Weierstrass Institute for
Applied Analysis and Stochastics

Finite Element Methods for the Simulation of Incompressible Flows

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Outline of the Lectures

1 A Model for Incompressible Flows

- conservation laws
  - conservation of linear momentum
  - conservation of mass
- flow variables
  - $\rho(t,\mathbf{x})$: density $[kg/m^3]$
  - $\mathbf{v}(t,\mathbf{x})$: velocity $[m/s]$
  - $P(t,\mathbf{x})$: pressure $[N/m^2]$

  assumed to be sufficiently smooth in
- $\Omega \subset \mathbb{R}^3$
- $[0,T]$

1 Conservation of Mass

- change of fluid in arbitrary volume V

$$-\frac{\partial}{\partial t}\underbrace{\int_V \rho \, d\mathbf{x}}_{\text{mass}} = \underbrace{\int_{\partial V} \rho \mathbf{v} \cdot \mathbf{n} \, ds}_{\text{transport through bdry}} = \int_V \nabla \cdot (\rho \mathbf{v}) \, d\mathbf{x}$$

- V arbitrary $\Longrightarrow$ continuity equation

$$\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0$$

- incompressibility ($\rho = \text{const}$)

$$\nabla \cdot \mathbf{v} = 0$$

1 Newton's Second Law of Motion

- Newton's second law of motion

$$\text{net force} = \text{mass} \times \text{acceleration}$$

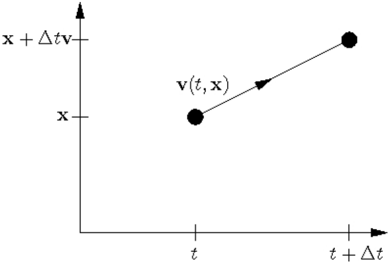
1 Newton's Second Law of Motion

- Newton's second law of motion

net force = mass $\times$ acceleration

- acceleration: using first order Taylor series expansion in time (board)

$$\frac{d\mathbf{v}}{dt}(t,\mathbf{x}) = \partial_t \mathbf{v}(t,\mathbf{x}) + (\mathbf{v}(t,\mathbf{x}) \cdot \nabla)\, \mathbf{v}(t,\mathbf{x})$$



movement of a particle

- acting forces on an arbitrary volume V:

  sum of external (body) forces

  - gravity

  and internal (molecular) forces

  - pressure
  - viscous drag that a 'fluid element' exerts on the 'adjacent element'
  - contact forces: act only on surface of 'fluid element'

$$\int_V \mathbf{F}(t,\mathbf{x})\, d\mathbf{x} + \int_{\partial V} \mathbf{t}(t,\mathbf{s})\, d\mathbf{s}$$

$\mathbf{t}$ $[N/m^2]$ – Cauchy stress vector

1 Newton's Second Law of Motion

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  sum of external (body) forces

  - gravity

  and internal (molecular) forces

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  - viscous drag that a 'fluid element' exerts on the 'adjacent element'
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$$\int_V \mathbf{F}(t,\mathbf{x})\, d\mathbf{x} + \int_{\partial V} \mathbf{t}(t,\mathbf{s})\, d\mathbf{s}$$

  $\mathbf{t}$ $[N/m^2]$ – Cauchy stress vector

- principle of Cauchy: internal contact forces depend (geometrically) only on the orientation of the surface

$$\mathbf{t} = \mathbf{t}(\mathbf{n})$$

  $\mathbf{n}$ – unit normal vector of the surface pointing outwards of V

1 Newton's Second Law of Motion

- it can be shown: conservation of linear momentum results in linear dependency on $\mathbf{n}$

$$\mathbf{t} = \mathbb{S}\mathbf{n}$$

  $\mathbb{S}(t,\mathbf{x})$ $[N/m^2]$ – stress tensor, dimension 3×3
- divergence theorem

$$\int_{\partial V} \mathbf{t}(t,\mathbf{s})\, d\mathbf{s} = \int_V \nabla \cdot \mathbb{S}(t,\mathbf{x})\, d\mathbf{x}$$

- momentum equation

$$\rho\left(\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = \nabla\cdot\mathbb{S} + \mathbf{F} \quad \forall\, t \in (0,T],\ \mathbf{x} \in \Omega$$

1 Newton's Second Law of Motion

- model for the stress tensor
  - torque
$$\mathbf{M_0} = \int_V \mathbf{r} \times \mathbf{F}\, d\mathbf{x} + \int_{\partial V} \mathbf{r} \times (\mathbb{S}\mathbf{n})\ d\mathbf{s} \quad [N\,m]$$
    at equilibrium is zero $\Longrightarrow$ symmetry $\mathbb{S} = \mathbb{S}^T$
  - decomposition
$$\mathbb{S} = \mathbb{V} + P\mathbb{I}$$
    $\mathbb{V}$ $[N/m^2]$ – viscous stress tensor
  - pressure P acts only normal to the surface, directed into V
$$-\int_{\partial V} P\mathbf{n}\, d\mathbf{s} = -\int_V \nabla P\, d\mathbf{x} = -\int_V \nabla \cdot (P\mathbb{I})\, d\mathbf{x}$$

- model for the stress tensor (cont.)
  - viscous stress tensor
    - friction between fluid particles can only occur if the particles move with different velocities
    - viscous stress tensor depends on gradient of velocity
    - because of symmetry: on symmetric part of the gradient: velocity deformation tensor

$$\mathbb{D}(\mathbf{v}) = \frac{\nabla \mathbf{v} + (\nabla \mathbf{v})^T}{2}$$

    - velocity not too large: dependency is linear (Newtonian fluids)

$$\mathbb{V} = 2\mu \mathbb{D}(\mathbf{v}) + \left(\zeta - \frac{2\mu}{3}\right)(\nabla \cdot \mathbf{v})\,\mathbb{I}$$

μ $[kg/(m\ s)]$ – dynamic or shear viscosity

ζ $[kg/(m\ s)]$ – second order viscosity

1 Navier–Stokes Equations

- general Navier–Stokes equations

$$\begin{aligned}
&\rho\left(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) \\
&-2\nabla\cdot\left(\mu\mathbb{D}(\mathbf{v})\right) - \nabla\cdot\left(\left(\zeta - \frac{2\mu}{3}\right)\nabla\cdot\mathbf{v}\mathbb{I}\right) + \nabla P &= \mathbf{F} \quad \text{in } (0,T]\times\Omega, \\
&\rho_t + \nabla\cdot(\rho\mathbf{v}) &= 0 \quad \text{in } (0,T]\times\Omega
\end{aligned}$$

1 Navier–Stokes Equations

- general Navier–Stokes equations

$$
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&-2\nabla\cdot\left(\mu\mathbb{D}(\mathbf{v})\right) - \nabla\cdot\left(\left(\zeta - \tfrac{2\mu}{3}\right)\nabla\cdot\mathbf{v}\mathbb{I}\right) + \nabla P = \mathbf{F} \quad \text{in } (0,T]\times\Omega, \\
&\rho_t + \nabla\cdot(\rho\mathbf{v}) = 0 \quad \text{in } (0,T]\times\Omega
\end{aligned}
$$

- incompressible flows: incompressible Navier–Stokes equations

$$
\begin{aligned}
\partial_t \mathbf{v} - 2\nu\nabla\cdot\mathbb{D}(\mathbf{v}) + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\frac{P}{\rho_0} &= \frac{\mathbf{F}}{\rho_0} \quad \text{in } (0,T]\times\Omega, \\
\nabla\cdot\mathbf{v} &= 0 \quad \text{in } (0,T]\times\Omega
\end{aligned}
$$

- Claude Louis Marie Henri Navier (1785 – 1836)
  George Gabriel Stokes (1819 – 1903)

1 Dimensionless Incompressible Navier–Stokes Equations

- dimensionless equations needed for (numerical) analysis and numerical simulations
- reference quantities of flow problem
  - $L\ [m]$ – a characteristic length scale
  - $U\ [m/s]$ – a characteristic velocity scale
  - $T^*\ [s]$ – a characteristic time scale
- transform of variables

$$\mathbf{x} = \frac{\mathbf{x}'}{L}, \quad \mathbf{u} = \frac{\mathbf{v}}{U}, \quad t = \frac{t'}{T^*}$$

- rescaling

$$\begin{aligned} \frac{L}{UT^*}\partial_t \mathbf{u} - \frac{2\nu}{UL}\nabla\cdot\mathbb{D}(\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\frac{P}{\rho_0 U^2} &= \frac{L}{\rho_0 U^2}\mathbf{F} \quad \text{in } (0,T]\times\Omega, \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } (0,T]\times\Omega, \end{aligned}$$

1 Dimensionless Incompressible Navier–Stokes Equations

- defining

$$p = \frac{P}{\rho_0 U^2}, \quad Re = \frac{UL}{\nu}, \quad St = \frac{L}{UT^*}, \quad \mathbf{f} = \frac{L}{\rho_0 U^2}\mathbf{F}$$

  p – new pressure
  Re – Reynolds number
  St – Strouhal number
  $\mathbf{f}$ – new right hand side

- result

$$\begin{aligned} St\,\partial_t \mathbf{u} - \frac{2}{Re}\nabla\cdot\mathbb{D}(\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } (0,T]\times\Omega, \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } (0,T]\times\Omega \end{aligned}$$

- generally $T^* = L/U \implies St = 1$

1 Dimensionless Incompressible Navier–Stokes Equations

- dimensionless Navier–Stokes equations
  - conservation of linear momentum
  - conservation of mass

$$\begin{aligned} \mathbf{u}_t - 2Re^{-1}\nabla\cdot\mathbb{D}(\mathbf{u}) + \nabla\cdot(\mathbf{u}\mathbf{u}^T) + \nabla p &= \mathbf{f} && \text{in } (0,T]\times\Omega \\ \nabla\cdot\mathbf{u} &= 0 && \text{in } [0,T]\times\Omega \\ \mathbf{u}(0,\mathbf{x}) &= \mathbf{u}_0 && \text{in } \Omega \end{aligned}$$

+ boundary conditions

- given:
  - $\Omega\subset\mathbb{R}^d, d\in\{2,3\}$: domain
  - T: final time
  - $\mathbf{u}_0$: initial velocity
  - boundary conditions

- to compute:
  - velocity $\mathbf{u}$, with

$$\mathbb{D}(\mathbf{u}) = \frac{\nabla\mathbf{u}+\nabla\mathbf{u}^T}{2},$$

velocity deformation tensor

  - pressure p

- parameter: Reynolds number Re

1 The Reynolds Number

- Reynolds number

$$\begin{aligned} Re &= \frac{LU}{\nu} \\ &= \frac{\text{convective forces}}{\text{viscous forces}} \end{aligned}$$

Osborne Reynolds (1842 – 1912)

- rough classification of flows:
  - Re small: steady-state flow field (if data do not depend on time)
  - Re larger: laminar time-dependent flow field
  - Re very large: turbulent flows

1 Dimensionless Incompressible Navier–Stokes Equations

- simplified form (for mathematics)

$$\begin{aligned}
\partial_t \mathbf{u} - 2\nu\nabla\cdot\mathbb{D}(\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } (0,T]\times\Omega, \\
\nabla\cdot\mathbf{u} &= 0 \quad \text{in } (0,T]\times\Omega
\end{aligned}$$

$\nu = Re^{-1}$ – dimensionless viscosity

1 Dimensionless Incompressible Navier–Stokes Equations

- simplified form (for mathematics)

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  $\nu = Re^{-1}$ – dimensionless viscosity
- alternative expression of viscous term (due to $\nabla\cdot\mathbf{u} = 0$)

$$2\nabla\cdot\mathbb{D}(\mathbf{u}) = \Delta\mathbf{u}$$

- alternative expression of convective term (due to $\nabla\cdot\mathbf{u} = 0$)

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\cdot(\mathbf{u}\mathbf{u}^T)$$

- special cases
  - steady-state Navier–Stokes equations: stationary flow fields

$$
\begin{aligned}
-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega \\
\nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega
\end{aligned}
$$

- special cases
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  - Oseen equations: convection field known (only for analysis)

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\mathbf{u}_0\cdot\nabla)\mathbf{u} + \nabla p + c\mathbf{u} &= \mathbf{f} \quad \text{in } \Omega \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega \end{aligned}$$

- special cases
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  - Stokes equations: no convection

$$\begin{aligned} -\Delta\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega \end{aligned}$$

- boundary conditions
  - Dirichlet boundary conditions (inflows)

$$\mathbf{u}(t,\mathbf{x}) = \mathbf{g}(t,\mathbf{x}) \text{ in } (0,T] \times \Gamma_{\text{diri}} \subset \Gamma$$

$\mathbf{g}(t,\mathbf{x}) = \mathbf{0}$ – no slip boundary condition (walls)

$$\mathbf{u}(t,\mathbf{x}) = \mathbf{0} \iff \mathbf{u}(t,\mathbf{x}) \cdot \mathbf{n} = 0,\ \mathbf{u}(t,\mathbf{x}) \cdot \mathbf{t}_1 = 0,\ \ \mathbf{u}(t,\mathbf{x}) \cdot \mathbf{t}_2 = 0$$

no penetration, no slip

- boundary conditions
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    no penetration, no slip
  - free slip boundary condition (e.g. symmetry planes)

$$\begin{aligned} \mathbf{u} \cdot \mathbf{n} &= g \quad \text{in } (0,T] \times \Gamma_{\text{slip}} \subset \Gamma, \\ \mathbf{n}^T \mathbb{S} \mathbf{t}_k &= 0 \quad \text{in } (0,T] \times \Gamma_{\text{slip}}, \quad 1 \le k \le d-1 \end{aligned}$$

- boundary conditions (cont.)
  - do-nothing boundary conditions (outflow)

$$\mathbb{S}\mathbf{n} = \mathbf{0} \ \text{ in } \ (0,T] \times \Gamma_{\text{outf}} \subset \Gamma$$

- boundary conditions (cont.)
  - do-nothing boundary conditions (outflow)

$$\mathbb{S}\mathbf{n} = \mathbf{0} \text{ in } (0,T] \times \Gamma_{\text{outf}} \subset \Gamma$$

  - periodic boundary conditions (only for analysis, $\Omega = (0,l)^d$)

$$\mathbf{u}(t,\mathbf{x}+l\mathbf{e}_i) = \mathbf{u}(t,\mathbf{x}) \quad \forall\ (t,\mathbf{x}) \in (0,T] \times \Gamma$$

- difficulties for mathematical analysis and numerical simulations
  - coupling of velocity and pressure
  - nonlinearity of the convective term
  - the convective term dominates the viscous term, i.e. ν is small

2 Linear Saddle Point Problems

- motivation
  - iterative solution of Navier–Stokes equations leads to linear system of equations
  - linear system have special form: saddle point problem
  - sufficient and necessary condition on unique solvability needed
  - can be derived in abstract form, see [1]

[1] Girault, Raviart: *Finite Element Methods for Navier-Stokes Equations* 1986

2 Linear Saddle Point Problems

- spaces: V, Q – real Hilbert spaces
- bilinear forms:

$$a(\cdot,\cdot) \ : \ V \times V \to \mathbb{R}, \quad b(\cdot,\cdot) \ : \ V \times Q \to \mathbb{R}$$

- linear problem: Find $(u,p) \in V \times Q$ such that for given $(f,r) \in V' \times Q'$

$$\begin{array}{rcll} a(u,v) + b(v,p) & = & \langle f, v \rangle_{V',V} & \forall\, v \in V, \\ b(u,q) & = & \langle r, q \rangle_{Q',Q} & \forall\, q \in Q \end{array}$$

- conditions on the spaces and bilinear forms necessary

- associated linear operators

$$A \in \mathscr{L}\left(V, V'\right) \quad \text{defined by} \quad \langle Au, v\rangle_{V',V} = a(u,v) \quad \forall\, u, v \in V$$
$$B \in \mathscr{L}\left(V, Q'\right) \quad \text{defined by} \quad \langle Bu, q\rangle_{Q',Q} = b(u,q) \quad \forall\, u \in V,\ \forall\, q \in Q$$

- dual operator: $B' \in \mathscr{L}(Q, V')$ defined by

$$\left\langle B'q, v\right\rangle_{V',V} = \langle Bv, q\rangle_{Q',Q} = b(v,q) \quad \forall\, v \in V,\ \forall\, q \in Q$$

- linear problem in operator form: Find $(u, p) \in V \times Q$ such that

$$\begin{array}{llcll} Au & +B'p & = & f & \text{in } V' \\ Bu & & = & r & \text{in } Q' \end{array}$$

- spaces
  - $V_0 := V(0) = \ker(B), \quad V = V_0^\perp \oplus V_0$
  - $\tilde{V}' = \{\phi \in V' \ : \ \langle \phi, v\rangle_{V',V} = 0 \ \ \forall\, v \in V_0\} \subset V'$
- inf-sup condition: The three following properties are equivalent:

  i) There exists a constant $\beta_{\text{is}} > 0$ such that

$$\inf_{q\in Q} \sup_{v\in V} \frac{b(v,q)}{\|v\|_V \, \|q\|_Q} \geq \beta_{\text{is}}.$$

  ii) The operator B' is an isomorphism from Q onto $\tilde{V}'$ and

$$\left\|B'q\right\|_{V'} \geq \beta_{\text{is}} \|q\|_Q \quad \forall\, q \in Q.$$

  iii) The operator B is an isomorphism from $V_0^\perp$ onto Q' and

$$\|Bv\|_{Q'} \geq \beta_{\text{is}} \|v\|_V \quad \forall\, v \in V_0^\perp.$$

2 The Inf-Sup Condition – Bilinear Form $b(\cdot,\cdot)$

- independently derived in [1,2]: Babuška–Brezzi condition
- sometimes: Ladyzhenskaya–Babuška–Brezzi condition, LBB condition
- it follows:

$$V(r) = \{v \in V \ : \ Bv = r\}$$

is not empty for all $r \in Q'$

[1] Babuška: Numer. Math. 20, 179–192, 1973

[2] Brezzi: Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge 8, 129–151, 1974

2 Unique Solution of Linear Saddle Point Problem

- sufficient and necessary conditions for unique solution of saddle point problem can be formulated with projection operator, see literature
- sufficient conditions
  - $a(\cdot,\cdot)$ is V_0-elliptic, i.e., there is a constant $\alpha > 0$ such that

$$a(v,v) \geq \alpha \|v\|_V^2 \quad \forall\, v \in V_0$$

  - $b(\cdot,\cdot)$ satisfies inf-sup condition

- for simplicity: Dirichlet boundary conditions on whole boundary
- velocity space

$$V = H_0^1(\Omega) = \left\{ \mathbf{v} \ : \ \mathbf{v} \in H^1(\Omega) \text{ with } \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega \right\}$$

with

$$(\mathbf{v}, \mathbf{w}) = \int_\Omega (\nabla \mathbf{v} \cdot \nabla \mathbf{w})(\mathbf{x})\, d\mathbf{x}, \quad \|\mathbf{v}\|_V := \|\nabla \mathbf{v}\|_{L^2(\Omega)}$$

dual space: $V' = H^{-1}(\Omega)$

- for simplicity: Dirichlet boundary conditions on whole boundary
- velocity space

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  dual space: $V' = H^{-1}(\Omega)$
- pressure space

$$Q = L_0^2(\Omega) = \left\{ q \; : \; q \in L^2(\Omega) \text{ with } \int_\Omega q(\mathbf{x}) \, d\mathbf{x} = 0 \right\}$$

  with

$$(q, r) = \int_\Omega (qr)(\mathbf{x}) \, d\mathbf{x}, \quad \|q\|_Q = \|q\|_{L^2(\Omega)}$$

- dual space: $Q' = Q$

- bilinear form for coupling velocity and pressure

$$b(\mathbf{v},q) = -\int_\Omega q\nabla\cdot\mathbf{v}\;d\mathbf{x} = -(\nabla\cdot\mathbf{v},q) \quad \mathbf{v}\in V,\ q\in Q$$

2 Continuous Incompressible Flow Problems

- bilinear form for coupling velocity and pressure

$$b(\mathbf{v},q) = -\int_\Omega q\nabla\cdot\mathbf{v}\,d\mathbf{x} = -(\nabla\cdot\mathbf{v},q) \quad \mathbf{v}\in V,\ q\in Q$$

- divergence operator

$$\text{div} \ : \ V \to \mathrm{range}(\text{div}), \quad \mathbf{v}\mapsto\nabla\cdot\mathbf{v}$$

- it can be shown: $\mathrm{range}(\text{div}) = Q'$
- associated linear operator: negative divergence operator

$$B\in\mathscr{L}(V,Q'), \quad B = -\text{div}$$

- dual operator: gradient operator

$$\text{grad} \; : \; Q \to \mathrm{range}(\text{grad}), \quad q \mapsto \nabla q$$

with

$$B' \in \mathscr{L}(Q, V'), \quad B' = \text{grad}$$

2 Continuous Incompressible Flow Problems

- dual operator: gradient operator

$$\text{grad} \; : \; Q \to \mathrm{range}(\text{grad}), \quad q \mapsto \nabla q$$

with

$$B' \in \mathscr{L}(Q, V'), \quad B' = \text{grad}$$

- kernel of B: space of weakly divergence-free functions

$$V_0 = V_{\mathrm{div}} = \{\mathbf{v} \in V \; : \; (\nabla \cdot \mathbf{v}, q) = 0 \; \forall \, q \in Q\}$$

- estimating divergence by gradient

$$\|\nabla \cdot \mathbf{v}\|_{L^2(\Omega)} \leq \sqrt{d}\,\|\nabla \mathbf{v}\|_{L^2(\Omega)} \quad \forall\, \mathbf{v} \in H^1(\Omega)$$

  - proof: board
  - estimate is sharp

- estimating divergence by gradient

$$\|\nabla \cdot \mathbf{v}\|_{L^2(\Omega)} \leq \sqrt{d}\, \|\nabla \mathbf{v}\|_{L^2(\Omega)} \quad \forall\, \mathbf{v} \in H^1(\Omega)$$

  - proof: board
  - estimate is sharp
- boundedness and continuity of $b(\cdot,\cdot)$

$$|b(\mathbf{v},q)| \leq \sqrt{d}\, \|\mathbf{v}\|_V \, \|q\|_Q$$

  - proof: board

- one can show: div is an isomorphism from $V_{\text{div}}^{\perp}$ onto Q
- corollary: each pressure is the divergence of a velocity field:
  for each $q \in Q$ there is a unique $\mathbf{v} \in V_{\text{div}}^{\perp} \subset V$ such that

$$\nabla \cdot \mathbf{v} = q \quad \text{and} \quad \|q\|_Q \le \sqrt{d}\,\|\mathbf{v}\|_V\,, \quad \|\mathbf{v}\|_V \le C\,\|q\|_Q$$

  with C independent of $\mathbf{v}$ and q
  - proof: board

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  with C independent of $\mathbf{v}$ and q
  - proof: board
- V and Q fulfill the inf-sup condition, i.e. there is a $\beta_{\text{is}} > 0$ such that

$$\inf_{q \in Q} \sup_{\mathbf{v} \in V} \frac{(\nabla \cdot \mathbf{v}, q)}{\|\mathbf{v}\|_V} \geq \beta_{\text{is}}$$

  - proof: board

2 Finite Element Spaces

- finite element spaces
  - V^h – finite element velocity space
  - Q^h – finite element pressure space
  - V^h/Q^h – pair
- conforming finite element spaces: $V^h \subset V$ and $Q^h \subset Q$

2 Finite Element Spaces

- finite element spaces
  - V^h – finite element velocity space
  - Q^h – finite element pressure space
  - V^h/Q^h – pair
- conforming finite element spaces: $V^h \subset V$ and $Q^h \subset Q$
- bilinear form $b^h \; : \; V^h \times Q^h \to \mathbb{R}$

$$b^h\left(\mathbf{v}^h, q^h\right) := -\sum_{K \in \mathscr{T}^h} \left(\nabla \cdot \mathbf{v}^h, q^h\right)_K$$

  - $\mathscr{T}^h$ – triangulation of Ω
  - $K \in \mathscr{T}^h$ – mesh cells
  - norm in V^h

$$\left\|\mathbf{v}^h\right\|_{V^h} = \sum_{K \in \mathscr{T}^h} \left(\nabla \mathbf{v}^h, \nabla \mathbf{v}^h\right)_K$$

2 Finite Element Spaces

- space of discretely divergence-free functions

$$V_{\text{div}}^h = \left\{ \mathbf{v}^h \in V^h \;:\; b^h\left(\mathbf{v}^h, q^h\right) = 0 \;\forall\, q^h \in Q^h \right\}$$

- generally $V_{\text{div}}^h \not\subset V_{\text{div}}$
  - finite element velocities not weakly or pointwise divergence-free
  - conservation of mass violated

2 Finite Element Spaces

- space of discretely divergence-free functions

$$V_{\text{div}}^h = \left\{ \mathbf{v}^h \in V^h \ : \ b^h\left(\mathbf{v}^h, q^h\right) = 0 \ \forall\, q^h \in Q^h \right\}$$

- generally $V_{\text{div}}^h \not\subset V_{\text{div}}$
  - finite element velocities not weakly or pointwise divergence-free
  - conservation of mass violated
- discrete inf-sup condition

$$\inf_{q^h \in Q^h} \sup_{\mathbf{v}^h \in V^h} \frac{b^h\left(\mathbf{v}^h, q^h\right)}{\|\mathbf{v}^h\|_{V^h} \|q^h\|_{L^2(\Omega)}} \geq \beta_{\text{is}}^h > 0$$

  - not inherited from inf-sup condition fulfilled by V and Q
  - discussion: board

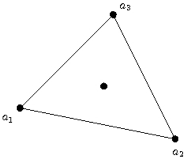
- Interpolation estimate for V_{div}^h. Let $\mathbf{v} \in V_{\mathrm{div}}$ and let the discrete inf-sup condition hold. Then

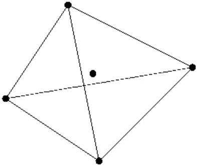
$$\inf_{\mathbf{v}^h \in V_{\mathrm{div}}^h} \left\| \nabla \left(\mathbf{v} - \mathbf{v}^h \right) \right\|_{L^2(\Omega)} \le \left(1 + \frac{\sqrt{d}}{\beta_{\mathrm{is}}^h} \right) \inf_{\mathbf{w}^h \in V^h} \left\| \nabla \left(\mathbf{v} - \mathbf{w}^h \right) \right\|_{L^2(\Omega)}$$

  - proof: board

2 Finite Elements

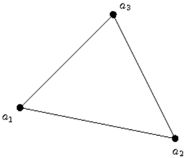
- piecewise constant finite elements P_0, (Q_0)

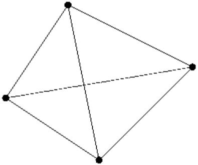




one degree of freedom (d.o.f.) per mesh cell

- continuous piecewise linear finite elements P_1

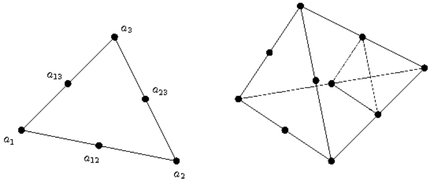




d d.o.f. per mesh cell

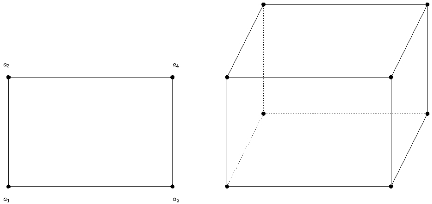
2 Finite Elements

- continuous piecewise quadratic finite elements P_2



$(d+1)(d+2)/2$ d.o.f. per mesh cell

- continuous piecewise bilinear finite elements Q_1

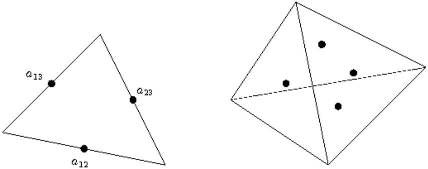


2^d d.o.f. per mesh cell

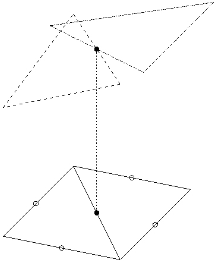
- and so on for continuous finite elements of higher order

2 Finite Elements

- nonconforming linear finite elements P_1^{nc}, Crouzeix, Raviart (1973)

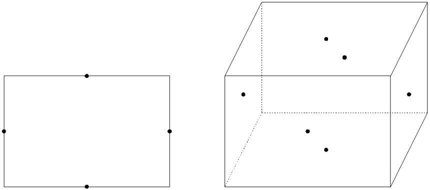


continuous only in barycenters of faces



$d+1$ d.o.f. per mesh cell

- rotated bilinear finite element Q_1^{rot}, Rannacher, Turek (1992)



  - continuous only in barycenters of faces
  - $2d$ d.o.f. per mesh cell

- discontinuous linear finite element P_1^{disc}
  - defined by integral nodal functionals
    e.g. $\varphi^h \in P_1^{\mathrm{disc}}$ if φ^h is linear on a mesh cell K (2d) and

$$\int_K \varphi^h(\mathbf{x})\, d\mathbf{x} = 0,\ \int_K x\varphi^h(\mathbf{x})\, d\mathbf{x} = 1,\ \int_K y\varphi^h(\mathbf{x})\, d\mathbf{x} = 0$$

  - $d+1$ d.o.f. per mesh cell

- criterion for violation of discrete inf-sup condition: there is non-trivial $q^h \in Q^h$ such that
$$b^h\left(\mathbf{v}^h, q^h\right) = 0 \quad \forall\, \mathbf{v}^h \in V^h$$
$\Longrightarrow$
$$\sup_{\mathbf{v}^h \in V^h} \frac{b^h\left(\mathbf{v}^h, q^h\right)}{\left\|\mathbf{v}^h\right\|_{V^h}} = 0$$

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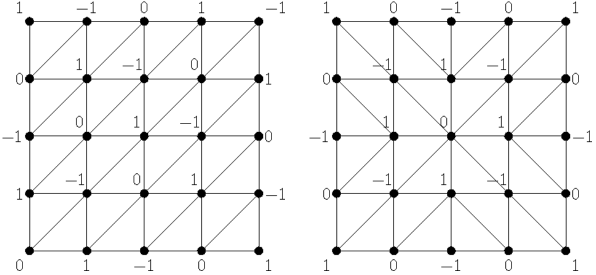
$$b^h\left(\mathbf{v}^h, q^h\right) = 0 \quad \forall\, \mathbf{v}^h \in V^h$$

$\Longrightarrow$

$$\sup_{\mathbf{v}^h \in V^h} \frac{b^h\left(\mathbf{v}^h, q^h\right)}{\|\mathbf{v}^h\|_{V^h}} = 0$$

- P_1/P_1 pair of finite element spaces violates discrete inf-sup condition
  - counter example: checkerboard instability, board



2 Finite Element Spaces

- other pairs which violated discrete inf-sup condition
  - P_1/P_0
  - Q_1/Q_0
  - P_k/P_k, $k \geq 1$
  - Q_k/Q_k, $k \geq 1$
  - $P_k/P_{k-1}^{\text{disc}}$, $k \geq 2$, on a special macro cell
- summary:
  - many easy to implement pairs violate discrete inf-sup condition
  - different finite element spaces for velocity and pressure necessary

2 Finite Element Spaces

- pairs which fulfill discrete inf-sup condition
  - P_k/P_{k-1}, Q_k/Q_{k-1}: Taylor–Hood finite elements [1]
    - proofs: 2D, $k = 2$ [2]
  - $Q_k/Q_{k-1}^{\mathrm{disc}}$
  - $P_k/P_{k-1}^{\mathrm{disc}}$, $k \geq d$, on very special meshes (Scott–Vogelius element)
  - $P_1^{\mathrm{bubble}}/P_1$, mini element
  - $P_k^{\mathrm{bubble}}/P_{k-1}^{\mathrm{disc}}$ [3]
  - P_1^{nc}/P_0, Crouzeix–Raviart element [4]
  - Q_1^{rot}/Q_0, Rannacher–Turek element [5]
  - $\vdots$

---

[1] Taylor, Hood: Comput. Fluids 1, 73-100, 1973

[2] Verfürth: RAIRO Anal. Numér. 18, 175–182, 1984

[3] Bernardi, Raugel: Math. Comp. 44, 71–79, 1985

[4] Crouzeix, Raviart: RAIRO. Anal. Numér. 7, 33–76, 1973

[5] Rannacher, Turek: Numer. Meth. Part. Diff. Equ. 8, 97–111, 1992

2 Finite Element Spaces

- techniques for proving the discrete inf-sup condition
  - construction of Fortin operator [1]
  - using projection to piecewise constant pressure [2]
  - macroelement technique [3]
  - survey in [4]

[1] Fortin: RAIRO Anal. Numér. 11, 341–354, 1977

[2] Brezzi, Bathe: Comput. Methods Appl. Mech. Engrg. 82, 27–57, 1990

[3] Stenberg: Math. Comput. 32, 9–23, 1984

[4] Boffi, Brezzi, Fortin: Lecture Notes in Mathematics 1939, Springer, 45–100, 2008

3 The Stokes Equations

- continuous equation

$$
\begin{aligned}
-\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\
\nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega
\end{aligned}
\tag{1}
$$

  for simplicity: homogeneous Dirichlet boundary conditions

- difficulty: coupling of velocity and pressure
- properties
  - linear
  - form

$$
\begin{aligned}
-\nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\
\nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega
\end{aligned}
$$

    becomes (1) by rescaling with new pressure, right hand side

3 The Stokes Equations

- weak form: Find $(\mathbf{u}, p) \in H_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$
\begin{array}{rcll}
(\nabla \mathbf{u}, \nabla \mathbf{v}) - (\nabla \cdot \mathbf{v}, p) & = & \langle \mathbf{f}, \mathbf{v} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} & \forall\, \mathbf{v} \in H_0^1(\Omega), \\
-(\nabla \cdot \mathbf{u}, q) & = & 0 & \forall\, q \in L_0^2(\Omega)
\end{array}
$$

- casting into abstract framework
  - spaces

$$
V = H_0^1(\Omega),\ \|\cdot\|_V = |\cdot|_{H^1(\Omega)}, \quad Q = L_0^2(\Omega),\ \|\cdot\|_Q = \|\cdot\|_{L^2(\Omega)}
$$

  - bilinear forms

$$
a(\mathbf{u}, \mathbf{v}) = (\nabla \mathbf{u}, \nabla \mathbf{v}), \quad b(\mathbf{v}, q) = -(\nabla \cdot \mathbf{v}, q)
$$

3 The Stokes Equations

- weak form: Find $(\mathbf{u}, p) \in H_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\begin{aligned} (\nabla \mathbf{u}, \nabla \mathbf{v}) - (\nabla \cdot \mathbf{v}, p) &= \langle \mathbf{f}, \mathbf{v} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \quad && \forall\, \mathbf{v} \in H_0^1(\Omega), \\ -(\nabla \cdot \mathbf{u}, q) &= 0 && \forall\, q \in L_0^2(\Omega) \end{aligned}$$

- casting into abstract framework
  - spaces

$$V = H_0^1(\Omega),\ \|\cdot\|_V = |\cdot|_{H^1(\Omega)}, \quad Q = L_0^2(\Omega),\ \|\cdot\|_Q = \|\cdot\|_{L^2(\Omega)}$$

  - bilinear forms

$$a(\mathbf{u}, \mathbf{v}) = (\nabla \mathbf{u}, \nabla \mathbf{v}), \quad b(\mathbf{v}, q) = -(\nabla \cdot \mathbf{v}, q)$$

- equivalent formulation: Find $(\mathbf{u}, p) \in V \times Q$ such that

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) - b(\mathbf{u}, q) = \langle \mathbf{f}, \mathbf{v} \rangle_{V', V} \quad \forall\, (\mathbf{v}, q) \in V \times Q$$

- V_{div} – space of weakly divergence-free functions
- associated problem: Find $\mathbf{u} \in V_{\mathrm{div}}$ such that

$$(\nabla \mathbf{u}, \nabla \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_{V',V} \quad \forall\, \mathbf{v} \in V_{\mathrm{div}}$$

- V_{div} – space of weakly divergence-free functions
- associated problem: Find $\mathbf{u} \in V_{\text{div}}$ such that

$$(\nabla \mathbf{u}, \nabla \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_{V',V} \quad \forall\, \mathbf{v} \in V_{\text{div}}$$

- existence and uniqueness of solution
  - $a(\cdot,\cdot)$ is V_{div}-elliptic

$$a(\mathbf{v},\mathbf{v}) = |\mathbf{v}|^2_{H^1(\Omega)} \quad \forall\, \mathbf{v} \in V \supset V_{\text{div}}$$

  - $b(\cdot,\cdot)$ satisfies inf-sup condition

- V_{div} – space of weakly divergence-free functions
- associated problem: Find $\mathbf{u} \in V_{\text{div}}$ such that

$$(\nabla \mathbf{u}, \nabla \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_{V',V} \quad \forall\, \mathbf{v} \in V_{\text{div}}$$

- existence and uniqueness of solution
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$$a(\mathbf{v}, \mathbf{v}) = |\mathbf{v}|^2_{H^1(\Omega)} \quad \forall\, \mathbf{v} \in V \supset V_{\text{div}}$$

  - $b(\cdot,\cdot)$ satisfies inf-sup condition
- stability of solution

$$\|\nabla \mathbf{u}\|_{L^2(\Omega)} \le \|\mathbf{f}\|_{H^{-1}(\Omega)}, \quad \|p\|_{L^2(\Omega)} \le \frac{2}{\beta_{\text{is}}} \|\mathbf{f}\|_{H^{-1}(\Omega)}$$

  - proof and discussion: board

3 Finite Element Methods for the Stokes Equations

- finite element problem: Find $(\mathbf{u}^h, p^h) \in V^h \times Q^h$ such that

$$\begin{aligned} a^h\left(\mathbf{u}^h, \mathbf{v}^h\right) + b^h\left(\mathbf{v}^h, p^h\right) &= \left(\mathbf{f}, \mathbf{v}^h\right) \quad \forall\, \mathbf{v}^h \in V^h, \\ b^h\left(\mathbf{u}^h, q^h\right) &= 0 \qquad\quad\ \ \forall\, q^h \in Q^h \end{aligned}$$

with

$$a^h\left(\mathbf{v}^h, \mathbf{w}^h\right) = \sum_{K \in \mathscr{T}^h} \left(\nabla \mathbf{v}^h, \nabla \mathbf{w}^h\right)_K, \quad b^h\left(\mathbf{v}^h, q^h\right) = -\sum_{K \in \mathscr{T}^h} \left(\nabla \cdot \mathbf{v}^h, q^h\right)_K$$

- finite element problem: Find $(\mathbf{u}^h, p^h) \in V^h \times Q^h$ such that

$$\begin{aligned} a^h\left(\mathbf{u}^h, \mathbf{v}^h\right) + b^h\left(\mathbf{v}^h, p^h\right) &= \left(\mathbf{f}, \mathbf{v}^h\right) \quad \forall\, \mathbf{v}^h \in V^h, \\ b^h\left(\mathbf{u}^h, q^h\right) &= 0 \qquad\quad \forall\, q^h \in Q^h \end{aligned}$$

with

$$a^h\left(\mathbf{v}^h, \mathbf{w}^h\right) = \sum_{K \in \mathscr{T}^h} \left(\nabla \mathbf{v}^h, \nabla \mathbf{w}^h\right)_K, \quad b^h\left(\mathbf{v}^h, q^h\right) = -\sum_{K \in \mathscr{T}^h} \left(\nabla \cdot \mathbf{v}^h, q^h\right)_K$$

- only conforming inf-sup stable finite element spaces
  - $V^h \subset V$ and $Q^h \subset Q$
  - $$\inf_{q^h \in Q^h} \sup_{\mathbf{v}^h \in V^h} \frac{b^h\left(\mathbf{v}^h, q^h\right)}{\|\mathbf{v}^h\|_{V^h} \|q^h\|_{L^2(\Omega)}} \geq \beta_{\text{is}}^h > 0$$

- existence and uniqueness of a solution
  - same proof as for continuous problem

- existence and uniqueness of a solution
  - same proof as for continuous problem
- stability

$$\left\|\nabla \mathbf{u}^h\right\|_{L^2(\Omega)} \le \|\mathbf{f}\|_{H^{-1}(\Omega)}, \quad \left\|p^h\right\|_{L^2(\Omega)} \le \frac{2}{\beta_{\mathrm{is}}^h} \|\mathbf{f}\|_{H^{-1}(\Omega)}$$

  - same proof as for continuous problem

- existence and uniqueness of a solution
  - same proof as for continuous problem
- stability

$$\left\|\nabla \mathbf{u}^h\right\|_{L^2(\Omega)} \leq \|\mathbf{f}\|_{H^{-1}(\Omega)}, \quad \left\|p^h\right\|_{L^2(\Omega)} \leq \frac{2}{\beta_{\mathrm{is}}^h} \|\mathbf{f}\|_{H^{-1}(\Omega)}$$

  - same proof as for continuous problem
- goal of finite element error analysis: estimate error by interpolation errors
  - interpolation errors depend only on finite element spaces, not on problem
  - estimates for interpolation error are known

- existence and uniqueness of a solution
  - same proof as for continuous problem
- stability

$$\left\|\nabla \mathbf{u}^h\right\|_{L^2(\Omega)} \le \|\mathbf{f}\|_{H^{-1}(\Omega)}, \quad \left\|p^h\right\|_{L^2(\Omega)} \le \frac{2}{\beta_{\text{is}}^h} \|\mathbf{f}\|_{H^{-1}(\Omega)}$$

  - same proof as for continuous problem
- goal of finite element error analysis: estimate error by interpolation errors
  - interpolation errors depend only on finite element spaces, not on problem
  - estimates for interpolation error are known
- reduction to a problem on the space of discretely divergence-free functions

$$a\left(\mathbf{u}^h, \mathbf{v}^h\right) = \left(\nabla \mathbf{u}^h, \nabla \mathbf{v}^h\right) = \left(\mathbf{f}, \mathbf{v}^h\right) \ \forall\, \mathbf{v}^h \in V_{\text{div}}^h$$

- finite element error estimate for the $L^2(\Omega)$ norm of the gradient of the velocity
  - $\Omega \subset \mathbb{R}^d$, bounded, polyhedral, Lipschitz-continuous boundary

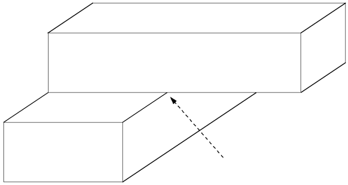
$$
\begin{aligned}
\left\| \nabla (\mathbf{u} - \mathbf{u}^h) \right\|_{L^2(\Omega)} \quad \leq \quad & 2 \left(1 + \frac{\sqrt{d}}{\beta_{\mathrm{is}}^h} \right) \inf_{\mathbf{v}^h \in V^h} \left\| \nabla (\mathbf{u} - \mathbf{v}^h) \right\|_{L^2(\Omega)} \\
& + \sqrt{d} \inf_{q^h \in Q^h} \left\| p - q^h \right\|_{L^2(\Omega)}
\end{aligned}
$$

  - proof: board

- finite element error estimate for the $L^2(\Omega)$ norm of the gradient of the velocity
  - $\Omega \subset \mathbb{R}^d$, bounded, polyhedral, Lipschitz-continuous boundary

$$\left\|\nabla\left(\mathbf{u}-\mathbf{u}^h\right)\right\|_{L^2(\Omega)} \leq 2\left(1+\frac{\sqrt{d}}{\beta_{\mathrm{is}}^h}\right)\inf_{\mathbf{v}^h\in V^h}\left\|\nabla\left(\mathbf{u}-\mathbf{v}^h\right)\right\|_{L^2(\Omega)} + \sqrt{d}\inf_{q^h\in Q^h}\left\|p-q^h\right\|_{L^2(\Omega)}$$

  - proof: board
- polyhedral domain in three dimensions which is not Lipschitz-continuous



- finite element error estimate for the $L^2(\Omega)$ norm of the pressure
  - same assumptions as for previous estimate

$$
\begin{aligned}
\left\|p - p^h\right\|_{L^2(\Omega)} \quad &\leq \quad \frac{2}{\beta_{\mathrm{is}}^h}\left(1 + \frac{\sqrt{d}}{\beta_{\mathrm{is}}^h}\right) \inf_{\mathbf{v}^h \in V^h} \left\|\nabla(\mathbf{u} - \mathbf{v}^h)\right\|_{L^2(\Omega)} \\
&\quad + \left(1 + \frac{2\sqrt{d}}{\beta_{\mathrm{is}}^h}\right) \inf_{q^h \in Q^h} \left\|p - q^h\right\|_{L^2(\Omega)}
\end{aligned}
$$

  - proof: board

- error of the velocity in the $L^2(\Omega)$ norm
  - by Poincaré inequality not optimal

$$\left\| \mathbf{u} - \mathbf{u}^h \right\|_{L^2(\Omega)} \leq C \left\| \nabla (\mathbf{u} - \mathbf{u}^h) \right\|_{L^2(\Omega)}$$

- error of the velocity in the $L^2(\Omega)$ norm
  - by Poincaré inequality not optimal

$$\left\|\mathbf{u}-\mathbf{u}^h\right\|_{L^2(\Omega)} \leq C\left\|\nabla(\mathbf{u}-\mathbf{u}^h)\right\|_{L^2(\Omega)}$$

- regular dual Stokes problem: For given $\hat{\mathbf{f}} \in L^2(\Omega)$, find $(\phi_{\hat{\mathbf{f}}}, \xi_{\hat{\mathbf{f}}}) \in V \times Q$ such that

$$\begin{aligned} -\Delta\phi_{\hat{\mathbf{f}}} + \nabla\xi_{\hat{\mathbf{f}}} &= \hat{\mathbf{f}} \quad \text{in } \Omega, \\ \nabla\cdot\phi_{\hat{\mathbf{f}}} &= 0 \quad \text{in } \Omega \end{aligned}$$

  - regular if mapping

$$\left(\phi_{\hat{\mathbf{f}}}, \xi_{\hat{\mathbf{f}}}\right) \mapsto -\Delta\phi_{\hat{\mathbf{f}}} + \nabla\xi_{\hat{\mathbf{f}}}$$

    is an isomorphism from $\left(H^2(\Omega)\cap V\right) \times \left(H^1(\Omega)\cap Q\right)$ onto $L^2(\Omega)$
  - Γ of class C^2
  - bounded, convex polygons in two dimensions

- finite element error estimate for the $L^2(\Omega)$ norm of the velocity
  - same assumptions as for previous estimates
  - dual Stokes problem regular with solution $(\phi_{\hat{\mathbf{f}}}, \xi_{\hat{\mathbf{f}}})$

$$
\begin{aligned}
&\left\|\mathbf{u}-\mathbf{u}^h\right\|_{L^2(\Omega)} \\
&\quad \leq \quad \sqrt{d}\left(\left\|\nabla\left(\mathbf{u}-\mathbf{u}^h\right)\right\|_{L^2(\Omega)} + \inf_{q^h\in Q^h}\left\|p-q^h\right\|_{L^2(\Omega)}\right) \\
&\qquad\quad \times \sup_{\hat{\mathbf{f}}\in L^2(\Omega)} \frac{1}{\left\|\hat{\mathbf{f}}\right\|_{L^2(\Omega)}}\left[\left(1+\frac{\sqrt{d}}{\beta_{\mathrm{is}}^h}\right)\inf_{\phi^h\in V^h}\left\|\nabla\left(\phi_{\hat{\mathbf{f}}}-\phi^h\right)\right\|_{L^2(\Omega)}\right. \\
&\qquad\qquad\qquad \left. + \inf_{r^h\in Q^h}\left\|\xi_{\hat{\mathbf{f}}}-r^h\right\|_{L^2(\Omega)}\right]
\end{aligned}
$$

  - proof: board (if time admits)

- finite element error estimates for conforming pairs of finite element spaces
  - same assumptions on domain as for previous estimates
  - solution sufficiently regular
  - h – mesh width of triangulation
  - spaces
    - P_k^{bubble}/P_k, $k=1$ (mini element),
    - P_k/P_{k-1}, Q_k/Q_{k-1}, $k \geq 2$ (Taylor–Hood element),
    - $P_k^{\text{bubble}}/P_{k-1}^{\text{disc}}$, $Q_k/P_{k-1}^{\text{disc}}$,$k \geq 2$

$$\begin{aligned} \left\|\nabla(\mathbf{u}-\mathbf{u}^h)\right\|_{L^2(\Omega)} &\leq Ch^k\left(\|\mathbf{u}\|_{H^{k+1}(\Omega)}+\|p\|_{H^k(\Omega)}\right) \\ \left\|p-p^h\right\|_{L^2(\Omega)} &\leq Ch^k\left(\|\mathbf{u}\|_{H^{k+1}(\Omega)}+\|p\|_{H^k(\Omega)}\right) \end{aligned}$$

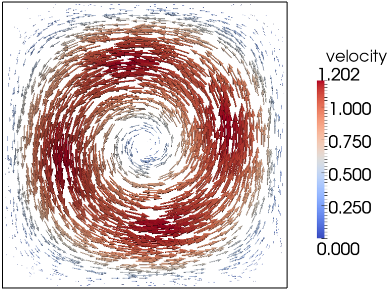
- finite element error estimates for conforming pairs of finite element spaces (cont.)
  - in addition: dual Stokes problem regular

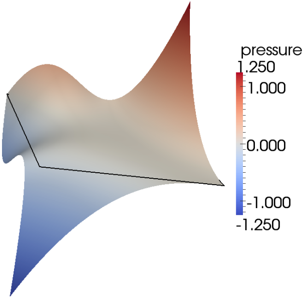
$$\left\|\mathbf{u}-\mathbf{u}^h\right\|_{L^2(\Omega)} \leq Ch^{k+1}\left(\|\mathbf{u}\|_{H^{k+1}(\Omega)}+\|p\|_{H^k(\Omega)}\right)$$

  - all C depend on the discrete inf-sup constant β_{is}^h

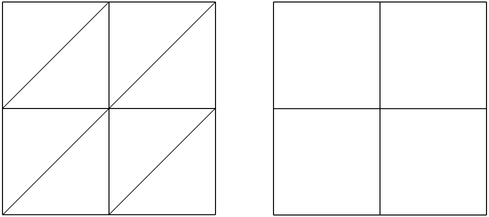
- analytical example which supports the error estimates
- prescribed solution

$$
\begin{aligned}
\mathbf{u} &= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \partial_y \psi \\ -\partial_x \psi \end{pmatrix} = 200 \begin{pmatrix} x^2(1-x)^2 y(1-y)(1-2y) \\ -x(1-x)(1-2x)y^2(1-y)^2 \end{pmatrix} \\
p &= 10\left(\left(x-\frac{1}{2}\right)^3 y^2 + (1-x)^3\left(y-\frac{1}{2}\right)^3\right)
\end{aligned}
$$



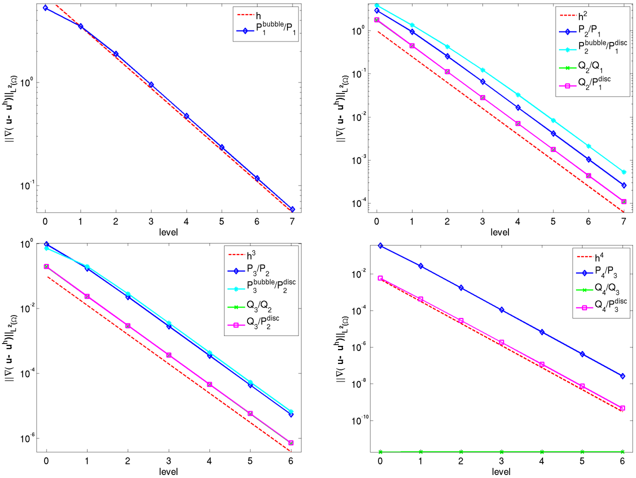


- initial grids (level 0)

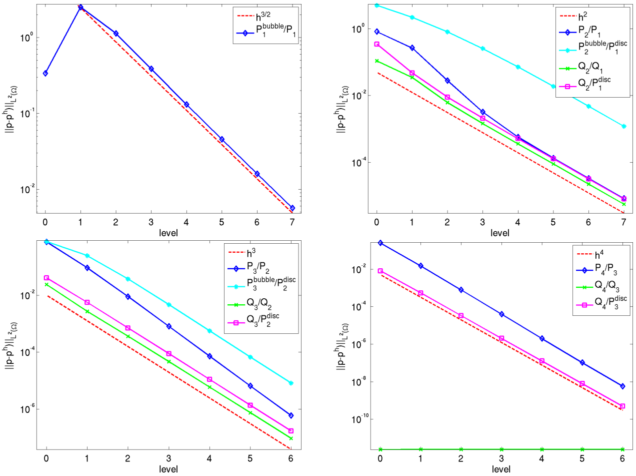


- red refinement

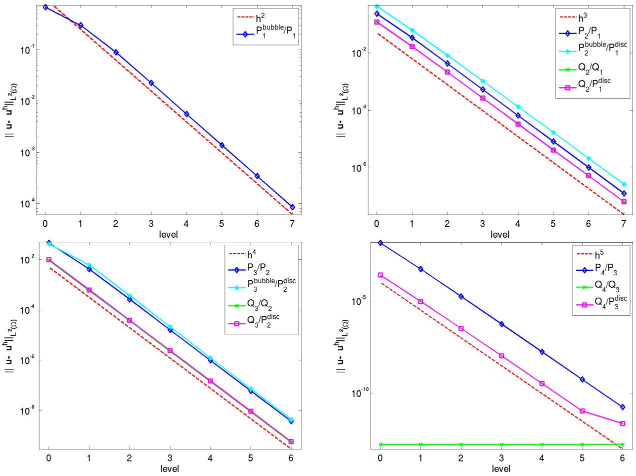
- convergence of the errors $\left\|\nabla(\mathbf{u}-\mathbf{u}^h)\right\|_{L^2(\Omega)}$ for different discretizations with different orders k



- convergence of the errors $\left\|p - p^h\right\|_{L^2(\Omega)}$ for different discretizations with different orders k



- convergence of the errors $\left\|\mathbf{u}-\mathbf{u}^h\right\|_{L^2(\Omega)}$ for different discretizations with different orders k



- implementation
  - vector-valued velocity space

$$
\begin{aligned}
V^h &= \text{span}\{\phi_i^h\}_{i=1}^{3N_v} \\
&= \text{span}\left\{\left\{\begin{pmatrix}\phi_i^h\\0\\0\end{pmatrix}\right\}_{i=1}^{N_v} \cup \left\{\begin{pmatrix}0\\\phi_i^h\\0\end{pmatrix}\right\}_{i=1}^{N_v} \cup \left\{\begin{pmatrix}0\\0\\\phi_i^h\end{pmatrix}\right\}_{i=1}^{N_v}\right\}
\end{aligned}
$$

  - pressure space

$$Q^h = \text{span}\{\psi_i^h\}_{i=1}^{N_p}$$

  - representation of unknown solution

$$\mathbf{u}^h = \sum_{j=1}^{3N_v} u_j^h \phi_j^h, \quad p^h = \sum_{j=1}^{N_p} p_j^h \psi_j^h$$

- pressure finite element space
  - standard basis functions not in $L^2_0(\Omega)$
  - it can be shown under mild assumptions that standard basis functions can be used as ansatz and test functions
  - computed pressure with standard basis functions has to be projected into $L^2_0(\Omega)$ at the end

- linear saddle point problem

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix}$$

with

$$\begin{aligned}
(A)_{ij} = a_{ij} &= \sum_{K\in\mathscr{T}^h} \left(\nabla\phi_j^h, \nabla\phi_i^h\right)_K, i,j = 1,\ldots,3N_v, \\
(B)_{ij} = b_{ij} &= -\sum_{K\in\mathscr{T}^h} \left(\nabla\cdot\phi_j^h, \psi_i^h\right)_K, i = 1,\ldots,N_p, j = 1,\ldots,3N_v, \\
(\underline{f})_i = f_i &= \sum_{K\in\mathscr{T}^h} \left(\mathbf{f}, \phi_i^h\right)_K, i = 1,\ldots,3N_v
\end{aligned}$$

- dimension (3d): $(3N_v + N_p) \times (3N_v + N_p)$

- matrix A
  - symmetric
  - positive definite
  - block-diagonal matrix

$$A = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{11} & 0 \\ 0 & 0 & A_{11} \end{pmatrix}$$

- matrix A
  - symmetric
  - positive definite
  - block-diagonal matrix

$$A = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{11} & 0 \\ 0 & 0 & A_{11} \end{pmatrix}$$

- $\left(\mathbb{D}\left(\mathbf{u}^h\right), \mathbb{D}\left(\mathbf{v}^h\right)\right)$ instead of $\left(\nabla \mathbf{u}^h, \nabla \mathbf{v}^h\right)$
  - equivalent only if $\mathbf{u}^h$ weakly divergence-free
  - generally not given for finite element velocities
  - not longer block-diagonal matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^T & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{pmatrix}$$

4 The Oseen Equations

- continuous equation

$$
\begin{aligned}
-\nu\Delta\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{u} + c\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\
\nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega
\end{aligned}
$$

  for simplicity: homogeneous Dirichlet boundary conditions

- difficulties:
  - coupling of velocity and pressure
  - dominating convection
- properties
  - linear

Carl Wilhelm Oseen (1879 – 1944)

- coefficients
  - $\nu > 0$
  - $\mathbf{b} \in W^{1,\infty}(\Omega)$, $\nabla \cdot \mathbf{b} = 0$
  - $c \in L^{\infty}(\Omega)$, $c(\mathbf{x}) \geq c_0 \geq 0$

- coefficients
  - $\nu > 0$
  - $\mathbf{b} \in W^{1,\infty}(\Omega)$, $\nabla \cdot \mathbf{b} = 0$
  - $c \in L^{\infty}(\Omega)$, $c(\mathbf{x}) \geq c_0 \geq 0$
- scaling of momentum equation: one of these possibilities
  - $\|\mathbf{b}\|_{L^{\infty}(\Omega)} = \mathscr{O}(1)$ if $\nu \leq \|\mathbf{b}\|_{L^{\infty}(\Omega)}$
  - $\nu = \mathscr{O}(1)$ if $\|\mathbf{b}\|_{L^{\infty}(\Omega)} \leq \nu$

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  - $\nu > 0$
  - $\mathbf{b} \in W^{1,\infty}(\Omega)$, $\nabla \cdot \mathbf{b} = 0$
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  - $\nu = \mathscr{O}(1)$ if $\|\mathbf{b}\|_{L^\infty(\Omega)} \leq \nu$
- interesting cases
  - ν of moderate size, $c = 0$

    in numerical solution of steady-state Navier–Stokes equations
  - ν of arbitrary size, $c = \mathscr{O}\left((\Delta t)^{-1}\right)$

    in numerical solution of time-dependent Navier–Stokes equations

- weak form

$$
\begin{aligned}
\nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + ((\mathbf{b}\cdot\nabla)\mathbf{u} + c\mathbf{u}, \mathbf{v}) - (\nabla\cdot\mathbf{v}, p) &= \langle \mathbf{f}, \mathbf{v}\rangle_{V',V} \quad \forall\, \mathbf{v} \in V, \\
-(\nabla\cdot\mathbf{u}, q) &= 0 \qquad\qquad\quad \forall\, q \in Q
\end{aligned}
$$

- bilinear forms

$$
\begin{aligned}
a &: V \times V \to \mathbb{R}, & a(\mathbf{u},\mathbf{v}) &= \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + ((\mathbf{b}\cdot\nabla)\mathbf{u} + c\mathbf{u}, \mathbf{v}), \\
b &: V \times Q \to \mathbb{R}, & b(\mathbf{v},q) &= -(\nabla\cdot\mathbf{v}, q)
\end{aligned}
$$

4 The Oseen Equations

- weak form

$$\begin{array}{rcll} \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + ((\mathbf{b}\cdot\nabla)\mathbf{u} + c\mathbf{u}, \mathbf{v}) - (\nabla\cdot\mathbf{v}, p) & = & \langle \mathbf{f}, \mathbf{v}\rangle_{V',V} & \forall\, \mathbf{v} \in V, \\ -(\nabla\cdot\mathbf{u}, q) & = & 0 & \forall\, q \in Q \end{array}$$

- bilinear forms

$$\begin{array}{rlrcl} a & : V \times V \to \mathbb{R}, & a(\mathbf{u},\mathbf{v}) & = & \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + ((\mathbf{b}\cdot\nabla)\mathbf{u} + c\mathbf{u}, \mathbf{v}), \\ b & : V \times Q \to \mathbb{R}, & b(\mathbf{v},q) & = & -(\nabla\cdot\mathbf{v}, q) \end{array}$$

- existence and uniqueness of solution
  - proof: board
  - essential condition

$$((\mathbf{b}\cdot\nabla)\mathbf{v}, \mathbf{v}) = 0 \quad \forall\, \mathbf{v} \in V$$

can be proved is $\mathbf{b}$ is weakly divergence-free and has zero trace on Γ

- stability of solution
  - dependency of bounds on coefficients is important
  - depending on regularity of data, different estimates possible
    - most general

$$\frac{\nu}{2}\|\nabla \mathbf{u}\|_{L^2(\Omega)}^2 + \left\|c^{1/2}\mathbf{u}\right\|_{L^2(\Omega)}^2 \le \frac{1}{2\nu}\|\mathbf{f}\|_{H^{-1}(\Omega)}^2$$

    - $\mathbf{f} \in L^2(\Omega)$ and $c_0 > 0$

$$\nu\|\nabla \mathbf{u}\|_{L^2(\Omega)}^2 + \frac{1}{2}\left\|c^{1/2}\mathbf{u}\right\|_{L^2(\Omega)}^2 \le \frac{1}{2c_0}\|\mathbf{f}\|_{L_2(\Omega)}^2$$

  - proof: board

- stability of solution
  - dependency of bounds on coefficients is important
  - depending on regularity of data, different estimates possible
    - most general

$$\frac{\nu}{2}\left\|\nabla\mathbf{u}\right\|_{L^2(\Omega)}^2+\left\|c^{1/2}\mathbf{u}\right\|_{L^2(\Omega)}^2\le\frac{1}{2\nu}\left\|\mathbf{f}\right\|_{H^{-1}(\Omega)}^2$$

    - $\mathbf{f}\in L^2(\Omega)$ and $c_0>0$

$$\nu\left\|\nabla\mathbf{u}\right\|_{L^2(\Omega)}^2+\frac{1}{2}\left\|c^{1/2}\mathbf{u}\right\|_{L^2(\Omega)}^2\le\frac{1}{2c_0}\left\|\mathbf{f}\right\|_{L_2(\Omega)}^2$$

  - proof: board
  - estimates for pressure with inf-sup condition
  - discussion: board

- Galerkin finite element method

$$\begin{aligned} a\left(\mathbf{u}^h,\mathbf{v}^h\right)+b\left(\mathbf{v}^h,p^h\right) &= \left(\mathbf{f},\mathbf{v}^h\right) & \forall\, \mathbf{v}^h \in V^h, \\ b\left(\mathbf{u}^h,q^h\right) &= 0 & \forall\, q^h \in Q^h \end{aligned}$$

  - homogeneous Dirichlet boundary conditions
  - conforming, inf-sup stable finite element spaces
- existence, uniqueness, stability like for continuous problem

- finite element error estimate for the $L^2(\Omega)$ norm of the gradient of the velocity
  - $\Omega \subset \mathbb{R}^d$, bounded, polyhedral, Lipschitz-continuous boundary
  - regularity of coefficients like stated above

$$
\begin{aligned}
& \nu^{1/2} \left\| \nabla \left(\mathbf{u} - \mathbf{u}^h \right) \right\|_{L^2(\Omega)} + \left\| c^{1/2} \left(\mathbf{u} - \mathbf{u}^h \right) \right\|_{L^2(\Omega)} \\
\leq \quad & C \left[\left(1 + \frac{1}{\beta_{\mathrm{is}}^h} \right) C_{\mathrm{os}} \inf_{\mathbf{v}^h \in V^h} \left\| \nabla (\mathbf{u} - \mathbf{v}^h) \right\|_{L^2(\Omega)} + \frac{1}{\nu^{1/2}} \inf_{q^h \in Q^h} \left\| p - q^h \right\|_{L^2(\Omega)} \right],
\end{aligned}
$$

where

$$
C_{\mathrm{os}} = \nu^{1/2} + \|c\|_{L^\infty(\Omega)}^{1/2} + \|\mathbf{b}\|_{L^\infty(\Omega)} \min \left\{ \frac{1}{\nu^{1/2}}, \frac{1}{c_0^{1/2}} \right\}
$$

  - C does not depend on coefficients and triangulation, but on Ω (Poincaré–Friedrichs inequality)

- finite element error estimate for the $L^2(\Omega)$ norm of the gradient of the velocity (cont.)
  - proof: principally same as for Stokes equations
  - estimates for convective term

$$\begin{aligned}\left|\left((\mathbf{b}\cdot\nabla)\,\eta,\phi^h\right)\right| &= \left|-\left((\mathbf{b}\cdot\nabla)\,\phi^h,\eta\right)\right| \le \|\mathbf{b}\|_{L^\infty(\Omega)}\left\|\nabla\phi^h\right\|_{L^2(\Omega)}\|\eta\|_{L^2(\Omega)} \\ &\le \frac{2}{\nu}\|\mathbf{b}\|^2_{L^\infty(\Omega)}\|\eta\|^2_{L^2(\Omega)} + \frac{\nu}{8}\left\|\nabla\phi^h\right\|^2_{L^2(\Omega)}\end{aligned}$$

or if $c_0 > 0$

$$\begin{aligned}\left|\left((\mathbf{b}\cdot\nabla)\,\eta,\phi^h\right)\right| &\le \|\mathbf{b}\|_{L^\infty(\Omega)}\|\nabla\eta\|_{L^2(\Omega)}\left\|\phi^h\right\|_{L^2(\Omega)} \\ &\le \frac{\|\mathbf{b}\|^2_{L^\infty(\Omega)}\|\nabla\eta\|^2_{L^2(\Omega)}}{c_0} + \frac{\left\|c^{1/2}\phi^h\right\|^2_{L^2(\Omega)}}{4}\end{aligned}$$

- finite element error estimate for the $L^2(\Omega)$ norm of the pressure
  - same assumptions as for previous estimate

$$
\begin{aligned}
\left\|p-p^h\right\|_{L^2(\Omega)} \leq \; & C\left[\frac{1}{\beta_{\mathrm{is}}^h}\left(1+\frac{1}{\beta_{\mathrm{is}}^h}\right) C_{\mathrm{os}}^2 \inf_{\mathbf{v}^h\in V^h}\left\|\nabla(\mathbf{u}-\mathbf{v}^h)\right\|_{L^2(\Omega)}\right. \\
& \left. + \left(1+\frac{1}{\beta_{\mathrm{is}}^h}+\frac{1}{\beta_{\mathrm{is}}^h}\frac{C_{\mathrm{os}}}{\nu^{1/2}}\right)\inf_{q^h\in Q^h}\left\|p-q^h\right\|_{L^2(\Omega)}\right],
\end{aligned}
$$

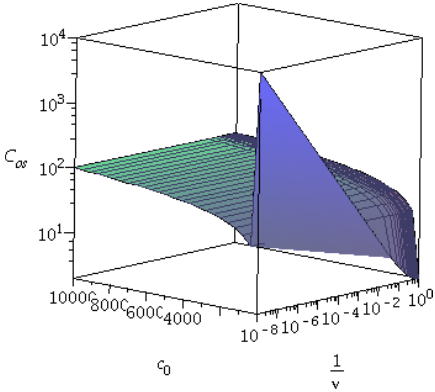
- proof: as for Stokes equations, with discrete inf-sup condition

- finite element error estimates for conforming pairs of finite element spaces
  - same assumptions on domain as for previous estimates
  - solution sufficiently regular
  - h – mesh width of triangulation
  - spaces
    - P_k^{bubble}/P_k, $k = 1$ (mini element),
    - P_k/P_{k-1}, Q_k/Q_{k-1}, $k \geq 2$ (Taylor–Hood element),
    - $P_k^{\text{bubble}}/P_{k-1}^{\text{disc}}$, $Q_k/P_{k-1}^{\text{disc}}$,$k \geq 2$

$$
\begin{aligned}
\left\|\nabla(\mathbf{u}-\mathbf{u}^h)\right\|_{L^2(\Omega)} &\leq \frac{C}{\nu^{1/2}} h^k \left(C_{\text{os}} \|\mathbf{u}\|_{H^{k+1}(\Omega)} + \frac{1}{\nu^{1/2}} \|p\|_{H^k(\Omega)} \right), \\
\left\|p-p^h\right\|_{L^2(\Omega)} &\leq C h^k \left(C_{\text{os}}^2 \|\mathbf{u}\|_{H^{k+1}(\Omega)} + \left(1 + \frac{C_{\text{os}}}{\nu^{1/2}}\right) \|p\|_{H^k(\Omega)} \right)
\end{aligned}
$$

4 The Oseen Equations – Galerkin FEM

- C_{os} for $\|\mathbf{b}\|_{L^\infty(\Omega)} = 1$



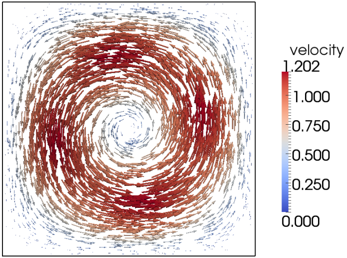
discussion: board

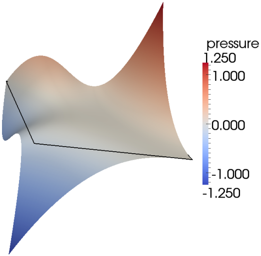
- error bounds not uniform for small ν or small time steps

4 The Oseen Equations – Galerkin FEM

- analytical example which supports the error estimates
- prescribed solution

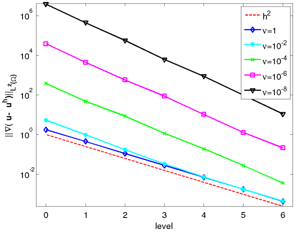
$$
\begin{aligned}
\mathbf{u} &= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \partial_y \psi \\ -\partial_x \psi \end{pmatrix} = 200 \begin{pmatrix} x^2(1-x)^2 y(1-y)(1-2y) \\ -x(1-x)(1-2x)y^2(1-y)^2 \end{pmatrix} \\
p &= 10\left(\left(x-\frac{1}{2}\right)^3 y^2 + (1-x)^3\left(y-\frac{1}{2}\right)^3\right)
\end{aligned}
$$

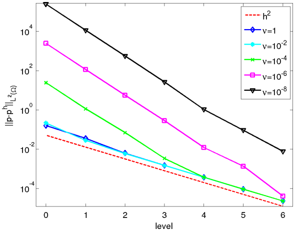




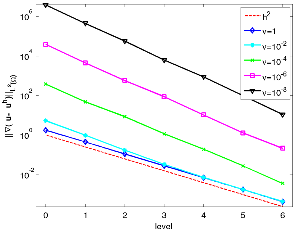
- $\mathbf{b} = \mathbf{u}$

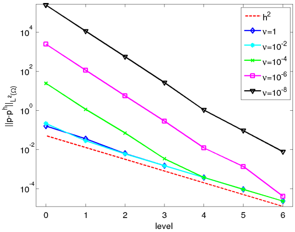
- Q_2/Q_1, convergence of errors for $c = 0$ and different values of ν



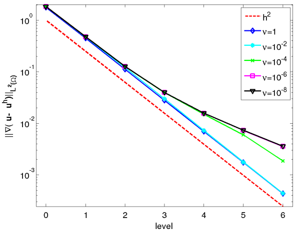


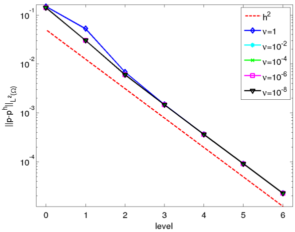
- Q_2/Q_1, convergence of errors for $c = 0$ and different values of ν



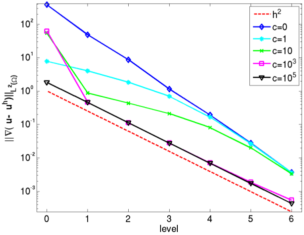


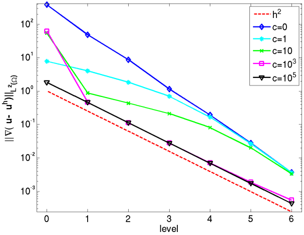
- Q_2/Q_1, convergence of errors for $c = 100$ and different values of ν



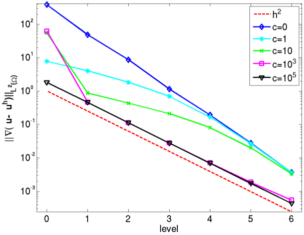


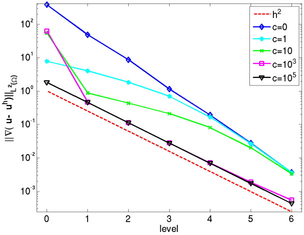
- Q_2/Q_1, convergence of errors for $\nu = 10^{-4}$ and different values of c





- Q_2/Q_1, convergence of errors for $\nu = 10^{-4}$ and different values of c





- summary
  - Galerkin discretization in some cases unstable

- principal idea
- given: linear partial differential equation in strong form

$$A_{\mathrm{str}} u_{\mathrm{str}} = f, \quad f \in L^2(\Omega)$$

- Galerkin discretization

$$a^h\left(u^h, v^h\right) = \left(f, v^h\right) \quad \forall\, v^h \in V^h$$

- needed: modification of strong operator $A_{\mathrm{str}}^h \;:\; V^h \to L^2(\Omega)$
- residual

$$r^h\left(u^h\right) = A_{\mathrm{str}}^h u^h - f \;\in L^2(\Omega)$$

- generally $r^h\left(u^h\right) \neq 0$

- principal idea (cont.)
- consider optimization problem

$$\arg\min_{u^h \in V^h} \left\| r^h\left(u^h\right) \right\|_{L^2(\Omega)}^2 = \arg\min_{u^h \in V^h} \left(r^h\left(u^h\right), r^h\left(u^h\right) \right)$$

- necessary condition for solution (board)

$$\left(r^h\left(u^h\right), A_{\mathrm{str}}^h v^h \right) = 0$$

- principal idea (cont.)
- consider optimization problem

$$\operatorname*{arg\,min}_{u^h \in V^h} \left\| r^h\left(u^h\right) \right\|^2_{L^2(\Omega)} = \operatorname*{arg\,min}_{u^h \in V^h} \left(r^h\left(u^h\right), r^h\left(u^h\right) \right)$$

- necessary condition for solution (board)

$$\left(r^h\left(u^h\right), A^h_{\mathrm{str}} v^h \right) = 0$$

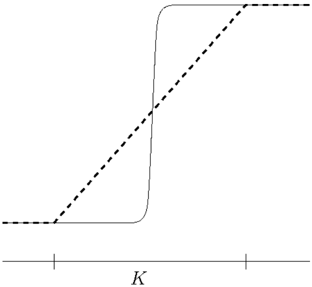
- generalization $\delta(\mathbf{x}) > 0$

$$\operatorname*{arg\,min}_{u^h \in V^h} \left\| \delta^{1/2} r^h\left(u^h\right) \right\|^2_{L^2(\Omega)} = \operatorname*{arg\,min}_{u^h \in V^h} \left(\delta r^h\left(u^h\right), r^h\left(u^h\right) \right)$$

with necessary condition

$$\left(\delta r^h\left(u^h\right), A^h_{\mathrm{str}} v^h \right) = 0$$

- principal idea (cont.)
- minimizing residual alone: not good



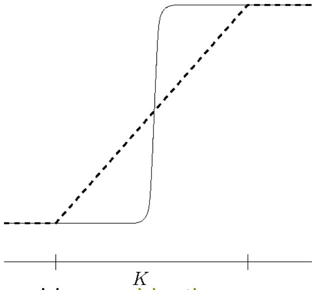
- solid line – function with layer
- dashed line – optimal piecewise linear approximation

- consider combination

$$a^h\left(u^h, v^h\right) + \left(\delta r^h\left(u^h\right), A_{\text{str}}^h v^h\right) = \left(f, v^h\right) \quad \forall\, v^h \in V^h$$

optimal choice of weighting function $\delta(\mathbf{x})$ by numerical analysis

- principal idea (cont.)
- minimizing residual alone: not good



- solid line – function with layer
- dashed line – optimal piecewise linear approximation

- consider combination

$$a^h\left(u^h, v^h\right) + \left(\delta r^h\left(u^h\right), A^h_{\text{str}} v^h\right) = \left(f, v^h\right) \quad \forall\, v^h \in V^h$$

optimal choice of weighting function $\delta(\mathbf{x})$ by numerical analysis

- example: Oseen equations, board

- SUPG/PSPG/grad-div stabilization
- find $\left(\mathbf{u}^h, p^h\right) \in V^h \times Q^h$ such that

$$A_{\text{spg}}\left(\left(\mathbf{u}^h, p^h\right), \left(\mathbf{v}^h, q^h\right)\right) = L_{\text{spg}}\left(\left(\mathbf{v}^h, q^h\right)\right) \quad \forall \left(\mathbf{v}^h, q^h\right) \in V^h \times Q^h,$$

with $A_{\text{spg}} \;:\; \left(V \times \tilde{Q}\right) \times \left(V \times \tilde{Q}\right) \to \mathbb{R}$

$$\begin{aligned} & A_{\text{spg}}\left((\mathbf{u}, p), (\mathbf{v}, q)\right) \\ = \quad & \nu\left(\nabla \mathbf{u}, \nabla \mathbf{v}\right) + \left((\mathbf{b} \cdot \nabla)\mathbf{u} + c\mathbf{u}, \mathbf{v}\right) - \left(\nabla \cdot \mathbf{v}, p\right) + \left(\nabla \cdot \mathbf{u}, q\right) \\ & + \sum_{K \in \mathscr{T}^h} \mu_K \left(\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}\right)_K + \sum_{E \in \mathscr{E}^h} \delta_E \left([|p|]_E, [|q|]_E\right)_E \\ & + \sum_{K \in \mathscr{T}^h} \left(-\nu \Delta \mathbf{u} + (\mathbf{b} \cdot \nabla)\mathbf{u} + c\mathbf{u} + \nabla p, \delta_K^v (\mathbf{b} \cdot \nabla)\mathbf{v} + \delta_K^p \nabla q\right)_K \end{aligned}$$

and $L_{\text{spg}} \;:\; \left(V \times \tilde{Q}\right) \to \mathbb{R}$

$$L_{\text{spg}}\left((\mathbf{v}, q)\right) = (\mathbf{f}, \mathbf{v}) + \sum_{K \in \mathscr{T}^h} \left(\mathbf{f}, \delta_K^v (\mathbf{b} \cdot \nabla)\mathbf{v} + \delta_K^p \nabla q\right)_K$$

4 The Oseen Equations – Residual-Based Stabilizations

- SUPG/PSPG/grad-div stabilization (cont.)
- finite element error analysis in [1]
- $\delta_K = \delta_K^v = \delta_K^p$ for all $K \in \mathscr{T}^h$

$$\delta = \max_{K \in \mathscr{T}^h} \delta_K, \quad \mu = \max_{K \in \mathscr{T}^h} \mu_K$$

[1] Tobiska, Verfürth, SINUM 33, 107–127, 1996

- SUPG/PSPG/grad-div stabilization (cont.)
- finite element error analysis in [1]
- $\delta_K = \delta_K^v = \delta_K^p$ for all $K \in \mathscr{T}^h$

$$\delta = \max_{K\in\mathscr{T}^h} \delta_K, \quad \mu = \max_{K\in\mathscr{T}^h} \mu_K$$

- no saddle point problem because of

$$-\sum_{E\in\mathscr{E}^h} \delta_E \left(\left[\left|p^h\right|\right]_E, \left[\left|q^h\right|\right]_E\right)_E - \sum_{K\in\mathscr{T}^h} \delta_K \left(\nabla p^h, \nabla q^h\right)_K$$

  - analysis for elliptic partial differential equations applicable
  - inf-sup stable spaces not necessary
  - choice of stabilization parameters affected by choice of finite element spaces

[1] Tobiska, Verfürth, SINUM 33, 107–127, 1996

- properties
  - consistency

$$A_{\text{spg}}\left((\mathbf{u},p),\left(\mathbf{v}^h,q^h\right)\right)=L_{\text{spg}}\left(\left(\mathbf{v}^h,q^h\right)\right),\quad \forall\left(\mathbf{v}^h,q^h\right)\in V^h\times Q^h$$

  - Galerkin orthogonality

$$A_{\text{spg}}\left(\left(\mathbf{u}-\mathbf{u}^h,p-p^h\right),\left(\mathbf{v}^h,q^h\right)\right)=0,\quad \forall\left(\mathbf{v}^h,q^h\right)\in V^h\times Q^h$$

- mesh-dependent norm

$$\|(\mathbf{v},q)\|_{\mathrm{spg}} = \left\{ \nu \|\nabla \mathbf{v}\|_{L^2(\Omega)}^2 + \left\| c^{1/2}\mathbf{v} \right\|_{L^2(\Omega)}^2 + \sum_{K\in\mathscr{T}^h} \mu_K \|\nabla\cdot\mathbf{v}\|_{L^2(K)}^2 \right.$$
$$\left. + \sum_{E\in\mathscr{E}^h} \delta_E \|[|q|]_E\|_{L^2(E)}^2 + \sum_{K\in\mathscr{T}^h} \delta_K \|(\mathbf{b}\cdot\nabla)\mathbf{v} + \nabla q\|_{L^2(K)}^2 \right\}^{1/2}$$

  - proof: board
  - additional control on error of
    - divergence
    - pressure jumps
    - streamline derivative + gradient of pressure
  - norm with pressure: later

- existence and uniqueness of a solution
  - assumptions

$$\mu_K \geq 0, \quad 0 < \delta_K \leq \min\left\{ \frac{h_K^2}{3\nu C_{\text{inv}}^2}, \frac{1}{3\|c\|_{L^\infty(K)}} \right\}$$

$\delta_E > 0$ if $Q^h \not\subset C(\overline{\Omega})$

  - proof: application of Lax–Milgram lemma
    - coercivity (board if time admits), $\forall \left(\mathbf{v}^h, q^h\right) \in V^h \times Q^h$

$$A_{\text{spg}}\left(\left(\mathbf{v}^h, q^h\right), \left(\mathbf{v}^h, q^h\right)\right) \geq \frac{1}{2}\left\|\left(\mathbf{v}^h, q^h\right)\right\|_{\text{spg}}^2$$

    - boundedness, $\forall \left(\mathbf{u}^h, p^h\right), \left(\mathbf{v}^h, q^h\right) \in V^h \times Q^h$

$$A_{\text{spg}}\left(\left(\mathbf{u}^h, p^h\right), \left(\mathbf{v}^h, q^h\right)\right) \leq C\left\|\left(\mathbf{u}^h, p^h\right)\right\|_{\text{spg}}\left\|\left(\mathbf{v}^h, q^h\right)\right\|_{\text{spg}}$$

using: all norms are equivalent in finite-dimensional spaces

- stability

$$\left\| \left(\mathbf{u}^h, p^h \right) \right\|_{\mathrm{spg}}^2 \le \frac{12}{5} \min \left\{ \frac{\|\mathbf{f}\|_{H^{-1}(\Omega)}^2}{\nu}, \frac{\|\mathbf{f}\|_{L_2(\Omega)}^2}{c_0} \right\} + 4 \sum_{K \in \mathscr{T}^h} \delta_K \|\mathbf{f}\|_{L^2(K)}^2$$

  - proof: as usual
  - estimate in stronger norm than for Galerkin finite element method
  - estimate for pressure with inf-sup condition possible

- norm for finite element error estimates

$$\|(\mathbf{v},q)\|_{\text{spg,p}} = \left(\|(\mathbf{v},q)\|_{\text{spg}} + w_{\text{pres}}^{-2}\,\|q\|_{L^2(\Omega)}^2\right)^{1/2}$$

with

$$w_{\text{pres}} = \max\left\{1, \nu^{-1/2}, \|c\|_{L^\infty(\Omega)}^{1/2}\right\}$$

for the interesting cases of small ν and large c: small contribution of the pressure

- norm for finite element error estimates

$$\|(\mathbf{v},q)\|_{\text{spg,p}} = \left(\|(\mathbf{v},q)\|_{\text{spg}} + w_{\text{pres}}^{-2}\,\|q\|_{L^2(\Omega)}^2\right)^{1/2}$$

with

$$w_{\text{pres}} = \max\left\{1, \nu^{-1/2}, \|c\|_{L^\infty(\Omega)}^{1/2}\right\}$$

for the interesting cases of small ν and large c: small contribution of the pressure

- first step: inf-sup conditions for A_{spg}

$$\inf_{\substack{(\mathbf{v}^h,q^h)\in V^h\times Q^h\\ \|(\mathbf{u}^h,p^h)\|_{\text{spg,p}}=1}} \ \sup_{\substack{(\mathbf{w}^h,r^h)\in V^h\times Q^h\\ \|(\mathbf{v}^h,q^h)\|_{\text{spg,p}}=1}} A_{\text{spg}}\left(\left(\mathbf{v}^h,q^h\right),\left(\mathbf{w}^h,r^h\right)\right) \geq \beta_{\text{spg}}$$

  - some conditions on stabilization parameters, e.g., $\delta_0 h_K^2 \leq \delta_K$
  - proof very technical
  - $\beta_{\text{spg}} = \mathcal{O}(\delta_0)$

- finite element error estimate

$$
\begin{aligned}
&\left\|\left(\mathbf{u}-\mathbf{u}^h, p-p^h\right)\right\|_{\mathrm{spg}} + \nu^{1/2}\left\|p-p^h\right\|_{L^2\Omega} \\
\leq \quad & C\left[h^k\left(\nu^{1/2}+\frac{\nu\delta^{1/2}}{h}+\frac{h}{\delta^{1/2}}+\delta^{1/2}+\frac{\mu\delta^{1/2}}{h}+\|c\|_{L^\infty(\Omega)}^{1/2} h+\delta\|c\|_{L^\infty(\Omega)} h\right)\right. \\
&\left.+h^{l+1}\left(\nu^{1/2}+\frac{\delta^{1/2}}{h}+\frac{1}{\nu^{1/2}}\left(\max\left\{1,\frac{\mu}{\nu}\right\}\right)^{-1/2}\right)\|p\|_{H^{l+1}(\Omega)}\right]
\end{aligned}
$$

  - $k \geq 1$, $l \geq 0$
  - C independent of the coefficients of the problem
  - proof: based on inf-sup condition

4 The Oseen Equations – Residual-Based Stabilizations

- optimal asymptotics for stabilization parameters, $\nu < h$ (board)
  - inf-sup stable discretizations with $k = l + 1$

$$\delta = \mathcal{O}\left(h^2\right), \quad \mu = \mathcal{O}\left(1\right) \quad \Longrightarrow \quad \text{order of error reduction: } k$$

- optimal asymptotics for stabilization parameters, $\nu < h$ (board)
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$$\delta = \mathscr{O}\left(h^2\right), \quad \mu = \mathscr{O}\left(1\right) \quad \Longrightarrow \quad \text{order of error reduction: } k$$

  - equal-order discretizations with $k = l \geq 1$

$$\mathscr{O}\left(\delta\right) = \mathscr{O}\left(\mu\right) = \mathscr{O}\left(h\right) \quad \Longrightarrow \quad \text{order of error reduction: } k + \frac{1}{2}$$

- optimal asymptotics for stabilization parameters, $\nu < h$ (board)
  - inf-sup stable discretizations with $k = l + 1$

$$\delta = \mathcal{O}\left(h^2\right), \quad \mu = \mathcal{O}\left(1\right) \quad \Longrightarrow \quad \text{order of error reduction: } k$$

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$$\mathcal{O}\left(\delta\right) = \mathcal{O}\left(\mu\right) = \mathcal{O}\left(h\right) \quad \Longrightarrow \quad \text{order of error reduction: } k + \frac{1}{2}$$

- optimal asymptotics for stabilization parameters, $\nu \geq h$
  - inf-sup stable discretizations with $k = l + 1$

$$\delta = \mathcal{O}\left(h^2\right), \quad \mu = \mathcal{O}\left(1\right) \quad \Longrightarrow \quad \text{order of convergence: } k$$

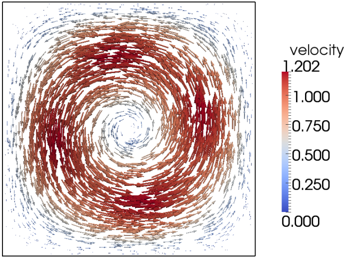
  - equal-order discretizations with $k = l \geq 1$

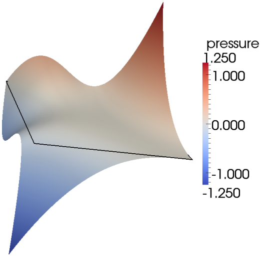
$$\delta = \mathcal{O}\left(h^2\right), \quad \mu \text{ arbitrary} \quad \Longrightarrow \quad \text{order of convergence: } k$$

4 The Oseen Equations – Residual-Based Stabilizations

- analytical example which supports the error estimates
- prescribed solution

$$
\begin{aligned}
\mathbf{u} &= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \partial_y \psi \\ -\partial_x \psi \end{pmatrix} = 200 \begin{pmatrix} x^2(1-x)^2 y(1-y)(1-2y) \\ -x(1-x)(1-2x)y^2(1-y)^2 \end{pmatrix} \\
p &= 10\left(\left(x-\frac{1}{2}\right)^3 y^2 + (1-x)^3\left(y-\frac{1}{2}\right)^3\right)
\end{aligned}
$$



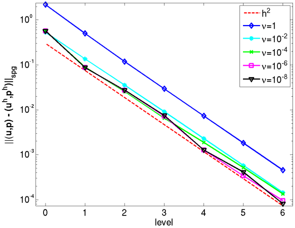


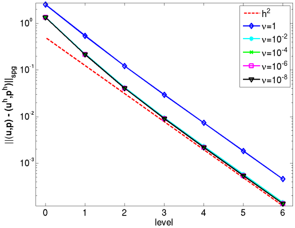
- $\mathbf{b} = \mathbf{u}$

- Q_2/Q_1 finite element
- stabilization parameters (based on numerical simulations from [1])

$$\mu_K = 0.2, \quad \delta_K = 0.1 h_K^2$$

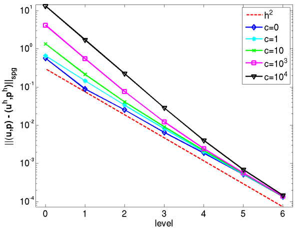
- convergence of errors for $c = 0$ and $c = 100$, different values of ν





[1] Matthies, Lube, Röhe, Comput. Methods Appl. Math. 9, 368 – 390, 2009

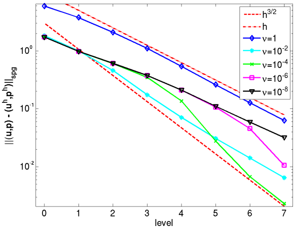
- Q_2/Q_1, convergence of errors for $\nu = 10^{-4}$ and different values of c

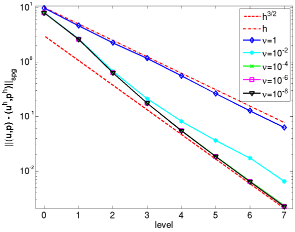


- P_1/P_1 finite element
- stabilization parameters

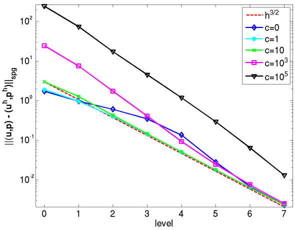
$$\delta_K = \begin{cases} 0.5h_K & \text{if } \nu < h_K, \\ 0.5h_K^2 & \text{else,} \end{cases} \qquad \mu_K = 0.5h_K$$

- convergence of errors for $c = 0$ and $c = 100$, different values of ν





- P_1/P_1, convergence of errors for $\nu = 10^{-4}$ and different values of c



- implementation: same approach as for Stokes equations
- grad-div term leads to matrix block

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^T & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{pmatrix} \quad \text{instead of} \quad \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{11} & 0 \\ 0 & 0 & A_{11} \end{pmatrix}$$

- PSPG term introduces pressure-pressure couplings
- SUPG term influences velocity-velocity coupling and the pressure (ansatz) - velocity (test) coupling
- final system

$$\begin{pmatrix} A & D \\ B & C \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{f_p} \end{pmatrix}$$

much more matrix blocks to store than for Galerkin FEM

- Summary and remarks
  - errors $\left\|(\mathbf{u},p)-(\mathbf{u}^h,p^h)\right\|_{\text{spg}}$ independent of ν
  - versions without pressure couplings available
    - only for inf-sup stable pairs of finite elements
    - easier to implement than SUPG/PSPG/grad-div stabilization
  - numerical analysis in [1,2,3]

[1] Tobiska, Verfürth, SINUM 33, 107–127, 1996

[2] Lube, Rapin, M3AS 16, 949–966, 2006

[3] Matthies, Lube, Röhe, Comput. Methods Appl. Math. 9, 368–390, 2009

5 The Stationary Navier–Stokes Equations

- continuous equation

$$
\begin{aligned}
-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega,\\
\nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega
\end{aligned}
$$

  for simplicity: homogeneous Dirichlet boundary conditions

- difficulties:
  - coupling of velocity and pressure
  - dominating convection
  - nonlinear

- different forms of the convective term

$$
\begin{array}{lcl}
(\mathbf{u}\cdot\nabla)\mathbf{u} & : & \text{convective form}, \\
\nabla\cdot\left(\mathbf{u}\mathbf{u}^T\right) & : & \text{divergence form}, \\
(\nabla\times\mathbf{u})\times\mathbf{u} & : & \text{rotational form}
\end{array}
$$

  - convective form and divergence form equivalent if $\nabla\cdot\mathbf{u}=0$ (board, if time permits)
  - convective form and rotational form

$$
(\nabla\times\mathbf{u})\times\mathbf{u}+\frac{1}{2}\nabla\left(\mathbf{u}^T\mathbf{u}\right)=(\mathbf{u}\cdot\nabla)\mathbf{u}
$$

definition of new pressure (Bernoulli pressure) in rotational form

$$
p_{\text{Bern}}=p+\frac{1}{2}\mathbf{u}^T\mathbf{u}
$$

5 The Stationary Navier–Stokes Equations

- variational form of the steady-state Navier–Stokes equations: Find $(\mathbf{u}, p) \in V \times Q$ such that

$$\begin{aligned} (\nu \nabla \mathbf{u}, \nabla \mathbf{v}) + ((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}) \\ -(\nabla \cdot \mathbf{u}, q) &= 0 \end{aligned}$$

  for all $(\mathbf{v}, q) \in V \times Q$
- equivalent: Find $(\mathbf{u}, p) \in V \times Q$ such that

$$(\nu \nabla \mathbf{u}, \nabla \mathbf{v}) + ((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, p) + (\nabla \cdot \mathbf{u}, q) = (\mathbf{f}, \mathbf{v})$$

  for all $(\mathbf{v}, q) \in V \times Q$

- properties of convective term
  - linear in each component (trilinear)
  - $\mathbf{u}, \mathbf{v}, \mathbf{w} \in H^1(\Omega)$, product rule

$$((\mathbf{u}\cdot\nabla)\mathbf{v}, \mathbf{w}) = \left(\nabla\cdot\left(\mathbf{v}\mathbf{u}^T\right), \mathbf{w}\right) - ((\nabla\cdot\mathbf{u})\mathbf{v}, \mathbf{w})$$

  - $\mathbf{u}, \mathbf{v}, \mathbf{w} \in H^1(\Omega)$, product rule

$$((\mathbf{u}\cdot\nabla)\mathbf{v}, \mathbf{w}) = (\mathbf{u}, \nabla(\mathbf{v}\cdot\mathbf{w})) - ((\mathbf{u}\cdot\nabla)\mathbf{w}, \mathbf{v})$$

- convective terms in the variational formulation
  - convective form
$$n_{\text{conv}}(\mathbf{u},\mathbf{v},\mathbf{w}) = ((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w})$$
  - divergence form
$$n_{\text{div}}(\mathbf{u},\mathbf{v},\mathbf{w}) = n_{\text{conv}}(\mathbf{u},\mathbf{v},\mathbf{w}) + ((\nabla\cdot\mathbf{u})\,\mathbf{v},\mathbf{w})$$
  - rotational form
$$n_{\text{rot}}(\mathbf{u},\mathbf{v},\mathbf{w}) = ((\nabla\times\mathbf{u})\times\mathbf{v},\mathbf{w})$$
    with momentum equation
$$(\nu\nabla\mathbf{u},\nabla\mathbf{v}) + n_{\text{rot}}(\mathbf{u},\mathbf{u},\mathbf{v}) - (\nabla\cdot\mathbf{v},p_{\text{Bern}}) = (\mathbf{f},\mathbf{v}) \quad \forall\,\mathbf{v}\in V,$$
  - skew-symmetric form (for $\mathbf{u}$ weakly divergence-free, $\mathbf{u}\cdot\mathbf{n}=0$ on Γ, board)
$$n_{\text{skew}}(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{1}{2}\left(n_{\text{conv}}(\mathbf{u},\mathbf{v},\mathbf{w}) - n_{\text{conv}}(\mathbf{u},\mathbf{w},\mathbf{v})\right)$$

- further properties of convective term
- vanishing
  - rotational and skew-symmetric form

$$n_{\mathrm{rot}}(\mathbf{u},\mathbf{v},\mathbf{v}) = n_{\mathrm{skew}}(\mathbf{u},\mathbf{v},\mathbf{v}) = 0$$

  - convective and divergence form: if $\mathbf{u}$ weakly divergence-free and $\mathbf{u}\cdot\mathbf{n} = 0$ on Γ

$$n_{\mathrm{conv}}(\mathbf{u},\mathbf{v},\mathbf{v}) = n_{\mathrm{div}}(\mathbf{u},\mathbf{v},\mathbf{v}) = 0$$

- further properties of convective term
- vanishing
  - rotational and skew-symmetric form

$$n_{\mathrm{rot}}(\mathbf{u},\mathbf{v},\mathbf{v}) = n_{\mathrm{skew}}(\mathbf{u},\mathbf{v},\mathbf{v}) = 0$$

  - convective and divergence form: if $\mathbf{u}$ weakly divergence-free and $\mathbf{u}\cdot\mathbf{n} = 0$ on Γ

$$n_{\mathrm{conv}}(\mathbf{u},\mathbf{v},\mathbf{v}) = n_{\mathrm{div}}(\mathbf{u},\mathbf{v},\mathbf{v}) = 0$$

- estimates: $\mathbf{u},\mathbf{v},\mathbf{w} \in H^1(\Omega)$

$$\begin{aligned} |n_{\mathrm{conv}}(\mathbf{u},\mathbf{v},\mathbf{w})| &\leq C\,\|\mathbf{u}\|_{H^1(\Omega)}\,\|\nabla\mathbf{v}\|_{L^2(\Omega)}\,\|\mathbf{w}\|_{H^1(\Omega)}\,, \\ |n_{\mathrm{skew}}(\mathbf{u},\mathbf{v},\mathbf{w})| &\leq C\,\|\mathbf{u}\|_{H^1(\Omega)}\,\|\mathbf{v}\|_{H^1(\Omega)}\,\|\mathbf{w}\|_{H^1(\Omega)} \end{aligned}$$

  - proof: board

- existence and uniqueness of a solution
  - $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, bounded domain with Lipschitz boundary
  - $\mathbf{f} \in H^{-1}(\Omega)$
  - then: existence
- main ideas of the proof
  - equivalent problem in the divergence-free subspace, only velocity
  - consider problem in finite dimensional spaces (Galerkin method)
  - fixed point equation, existence of a solution of the finite dimensional problems: fixed point theorem of Brouwer
  - dimension of the spaces $\to \infty$: show subsequence of the solutions tends to a solution of the problem in the divergence-free subspace
  - existence of the pressure: inf-sup condition

- existence and uniqueness of a solution (cont.)
  - ν sufficiently large, i.e.

$$\|\mathbf{f}\|_{H^{-1}(\Omega)} \sup_{\mathbf{u},\mathbf{v},\mathbf{w}\in V} \frac{((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w})}{\|\nabla\mathbf{u}\|_{L^2(\Omega)}\|\nabla\mathbf{v}\|_{L^2(\Omega)}\|\nabla\mathbf{w}\|_{L^2(\Omega)}} < \nu^2$$

  - then: uniqueness
- main idea of the proof
  - construct a contraction, apply Banach's fixed point theorem

5 The Stationary Navier–Stokes Equations

- existence and uniqueness of a solution (cont.)
  - ν sufficiently large, i.e.

$$\|\mathbf{f}\|_{H^{-1}(\Omega)} \sup_{\mathbf{u},\mathbf{v},\mathbf{w}\in V} \frac{((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w})}{\|\nabla\mathbf{u}\|_{L^2(\Omega)}\|\nabla\mathbf{v}\|_{L^2(\Omega)}\|\nabla\mathbf{w}\|_{L^2(\Omega)}} < \nu^2$$

  - then: uniqueness
- main idea of the proof
  - construct a contraction, apply Banach's fixed point theorem
- numerical simulations
  - case of unique solution is of interest
  - steady-state solutions unstable in non-unique case, solve time-dependent solution

- stability

$$
\begin{aligned}
\|\nabla \mathbf{u}\|_{L^2(\Omega)} &\le \frac{1}{\nu}\|\mathbf{f}\|_{H^{-1}(\Omega)}, \\
\|p\|_{L^2(\Omega)} &\le \frac{1}{\beta_{\mathrm{is}}}\left(2\|\mathbf{f}\|_{H^{-1}(\Omega)} + \frac{C}{\nu^2}\|\mathbf{f}\|_{H^{-1}(\Omega)}^2\right)
\end{aligned}
$$

  - proof: as usual, using

$$n(\mathbf{u},\mathbf{u},\mathbf{u}) = 0$$

- Galerkin finite element method

$$
\begin{array}{rcll}
\nu\left(\nabla \mathbf{u}^h, \nabla \mathbf{v}^h\right) + n\left(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h\right) - \left(\nabla \cdot \mathbf{v}^h, p^h\right) & = & \left(\mathbf{f}, \mathbf{v}^h\right) & \forall\ \mathbf{v}^h \in V^h, \\
-\left(\nabla \cdot \mathbf{u}^h, q^h\right) & = & 0 & \forall\ q^h \in Q^h,
\end{array}
$$

- inf-sup stable pair of finite element spaces

- Galerkin finite element method

$$
\begin{array}{rcll}
\nu\left(\nabla \mathbf{u}^h, \nabla \mathbf{v}^h\right) + n\left(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h\right) - \left(\nabla\cdot \mathbf{v}^h, p^h\right) & = & \left(\mathbf{f}, \mathbf{v}^h\right) & \forall\ \mathbf{v}^h \in V^h, \\
-\left(\nabla\cdot \mathbf{u}^h, q^h\right) & = & 0 & \forall\ q^h \in Q^h,
\end{array}
$$

- inf-sup stable pair of finite element spaces
- finite element error analysis for $n_{\text{skew}}(\cdot,\cdot,\cdot)$

$$
n_{\text{skew}}\left(\mathbf{u}^h, \mathbf{v}^h, \mathbf{v}^h\right) = \frac{1}{2}\left(n_{\text{conv}}\left(\mathbf{u}^h, \mathbf{v}^h, \mathbf{v}^h\right) - n_{\text{conv}}\left(\mathbf{u}^h, \mathbf{v}^h, \mathbf{v}^h\right)\right) = 0
$$

note that in general $\mathbf{u}^h \notin V_{\text{div}} \quad \Longrightarrow$

$$
n_{\text{conv}}\left(\mathbf{u}^h, \mathbf{v}^h, \mathbf{v}^h\right) \neq 0, \quad n_{\text{div}}\left(\mathbf{u}^h, \mathbf{v}^h, \mathbf{v}^h\right) \neq 0
$$

- Galerkin finite element method

$$\begin{aligned} \nu\left(\nabla \mathbf{u}^h, \nabla \mathbf{v}^h\right) + n\left(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h\right) - \left(\nabla\cdot\mathbf{v}^h, p^h\right) &= \left(\mathbf{f}, \mathbf{v}^h\right) \quad \forall\, \mathbf{v}^h \in V^h, \\ -\left(\nabla\cdot\mathbf{u}^h, q^h\right) &= 0 \qquad\qquad \forall\, q^h \in Q^h, \end{aligned}$$

- inf-sup stable pair of finite element spaces
- finite element error analysis for $n_{\text{skew}}(\cdot,\cdot,\cdot)$

$$n_{\text{skew}}\left(\mathbf{u}^h, \mathbf{v}^h, \mathbf{v}^h\right) = \frac{1}{2}\left(n_{\text{conv}}\left(\mathbf{u}^h, \mathbf{v}^h, \mathbf{v}^h\right) - n_{\text{conv}}\left(\mathbf{u}^h, \mathbf{v}^h, \mathbf{v}^h\right)\right) = 0$$

  note that in general $\mathbf{u}^h \notin V_{\text{div}} \quad \Longrightarrow$

$$n_{\text{conv}}\left(\mathbf{u}^h, \mathbf{v}^h, \mathbf{v}^h\right) \neq 0, \quad n_{\text{div}}\left(\mathbf{u}^h, \mathbf{v}^h, \mathbf{v}^h\right) \neq 0$$

- same as for continuous problem:
  - existence, uniqueness
  - stability

- Finite element error estimate for the $L^2(\Omega)$ norm of the gradient of the velocity
  - $\Omega \subset \mathbb{R}^d$ bounded domain with polyhedral boundary
  - $\nu^{-2} \|\mathbf{f}\|_{H^{-1}(\Omega)}$ be sufficiently small such that unique solution
  - inf-sup stable finite element spaces $V^h \times Q^h$

$$
\begin{aligned}
&\left\|\nabla(\mathbf{u}-\mathbf{u}^h)\right\|_{L^2(\Omega)} \\
&\quad \le C\left(\left(1+\frac{1}{\nu^2}\|\mathbf{f}\|_{H^{-1}(\Omega)}\right)\left(1+\frac{1}{\beta_{\mathrm{is}}^h}\right)\inf_{\mathbf{v}^h\in V^h}\left\|\nabla\left(\mathbf{u}-\mathbf{v}^h\right)\right\|_{L^2(\Omega)}\right. \\
&\qquad \left. +\frac{1}{\nu}\inf_{q^h\in Q^h}\left\|p-q^h\right\|_{L^2(\Omega)}\right)
\end{aligned}
$$

  - proof: main ideas and treatment of nonlinear term: board

- Finite element error estimate for the $L^2(\Omega)$ norm of the pressure

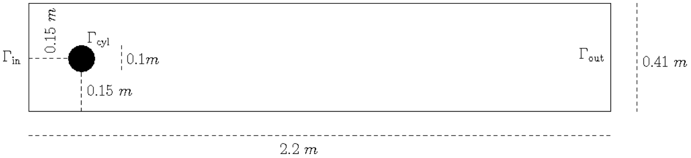
$$
\begin{aligned}
&\left\|p-p^h\right\|_{L^2(\Omega)} \\
&\leq \quad C\frac{\nu}{\beta_{\text{is}}^h}\left(\left(1+\frac{1}{\nu^2}\left\|\mathbf{f}\right\|_{H^{-1}(\Omega)}\right)^2\left(1+\frac{1}{\beta_{\text{is}}^h}\right)\inf_{\mathbf{v}^h\in V^h}\left\|\nabla\left(\mathbf{u}-\mathbf{v}^h\right)\right\|_{L^2(\Omega)}\right. \\
&\qquad \left. +C\frac{\nu}{\beta_{\text{is}}^h}\left(1+\frac{1}{\nu^2}\left\|\mathbf{f}\right\|_{H^{-1}(\Omega)}\right)\inf_{q^h\in Q^h}\left\|p-q^h\right\|_{L^2(\Omega)}\right)
\end{aligned}
$$

- Finite element error estimate for the $L^2(\Omega)$ norm of the pressure

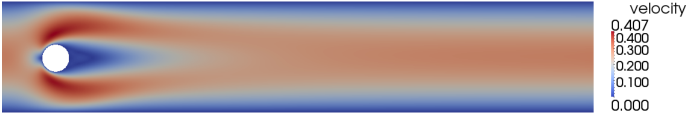
$$
\begin{aligned}
&\left\| p - p^h \right\|_{L^2(\Omega)} \\
&\quad\leq \quad C\frac{\nu}{\beta_{\text{is}}^h}\left(\left(1+\frac{1}{\nu^2}\left\|\mathbf{f}\right\|_{H^{-1}(\Omega)}\right)^2\left(1+\frac{1}{\beta_{\text{is}}^h}\right)\inf_{\mathbf{v}^h\in V^h}\left\|\nabla\left(\mathbf{u}-\mathbf{v}^h\right)\right\|_{L^2(\Omega)}\right. \\
&\qquad\qquad \left. +C\frac{\nu}{\beta_{\text{is}}^h}\left(1+\frac{1}{\nu^2}\left\|\mathbf{f}\right\|_{H^{-1}(\Omega)}\right)\inf_{q^h\in Q^h}\left\|p-q^h\right\|_{L^2(\Omega)}\right)
\end{aligned}
$$

- analytical results can be supported numerically by analytical test examples

- Example: steady-state flow around a cylinder at $Re = 20$
  - domain

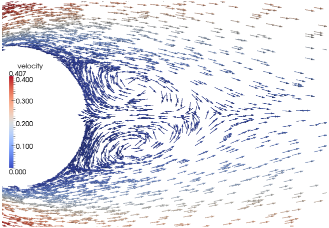


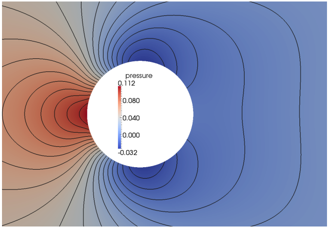
  - velocity



  - pressure

- Example: steady-state flow around a cylinder at $Re = 20$
  - at the cylinder





- important: drag and lift coefficient at the cylinder

$$
\begin{aligned}
c_{\text{drag}} &= \frac{2}{\rho d U_{\text{mean}}^2} \int_{\Gamma_{\text{cyl}}} \left(\mu \frac{\partial \mathbf{v_t}}{\partial \mathbf{n}} n_y - P n_x \right) ds, \\
c_{\text{lift}} &= -\frac{2}{\rho d U_{\text{mean}}^2} \int_{\Gamma_{\text{cyl}}} \left(\mu \frac{\partial \mathbf{v_t}}{\partial \mathbf{n}} n_x + P n_y \right) ds
\end{aligned}
$$

- important: drag and lift coefficient at the cylinder

$$
\begin{aligned}
c_{\text{drag}} &= \frac{2}{\rho d U_{\text{mean}}^2} \int_{\Gamma_{\text{cyl}}} \left(\mu \frac{\partial \mathbf{v_t}}{\partial \mathbf{n}} n_y - P n_x \right) \, ds, \\
c_{\text{lift}} &= -\frac{2}{\rho d U_{\text{mean}}^2} \int_{\Gamma_{\text{cyl}}} \left(\mu \frac{\partial \mathbf{v_t}}{\partial \mathbf{n}} n_x + P n_y \right) \, ds
\end{aligned}
$$

- reformulation with volume integrals possible, long but elementary derivation, e.g.

$$
c_{\text{drag}} = -\frac{2U^2}{dU_{\text{mean}}^2} \big((\nu \nabla \mathbf{u}, \nabla \mathbf{w}_d) + n(\mathbf{u}, \mathbf{u}, \mathbf{w}_d) - (\nabla \cdot \mathbf{w}_d, p) - (\mathbf{f}, \mathbf{w}_d) \big)
$$

for any function $\mathbf{w}_d \in H^1(\Omega)$ with $\mathbf{w}_d = \mathbf{0}$ on $\Gamma \setminus \Gamma_{\text{cyl}}$ and $\mathbf{w}_d|_{\Gamma_{\text{cyl}}} = (1,0)^T$

- reference values
  - [1] : compiled from simulations of different groups

$$c_{\text{drag,ref}} \in [5.57, 5.59], \quad c_{\text{lift,ref}} \in [0.104, 0.110]$$

  - [2] : do-nothing conditions at outlet

$$c_{\text{drag,ref}} = 5.57953523384, \quad c_{\text{lift,ref}} = 0.010618948146$$

  - [3] : Dirichlet conditions at outlet

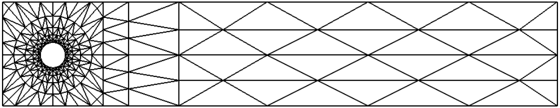
$$c_{\text{drag,ref}} = 5.57953523384, \quad c_{\text{lift,ref}} = 0.010618937712$$

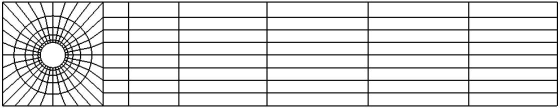
[1] Schäfer, Turek, Notes on Numerical Fluid Mechanics 52, 547–566, 1996

[2] Nabh, PhD thesis, Heidelberg, 1998

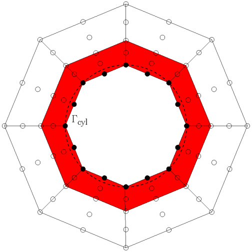
[3] J., Matthies, IJNMF 37, 885–903, 2001

- initial grids

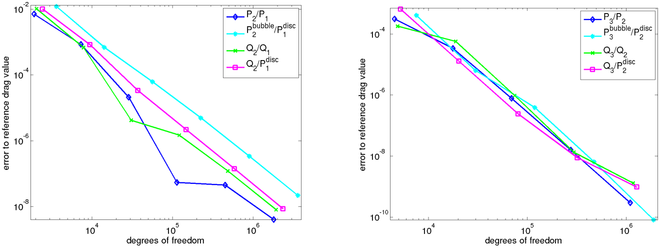




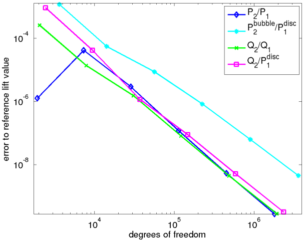
- patch for test function in computation of coefficients, Q_2

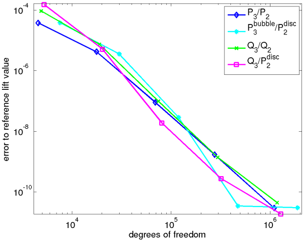


- convective form of convective term
- do-nothing boundary conditions
- convergence of drag coefficient



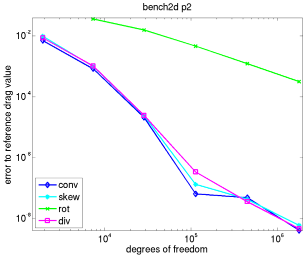
- convergence of lift coefficient

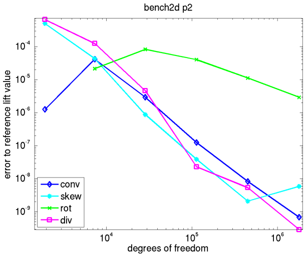




5 The Stationary Navier–Stokes Equations – Galerkin FEM

- preliminary results: different forms of the convective term, P_2/P_1





- rotational form
  - reconstructed pressure has boundary layers, inaccurate results

- schemes for solving the nonlinearity
- fixed point iteration

$$\begin{pmatrix} \mathbf{u}^{(m+1)} \\ p^{(m+1)} \end{pmatrix} = \begin{pmatrix} \mathbf{u}^{(m)} \\ p^{(m)} \end{pmatrix} - \vartheta \mathbf{N}_{\text{lin}}^{-1} \left(\begin{pmatrix} (\mathbf{f}, \mathbf{v}) - \mathbf{N}\left(\mathbf{u}^{(m)}; \mathbf{u}^{(m)}, p^{(m)}\right) \\ 0 \end{pmatrix} \right)$$

with

$$\mathbf{N}(\mathbf{w}; \mathbf{u}, p) = \begin{pmatrix} a(\mathbf{u}, \mathbf{v}) + n(\mathbf{w}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) \\ b(\mathbf{u}, q) \end{pmatrix}$$

$\mathbf{N}_{\text{lin}}$ – linear operator
$\vartheta \in (0, 1]$ – damping factor

- fixed point iteration
  - linear system to be solved

$$\mathbf{N}_{\text{lin}}\begin{pmatrix}\delta\mathbf{u}^{(m+1)}\\ \delta p^{(m+1)}\end{pmatrix} = \left(\begin{pmatrix}(\mathbf{f},\mathbf{v}) - \mathbf{N}\left(\mathbf{u}^{(m)};\mathbf{u}^{(m)},p^{(m)}\right)\\ 0\end{pmatrix}\right)$$

  - setting

$$\begin{pmatrix}\delta\mathbf{u}^{(m+1)}\\ \delta p^{(m+1)}\end{pmatrix} = \begin{pmatrix}\tilde{\mathbf{u}}^{(m+1)} - \mathbf{u}^{(m)}\\ \tilde{p}^{(m+1)} - p^{(m)}\end{pmatrix},$$

then

$$\mathbf{N}_{\text{lin}}\begin{pmatrix}\tilde{\mathbf{u}}^{(m+1)}\\ \tilde{p}^{(m+1)}\end{pmatrix} = \left(\begin{pmatrix}(\mathbf{f},\mathbf{v}) - \mathbf{N}\left(\mathbf{u}^{(m)};\mathbf{u}^{(m)},p^{(m)}\right)\\ 0\end{pmatrix}\right) + \mathbf{N}_{\text{lin}}\begin{pmatrix}\mathbf{u}^{(m)}\\ p^{(m)}\end{pmatrix}$$

5 The Stationary Navier–Stokes Equations – Galerkin FEM

- iteration with Stokes equations

$$\mathbf{N}_{\text{lin}} = \mathbf{N}\left(\mathbf{0};\tilde{\mathbf{u}}^{(m+1)},\tilde{p}^{(m+1)}\right)$$

- then

$$\begin{aligned}
&\begin{pmatrix} a(\tilde{\mathbf{u}}^{(m+1)},\mathbf{v}) + b(\mathbf{v},\tilde{p}^{(m+1)}) \\ b(\tilde{\mathbf{u}}^{(m+1)},q) \end{pmatrix} \\
&= \quad \begin{pmatrix} (\mathbf{f},\mathbf{v}) - a(\mathbf{u}^{(m)},\mathbf{v}) - n(\mathbf{u}^{(m)},\mathbf{u}^{(m)},\mathbf{v}) - b(\mathbf{v},\tilde{p}^{(m)}) \\ -b(\tilde{\mathbf{u}}^{(m)},q) \end{pmatrix} \\
&\qquad + \begin{pmatrix} a(\mathbf{u}^{(m)},\mathbf{v}) + b(\mathbf{v},\tilde{p}^{(m)}) \\ b(\tilde{\mathbf{u}}^{(m)},q) \end{pmatrix} = \begin{pmatrix} (\mathbf{f},\mathbf{v}) - n(\mathbf{u}^{(m)},\mathbf{u}^{(m)},\mathbf{v}) \\ 0 \end{pmatrix}
\end{aligned}$$

  - converges only if ν is sufficiently large
  - not recommended

- iteration with Oseen-type equations, Picard iteration

$$\mathbf{N}_{\text{lin}} = \mathbf{N}\left(\mathbf{u}^{(m)}; \tilde{\mathbf{u}}^{(m+1)}, \tilde{p}^{(m+1)}\right)$$

- then

$$\begin{aligned}
&\begin{pmatrix} a(\tilde{\mathbf{u}}^{(m+1)},\mathbf{v}) + n(\mathbf{u}^{(m)},\tilde{\mathbf{u}}^{(m+1)},\mathbf{v}) + b(\mathbf{v},\tilde{p}^{(m+1)}) \\ b(\tilde{\mathbf{u}}^{(m+1)},q) \end{pmatrix} \\
= \quad & \begin{pmatrix} (\mathbf{f},\mathbf{v}) - a(\mathbf{u}^{(m)},\mathbf{v}) - n(\mathbf{u}^{(m)},\mathbf{u}^{(m)},\mathbf{v}) - b(\mathbf{v},\tilde{p}^{(m)}) \\ -b(\tilde{\mathbf{u}}^{(m)},q) \end{pmatrix} \\
& + \begin{pmatrix} a(\mathbf{u}^{(m)},\mathbf{v}) + n(\mathbf{u}^{(m)},\mathbf{u}^{(m)},\mathbf{v}) + b(\mathbf{v},\tilde{p}^{(m)}) \\ b(\tilde{\mathbf{u}}^{(m)},q) \end{pmatrix} \\
= \quad & \begin{pmatrix} (\mathbf{f},\mathbf{v}) \\ 0 \end{pmatrix}
\end{aligned}$$

- iteration with Oseen-type equations, Picard iteration (cont.)
- different forms of nonlinear term

$$\begin{aligned}
n_{\text{conv}}\left(\mathbf{u}^{(m)},\tilde{\mathbf{u}}^{(m+1)},\mathbf{v}\right) &= \left(\left(\mathbf{u}^{(m)}\cdot\nabla\right)\tilde{\mathbf{u}}^{(m+1)},\mathbf{v}\right),\\
n_{\text{div}}\left(\mathbf{u}^{(m)},\tilde{\mathbf{u}}^{(m+1)},\mathbf{v}\right) &= \left(\left(\mathbf{u}^{(m)}\cdot\nabla\right)\tilde{\mathbf{u}}^{(m+1)},\mathbf{v}\right)+\left(\left(\nabla\cdot\mathbf{u}^{(m)}\right)\tilde{\mathbf{u}}^{(m+1)},\mathbf{v}\right),\\
n_{\text{rot}}\left(\mathbf{u}^{(m)},\tilde{\mathbf{u}}^{(m+1)},\mathbf{v}\right) &= \left(\left(\nabla\times\mathbf{u}^{(m)}\right)\times\tilde{\mathbf{u}}^{(m+1)},\mathbf{v}\right)\\
n_{\text{skew}}\left(\mathbf{u}^{(m)},\tilde{\mathbf{u}}^{(m+1)},\mathbf{v}\right) &= \frac{1}{2}\Big[\left(\left(\mathbf{u}^{(m)}\cdot\nabla\right)\tilde{\mathbf{u}}^{(m+1)},\mathbf{v}\right)\\
&\quad -\left(\left(\mathbf{u}^{(m)}\cdot\nabla\right)\mathbf{v},\tilde{\mathbf{u}}^{(m+1)}\right)\Big]
\end{aligned}$$

  - discussion: board
- widely used

- Newton's method
- linear operator is derivative of the nonlinear operator at the current position

$$\mathbf{N}_{\text{lin}} = D\mathbf{N}\begin{pmatrix}\mathbf{u}^{(m)} \\ p^{(m)}\end{pmatrix}$$

  - with Gâteaux derivative at $(\mathbf{u}, p)^T$

$$\begin{aligned} D\mathbf{N}\begin{pmatrix}\mathbf{u} \\ p\end{pmatrix} &= \lim_{\varepsilon\to 0}\frac{\mathbf{N}(\mathbf{u}+\varepsilon\phi;\mathbf{u}+\varepsilon\phi, p+\varepsilon\psi)-\mathbf{N}(\mathbf{u};\mathbf{u},p)}{\varepsilon} \\ &= \mathbf{N}(\phi;\mathbf{u},p)+\mathbf{N}(\mathbf{u};\phi,p)+\mathbf{N}(\mathbf{u},\mathbf{u},\psi) \end{aligned}$$

  - inserting and collecting terms (board)

$$\begin{pmatrix} a(\tilde{\mathbf{u}}^{(m+1)},\mathbf{v})+n(\mathbf{u}^{(m)},\tilde{\mathbf{u}}^{(m+1)},\mathbf{v})+n(\tilde{\mathbf{u}}^{(m+1)},\mathbf{u}^{(m)},\mathbf{v})+b(\mathbf{v},\tilde{p}^{(m+1)}) \\ b(\tilde{\mathbf{u}}^{(m+1)},q) \end{pmatrix}$$

$$= \begin{pmatrix} (\mathbf{f},\mathbf{v})+n(\mathbf{u}^{(m)},\mathbf{u}^{(m)},\mathbf{v}) \\ 0 \end{pmatrix}$$

- Newton's method (cont.)
  - order of convergence of Newton's method expected to be better than of the Picard iteration if
    - the solution $(\mathbf{u}, p)$ is sufficiently smooth
    - the linear systems are solved sufficiently accurately
  - properties of term $n(\tilde{\mathbf{u}}^{(m+1)}, \mathbf{u}^{(m)}, \mathbf{v})$ not clear
  - sometimes used in practice

- implementation
  - same principal approach as for Stokes and Oseen equations
  - inf-sup stable finite elements lead to linear saddle point problems in fixed point iteration

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix}$$

- implementation
  - same principal approach as for Stokes and Oseen equations
  - inf-sup stable finite elements lead to linear saddle point problems in fixed point iteration

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix}$$

  - convective form of convective term
    - Picard iteration

$$A = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{11} & 0 \\ 0 & 0 & A_{11} \end{pmatrix}$$

    - Newton iteration

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

- residual-based (and other) stabilizations possible
  - better: solve time-dependent problem

- continuous equation

$$
\begin{aligned}
\partial_t \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } (0,T] \times \Omega, \\
\nabla \cdot \mathbf{u} &= 0 && \text{in } [0,T] \times \Omega, \\
\mathbf{u}(0,\cdot) &= \mathbf{u}_0 && \text{in } \Omega,
\end{aligned}
$$

with

$$
\mathbf{u} = \mathbf{0} \text{ in } [0,T] \times \Gamma
$$

- weak or variational formulation obtained by
  - multiply Navier–Stokes equations with a suitable test function φ
  - integrate on $(0,T)\times\Omega$
  - apply integration by parts
- weak or variational formulation: find $\mathbf{u}\ :\ (0,T]\to H_0^1(\Omega)$ such that

$$\int_0^T \Big[-(\mathbf{u},\partial_t\varphi)+Re^{-1}(\nabla\mathbf{u},\nabla\varphi)+((\mathbf{u}\cdot\nabla)\mathbf{u},\varphi)\Big](\tau)\,d\tau$$
$$=\quad\int_0^T(\mathbf{f},\varphi)(\tau)\,d\tau+(\mathbf{u}_0,\varphi(0,\cdot))$$

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  - multiply Navier–Stokes equations with a suitable test function φ
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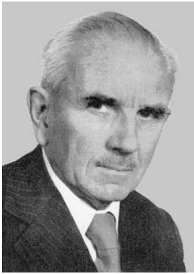
- weak or variational formulation: find $\mathbf{u} \;:\; (0,T] \to H_0^1(\Omega)$ such that

$$\int_0^T \Big[-(\mathbf{u}, \partial_t \varphi) + Re^{-1}(\nabla \mathbf{u}, \nabla \varphi) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \varphi) \Big](\tau)\, d\tau$$
$$= \int_0^T (\mathbf{f}, \varphi)(\tau)\, d\tau + (\mathbf{u}_0, \varphi(0,\cdot))$$

- properties
  - no time derivative with respect to $\mathbf{u}$
  - no second order space derivative with respect to $\mathbf{u}$
  - the pressure vanished because

$$\int_\Omega \nabla p \cdot \varphi \, d\mathbf{x} = (\nabla p, \varphi) = \int_\Gamma p(\mathbf{s}) \underbrace{\varphi(\mathbf{s})}_{=\mathbf{0}} \cdot \mathbf{n}(\mathbf{s})\, d\mathbf{s} - (p, \underbrace{\nabla\cdot\varphi}_{=0}) = 0$$

- mathematical analysis
  - 2d: existence and uniqueness of weak solution, Leary (1933), Hopf (1951)
  - 3d: existence of weak solution, Leary (1933), Hopf (1951)
- Jean Leray (1906 – 1998) Eberhard Hopf (1902 – 1983)



Uniqueness of weak solution of 3d Navier–Stokes equations is open problem !

- different form of the variational formulation

$$
\begin{aligned}
(\partial_t \mathbf{u},\mathbf{v}) + \nu\,(\nabla\mathbf{u},\nabla\mathbf{v}) + n\,(\mathbf{u},\mathbf{u},\mathbf{v}) - (\nabla\cdot\mathbf{v},p) &= (\mathbf{f},\mathbf{v}) \quad \forall\,\mathbf{v}\in V,\\
-(\nabla\cdot\mathbf{u},q) &= 0 \qquad\quad \forall\,q\in Q,
\end{aligned}
$$

and $\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x})$

- different form of the variational formulation

$$\begin{array}{rcll} (\partial_t \mathbf{u}, \mathbf{v}) + \nu (\nabla \mathbf{u}, \nabla \mathbf{v}) + n(\mathbf{u}, \mathbf{u}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, p) & = & (\mathbf{f}, \mathbf{v}) & \forall\, \mathbf{v} \in V, \\ -(\nabla \cdot \mathbf{u}, q) & = & 0 & \forall\, q \in Q, \end{array}$$

  and $\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x})$
- stability of velocity (board)

$$\|\mathbf{u}(T)\|_{L^2(\Omega)}^2 + \nu \|\nabla \mathbf{u}\|_{L^2(0,T;L^2(\Omega))}^2 \le \|\mathbf{u}(0)\|_{L^2(\Omega)}^2 + \frac{1}{\nu} \|\mathbf{f}\|_{L^2(0,T;H^{-1}(\Omega))}^2$$

- implicit θ-schemes as semi discretization in time
  - $\Delta t_{n+1} = t_{n+1} - t_n$
  - subscript k for quantities at time level k

$$
\begin{aligned}
\mathbf{u}_{k+1} + \theta_1 \Delta t_{n+1} \left[-\nu \Delta \mathbf{u}_{k+1} + (\mathbf{u}_{k+1} \cdot \nabla)\, \mathbf{u}_{k+1} \right] + \Delta t_{k+1} \nabla p_{k+1} & \\
= \quad & \mathbf{u}_k - \theta_2 \Delta t_{n+1} \left[-\nu \nabla \cdot \Delta \mathbf{u}_k + (\mathbf{u}_k \cdot \nabla)\, \mathbf{u}_k \right] + \theta_3 \Delta t_{n+1} \mathbf{f}_k \\
& + \theta_4 \Delta t_{n+1} \mathbf{f}_{k+1}, \\
\nabla \cdot \mathbf{u}_{k+1} \quad = \quad & 0,
\end{aligned}
$$

- implicit θ-schemes as semi discretization in time
  - $\Delta t_{n+1} = t_{n+1} - t_n$
  - subscript k for quantities at time level k

$$
\begin{aligned}
\mathbf{u}_{k+1} + \theta_1 \Delta t_{n+1}\left[-\nu\Delta\mathbf{u}_{k+1} + (\mathbf{u}_{k+1}\cdot\nabla)\,\mathbf{u}_{k+1}\right] + \Delta t_{k+1}\nabla p_{k+1} & \\
= \quad & \mathbf{u}_k - \theta_2\Delta t_{n+1}\left[-\nu\nabla\cdot\Delta\mathbf{u}_k + (\mathbf{u}_k\cdot\nabla)\,\mathbf{u}_k\right] + \theta_3\Delta t_{n+1}\mathbf{f}_k \\
& + \theta_4\Delta t_{n+1}\mathbf{f}_{k+1}, \\
\nabla\cdot\mathbf{u}_{k+1} \quad = \quad & 0,
\end{aligned}
$$

- one-step θ-schemes: $n = k$

	θ_1	θ_2	θ_3	θ_4	t_k	t_{k+1}	Δt_{k+1}	order
forward Euler scheme	0	1	1	0	t_n	t_{n+1}	Δt_{n+1}	
backward Euler scheme (BWE)	1	0	0	1	t_n	t_{n+1}	Δt_{n+1}	1
Crank–Nicolson scheme (CN)	0.5	0.5	0.5	0.5	t_n	t_{n+1}	Δt_{n+1}	2

- fractional-step θ-scheme [1]
  - three-step scheme
  - two variants

$$\theta = 1 - \frac{\sqrt{2}}{2}, \quad \tilde{\theta} = 1 - 2\theta, \quad \tau = \frac{\tilde{\theta}}{1-\theta}, \quad \eta = 1 - \tau$$

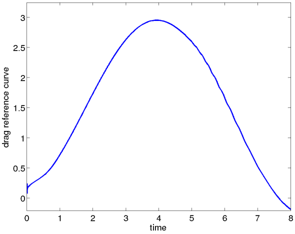
	θ_1	θ_2	θ_3	θ_4	t_k	t_{k+1}	Δt_{k+1}	order
FS0	$\tau\theta$	$\eta\theta$	$\eta\theta$	$\tau\theta$	t_n	$t_n + \theta\Delta t_{n+1}$	$\theta\Delta t_{n+1}$	
	$\eta\tilde{\theta}$	$\tau\tilde{\theta}$	$\tau\tilde{\theta}$	$\eta\tilde{\theta}$	$t_n + \theta\Delta t_{n+1}$	$t_{n+1} - \theta\Delta t_{n+1}$	$\tilde{\theta}\Delta t_{n+1}$	2
	$\tau\theta$	$\eta\theta$	$\eta\theta$	$\tau\theta$	$t_{n+1} - \theta\Delta t_{n+1}$	t_{n+1}	$\theta\Delta t_{n+1}$	
FS1	$\tau\theta$	$\eta\theta$	θ	0	t_n	$t_n + \theta\Delta t_{n+1}$	$\theta\Delta t_{n+1}$	
	$\eta\tilde{\theta}$	$\tau\tilde{\theta}$	0	$\tilde{\theta}$	$t_n + \theta\Delta t_{n+1}$	$t_{n+1} - \theta\Delta t_{n+1}$	$\tilde{\theta}\Delta t_{n+1}$	2
	$\tau\theta$	$\eta\theta$	θ	0	$t_{n+1} - \theta\Delta t_{n+1}$	t_{n+1}	$\theta\Delta t_{n+1}$	

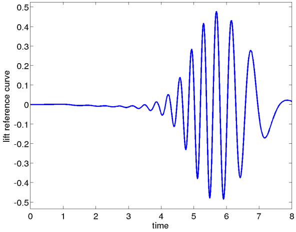
[1] Bristeau, Glowinski, Periaux: Finite elements in physics, North-Holland, 73–187, 1986

- popular approaches: BWE, CN
- stability
  - BWE, FS0, FS1: strongly A-stable
  - CN: A-stable
- FS1 less expensive than FS0 if computation of right hand side costly

- popular approaches: BWE, CN
- stability
  - BWE, FS0, FS1: strongly A-stable
  - CN: A-stable
- FS1 less expensive than FS0 if computation of right hand side costly
- number of papers with finite element error estimates available
  - proofs become long
  - same techniques as for steady-state problems + Gronwall's lemma

- flow around a cylinder
  - reference curves for drag and lift [1]





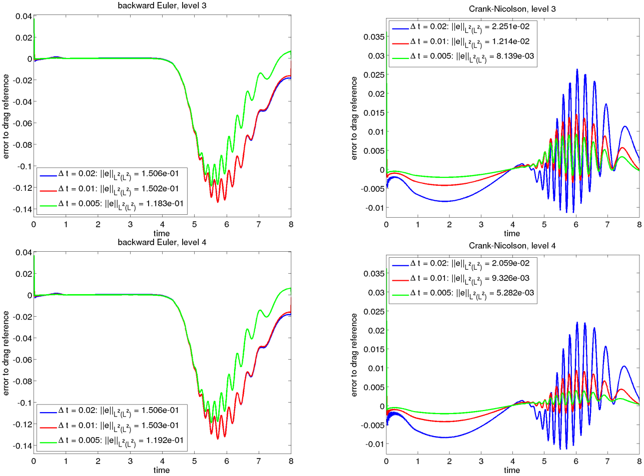
[1] J., Rang, CMAME 199, 514–524, 2010

- refinement in space with Q_2/P_1^{disc}

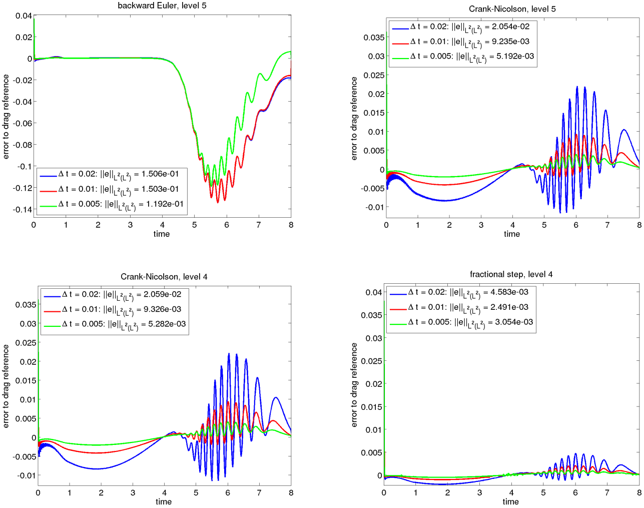
	P_2/P_1			Q_2/P_1^{disc}		
level	velocity	pressure	all	velocity	pressure	all
3	25 408	3248	28 656	27 232	9984	37 216
4	100 480	12 704	113 184	107 712	39 936	147 648
5	399 616	50 240	449 856	428 416	159 744	588 160

- refinement in time: $\Delta t \in \{0.02, 0.01, 0.005\}$

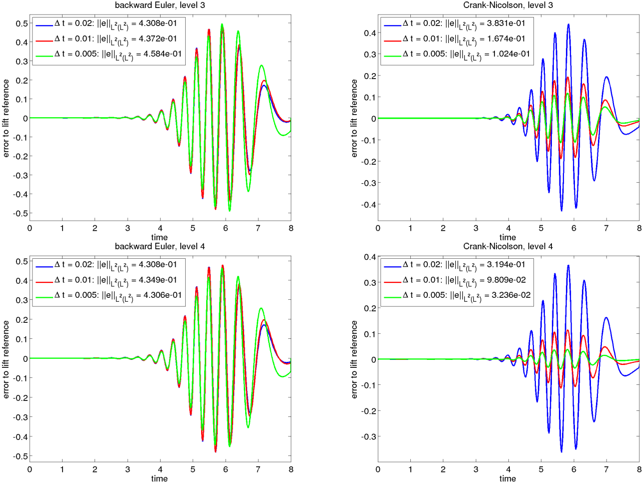
- error to the reference curve for the drag coefficient



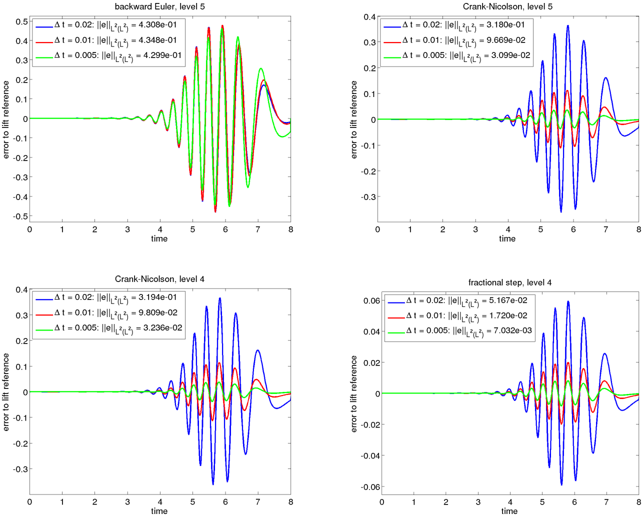
- error to the reference curve for the drag coefficient



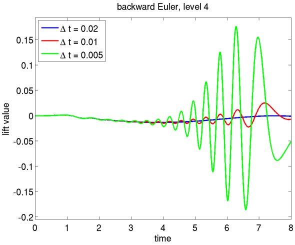
- error to the reference curve for the lift coefficient

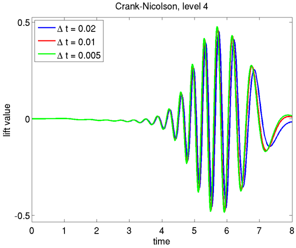


- error to the reference curve for the lift coefficient



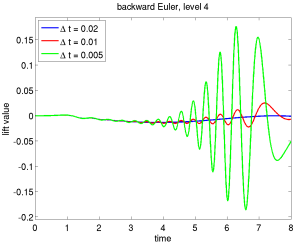
- temporal evolution of lift coefficient

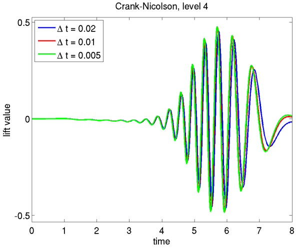




6 The Time-Dependent NSE – Laminar Flows

- temporal evolution of lift coefficient





- BWE much to inaccurate (dissipative)

- projection method
  - motivation: schemes without need to solve (nonlinear) saddle point problems
  - survey in [1]

[1] Guermond, Minev, Shen, CMAME 195, 6011–6045, 2006

- idea: decoupled NSE to obtain separate equations for velocity and pressure
  - approximation of time derivative given (q-step scheme)

$$\partial_t \mathbf{u}\left(t_{n+1}\right) \approx \frac{1}{\Delta t}\left(\tau_q \mathbf{u}_{n+1} + \sum_{i=0}^{q-1} \tau_j \mathbf{u}_{n-j}\right), \quad \sum_{i=0}^{q} \tau_j = 0$$

  - equation for intermediate velocity: given $\hat{p}$ or $\nabla \hat{p}$

$$\frac{1}{\Delta t}\left(\tau_q \tilde{\mathbf{u}}_{n+1} + \sum_{i=0}^{q-1} \tau_j \mathbf{u}_{n-j}\right) - \nu \Delta \tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}} = \mathbf{f} - \nabla \hat{p} \quad \text{in } (0,T] \times \Omega$$

  - correction step for divergence-free velocity

$$\begin{aligned} \frac{1}{\Delta t}\left(\tau_q \mathbf{u}_{n+1} - \tau_q \tilde{\mathbf{u}}_{n+1}\right) + \nabla \varphi\left(\tilde{\mathbf{u}}\right) + \nabla p &= \nabla \hat{p} \quad \text{in } (0,T] \times \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in } [0,T] \times \Omega \end{aligned}$$

$\varphi(\cdot)$ – given function

- velocity computed in projection step is $L^2(\Omega)$ projection of $\tilde{\mathbf{u}}_{n+1}$ into

$$H_{\mathrm{div}}(\Omega) = \left\{ \mathbf{v} \in L^2(\Omega),\ \nabla\cdot\mathbf{v} \in L^2(\Omega)\ :\ \nabla\cdot\mathbf{v} = 0 \text{ and } \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma \right\}$$

- non-incremental pressure-correction scheme
  - $\hat{p} = 0$, $\varphi(\cdot) = 0$
  - proposed in [1,2]
  - with backward Euler
- intermediate velocity

$$\tilde{\mathbf{u}}_{n+1} + \Delta t_{n+1}\left(-\nu\Delta\tilde{\mathbf{u}}_{n+1} + (\tilde{\mathbf{u}}_n \cdot \nabla)\tilde{\mathbf{u}}_{n+1}\right) = \mathbf{u}_n + \Delta t_{n+1}\mathbf{f}_{n+1} \quad \text{in } \Omega$$

with $\tilde{\mathbf{u}}_{n+1} = \mathbf{0}$ on Γ

- projection step

$$\begin{aligned} \mathbf{u}_{n+1} + \Delta t_{n+1}\nabla p_{n+1} &= \tilde{\mathbf{u}}_{n+1} && \text{in } \Omega, \\ \nabla \cdot \mathbf{u}_{n+1} &= 0 && \text{in } \Omega, \\ \mathbf{u}_{n+1} \cdot \mathbf{n} &= 0 && \text{on } \Gamma \end{aligned}$$

[1] Chorin, Math. Comp. 22, 745–762, 1968

[2] Temam, Arch. Rational Mech. Anal. 33, 377–385, 1969

- non-incremental pressure-correction scheme (cont.)
- taking divergence of projection step

$$\nabla \cdot \nabla p_{n+1} = \Delta p_{n+1} = \frac{1}{\Delta t_{n+1}} \nabla \cdot \tilde{\mathbf{u}}_{n+1}$$

  - Poisson equation for the pressure
  - boundary condition

$$\nabla p_{n+1} \cdot \mathbf{n} = -\frac{1}{\Delta t_{n+1}} \left(\mathbf{u}_{n+1} - \tilde{\mathbf{u}}_{n+1} \right) \cdot \mathbf{n} = 0$$

- non-incremental pressure-correction scheme (cont.)
- taking divergence of projection step

$$\nabla \cdot \nabla p_{n+1} = \Delta p_{n+1} = \frac{1}{\Delta t_{n+1}} \nabla \cdot \tilde{\mathbf{u}}_{n+1}$$

  - Poisson equation for the pressure
  - boundary condition

$$\nabla p_{n+1} \cdot \mathbf{n} = -\frac{1}{\Delta t_{n+1}} \left(\mathbf{u}_{n+1} - \tilde{\mathbf{u}}_{n+1}\right) \cdot \mathbf{n} = 0$$

- error estimates: $(\overline{\mathbf{u}}, \overline{p})$ result of projection step

$$\|p - \overline{p}\|_{l^\infty\left(0,T;L^2(\Omega)\right)} + \|\mathbf{u} - \tilde{\mathbf{u}}\|_{l^\infty\left(0,T;H^1(\Omega)\right)} \le C\left(\mathbf{u}, p, T\right) \Delta t^{1/2}$$

  if in addition domain has regularity property

$$\|\mathbf{u} - \overline{\mathbf{u}}\|_{l^\infty\left(0,T;L^2(\Omega)\right)} + \|\mathbf{u} - \tilde{\mathbf{u}}\|_{l^\infty\left(0,T;L^2(\Omega)\right)} \le C\left(\mathbf{u}, p, T\right) \Delta t$$

- non-incremental pressure-correction scheme (cont.)
  - inf-sup stable finite elements not necessary
  - however, spurious oscillations may appear if the time step becomes too small
  - low orders of convergence
  - splitting error is $\mathcal{O}(\Delta t) \Longrightarrow$ first order time stepping scheme sufficient
  - artificial Neumann boundary condition for the pressure induces a numerical boundary layer

- standard incremental pressure-correction scheme
  - $\hat{p} = p_n$, $\varphi(\cdot) = 0$
  - with BDF2
- intermediate velocity

$$
\begin{aligned}
3\tilde{\mathbf{u}}_{n+1} + 2\Delta t\left(-\nu\Delta\tilde{\mathbf{u}}_{n+1} + (\tilde{\mathbf{u}}_n\cdot\nabla)\tilde{\mathbf{u}}_{n+1}\right) & \\
= \quad 4\mathbf{u}_n - \mathbf{u}_{n-1} + 2\Delta t\left(\mathbf{f}_{n+1} - \nabla p_n\right) & \quad \text{in } \Omega,
\end{aligned}
$$

with $\tilde{\mathbf{u}}_{n+1} = \mathbf{0}$ on Γ

- projection step

$$
\begin{aligned}
3\mathbf{u}_{n+1} + 2\Delta t\nabla\left(p_{n+1} - p_n\right) &= 3\tilde{\mathbf{u}}_{n+1} && \text{in } \Omega,\\
\nabla\cdot\mathbf{u}_{n+1} &= 0 && \text{in } \Omega,\\
\mathbf{u}_{n+1}\cdot\mathbf{n} &= 0 && \text{on } \Gamma
\end{aligned}
$$

- standard incremental pressure-correction scheme (cont.)
- taking divergence of projection step

$$\Delta\left(p_{n+1}-p_n\right)=\frac{3}{2\Delta t}\nabla\cdot\tilde{\mathbf{u}}_{n+1}\quad\text{in }\Omega$$

  - Poisson equation for the pressure update
  - boundary condition

$$\nabla\left(p_{n+1}-p_n\right)\cdot\mathbf{n}=0\quad\text{on }\Gamma$$

- standard incremental pressure-correction scheme (cont.)
- taking divergence of projection step

$$\Delta\left(p_{n+1}-p_n\right)=\frac{3}{2\Delta t}\nabla\cdot\tilde{\mathbf{u}}_{n+1}\quad \text{in } \Omega$$

  - Poisson equation for the pressure update
  - boundary condition

$$\nabla\left(p_{n+1}-p_n\right)\cdot\mathbf{n}=0\quad \text{on } \Gamma$$

- error estimates, with appropriate initial step, $(\overline{\mathbf{u}},\overline{p})$ result of projection step

$$\|p-\overline{p}\|_{l^\infty\left(0,T;L^2(\Omega)\right)}+\|\mathbf{u}-\tilde{\mathbf{u}}\|_{l^\infty\left(0,T;H^1(\Omega)\right)}\le C\left(\mathbf{u},p,T\right)\Delta t$$

  if in addition domain has regularity property

$$\|\mathbf{u}-\overline{\mathbf{u}}\|_{l^\infty\left(0,T;L^2(\Omega)\right)}+\|\mathbf{u}-\tilde{\mathbf{u}}\|_{l^2\left(0,T;L^2(\Omega)\right)}\le C\left(\mathbf{u},p,T\right)\Delta t^2$$

- standard incremental pressure-correction scheme (cont.)
  - similar estimates for Crank–Nicolson scheme
  - splitting error is $\mathcal{O}\left(\Delta t^2\right) \Longrightarrow$ second order time stepping scheme sufficient
  - artificial Neumann boundary condition for the pressure induces a numerical boundary layer

- rotational incremental pressure-correction scheme
  - $\hat{p} = p_n$, $\varphi(\tilde{\mathbf{u}}) = \nu \nabla \cdot \tilde{\mathbf{u}}_{n+1}$
  - with BDF2
- intermediate velocity

$$
\begin{aligned}
&3\tilde{\mathbf{u}}_{n+1} + 2\Delta t \left(-\nu \Delta \tilde{\mathbf{u}}_{n+1} + (\tilde{\mathbf{u}}_n \cdot \nabla)\tilde{\mathbf{u}}_{n+1}\right) \\
&\quad = \quad 4\mathbf{u}_n - \mathbf{u}_{n-1} + 2\Delta t \left(\mathbf{f}_{n+1} - \nabla p_n\right) \quad \text{in } \Omega,
\end{aligned}
$$

with $\tilde{\mathbf{u}}_{n+1} = \mathbf{0}$ on Γ

- projection step

$$
\begin{aligned}
3\mathbf{u}_{n+1} + 2\Delta t \nabla \left(p_{n+1} - p_n\right) &= 3\tilde{\mathbf{u}}_{n+1} - 2\nu \Delta t \nabla \left(\nabla \cdot \tilde{\mathbf{u}}_{n+1}\right) && \text{in } \Omega, \\
\nabla \cdot \mathbf{u}_{n+1} &= 0 && \text{in } \Omega, \\
\mathbf{u}_{n+1} \cdot \mathbf{n} &= 0 && \text{on } \Gamma
\end{aligned}
$$

- rotational incremental pressure-correction scheme (cont.)
- taking divergence of projection step

$$\Delta \tilde{p}_n = \frac{3}{2\Delta t} \nabla \cdot \tilde{\mathbf{u}}_{n+1} \quad with \quad \tilde{p}_n = p_{n+1} - p_n + \nu \nabla \cdot \tilde{\mathbf{u}}_{n+1}$$

  - Poisson equation for the modified pressure
  - boundary condition

$$\nabla p_{n+1} \cdot \mathbf{n} = \left(\mathbf{f}_{n+1} - \nu \nabla \times \nabla \times \mathbf{u}_{n+1}\right) \cdot \mathbf{n} \text{ on } \Gamma$$

- rotational incremental pressure-correction scheme (cont.)
- taking divergence of projection step

$$\Delta \tilde{p}_n = \frac{3}{2\Delta t} \nabla \cdot \tilde{\mathbf{u}}_{n+1} \quad with \quad \tilde{p}_n = p_{n+1} - p_n + \nu \nabla \cdot \tilde{\mathbf{u}}_{n+1}$$

  - Poisson equation for the modified pressure
  - boundary condition

$$\nabla p_{n+1} \cdot \mathbf{n} = \left(\mathbf{f}_{n+1} - \nu \nabla \times \nabla \times \mathbf{u}_{n+1}\right) \cdot \mathbf{n} \text{ on } \Gamma$$

- error estimates, with appropriate initial step, $(\overline{\mathbf{u}}, \overline{p})$ result of projection step

$$\|p - \overline{p}\|_{l^2\left(0,T;L^2(\Omega)\right)} + \|\mathbf{u} - \tilde{\mathbf{u}}\|_{l^2\left(0,T;H^1(\Omega)\right)} + \|\mathbf{u} - \overline{\mathbf{u}}\|_{l^2\left(0,T;H^1(\Omega)\right)} \le C\left(\mathbf{u}, p, T\right) \Delta t^{3/2}$$

  if in addition domain has regularity property

$$\|\mathbf{u} - \overline{\mathbf{u}}\|_{l^2\left(0,T;L^2(\Omega)\right)} + \|\mathbf{u} - \tilde{\mathbf{u}}\|_{l^2\left(0,T;L^2(\Omega)\right)} \le C\left(\mathbf{u}, p, T\right) \Delta t^2$$

- rotational incremental pressure-correction scheme (cont.)
  - equivalent formulation of velocity step

$$
\begin{aligned}
3\mathbf{u}_{n+1} + 2\Delta t\,(\nu \nabla \times \nabla \times \mathbf{u}_{n+1} + (\tilde{\mathbf{u}}_n \cdot \nabla)\tilde{\mathbf{u}}_{n+1} + \nabla p_{n+1}) & \\
= \quad 4\mathbf{u}_n - \mathbf{u}_{n-1} + 2\Delta t \mathbf{f}_{n+1} & \quad \text{in } \Omega, \\
\nabla \cdot \mathbf{u}_{n+1} \quad = \quad 0 & \quad \text{in } \Omega
\end{aligned}
$$

  - boundary condition for the pressure is consistent, can be derived from the Navier–Stokes equations

- only $\tilde{\mathbf{u}}_{n+1}$ needed in implementation
- first experience with non-incremental and standard incremental scheme: very inaccurate at boundaries (bad drag and lift coefficients)

Thank you for your attention !

http://www.wias-berlin.de/people/john/