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We will learn about : Basics of functions of several variables. In this lecture:

1.1 A sequence in the Euclidean space and its application

Using these notation :

- $\mathbb{N}$: set of natural number ($\mathbb{N} = \{1, 2, 3, \dots\}$)
- $\mathbb{Z}$: set of integers ($\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$)
- $\mathbb{Q}$: set of rational number ($\mathbb{Q} = \{0, \pm 1, \pm 2, \frac{2}{3}, \dots\}$)
- $\mathbb{R}$: set of real number
- $\mathbb{C}$: set of complex number

Definition 1. *A sequence $(x_n)_{n=1}^{\infty}$ is an assignment of (real) number $x_n \in \mathbb{R}$ to natural number $n \in \mathbb{N}$ ($x_n \in \mathbb{R}$).*
Example : $x_n = \frac{1}{n}$. $x_1 = 1, x_2 = \frac{1}{2}, \dots$

Definition 2. *A subsequence of a sequence $(x_n)_{n=1}^{\infty}$ is a sequence $(y_j)_{j=1}^{\infty}$ defined by $y_j = x_{n_j}$ for some sequence $(n_j)_{j=1}^{\infty}$ in $\mathbb{N}$ such that $n_j < n_{j+1}$ $(j = 1, 2, \dots)$.*
Example : sequence $(x_n)_{n=1}^{\infty} = 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{100}$, takes $n_1 = 1, n_2 = 3, n_3 = 5, n_4 = 100$
subsequence $(x_{n_j})_{j=1}^{\infty} = x_{n_1}, x_{n_2}, x_{n_3}, x_{n_4} = 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{100}$.

Definition 3. *Let $(x_n)_{n=1}^{\infty}$ be a sequence converges to $\alpha \in \mathbb{R}$ if for any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $n > N$, $|x_n - \alpha| < \epsilon$.*
In the mathematical symbol $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $n > N$, $|x_n - \alpha| < \epsilon$ for $n > N$.
In this case we write, $\lim\limits_{n \to \infty}$ or $x_n \to \alpha$ $(n \to \infty)$

Example 1.

Theorem 1. *$(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}$ is sequence. Suppose $x_n \to \alpha$ and $y_n \to \beta$ as $n \to \infty$.*

1. *$x_n \pm y_n \to \alpha \pm \beta$, $(n \to \infty)$*
2. *$x_n \cdot y_n \to \alpha \cdot \beta$, $(n \to \infty)$*
3. *if $\beta \neq 0$, $\frac{x_n}{y_n} \to \frac{\alpha}{\beta}$, $(n \to \infty)$*

Remark 1. *On 3, $\frac{x_n}{y_n}$ is not defined for all $n \in \mathbb{N}$ because $y_n = 0$ possibly for some $n \in \mathbb{N}$. But, since $y_n \to \beta \neq 0$, y_n eventually is not 0. Hence $\frac{x_n}{y_n}$ is defined eventually.*

Theorem 2. *$(x_n)_{n=1}^{\infty}$ a sequence. If $(x_n)_{n=1}^{\infty}$ converges to $\alpha \in \mathbb{R}$, any subsequence of $(x_n)_{n=1}^{\infty}$ converges to $\alpha \in \mathbb{R}$.*
$\because$ Let $(x_n)_{n=1}^{\infty}$ be a subsequence. Because $x_n \to \alpha (n \to \infty), \forall \epsilon > 0, \exists N \in \mathbb{N}$. Take $J_0 \in \mathbb{N}$ such that $n_j > N_\theta$ for all $y > J_0$. Then $|x_{n_{j > N_\theta}} - \alpha| < \epsilon$ for $j > J_0$. $x_{n_j} \to \alpha (j \to \infty), (n_j)_{j=1}^{\infty}$ also a sequence, $n_j \in \mathbb{N}, n_j < n_{j+1}$.

Completeness Axiom. Let $(x_n)_{n=1}^{\infty}$ be a monotonically increasing (decreasing) sequence (i.e. $x_n \leq x_{n+1}, n \in \mathbb{N}$). Suppose that there is an $M \in \mathbb{R}$ such that $x_n \leq M (n \in \mathbb{N})$ $(x_n \geq M)$. Then, $(x_n)_{n=1}^{\infty}$ converges ($\exists \alpha \in \mathbb{R}$ such that $x_n \to \alpha$)

Theorem 3. ***Bolzano-Weirstrass.*** *$(x_n)_{n=1}^{\infty}$ a sequence in $\mathbb{R}$. Suppose $(x_n)_{n=1}^{\infty}$ is bounded in the sense that $|x_n| \leq M, \forall n \in \mathbb{N}$. Then $(x_n)_{n=1}^{\infty}$ contains a convergent subsequence.*
x_n is a peak of $(x_n)_{n=1}^{\infty}$ if $x_n > x_m$ for $m > n$.

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2.1 n-dimensional space

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, \dots, x_n) | x_i \in \mathbb{R}\}$.
Takes $n = 2$, $\mathbb{R}^2 \Leftrightarrow$ plane, we have $P(a, b)$.
For $n = 3$, we have $P(a, b, c)$.

Definition 4. *$P_m = (x_1^m, \dots, x_n^m) \in \mathbb{R}^n$, and $\{P_m\}_{m=1}^{\infty}$: a sequence in $\mathbb{R}^n$.*
$\{P_m\}$ converges to $A = (a_1, \dots, a_n) \in \mathbb{R}^n$, if $\forall k = 1, \dots, n$, $x_k^m \to a_k$ as $n \to \infty$.

Definition 5. ***Inner product and norm.***
$\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$. We can define :
*$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n$; **inner product***
*$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$; **norm***

Example 2. *$\mathbf{x} \cdot \mathbf{y} = 0 \Leftrightarrow \mathbf{x}$ is perpendicular to $\mathbf{y}$*
Takes $n = 0$ then

$$\begin{aligned} x_1 y_1 + x_2 y_2 &= 0 \\ x_1 y_1 &= -x_2 y_2 \\ \frac{y_1}{y_2} &= -\frac{x_2}{x_1} \\ \text{then } (x_1, x_2) = c \cdot (-y_2, y_1) \end{aligned}$$

pict :

Example 3. *$\|\mathbf{x}\| = 0 \Leftrightarrow x = 0$*
$(\Rightarrow)$ $0 = \|x\|^2 = x_1^2 + \cdots + x_n^2$, then $x_1^2 = 0$ $(\forall i = 1, \dots, n)$ and finally $x_1 = 0$.

Notes 1. *$\|x\|$ is the distance between $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^n$ and $\mathbf{x} = (x_1, \dots, x_n)$.*
For notation, we will use $P, Q \in \mathbb{R}^n$ as points and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ as vectors.
We also use $\|x - y\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$ as distance between $\mathbf{x}$ and $\mathbf{y}$.
$\|P - Q\|$ is distance between P and Q.

$$\mathbf{x} \pm \mathbf{y} = (x_1 \pm y_1, \dots, x_n \pm y_n)$$

$$P = (p_1, \dots, p_n), Q = (q_1, \dots, q_n), \text{ then } P + Q = (p_1 + q_1, \dots, p_n + q_n)$$

$$\alpha \in \mathbb{R},\ \alpha\mathbf{x} = (\alpha x_1, \dots, \alpha x_n), \alpha P = (\alpha p_1, \dots, \alpha p_n)$$

$$\{P_m\}_{m=1}^{\infty} : \text{a sequence in } \mathbb{R}^n,\ P_m \to A \Leftrightarrow \|P_m - A\| \to 0$$

Theorem 4. ***Cauchy-Schwarz inequality.*** *For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,*

$$| \mathbf{x} \cdot \mathbf{y} | \leq \|x\| \|y\|$$

"=" $\Rightarrow a\mathbf{y} = b\mathbf{x}$ for some $a, b \in \mathbb{R}$.
$\because$ We may assume $\mathbf{x} \neq \emptyset$, $\forall t \in \mathbb{R}$.

$$0 \leq \|t\mathbf{x} + \mathbf{y}\| = (t\mathbf{x} + \mathbf{y})(t\mathbf{x} + \mathbf{y}) = t^2\|\mathbf{x}\|^2 + 2t(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2$$

$$D/4 \leq 0$$

Theorem 5. ***Bolzano=Weierstrass.*** *Let $(P_m)_{m=1}^{\infty} \subset \mathbb{R}^n$ be a sequence. Suppose that $(P_m)_{m=1}^{\infty}$ is bounded. In the sense that $\|P_m\| \leq M (m \in \mathbb{N})$ for some $M \geq 0$. Then $(P_m)_{m=1}^{\infty}$ contains a convergent subsequence.*

Definition 6. ***Ball.*** *$A \in \mathbb{R}^n, R > 0$*

$$\mathbf{B}(A, R) = \{P \in \mathbb{R}^n | \|P - A\| < R\}; \text{ open ball of center } A \text{ with radius } R$$

$$\overline{\mathbf{B}}(A, R) = \{P \in \mathbb{R}^n | \|P - A\| \leq R\}; \text{ closed ball}$$

Definition 7.

1. *$E \subset \mathbb{R}^n$ is said to be **an open set** if $E = \emptyset$ or $\forall A \in E, \exists R > 0$ such that $\mathbf{B}(A, R) \subset E$.*
2. *$E \subset \mathbb{R}^n$ is said to be **a closed set** if $E^c \in \mathbb{R}^n$ E is an open set.*
   E : open, then neighbor in any point

Definition 8. ***Accumulation point.*** *$E \subset \mathbb{R}^n$; a set. $A \in \mathbb{R}^n$ is called **an accumulation point** of E if $\forall R > 0, (\mathbf{B}(A, R) - \{A\}) \bigcap E \neq \emptyset$.*

Notes 2. *$E \subset \mathbb{R}^n$ is closed if and only if E contains any accumulation point of E .* ***Homework report, prove this***

Notes 3. *1. Both $\emptyset$ and $\mathbb{R}^n$ are open and closed*

2. *$\{E_\lambda \lambda \in A\}$; a collection of open sets $\Rightarrow$ union $\lambda \in AE_\lambda$ is also open*

3. *$\{E_\lambda\}_{\lambda=1}^N$, a finite collection of open sets $\Rightarrow$ irisan $_{lamda=1}^N E_\lambda$ is also open.*

4. ***Rephrase of Bolzano Weierstrass theorem**. $E \subset \mathbb{R}^n$; a **bounded closed set** $\Leftrightarrow$ E is a closed set such that $E \subset \mathbf{B}(\mathbf{0}, R)$ for some $R > 0$. E ; a bounded closed set then any sequence of E contains a convergent subsequence whose limit is in E.*

Definition 9. *A bounded closed set in $\mathbb{R}^n$ is called **compact**.*

Example 4. *$\overline{\mathbf{B}}(A, R)$ is compact.* ***Report! prove this***

2.2 Continuity and differentiability of a function

2.2.1 Continuity

E : a set in $\mathbb{R}^n$ and f : is a function of E (real valued function).
i.e. f is an assignment a (real) number to a point in E.

Definition 10. *1. f is **continuous at** $A \in E$ if $\forall (P_m)_{m=1}^\infty \subset E$: sequence with $P_m \to A$ $(m \to \infty)$*

$$f(P_m) \to f(A) \ (m \to \infty)$$

2. *f is **continuous on E** if f is continuous at any point of E.*

2.2.2 Basic of continuous function on an interval in $\mathbb{R}$

Theorem 6. ***Intermediate value theorem**. f : function on a closed interval $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$. Suppose that $f(a) \leq f(b)$. Then, $\forall \gamma$ with $f(a) \leq \gamma \leq f(b)$, $\exists c \in [a, b]$ with $f(c) = \gamma$.*

Theorem 7. ***Extreme value theorem**. f is a continuous function on a closed interval $[a, b]$. Then, f attains a maximum and a minimum on $[a, b]$.*

Name : Afifah Maya Iknaningrum (1715011053)

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1. $E \subset \mathbb{R}^n$ is closed if and only if E contains any accumulation point of E
2. $\overline{\mathbf{B}}(A, R)$ is compact.

Proof :

1. ($\Rightarrow$) if E is closed then E contains all of its accumulation point. Let x accumulation point of E, $x \in E$ and E is closed then E^c is open .

   Let $x \in E^c$ and $R > 0 \Rightarrow \forall x \in E^c, \exists B(x, R)$ such that $\forall y \in B(x, R) \Rightarrow y \in E^c$.
   Suppose x is accumulation point of E that is not in E.
   Then, $\forall e \in B(x, R), \exists y \neq x$ with $y \in e \bigcap E$.
   $y \in e \bigcap E \Rightarrow y \notin E^c$ contradiction.

   ($\Leftarrow$) E contains all of its accumulation point then E is closed.

   Suppose E contains all of its accumulation point. Suppose E^c is not open.
   $\exists x \in E^c$ such that $\forall e \in B(x, R), R > 0, \exists y \in e$ that also in E.
   Its contradict the premise, because x is accumulation point.

2. Suppose $x \notin \overline{B}(A, R) \Rightarrow \|x - A\| > R$.
   So let $\|x - A\| - R = \epsilon > 0$.
   Consider $y \in B(x, \epsilon/2)$,

$$
\begin{aligned}
\|y - A\| &\geq \|x - A\| - \|y - x\| \\
\|y - A\| &\geq R + \epsilon - (\epsilon/2) \\
\|y - A\| &\geq R + (\epsilon/2) \\
\|y - A\| &> R
\end{aligned}
$$

   shows that $y \in \overline{B}(A, R)$. Hence $B(x, \epsilon/2)$ subset of $\overline{B}(A, R)^c$.
   Because $\overline{B}(A, R)^c$ hence $\overline{B}(A, R)$ is closed.

   By definition, $\overline{B}(A, R) = \{x \in \mathbb{R}^n | \|x - A\| \leq R\}$
   Then $\forall x \in \overline{B}(A, R)$ we can find

$$
\begin{aligned}
\|x - A\| &\leq R \\
-R \leq \quad |x - A| &\leq R. \qquad (1)
\end{aligned}
$$

   shows that $\overline{B}(A, R)$ is bounded.

   Because closed and bounded, $\overline{B}(A, R)$ is compact.

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f is continuous function on $[a,b] = \{x \in \mathbb{R} | a \leq x \leq b\}$

Theorem 8. ***Intermediate Value Theorem.*** *Suppose $f(a) \leq f(b)$ then $\forall \gamma \in \mathbb{R}$ with $f(a) \leq \gamma \leq f(b)$, $\exists c\ in [a,b]$ such that $f(c) = \gamma$.*

Theorem 9. ***Extreme value theorem.*** *f attains a maximum and a minimum on $[a,b]$.*

3.1 Differentiable function on intervals

f : function defined around $x = a \in \mathbb{R}$.
f is differentiable at $x = a$ if the limit $f\prime(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exist.

f : function on $(a,b) = \{x \in \mathbb{R} | a < x < b\}$
f is differentiable on (a,b) if f is differentiable at any point of (a,b).

Properties : if f is differentiable at $x = a$ then f is continuous at $x = a$.
$\because f(a+h) = f(a) + h\dfrac{f(a+h) - f(a)}{h} = f(a) + hf\prime(a)$. Because $h \to 0$ then $f(a+h) \to f(a)$.

Theorem 10. ***Rolle's theorem.*** *f : continue on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b)$ then $\exists c \in (a,b)$ such that $f\prime(c) = 0$.*
$\because$ if f is a constant function, $f\prime(x) = 0, \forall x \in (a,b)$. Suppose that f is not a constant function, by extreme value theorem, f attain max at $x = c_1$ and min at $x = c_2$ with $c_1 \neq c_2$.
(Otherwise max $= f(c_1) = f(c_2) =$ min)

$\forall x \in [a,b], f(c_2) \leq$ min $\leq f(x) \leq$ max $\leq f(a)$. We may assume $c_1 \in (a,b)$.
(Otherwise, consider $-f$ instead f; $(-f)\prime(a) = \lim_{h \to 0} \dfrac{-f(a+h) - (-f(a))}{h} = -f\prime(a)$)

$$\frac{f(c_1+h) - f(c_1)}{h} \leq 0, h < 0. \quad \textit{for } h \to 0, \quad f\prime(c_1) \leq 0$$
$$\frac{f(c_1+h) - f(c_1)}{h} \geq 0, h > 0. \quad \textit{for } h \to 0, \quad f\prime(c_1) \geq 0$$

Then, we can conclude that $f\prime(c_1) = 0$

Theorem 11. ***Meanvalue theorem.*** *f : continuous on $[a,b]$ and differntiable on (a,b).*
$\exists c \in (a,b)$ such that $\dfrac{f(b) - f(a)}{b-a} = f\prime(c)$.
$\because$ consider $F(x) = f(x) - f(a) - \dfrac{f(b) - f(a)}{b-a}(x-a)$ and apply the Rolle's theorem.

3.2 Basic of function of several variables

$D \subset \mathbb{R}^n$ is a domain $\Leftrightarrow$ D is open. Any two points of D are connected by a polygonal arc in D. We can consider a ball $\mathbf{B}(P,R)$.
Note : From now on, we discuss with $\mathbb{R}^2$ for simplicity.

3.2.1 The partial derivative at $P(a,b)$

1D : $f\prime(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$.
2D : $f_x(a,b) = \dfrac{\partial f}{\partial x}(a,b) = \lim_{h \to 0} \dfrac{f(a+h,b) - f(a,b)}{h}$.
$f_y(a,b) = \dfrac{\partial f}{\partial y}(a,b) = \lim_{h \to 0} \dfrac{f(a,b+h) - f(a,b)}{h}$

Definition 11. *f is partially differentiable at $P(a,b)$ if $f_x(a,b), f_y(a,b)$ exist. And f is partially differentiable on D if f is partially differentiable at any point on D.*

3.2.2 Landau symbol

O : big o and o : small o describe the behavior of function.