Report Basic of Applied Analysis

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Problem 1 : Let $N \in \mathbb{N}, N \geq 2$ and $\alpha > 0$ be a given and consider the $(N-1) \times (N-1)$ matrix $A = (a_{ij})$ that is the finite difference discretization of the problem

$$\begin{cases} \alpha u - u'' = f & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

with entries

$$a_{ij} = \begin{cases} \alpha + \dfrac{2}{h^2} & , i = j \\ -\dfrac{1}{h^2} & , |i-j| = 1 \\ 0 & , \text{ otherwise} \end{cases}$$

where $h = \dfrac{1}{N}$

(a) Eigenvalue and eigenvector of A.
We have

$$Av = \lambda v$$

for $\lambda \in \mathbb{R}, v \neq 0$. Let eigenvector $v \in \mathbb{R}^{N-1}$ of the form $v_i = \sin \dfrac{m\pi i}{N}$. Such that for every i, we have

$$\begin{aligned}
& \sum_j a_{ij} v_j && = \lambda v_i \\
\Leftrightarrow \quad & (\alpha + \frac{2}{h^2}) v_i - \frac{1}{h^2} v_{i-1} - \frac{1}{h^2} v_{i+1} && = \lambda v_i \\
\Leftrightarrow \quad & (\alpha + \frac{2}{h^2} - \lambda) v_i - \frac{1}{h^2}(v_{i+1} + v_{i-1}) && = 0 \\
\Leftrightarrow \quad & (\alpha + \frac{2}{h^2} - \lambda) \sin(\frac{m\pi i}{N}) - \frac{1}{h^2}\Big(\sin(\frac{m\pi(i+1)}{N}) + \sin(\frac{m\pi(i-1)}{N})\Big) && = 0 \\
\Leftrightarrow \quad & (\alpha + \frac{2}{h^2} - \lambda) \sin(\frac{m\pi i}{N}) - \frac{1}{h^2}\Big(2\sin(\frac{m\pi i}{N}) \cos(\frac{m\pi}{N})\Big) && = 0 \\
\Leftrightarrow \quad & \sin(\frac{m\pi i}{N})\big(\alpha + \frac{2}{h^2} - \lambda - \frac{2}{h^2}\cos(\frac{m\pi}{N})\big) && = 0 \\
\Leftrightarrow \quad & \alpha + \frac{2}{h^2} - \lambda - \frac{2}{h^2}\cos(\frac{m\pi}{N}) && = 0 \\
\Leftrightarrow \quad & \lambda = \alpha + 2N^2(1 - \cos(\frac{m\pi}{N})) &&
\end{aligned}$$

Then, we obtain the eigenvalue $\lambda = \alpha + 2N^2(1 - \cos(\dfrac{m\pi}{N})) = \alpha + \dfrac{2}{h^2}(1 - \cos(m\pi h))$ and eigenvector $v \in \mathbb{R}^{N-1}$ of the form $v_i = \sin \dfrac{m\pi i}{N} = \sin(m\pi h i)$

(b) Spectral radius $\sigma(A)$.
Because $\lambda = \alpha + 2N^2(1 - \cos(\dfrac{m\pi}{N}))$, using Taylor expansion, we obtain

$$\begin{aligned}
\sigma(A) &= max\{|\lambda|\} \\
&= \alpha + 2N^2(1 - (1 - \frac{1}{2}(\frac{m\pi}{N})^2)) \\
&= \alpha + (m\pi)^2
\end{aligned}$$

(c) Eigenvalue and eigenvector of the Jacobi iteration matrix $R = -D^{-1}(L+U)$.
First, for eigenvector v and eigenvalue λ we have

$$\begin{aligned} Rv &= \lambda v \\ -D^{-1}(L+U)v &= \lambda v \\ -(L+U)v &= \lambda Dv \end{aligned}$$

Such that for each i, with $A = (a_{ij})$ and $v_i = \sin\frac{m\pi x}{N}$ we obtain

$$\begin{aligned} & \frac{1}{h^2}v_{i-1} + \frac{1}{h^2}v_{i+1} && = \lambda(\alpha + \frac{2}{h^2})v_i \\ \Leftrightarrow\ & \frac{1}{h^2}\Big(\sin(\frac{m\pi(i-1)}{N}) + \sin(\frac{m\pi(i+1)}{N})\Big) && = \lambda(\alpha + \frac{2}{h^2})\sin\frac{m\pi i}{N} \\ \Leftrightarrow\ & \frac{1}{h^2}\Big(2\sin(\frac{m\pi i}{N})\cos(\frac{m\pi}{N})\Big) && = \lambda(\alpha + \frac{2}{h^2})\sin\frac{m\pi i}{N} \\ \Leftrightarrow\ & \lambda && = \frac{2N^2}{\alpha + 2N^2}\cos(\frac{m\pi}{N}) \end{aligned}$$

Then, we obtain the eigenvalue $\lambda = \frac{2N^2}{\alpha+2N^2}\cos(\frac{m\pi}{N}) = \frac{2}{\alpha h^2 + 2}\cos(m\pi h)$ and eigenvector $v \in \mathbb{R}^{N-1}$ of the form $v_i = \sin\frac{m\pi i}{N} = \sin(m\pi h i)$

(d) The spectral radius $\sigma(R)$
Because $\lambda = \frac{2N^2}{\alpha+2N^2}\cos(\frac{m\pi}{N})$, using Taylor expansion, we obtain

$$\begin{aligned} \sigma(R) &= max\{|\lambda|\} \\ &= \frac{2N^2}{\alpha+2N^2}(1 - \frac{1}{2}(\frac{m\pi}{N})^2) \\ &= \frac{2N^2}{\alpha+2N^2} - \frac{(m\pi)^2}{\alpha+2N^2} \\ &= \frac{2N^2 - (m\pi)^2}{\alpha+2N^2} \\ &= \frac{2-(m\pi h)^2}{\alpha h^2 + 2} \end{aligned}$$

Problem 2 : Let $N, h = \frac{1}{N}, \alpha$ and A as in the Problem 1

(a) Exact solution $u : [0,1] \to \mathbb{R}$ of

$$\begin{cases} \alpha u - u'' = sin(\pi x) & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

First, we look for the solution of homogeneous part, such that we obtain

$$\lambda^2 - \alpha = 0 \Leftrightarrow \lambda_{1,2} = \pm\sqrt{\alpha}$$

Then, from the homogeneous part we get

$$u = c_1 e^{\sqrt{\alpha}x} + c_2 e^{-\sqrt{\alpha}x}$$

Subtitute it into the boundary condition,

$$u(0) = c_1 + c_2 = 0 \Leftrightarrow c_1 = -c_2$$

$$u(1) = c_1 e^{\sqrt{\alpha}} + c_2 e^{-\sqrt{\alpha}} = -c_2 e^{\sqrt{\alpha}} + c_2 e^{-\sqrt{\alpha}} = c_2(e^{-\sqrt{\alpha}} - e^{\sqrt{\alpha}}) = 0 \Leftrightarrow c_2 = 0 \Leftrightarrow c_1 = -c_2 = 0$$

we obtain $u = c_1 e^{\sqrt{\alpha}x} + c_2 e^{-\sqrt{\alpha}x} = 0$.
After that, solving the nonhomogenous part by subtitute general solution in form $u = A\sin(\pi x) + B\cos(\pi x)$ to the problem, such that

$$\begin{aligned} & \alpha A\sin(\pi x) + \alpha B\cos(\pi x) + A\pi^2\sin(\pi x) + B\pi^2\cos(\pi x) && = \sin(\pi x) \\ \Leftrightarrow\ & (\alpha + \pi^2)(A\sin(\pi x) + B\cos(\pi x)) && = \sin(\pi x) \end{aligned}$$

Set the coeffiient equal, we obtain

$$(\alpha+\pi^2)A = 1 \text{ and } (\alpha+\pi^2)B = 0 \Leftrightarrow B = 0 \text{ and } A = \frac{1}{\alpha+\pi^2}$$

Such that solution $u = \dfrac{\sin(\pi x)}{\alpha+\pi^2}$. Adding the solution from the homogeneous and non-homogeneous, we get exact solution

$$u(x) = \frac{\sin(\pi x)}{\alpha+\pi^2}$$

(b) Exact solution $v \in R^{N-1}$ with $b_i = sin(\pi h i)$, $i = 1, \ldots, N-1$ of

$$Av = b$$

or we want to solve

$$(\alpha + \frac{2}{h^2})v_i - \frac{1}{h^2}(v_{i+1} + v_{i-1}) = \sin(\pi h i)$$

For the solution of homogeneous part, it is the same as the problem 2(a), that $v = 0$. For the nonhomogeneous part, we assume the solution has form $v_i = A\sin(\pi h i)$ such that

$$\begin{aligned}
& (\alpha + \frac{2}{h^2})A\sin(\pi h i) - \frac{1}{h^2}(A\sin(\pi h(i+1)) + A\sin(\pi h(i-1))) && = \sin(\pi h i) \\
\Leftrightarrow \quad & (\alpha + \frac{2}{h^2})A\sin(\pi h i) - \frac{2}{h^2}A\sin(\pi h i)\cos(\pi h) && = \sin(\pi h i) \\
\Leftrightarrow \quad & (\alpha + \frac{2}{h^2} - \frac{2}{h^2}\cos(\pi h))A\sin(\pi h i) && = \sin(\pi h i) \\
\Leftrightarrow \quad & A = \frac{1}{\alpha + 2N^2(1-\cos(\frac{\pi}{N}))} \\
\Leftrightarrow \quad & A = \frac{h^2}{\alpha h^2 + 2(1-\cos(\pi h))}
\end{aligned}$$

Then adding the solution of homogen and nonhomogen part, we obtain

$$v_i = \frac{h^2}{\alpha h^2 + 2(1-\cos(\pi h))}\sin(\pi h i)$$

(c) Assuming N is even, the explicit formula for

$$\epsilon(h) := \max_{1\le i\le N-1} |u(hi) - vi|$$

as a function of $h = \dfrac{1}{N}$ and the leading order term in the Taylor expansion of $\epsilon(h)$ at $h = 0$.
The explicit formula of

$$\begin{aligned}
\epsilon(h) \quad := & \max_{1\le i\le N-1} |u(hi) - vi| \\
= & \max_{1\le i\le N-1} |\frac{\sin(\pi h i)}{\alpha+\pi^2} - \frac{h^2}{\alpha h^2 + 2(1-\cos(\pi h))}\sin(\pi h i)| \\
= & \max_{1\le i\le N-1} |(\frac{1}{\alpha+\pi^2} - \frac{h^2}{\alpha h^2 + 2(1-\cos(\pi h))})\sin(\pi h i)| \\
= & \max_{1\le i\le N-1} |(\frac{1}{\alpha+\pi^2} - \frac{h^2}{\alpha h^2 + 2(1-\cos(\pi h))})| \max_{1\le i\le N-1} |\sin(\pi h i)| \\
= & |(\frac{1}{\alpha+\pi^2} - \frac{h^2}{\alpha h^2 + 2(1-\cos(\pi h))})|
\end{aligned}$$

Taking Taylor expansion for $cos(\pi h) = 1 - \dfrac{(\pi h)^2}{2} + \sum_{n=2}^{\infty}(-1)^n \dfrac{(\pi h)^{2n}}{(2n)!}$. We obtain error estimate

$$\epsilon(h) = |(\frac{1}{\alpha+\pi^2} - \frac{1}{\alpha + \pi - 2\sum_{n=2}^{\infty}(-1)^n \frac{(\pi h)^{2n}}{(2n)!}})|$$

Only taking sum of $n = 2$, we obtain

$$\epsilon(h) = |(\frac{1}{\alpha+\pi^2} - \frac{1}{\alpha + \pi^2 - \frac{(\pi h)^4}{12}})|$$

With the leading order term in the Taylor expansion of $\epsilon(h)$ at $h = 0$ it is obvious that $\epsilon(0) = 0$

Problem 3 : For given N and $b_{i,j} \in \mathbb{R},\ i,j = 1,2,\ldots,N-1$, consider the system of linear equation

$$\begin{cases} -v_{i-1,j} - v_{i,j-1} - v_{i+1,j} - v_{i,j+1} - 4v_{i,j} = b_{i,j} & i,j = 1,\ldots,N-1 \\ v_{0,j} = v_{N,j} = v_{i,0} = v_{i,N} = 0 & i,j = 1,\ldots,N-1 \end{cases}$$

for unknown $v_{i,j}$

(a) Eigenvalue and eigenvector of matrix A for the system above.
We will look for eigenvalue λ and eigenvector w in $Aw = \lambda w$. Using the system with $w = v_i\tilde{v}_j$ with $v_i = \sin(m\pi hi)$,

$$\begin{aligned} & -v_{i-1}\tilde{v}_j - v_i\tilde{v}_{j-1} - v_{i+1}\tilde{v}_j - v_i\tilde{v}_{j+1} + 4v_i\tilde{v}_j && = \lambda v_i\tilde{v}_j \\ \Leftrightarrow\ & -\sin(m\pi h(i-1))\sin(m\pi hj) - \sin(m\pi hi)\sin(m\pi h(j-1)) - \sin(m\pi h(i+1))\sin(m\pi hj) && \\ & -\sin(m\pi hi)\sin(m\pi h(j+1)) + 4\sin(m\pi hi)\sin(m\pi hj) && = \lambda\sin(m\pi hi)\sin(m\pi hj) \\ \Leftrightarrow\ & -4\sin(m\pi hi)\sin(m\pi hj)\cos(m\pi h) + (4-\lambda)\sin(m\pi hi)\sin(m\pi hj) && = 0 \\ \Leftrightarrow\ & \lambda = 4(1-\cos(m\pi h)) && \end{aligned}$$

Then, we obtain the eigenvalue $\lambda = 4(1-\cos(m\pi h))$ and eigenvector of the form $w_i = v_i\tilde{v}_j = \sin(m\pi hi)\sin(m\pi hj)$.

(b) Implement a program with initial guess $v_{i,j}^{(0)} = 0$ that solve using

(i) the Jacobi method

(ii) the Gauss-Seidel method

(iii) SOR method given ω

(c) $v^{(k)}$ is approximate solution after k iteration. Set the right hand side to

$$b_{i,j} = \frac{\sin(\frac{\pi i}{N})\sin(\frac{\pi j}{N})}{N^2}\ i = 1,\ldots,N-1$$

for each method for $N = 10, 20, 50$. (see the program)

(d) The optimum ω for SOR. (see the program)