6.1 Remarks on numerical computation

We have shown, using the theory of monotone operators, that the following nonlinear problem of stationary magnetic field has a unique weak solution:

$$-\text{div}\Big(\nu(x, \|\nabla u(x)\|^2)\nabla u(x)\Big) = f(x) \qquad \text{for } x \in \Omega \tag{1}$$

$$u(x) = 0 \qquad \text{on } \partial\Omega. \tag{2}$$

The weak solution is defined as a function $u \in H_0^1(\Omega)$ which satisfies

$$\langle Au, \varphi \rangle = (f, \varphi)_0 \qquad \forall \varphi \in H_0^1(\Omega),$$

where the operator $A : H_0^1(\Omega) \to H_0^1(\Omega)$ is defined by

$$\langle Au, \varphi \rangle = \int_\Omega \nu\big(x, |\nabla u(x)|^2\big)\nabla u(x) \cdot \nabla\varphi(x)\, dx, \qquad u, \varphi \in H_0^1(\Omega).$$

Here $\langle \cdot, \cdot \rangle$ is the standard inner product in $H^1(\Omega)$. For a fixed $u \in H_0^1(\Omega)$ the right-hand side of the above identity represents a continuous linear functional on $H_0^1(\Omega)$ (with respect to φ). Hence, such an Au exists and is unique according to Riesz theorem.

In order to express the nonlinearity explicitly, let us introduce the following form

$$a(w; u, \varphi) = \int_\Omega \nu\big(x, |\nabla w(x)|^2\big)\nabla u(x) \cdot \nabla\varphi(x)\, dx, \qquad u, w, \varphi \in V,$$

where we use the notation $V = H_0^1(\Omega)$. It can be check by the assumptions that this integral is finite.

Definition (Galerkin approximation) Let $V_h \subset V$ be a nonempty finite-dimensional subspace of V. Then a function $u_h \in V_h$ is called a **Galerkin approximation** to the solution of $Au = f$ if

$$\langle Au_h, \varphi_h \rangle = \langle f, \varphi_h \rangle \qquad \forall \varphi_h \in V_h. \tag{3}$$

The space V_h can be, for example, a space of polynomials up to a certain order, or a finite element space, which we shall discuss later.

The theorem on existence and uniqueness of solution that we have proved before can be applied on the above defined u_h, too. The following theorem addresses the question of how far are the approximate Galerkin solutions from the solution of the original problem. It turns out that this question of convergence can be reduced to the examination of the approximation properties of the system $\{V_h\}$ in V. For example, if the solution is smooth enough then we can even obtain a certain speed of convergence.

Theorem (Error of Galerkin approximations)

Let $A : V \to V$ be a strongly monotone and Lipschitz continuous operator and $V_h \neq \emptyset$ be a finite-dimensional subspace of V. Then there exists a constant C independent on V_h such that

$$\|u - u_h\|_V \leq C \inf_{v_h \in V_h} \|u - v_h\|_V,$$

where u is the solution to $Au = f$ and u_h is the Galerkin approximation (3).

Proof. From the existence theorem we know that u and u_h exist and are unique. Then subtracting

$$\langle Au, \varphi \rangle = \langle f, \varphi \rangle \qquad \forall \varphi \in V$$

$$\langle Au_h, \varphi \rangle = \langle f, \varphi \rangle \qquad \forall \varphi \in V_h$$

and selecting $\varphi = u_h - v_h \in V_h$ with $v_h \in V_h$, we see that

$$\langle Au - Au_h, u_h - v_h \rangle = 0 \qquad \forall v_h \in V_h.$$

Hence, using the monotonicity and Lipschitz continuity of A we have

$$\begin{aligned}
\eta \|u - u_h\|_V^2 &\leq \langle Au - Au_h, u - u_h \rangle + \langle Au - Au_h, u_h - v_h \rangle = \langle Au - Au_h, u - v_h \rangle \\
&\leq \|Au - Au_h\|_V \|u - v_h\|_V \\
&\leq L\|u - u_h\|_V \|u - v_h\|_V \qquad \forall v_h \in V_h.
\end{aligned}$$

Some methods of numerical computation of Galerkin solutions

$$a(u_h; u_h, \varphi_h) = \langle Au_h, \varphi_h \rangle = (f, \varphi_h)_0 \qquad \forall \varphi_h \in V_h$$

- **Method of successive approximations**

  1. Choose $w_0 \in V_h$ arbitrarily.
  2. Compute $w_k + 1$ for $k = 0, 1, 2, \dots$ by repeating the solution of the following **linear** problem

  $$a(w_k; w_{k+1}, \varphi_h) = (f, \varphi_h)_0 \qquad \forall \varphi_h \in V_h \tag{4}$$

  Such a $w_{k+1} \in V_h$ uniquely exists by Lax-Milgram lemma.

- **Minimization method**: minimize the following non-quadratic convex functional on V_h:

  $$J(u) = \frac{1}{2} \int_\Omega \mathcal{N}(x, |\nabla u(x)|^2)\, dx - \int_\Omega f u \, dx$$

  Here $\mathcal{N}$ is a primitive function to ν, i.e., $d\mathcal{N}/ds(x, s) = \nu(x, s)$. There are several methods how to solve the minimization problem numerically:

  – Newton's method
  – generalized conjugate gradient method
  – relaxation method
  – method of successive approximations - this amounts to solving the linear problem (4) by minimization, i.e., minimizing

  $$J_k(w) = \frac{1}{2} a(w_k; w, w) - (f, w)_0 - \frac{1}{2} a(w_k; w_k, w_k) + (f, w_k)_0 + J(w_k),$$

  where the last three constant terms do not have any influence on the minimization but are selected so that $J_k(w_k) = J(w_k)$ for all k. Moreover, it can be shown that the directional derivative of J_k and J is the same at the point w_k. This yields a nice geometric interpretation of the successive approximation method.

6.2 Generalization of the existence theorem

The assumptions of strong monotonicity and Lipschitz continuity in the previous existence theorem are quite limiting. The existence of solution can be proved under much weaker assumptions. One of the possible forms of the theorem is as follows:

Theorem (Main theorem on monotone operators)

Let V be a separable Hilbert space and let $A : V \to V$ be monotone, demi-continuous and coercive operator. Then the set of solutions to $Au = f$ is nonempty, convex and closed for each $f \in V$.

The theorem can be proved constructively by constructing Galerkin approximations (possible because they are defined only on a finite-dimensional subspace) and showing that they converge to a solution of the original problem.

We shall not give the proof but only explain the new terms that appear in the statement of the theorem. We assume that V is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

- Separable space: V is separable if it contains a countable dense subset. The space $H^1(\Omega)$ is separable. An example of a countable dense subset is the set of all polynomials with rational coefficients.

- Coercive operator: $A : V \to V$ is called coercive if

$$\lim_{\|v\|_V \to \infty} \frac{\langle Av, v \rangle}{\|v\|_V} = \infty.$$

  It is clear that a strongly monotone operator is coercive because from the strong monotonicity we have

$$\langle Av - A0, v \rangle \geq \eta \|v\|_V^2 \qquad \forall v \in V$$

  and thus

$$\frac{\langle Av, v \rangle}{\|v\|_V} \geq \frac{\langle A0, v \rangle}{\|v\|_V} + \eta \|v\|_V \geq -\|A0\|_V + \eta \|v\|_V \to \infty \quad \text{for} \quad \|v\|_V \to \infty.$$

- Demi-continuous operator : $A : V \to V$ is demi-continuous if

$$v_k \to v \quad \Rightarrow \quad Av_k \rightharpoonup Av$$

  Here, the arrow $\rightharpoonup$ means **weak convergence** defined as follows:

$$v_k \rightharpoonup v \quad \Leftrightarrow \quad F(v_k) \to F(v) \quad \forall F \in V^*$$

  V^* is the dual space of V, i.e., the space of all continuous linear functionals on V.

  Let us state several basic facts about weak convergence:

  1. Strong convergence implies weak convergence but the converse is not true unless the space is finite-dimensional. An example can be the sequence $\{\sin kx\}_{k=1}^{\infty}$ in $L^2(0, \pi)$ which converges weakly to 0 but does not converge in the norm of $L^2(0, \pi)$.
  2. Uniform boundedness principle: If $\{v_k\} \subset V$ is a sequence such that $F(v_k)$ is bounded for every $F \in V^*$, then $\{\|v_k\|_V\}$ is also bounded.
  3. The following two statements are equivalent

     (i) v_k converges weakly to v in V

     (ii) $\{\|v_k\|_V\}$ is bounded and $\lim_{k\to\infty} \langle v_k, f \rangle = \langle v, f \rangle$ for all f from a set which is dense in V.