תרגיל מס.4

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1 שאלה 1

1.1 א

$$
\begin{aligned}
Y &= g(X) = \frac{1}{X} \\
X &= g'(Y) = \frac{1}{Y} \\
f_Y(y) &= f_x\left(g^{-1}(y)\right) \cdot \frac{\partial g^{-1}(y)}{\partial y} \\
&= \begin{cases} \frac{1}{\Gamma(\alpha)} \cdot \beta^{mga} \left(\frac{1}{y}\right)^{\alpha-1} e^{-\frac{\beta}{y}} \cdot \left(-\frac{1}{y^2}\right) & 0 < y < \infty \\ 0 & otherwise \end{cases}
\end{aligned}
$$

1.2 ב

$$
\begin{aligned}
E(y) &= -\int_{\infty}^{0} \frac{1}{\Gamma(\alpha)} \beta^{\alpha} \left(\frac{1}{y}\right)^{\alpha-1} e^{-\frac{\beta}{y}} \cdot y \cdot \frac{1}{y^2} dy \\
&= \int_{0}^{\infty} \frac{1}{\Gamma(\alpha)} \beta^{\alpha} \left(\frac{1}{y}\right)^{\alpha} e^{-\frac{\beta}{y}} \cdot dy \\
\xi &= \frac{1}{y} \\
dy &= -y^2 d\xi \\
E(y) &= -\int_{\infty}^{0} \frac{1}{\Gamma(\alpha)} \cdot \beta^{\alpha} \cdot \left(\frac{1}{y}\right)^{\alpha} e^{-\frac{\beta}{y}} dy \\
&= \int_{0}^{\infty} \frac{1}{\Gamma(\alpha)} \cdot \beta^{\alpha} \cdot (\xi)^{\alpha-2} e^{-\beta\xi} d\xi \\
&= \frac{\beta^{\alpha}\Gamma(\alpha-1)}{\beta^{\alpha-1}\Gamma(\alpha)} \underbrace{\int_{0}^{\infty} \frac{1}{\Gamma(\alpha-1)} \beta^{\alpha-1} \xi^{\alpha-2} e^{-\beta\xi} d\xi}_{1} \\
&= \frac{\beta\Gamma(\alpha-1)}{(\alpha-1)\Gamma(\alpha-1)} \\
&= \frac{\beta}{\alpha-1}
\end{aligned}
$$

1.3 ג

$$
\begin{aligned}
E\left(Y^2\right) &= -\int_{\infty}^{0} \frac{1}{\Gamma(\alpha)}\beta^{\alpha}\left(\frac{1}{y}\right)^{\alpha-1} e^{-\frac{\beta}{y}} \cdot y^2 \frac{1}{y^2} dy \\
&= \int_{0}^{\infty} \frac{1}{\Gamma(\alpha)}\beta^{\alpha}\left(\frac{1}{y}\right)^{\alpha-1} e^{-\frac{\beta}{y}} dy \\
&= -\int_{\infty}^{0} \frac{1}{\Gamma(\alpha)}\beta^{\alpha}\left(\frac{1}{y}\right)^{\alpha-1} e^{\frac{-\beta}{y}} y^2 dy \\
&= \int_{0}^{\infty} \frac{1}{\Gamma(\alpha)}\beta^{\alpha}\xi^{\alpha-3}e^{-\beta\xi}d\xi \\
&= \frac{\beta^{\alpha}\Gamma(\alpha-2)}{\beta^{\alpha-2}\Gamma(\alpha)} \cdot \underbrace{\int_{0}^{\infty} \frac{1}{\Gamma(\alpha-2)}\beta^{\alpha-2}\xi^{\alpha-3}e^{-\beta\xi}d\xi}_{=1} \\
&= \frac{\beta^2\Gamma(\alpha-2)}{(\alpha-1)(\alpha-2)\Gamma(\alpha-2)} \\
&= \frac{\beta^2}{(\alpha-1)(\alpha-2)}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{var}(Y) &= E\left(Y^2\right) - E^2(Y) \\
&= \frac{\beta^2}{(\alpha-1)(\alpha-2)} - \frac{\beta^2}{(\alpha-1)^2} \\
&= \frac{\beta^2(\alpha-1)-(\alpha-2)\beta^2}{(\alpha-1)^2(\alpha-2)} \\
&= \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}
\end{aligned}
$$

2 שאלה מס.2

2.1 א

$$
\begin{aligned}
X_i &\sim N\left(\mu, \sigma^2\right) \\
Y &= \sum_{i=1}^{n} x_i
\end{aligned}
$$

נשתמש בפונ' יוצרת מומנטים של x_i כלומר $m_{x_i}(t) = e^{\mu t + \frac{\sigma^2}{2}t^2}$

$$
\begin{aligned}
m_Y(t) &= \sum_{i=1}^{n} m_i(t) \\
&= \prod_{i=1}^{n} m_{x_i}(t) \\
&= \left(m_{x_1}(t)\right)^n \\
&= \left(e^{\mu t + \frac{\sigma^2}{2} t^2}\right)^n \\
&= e^{n\mu t + n\frac{\sigma^2}{2} t^2}
\end{aligned}
$$

רואים שזה משתנה מקרי נורמלי.

2.2 ב

$$
x_1 \sim Gamma \Rightarrow m_{x_1}(t) = \left(1 - \frac{t}{\beta}\right)^{-\alpha}
$$

$$
\begin{aligned}
M_Y &= \prod_{i=1}^{n} m_{x_i}(t) = \left(m_{x_1}(t)\right)^n = \left(1 - \frac{t}{\beta}\right)^{-(n\alpha)} \\
Y &\sim Gamma\left(n\alpha, \beta\right)
\end{aligned}
$$

3 שאלה מס.3

$$
\begin{array}{rcl}
x_1, x_2 \ldots x_{30} & \sim & exp\left(\lambda\right) \\
\mu & = & 20 \\
\sigma^2 & = & 40 \\
Y & = & \sum_{i=1}^{30} x_i \sim N\left(30\mu, 30 \cdot \sigma^2\right) \\
 & \sim & N\left(600, 1200\right) \\
Z & = & \dfrac{Y - \mu}{\sqrt{\sigma^2}} \\
P\left(Y < c\right) & = & 0.05 \\
P\left(z < \dfrac{c - 600}{\sqrt{12000}}\right) & = & 0.05 \\
-1.56 < & z & < -1.55 \\
1.56 < & \frac{c-600}{\sqrt{12000}} & < -1.55 \\
429.11 < & a & < 430.206
\end{array}
$$

4 שאלה מס.4

4.1 א

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\begin{aligned}
x_1 &\sim N(75,20)\\
\overline{x} &= \frac{1}{25}\sum_{i=1}^{25} x_i\\
\overline{x} &= \sim N\left(75,\frac{20}{25}\right)\\
Z &= \frac{\overline{x}-75}{\sqrt{\frac{20}{25}}}\\
P(\overline{x}>78) &= 1-P(\overline{x}<78)\\
&= 1-P\left(z<\frac{78-75}{\sqrt{\frac{20}{25}}}\right)\\
&= 1-P\left(z<\frac{3}{\sqrt{0.8}}\right)\\
&= 1-P(z<3.35)\\
&= 1-0.9996\\
&= 0.0004
\end{aligned}
$$

4.2 ב

$$
\begin{aligned}
P(74<\overline{x}<76) &= P\left(\frac{74-75}{\sqrt{\frac{20}{25}}}<z<\frac{76-75}{\sqrt{\frac{20}{25}}}\right)\\
&= P(-1.11<z<1.11)\\
&= \phi(1.11)-(1-\phi(1.11)) =\\
&= 2\cdot\phi(1.11)-1\\
&= 2\cdot 0.8665-1\\
&= 0.733
\end{aligned}
$$

4.3 ג

$$
\begin{aligned}
\overline{x}_n &= \sum_{i=1}^{n} x_i \sim N\left(\mu, \frac{\sigma^2}{n}\right) = N\left(75, \frac{20}{n}\right) \\
P\left(\overline{x} > 77\right) &\leq 0.05 \\
1 - P\left(x < 77\right) &\leq 0.05 \\
P\left(x < 77\right) &\leq 0.95 \\
P\left(x < 77\right) &= P\left(z < \frac{77-75}{\sqrt{\frac{20}{n}}}\right) > 0.95 \\
z_{0.95} &> 1.65 \\
\frac{77-75}{\sqrt{\frac{20}{n}}} &\geq 1.65 \\
n &\geq 13.6 \geq 4
\end{aligned}
$$

5 שאלה 5

5.1 ג

$$
\begin{aligned}
E\left(x\right) &= 0.2 \cdot 2 + 0.3 \cdot 3 + 0.4 \cdot 4 + 0.1 \cdot 5 = 3.4 \\
E\left(x^2\right) &= 0.2 \cdot 2^2 + 0.3 \cdot 3^2 + 0.4 \cdot 4^2 + 0.1 \cdot 5^2 = 0.84 \\
\mathrm{var}\left(x\right) &= 12.4 - 3.4^2 = 0.84
\end{aligned}
$$

5.2 ב

$$
\begin{aligned}
\overline{x} &= \frac{1}{50}\sum_{i=1}^{50} x_i \sim N\left(3.4, \frac{0.84}{50}\right) = N\left(3.4, 0.0168\right) \\
P\left(\overline{x} > 3.7\right) &= P\left(\frac{x-3.4}{\sqrt{0.0168}} \geq \frac{3.7-3.4}{\sqrt{0.0168}}\right) \\
&= 1 - P\left(z < \frac{3.7-3.4}{\sqrt{0.0168}}\right) \\
&= 1 - P\left(z \leq 2.31\right) \\
&= 1 - 0.9896 \\
&= 0.0104
\end{aligned}
$$

$$\begin{aligned}\overline{x} &= \sum_{i=1}^{50} x_i \Rightarrow \overline{x} \sim N\left(3.4\cdot 50, 50\cdot 0.84\right)\\ \overline{x} &\sim N\left(170, 42\right)\end{aligned}$$

$$\begin{aligned}P\left(160 < x < 200\right) &= P\left(\frac{160-170}{\sqrt{42}} < z < \frac{200-170}{\sqrt{42}}\right)\\ &= \phi\left(4.67\right) - \phi\left(1.54\right)\\ &= 0.9382\end{aligned}$$

6 שאלה 6

6.1 א

$$\begin{aligned}f_{x_i}\left(t\right) &= \frac{1}{b-a}\\ F_{x_i}\left(t\right) &= \frac{t}{b-a}\\ F_T\left(t\right) &= P\left(\max x_i < t\right)\\ &= P\left(x_1 < t, x_2 < t, x_3 < t \ldots x_n < t\right)\\ &= \left(P\left(x_1 \le t\right)\right)^2\\ &= \left(f_{x_i}\left(t\right)\right)^n\\ &= \left(\frac{t}{b-a}\right)^2\end{aligned}$$

6.2 ב

$$\begin{aligned}F_z\left(t\right) &= P\left(\min_i x_i \le t\right)\\ &= P\left(\text{at least one is smaller than t}\right)\\ &= 1 - P\left(\text{all greater than t}\right)\\ &= 1 - P\left(x > t\right)^n\\ &= 1 - \left(\frac{b-t}{b-a}\right)^n\\ f_z\left(t\right) &= n\left(\frac{b-t}{b-a}\right)^{n-1}\end{aligned}$$

$$
\begin{aligned}
E\left(Y\right) &= \int_a^b t \cdot n \frac{t^{n-1}}{b-a} dt \\
&= \frac{n}{b-a} \int_a^b t^n dt \\
&= \frac{nt^{n4}}{\left(n-1\right)\left(n+1\right)} |_a^b \\
&= \frac{n\left(b^{n+1}-a^{n+1}\right)}{\left(b-a\right)\left(n+1\right)}
\end{aligned}
$$

$$
\begin{aligned}
E\left(z\right) &= \int_0^n \left(\frac{b-t}{b-a}\right) t dt \\
&= \frac{b}{\left(b-a\right)^{n-1}} \int_a^b t\left(b-t\right)^{n-1} dt \\
&= \frac{n}{\left(b-a\right)^{n01}} \left[t \cdot \frac{\left(b-t\right)^n}{n} |_a^b + \int_a^b \frac{\left(b-t\right)^4}{n} dt \right] \\
&= \left(b-a\right) + \frac{\left(b-a\right)^2}{n+1}
\end{aligned}
$$