תרגיל 3

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1 שאלה 1

נתון שדה

$$\vec{E}=\begin{cases} r\le R & \left(\frac{\alpha}{r^2}+\beta r^2\right)\hat{r} \\ r>R & \frac{\alpha r+2\beta R^4}{r^2}\hat{r}\end{cases}$$

1.1 א

אזי מחשבים את צפיפות המטען:

$$\begin{aligned}
\frac{\rho}{\varepsilon_0} &= \vec{\nabla}\vec{E} \\
\frac{\rho}{\varepsilon_0} &= \begin{cases} r\le R & -2\frac{\alpha}{r^3}+2\beta r+\frac{2}{r}\left(\frac{\alpha}{r^2}+\beta r^2\right) \\ r>R & -2\left(\frac{\alpha+2\beta R^4}{r^3}\right)+\frac{2\alpha+4\beta R^4}{r^3}\end{cases} \\
\rho &= \begin{cases} r\le R & -2\frac{\alpha\varepsilon_0}{r^3}+2\beta\varepsilon_0 r+\frac{2\varepsilon_0}{r}\left(\frac{\alpha}{r^2}+\beta r^2\right) \\ r>R & -2\varepsilon_0\left(\frac{\alpha+2\beta R^4}{r^3}\right)+\frac{2\varepsilon_0\alpha+4\beta\varepsilon_0 R^4}{r^3}\end{cases}
\end{aligned}$$

1.2 ב

נבנה קליפה גאוסית כדורית סביב $(0,0,0)$ בעלת רדיוס l

$$\begin{aligned}
\lim_{l\to 0}\iint\vec{E}d\vec{s} &= \iiint\vec{\nabla}\vec{E}dV \\
&= \iiint\frac{\rho}{\varepsilon_0}dV \\
&\neq 0
\end{aligned}$$

לכן קיים מטען נקודתי בראשית

1.3 ג

$$\varphi=\int_{\infty}^{r}\vec{E}d\vec{x}=\begin{cases}r\leq R & -\left(-\frac{\alpha}{r}+\frac{\beta r^{3}}{3}\right)\\ r>R & \left(\frac{\alpha+2\beta R^{4}}{r}\right)+c\end{cases}$$

אזי צריך למצא את c. יודעים כי $\lim_{r\to R}\varphi\left(r\right)=\varphi\left(R\right)$ לכן

$$\begin{aligned}-\left(-\frac{\alpha}{R}+\frac{\beta R^{3}}{3}\right) &= \left(\frac{\alpha+2\beta R^{4}}{R}\right)+c\\ c &= -\left(-\frac{\alpha}{R}+\frac{\beta R^{3}}{3}\right)-\left(\frac{\alpha+2\beta R^{4}}{R}\right)\\ &= \frac{\alpha}{R}-\frac{\beta R^{3}}{3}-\frac{\alpha+2\beta R^{4}}{R}\\ &= \frac{3\alpha-\beta R^{4}-3\alpha-6\beta R^{4}}{3R}\\ &= \frac{-\beta R^{4}-6\beta R^{4}}{3R}\\ &= \frac{-7\beta R^{3}}{3}\end{aligned}$$

לכן

$$\varphi=\begin{cases}r\leq R & -\left(-\frac{\alpha}{r}+\frac{\beta r^{3}}{3}\right)\\ r>R & \left(\frac{\alpha+2\beta R^{4}}{r}\right)+\frac{-7\beta R^{3}}{3}\end{cases}$$

1.4 ד

$$\begin{aligned}\varphi\left(\frac{R}{2}\right) &= \frac{2\alpha}{R}-\frac{8\beta R^{3}}{3}\\ \varphi\left(2R\right) &= \frac{\alpha+2\beta R^{4}}{2R}-\frac{7\beta R^{3}}{3}\\ \frac{U}{q} &= \varphi\left(2R\right)-\varphi\left(\frac{R}{2}\right)\\ &= \frac{\alpha+2\beta R^{4}}{2R}-\frac{7\beta R^{3}}{3}-\left(\frac{2\alpha}{R}-\frac{8\beta R^{3}}{3}\right)\\ &= \frac{\alpha+2\beta R^{4}}{2R}-\frac{7\beta R^{3}}{3}-\frac{2\alpha}{R}+\frac{8\beta R^{3}}{3}\end{aligned}$$

2 שאלה 2

2.1 א

$$
\begin{aligned}
\varphi_q &= \frac{kq}{r}\\
\varphi_{dq} &= \frac{kdq}{r}\\
&= \frac{k\lambda\left(z\right)dz}{r}\\
&= \frac{ka\left|z\right|dz}{r}
\end{aligned}
$$

אזי כדי למצא הפוטנציאל סביב הטיל:

$$
\begin{aligned}
\varphi &= \int\limits_{-L}^{L}\frac{ka\left|z\right|}{\sqrt{r^2+z^2}}dz\\
&= 2ka\int\limits_{0}^{L}\frac{z}{\sqrt{r^2+z^2}}dz\\
&= 2ka\left(r-\sqrt{r^2+L^2}\right)
\end{aligned}
$$

2.2 ב

השדה בכיוון $\hat{r}$ שזה מאונך ל $\hat{z}$

2.3 ג

$$
\begin{aligned}
\vec{\nabla}\varphi &= \frac{\partial\varphi}{\partial r}\hat{r}\\
&= 2ka\left(1-\frac{r}{\sqrt{r^2+L^2}}\right)\hat{r}
\end{aligned}
$$

3 שאלה 3

3.1 א

$$
\begin{aligned}
\vec{f_1} &= \left(x^2y,2y+1,z\right)\\
\vec{\nabla}\vec{f_1} &= 2xy+3
\end{aligned}
$$

3.2 ב

$$
\begin{aligned}
\vec{f_2} &= \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}, z\right) \\
&= \left(\frac{1}{\sqrt{x^2+y^2}} \cdot \frac{x}{\sqrt{x^2+y^2}}, \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{y}{\sqrt{x^2+y^2}}, z\right) \\
&= \left(\frac{1}{r} \cdot \cos\phi, \frac{1}{r} \cdot \sin\phi, z\right)_{(x,y,z)} \\
&= \left(\sqrt{\left(\frac{\cos\phi}{r}\right)^2 + \left(\frac{\sin\phi}{r}\right)^2}, \arctan\left(\frac{\sin\phi}{\cos\phi}\right), z\right)_{(\rho,\phi,z)} \\
&= \left(\frac{1}{r}, \phi, z\right)_{(\rho,\phi,z)}
\end{aligned}
$$

$$
\begin{aligned}
\vec{\nabla}\vec{f_2} &= \frac{1}{r} \cdot \frac{\partial\left(r \cdot \frac{1}{r}\right)}{\partial r} + \frac{1}{r}\frac{\partial\phi}{\partial\phi} + \frac{\partial z}{\partial z} \\
&= 0 + \frac{1}{r} + 1
\end{aligned}
$$

3.3 ג

$$\begin{aligned}
\vec{f}_2 &= \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}, z\right) \\
&= \left(\frac{1}{\sqrt{x^2+y^2}} \cdot \frac{x}{\sqrt{x^2+y^2}}, \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{y}{\sqrt{x^2+y^2}}, z\right) \\
&= \left(\frac{1}{r\sin\theta} \cdot \frac{r\sin\theta\cos\phi}{r\sin\theta}, \frac{1}{r\sin\theta} \cdot \frac{r\sin\theta\sin\phi}{r\sin\theta}, r\cos\theta\right)_{(x,y,z)} \\
&= \left(\frac{\cos\phi}{r\sin\theta}, \frac{\sin\phi}{r\sin\theta}, r\cos\theta\right)_{(x,y,z)} \\
& \begin{pmatrix} \sqrt{\frac{\cos^2\phi}{r^2\sin^2\theta} + \frac{\sin^2\phi}{r^2\sin^2\theta} + r^2\cos^2\theta} \\ \arctan\left(\frac{\left(\frac{\sin\phi}{r\sin\theta}\right)}{\left(\frac{\cos\phi}{r\sin\theta}\right)}\right) \\ \arccos\left(\frac{r\cos\theta}{\sqrt{\frac{\cos^2\phi}{r^2\sin^2\theta} + \frac{\sin^2\phi}{r^2\sin^2\theta} + r^2\cos^2\theta}}\right) \end{pmatrix}_{(r,\theta\phi)} \\
&= \left(\sqrt{\frac{1}{r^2\sin^2\theta} + r^2\cos^2\theta}, \arctan\left(\frac{\sin\phi}{\cos\phi}\right), \arccos\left(\frac{r\cos\theta}{\sqrt{\frac{1}{r^2\sin^2\theta} + r^2\cos^2\theta}}\right)\right) \\
&= \left(\sqrt{\frac{1+r^3\sin^2\theta\cos^2\theta}{r^2\sin^2\theta}}, \phi, \frac{r\cos\theta}{\sqrt{\frac{1+r^3\sin^2\theta\cos^2\theta}{r^2\sin^2\theta}}}\right)
\end{aligned}$$

אזי

$$\begin{aligned}
\vec{\nabla} \cdot \vec{f}_3 &= \frac{1}{r^2}\frac{\partial\left(r^2 A_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \\
&= \frac{1}{r^2}\frac{\partial\left(\sqrt{\frac{1+r^3\sin^2\theta\cos^2\theta}{r\sin^2\theta}}\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial\left(\frac{r\cos\theta}{\sqrt{\frac{1+r^3\sin^2\theta\cos^2\theta}{r^2\sin^2\theta}}}\right)}{\partial\phi} \\
&= \frac{2\,r^3\,\cos^2 t\,\sin^2 t - 1}{2\,r^4\sqrt{\frac{r^3\,\cos^2 t\,\sin^2 t+1}{r}}\,\left|\sin t\right|} + \frac{1}{r\sin\theta}\left(1+\theta\cos\theta\right) + \frac{1}{r\sin\theta}\cdot 0 \\
&= \frac{2\,r^3\,\cos^2 t\,\sin^2 t - 1}{2\,r^4\sqrt{\frac{r^3\,\cos^2 t\,\sin^2 t+1}{r}}\,\left|\sin t\right|} + \frac{1}{r\sin\theta}\left(1+\theta\cos\theta\right)
\end{aligned}$$