3D Incompressible Navier-Stokes

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Abstract

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1 Introduction

Navier-Stokes equation is considered as one of nonlinear equations that holds important rule in fluid mechanics. It can describe interesting phenomena of flow of viscous fluid like tornado. By understanding the behavior of tornadoes, we can study how to prevent greater loss when its happened.

In this research, using Finite Element Method (FEM) and software FreeFEM++, the 3D simulations of tornadoes is disscussed. First, considering cubes domain, better choice of time-step and time-discretization method is determined. Then changing the domain, the simulation is done in the cylindrical domain. Given initial condition with swirl and without swirl for tornado, simulation in cylindrical and curved cylindrical domain is disscussed.

1.1 Problem

Solving Navier-Stokes equations in FreeFEM++, the weak formulation of the equation is needed. In this research, the Incompressible condition term $\nabla \cdot u = 0$ is considered.

1.1.1 Strong formulation

We want to find

$$(u, p) : \Omega \times (0, T) \to \mathbb{R}^3 \times \mathbb{R}$$

where u is unknown velocity and p is unknown pressure with $\nu > 0$ is a viscosity, such that

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \triangle u + \nabla p = f & \text{in } \Omega \times (0, T) \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u = u^0 & \text{in } \Omega, \text{ at } t = 0 \end{cases} \tag{1}$$

where $f : \Omega \times (0, T) \to \mathbb{R}^3$ and $u^0 : \Omega \to \mathbb{R}^3$ are given functions.

1.1.2 Weak formulation

The weak formulation for equation (1) is shown below. We want to find $\{(u,p)(t) \in V \times Q; t \in (0.T)\}$ such that for $t \in (0,T)$

$$\begin{cases} \big(\frac{\partial u}{\partial t} + (u \cdot \nabla)u, v\big) + a(u,v) + b(v,p) + b(u,q) = (f,v) & ,\forall (v,q) \in V \times Q \\ u = u^0 & ,t=0 \end{cases} \tag{2}$$

where

$$\begin{aligned} a(u,v) &= \nu \int_\Omega \nabla u : \nabla v \; dx \\ b(v,q) &= -\int_\Omega (\nabla \cdot v) q \; dx \\ V &= H_0^1(\Omega, \mathbb{R}^d) = H_0^1(\Omega)^d \\ Q &= \{q \in L^2(\Omega); \int_\Omega q \; dx = 0\}. \end{aligned}$$

1.2 Time discretization

Before applying to FreeFEM++, we need to discritize $\Big(\frac{\partial u_i}{\partial t} + (u \cdot \nabla)u_i\Big)$ part, with dt as time increment and convect field

$$X_1(u_i^{n-1}, dt)(x) = x - u_i^{n-1}(x) \; dt.$$

1.2.1 First order

Using backward scheme,

$$\frac{\partial u_i}{\partial t} + (u \cdot \nabla)u_i \approx \frac{u_i^n - u_i^{n-1}(X_1(u_i^{n-1}, dt))}{dt} + O(dt)$$

1.2.2 Second order

Using Adam-Bashforth method,

$$\frac{\partial u_i}{\partial t} + (u \cdot \nabla)u_i \approx \frac{3u_i^n - 4u_i^{n-1}(X_1(\tilde{u}_i^{n-1}, dt)) + u_i^{n-2}(X_1(\tilde{u}_i^{n-1}, 2dt))}{2 \; dt} + O(dt^2)$$

with $\tilde{u}_i^{n-1} = 2u_i^{n-1} - u_i^{n-2}$.

1.3 Stabilization term

With $\delta > 0$ and h as mesh size, the stabilization term used is

$$C_i(p,q) = \delta \sum_k h_k^2 (\nabla p, \nabla q)_k$$

1.4 Error estimate

1.4.1 L^2 norm

$$\|u_h^n - u^n\|_{\ell^\infty(L^2)} = max \; \|u_h^n - u^n\|_{L^2}$$

with $O(h^2)$.

1.4.2 H^1 norm

$$\|u_h^n - u^n\|_{\ell^\infty(H^1)} = max\sqrt{\|u_h^n - u^n\|^2_{L^2(\Omega)} + \|\nabla(u_h^n - u^n)\|^2_{L^2(\Omega)}}$$

with $O(h)$.

2 Simulation with artificial solution

2.1 Cubic and cylindrical domain

With the exact solution

$$\begin{aligned} u &= (u_1, u_2, u_3) \\ u_1 &= -\cos(x_1)\sin(x_2)\cos(x_3)e^{-2t} \\ u_2 &= \sin(x_1)\cos(x_2)\cos(x_3)e^{-2t} \\ u_3 &= 0 \\ p &= \frac{-1}{4}e^{-4t}(\cos(2x_1) + \cos(2x_2) + \cos(2x_3)) \end{aligned}$$

such that equation (1) is satisfied with $f = (f_1, f_2, f_3)$

$$\begin{aligned} f_1 &= -\cos(x_1)\sin(x_2)\cos(x_3)e^{-2t} \\ f_2 &= -\sin(x_1)\cos(x_2)\cos(x_3)e^{-2t} \\ f_3 &= -(\frac{1}{4})e^{-4t}\sin(2x_3)(2\cos(2x_3) + 1) \end{aligned}$$

2.1.1 Cubic domain

For cubic domain with $z \in [0, 1]$, comparing time discretization first order and second order with L^2 and H^1 norm. For the first order in time with $dt = h = 1/n$,

$$\begin{aligned} L^2 &= O(dt) + O(h^2) = O(h) + O(h^2) = O(h) \\ H^1 &= O(dt) + O(h) = O(h) + O(h) = O(h) \end{aligned}$$

are expected. Then, for the second order in time with $dt = \dfrac{\sqrt{h}}{16}$,

$$\begin{aligned} L^2 &= O(dt^2) + O(h^2) = O(h) + O(h^2) = O(h) \\ H^1 &= O(dt^2) + O(h) = O(h) + O(h) = O(h) \end{aligned}$$

is expected. The result from the computation is shown in Figure (1).

From the plot, we can see that Adam-Bashforth method gives second order error. Other words, this method will provide better solution with smaller error increment.

2.1.2 Cylindrical domain

Choosing $a = 1/8$, $\epsilon_i = 1$, $\beta_i = 1$ for $(i = 1, \ldots, 6)$.

$$\Omega = \{x = (x, y, z) \in \mathbb{R}^3; -a \le z \le 4a, \sqrt{x^2 + y^2} < 1\}$$

and $u = 0$ on the boundary, cylindrical domain with with ratio 1, 0.4, and 0.1 is build. We look for L^2 and H^1 norm for second order in time such that $L^2 = O(h)$ and $H^1 = O(h)$ is expected for $dt = \dfrac{\sqrt{h}}{16}$.

From the Figure (2), the L^2 norm is seems correct before bigger dt. But, H^1 norm is still not correct. I expect, the definition space for the exact in the program is playing part for the decreasing order.

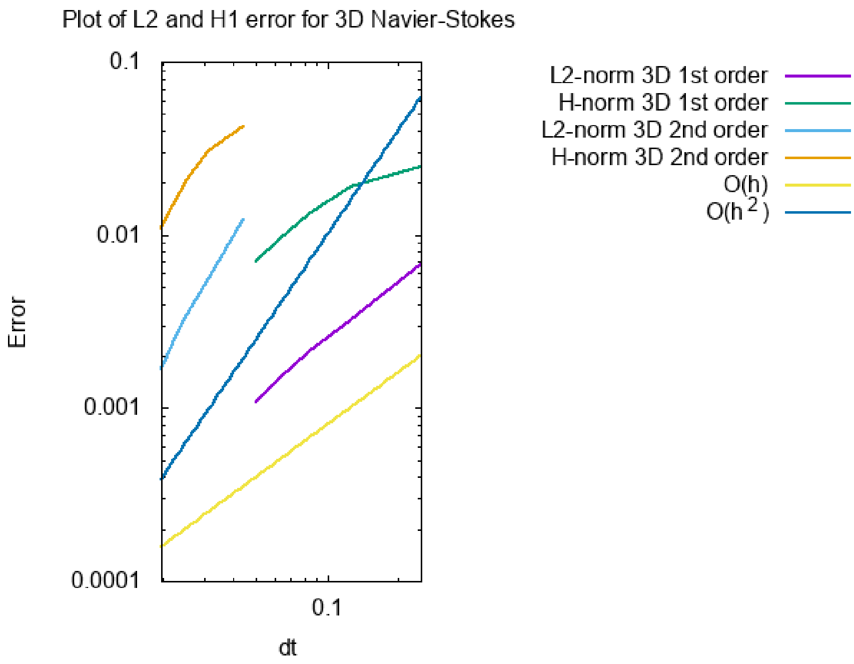

Figure 1: Compare L^2 and H^1 norm in cubic domain

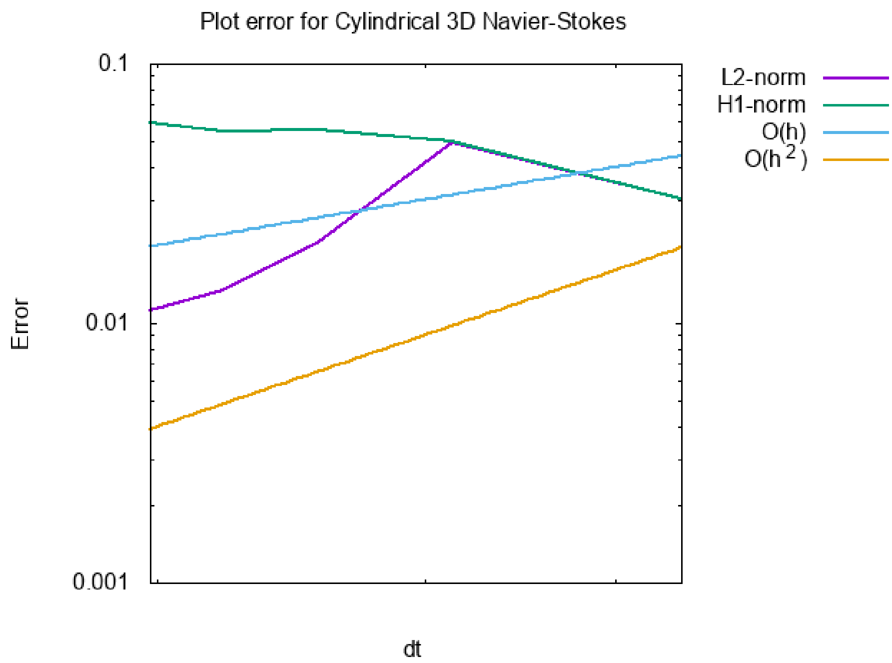

Figure 2: Compare L^2 and H^1 norm

2.2 Cylindrical and curved cylindrical domain with initial

Choose $a = 1/8$, $\epsilon_i = 1$, $\beta_i = 1$ for $(i = 1, \dots, 6)$.

$$\Omega = \{x = (x, y, z) \in \mathbb{R}^3; -a \le z \le 4a, \sqrt{x^2 + y^2} < 1\}$$

and $u = 0$ on the boundary. We have the initial

$$\begin{aligned}
\psi(a, \epsilon, \sigma) &= (a^2 + \epsilon)^\sigma \\
u_z &= \psi(r, \epsilon_1, -\beta_1)\psi(z, \epsilon_2, -\beta_2) \\
\rho &= \psi(r, \epsilon_3, -\beta_3)\psi(z, \epsilon_4, \beta_4) \\
u_0 &= \psi(r, \epsilon_5, -\beta_5)\psi(z, \epsilon_6, -\beta_6) && \text{(with swirl)} \\
u_0 &= 0 && \text{(no swirl)} \\
u_r &= sign(z)\rho u_z
\end{aligned}$$

2.2.1 Cylindrical domain

The error shown in the Figure (3) is **still not correc**t. I expect it happened because the artificial solution is not fit with the condition for given initial condition.

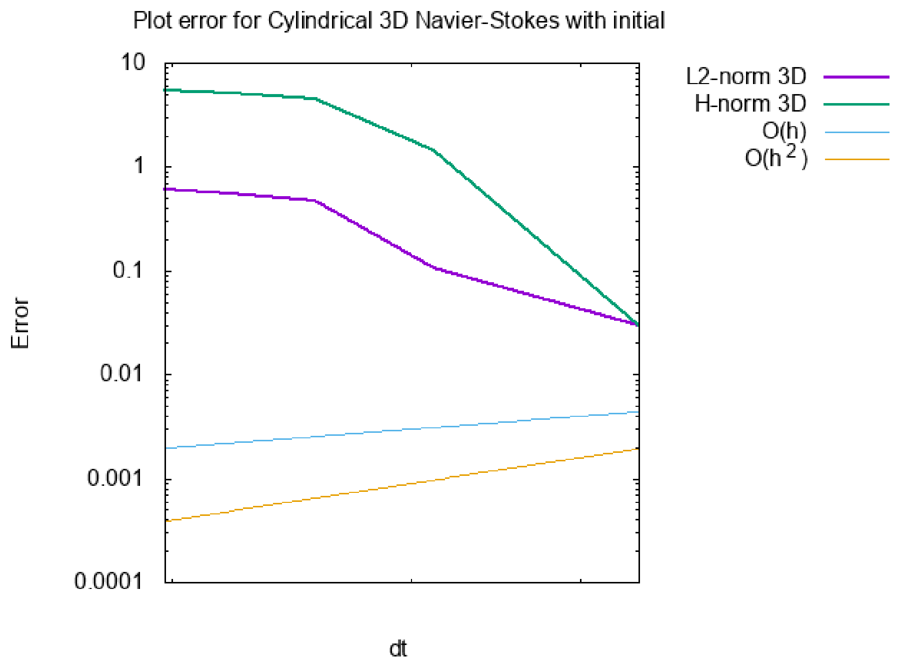

Figure 3: Compare L^2 and H^1 norm

2.2.2 Curved cylindrical domain

The curved domain is buid by the As the simulation in cylindrical domain, the error shown in the Figure (5) is **still not correct**. I expect it happened because the artificial solution is not fit with the condition for given initial condition.

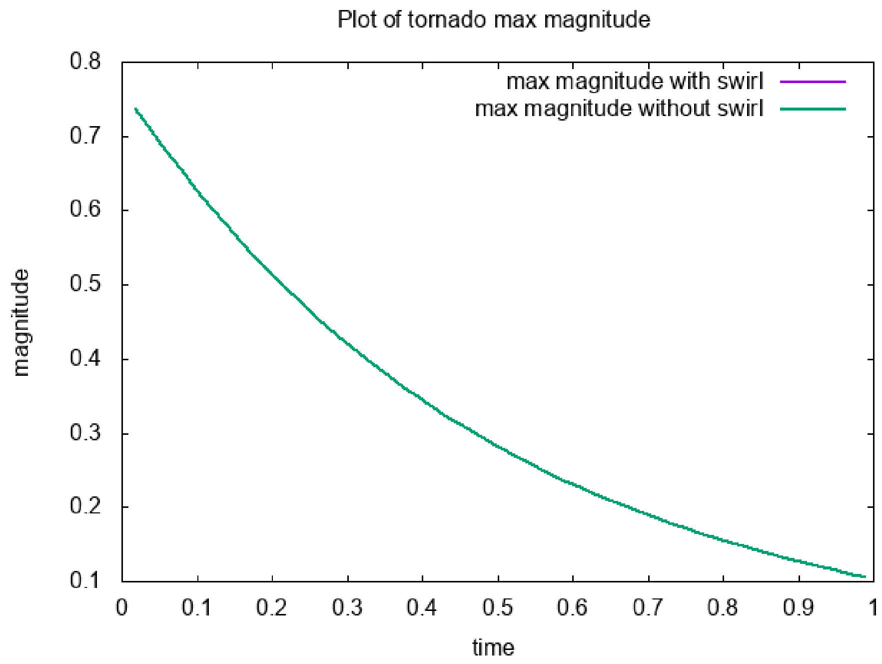

Figure 4: Maximum of u

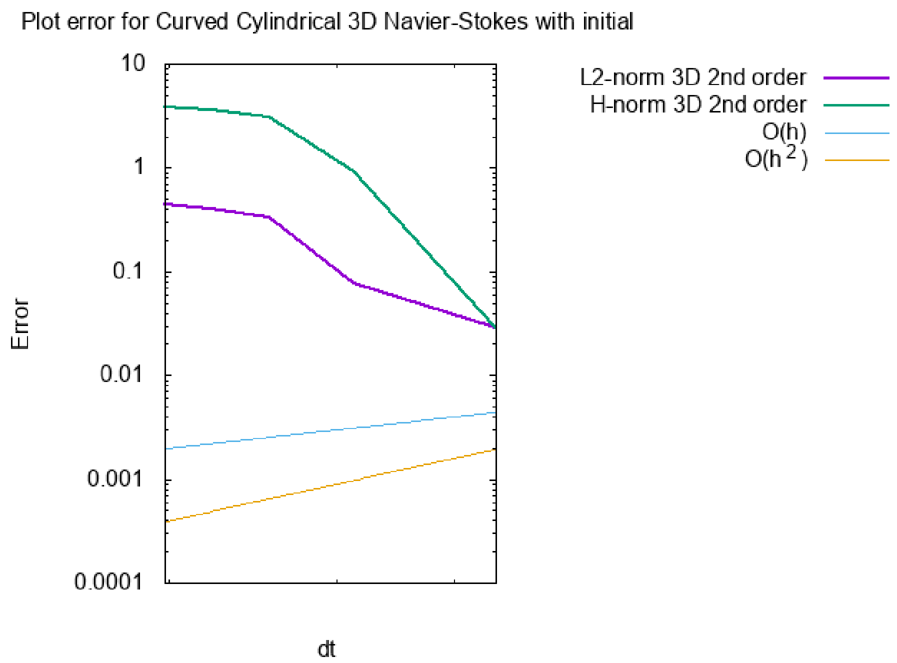

Figure 5: Compare L^2 and H^1 norm

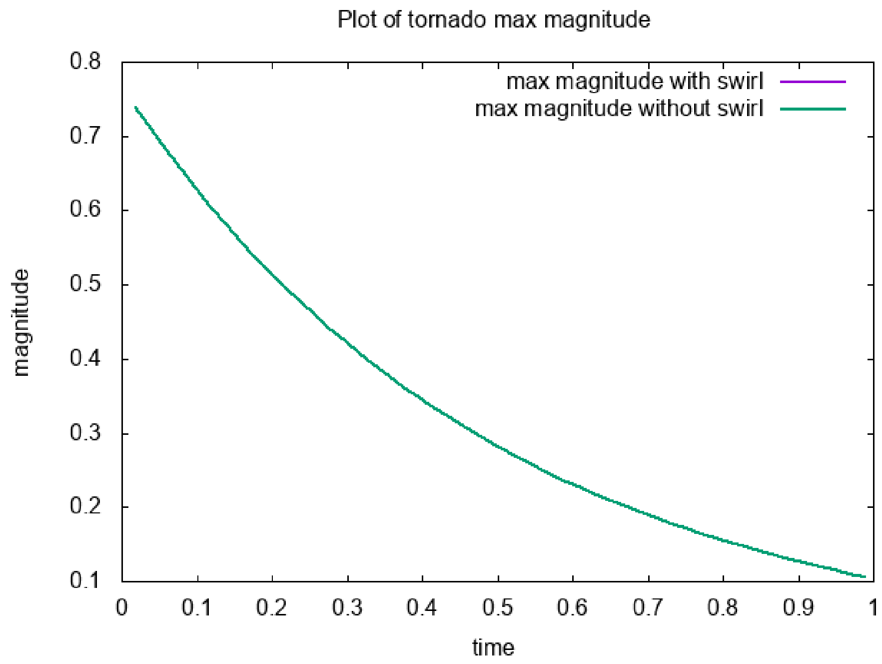

Figure 6: Maximum of u

Even through the simulation of artificial exact solution is done. There are still some doubts on the solution or the boundary conditioning. But in the movie, we could see the swirl is happening

3 Simulation of Tornadoes

3.1 Cylindrical domain

3.1.1 With swirl

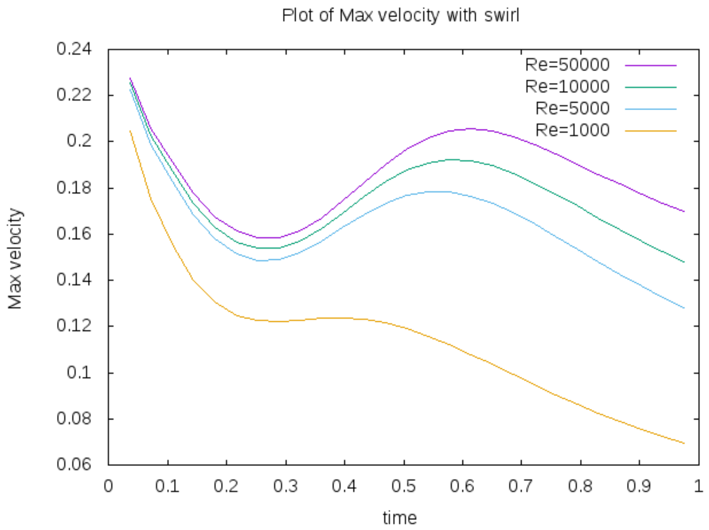

Figure 7:

3.1.2 Without swirl

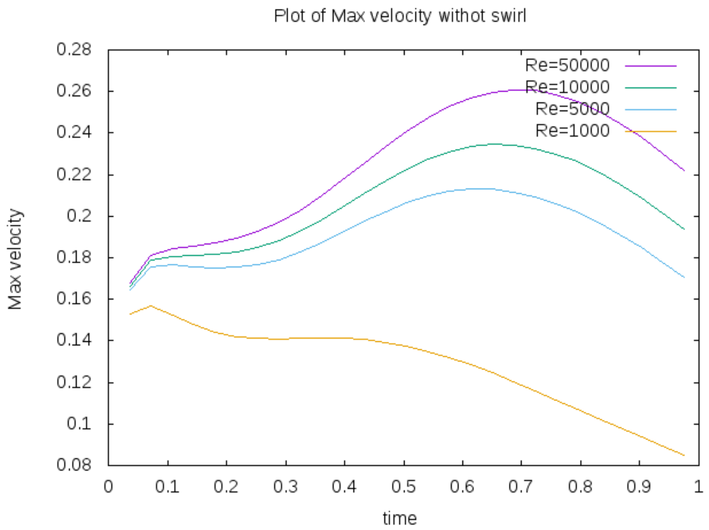

Figure 8:

3.2 Curved cylindrical domain

3.2.1 With swirl

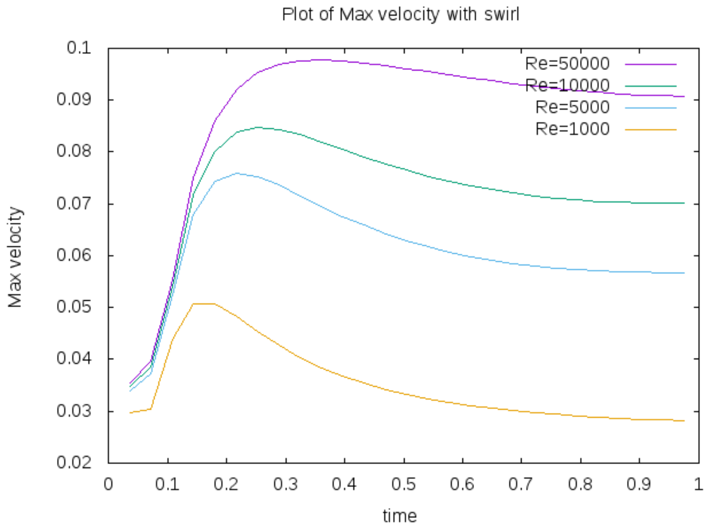

Figure 9:

3.2.2 Without swirl

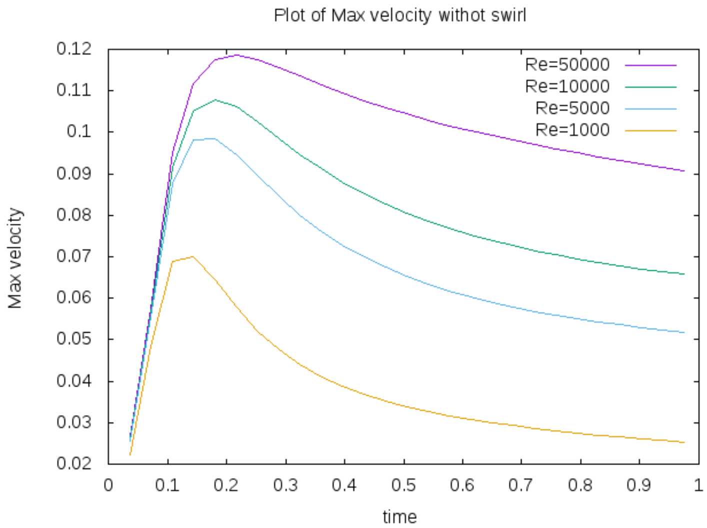

Figure 10: