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We will learn about : Basics of functions of several variables. In this lecture:

## 1.1 A sequence in the Euclidean space and its application

Using these notation :

- $\mathbb{N}$  : set of natural number ( $\mathbb{N} = \{1, 2, 3, \dots\}$ )
- $\mathbb{Z}$  : set of integers ( $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ )
- $\mathbb{Q}$  : set of rational number ( $\mathbb{Q} = \{0, \pm 1, \pm 2, \frac{2}{3}, \dots\}$ )
- $\mathbb{R}$  : set of real number
- $\mathbb{C}$  : set of complex number

**Definition 1.** A sequence  $(x_n)_{n=1}^{\infty}$  is an assignment of (real) number  $x_n \in \mathbb{R}$  to natural number  $n \in \mathbb{N}$  ( $x_n \in \mathbb{R}$ ).

Example :  $x_n = \frac{1}{n}$ .  $x_1 = 1, x_2 = \frac{1}{2}, \dots$

**Definition 2.** A subsequence of a sequence  $(x_n)_{n=1}^{\infty}$  is a sequence  $(y_j)_{j=1}^{\infty}$  defined by  $y_j = x_{n_j}$  for some sequence  $(n_j)_{j=1}^{\infty}$  in  $\mathbb{N}$  such that  $n_j < n_{j+1}$  ( $j = 1, 2, \dots$ ).

Example : sequence  $(x_n)_{n=1}^{\infty} = 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{100}$ , takes  $n_1 = 1, n_2 = 3, n_3 = 5, n_4 = 100$

subsequence  $(x_{n_j})_{j=1}^{\infty} = x_{n_1}, x_{n_2}, x_{n_3}, x_{n_4} = 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{100}$ .

**Definition 3.** Let  $(x_n)_{n=1}^{\infty}$  be a sequence converges to  $\alpha \in \mathbb{R}$  if for any  $\epsilon > 0$ , there is an  $N \in \mathbb{N}$  such that  $n > N$ ,  $|x_n - \alpha| < \epsilon$ .

In the mathematical symbol  $\forall \epsilon > 0, \exists N \in \mathbb{N}$  such that  $n > N$ ,  $|x_n - \alpha| < \epsilon$  for  $n > N$ .

In this case we write,  $\lim_{n \rightarrow \infty} x_n$  or  $x_n \rightarrow \alpha$  ( $n \rightarrow \infty$ )

**Example 1.**

**Theorem 1.**  $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}$  is sequence. Suppose  $x_n \rightarrow \alpha$  and  $y_n \rightarrow \beta$  as  $n \rightarrow \infty$ .

1.  $x_n \pm y_n \rightarrow \alpha \pm \beta$ , ( $n \rightarrow \infty$ )
2.  $x_n \cdot y_n \rightarrow \alpha \cdot \beta$ , ( $n \rightarrow \infty$ )
3. if  $\beta \neq 0$ ,  $\frac{x_n}{y_n} \rightarrow \frac{\alpha}{\beta}$ , ( $n \rightarrow \infty$ )

**Remark 1.** On 3,  $\frac{x_n}{y_n}$  is not defined for all  $n \in \mathbb{N}$  because  $y_n = 0$  possibly for some  $n \in \mathbb{N}$ . But, since  $y_n \rightarrow \beta \neq 0$ ,  $y_n$  eventually is not 0. Hence  $\frac{x_n}{y_n}$  is defined eventually.

**Theorem 2.**  $(x_n)_{n=1}^{\infty}$  a sequence. If  $(x_n)_{n=1}^{\infty}$  converges to  $\alpha \in \mathbb{R}$ , any subsequence of  $(x_n)_{n=1}^{\infty}$  converges to  $\alpha \in \mathbb{R}$ .  
 $\therefore$  Let  $(x_n)_{n=1}^{\infty}$  be a subsequence. Because  $x_n \rightarrow \alpha$  ( $n \rightarrow \infty$ ),  $\forall \epsilon > 0, \exists N \in \mathbb{N}$ . Take  $J_0 \in \mathbb{N}$  such that  $n_j > N_\theta$  for all  $j > J_0$ . Then  $|x_{n_j} - \alpha| < \epsilon$  for  $j > J_0$ .  $x_{n_j} \rightarrow \alpha$  ( $j \rightarrow \infty$ ),  $(n_j)_{j=1}^{\infty}$  also a sequence,  $n_j \in \mathbb{N}, n_j < n_{j+1}$ .

**Completeness Axiom.** Let  $(x_n)_{n=1}^{\infty}$  be a monotonically increasing (decreasing) sequence (i.e.  $x_n \leq x_{n+1}, n \in \mathbb{N}$ ). Suppose that there is an  $M \in \mathbb{R}$  such that  $x_n \leq M$  ( $n \in \mathbb{N}$ ) ( $x_n \geq M$ ). Then,  $(x_n)_{n=1}^{\infty}$  converges ( $\exists \alpha \in \mathbb{R}$  such that  $x_n \rightarrow \alpha$ )

**Theorem 3. Bolzano-Weirstrass.**  $(x_n)_{n=1}^{\infty}$  a sequence in  $\mathbb{R}$ . Suppose  $(x_n)_{n=1}^{\infty}$  is bounded in the sense that  $|x_n| \leq M, \forall n \in \mathbb{N}$ . Then  $(x_n)_{n=1}^{\infty}$  contains a convergent subsequence.  
 $x_n$  is a peak of  $(x_n)_{n=1}^{\infty}$  if  $x_n > x_m$  for  $m > n$ .

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### 2.1 n-dimensional space

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, \dots, x_n) | x_i \in \mathbb{R}\}$ .

Takes  $n = 2$ ,  $\mathbb{R}^2 \Leftrightarrow$  plane, we have  $P(a, b)$ .

For  $n = 3$ , we have  $P(a, b, c)$ .

**Definition 4.**  $P_m = (x_1^m, \dots, x_n^m) \in \mathbb{R}^n$ , and  $\{P_m\}_{m=1}^\infty$  : a sequence in  $\mathbb{R}^n$ .  
 $\{P_m\}$  converges to  $A = (a_1, \dots, a_n) \in \mathbb{R}^n$ , if  $\forall k = 1, \dots, n$ ,  $x_k^m \rightarrow a_k$  as  $n \rightarrow \infty$ .

**Definition 5. Inner product and norm.**

$\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ . We can define :

$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n$  ; **inner product**

$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$  ; **norm**

**Example 2.**  $\mathbf{x} \cdot \mathbf{y} = 0 \Leftrightarrow \mathbf{x}$  is perpendicular to  $\mathbf{y}$

Takes  $n = 0$  then

$$\begin{aligned} x_1 y_1 + x_2 y_2 &= 0 \\ x_1 y_1 &= -x_2 y_2 \\ \frac{y_1}{y_2} &= -\frac{x_2}{x_1} \\ \text{then } (x_1, x_2) &= c \cdot (-y_2, y_1) \end{aligned}$$

pict :

**Example 3.**  $\|\mathbf{x}\| = 0 \Leftrightarrow x = 0$

$(\Rightarrow) 0 = \|x\|^2 = x_1^2 + \cdots + x_n^2$ , then  $x_i^2 = 0$  ( $\forall i = 1, \dots, n$ ) and finally  $x_i = 0$ .

**Notes 1.**  $\|\mathbf{x}\|$  is the distance between  $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^n$  and  $\mathbf{x} = (x_1, \dots, x_n)$ .

For notation, we will use  $P, Q \in \mathbb{R}^n$  as points and  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  as vectors.

We also use  $\|x - y\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$  as distance between  $\mathbf{x}$  and  $\mathbf{y}$ .

$\|P - Q\|$  is distance between  $P$  and  $Q$ .

$$\mathbf{x} \pm \mathbf{y} = (x_1 \pm y_1, \dots, x_n \pm y_n)$$

$$P = (p_1, \dots, p_n), Q = (q_1, \dots, q_n), \text{ then } P + Q = (p_1 + q_1, \dots, p_n + q_n)$$

$$\alpha \in \mathbb{R}, \alpha \mathbf{x} = (\alpha x_1, \dots, \alpha x_n), \alpha P = (\alpha p_1, \dots, \alpha p_n)$$

$$\{P_m\}_{m=1}^\infty : \text{a sequence in } \mathbb{R}^n, P_m \rightarrow A \Leftrightarrow \|P_m - A\| \rightarrow 0$$

**Theorem 4. Cauchy-Schwarz inequality.** For any  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ ,

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|$$

"="  $\Rightarrow a\mathbf{y} = b\mathbf{x}$  for some  $a, b \in \mathbb{R}$ .

$\therefore$  We may assume  $\mathbf{x} \neq \emptyset, \forall t \in \mathbb{R}$ .

$$0 \leq \|t\mathbf{x} + \mathbf{y}\|^2 = (t\mathbf{x} + \mathbf{y}) \cdot (t\mathbf{x} + \mathbf{y}) = t^2 \|\mathbf{x}\|^2 + 2t(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2$$

$$D/4 \leq 0$$

**Theorem 5. Bolzano=Weierstrass.** Let  $(P_m)_{m=1}^\infty \subset \mathbb{R}^n$  be a sequence. Suppose that  $(P_m)_{m=1}^\infty$  is bounded. In the sense that  $\|P_m\| \leq M$  ( $m \in \mathbb{N}$ ) for some  $M \geq 0$ . Then  $(P_m)_{m=1}^\infty$  contains a convergent subsequence.

**Definition 6. Ball.**  $A \in \mathbb{R}^n, R > 0$

$$\mathbf{B}(A, R) = \{P \in \mathbb{R}^n | \|P - A\| < R\}; \text{ open ball of center } A \text{ with radius } R$$

$$\overline{\mathbf{B}}(A, R) = \{P \in \mathbb{R}^n | \|P - A\| \leq R\}; \text{ closed ball}$$

**Definition 7.** 1.  $E \subset \mathbb{R}^n$  is said to be **an open set** if  $E = \emptyset$  or  $\forall A \in E, \exists R > 0$  such that  $\mathbf{B}(A, R) \subset E$ .

2.  $E \subset \mathbb{R}^n$  is said to be **a closed set** if  $E^c \in \mathbb{R}^n$   $E$  is an open set.

$E$  : open, then neighbor in any point

**Definition 8. Accumulation point.**  $E \subset \mathbb{R}^n$ ; a set.  $A \in \mathbb{R}^n$  is called **an accumulation point** of  $E$  if  $\forall R > 0, (\mathbf{B}(A, R) - \{A\}) \cap E \neq \emptyset$ .

**Notes 2.**  $E \subset \mathbb{R}^n$  is closed if and only if  $E$  contains any accumulation point of  $E$ . **Homework report, prove this**

**Notes 3.** 1. Both  $\emptyset$  and  $\mathbb{R}^n$  are open and closed

2.  $\{E_\lambda \mid \lambda \in A\}$ ; a collection of open sets  $\Rightarrow$  union  $\bigcup_{\lambda \in A} E_\lambda$  is also open

3.  $\{E_\lambda\}_{\lambda=1}^N$ , a finite collection of open sets  $\Rightarrow$   $\bigcup_{\lambda=1}^N E_\lambda$  is also open.

4. **Rephrase of Bolzano Weierstrass theorem.**  $E \subset \mathbb{R}^n$ ; a **bounded closed set**  $\Leftrightarrow E$  is a closed set such that  $E \subset B(0, R)$  for some  $R > 0$ .  $E$ ; a bounded closed set then any sequence of  $E$  contains a convergent subsequence whose limit is in  $E$ .

**Definition 9.** A bounded closed set in  $\mathbb{R}^n$  is called **compact**.

**Example 4.**  $\overline{B}(A, R)$  is compact. **Report! prove this**

## 2.2 Continuity and differentiability of a function

### 2.2.1 Continuity

$E$ : a set in  $\mathbb{R}^n$  and  $f$ : is a function of  $E$  (real valued function).

i.e.  $f$  is an assignment a (real) number to a point in  $E$ .

**Definition 10.** 1.  $f$  is **continuous at**  $A \in E$  if  $\forall (P_m)_{m=1}^\infty \subset E$ : sequence with  $P_m \rightarrow A$  ( $m \rightarrow \infty$ )

$$f(P_m) \rightarrow f(A) \quad (m \rightarrow \infty)$$

2.  $f$  is **continuous on**  $E$  if  $f$  is continuous at any point of  $E$ .

### 2.2.2 Basic of continuous function on an interval in $\mathbb{R}$

**Theorem 6. Intermediate value theorem.**  $f$ : function on a closed interval  $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ . Suppose that  $f(a) \leq f(b)$ . Then,  $\forall \gamma$  with  $f(a) \leq \gamma \leq f(b)$ ,  $\exists c \in [a, b]$  with  $f(c) = \gamma$ .

**Theorem 7. Extreme value theorem.**  $f$  is a continuous function on a closed interval  $[a, b]$ . Then,  $f$  attains a maximum and a minimum on  $[a, b]$ .

## REPORT 1

1.  $E \subset \mathbb{R}^n$  is closed if and only if  $E$  contains any accumulation point of  $E$
2.  $\overline{B}(A, R)$  is compact.

### Proof :

1. ( $\Rightarrow$ ) if  $E$  is closed then  $E$  contains all of its accumulation point. Let  $x$  accumulation point of  $E$ ,  $x \in E$  and  $E$  is closed then  $E^c$  is open .

Let  $x \in E^c$  and  $R > 0 \Rightarrow \forall x \in E^c, \exists B(x, R)$  such that  $\forall y \in B(x, R) \Rightarrow y \in E^c$ .

Suppose  $x$  is accumulation point of  $E$  that is not in  $E$ .

Then,  $\forall e \in B(x, R), \exists y \neq x$  with  $y \in e \cap E$ .

$y \in e \cap E \Rightarrow y \notin E^c$  contradiction.

( $\Leftarrow$ )  $E$  contains all of its accumulation point then  $E$  is closed.

Suppose  $E$  contains all of its accumulation point. Suppose  $E^c$  is not open.

$\exists x \in E^c$  such that  $\forall e \in B(x, R), R > 0, \exists y \in e$  that also in  $E$ .

Its contradict the premise, because  $x$  is accumulation point.

2. Suppose  $x \notin \overline{B}(A, R) \Rightarrow \|x - A\| > R$ .  
So let  $\|x - A\| - R = \epsilon > 0$ .  
Consider  $y \in B(x, \epsilon/2)$ ,

$$\begin{aligned}\|y - A\| &\geq \|x - A\| - \|y - x\| \\ \|y - A\| &\geq R + \epsilon - (\epsilon/2) \\ \|y - A\| &\geq R + (\epsilon/2) \\ \|y - A\| &> R\end{aligned}$$

shows that  $y \in \overline{B}(A, R)$ . Hence  $B(x, \epsilon/2)$  subset of  $\overline{B}(A, R)^c$ .

Because  $\overline{B}(A, R)^c$  hence  $\overline{B}(A, R)$  is closed.

By definition,  $\overline{B}(A, R) = \{x \in \mathbb{R}^n \mid \|x - A\| \leq R\}$

Then  $\forall x \in \overline{B}(A, R)$  we can find

$$\begin{aligned}\|x - A\| &\leq R \\ -R &\leq \|x - A\| \leq R.\end{aligned}\tag{1}$$

shows that  $\overline{B}(A, R)$  is bounded.

Because closed and bounded,  $\overline{B}(A, R)$  is compact.

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$f$  is continuous function on  $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$

**Theorem 8. Intermediate Value Theorem.** Suppose  $f(a) \leq f(b)$  then  $\forall \gamma \in \mathbb{R}$  with  $f(a) \leq \gamma \leq f(b)$ ,  $\exists c$  in  $[a, b]$  such that  $f(c) = \gamma$ .

**Theorem 9. Extreme value theorem.**  $f$  attains a maximum and a minimum on  $[a, b]$ .

#### 3.1 Differentiable function on intervals

$f$  : function defined around  $x = a \in \mathbb{R}$ .

$f$  is differentiable at  $x = a$  if the limit  $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$  exist.

$f$  : function on  $(a, b) = \{x \in \mathbb{R} | a < x < b\}$

$f$  is differentiable on  $(a, b)$  if  $f$  is differentiable at any point of  $(a, b)$ .

**Properties :** if  $f$  is differentiable at  $x = a$  then  $f$  is continuous at  $x = a$ .

$\because f(a+h) = f(a) + h \frac{f(a+h) - f(a)}{h} = f(a) + hf'(a)$ . Because  $h \rightarrow 0$  then  $f(a+h) \rightarrow f(a)$ .

**Theorem 10. Rolle's theorem.**  $f$  : continue on  $[a, b]$  and differentiable on  $(a, b)$ . If  $f(a) = f(b)$  then  $\exists c \in (a, b)$  such that  $f'(c) = 0$ .

$\because$  if  $f$  is a constant function,  $f'(x) = 0, \forall x \in (a, b)$ . Suppose that  $f$  is not a constant function, by extreme value theorem,  $f$  attain max at  $x = c_1$  and min at  $x = c_2$  with  $c_1 \neq c_2$ .

(Otherwise max =  $f(c_1) = f(c_2) = \min$ )

$\forall x \in [a, b], f(c_2) \leq \min \leq f(x) \leq \max \leq f(a)$ . We may assume  $c_1 \in (a, b)$ .

(Otherwise, consider  $-f$  instead  $f$ ;  $(-f)'(a) = \lim_{h \rightarrow 0} \frac{-f(a+h) - (-f(a))}{h} = -f'(a)$ )

$$\begin{aligned} \frac{f(c_1+h) - f(c_1)}{h} &\leq 0, h < 0. \quad \text{for } h \rightarrow 0, \quad f'(c_1) \leq 0 \\ \frac{f(c_1+h) - f(c_1)}{h} &\geq 0, h > 0. \quad \text{for } h \rightarrow 0, \quad f'(c_1) \geq 0 \end{aligned}$$

Then, we can conclude that  $f'(c_1) = 0$

**Theorem 11. Meanvalue theorem.**  $f$  : continuous on  $[a, b]$  and differentiable on  $(a, b)$ .

$\exists c \in (a, b)$  such that  $\frac{f(b) - f(a)}{b - a} = f'(c)$ .

$\because$  consider  $F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$  and apply the Rolle's theorem.

#### 3.2 Basic of function of several variables

$D \subset \mathbb{R}^n$  is a domain  $\Leftrightarrow D$  is open. Any two points of  $D$  are connected by a polygonal arc in  $D$ . We can consider a ball  $\mathbf{B}(P, R)$ .

**Note :** From now on, we discuss with  $\mathbb{R}^2$  for simplicity.

##### 3.2.1 The partial derivative at $P(a, b)$

$$1D : f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

$$2D : f_x(a, b) = \frac{\partial f}{\partial x}(a, b) = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h}$$

$$f_y(a, b) = \frac{\partial f}{\partial y}(a, b) = \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a, b)}{h}$$

**Definition 11.**  $f$  is partially differentiable at  $P(a, b)$  if  $f_x(a, b), f_y(a, b)$  exist. And  $f$  is partially differentiable on  $D$  if  $f$  is partially differentiable at any point on  $D$ .

##### 3.2.2 Landau symbol

$O$  : big  $o$  and  $o$  : small  $o$  describe the behavior of function.

**REPORT (Afifah Maya Iknaningrum / 1715011053)**

Using chain rule, calculate the derivatives

1. Let  $y = f(x) = x^2$ ,  $z = g(y) = y^3 + 2y$ . Calculate  $\frac{d(g \circ f)}{dx}$ .

$$\begin{aligned}\frac{d(g \circ f)}{dx} &= \frac{dg}{dy} \cdot \frac{df}{dx} \\ &= (3(f(x))^2 + 2)(2x) \\ &= (3(x^2)^2 + 2)(2x) \\ &= (3x^4 + 2)(2x) \\ &= 6x^5 + 4x\end{aligned}$$

2. Let  $y = f(x) = x^3 + 2x$ ,  $z = g(y) = y^2 + 3y$ . Calculate  $\frac{d(g \circ f)}{dx}$ .

$$\begin{aligned}\frac{d(g \circ f)}{dx} &= \frac{dg}{dy} \cdot \frac{df}{dx} \\ &= (2(f(x)) + 3)(3x^2 + 2) \\ &= (2(x^3 + 2x) + 3)(3x^2 + 2) \\ &= (2x^3 + 4x + 3)(3x^2 + 2) \\ &= 6x^5 + 4x^3 + 12x^3 + 8x + 9x^2 + 6 \\ &= 6x^5 + 16x^3 + 9x^2 + 8x + 6\end{aligned}$$

3. Let  $\gamma(t) = (t^2, t^3 + t)$ ,  $z = f(x, y) = x^3y$ . Calculate  $\frac{d(f \circ \gamma)}{dt}$ .

$$\begin{aligned}\frac{d(f \circ \gamma)}{dt} &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \\ &= (3x^2y)(2t) + (x^3)(3t^2 + 1) \\ &= (3(t^2)^2(t^3 + t))(2t) + ((t^2)^3)(3t^2 + 1) \\ &= 6t^5(t^3 + t) + t^6(3t^2 + 1) \\ &= 6t^8 + 6t^6 + 3t^8 + t^6 \\ &= 9t^8 + 7t^6\end{aligned}$$

4. Let  $\gamma(t) = (t, t^2 + t)$ ,  $z = f(x, y) = xe^y$ . Calculate  $\frac{d(f \circ \gamma)}{dt}$ .

$$\begin{aligned}\frac{d(f \circ \gamma)}{dt} &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \\ &= (e^y)(1) + (xe^y)(2t + 1) \\ &= e^{t^2+t} + te^{t^2+t}(2t + 1) \\ &= e^{t^2+t} + (2t^2 + t)e^{t^2+t} \\ &= (2t^2 + t + 1)e^{t^2+t}\end{aligned}$$

5. Let  $(u, v) = f(x, y) = (ax + by, cx + dy)$ ,  $(z, w) = g(u, v) = (pu + qv, ru + sv)$ . Calculate  $J(g \circ f)$  and  $Jac(f)$ .

$$\begin{aligned}J(g \circ f) &= J(g)(f) J(f) \\ &= \begin{pmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \\ \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \\ &= \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} ap + cq & bp + dq \\ ar + cs & br + ds \end{pmatrix} \\ Jac(f) &= \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc\end{aligned}$$

6. Let  $(u, v) = f(x, y) = (x^y, x^5 y^2)$ ,  $(z, w) = g(u, v) = (u^2, v^3)$ . Calculate  $J(g \circ f)$  and  $Jac(f)$ .

$$\begin{aligned}
 J(g \circ f) &= J(g)(f) J(f) \\
 &= \begin{pmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \\ \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \\
 &= \begin{pmatrix} 2u & 0 \\ 0 & 3v^2 \end{pmatrix} \begin{pmatrix} 2xy & x^2 \\ 5x^4 y^2 & x^5 2y \end{pmatrix} \\
 &= \begin{pmatrix} 2(x^2 y) & 0 \\ 0 & 3(x^5 y^2)^2 \end{pmatrix} \begin{pmatrix} 2xy & x^2 \\ 5x^4 y^2 & 2x^5 y \end{pmatrix} \\
 &= \begin{pmatrix} 2x^2 y & 0 \\ 0 & 3x^{10} y^4 \end{pmatrix} \begin{pmatrix} 2xy & x^2 \\ 5x^4 y^2 & 2x^5 y \end{pmatrix} \\
 &= \begin{pmatrix} 4x^3 y^2 & 2x^4 y \\ 15x^{14} y^6 & 6x^{15} y^5 \end{pmatrix} \\
 Jac(f) &= \det \begin{pmatrix} 2xy & x^2 \\ 5x^4 y^2 & 2x^5 y \end{pmatrix} \\
 &= (2xy)(2x^5 y) - (5x^4 y^2)(x^2) \\
 &= 4x^6 y^2 - 5x^6 y^2 \\
 &= -x^6 y^2
 \end{aligned}$$

7. Let  $(x, y) = f(r, \theta) = (r \cos \theta, r \sin \theta)$ ,  $(z, w) = g(x, y) = (x^2, y)$ . Calculate  $J(g \circ f)$  and  $Jac(f)$ .

$$\begin{aligned}
 J(g \circ f) &= J(g)(f) J(f) \\
 &= \begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} \\
 &= \begin{pmatrix} 2x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \\
 &= \begin{pmatrix} 2r \cos \theta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \\
 &= \begin{pmatrix} 2r \cos^2 \theta & -2r^2 \cos \theta \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \\
 &= \begin{pmatrix} 2r \cos^2 \theta & -r^2 \sin 2\theta \\ \sin \theta & r \cos \theta \end{pmatrix} \\
 Jac(f) &= \det \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \\
 &= r \cos^2 \theta - (-r \sin^2 \theta) \\
 &= r(\cos^2 \theta + \sin^2 \theta) \\
 &= r
 \end{aligned}$$