

# 1 09-04-2018

We will learn about : Basics of functions of several variables. In this lecture:

## 1.1 A sequence in the Euclidean space and its application

Using these notation :

- $\mathbb{N}$  : set of natural number ( $\mathbb{N} = \{1, 2, 3, \dots\}$ )
- $\mathbb{Z}$  : set of integers ( $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ )
- $\mathbb{Q}$  : set of rational number ( $\mathbb{Q} = \{0, \pm 1, \pm 2, \frac{2}{3}, \dots\}$ )
- $\mathbb{R}$  : set of real number
- $\mathbb{C}$  : set of complex number

**Definition 1.** A sequence  $(x_n)_{n=1}^{\infty}$  is an assignment of (real) number  $x_n \in \mathbb{R}$  to natural number  $n \in \mathbb{N}$  ( $x_n \in \mathbb{R}$ ).

Example :  $x_n = \frac{1}{n}$ .  $x_1 = 1, x_2 = \frac{1}{2}, \dots$

**Definition 2.** A subsequence of a sequence  $(x_n)_{n=1}^{\infty}$  is a sequence  $(y_j)_{j=1}^{\infty}$  defined by  $y_j = x_{n_j}$  for some sequence  $(n_j)_{j=1}^{\infty}$  in  $\mathbb{N}$  such that  $n_j < n_{j+1}$  ( $j = 1, 2, \dots$ ).

Example : sequence  $(x_n)_{n=1}^{\infty} = 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{100}$ , takes  $n_1 = 1, n_2 = 3, n_3 = 5, n_4 = 100$

subsequence  $(x_{n_j})_{j=1}^{\infty} = x_{n_1}, x_{n_2}, x_{n_3}, x_{n_4} = 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{100}$ .

**Definition 3.** Let  $(x_n)_{n=1}^{\infty}$  be a sequence converges to  $\alpha \in \mathbb{R}$  if for any  $\epsilon > 0$ , there is an  $N \in \mathbb{N}$  such that  $n > N$ ,  $|x_n - \alpha| < \epsilon$ .

In the mathematical symbol  $\forall \epsilon > 0, \exists N \in \mathbb{N}$  such that  $n > N$ ,  $|x_n - \alpha| < \epsilon$  for  $n > N$ .

In this case we write,  $\lim_{n \rightarrow \infty} x_n$  or  $x_n \rightarrow \alpha$  ( $n \rightarrow \infty$ )

**Example 1.**

**Theorem 1.**  $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}$  is sequence. Suppose  $x_n \rightarrow \alpha$  and  $y_n \rightarrow \beta$  as  $n \rightarrow \infty$ .

1.  $x_n \pm y_n \rightarrow \alpha \pm \beta$ , ( $n \rightarrow \infty$ )
2.  $x_n \cdot y_n \rightarrow \alpha \cdot \beta$ , ( $n \rightarrow \infty$ )
3. if  $\beta \neq 0$ ,  $\frac{x_n}{y_n} \rightarrow \frac{\alpha}{\beta}$ , ( $n \rightarrow \infty$ )

**Remark 1.** On  $\frac{x_n}{y_n}$  is not defined for all  $n \in \mathbb{N}$  because  $y_n = 0$  possibly for some  $n \in \mathbb{N}$ . But, since  $y_n \rightarrow \beta \neq 0$ ,  $y_n$  eventually is not 0. Hence  $\frac{x_n}{y_n}$  is defined eventually.

**Theorem 2.**  $(x_n)_{n=1}^{\infty}$  a sequence. If  $(x_n)_{n=1}^{\infty}$  converges to  $\alpha \in \mathbb{R}$ , any subsequence of  $(x_n)$

# 2 16-04-2018

## 2.1 n-dimensional space

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R} = \{(x_1, \dots, x_n) | x_i \in \mathbb{R}\}$ .

Takes  $n = 2$ ,  $\mathbb{R}^2 \Leftrightarrow$  plane, we have  $P(a, b)$ .

For  $n = 3$ , we have  $P(a, b, c)$ .

**Definition 4.**  $P_m = (x_1^m, \dots, x_n^m) \in \mathbb{R}^n$ , and  $\{P_m\}_{m=1}^{\infty}$  : a sequence in  $\mathbb{R}^n$ .  $\{P_m\}$  converges to  $A = (a_1, \dots, a_n) \in \mathbb{R}^n$ , if  $\forall k = 1, \dots, n$ ,  $x_k^m \rightarrow a_k$  as  $n \rightarrow \infty$ .

**Definition 5. Inner product and norm.**

$\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ . We can define :

$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \dots + x_n y_n$  ; **inner product**

$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$  ; **norm**

**Example 2.**  $\mathbf{x} \cdot \mathbf{y} = 0 \Leftrightarrow \mathbf{x}$  is perpendicular to  $\mathbf{y}$

Takes  $n = 0$  then

$$\begin{aligned} x_1 y_1 + x_2 y_2 &= 0 \\ x_1 y_1 &= -x_2 y_2 \\ \frac{y_1}{y_2} &= -\frac{x_2}{x_1} \\ \text{then } (x_1, x_2) &= c \cdot (-y_2, y_1) \end{aligned}$$

*pict :*

**Example 3.**  $\|\mathbf{x}\| = 0 \Leftrightarrow x = 0$

$(\Rightarrow) 0 = \|x\|^2 = x_1^2 + \dots + x_n^2$ , then  $x_1^2 = 0$  ( $\forall i = 1, \dots, n$ ) and finally  $x_1 = 0$ .

**Notes 1.**  $\|x\|$  is the distance between  $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^n$  and  $\mathbf{x} = (x_1, \dots, x_n)$ .

For notation, we will use  $P, Q \in \mathbb{R}^n$  as points and  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  as vectors.

We also use  $\|x - y\| = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$  as distance between  $\mathbf{x}$  and  $\mathbf{y}$ .

$\|P - Q\|$  is distance between  $P$  and  $Q$ .

$$\mathbf{x} \pm \mathbf{y} = (x_1 \pm y_1, \dots, x_n \pm y_n)$$

$$P = (p_1, \dots, p_n), Q = (q_1, \dots, q_n), \text{ then } P + Q = (p_1 + q_1, \dots, p_n + q_n)$$

$$\alpha \in \mathbb{R}, \alpha \mathbf{x} = (\alpha x_1, \dots, \alpha x_n), \alpha P = (\alpha p_1, \dots, \alpha p_n)$$

$$\{P_m\}_{m=1}^\infty : \text{a sequence in } \mathbb{R}^n, P_m \rightarrow A \Leftrightarrow \|P_m - A\| \rightarrow 0$$

**Theorem 3. Cauchy-Schwarz inequality.** For any  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ ,

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|$$

"="  $\Rightarrow a\mathbf{y} = b\mathbf{x}$  for some  $a, b \in \mathbb{R}$ .

$\therefore$  We may assume  $\mathbf{x} \neq \emptyset, \forall t \in \mathbb{R}$ .

$$0 \leq \|t\mathbf{x} + \mathbf{y}\|^2 = (t\mathbf{x} + \mathbf{y}) \cdot (t\mathbf{x} + \mathbf{y}) = t^2 \|\mathbf{x}\|^2 + 2t(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2$$

$$D/4 \leq 0$$

**Theorem 4. Bolzano=Weierstrass.** Let  $(P_m)_{m=1}^\infty \subset \mathbb{R}^n$  be a sequence. Suppose that  $(P_m)_{m=1}^\infty$  is bounded. In the sense that  $\|P_m\| \leq M$  ( $m \in \mathbb{N}$ ) for some  $M \geq 0$ . Then  $(P_m)_{m=1}^\infty$  contains a convergent subsequence.

**Definition 6. Ball.**  $A \in \mathbb{R}^n, R > 0$

$$\mathbf{B}(A, R) = \{P \in \mathbb{R}^n \mid \|P - A\| < R\}; \text{ open ball of center } A \text{ with radius } R$$

$$\bar{\mathbf{B}}(A, R) = \{P \in \mathbb{R}^n \mid \|P - A\| \leq R\}; \text{ closed ball}$$

**Definition 7.** 1.  $E \subset \mathbb{R}^n$  is said to be **an open set** if  $E = \emptyset$  or  $\forall A \in E, \exists R > 0$  such that  $\mathbf{B}(A, R) \subset E$ .

2.  $E \subset \mathbb{R}^n$  is said to be **a closed set** if  $E^c = \mathbb{R}^n \setminus E$  is an open set.

$E$  : open, then neighbor in any point

**Definition 8. Accumulation point.**  $E \subset \mathbb{R}^n$ ; a set.  $A \in \mathbb{R}^n$  is called **an accumulation point** of  $E$  if  $\forall R > 0, (\mathbf{B}(A, R) - \{A\}) \cap E \neq \emptyset$ .

**Remark 2.** ini notes.  $E \subset \mathbb{R}^n$  is closed if and only if  $E$  contains any accumulation point of  $E$ . **Homework report, prove this**

**Remark 3.** note juga.

1. Both  $\emptyset$  and  $\mathbb{R}^n$  are open and closed

2.  $\{E_\lambda \mid \lambda \in A\}$ ; a collection of open sets  $\Rightarrow$  union  $\lambda \in A E_\lambda$  is also open

3.  $\{E_\lambda\}_{\lambda=1}^N$ , a finite collection of open sets  $\Rightarrow$  irisan  $\bigcap_{\lambda=1}^N E_\lambda$  is also open.

4. **Rephrase of Bolzano Weierstrass theorem.**  $E \subset \mathbb{R}^n$ ; a **ounded closed set**  $\Leftrightarrow E$  is a closed set such that  $E \subset \mathbf{B}(\mathbf{x}, R)$  for some  $R > 0$ .  $E$ ; a bounded closed set then any sequence of  $E$  contains a convergent subsequence whose limit is in  $E$ .

**Definition 9.** A bounded closed set in  $\mathbb{R}^n$  is called **compact**.

**Example 4.**  $\bar{\mathbf{B}}(A, R)$  is compact. **Report! prove this**

## 2.2 Continuity and differentiability of a function

### 2.2.1 Continuity

$E$  : a set in  $\mathbb{R}^n$  and  $f$  : is a function of  $E$  (real valued function). i.e.  $f$  is an assignment a (real) number to a point in  $E$ .

**Definition 10.** 1.  $f$  is *continuous at*  $A \in E$  if  $\forall (P_m)_{m=1}^{\infty} \subset E$  : sequence with  $P_m \rightarrow A$  ( $m \rightarrow \infty$ )

$$f(P_m) \rightarrow f(A) \quad (m \rightarrow \infty)$$

2.  $f$  is *continuous on*  $E$  if  $f$  is continuous at any point of  $E$ .

### 2.2.2 Basic of continuous function on an interval in $\mathbb{R}$

**Theorem 5. Intermediate value theorem.**  $f$  : function on a closed interval  $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$ . Suppose that  $f(a) \leq f(b)$ . Then,  $\forall \gamma$  with  $f(a) \leq \gamma \leq f(b)$ ,  $\exists c \in [a, b]$  with  $f(c) = \gamma$ .

**Theorem 6. Extreme value theorem.**  $f$  is a continuous function