6 Existence for the nonlinear problem

We study the nonlinear problem of stationary magnetic field

$$-\mathrm{div}\Big(\nu(x, \|\nabla u(x)\|^2)\nabla u(x)\Big) = f(x) \qquad \text{for } x \in \Omega \tag{1}$$

$$u(x) = 0 \qquad \text{on } \partial\Omega \tag{2}$$

We shall solve this problem in the following steps:

- Write the problem in the operator form $A(u) = f$, where $A : H \to H$ is a nonlinear operator expressing the left-hand side of the equation. In the sequel, we shall simply write Au instead of $A(u)$.
- Proof an existence theorem for this abstract operator equation (using monotone operator theory).
- Check that the assumptions of the existence theorem for our problem are satisfied.

Now we prove the existence of a unique solution to the problem $Au = f$:

Theorem. Let $A : H \to H$ be strongly monotone with respect to H (with constant η) and let A satisfy Lipschitz condition (with constant L). Then for each $f \in H$ there exists a unique solution u of the problem

$$Au = f \qquad \text{in } H.$$

Proof of Theorem.

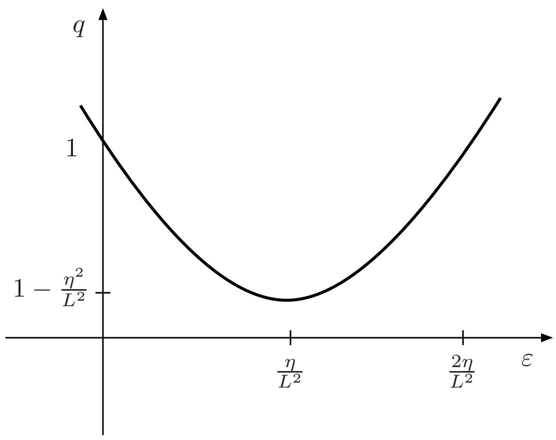
Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. The idea of the proof is to look for a fixed point of the mapping $T_\varepsilon : H \to H$ defined by

$$T_\varepsilon(v) = v - \varepsilon\big(Av - f\big), \qquad v \in H.$$

We want to show that if we choose ε appropriately, then T_ε is contractive. To this end, we compute

$$\begin{aligned}
\|T_\varepsilon(v) - T_\varepsilon(w)\|_H^2 &= \big\langle v - \varepsilon(Av - f) - w + \varepsilon(Aw - f), v - \varepsilon(Av - f) - w + \varepsilon(Aw - f)\big\rangle \\
&= \big\langle v - w - \varepsilon(Av - Aw), v - w - \varepsilon(Av - Aw)\big\rangle \\
&= \|v - w\|_H^2 + \varepsilon^2\|Av - Aw\|_H^2 - 2\varepsilon\langle Av - Aw, v - w\rangle \\
&\le \|v - w\|_H^2 + \varepsilon^2 L^2\|v - w\|_H^2 - 2\varepsilon\eta\|v - w\|_H^2 \\
&= (\varepsilon^2 L^2 - 2\varepsilon\eta + 1)\|v - w\|_H^2.
\end{aligned}$$

We denote the expression in the brackets as q: $q(\varepsilon) = L^2\varepsilon^2 - 2\eta\varepsilon + 1$. We want to find values of ε for which $q(\varepsilon)$ falls in the interval $(0, 1)$. By examining the function q we find that it has the following behaviour:



The graph is a parabola with minimum at $\varepsilon = \eta/L^2$ and equal to 1 at the points $\varepsilon = 0$ and $\varepsilon = 2\eta/L^2$. The minimum value is $q(\eta/L^2) = 1 - \frac{\eta^2}{L^2}$ and because of the estimate

$$\eta\|v - w\|_H^2 \leq \langle Av - Aw, v - w\rangle \leq \|Av - Aw\|_H\|v - w\|_H \leq L\|v - w\|_H^2$$

we see that $\eta \leq L$ and thus the minimum value of q is nonnegative.

To summarize, we found out that for each $\varepsilon \in (0, \frac{2\eta}{L^2})$ the value $q(\varepsilon)$ belongs to the interval $[0, 1)$ and thus T_ε is contractive for such values of ε.
(**Note:** To get the fastest convergence in numerical computations, it is reasonable to set $\varepsilon = \eta/L^2$.)

Review. Fixed point theorems

Banach's Fixed Point Theorem Let X be a Banach space and assume $F : X \to X$ is a mapping satisfying

$$\|F(u) - F(v)\| \leq \gamma\|u - v\| \qquad u, v \in X$$

for some constant $\gamma < 1$ (i.e., F is a strict **contraction**). Then F has a unique fixed point.

Proof: Fix any point $u_0 \in X$ and iteratively define $u_{k+1} = F(u_k)$ for $k = 0, 1, \ldots$. Then

$$\|F(u_{k+1}) - F(u_k)\| < \gamma\|u_{k+1} - u_k\| = \gamma\|F(u_k) - F(u_{k-1})\|,$$

and so

$$\|F(u_{k+1}) - F(u_k)\| \leq \gamma^k\|F(u_0) - u_0\|, \qquad k = 1, 2, \ldots.$$

Consequently, for $k \geq l$,

$$\|u_k - u_l\| = \|F(u_{k-1}) - F(u_{l-1})\| \leq \sum_{j=l-1}^{k-2}\|F(u_{j+1}) - F(u_j)\| \leq \|F(u_0) - u_0\|\sum_{j=l-1}^{k-2}\gamma^j.$$

Hence $\{u_k\}$ is a Cauchy sequence in X and since X is a Banach space, there exists a $u \in X$ so that $u_k \to u$ in X. Then $F(u) = u$ and the fixed point is unique.

Schauder's Fixed Point Theorem Let X be a Banach space. Suppose $K \subset X$ is compact and convex and assume also that $F : K \to K$ is continuous. Then F has a fixed point in K.

Schaefer's Fixed Point Theorem Let X be a Banach space. Suppose $F : X \to X$ is a continuous and compact mapping. Assume further that the set

$$\{u \in X\ ;\quad u = \lambda F(u) \text{ for some } \quad \lambda \in [0, 1]\}$$

is bounded. Then F has a fixed point.

By the Banach fixed point theorem, there exists a unique fixed point $v \in H$ of T_ε, i.e., a unique point satisfying

$$T_\varepsilon(v) = v.$$

But since $T_\varepsilon(v) = v - \varepsilon(Av - f)$, this fixed point is also the unique solution of $Av = f$.

Remarks.

- If the operator A is not Lipschitz continuous, the solution to $A(u) = f$ may not exists, even if A is monotone. A simple example in $H = \mathbb{R}$ is the function $A(u) = u + \operatorname{sign} u$ and $f = \frac{1}{2}$.
- If A is linear, then the strong monotonicity reduces to coercivity:

$$\langle Av, v\rangle \geq \eta \|v\|_H^2 \qquad \forall v \in H$$

and Lipschitz continuity reduces to usual continuity (or boundedness):

$$\|Av\|_H \leq L\|v\|_H \qquad \forall v \in H.$$

Let us go back to our specific problem of nonlinear magnetic field (1), (2). Here we choose $H = H_0^1(\Omega)$ and the weak formulation for this problem reads

Weak solution

Find $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega \nu\big(x, |\nabla u(x)|^2\big)\nabla u(x) \cdot \nabla\varphi(x)\, dx = \int_\Omega f(x)\varphi(x)\, dx \tag{3}$$

for all test functions $\varphi \in H_0^1(\Omega)$.

Recall that the function ν is given by

$$\nu(x,\eta) = \begin{cases} \nu_1(\eta) & \text{for } x \in \Omega_1 = \text{ ferromagnetic materials} \\ \nu_0 & \text{for } x \in \Omega_0 = \text{ other materials (copper wires, insulators, air, etc.)} \end{cases}$$

where $\nu_0 = 1/\mu_0$ with $\mu_0 = 4\pi \times 10^{-7}$ Tm/A, the permeability of vacuum, and ν_1 is a nondecreasing function satisfying

$$C_0 \leq \nu_1(\eta) \leq C_1, \qquad C_0, C_1 > 0,$$
$$|\vartheta \nu_1'(\eta)| \leq C_2, \qquad \eta \geq \vartheta \geq 0,\ C_2 > 0.$$

Therefore, the integral on the left-hand side of the weak equation is finite.

We set $H = H_0^1(\Omega)$ and define the nonlinear operator $A : H \to H$ by

$$\langle Au, \varphi\rangle = \int_\Omega \nu\big(x, |\nabla u(x)|^2\big)\nabla u(x) \cdot \nabla\varphi(x)\, dx.$$

Here $\langle \cdot, \cdot\rangle$ is the standard inner product in $H^1(\Omega)$. For a fixed $u \in H$ the right-hand side of the above identity represents a continuous linear functional on H (with variable φ). Hence, such an Au exists and is unique according to Riesz theorem.

We check that A is Lipschitz continuous. For simplicity, set $\xi := \nabla u(x)$, $\sigma := \nabla v(x)$, $\vartheta := \nabla\varphi(x) \in \mathbb{R}^2$. We want to show that

$$|\langle Au, \varphi\rangle - \langle Av, \varphi\rangle| = \left| \int_\Omega \Big(\nu(|\xi|^2)\xi - \nu(|\sigma|^2)\sigma\Big) \cdot \vartheta\, dx \right| \leq L\|u - v\|_{H^1(\Omega)} \|\varphi\|_{H^1(\Omega)}$$

To this end we write the integrand conveniently as

$$\Big(\nu(|\xi|^2)\xi - \nu(|\sigma|^2)\sigma\Big) \cdot \vartheta = g(1) - g(0),$$

where $g \in C^1([0,1])$ is the function

$$g(t) = \nu\big(|\sigma + t(\xi - \sigma)|^2\big)(\sigma + t(\xi - \sigma)) \cdot \vartheta.$$

We compute the derivative of g:

$$g'(t) = \nu\big(|\sigma + t(\xi-\sigma)|^2\big)(\xi - \sigma)\cdot\vartheta + \nu'\big(|\sigma + t(\xi-\sigma)|^2\big)\cdot 2\big[(\sigma + t(\xi-\sigma))\cdot(\xi-\sigma)\big](\sigma + t(\xi-\sigma))\cdot\vartheta.$$

Then we can write

$$\begin{aligned}
|g(1) - g(0)| &= \Big|\int_0^1 g'(t)\,dt\Big| \\
&\le \int_0^1 \big\{C_1|\xi-\sigma|\cdot|\vartheta| + 2b'(|\sigma + t(\xi-\sigma)|^2)\cdot|\sigma + t(\xi-\sigma)|^2|\xi-\sigma|\cdot|\vartheta|\big\}\,dt \le (C_1 + 2C_2)|\xi-\sigma|\cdot|\vartheta|.
\end{aligned}$$

This estimate finally yields

$$\begin{aligned}
|\langle Au,\varphi\rangle - \langle Av,\varphi\rangle| &\le \int_\Omega (C_1 + 2C_2)|\nabla(u-v)(x)|\cdot|\nabla\varphi(x)|\,dx \\
&\le (C_1 + 2C_2)\Big(\int_\Omega |\nabla(u-v)|^2\,dx\Big)^{1/2}\Big(\int_\Omega |\nabla\varphi|^2\,dx\Big)^{1/2} \\
&= L|u-v|_{H^1(\Omega)}|\varphi|_{H^1(\Omega)} \\
&\le L\|u-v\|_{H^1(\Omega)}\|\varphi\|_{H^1(\Omega)}
\end{aligned}$$

In a similar way it can be checked that A is strongly monotone. We want to show that

$$\langle Au - Av, u - v\rangle \ge \eta\|u-v\|^2_{H^1(\Omega)}.$$

Let us put $w = u - v$ and define

$$h(t) = \int_{\Omega_1} \nu_1(|\nabla(v + tw)|^2)\nabla(v + tw)\cdot\nabla w\,dx, \qquad t \in [0,1].$$

Since we assume that the function ν_1 is nondecreasing, its derivative is non-negative and we obtain

$$\begin{aligned}
h'(t) &= \int_{\Omega_1} \Big\{\nu_1(|\nabla(v+tw)|^2)|\nabla w|^2 + 2\nu_1'(|\nabla(v+tw)|^2)\big|\nabla(v+tw)\cdot\nabla w\big|^2\Big\}\,dx \\
&\ge \int_{\Omega_1} \nu_1(|\nabla(v+tw)|^2)|\nabla w|^2\,dx \\
&\ge C_0\|\nabla w\|^2_{L^2(\Omega)}.
\end{aligned}$$

Using this result we can write

$$\begin{aligned}
\langle Au - Av, u - v\rangle &= \int_{\Omega_0\cup\Omega_1} \nu(|\nabla u|^2)\nabla u\cdot\nabla w\,dx - \int_{\Omega_0\cup\Omega_1} \nu(|\nabla v|^2)\nabla v\cdot\nabla w\,dx \\
&= \int_{\Omega_0} \nu_0\nabla u\cdot\nabla w\,dx - \int_{\Omega_0} \nu_0\nabla v\cdot\nabla w\,dx + h(1) - h(0) \\
&= \int_{\Omega_0} \nu_0\nabla w\cdot\nabla w\,dx + \int_0^1 h'(t)\,dt \\
&\ge (\nu_0 + C_0)\|\nabla w\|^2_{L^2(\Omega)} \\
&\ge C\|w\|^2_{H^1(\Omega)}.
\end{aligned}$$

In the last estimate, Friedrichs inequality was used (note that $w \in H^1_0(\Omega)$).

Thus, we can apply the existence theorem and conclude that our problem has a unique weak solution.