1 10-04-18

1.1 Algebraic axioms for real numbers

Two binary operations, $+$ addition and $\cdot$ multiplication on $\mathbb{R}$ are defined and have the following propoerties for all $x, y, z \in \mathbb{R}$:

1. $x + (y + z) = (x + y) + z$. Associative law for addition.
2. $\exists 0$ such that $x + 0 = 0 + x = x$. Existence of additive identity.
3. There exist an element $-x \in \mathbb{R}$ such that $x + (-x) = (-x) + x = 0$. Existence of additive inverse.
4. $x + y = y + x$. Commutative law for addition.
5. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$. Associative law for multiplication.
6. $\exists 1 \neq 0$ such that $x \cdot 1 = 1 \cdot x = x$. Existence of multiplicative identity.
7. If $x \neq 0$, then there exist an element $x^{-1} \in \mathbb{R}$ such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$. Existence of multiplicative inverse.
8. $x \cdot y = y \cdot x$. Commutative law for multiplication.
9. $x \cdot (y + z) = x \cdot y + x \cdot z$. Distributive law.

In the language of algebra, axioms above state that $\mathbb{R}$ with addition and multiplication is a **field**.

1.2 The order axioms for real number

A binary relation $\leq$ on $\mathbb{R}$ is defined and satisfies the following properties for all $x, y, z \in \mathbb{R}$.

1. $x \leq x$. Reflexivity.
2. If $x \leq y,\ y \leq x$ then $x = y$. Antisymmetry.
3. If $x \leq y,\ y \leq z$ then $x \leq z$. Transitivity.
4. Either $x \leq y$ or $y \leq x$. Totality.
5. If $x \leq y$, then $x + y \leq y + z$
6. If $0 \leq x$ and $0 \leq y$, then $0 \leq x \cdot y$.

2 17-04-18

2.1 Real Number

$\mathbb{Q} = \{\frac{n}{m} | n, m \in \mathbb{Z}, m \neq 0\}$. We have $p, q \in \mathbb{Q}$, then

$$p + q = \frac{n}{m} + \frac{k}{l} = \frac{kn + ml}{mk};\ pq = \frac{nl}{mk};\ p \geq q \Leftrightarrow p - q \geq 0$$

For $+, \times, \geq$ satisfy A1-A15.

Remark 1. *$\mathbb{Q}$ is incomplete in the following sense. There is no $r \in \mathbb{Q}$ such that $r^2 = 2$. Remember Phytagoras theorem, $a^2 + b^2 = c^2$. Pict : $\because$ if $c \in \mathbb{Q}$, then $c = \frac{n}{m}$ $(n, m \in \mathbb{Z}, m \neq 0)$. We may assume that either m or n is odd.*

$$c^2 = 2 \to (\frac{n}{m})^2 = 2 \to n^2 = 2m^2$$

case 1 : *n is odd $\Rightarrow$ odd = even (impossible)*
case 2 : *n is even $\Rightarrow$ m is odd (from assumtion) $\Rightarrow$ n^2 can be devided by 4 but $2m^2$ can not devided by 4 (contradiction)*

Question : How to fill the gap of $\mathbb{Q}$? Answer : Idea of Weirstrass (supreme axioms)

Definition 1. *$A \subset \mathbb{R}$.*

- *A is bounded from above $\Leftrightarrow \exists b \in \mathbb{R}$ such that $a \leq b$ $(\forall a \in A)$. such b is called upper bound of A.*
- *A is bounded from below $\Rightarrow \exists b\prime \in \mathbb{R}$ such that $a \geq b\prime$ $(\forall a \in A)$. Such $b\prime$ is called lower bound of A*
- *$\alpha = supA$*
  $\Leftrightarrow$ the minimum of the set of upper bound
  $\Leftrightarrow$ 1. α is an upper bound of A ; 2. if b is an upper bound of A, then $\alpha \leq b$.
- *$\beta = infA \Leftrightarrow$ the maximum of the set of lower bounds of A.*

Remark 2. *$supA(infA)$ is uniquely determined if it exist. For example, $sup\mathbb{Q}(inf\mathbb{Q})$ does not exist. $\because$ $\mathbb{Q}$ is not bounded from above (below)*

Remark 3. ***Completeness axioms.*** *Every nonempty subset of $\mathbb{R}$ which is bounded from above (below) has a supremum (infimum) in $\mathbb{R}$*

2.2 Real sequence

Definition 2. *For $x \in \mathbb{R}$, $|x| = \begin{cases} x & , x \geq 0 \\ -x & , x \leq 0 \end{cases}$*

Remark 4.

- $|x| \geq 0,\ |x| = 0 \Leftrightarrow x = 0$
- $|xy| = |x||y|$
- $|x + y| \leq |x| + |y|$ *(triangle inequality)*

An infinite sequence of $\mathbb{R} \Leftrightarrow a : \mathbb{N} \to \mathbb{R}$ usually we write $a_n = a(n),\ n \in \mathbb{N}$ or $\{a_n\}_{n \in \mathbb{N}}$ or $a_1, a_2, \ldots$.
Question : Limiting behavior of a_n as n increases ?
Answer : $a_n \to l,\ n \to \infty \Leftrightarrow$ as n become larger and larger, the value a_n become arbitrarily close to l.

Definition 3. *$\epsilon - N$ **definition of the limit**. $\{a_n\}$ converges to $l \in \mathbb{R} \Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $|a_n - l| < \epsilon, \forall n \geq N$. We write $\lim_{n \to \infty} a_n = l$.*

Definition 4.

- *$a_n \to +\infty \Leftrightarrow \forall M > 0.\ \exists N \in \mathbb{N}$ such that $a_n > M$ $(\forall n \geq N)$*
- *$a_n \to -\infty \Leftrightarrow \forall M > 0.\ \exists N \in \mathbb{N}$ such that $a_n < -M$ $(\forall n \geq N)$*

Remark 5. *A convergent sequence has a unique limit.*

$$\because \qquad \epsilon = \frac{1}{2}|l - l\prime| > 0$$
$$\exists N \in \mathbb{N} \text{ such that } |a_n - l| < \epsilon,\ (\forall n \geq N)$$
$$\exists N\prime \in \mathbb{N} \text{ such that } |a_n - l\prime| < \epsilon,\ (\forall n \geq N\prime)$$

Set $\tilde{N} = max\{N, N\prime\} \in \mathbb{N}$. For $n \geq \tilde{N} \Rightarrow |a_n - l| < \epsilon,\ |a_n - l\prime| < \epsilon$ is impossible.

REPORT 1
Afifah Maya Iknaningrum (1715011053)

1. Problem : Let $\{a_n\}, \{b_n\}, \{c_n\}$ be a real sequence. Suppose that for every $n \in \mathbb{N}$, we have

$$b_n \leq a_n \leq c_n$$

and also suppose that

$$\lim_{n\to\infty} b_n = l = \lim_{n\to\infty} c_n$$

for some $l \in \mathbb{R}$. Then

$$\lim_{n\to\infty} a_n = l.$$

Answer : By definition of limit, $\forall \epsilon > 0,\ \exists N_1, N_2 \in \mathbb{N}$ such that for $l \in \mathbb{R}$

$$|b_n - l| < \epsilon,\ \forall n \geq N_1,$$

$$|c_n - l| < \epsilon,\ \forall n \geq N_2.$$

Then we can obtain

$$|b_n - l| < \epsilon \Leftrightarrow -\epsilon < b_n - l < \epsilon \Leftrightarrow l - \epsilon < b_n < l + \epsilon,$$
$$|c_n - l| < \epsilon \Leftrightarrow -\epsilon < c_n - l < \epsilon \Leftrightarrow l - \epsilon < c_n < l + \epsilon.$$

Take $N = max\{N_1, N_2\}$, then $\forall n > N$

$$\begin{aligned} & b_n \leq a_n \leq c_n \\ \Leftrightarrow\quad & l - \epsilon < b_n \leq a_n \leq c_n < l + \epsilon \\ \Leftrightarrow\quad & l - \epsilon < a_n < l + \epsilon \\ \Leftrightarrow\quad & -\epsilon < a_n - l < \epsilon \\ \Leftrightarrow\quad & |a_n - l| < \epsilon. \end{aligned}$$

It is proved that $\forall \epsilon > 0,\ \exists N \in \mathbb{N}$ such that for $l \in \mathbb{R}$

$$|a_n - l| < \epsilon,\ \forall n \geq N$$

or we can write

$$\lim_{n\to\infty} a_n = l.$$

$\square$

2. (a) Problem : If a sequence of real numbers converges, then it is bounded.
Answer : Let $\{x_n\}$ be a sequence in real number. Suppose $\{x_n\}$ is converge to $a \in \mathbb{R}$ as $n \to \infty$. Then $\forall \epsilon > 0,\ \exists N \in \mathbb{N}$ such that $\forall n > N$,

$$|x_n - a| < \epsilon.$$

From triangle inequality we obtain

$$\begin{aligned} |x_n - a| &< \epsilon \\ |x_n| - |a| &< \epsilon \\ |x_n| &< \epsilon + |a|. \end{aligned}$$

Takes $M = max\{\epsilon + |a|, x_1, x_2, \ldots, x_N\}$, we obtain

$$|x_n| \leq M.$$

It shows that $\forall \epsilon > 0, \exists M > 0$ such that $|x_n| \leq M,\ \forall n$ or it is proved that $\{x_n\}$ is bounded. □

(b) Problem : If a sequence of real numbers converge, then it is a Cauchy sequence.
Answer : Let $\{x_n\}, \{x_m\}$ be a sequence in real number. Suppose $\{x_n\}, \{x_m\}$ is converge to $a \in \mathbb{R}$ as $n \to \infty$. Then $\forall \epsilon > 0,\ \exists N_1, N_2 \in \mathbb{N}$ such that $\forall n > N_1$,

$$|x_n - a| < \frac{\epsilon}{2}$$

and $\forall m > N_2$,

$$|x_m - a| < \frac{\epsilon}{2}.$$

Takes $N = max\{N_1, N_2\}$ then $\forall n, m > N$

$$\begin{aligned} |x_n - x_m| &\leq |x_n - a + a - x_m| \\ &\leq |x_n - a| + |x_m - a| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

Then, it is proved that $\forall \epsilon > 0,\ \exists N \in \mathbb{N}$ such that for $n, m > N$

$$|x_n - x_m| < \epsilon$$

or it is Cauchy sequence. □

3 07-05-2018

3.1 Landau Symbol

Symbol for representing the behavior of functions. O : big o and o : small o.
Let f, g be function around $x = a \in \mathbb{R}$ (or $x > M$ for some $M \in \mathbb{R}$)

- $f(x) = O(g(x))$ as $x \to a$ if $\exists \delta > 0, \exists A > 0$ such that $|f(x)| \leq A\ g(x)$ for $0 < |x - a| < \delta$.
  Means : eventually the graph $f(x)$ is below of $y = A\ g(x)$.

- $f(x) = O(g(x))$ as $x \to a$ if $\exists m > M, \exists A > 0$ such that $|f(x)| \leq A\ g(x)$ for $x > m$.
  Means : $|f(x)$ is eventually *dominated* by linear function as $x \to \infty$

Example :
$f(x) = O(x^2)$, $(x \to \infty)$ then $f(x)$ is eventually dominated by a quadratic function as $x \to \infty$.
$f(x) = O(1)$ as $x \to \infty$ then $f(x)$ is a bounded function around ∞

Explanation behavior : $f(x)$ is a polynomial time behaviour as $x \to \infty$. $f(x) = O(e^{ax})$, $f(x) = o(x^n)$ for some $n \in \mathbb{N}$.

- $f(x) = o(g(x))$, $(x \to \infty)$ if $\forall \epsilon > 0, \exists \delta > 0$ such that $|f(x)| \leq \epsilon g(x), 0 < |x - a| < \delta$
  (or equivalently, if $g(x) \neq 0$, $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)} = 0$)

- $f(x) = o(g(x))$, $(x \to \infty)$ if $\forall \epsilon > 0, \exists m > 0$ such that $x > m \Rightarrow |f(x)| \leq \epsilon g(x)$
  (or equivalently, if $g(x) \neq 0$, $\displaystyle\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$)

- $f(x) = o(x)$ as $x \to \infty \Leftrightarrow \displaystyle\lim_{x \to \infty} \frac{f(x)}{x} = 0$

- $f(x) = o(x)$ as $x \to 0 \Leftrightarrow \displaystyle\lim_{x \to 0} \frac{f(x)}{x} = 0$

- $f(x) = o(1)$ as $x \to \infty \Leftrightarrow \displaystyle\lim_{x \to \infty} \frac{f(x)}{1} = 0 \Leftrightarrow \lim_{x \to \infty} f(x) = 0$

- $f(x) = o(1)$ as $x \to a$ $(a \in \mathbb{R}) \Leftrightarrow \displaystyle\lim_{x \to a} f(x) = 0$

Remark 6. 1. *$f(x)$ is continuous at $x = a \Leftrightarrow \displaystyle\lim_{x \to \infty} f(x) = f(a)$ iff $\Leftrightarrow \lim_{x \to a}(f(x) - f(a)) = 0$*
by previous, $\Leftrightarrow f(x) - f(a) = o(1)$ as $x \to a \Leftrightarrow f(x) = f(a) + o(1)$ as $x \to a$

2. *$f(x)$ is differentiable as $x = a \Leftrightarrow \displaystyle\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$ iff $\Leftrightarrow \displaystyle\frac{f(a+h) - f(a)}{h} = f'(a) + o(1)$ as $h \to 0$*
$\Leftrightarrow f(a+h) = f(a) + f'(a)\ h + o(h)$ as $h \to 0$
note : $o(h) \Leftrightarrow \displaystyle\frac{o(h)}{h} = 0 \Rightarrow o(1)\ h \Rightarrow \frac{o(1)\ h}{h} = o(1) \to 0$ as $h \to 0$

D : domain in $\mathbb{R}^2$ and f : function on D

$$f_x(a, b) = \frac{\partial f}{\partial x}(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

$$f_y(a, b) = \frac{\partial f}{\partial y}(a, b) = \lim_{k \to 0} \frac{f(a, b+k) - f(a, b)}{k}$$

$$f(a+h, b) = f(a, b) + f_x(a, b)h + o(h) \text{ as } h \to 0$$

$$f(a, b+k) = f(a, b) + f_y(a, b)k + o(k) \text{ as } k \to 0$$

REPORT 2

Afifah Maya Iknaningrum (1715011053)

Problem 1 : Let $C([a,b])$ be the set of all continuous functions $f : [a,b] \to \mathbb{R}$ and define

$$d_2(f,g) := \Big[\int_a^b \big(f(x)-g(x)\big)^2 dx\Big]^{1/2}$$

for $f, g \in C([a,b])$. Show that $(C([a,b]), d_2)$ is a metric space.

Answer :
To proof that $(C([a,b]), d_2)$ is a metric space, we need to proof :

1. $d_2(f,g) \geq 0$ and $d_2(f,g) = 0 \Leftrightarrow f = g$.

   Proof :
   By the definitions of $d_2(f,g)$, it is obvious that the value of integral is always positive. So it is proved that $d_2(f,g) \geq 0$.
   Then,
   $(\Rightarrow)$ We have

   $$d_2(f,g) = \Big[\int_a^b \big(f(x)-g(x)\big)^2 dx\Big]^{1/2} = 0$$

   The only possible answer will be

   $$f(x) - g(x) = 0 \text{ or } f(x) = g(x)$$

   $(\Leftarrow)$ We have $f(x) = g(x)$, using the definition of $d_2(f,g)$

   $$\begin{aligned} d_2(f,g) &= \Big[\int_a^b \big(f(x)-g(x)\big)^2 dx\Big]^{1/2} \\ &= \Big[\int_a^b 0 \; dx\Big]^{1/2} \\ &= 0 \end{aligned}$$

2. $d_2(f,g) = d_2(g,f)$.

   Proof :

   $$\begin{aligned} d_2(f,g) &= \Big[\int_a^b \big(f(x)-g(x)\big)^2 dx\Big]^{1/2} \\ &= \Big[\int_a^b \big(g(x)-f(x)\big)^2 dx\Big]^{1/2} \\ &= d_2(g,f) \end{aligned}$$

3. $d_2(f,g) \leq d_2(f,h) + d_2(h,g)$.

   Proof :
   Using fact that

   $$\int \big(a+b\big)^2 = \int a^2 + \int b^2 + 2\int ab$$

   and via Schwartz inequality

   $$\int ab \leq \sqrt{\int a^2}\sqrt{\int b^2}$$

   then

   $$\int (a+b)^2 \leq \int a^2 + \int b^2 + 2\sqrt{\int a^2}\sqrt{\int b^2} = \Big(\sqrt{\int a^2} + \sqrt{\int b^2}\Big)^2$$

Using these fact with $a = f - h$ and $b = h - g$,

$$\begin{aligned}
d_2(f,g) &= \left[\int_a^b \big(f(x) - g(x)\big)^2 dx\right]^{1/2} \\
&= \left[\int_a^b \big(f(x) - h(x) + h(x) - g(x)\big)^2 dx\right]^{1/2} \\
&\leq \left(\left[\int_a^b \big(f(x) - h(x)\big)^2 dx\right]^{1/2} + \left[\int_a^b \big(h(x) - g(x)\big)^2 dx\right]^{1/2}\right)^{2(1/2)} \\
&\leq \left[\int_a^b \big(f(x) - h(x)\big)^2 dx\right]^{1/2} + \left[\int_a^b \big(h(x) - g(x)\big)^2 dx\right]^{1/2} \\
&\leq d_2(f,h) + d_2(h,g)
\end{aligned}$$

□

Problem 2 : Let (X, d) be a metric space. Prove that the function

$$\tilde{d}(x,y) := \frac{d(x,y)}{1 + d(x,y)} \ , \quad (x, y \in X)$$

is also a metric on X.

Answer :
It is known that (X, d) is metric space. Then for $x, y, z \in X$ we have the following

1. $d(x,y) \geq 0$ and $d(x,y) = 0 \Leftrightarrow x = y$
2. $d(x,y) = d(y,x)$
3. $d(x,z) \leq d(x,y) + d(y,z)$.

We want to proof that $\tilde{d}(x,y) := \dfrac{d(x,y)}{1 + d(x,y)}$ is also a metric space. We need to proof :

1. $\tilde{d}(x,y) \geq 0$ and $\tilde{d}(x,y) = 0 \Leftrightarrow x = y$.

   Proof :
   Using the fact that $d(x,y) \geq 0$, it is obvious that

   $$\tilde{d}(x,y) = \frac{d(x,y)}{1 + d(x,y)} \geq 0.$$

   $(\Rightarrow)$

   $$\tilde{d}(x,y) = \frac{d(x,y)}{1 + d(x,y)} = 0$$

   only possible if $\tilde{d}(x,y) = 0$. Using properties of (X, d),

   $$d(x,y) = 0 \Leftrightarrow x = y$$

   , then it proved that $\tilde{d}(x,y) = 0 \Leftrightarrow x = y$.
   $(\Leftarrow)$ For $x = y$, using the fact $d(x,y) = 0 \Leftrightarrow x = y$, we have

   $$\tilde{d}(x,y) = \frac{d(x,y)}{1 + d(x,y)} = 0$$

2. $\tilde{d}(x,y) = \tilde{d}(y,x)$.

   Proof :
   Because (X, d) is metric space, then $d(x,y) = d(y,x)$ is hold. Such that

   $$\begin{aligned}
   \tilde{d}(x,y) &= \frac{d(x,y)}{1 + d(x,y)} \\
   &= \frac{d(y,x)}{1 + d(y,x)} \\
   &= \tilde{d}(y,x)
   \end{aligned}$$

3. $\tilde{d}(x,z) \leq \tilde{d}(x,y) + \tilde{d}(y,z)$.

<u>Proof :</u>
Using triangle inequality of (X,d),

$$\begin{aligned}
\tilde{d}(x,z) &= \frac{d(x,z)}{1+d(x,z)} \\
&\leq \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)} \\
&\leq \frac{d(x,y)}{1+d(x,y)+d(y,z)} + \frac{d(y,z)}{1+d(x,y)+d(y,z)} \\
&\leq \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} \\
&\leq \tilde{d}(x,y) + \tilde{d}(y,z)
\end{aligned}$$

□

REPORT 3
Afifah Maya Iknaningrum (1715011053)

Problem 1 : Prove that a subset of a metric space is open if and only if it is a union of open balls.

Answer :

($\Rightarrow$)Suppose G in (X, d) is open. If G is empty, there no open balls contained in it. Thus union of an empty class, which is empty and therefore equal to G. If G is nonempty, then G is open such that $\forall x \in G, \exists r > 0, B_r(x) \subset G$ then $G = \bigcup B_r(x)$.

($\Leftarrow$) In metric space, it is known that every open ball is open set. And, union of open set is open.
Let $G = \bigcup\limits_{\alpha \in \Lambda} B_r(\alpha)$ for $\alpha \in G, \exists r > 0$. If G is empty, then it is open. So we assume G is nonempty. Consider $y \in G$, then $y \in B_r(\alpha)$ for some $\alpha \in \Lambda$. Since $B_r(\alpha)$ is open, $\exists r > 0$ such that $B_r(y) \subseteq B_r(\alpha) \subseteq G$.
Thus $\forall y \in G, \exists r > 0$ such that $B_r(y) \subseteq G$. Consequently, G is open.

Problem 2 : Let $C([0,1])$ be the set of all continuous function $f : [0,1] \to \mathbb{R}$ and define

$$d_1(f, g) := \int_0^1 |f(x) - g(x)| \; dx$$

for $f, g \in C([0,1])$. Show that $(C([0,1]), d_1)$ is not complete.
Hint : Consider the sequence $\{f_n\}_{n \geq 3}$ defined by

$$f_n(x) = \begin{cases} 0 & , 0 \leq x < \frac{1}{2} - \frac{1}{n}, \\ n(x + \frac{1}{n} - \frac{1}{2}) & , \frac{1}{2} - \frac{1}{n} \leq x < \frac{1}{2} \\ 1 & , \frac{1}{2} \leq x \leq 1 \end{cases}$$

Answer :

Considering the sequence $\{f_n\}_{n \geq 3}$ above, then

$$\|f_n - f_m\| = \Big(\int_{1/2 - 1/n}^{1/2} \|f_n(x) - f_m(x)\| \; dx \Big) \leq \big(\frac{-1}{n}\big) \to 0$$

so f_n is Cauchy. Suppose f_n has limit $f \in C([0,1])$. Then

$$\int_{1/2}^{1} |f(x) - f_n(x)| \; dx \leq \|f - f_n\| \to 0$$

so $f(x) = 1$ on $[1/2, 1]$. Similarly we see $f(x) = 0$ on $[0, 1/2]$ which is contradiction.