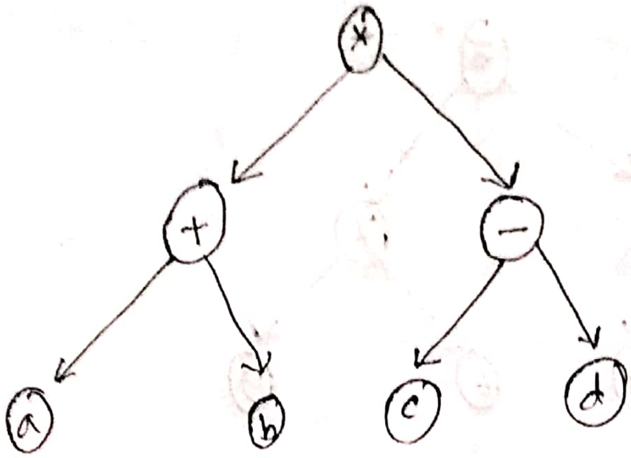
Topic 5.4: Tree Traversals and some use of Trees

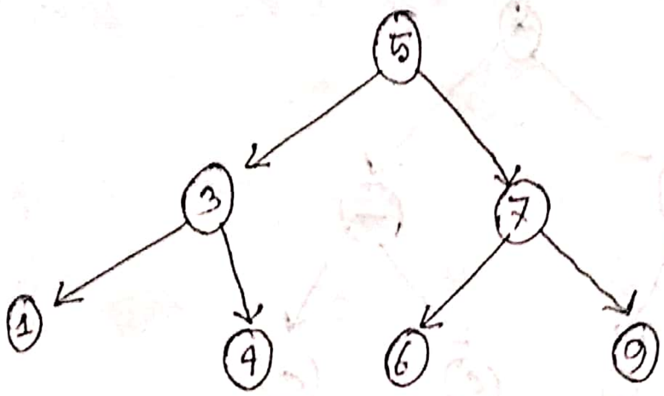
✓ We may have a binary tree to represent an arithmetic expression.



- In-order traversal yields [a + b * c − d].
- It stands for the infix notation
  (a+b) * (c−d).

- Pre-order traversal yields [* + a b − c d].
- It stands for the prefix notation
  *(+(ab) − (cd)).
- Post-order traversal yields [ab + cd − *].
- It stands for the postfix notation
  ((ab) + (cd) −) * .

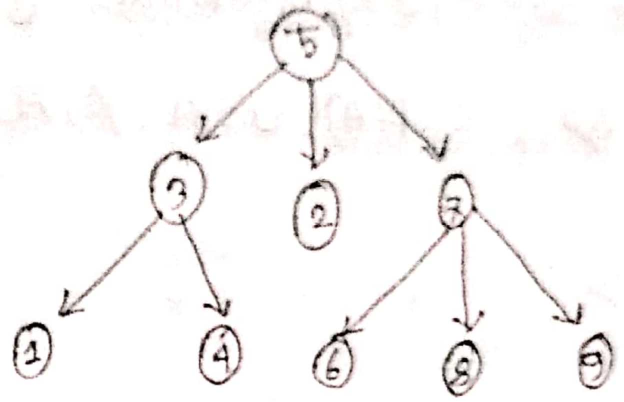
✓ We may also think of a 'balanced' binary search tree holding search keys in an orderly fashion.



- We may now think of searching for, say, 4 or 8.
- The search space is reduced to a great extent.
- Note that in-order traversal of the tree returns the keys in an order.

1 3 4 5 6 7 9

✓ We may think of a tree for visiting nodes in depth-first or breadth-first fashion to search for a given value.



- Depth-first strategy may yield 1, 4, 2, 6, 8, 9, 3, 7, 5.
- Breadth-first strategy may yield 5, 3, 2, 7, 1, 4, 6, 8, 9.

Lecture-30 (11-4-19)

Chapter-6. Algebraic Structures and the theory of Groups

Topic-6.1. Basic Concepts

✓ Algebra/ Algebraic structure/ Algebraic System

✓ Examples: $\langle Z, +, 0\rangle$, $\langle P(A), \cup, \cap, \phi, A\rangle$, $\langle R, +, \times, -, 0, 1\rangle$

Unary

* $\langle Z, +, 0\rangle$

$Z = \{0, 1, -1, 2, -2, \ldots\}$

$(-1) + 2 = 1$

$-1, 2 \in Z$

$1 \in Z$

$(-1) + 1 = 0$

$1 + (-1) = 0$

* $A = \{a, b, 3\}$

$P(A) = \{\phi, \{a\}, \{b\}, \{3\}, \{a, b\}, \{a, 3\}, \{b, 3\}, \{a, b, 3\}\}$

$\{a, b\} \cup \{a, 3\} = \{a, b, 3\}$

$\{a, b\} \cap \{a, 3\} = \{a\}$

$\{a, b\} \cup \phi = \{a, b\}$

$\{a, b\} \cap A = \{a, b\}$

$1.5 \times 1 = 1.5$

$1 \times 1.5 = 1.5$

$5 + (-5) = 0$

$-5 + 5 = 0$

✓ An algebra has

- a set, S, called the carrier of the algebra.
- one or more operations, o_i, such that

$$o_i : S^m \rightarrow S, \text{ for } m \in Z^+;$$

$$\langle Z, +, 0 \rangle$$

$$5 + 3 = 8$$

$$+ : Z^2 \rightarrow Z$$

$$Z^2 = Z \times Z$$

$$(5, 3) \in Z^2$$

$$8 \in Z$$

- usually, one or more constants, k_i, that are distinguished elements of S.

* And the carrier is closed under all the operations of the algebra.

✓ Signature of an algebra:

- Example: $\langle P(A), \cup, \cap, \emptyset, A \rangle$ and $\langle R, +, \times, 0, 1 \rangle$ have the same signature.
- Number and arity of the operations, along with the constants define the signature of algebra.

✓ subalgebra:

- Examples: $\langle N, +, 0 \rangle$ of $\langle Z, +, 0 \rangle$;

  $\langle Z, \times, 1 \rangle$ of $\langle R, \times, 1 \rangle$.

  $N \subseteq Z$, $Z \subseteq R$
- The carrier is a subset;
- The operations and constants are 'similar'

✓ Class of algebras

• Example:

Boolean algebras:

$\langle P(A), \cup, \cap, ', \emptyset, A \rangle$

$\langle \{F, T\}, \vee, \wedge, \neg, F, T \rangle$

• Have the same signature;

• The operations obey the same axioms or rules like commutativity, associativity, etc.

✓ Algebraic systems involving common sets of number:

* Natural number system:

• $\langle N, +, \times, 0, 1 \rangle$

• commutativity & Associativity of + & ×, and Distributivity of × over +;

• Both sided identity elements for + and ×.

$5 + 3 = 3 + 5$

$5 \times (3 + 4) = 5 \times 3 + 5 \times 4$

* System of integers:

- $\langle Z, +, \times, -, 0, 1 \rangle$
- commutativity & Associativity of + & $\times$, and distributivity of $\times$ over +;
- Both sided identity elements for + and $\times$;
- Both sided additive inverse elements.

* System of real number:

$\langle R, +, \times, -, {}^{-1}, 0, 1 \rangle$ [$^{-1}$ not defined for 0]

- commutativity & Associativity of + & $\times$, and distributivity of $\times$ over +;
- Both sided identity elements for + and $\times$;
- Both sided additive inverse element;

• Both sided multiplicative inverse element except of the number 0.