תרגיל מס.2

עפיף חלומה 302323001

23 במרץ 2010

1 שאלה 1

1.1 א

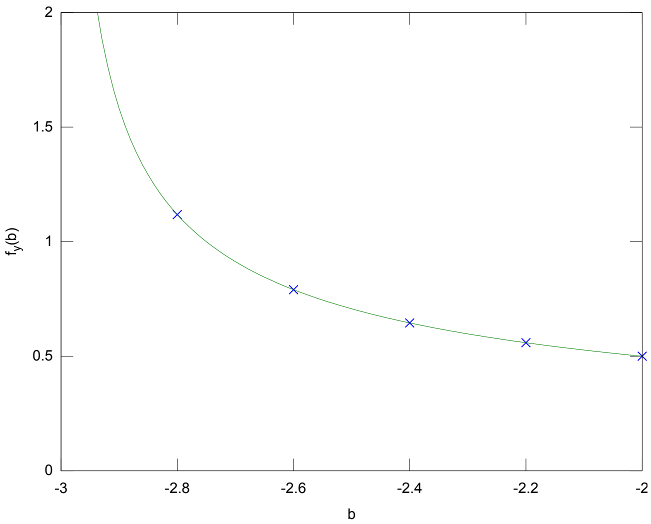
$$
\begin{aligned}
P\left(y \leq a\right) &= P\left(x^{2}-3 \leq a\right) \\
&= P\left(x^{2} \leq a+3\right) \\
&= P\left(-\sqrt{a+3} \leq x \leq \sqrt{a+3}\right) \\
&= F_{x}\left(\sqrt{a+3}\right)-\underbrace{F\left(-\sqrt{a+3}\right)}_{0} \\
&= F_{x}\left(\sqrt{a+3}\right) \\
F_{x}\left(t\right) &= \begin{cases} 1 & t \geq 1 \\ t & 0 \leq t \leq 1 \\ 0 & t \leq 0 \end{cases}
\end{aligned}
$$

אזי

$$
\begin{aligned}
P\left(y \leq a\right) &= \begin{cases} 1 & a \geq -2 \\ \sqrt{a+3} & -3 \leq a \leq 2 \\ 0 & a \leq -3 \end{cases} \\
f_{y}\left(b\right) &= \frac{\partial F\left(b\right)}{\partial b} \\
&= \begin{cases} \frac{1}{2\sqrt{b+3}} & -3 < b < -2 \\ 0 & otherwise \end{cases}
\end{aligned}
$$

1.2 ב

$$\begin{aligned} f_y(-3) &= \infty \\ f_y(-2.8) &= 1.12 \\ f_y(-2.6) &= 0.79 \\ f_y(-2.4) &= 0.65 \\ f_y(-2.2) &= 0.56 \\ f_y(-2) &= 0.5 \end{aligned}$$



איור 1: f_y

1.3 ג

$$\int_{-\infty}^{\infty} b f_w(b)\,\partial b = \int_{-\infty}^{\infty} b \frac{f_v\left(h^{-1}(b)\right)}{\left|h'\left(h^{-1}(b)\right)\right|}\partial b$$

נתון כי הפונק' מונוטונית לכן נשתמש במשפט

$$\begin{aligned} f_w(b) &= f_v\left(h^{-1}(b)\right)\cdot\frac{1}{\left|h'\left(h^{-1}(b)\right)\right|} \\ b = h(a) &\Rightarrow db = h'(a)\,da \end{aligned}$$

$$\begin{aligned} f_w(b) &= \int_{-\infty}^{\infty} b \frac{f_v\left(h^{-1}(b)\right)}{\left|h'\left(h^{-1}(b)\right)\right|} db \\ &= \int_{-\infty}^{\infty} h(a)\,\frac{f_v(a)}{h'(a)} h'(a)\,da \end{aligned}$$

כאשר $h\left(a\right)$ מונוטונית יורדת אזי $h'\left(a\right)<0$

$$\begin{aligned}\int_{-\infty}^{\infty}h\left(a\right)f_{v}\left(a\right)\frac{h'\left(a\right)}{\left|h'\left(a\right)\right|}da &= \int_{-\infty}^{\infty}-h\left(a\right)f_{v}\left(a\right)da\\ &= \int_{\infty}^{-\infty}-h\left(a\right)f_{v}\left(a\right)da\\ &= \int_{-\infty}^{\infty}f_{v}\left(a\right)h\left(a\right)da\end{aligned}$$

כאשר $h\left(a\right)$ עולה $\Leftarrow$ $0<h'\left(a\right)$ אזי

$$\int_{-\infty}^{\infty}h\left(a\right)f_{v}\left(a\right)\frac{h'\left(a\right)}{\left|h\left(a\right)\right|}da=\int_{-\infty}^{\infty}h\left(a\right)f_{v}\left(a\right)da$$

משל.

2 שאלה 2

2.1 א

$$\begin{aligned}X &\sim N\left(100,2000\right)\\ f_{x}\left(x\right) &= \frac{1}{\sqrt{2\cdot2000\cdot\pi}}\exp\left\{ \frac{-\left(x-100\right)^{2}}{2\cdot2000}\right\}\\ \mu &= 100\\ \sigma^{2} &= 2000\end{aligned}$$

אם נגדיר $y=\frac{x-\mu}{\sigma}=\frac{x-100}{\sqrt{2000}}$ מקבלים:

$$\begin{aligned}f_{y} &= \frac{1}{\sqrt{2\pi}}\exp\left\{ \frac{-y^{2}}{2}\right\}\\ \mu_{y} &= 0\\ \sigma_{y} &= 1\end{aligned}$$

אזי מצאנו $g\left(x\right)=\frac{x-\mu}{\sigma}=\frac{x-100}{\sqrt{2000}}$ כדרוש.

2.2 ב

$$\begin{aligned}
F_x(300) &= P(x \le t) \\
&= P\left(\frac{x-\mu}{\sigma} \le \frac{t-\mu}{\sigma}\right) \\
&= P\left(y \le \frac{300-100}{\sqrt{2000}}\right) \\
&= P\left(y \le \frac{200}{\sqrt{2000}}\right) \\
&= F_y\left(\frac{200}{\sqrt{2000}}\right) \\
c &= \frac{200}{\sqrt{2000}}
\end{aligned}$$

2.3 ג

$$\begin{aligned}
w_1 &\sim \exp(\lambda) \\
z &= 2w \\
F_z(t) &= P(z \le t) \\
&= P(2w_1 \le t) \\
&= P\left(w_1 \le \frac{t}{2}\right) \\
F_z(t) &= \begin{cases} 1-e^{-\frac{t\lambda}{2}} & t \ge 0 \\ 0 & t < 0 \end{cases} \\
f_z(t) &= \begin{cases} \frac{\lambda}{2}e^{-\left(\frac{\lambda}{2}\right)t} & t \ge 0 \\ 0 & t < 0 \end{cases} \\
Z &\sim \exp\left(\frac{\lambda}{2}\right)
\end{aligned}$$

2.4 ד

כאשר w_1, w_2 בלתי תלויים

$$\begin{aligned}
V &= w_1 + w_2 \\
w_1, w_2 &\sim \exp(\lambda)
\end{aligned}$$

הוכחנו בכיתה כי $\mathrm{var}(t) = E\left(t^2\right) - E^2(t)$ אזי

$$
\begin{aligned}
\operatorname{var}(V) &= \operatorname{var}(w_1+w_2)\\
&= E\left(w_1^2+2w_1w_2+w^2\right)-\left(E\left(w_1+w_2\right)\right)^2\\
&= E\left(w_1^2\right)+2E\left(w_1w_2\right)+E\left(w_2^2\right)-\left(E\left(w_1\right)+E\left(w_2\right)\right)^2\\
&= E\left(w_1^2\right)+2E\left(w_1w_2\right)+E\left(w_2^2\right)-E^2\left(w_1\right)-2E\left(w_1\right)E\left(w_2\right)-E^2\left(w_2\right)\\
&= \left(E\left(w_1^2\right)-E^2\left(w_1\right)\right)+\left(E\left(w_2^2\right)-E^2\left(w_2\right)\right)+2\left(E\left(w_1w_2\right)-E\left(w_1\right)E\left(w_2\right)\right)\\
&= \operatorname{var}(w_1)+\operatorname{var}(w_2)+\underbrace{2\operatorname{cov}(w_1,w_2)}_{0}\\
&= \frac{1}{\lambda^2}+\frac{1}{\lambda^2}=\frac{2}{\lambda^2}
\end{aligned}
$$

3 שאלה 3

3.1 א

$$
\begin{aligned}
\int_{-\infty}^{\infty} f_x(a)\,da &= 1\\
B\int_0^{\infty} e^{-5t}dt+\frac{1}{3}\int_0^{\infty}\delta\left(t-\frac{1}{8}\right)dt+\frac{1}{2}\int_0^{\infty}\delta\left(t-\frac{1}{2}\right)dt &= 1\\
-\frac{1}{5}B\left[e^{-5t}\right]_0^{\infty}+\frac{1}{3}+\frac{1}{2} &= 1\\
-\frac{1}{5}B\left(0-1\right)+\frac{1}{3}+\frac{1}{2} &= 1\\
B &= \frac{5}{6}
\end{aligned}
$$

3.2 ב

$$
\begin{aligned}
E(x) &= \int_{-\infty}^{\infty} tf(x)\,dt\\
&= \int_0^{\infty}\frac{5}{6}te^{-5t}+\frac{1}{3}\delta\left(t-\frac{1}{8}\right)\cdot t+\frac{t}{2}\delta\left(t-\frac{1}{2}\right)dt\\
&= \frac{5}{6}\cdot\frac{1}{25}+\frac{1}{24}+\frac{1}{4}=\frac{14}{40}
\end{aligned}
$$

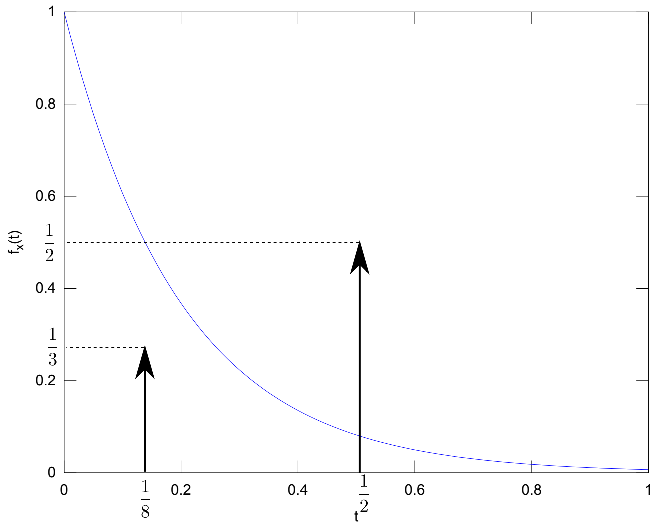
3.3 ג

$$
\begin{aligned}
E\left(x^2\right) &= \int_0^\infty t^2 f_x(t)\,dt \\
&= \int_0^\infty t^2\left(\frac{5}{6}e^{5t}+\frac{1}{3}\delta\left(t-\frac{1}{8}\right)+\frac{1}{2}\delta\left(t-\frac{1}{2}\right)\right)dt \\
\int t^2 e^{-5t}dt &= \underbrace{-\frac{1}{5}t^2e^{-5t}|_0^\infty}_{0}+\frac{2}{5}\int_0^\infty te^{-5t}=\frac{2}{5^3}=\frac{2}{125} \\
E\left(x^2\right) &= \frac{1}{64}\cdot\frac{1}{3}+\frac{1}{4}\cdot\frac{1}{2}+\frac{2}{125}\approx 0.146 \\
\mathrm{var}(x) &= E\left(x^2\right)-E^2(x)\approx 0.041
\end{aligned}
$$

3.4 ד

$$
\begin{aligned}
P\left(\frac{1}{4}\le x\le\frac{3}{4}\right) &= F_x\left(\frac{3}{4}\right)-F_x\left(\frac{1}{4}\right)+\frac{1}{2}=P\left(\frac{1}{2}\right) \\
&= \frac{1}{6}\left(e^{-\frac{5}{4}}-e^{-\frac{15}{4}}\right)+\frac{1}{2} \\
&\approx 0.544
\end{aligned}
$$

3.5 ה



איור 2: f_x