תרגיל מס.3

עפיף חלומה 302323001

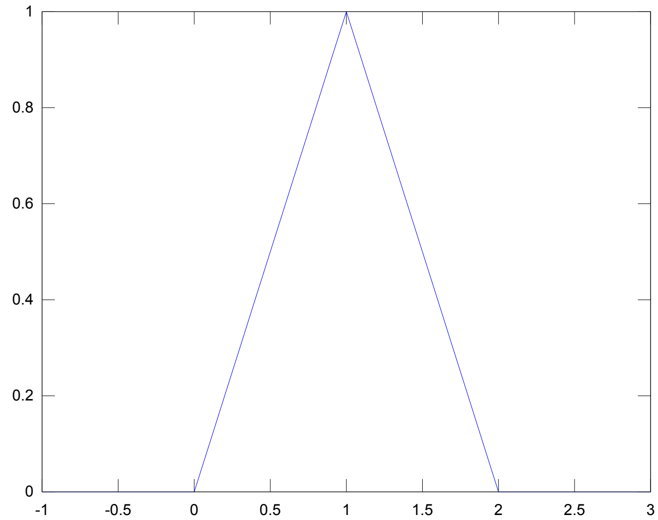
13 באפריל 2010

1 שאלה 1

$$
\begin{aligned}
X, Y &\sim \mathcal{U}(0,1) \\
Z &= X+Y \\
W &= X-Y
\end{aligned}
$$

1.1 א

$$
\begin{aligned}
f_Z &= f_x * f_y \\
&= \int_{-\infty}^{\infty} f_x(\tau) f_y(t-\tau) d\tau \\
&= \int_{-\infty}^{\infty} f_x(\tau) f_y(t-\tau) d\tau \\
&= \begin{cases} 0 & t \le 0 \\ \int_0^t 1dt & 0 < t < 1 \\ \int_{t-1}^1 1dt & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases} \\
&= \begin{cases} 0 & t \le 0 \\ t & 0 < t < 1 \\ 2-t & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases}
\end{aligned}
$$



איור 1: f_Z

1.2 ב

$$
\begin{aligned}
F_Z(t) &= \int_{-\infty}^{t} f_Z(\tau)\, d\tau \\
&= \begin{cases} 0 & t \le 0 \\ \int_0^t \tau d\tau & 0 < t < 1 \\ \int_0^1 \tau d\tau + \int_1^t (2-\tau)\, d\tau & 1 \le t < 2 \\ \int_0^1 \tau d\tau + \int_1^2 (2-\tau)\, d\tau & t \ge 2 \end{cases} \\
&= \begin{cases} 0 & t \le 0 \\ \frac{1}{2}t^2 & 0 < t < 1 \\ \frac{1}{2} + \left(-\frac{t^2}{2} + 2t - \frac{3}{2}\right) & 1 \le t < 2 \\ 1 & t \ge 2 \end{cases}
\end{aligned}
$$

1.3 ג

$$
\begin{aligned}
f_{W|Z}(b|a) &= f(X-Y=b|X+Y=a) \\
&= f(X-Y=b|X=a-Y) \\
&= f(a-Y-Y=b) \\
&= f(-2Y=b-a) \\
&= f\left(Y=\frac{a-b}{2}\right) \\
&= \begin{cases} 0 & \frac{a-b}{2}<0 \\ 1 & 0\le\frac{a-b}{2}\le 1 \\ 0 & \frac{a-b}{2}>1 \end{cases} \\
&= \begin{cases} 0 & a<b \\ 1 & 0\le a\le 2+b \\ 0 & a>2+b \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
f_{W|Z}(b|a) &= f(X-Y=b|X+Y=a) \\
&= f(X-Y=b|Y=a-X) \\
&= f(X-a+X=b) \\
&= f(2X=a+b) \\
&= f\left(X=\frac{a+b}{2}\right) \\
&= \begin{cases} 0 & \frac{a+b}{2}<0 \\ 1 & 0\le\frac{a+b}{2}\le 1 \\ 0 & \frac{a+b}{2}>1 \end{cases} \\
&= \begin{cases} 0 & a<-b \\ 1 & 0\le a\le 2-b \\ 0 & a>2+b \end{cases}
\end{aligned}
$$

1.4 ד

$$\begin{aligned} f_{Z,W}(a,b) &= f_{W|Z}(b,a)\cdot f_Z(a) \\ &= \left(\begin{cases} 0 & \frac{a-b}{2}<0 \\ 1 & 0\le\frac{a-b}{2}\le 1 \\ 0 & \frac{a-b}{2}>1 \end{cases}\right)\left(\begin{cases} 0 & a\le 0 \\ \frac{1}{2}a^2 & 0<a<1 \\ \frac{a^2}{2}-2a+2 & 1\le a<2 \\ 0 & a\ge 2 \end{cases}\right) \\ &= \begin{cases} \frac{1}{2}a^2 & (0<a<1)\,\&\&\left(0\le\frac{a-b}{2}\le 1\right) \\ \frac{a^2}{2}-2a+2 & (1\le a<2)\,\&\&\left(0\le\frac{a-b}{2}\le 1\right) \\ 0 & otherwise \end{cases} \\ &= \begin{cases} \frac{1}{2}a^2 & (0<a<1)\,\&\&\,(0\le a-b\le 2) \\ \frac{a^2}{2}-2a+2 & (1\le a<2)\,\&\&\,(0\le a-b\le 2) \\ 0 & otherwise \end{cases} \end{aligned}$$

1.5 ה

$$f_{Z,W}(a,b) = \begin{cases} \frac{1}{2}a^2 & (0<a<1)\,\&\&\,(0\le a-b\le 2) \\ \frac{a^2}{2}-2a+2 & (1\le a<2)\,\&\&\,(0\le a-b\le 2) \\ 0 & otherwise \end{cases}$$

2 שאלה 2

2.1 א

$$f_{X,Y}(a,b) = \begin{cases} 0.6\left(a^2+b^2+2b\right) & 0\le a,b\le 1 \\ 0 & otherwise \end{cases}$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} da \int_{-\infty}^{\infty} db \left(0.6\left(a^2+b^2+2b\right)\right) &\stackrel{?}{=} 1 \\
0.6\int_{0}^{1} da \int_{0}^{1} db \left(a^2+b^2+2b\right) &\stackrel{?}{=} 1 \\
0.6\int_{0}^{1} da \left(a^2 b+\frac{b^3}{3}+b^2\right)|_0^1 &\stackrel{?}{=} 1 \\
\frac{6}{10}\int_{0}^{1} da \left(a^2+\frac{1}{3}+1\right) &\stackrel{?}{=} 1 \\
\frac{3}{5}\left(\frac{a^3}{3}+\frac{1}{3}a+a\right)|_0^1 &\stackrel{?}{=} 1 \\
\frac{3}{5}\left(\frac{5}{3}\right) &\stackrel{?}{=} 1 \\
1 &\stackrel{\checkmark}{=} 1
\end{aligned}
$$

נניח כי קיימים a, b כך ש $f_{x,y}\left(a, b\right) < 0$ אזי

$$
\begin{aligned}
0.6\left(a^2+b^2+2b\right) &< 0 \\
a^2+b^2+2b &< 0 \\
a^2+\left(b+1\right)^2-1 &< 0 \\
\left(b+1\right)^2-1 < a^2+\left(b+1\right)^2-1 &< 0 \\
\left(b+1\right)^2-1 &< 0 \\
\left(b+1\right)^2 &< 1
\end{aligned}
$$

ניתן להזניח את a^2 כי זה אף פעם לא יהיה שלילי. אזי יש שתי תנאים:

$$
\begin{array}{ccccc}
-1 & < & b+1 & < & 1 \\
-2 & < & b & < & 0
\end{array}
$$

בסתירה לזה ש $0 < b < 1$

2.2 ב

$$\begin{aligned}
f_X(a) &= \int_{-\infty}^{\infty} f_{X,Y}(a,b)\,\partial b \\
&= \int_0^1 0.6\left(a^2+b^2+2b\right)db \\
&= \left(0.2a^3b+0.6ab^3+1.2ab^2\right)|_0^1 \\
&= 0.6a^2+0.8 \\
f_Y(b) &= \int_{-\infty}^{\infty} f_{X,Y}(a,b)\,\partial a \\
&= \int_0^1 0.6\left(a^2+b^2+2b\right)da \\
&= \left(0.15a^4+0.3a^2b^2+0.6a^2b\right)|_0^1 \\
&= 0.6b^2+1.2b+0.2
\end{aligned}$$