

Progress Report

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3D Incompressible Navier-Stokes

Strong Form

We want to find

$$(u, p) : \Omega \times (0, T) \rightarrow \mathbb{R}^3 \times \mathbb{R}$$

where u is unknown velocity and p is unknown pressure such that

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f & \text{in } \Omega \times (0, T) \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u = u^0 & \text{in } \Omega, \text{ at } t = 0 \end{cases} \quad (1)$$

where $f : \Omega \times (0, T) \rightarrow \mathbb{R}^3$ and $u^0 : \Omega \rightarrow \mathbb{R}^3$ are given functions, choosing $\nu > 0$, $\nu = 1$ is a viscosity.

Weak Form

We want to find $\{(u, p)(t) \in V \times Q; t \in (0, T)\}$ such that for $t \in (0, T)$

$$\begin{cases} \left(\frac{\partial u}{\partial t} + (u \cdot \nabla)u, v \right) + a(u, v) + b(v, p) + b(u, q) = (f, v) & , \\ \forall (v, q) \in V \times Q \\ u = u^0, & t = 0 \end{cases}$$

$$a(u, v) = \nu \int_{\Omega} \nabla u : \nabla v \, dx$$

$$b(v, q) = - \int_{\Omega} (\nabla \cdot v) q \, dx$$

$$V = H_0^1(\Omega, \mathbb{R}^d) = H_0^1(\Omega)^d$$

$$Q = \{q \in L^2(\Omega); \int_{\Omega} q \, dx = 0\}.$$

3D Discretization

First order in time

Before applying to FreeFEM++, we need to discretize

$\frac{\partial u_i}{\partial t} + (u \cdot \nabla) u_i$ part, where dt as time increment.

$$\frac{\partial u_i}{\partial t} + (u \cdot \nabla) u_i \approx \frac{u_i^n - u_i^{n-1}(X_1(u^{n-1}, dt))}{dt} + O(dt)$$

Second order in time / Adam-Bashforth Method

$$\frac{\partial u_i}{\partial t} + (u \cdot \nabla) u_i \approx \frac{3u_i^n - 4u_i^{n-1}(X_1(\tilde{u}^{n-1}, dt)) + u_i^{n-2}(X_1(\tilde{u}^{n-1}, 2dt))}{2 dt} + O(dt^2)$$

where

$$X_1(u^{n-1}, dt)(x) = x - u^{n-1}(x) dt$$

$$\tilde{u}_i^{n-1} = 2u_i^{n-1} - u_i^{n-2}$$

with stabilization term

With $\delta > 0$ and h as mesh size

$$C_i(p, q) = \delta \sum_k h_k^2 (\nabla p, \nabla q)_k$$

L^2

$$\|u_h^n - u^n\|_{L^\infty(L^2)} = \max \|u_h^n - u^n\|_{L^2}$$

H^1

$$\|u_h^n - u^n\|_{L^\infty(H^1)} = \max \sqrt{\|u_{h_1}^n - u_1^n\|_{L^2(\Omega)}^2 + \|\nabla(u_{h_1}^n - u_1^n)\|_{L^2(\Omega)}^2}$$

Exact solution

$$u = (u_1, u_2, u_3)$$

$$u_1 = -\cos(x_1) \sin(x_2) \cos(x_3) e^{-2t}$$

$$u_2 = \sin(x_1) \cos(x_2) \cos(x_3) e^{-2t}$$

$$u_3 = 0$$

$$p = \frac{-1}{4} e^{-4t} (\cos(2x_1) + \cos(2x_2) + \cos(2x_3))$$

such that equation (1) is satisfied with $f = (f_1, f_2, f_3)$. With

$$f_1 = -\cos(x_1) \sin(x_2) \cos(x_3) e^{-2t},$$

$$f_2 = -\sin(x_1) \cos(x_2) \cos(x_3) e^{-2t}, \text{ and}$$

$$f_3 = -\left(\frac{1}{4}\right) e^{-4t} \sin(2x_3) (2 \cos(2x_3) + 1)$$

Error Estimate L_2

With $c = \frac{\sqrt{2}}{4}$, we choose $dt = c\sqrt{h} = \frac{c}{\sqrt{n}}$ such that

$$O(dt^2) + O(h) = O(h) + O(h) = O(h)$$

Error Estimate H_1

With $c = \frac{\sqrt{2}}{4}$, we choose $dt = c\sqrt{h} = \frac{c}{\sqrt{n}}$ such that

$$O(dt^2) + O(h^2) = O(h) + O(h^2) = O(h)$$

Plot of L2 and H1 error for 3D Navier-Stokes

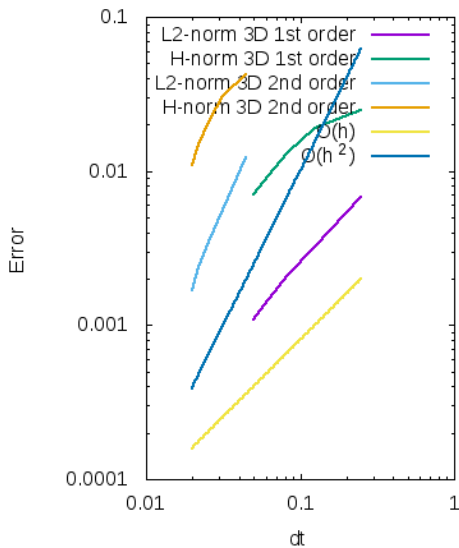


Figure:

Using the first order in time for the first iteration, and then second order in time for the rest, we obtain :

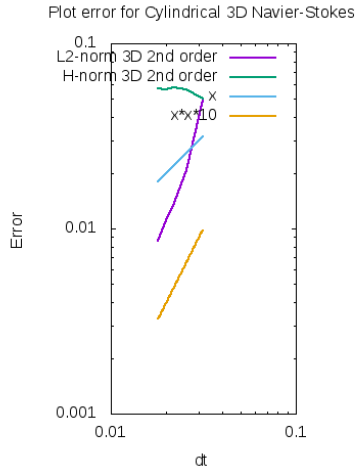


Figure:

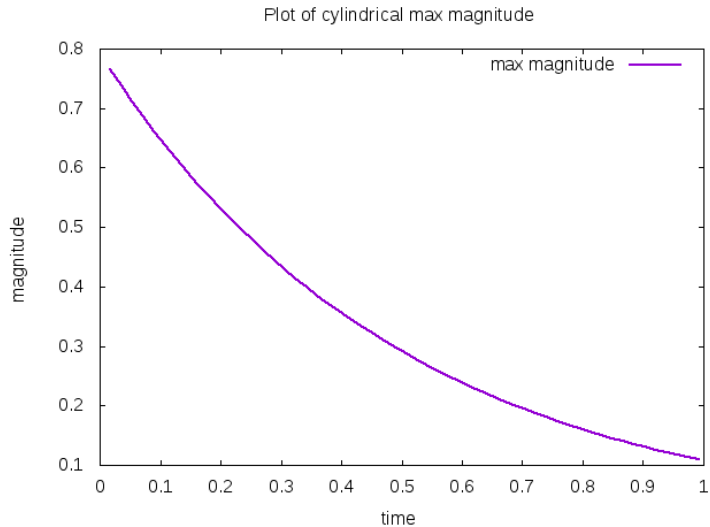


Figure:

Tornado simulation on cylindrical domain

Domain and initial condition

Taking $a = 1/8$, $\epsilon_i = 1$, $\beta_i = 1$ ($i = 1, \dots, 6$), with domain $\Omega = \{x = (x, y, z) \in \mathbb{R}^3; -a \leq z \leq 4a, \sqrt{x^2 + y^2} < 1\}$ and $u = 0$ on boundary.

$$\psi(a, \epsilon, \sigma) = (a^2 + \epsilon)^\sigma$$

$$u_z = \psi(r, \epsilon_1, -\beta_1)\psi(z, \epsilon_2, -\beta_2)$$

$$\rho = \psi(r, \epsilon_3, -\beta_3)\psi(z, \epsilon_4, \beta_4)$$

$$u_\theta = \psi(r, \epsilon_5, -\beta_5)\psi(z, \epsilon_6, -\beta_6) \quad (\text{with swirl})$$

$$u_\theta = 0 \quad (\text{no swirl})$$

$$u_r = \text{sign}(z)\rho u_z$$

Tornado simulation on cylindrical and curved cylindrical domain

Exact solution

$$\begin{aligned}u &= (u_1, u_2, u_3) \\u_1 &= -\cos(x_1) \sin(x_2) e^{-2t} \\u_2 &= \sin(x_1) \cos(x_2) e^{-2t} \\u_3 &= 0 \\p &= \frac{-1}{4} e^{-4t} (\cos(2x_1) + \cos(2x_2))\end{aligned}$$

such that equation (1) is satisfied with $f = (f_1, f_2, f_3)$. With $f_1 = 0$, $f_2 = 0$, and $f_3 = 0$. Taking $\nu = 2/3$ which satisfy the strong formulation

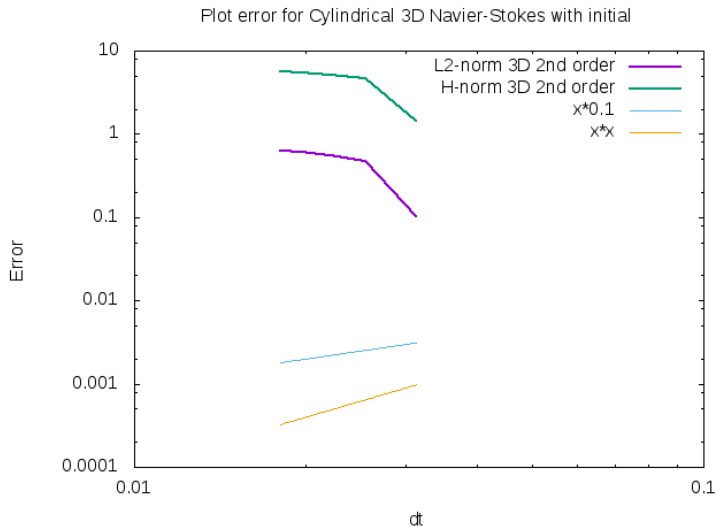


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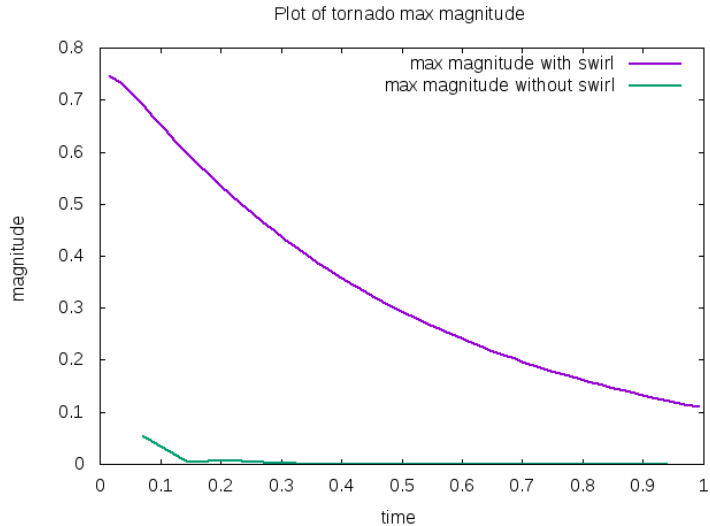
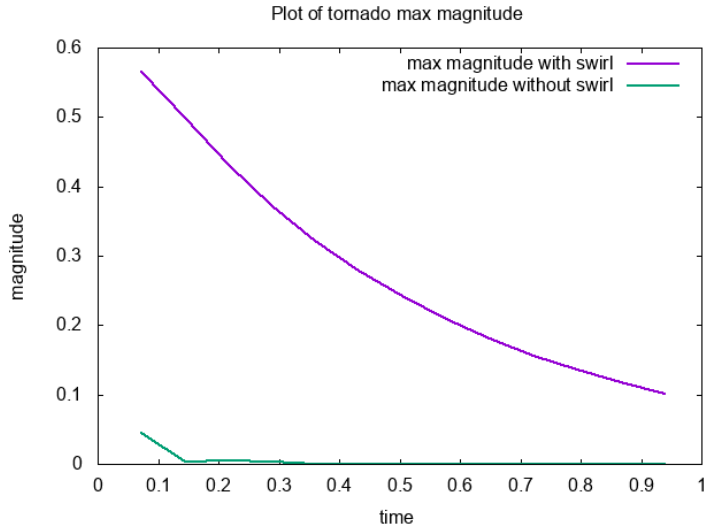


Figure: Max v of tornado simulation every time step

Tornado simulation on curved cylindrical domain

Using FreeFEM++ (applying Kazunori's ideas)



Plot error for Curved Cylindrical 3D Navier-Stokes with initial

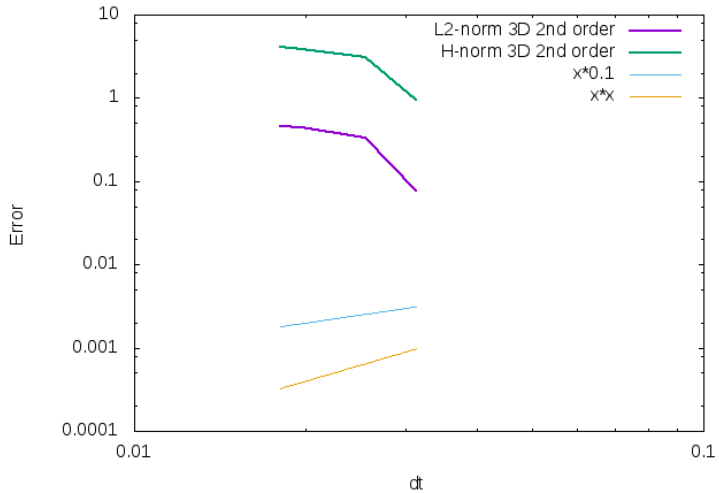


Figure: