תרגיל מס.2

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שאלה 1 1

$$f(x)=\begin{cases}\frac{\cos(x)}{\sqrt{2}} & a<x<b\\ 0 & otherwise\end{cases}$$

נתון:

$$\begin{aligned}\int_{-\infty}^{\infty}f(x)\,dx &= 1\\ P\left(x>\frac{\pi}{6}\right) &= \frac{2-\sqrt{2}}{4}\end{aligned}$$

$$\begin{aligned}\frac{1}{\sqrt{2}}\int_{a}^{b}\cos(x)\,dx &= 1\\ \frac{1}{\sqrt{2}}\left(\sin(b)-\sin(a)\right) &= 1\\ \sin(b)-\sin(a) &= \sqrt{2}\end{aligned}$$

$$\begin{aligned}
P\left(x > \frac{\pi}{6}\right) &= \frac{2-\sqrt{2}}{4} \\
1 - P\left(x < \frac{\pi}{6}\right) &= \frac{2-\sqrt{2}}{4} \\
1 - \int_a^{\pi/6} f(x)\,dx &= \frac{2-\sqrt{2}}{4} \\
\frac{1}{\sqrt{2}}\left(\sin\left(\frac{\pi}{6}\right) - \sin(a)\right) &= 1 - \frac{2-\sqrt{2}}{4} \\
\sin\left(\frac{\pi}{6}\right) - \sin(a) &= \frac{2+\sqrt{2}}{4}\sqrt{2} \\
-\sin(a) &= \frac{2\sqrt{2}+2}{4} - \frac{1}{2} \\
\sin(a) &= -\frac{\sqrt{2}}{2} \\
a &= \arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}
\end{aligned}$$

$$\begin{aligned}
\sin(b) - -\frac{\pi}{4} &= \sqrt{2} \\
\sin(b) &= \sqrt{2} - \frac{\pi}{4} \\
b &= \arcsin\left(\sqrt{2} - \frac{\pi}{4}\right)
\end{aligned}$$

ב 1.1

$$\begin{aligned}
E[x] &= \int_a^b x f(x)\,dx \\
&= \frac{1}{\sqrt{2}} \int_{-\frac{\pi}{4}}^{\arcsin\left(\sqrt{2}-\frac{\pi}{4}\right)} x\cos(x)\,dx \\
&= \frac{1}{\sqrt{2}} \left[(x\sin(x) + \cos(x))\right]_{-\frac{\pi}{4}}^{\arcsin\left(\sqrt{2}-\frac{\pi}{4}\right)} \\
&= -0.0405 \\
E\left[x^2\right] &= \int_a^b x^2 f(x)\,dx \\
&= \frac{1}{\sqrt{2}} \int_{-\frac{\pi}{4}}^{\arcsin\left(\sqrt{2}-\frac{\pi}{4}\right)} x^2\cos(x)\,dx \\
&= 0.1579 \\
\mathrm{var}x &= E\left[x^2\right] - E^2[x] \\
&= 0.156
\end{aligned}$$

שאלה 2 2

$$f(x)=\begin{cases}ax+bx^2 & 0<x<1\\ 0 & otherwise\end{cases}$$

נתון $E[x]=0.6$

$$\begin{aligned}\int_{-\infty}^{\infty}f(x)\,dx &= 1\\ \left[\frac{ax^2}{2}+\frac{bx^3}{3}\right]_0^1 &= 1\\ \frac{a}{2}+\frac{b}{3} &= 1\\ a &= 2-\frac{2b}{3}\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{\infty}xf(x)\,dx &= 0.6\\ \int_0^1 ax^2+bx^3dx &= 0.6\\ \left[\frac{ax^3}{3}+\frac{bx^4}{4}\right]_0^1 &= 0.6\\ \frac{a}{3}+\frac{b}{4} &= 0.6\\ a+\frac{3b}{4} &= 1.8\\ 2-\frac{2b}{3}+\frac{3b}{4} &= 1.8\\ \frac{b}{12} &= -0.2\\ b &= -2.4\\ a &= 3.6\end{aligned}$$

ב 2.1

$$\begin{aligned}E\left[x^2\right] &= \int_0^1 xf(x)\,dx\\ &= \int_0^1 ax^2+bx^3dx\\ &= 0.42\\ \operatorname{var}[x] &= E\left[x^2\right]-E[x]\\ &= 0.06\end{aligned}$$

שאלה 3 3

$$X \sim exp\left(\lambda\right)$$

נגדיר $Y = \{X|x > k\}$ ונחשב פונקצית ההתפלגות התצטברת:

$$\begin{aligned} F_Y\left(y\right) &= P\left(x < y|x > k\right) \\ &= \frac{F_x\left(y\right) - F_x\left(k\right)}{1 - F_x\left(k\right)} \\ &= \frac{1 - e^{-\lambda y} - \left(1 - e^{-\lambda k}\right)}{1 - \left(1 - e^{-\lambda k}\right)} \\ &= \frac{-e^{-\lambda y} + e^{-\lambda k}}{e^{-\lambda k}} \\ &= 1 - \frac{e^{-\lambda y}}{e^{-\lambda k}} \\ &= 1 - e^{-\lambda y + \lambda k} \\ &= 1 - e^{\lambda(k-y)} \end{aligned}$$

שאלה 4 4

$$E\left[x\right] = \int_0^\infty \left(1 - F\left(x\right)\right) dx$$

צ"ל כי

$$\int_0^\infty \left(1 - F\left(x\right)\right) dx = \int_0^\infty \int_x^\infty f\left(t\right) dtdx$$

יודעים כי

$$
\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(x')\,dx' \\
F(x) &= \int_{-\infty}^{x} f(x')\,dx' + \int_{x}^{\infty} f(x')\,dx' - \int_{x}^{\infty} f(x')\,dx' \\
F(x) &= \int_{-\infty}^{\infty} f(x')\,dx' - \int_{x}^{\infty} f(x')\,dx' \\
F(x) &= 1 - \int_{x}^{\infty} f(x')\,dx' \\
1 - F(x) &= \int_{x}^{\infty} f(x')\,dx' \\
\int_{0}^{\infty} 1 - F(x)\,dx &= \int_{0}^{\infty}\int_{x}^{\infty} f(x')\,dx'dx \\
&= \int_{0}^{\infty} x' f(x')\,dx' \\
&= E(x)
\end{aligned}
$$

משל

שאלה 5 5

מ"מ מעריכי

$$
\begin{aligned}
X &\sim exp(\lambda) \\
Y &= \sqrt[n]{X}, x > 0 \\
f_y(y) &= \frac{\partial}{\partial y} 1 - e^{-\lambda y^n} \\
&= e^{-\lambda y^n} \cdot \lambda n y^{n-1} \\
F_y(y) &= P(Y \le y) \\
&= P\left(\sqrt[n]{X} \le y\right) \\
&= P(X \le y^n) \\
&= 1 - e^{-\lambda y^n}
\end{aligned}
$$