

1 2D Incompressible Navier Stokes Equation

1.1 Problem

We want to find

$$(u, p) : \Omega \times (0, T) \rightarrow \mathbb{R}^d \times \mathbb{R}$$

where u is unknown velocity and p is unknown pressure such that

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f & \text{in } \Omega \times (0, T) \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u = u^0 & \text{in } \Omega, \text{ at } t = 0 \end{cases} \quad (1)$$

where $f : \Omega \times (0, T) \rightarrow \mathbb{R}^d$ and $u^0 : \Omega \rightarrow \mathbb{R}^d$ are given functions, $\nu > 0$ is a viscosity.

1.2 Weak Form

The weak formulation for equation (1) is shown below. We want to find $\{(u, p)(t) \in V \times Q; t \in (0, T)\}$ such that for $t \in (0, T)$

$$\begin{cases} \left(\frac{\partial u}{\partial t} + (u \cdot \nabla)u, v \right) + a(u, v) + b(v, p) + b(u, q) = (f, v) & , \forall (v, q) \in V \times Q \\ u = u^0 & , t = 0 \end{cases} \quad (2)$$

where

$$\begin{aligned} a(u, v) &= \nu \int_{\Omega} \nabla u : \nabla v \, dx \\ b(v, q) &= - \int_{\Omega} (\nabla \cdot v) q \, dx \\ V &= H_0^1(\Omega, \mathbb{R}^d) = H_0^1(\Omega)^d \\ Q &= \{q \in L^2(\Omega); \int_{\Omega} q \, dx = 0\}. \end{aligned}$$

1.3 Simulation

Below, is the exact solution to check if the program is working.

$$\begin{aligned} u &= (u_1, u_2) \\ u_1 &= -\cos(x_1)\sin(x_2)e^{-4t} \\ u_2 &= -\sin(x_1)\cos(x_2)e^{-4t} \\ p &= \frac{1}{4}(\cos(2x_1) + \cos(2x_2))e^{-4t} \end{aligned}$$