1 2D Incompressible Navier Stokes Equation

1.1 Problem

We want to find

$$(u, p) : \Omega \times (0, T) \to \mathbb{R}^d \times \mathbb{R}$$

where u is unknown velocity and p is unknown pressure such that

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \triangle u + \nabla p = f & \text{in } \Omega \times (0, T) \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u = u^0 & \text{in } \Omega, \text{ at } t = 0 \end{cases} \tag{1}$$

where $f : \Omega \times (0, T) \to \mathbb{R}^d$ and $u^0 : \Omega \to \mathbb{R}^d$ are given functions, $\nu > 0$ is a viscosity.

1.2 Weak Form

The weak formulation for equation (1) is shown below. We want to find $\{(u, p)(t) \in V \times Q; t \in (0.T)\}$ such that for $t \in (0, T)$

$$\begin{cases} (\dfrac{\partial u}{\partial t} + (u \cdot \nabla)u, v) + a(u, v) + b(v, p) + b(u, q) = (f, v) & , \forall (v, q) \in V \times Q \\ u = u^0 & , t = 0 \end{cases} \tag{2}$$

where

$$\begin{aligned} a(u, v) &= \nu \int_\Omega \nabla u : \nabla v \; dx \\ b(v, q) &= -\int_\Omega (\nabla \cdot v) q \; dx \\ V &= H_0^1(\Omega, \mathbb{R}^d) = H_0^1(\Omega)^d \\ Q &= \{q \in L^2(\Omega); \int_\Omega q \; dx = 0\}. \end{aligned}$$

1.3 Error estimate

To estimate the error, we use

$$\begin{aligned} E(h, dt) &= max \; \|u_h^n - u^n\| \\ &= max \; \{\|u_{h_1}^n - u_1^n\|_{L^2(\Omega)}^2 + \|u_{h_2}^n - u_2^n\|_{L^2(\Omega)}^2\}^{1/2} \\ &= max \; \{\int_\Omega (u_{h_1}^n - u_1^n)^2 \; dx + \int_\Omega (u_{h_2}^n - u_2^n)^2 \; dx\}^{1/2} \end{aligned}$$

1.4 Simulation

Below, is the exact solution to check if the program is working.

$$\begin{aligned} u &= (u_1, u_2) \\ u_1 &= -cos(x_1) sin(x_2) e^{-4t} \\ u_2 &= -sin(x_1) cos(x_2) e^{-4t} \\ p &= \frac{1}{4}(cos(2x_1) + cos(2x_2)) e^{-4t} \end{aligned}$$

such that equation (1) is satisfied with $f = (f_1, f_2)$.
With $f_1 = -e^{-4t} sin(2x_1)$ and $f_2 = -e^{-4t} sin(2x_2)$