Prob - 1: Solve the boundary value problem $\frac{\partial U}{\partial t} = \frac{\partial}{\partial x^2}$
$(0,t) = 1$, $U(\pi, t) = 3$, $U(x,0) = 2$ where $0 < x < \pi$, $t > 0$.

Sol^n: The given partial differential equ^n is

$$\frac{\partial U}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \cdots\cdots (1)$$

Taking the finite Fourier sine transform of both sides of (1) we get,

$$\int_0^{\pi} \frac{\partial U}{\partial t} \sin \frac{n\pi x}{\pi}\, dx = \int_0^{\pi} \frac{\partial^2 U}{\partial x^2} \sin \frac{n\pi x}{\pi}\, dx$$

Ans $\Rightarrow$ $$\int_0^{\pi} \frac{\partial U}{\partial t} \sin nx\, dx = \int_0^{\pi} \frac{\partial^2 U}{\partial x^2} \sin nx\, dx \quad \cdots\cdots (2)$$

Let $u = u(n,t) = \int_0^{\pi} U(x,t) \sin nx\, dx$

then $$\frac{du}{dt} = \int_0^{\pi} \frac{\partial U}{\partial t} \sin nx\, dx$$

$$= \int_0^{\pi} \frac{\partial^2 U}{\partial x^2} \sin nx\, dx \quad [\text{using } (2)]$$

$$= \left[\sin nx \frac{\partial U}{\partial x}\right]_0^{\pi} - n\int_0^{\pi} \cos nx \frac{\partial U}{\partial x}\, dx \quad [\text{on integrating by parts}]$$

$$= 0 - n\left[\cos nx\, U(x,t)\right]_0^{\pi} - n^2 \int_0^{\pi} \sin nx\, U(x,t)\, dx$$

$$= -n\left[\cos n\pi\, U(\pi,t) - U(0,t)\right] - n^2 \int_0^{\pi} U(x,t) \sin nx\, dx$$

$W = 1$

$$\frac{dy}{dx} + P(x)y = Q(x)$$ (3)

Signature of the Invigilator

Additional Sheet

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$Sol^n: y \times IF$

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$= -n[3\cos n\pi - 1] - n^2 u$ [using boundary $cond^n$]

$$\therefore \frac{du}{dt} = n[1 - 3\cos n\pi] - n^2 u$$

$$\text{or, } \frac{du}{dt} + n^2 u = n[1 - 3\cos n\pi] \text{ ......(3)}$$

which is a linear differential eqn^n of first order.

$$\boxed{I.F. = e^{\int n^2 dt} = e^{n^2 t}}$$

Therefore solution of (3) is,

mult (3) by I.F.

$$u e^{n^2 t} = n(1 - 3\cos n\pi)\int e^{n^2 t} dt$$

$$= \frac{n(1 - 3\cos n\pi)}{n^2} e^{n^2 t} + A$$

$$= \frac{1 - 3\cos n\pi}{n} e^{n^2 t} + A$$

$$\text{or, } u = u(n,t) = \frac{1 - 3\cos n\pi}{n} + A e^{-n^2 t} \text{ ......(4)}$$

$$\text{When } t = 0,\ u(n,0) = \frac{1 - 3\cos n\pi}{n} + A \text{ ......(5)}$$

$$\text{Now, } u = u(n,t) = \int_0^{\pi} U(x,t)\sin nx\,dx$$

$$\therefore u(n,0) = \int_0^{\pi} U(x,0)\sin nx\,dx$$

~~$\frac{du}{dt}e^{n^2 t} + n^2 e^{n^2 t} u = e^{n^2 t}[1 - 3\cos n\pi]$~~

~~$\int \frac{d}{dt}(e^{n^2 t} u)\,dt = (1 - 3\cos n\pi)\int e^{n^2 t}$~~

$I.F = e^{n^2 t}$

$Sol^n: u \times IF$

3

$$= \int_0^{\pi} 2\sin nx\,dx = -\frac{2}{n}\left[\cos nx\right]_0^{\pi}$$

$$= -\frac{2}{n}(\cos n\pi - 1) = \frac{2}{n}(1-\cos n\pi)$$

Thus from (5), we get

$$\frac{2}{n}(1-\cos n\pi) = \frac{1-3\cos n\pi}{n} + A$$

$$\Rightarrow A = \frac{1}{n}(2-2\cos n\pi - 1 + 3\cos n\pi)$$

$$\Rightarrow A = \frac{1}{n}(1+\cos n\pi)$$

Putting the value of A in (4), we get

$$u = u(n,t) = \frac{1-3\cos n\pi}{n} + \frac{1}{n}(1+\cos n\pi)e^{-n^2 t}$$

Taking inverse finite Fourier sine transform we have

$$U(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-3\cos n\pi}{n}\sin\frac{n\pi x}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(1+\cos n\pi)e^{-n^2 t}\sin\frac{n\pi x}{\pi}.$$

$$\left[U(x,t) = \frac{2}{l}\sum u(n,t)\sin\frac{n\pi x}{l} \right]$$

$$\left[\because F(x) = \frac{2}{l}\sum_{n=1}^{\infty} f_s(n)\frac{\sin n\pi x}{l}\right]$$

$u(n,t)$ on

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-3\cos n\pi}{n}\sin nx + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(1+\cos n\pi)e^{-n^2 t}\sin nx$$

(Ans

$$\boxed{\text{(Note: If } u \text{ at } x=0 \text{ is given, take Fourier sine transform and if } \frac{\partial u}{\partial x} \text{ at } x=0 \text{ is given, use Fourier cosine transform)}}$$

⊛ Prob-3: Use finite Fourier transform to solve
$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$; $U(0,t)=0$; $U(\pi,t)=0$, $U(x,0)=2x$
$0<x<\pi$, $t>0$. Write the physical interpretation.

Solⁿ: The given partial differential eqⁿ is

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \cdots\cdots (1)$$

Taking the finite Fourier sine transform of both sides of (1), we get

$$\int_0^\pi \frac{\partial U}{\partial t} \sin nx\,dx = \int_0^\pi \frac{\partial^2 U}{\partial x^2} \sin nx\,dx \cdots\cdots (2)$$

Let, $u = u(n,t) = \int_0^\pi U(x,t) \sin nx\,dx$

then $\frac{du}{dt} = \int_0^\pi \frac{\partial U}{\partial t} \sin nx\,dx$

$$= \int_0^\pi \frac{\partial^2 U}{\partial x^2} \sin nx\,dx \quad [\text{using } (2)]$$

$$= \left[\sin nx \frac{\partial U}{\partial x}\right]_0^\pi - n\int_0^\pi \cos nx \cdot \frac{\partial U}{\partial x}\,dx$$

$$= 0 - n\int_0^\pi \cos nx \frac{\partial U}{\partial x}\,dx$$

$$= -n\left[\cos nx\, U(x,t)\right]_0^\pi - n^2\int_0^\pi \sin nx\, U(x,t)\,dx$$

$$= 0 - n^2\int_0^\pi U(x,t) \sin nx\,dx \text{ ; Since } U(\pi,t)=U$$

$$= -n^2 u \text{ ; since } u = \int_0^\pi U(x,t) \sin nx\,dx$$

$$\therefore \frac{du}{dt} = -n^2 u$$

$$\Rightarrow \frac{du}{u} = -n^2 dt$$

Integrating both sides we get,

$\log u = -n^2 t + \log A$; where A is any arbitrary const.

or, $\log u = \log e^{-n^2 t} + \log A = \log A e^{-n^2 t}$

$$\therefore u = A e^{-n^2 t} \quad \ldots\ldots (3)$$

Now, $u = u(n,t) = \int_0^{\pi} U(x,t) \sin nx \, dx$

$$\therefore u(n,0) = \int_0^{\pi} U(x,0) \sin nx \, dx$$

$$= \int_0^{\pi} 2x \sin nx \, dx \text{ ; Since } U(x,0) = 2x$$

$$= \left[2x \frac{-\cos nx}{n}\right]_0^{\pi} - (2)\left[\frac{-\sin nx}{n^2}\right]_0^{\pi}$$

$$= -\frac{2\pi}{n}\cos n\pi + 0 + \frac{2}{n^2}\left[\sin nx\right]_0^{\pi}$$

$$= -\frac{2\pi}{n}\cos n\pi \qquad \boxed{\therefore u(n,0) = -\frac{2\pi}{n}\cos n\pi}$$

Putting the value of A in (3) we get,

$$u(n,t) = u = -\frac{2\pi}{n}\cos n\pi \, e^{-n^2 t}$$

Applying the inversion formula for finite Fourier sine transform, we get

$\therefore\ u = A e^{-\frac{n^2\pi^2 t}{36}} \quad \ldots\ldots (3)$

When $t=0$, $u(n,0) = Ae^0 = A$

$$\therefore \boxed{A = u(n,0)} \quad \ldots\ldots (4)$$

Now, $u(n,t) = \int_0^6 U(x,t) \sin \frac{n\pi x}{6}\, dx$

$$\Rightarrow u(n,0) = \int_0^6 U(x,0) \sin \frac{n\pi x}{6}\, dx$$

$$= \int_0^3 U(x,0) \sin \frac{n\pi x}{6}\, dx + \int_3^6 U(x,0) \sin \frac{n\pi x}{6}\, dx$$

$$= \int_0^3 1 . \sin \frac{n\pi x}{6}\, dx + \int_3^6 0 . \sin \frac{n\pi x}{6}\, dx$$

$$= \int_0^3 \sin \frac{n\pi x}{6}\, dx = \left[- \frac{\cos \frac{n\pi x}{6}}{\frac{n\pi}{6}} \right]_0^3$$

$$= - \frac{6}{n\pi} \left[\cos \frac{n\pi}{2} - 1 \right]$$

$$= \frac{6}{n\pi} \left[1 - \cos \frac{n\pi}{2} \right]$$

Thus from (4), we have

$$A = \frac{6}{n\pi} \left(1 - \cos \frac{n\pi}{2} \right) \quad \ldots\ldots (5)$$

Putting the value of A in (3) we get

$$u(n,t) = \frac{6}{n\pi} \left(1 - \cos \frac{n\pi}{2} \right) e^{-\frac{n^2\pi^2 t}{36}}$$

$$U(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(-\frac{2\pi}{n}\cos n\pi\, e^{-n^2 t}\right)\sin nx.$$

For physical interpretation, $U(x,t)$ may be regarded as the temperature at any pt. x at an instant of time t in a solid bounded by the planes $x=0$ and $x=\pi$. The boundary conditions $U(0,t)=0$ and $U(\pi,t)=0$ give the zero temperature at the ends while $U(x,0)=2x$ represents that the initial temperature is a function of x.

H.W. ⊛ Prob-4: Solve $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$, $0<x<6$; $t>0$ subject to the conditions $U(0,t)=0$; $U(6,t)=0$; $U(x,0)=\begin{cases} 1, & 0<x<3 \\ 0, & 3<x<6 \end{cases}$ and interpret physically.

Solⁿ: The given partial differential eqnⁿ is

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \quad \cdots\cdots (1)$$

Taking the finite Fourier sine transform (with $l=6$) of both sides of (1) we get,

$$\int_0^6 \frac{\partial U}{\partial t}\sin\frac{n\pi x}{6}\,dx = \int_0^6 \frac{\partial^2 U}{\partial x^2}\sin\frac{n\pi x}{6}\,dx \quad \cdots\cdots (2)$$

Let, $u = u(n,t) = \int_0^6 U(x,t)\sin\frac{n\pi x}{6}\,dx$

ing the inverse Fourier sine transform we get

$$U(x,t) = \frac{2}{6}\sum_{n=1}^{\infty} \frac{6}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)e^{-\frac{n^2\pi^2 t}{36}} \sin\frac{n\pi x}{6}$$

$$\Rightarrow U(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left(1-\cos\frac{n\pi}{2}\right)e^{-\frac{n^2\pi^2 t}{36}} \sin\frac{n\pi x}{6}$$

Physical Interpretation: Physically $U(x,t)$ represents the temperature at any pt. x at any time t in a bar with the ends $x=0$ and $x=6$ kept at zero temperature which is insulated laterally. Initially the temperature in the half bar from $x=0$ to $x=3$ is constant equal to 1 unit while the half bar from $x=3$ to $x=6$ is at zero temperature.

⊛ Heat equation $\rightarrow$ $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

⊛ Prob. Solve the eqn $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ subject to the condn

(i) $u=0$ when $x=0,\ t>0$

(ii) $u = \begin{cases} 1, & 0<x<1 \\ 0, & x>1 \end{cases}$ when $t=0$

(iii) $u(x,t)$ is bounded.

(Note: If u at $x=0$ is given, take Fourier sine transform and if $\frac{\partial u}{\partial x}$ at $x=0$ is given, use Fourier cosine transform)

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Ex-2: Prove that the solution of the boundary value problem $\frac{\partial U}{\partial t} = 3\frac{\partial^2 U}{\partial x^2}$; $U(0,t) = U(2,t) = 0$, $t > 0$

$U(x,0) = x$, $0 < x < 2$ is $U(x,t) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin\frac{n\pi x}{2} e^{-\frac{3}{4}n^2\pi^2 t}$.

Proof: The given partial differential eqn is

$$\frac{\partial U}{\partial t} = 3\frac{\partial^2 U}{\partial x^2} \quad \cdots\cdots (1)$$

Taking the finite Fourier sine transform (with $l=2$) of both sides of (1), we get

$$\int_0^2 \frac{\partial U}{\partial t} \sin\frac{n\pi x}{2}\, dx = \int_0^2 3\frac{\partial^2 U}{\partial x^2} \sin\frac{n\pi x}{2}\, dx \quad \cdots\cdots (2)$$

$-2 < x < 2$

$T = 4$

$w = \frac{2\pi}{4}$

$= \pi/2$

Let, $u = u(n,t) = \int_0^2 U(x,t) \sin\frac{n\pi x}{2}\, dx$

then $\frac{du}{dt} = \int_0^2 \frac{\partial U}{\partial t} \sin\frac{n\pi x}{2}\, dx$

$$= \int_0^2 3\frac{\partial^2 U}{\partial x^2} \sin\frac{n\pi x}{2}\, dx \quad [\text{using } (2)]$$

$$= 3\left[\sin\frac{n\pi x}{2}\cdot\frac{\partial U}{\partial x}\right]_0^2 - \frac{3n\pi}{2}\int_0^2 \cos\frac{n\pi x}{2}\cdot\frac{\partial U}{\partial x}\, dx$$

$$= 0 - \frac{3n\pi}{2}\left[\cos\frac{n\pi x}{2}\cdot U(x,t)\right]_0^2 - \frac{3n^2\pi^2}{4}\int_0^2 \sin\frac{n\pi x}{2}\cdot U(x,t)\, dx$$

$$= 0 - \frac{3n\pi}{2}\left[\cos n\pi \cdot U(2,t) - U(0,t)\right] - \frac{3n^2\pi^2}{4}\int_0^2 \sin\frac{n\pi x}{2}\, U(x,t)\, dx$$

$$= 0 - \frac{3n^2\pi^2}{4}\int_0^2 U(x,t) \sin\frac{n\pi x}{2}\, dx \quad [\text{using boundary cond}^n]$$

$U(0,t) = U(2,t) =$

$n=1$

$$= \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin\frac{n\pi x}{2} \cdot e^{-\frac{3}{4}n^2\pi^2 t}$$

which is the required solⁿ.

(Ans.)