Nonlinear PDEs

6th lecture

Monotone operators

Definition Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. We say that an operator $A : H \to H$ is

- **monotone** if
$$\langle Au - Av, u - v \rangle \geq 0 \qquad \forall u, v \in H$$
- **strongly monotone** if there exists $\eta > 0$ such that
$$\langle Au - Av, u - v \rangle \geq \eta \|u - v\|_H^2 \qquad \forall u, v \in H.$$

We say that A satisfies **Lipschitz condition** with constant L if
$$\|Au - Av\|_H \leq L\|u - v\|_H \qquad \forall u, v \in H.$$

- non-decreasing function
- positive-definite matrix
- continuous coercive bilinear form
- some differential operators

Today's contents

We shall solve this problem in the following steps:

- Write the problem in the operator form $A(u) = f$, where $A : H \to H$ is a nonlinear operator expressing the left-hand side of the equation.
- Proof an existence theorem for this abstract operator equation (using monotone operator theory).
- Check that the assumptions of the existence theorem for our problem are satisfied.

Theorem

Theorem. Let $A : H \to H$ be strongly monotone with respect to H (with constant η) and let A satisfy Lipschitz condition (with constant L). Then for each $f \in H$ there exists a unique solution u of the problem

$$Au = f \qquad \text{in } H.$$

Fixed point theorems

Banach's Fixed Point Theorem Let X be a Banach space and assume $F : X \to X$ is a mapping satisfying

$$\|F(u) - F(v)\| \leq \gamma \|u - v\| \qquad u, v \in X$$

for some constant $\gamma < 1$ (i.e., F is a strict **contraction**). Then F has a unique fixed point.

Schauder's Fixed Point Theorem Let X be a Banach space. Suppose $K \subset X$ is compact and convex and assume also that $F : K \to K$ is continuous. Then F has a fixed point in K.

Schaefer's Fixed Point Theorem Let X be a Banach space. Suppose $F : X \to X$ is a continuous and compact mapping. Assume further that the set

$$\{u \in X \; ; \quad u = \lambda F(u) \; \text{ for some } \; \lambda \in [0,1]\}$$

is bounded. Then F has a fixed point.

Weak solution

Find $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega \nu(x, |\nabla u(x)|^2) \nabla u(x) \cdot \nabla \varphi(x)\, dx = \int_\Omega f(x)\varphi(x)\, dx$$

for all test functions $\varphi \in H_0^1(\Omega)$.

The function ν is given by

$$\nu(x,\eta) = \begin{cases} \nu_1(\eta) & \text{for } x \in \Omega_1 = \text{ ferromagnetic materials} \\ \nu_0 & \text{for } x \in \Omega_0 = \text{ other materials (copper wires, insulators, air, etc.)} \end{cases}$$

where $\nu_0 = 1/\mu_0$ with $\mu_0 = 4\pi \times 10^{-7}$ Tm/A, the permeability of vacuum, and ν_1 is a nondecreasing function satisfying

$$\begin{aligned} &C_0 \le \nu_1(\eta) \le C_1, && C_0, C_1 > 0, \\ &|\vartheta \nu_1'(\eta)| \le C_2, && \eta \ge \vartheta \ge 0,\ C_2 > 0. \end{aligned}$$

Riesz theorem

A mapping $A : X \to Y$ (X, Y are normed spaces) is a **linear operator** provided

$$A(\lambda u + \mu v) = \lambda A(u) + \mu A(v) \qquad \forall u, v \in X, \;\; \forall \lambda, \mu \in \mathbb{R}$$

A linear operator $A : X \to Y$ is **bounded** if

$$\|A\| = \sup_{\|u\|_X \leq 1} \|A(u)\|_Y = \sup_{u \in X} \frac{\|A(u)\|_Y}{\|u\|_X} \; < \; \infty$$

If $Y = \mathbb{R}$, then we call the operator $A : X \to \mathbb{R}$ a **functional**.

X^* = collection of all bounded linear functionals on X = **dual space** of X

Theorem (Riesz representation theorem) Let H be a real Hilbert space. Then for each $F \in H^*$ there exists a unique element $f \in H$ such that

$$F(v) = (f, v) \qquad \forall v \in H$$

The mapping $F \mapsto f$ is a linear isomorphism of H^* onto H.

Friedrichs' inequality

Friedrichs inequality. Let Ω be a domain with lipschitz boundary. Then there is a constant C_F such that

$$\|v\|_{H^1(\Omega)} \leq C_F |v|_{H^1(\Omega)} \qquad \forall v \in H_0^1(\Omega).$$

Here $|\cdot|_{H^1(\Omega)}$ is the seminorm defined as

$$|v|_{H^1(\Omega)} = \left(\int_\Omega |\nabla v|^2 \right)^{1/2}.$$

The fact that v has zero trace on the boundary is important!

This inequality says that the norm and seminorm on $H^1(\Omega)$ are equivalent norms for functions from $H_0^1(\Omega)$:

$$c\|v\|_{H^1(\Omega)} \leq |v|_{H^1(\Omega)} \leq \|v\|_{H^1(\Omega)} \qquad \forall v \in H_0^1(\Omega).$$