תרגיל מס.1

עפיף חלומה 302323001

10 בנובמבר 2009

1 שאלה 1

$$\begin{aligned} V &= IR + \dot{I}L + \frac{Q}{C} \\ 0 &- \dot{I}R + L\ddot{I} + \frac{1}{C}I \end{aligned}$$

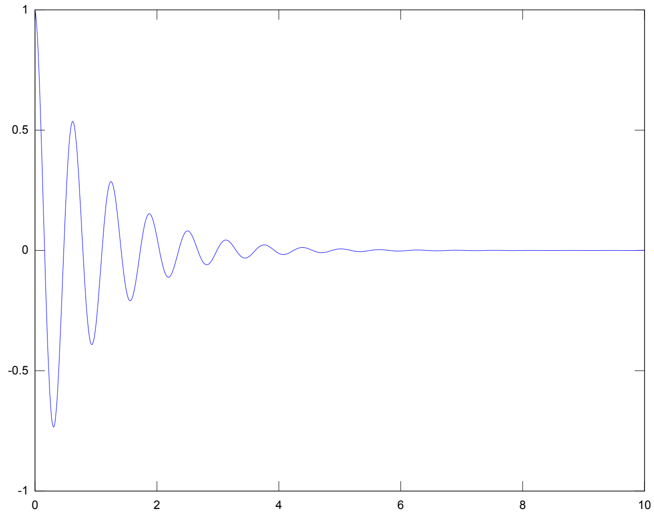
ננחש פתרון $I\left(t\right) = e^{ist}$ אזי

$$\begin{aligned} Ls^2 - isR - \frac{1}{C} &= 0 \\ s_{1,2} &= \frac{iR \pm \sqrt{R^2 + \frac{4L}{C}}}{2L} \\ &= \frac{iR}{2L} \pm \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \\ \eta &= \frac{R}{2L} \\ \omega_{1,2} &= \pm\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \end{aligned}$$

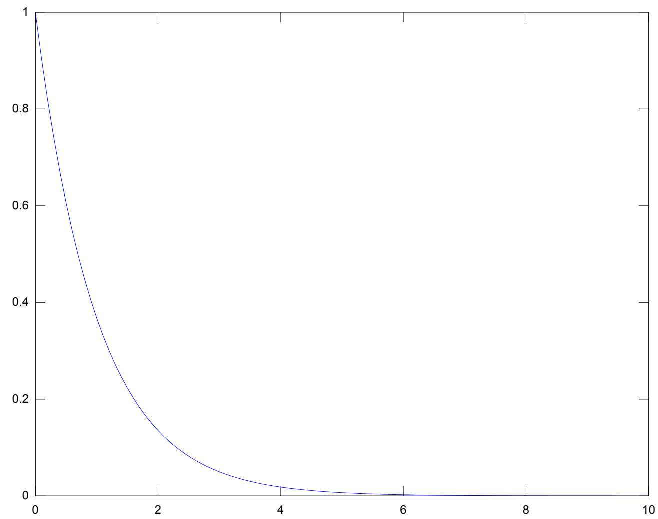
נציב את הערכים שקיבלנו ונקבל פתרון סופי:

$$I\left(t\right) = e^{-\frac{R}{2L}t}\left(Ae^{i\sqrt{-\frac{R^2}{4L^2} - \frac{1}{LC}}} + Be^{-i\sqrt{-\frac{R^2}{4L^2} - \frac{1}{LC}}}\right)$$

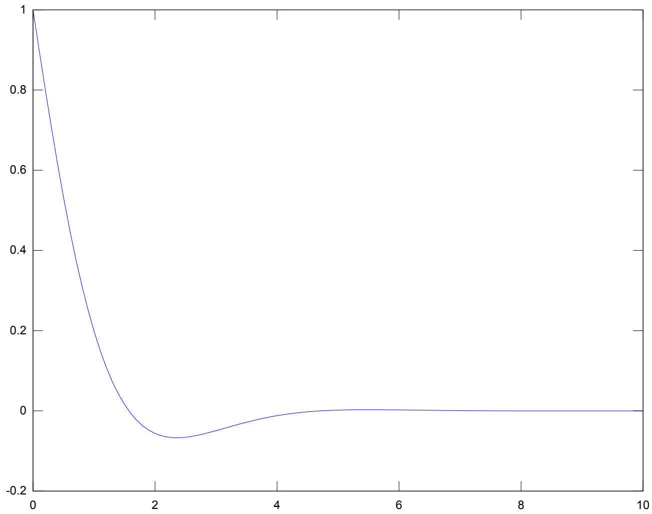
נסמן $\Delta = -\frac{R^2}{4L^2} - \frac{1}{LC}$ אזי מקבלים המקרים הבאים:



איור 1: $\Delta > 0$



איור 2: $\Delta = 0$



איור 3: $\Delta < 0$

2 שאלה 2

נשתמש בקירובים $\sin\alpha \approx \alpha, \cos\alpha \approx 1$
מגיאומטריה רואים כי

$$
\begin{aligned}
\frac{y_1}{l} &= \tan\alpha_1 \approx \alpha_1 \\
\alpha_2 &= \frac{y_2 - y_1}{l} \\
\alpha_3 &= \frac{h - y_2}{l}
\end{aligned}
$$

$$
\begin{aligned}
-T\sin\alpha_1 + T\sin\alpha_2 &= m\ddot{y}_1 \\
-T\sin\alpha_2 + T\sin\alpha_3 &= m\ddot{y}_2 \\
T\left[\frac{y_2 - y_1}{l} - \frac{y_1}{l}\right] &= m\ddot{y}_1 \\
T\left[\frac{h - y_1}{l} - \frac{y_2 - y_1}{l}\right] &= m\ddot{y}_2
\end{aligned}
$$

רושמים במטריצות:

$$
\begin{pmatrix} 0 \\ \frac{Th}{ml} \end{pmatrix} + \begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{pmatrix} = \frac{T}{lm}\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}
$$

פותרים את המשוואה ההומוגינית.מוצאים את הערכים העצמיים:

$$\begin{aligned}
\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} &= 0 \\
(2+\lambda)^2 + 1 &= 0 \\
\lambda^2 + 4\lambda + 3 &= 0 \\
\lambda_{1,2} &= -3, -1
\end{aligned}$$

עבור $\lambda = -3$:

$$\begin{aligned}
\begin{pmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \vec{v}
\end{aligned}$$

עבור $\lambda = -1$:

$$\begin{aligned}
\begin{pmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \vec{u}
\end{aligned}$$

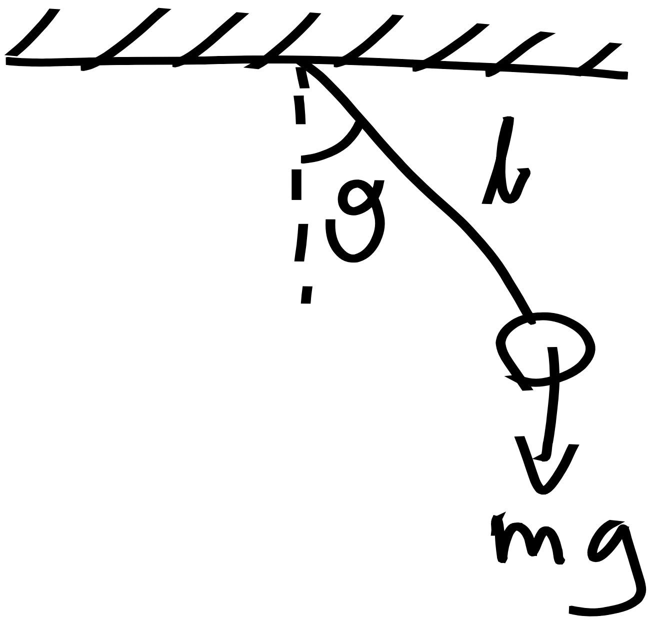
פתרון פרטי:

$$\begin{aligned}
A &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
\begin{pmatrix} \ddot{S}_1 \\ \ddot{S}_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} &= \begin{pmatrix} 0 \\ \frac{Th}{ml} \end{pmatrix} \\
\ddot{S}_1 &= -\ddot{S}_2 \\
\ddot{S}_1 &= -\frac{Th}{ml} \\
S_1 &= -\frac{Th}{\omega^2 ml}
\end{aligned}$$

$$\begin{aligned}
\begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} S_1 + \begin{pmatrix} 1 \\ -1 \end{pmatrix} S_2 = \begin{pmatrix} 0 \\ 2S_1 \end{pmatrix} \\
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} Ae^{i\sqrt{\frac{T}{lm}}\cdot t} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} Be^{i\sqrt{\frac{3T}{lm}}\cdot t} + \begin{pmatrix} 0 \\ -\frac{Th_0}{lm\omega^2}\cos\omega t \end{pmatrix} \\
y_1(0) &= 0 \\
y_2(0) &= 0 \\
\dot{y}_1(0) &= 0 \\
\dot{y}_2(0) &= 0
\end{aligned}$$

3 שאלה 3

3.1 א



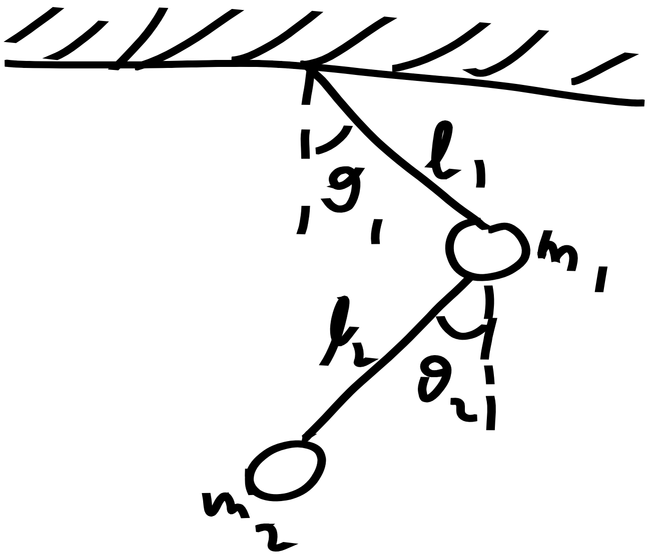
איור 4: כוחות במטולטלת

$$
\begin{aligned}
T\cos\theta &= mg \\
m\ddot{x} &= -T\sin\theta \\
m\ddot{x} &= -\frac{mg\sin\theta}{\cos\theta} \\
\ddot{x} &= -g\underbrace{\tan\theta}_{\frac{\sin\theta}{\cos\theta}=\frac{\theta}{1}} \\
\frac{\partial^2 x}{\partial t^2} &= -g\theta \\
\frac{\partial^2\left(l\sin\theta\right)}{\partial t^2} &= -g\theta \\
\frac{\partial^2\left(l\theta\right)}{\partial t^2} &= -g\theta \\
\ddot{\theta} &= -\frac{g}{l}\theta \\
\ddot{\theta}+\frac{g}{l}\theta &= 0
\end{aligned}
$$

מנחשים פתרון $\theta = A\cos\left(\omega\theta\right) + B\sin\left(\omega\theta\right)$

$$
\begin{aligned}
-A\omega^2\cos(\omega\theta)-B\omega^2\sin(\omega\theta)+ \\
+\tfrac{g}{l}\left(A\cos(\omega\theta)+B\sin(\omega\theta)\right) &= 0 \\
\omega &= \sqrt{\frac{g}{l}} \\
\theta &= A\cos\left(\sqrt{\frac{g}{l}}\cdot\theta\right)+B\sin\left(\sqrt{\frac{g}{l}}\cdot\theta\right)
\end{aligned}
$$

3.2 ב



איור 5: שתי מטוטלות

$$
\begin{aligned}
T_1\cos\theta_1 &= m_1g+T_2\cos\theta_2 \\
T_1 &= \frac{m_1g+T_2\cos\theta_2}{\cos\theta_1} \\
T_2\cos\theta_2 &= m_2g \\
T_2 &= \frac{m_2g}{\cos\theta_2} \\
T_1 &= \frac{m_1g+\frac{m_2g}{\cos\theta_2}\cos\theta_2}{\cos\theta_1} \\
T_2 &= \frac{m_2g}{1}=m_2g \\
T_1 &= \frac{m_1g+\frac{m_2g}{1}\cdot 1}{1} \\
&= m_1g+m_2g
\end{aligned}
$$

אזי מציבים את זה בציר x שלנו:

$$\begin{aligned}
m_1\ddot{x}_1 &= -T_1\sin\theta_1 - T_2\sin\left(-\theta_2\right)\\
&= -T_1\sin\theta_1 + T_2\sin\theta_2\\
m_2\left(\ddot{x}_1+\ddot{x}_2\right) &= -T_2\sin\theta_2\\
m_1 l\ddot{\theta}_1 &= -\left(m_1 g + m_2 g\right)\theta_1 + m_2 g\theta_2\\
m_2\left(l_2\ddot{\theta}_2 + l_1\ddot{\theta}_1\right) &= -m_2 g\theta_2\\
\ddot{\theta}_1 &= -\frac{\left(m_1+m_2\right)g}{m_1 l_1}\theta_1 + \frac{m_2 g}{m_1 l_1}\theta_2\\
m_2 l_2\ddot{\theta}_2 + m_2 l_1\ddot{\theta}_1 &= -m_2 g\theta_2\\
m_2 l_2\ddot{\theta}_2 &= -m_2 g\theta_2 - m_2 l_1\ddot{\theta}_1\\
\ddot{\theta}_2 &= -\frac{m_2 g}{m_2 l_2}\theta_2 - \frac{m_2 l_1}{m_2 l_2}\ddot{\theta}_1\\
&= -\frac{m_2 g}{m_2 l_2}\theta_2 - \frac{m_2 l_1}{m_2 l_2}\left(-\frac{\left(m_1+m_2\right)g}{m_1 l_1}\theta_1 + \frac{m_2 g}{m_1 l_1}\theta_2\right)\\
&= -\frac{m_2 g}{m_2 l_2}\theta_2 - \left(-\frac{\left(m_1+m_2\right)g}{m_1 l_2}\theta_1 + \frac{m_2 g}{m_1 l_2}\theta_2\right)\\
&= -\frac{m_2 g}{m_2 l_2}\theta_2 + \frac{\left(m_1+m_2\right)g}{m_1 l_2}\theta_1 - \frac{m_2 g}{m_1 l_2}\theta_2\\
&= \frac{\left(m_1+m_2\right)g}{m_1 l_2}\theta_1 - \frac{m_2 g}{m_1 l_2}\theta_2 - \frac{m_2 g}{m_2 l_2}\theta_2\\
&= \frac{\left(m_1+m_2\right)g}{m_1 l_2}\theta_1 - \frac{\left(m_2+m_2\right)g}{m_1 l_2}\theta_2
\end{aligned}$$

3.3 ג

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} &= \begin{pmatrix}\frac{-\left(m_1+m_2\right)g}{m_1 l_1} & \frac{m_2 g}{m_1 l_1}\\ \frac{\left(m_1+m_2\right)g}{m_1 l_2} & \frac{-\left(m_2+m_2\right)g}{m_1 l_2}\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}\\
\frac{\partial^2}{\partial t^2}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} &= \begin{pmatrix}\frac{-2mg}{ml} & \frac{mg}{ml}\\ \frac{2mg}{ml} & \frac{-2mg}{ml}\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}\\
\frac{\partial^2}{\partial t^2}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} &= \frac{g}{l}\begin{pmatrix}-2 & 1\\ 2 & -2\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
|A - I\lambda| &= 0 \\
\begin{vmatrix} -2-\lambda & 1 \\ 2 & -2-\lambda \end{vmatrix} &= 0 \\
(-2-\lambda)^2 - 2 &= 0 \\
\lambda^2 + 4\lambda + 4 - 2 &= 0 \\
\lambda^2 + 4\lambda + 2 &= 0 \\
\lambda &= \frac{-4 \pm \sqrt{16 - 4 \cdot 2}}{2} \\
&= \frac{-4 \pm \sqrt{8}}{2} \\
&= -2 + \sqrt{2}, -2 - \sqrt{2}
\end{aligned}
$$

עבור $\lambda_1 = \frac{g}{l}\left(-2+\sqrt{2}\right)$ מקבלים:

$$
\begin{aligned}
\begin{pmatrix} -2+2-\sqrt{2} & 1 \\ 2 & -2+2-\sqrt{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} -\sqrt{2} & 1 \\ 2 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}
\end{aligned}
$$

עבור $\lambda_2 = \frac{g}{l}\left(-2-\sqrt{2}\right)$ מקבלים:

$$
\begin{aligned}
\begin{pmatrix} -2+2+\sqrt{2} & 1 \\ 2 & -2+2+\sqrt{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} \sqrt{2} & 1 \\ 2 & \sqrt{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}
\end{aligned}
$$

קיבלנו ווקטורים עצמיים $\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$ אזי

$$
\begin{aligned}
z &= Sy \\
&= \begin{pmatrix} 1 & 1 \\ \sqrt{2} & -\sqrt{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\
&= \begin{pmatrix} y_1 + y_2 \\ \sqrt{2}y_1 - \sqrt{2}y_2 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\begin{pmatrix}\ddot{z}_1\\ \ddot{z}_2\end{pmatrix} &= \begin{pmatrix}-2+\sqrt{2} & 0\\ 0 & -2-\sqrt{2}\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix}\\
&= \begin{pmatrix}A_1e^{-\sqrt{\frac{g}{l}\left(-2+\sqrt{2}\right)}\cdot t}+B_1e^{\sqrt{\frac{g}{l}\left(-2+\sqrt{2}\right)}\cdot t}\\ A_2e^{-\sqrt{\frac{g}{l}\left(-2-\sqrt{2}\right)}\cdot t}+B_2e^{\sqrt{\frac{g}{l}\left(-2-\sqrt{2}\right)}\cdot t}\end{pmatrix}
\end{aligned}
$$

נחזור ל y_1, y_2:

$$
\begin{aligned}
\begin{pmatrix}z_1\\ z_2\end{pmatrix} &= \begin{pmatrix}y_1+y_2\\ \sqrt{2}y_1-\sqrt{2}y_2\end{pmatrix}\\
\begin{pmatrix}y_1\\ y_2\end{pmatrix} &= \begin{pmatrix}\frac{1}{4}\left(2z_1+\sqrt{2}z_2\right)\\ \frac{\sqrt{2}z_1-z_2}{2\sqrt{2}}\end{pmatrix}\\
&= \begin{pmatrix}\frac{1}{2}\left(A_1e^{-\sqrt{\frac{g}{l}\left(-2+\sqrt{2}\right)}\cdot t}+B_1e^{\sqrt{\frac{g}{l}\left(-2+\sqrt{2}\right)}\cdot t}\right)+\\ +\frac{\sqrt{2}}{4}\left(A_2e^{-\sqrt{\frac{g}{l}\left(-2-\sqrt{2}\right)}\cdot t}+B_2e^{\sqrt{\frac{g}{l}\left(-2-\sqrt{2}\right)}\cdot t}\right)\\ \frac{\sqrt{2}\left(A_1e^{-\sqrt{\frac{g}{l}\left(-2+\sqrt{2}\right)}\cdot t}+B_1e^{\sqrt{\frac{g}{l}\left(-2+\sqrt{2}\right)}\cdot t}\right)}{2\sqrt{2}}-\\ -\frac{\left(A_2e^{-\sqrt{\frac{g}{l}\left(-2-\sqrt{2}\right)}\cdot t}+B_2e^{\sqrt{\frac{g}{l}\left(-2-\sqrt{2}\right)}\cdot t}\right)}{2\sqrt{2}}\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\omega_1 &= \sqrt{\frac{g}{l}\left(-2+\sqrt{2}\right)}\\
\omega_2 &= \sqrt{\frac{g}{l}\left(-2-\sqrt{2}\right)}
\end{aligned}
$$