Report Topics in Computational Science

Afifah Maya Iknaningrum
1715011053

June 15, 2018

Problem 1 :
Let $m > 1$ and suppose that u is a twice continuously differentiable positive solution in the pressure porous medium equation

$$u_t - \Delta(u^m) = 0, \ \ x \in \mathbb{R}, t > 0. \tag{1}$$

Show that $p(x,t) := \dfrac{m}{m-1} u^{m-1}(x,t)$ is a solution of the porous medium equation in the pressure form

$$p_t - (m-1)p\Delta p - |\nabla p|^2 = 0 \tag{2}$$

Answer 1 :
Substitute

$$p(x,t) := \frac{m}{m-1} u^{m-1}(x,t) \tag{3}$$

, to equation (2), we obtain

$$\begin{aligned}
& p_t - (m-1)p\Delta p - |\nabla p|^2 \\
= \ & \Big(\frac{m}{m-1}(m-1)u^{m-2}u_t\Big) - \Big((m-1)\frac{m}{m-1}u^{m-1}m(m-2)u^{m-3}u_x^2 + m^2u^{m-2}u_{xx}\Big) - |mu^{m-2}u_x|^2 \\
= \ & mu^{m-2}u_t - (m^3 + m^2 - 2m)u^{2m-4}u_x^2 - m^3u^{2m-3}u_{xx}
\end{aligned} \tag{4}$$

Remember we have

$$u_t - \Delta(u^m) = u_t - (m^2 - m)u^{m-2}u_x^2 - m^2u^{m-1}u_{xx} = 0. \tag{5}$$

Then the equation (4) can be modified such that it is contain equation (5) such that

$$\begin{aligned}
& mu^{m-2}u_t - (m^3 + m^2 - 2m)u^{2m-4}u_x^2 - m^3u^{2m-3}u_{xx} \\
= \ & mu^{m-2}\big(u_t - (m^2 - m)u^{m-2}u_x^2 - m^2u^{m-1}u_{xx}\big) + \big((m^3 - m^2) - (m^3 - 2m^2) - m^2\big)u^{2m-4}u_x^2 + (m^3 - m^3)u^{2m-3}u_{xx} \\
= \ & 0
\end{aligned}$$

It is shown that $p(x,t)$ as defined above, is a solution of the porous medium equation in the pressure form (2).

Problem 2 :
For dimension $n \in \mathbb{N}$ and constant $m > 1, C > 0, \alpha > 0, \beta > 0$. Define the function $u : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ as

$$u(x,t) = t^{-\alpha}\Big(\max\big(C - \frac{\beta(m-1)}{2m}\frac{|x|^2}{t^{2\beta}}, 0\big)\Big)^{\frac{1}{m-1}}, \; x \in \mathbb{R}^n, t > 0$$

where $|x| := (\sum_{i=1}^n x_i^2)^{1/2}$

1. Find α and β in terms of m and n so that u is a solution of (1) in the set

$$\{(x,t) : x \in \mathbb{R}^n, t > 0, u(x,t) > 0\}.$$

Answer : We have p as defined in (3). Because the set, we should take $u(x,t) > 0$ such that

$$\begin{aligned}
u(x,t) &= t^{-\alpha}\big(C - \frac{\beta(m-1)}{2m}\frac{|x|^2}{t^{2\beta}}\big)^{\frac{1}{m-1}} \\
p(x,t) &= \frac{m}{m-1}t^{-\alpha(m-1)}\big(C - \frac{\beta(m-1)}{2m}\frac{|x|^2}{t^{2\beta}}\big) \\
p_t(x,t) &= -\alpha m t^{-\alpha(m-1)-1}\big(C - \frac{\beta(m-1)}{2m}\frac{|x|^2}{t^{2\beta}}\big)^{\frac{1}{m-1}} + \beta^2|x|^2 t^{-\alpha(m-1)-2\beta-1} \\
&= -\frac{\alpha}{t}(m-1)p(x,t) + \beta^2|x|^2 t^{-\alpha(m-1)-2\beta-1} \\
\nabla p(x,t) &= -\beta t^{-\alpha(m-1)-2\beta} x \\
\Delta p(x,t) &= -\beta t^{-\alpha(m-1)-2\beta} n
\end{aligned}$$

From the information we had above, we can subtitute it to (2) such that

$$\begin{aligned}
& p_t - (m-1)p\Delta p - |\nabla p|^2 \\
= \;& \Big(-\frac{\alpha}{t}(m-1)p(x,t) + \beta^2|x|^2 t^{-\alpha(m-1)-2\beta-1}\Big) - \Big((m-1)p(x,t)\big(-\beta t^{-\alpha(m-1)-2\beta} n\big)\Big) - |\big(-\beta t^{-\alpha(m-1)-2\beta} x\big)|^2 \\
= \;& \Big(-\frac{\alpha}{t} + \beta n t^{-\alpha(m-1)-2\beta}\Big)(m-1)p(x,t) + \beta^2|x|^2\Big(t^{-\alpha(m-1)-2\beta-1} - t^{-2\alpha(m-1)-4\beta}\Big) \\
= \;& 0
\end{aligned}$$

Because $\alpha, \beta, t, C > 0$, $m > 1$ and by $u(x,t) > 0$, then $p(x,t) > 0$. Then to make the equation above holds. We can let

$$\begin{aligned}
-\alpha t^{-1} + \beta n t^{-\alpha(m-1)-2\beta} &= 0 \\
\Leftrightarrow \quad \alpha t^{-1} &= \beta n t^{-\alpha(m-1)-2\beta} \\
\Leftrightarrow \quad \alpha = \beta n \quad & \text{and } t^{-1} = t^{-\alpha(m-1)-2\beta} \\
\Leftrightarrow \quad \alpha = \beta n \quad & \text{and } \beta = \frac{1}{n(m-1)+2} \\
\Leftrightarrow \quad \alpha = \frac{n}{n(m-1)+2} \quad & \text{and } \beta = \frac{1}{n(m-1)+2}
\end{aligned}$$

Substitute α and β we obtain into

$$\begin{aligned}
t^{-\alpha(m-1)-2\beta-1} - t^{-2\alpha(m-1)-4\beta} &= 0 \\
\Leftrightarrow \quad -\alpha(m-1) - 2\beta - 1 &= -2\alpha(m-1) - 4\beta \\
\Leftrightarrow \quad -\frac{n}{n(m-1)+2}(m-1) - 2\frac{1}{n(m-1)+2} - 1 &= -2\frac{n}{n(m-1)+2}(m-1) - 4\frac{1}{n(m-1)+2} \\
\Leftrightarrow \quad -n(m-1) - 2 - (n(m-1)+2) &= -2n(m-1) - 4 \\
\Leftrightarrow \quad -2nm + 2n - 4 &= -2nm + 2n - 4
\end{aligned}$$

So the $\alpha = \dfrac{n}{n(m-1)+2}$ and $\beta = \dfrac{1}{n(m-1)+2}$ we got is satisfy the u solution of (1).

2. For given $t > 0$, the set $\Omega(t) := \{x \in \mathbb{R}^n : u(x,t) > 0\}$ is an n-dimensional ball. Find its radius $r = r(t)$. Find $\lim_{t\to 0+} r(t)$ and $\lim_{t\to\infty} r(t)$.

Answer :
Because of the definition of $\Omega(t)$, then we take

$$u(x,t) = t^{-\alpha}\big(C - \frac{\beta(m-1)}{2m}\frac{|x|^2}{t^{2\beta}}\big)^{\frac{1}{m-1}}$$

Then, $u(x,t)$ should be positive for $x \in \mathbb{R}^n$ such that

$$u(-x,t) = u(x,t).$$

So we can see that the center of ball is 0 and the $|x| = r$. To find the radius, we can use the fact that

$$\begin{aligned}
C - \frac{\beta(m-1)}{2m}\frac{r^2}{t^{2\beta}} &= 0 \\
\Leftrightarrow \qquad \frac{r^2}{t^{2\beta}} &= \frac{2Cm}{\beta(m-1)} \\
\Leftrightarrow \qquad r(t) &= \sqrt{\frac{2Cmt^{2\beta}}{\beta(m-1)}}
\end{aligned}$$

We could find

$$\lim_{t\to 0+} r(t) = \lim_{t\to 0+} \sqrt{\frac{2Cmt^{2\beta}}{\beta(m-1)}} = 0$$

and

$$\lim_{t\to\infty} r(t) = \lim_{t\to\infty} \sqrt{\frac{2Cmt^{2\beta}}{\beta(m-1)}} = \infty$$

3. For $n=2$ and the constants α and β from (1), compute the value of

$$M(t) := \int_{\mathbb{R}^n} u(x,t)\ dx$$

for $t>0$. Does the solution conserve mass $M(t)$?

Answer :

$$\begin{aligned}
M(t) &:= \int_{\mathbb{R}^2} u(x,t)\ dx \\
&= \int_{x_1}\int_{x_2} u(x,t)\ dx_1 dx_2 \\
&= \int_\theta \int_r rt^{-\alpha}\big(C - \frac{\beta(m-1)}{2m}\frac{r^2}{t^{2\beta}}\big)^{\frac{1}{m-1}}\ drd\theta \\
&= \int_0^{2\pi}\int_0^{r(t)} t^{-\alpha}\big(Cr - \frac{\beta(m-1)}{2m}\frac{r^3}{t^{2\beta}}\big)^{\frac{1}{m-1}}\ drd\theta \\
&= t^{-\alpha}\int_0^{2\pi}\Big[\frac{Cr^2}{2} - \frac{\beta(m-1)r^4}{8mt^{2\beta}}\Big]_0^{r(t)} d\theta \\
&= t^{-\alpha}\int_0^{2\pi} \frac{C\big(\frac{2Cmt^{2\beta}}{\beta(m-1)}\big)}{2} - \frac{\beta(m-1)\big(\frac{2Cmt^{2\beta}}{\beta(m-1)}\big)^2}{8mt^{2\beta}} d\theta \\
&= t^{-\alpha}\int_0^{2\pi} \frac{C^2mt^{2\beta}}{2\beta(m-1)} d\theta \\
&= t^{-\alpha}\pi\frac{C^2mt^{2\beta}}{\beta(m-1)}
\end{aligned}$$

Substitute $\alpha = \dfrac{n}{n(m-1)+2}$ and $\beta = \dfrac{1}{n(m-1)+2}$ we got

$$M(t) = \Big(t^{-\big(\frac{n}{n(m-1)+2}\big)}\pi\Big)\Big(\frac{C^2mt^{2\big(\frac{1}{n(m-1)+2}\big)}}{\big(\frac{1}{n(m-1)+2}\big)(m-1)}\Big) = \frac{2C^2m^2\pi}{m-1}$$

As we can see, $M(t)$ is constant and independent of t, so the solution is conserve mass $M(t)$.