Progress Report

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3D Incompressible Navier-Stokes

Strong Form

We want to find

$$(u, p) : \Omega \times (0, T) \to \mathbb{R}^3 \times \mathbb{R}$$

where u is unknown velocity and p is unknown pressure such that

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \triangle u + \nabla p = f & \text{in } \Omega \times (0, T) \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u = u^0 & \text{in } \Omega, \text{ at } t = 0 \end{cases} \tag{1}$$

where $f : \Omega \times (0, T) \to \mathbb{R}^3$ and $u^0 : \Omega \to \mathbb{R}^3$ are given functions, choosing $\nu > 0, \nu = 1$ is a viscosity.

Weak Form

We want to find $\{(u,p)(t) \in V \times Q; t \in (0.T)\}$ such that for $t \in (0, T)$

$$\begin{cases} (\frac{\partial u}{\partial t} + (u \cdot \nabla)u, v) + a(u,v) + b(v,p) + b(u,q) = (f,v) & , \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall (v,q) \in V \times Q & \\ u = u^0, & t = 0 \end{cases}$$

$$\begin{aligned} a(u,v) &= \nu \int_\Omega \nabla u : \nabla v \; dx \\ b(v,q) &= -\int_\Omega (\nabla \cdot v) q \; dx \\ V &= H_0^1(\Omega, \mathbb{R}^d) = H_0^1(\Omega)^d \\ Q &= \{q \in L^2(\Omega); \int_\Omega q \; dx = 0\}. \end{aligned}$$

3D Discretization

First order in time

Before applying to FreeFEM++, we need to discritize $\dfrac{\partial u_i}{\partial t} + (u \cdot \nabla)u_i$ part, where dt as time increment.

$$\frac{\partial u_i}{\partial t} + (u \cdot \nabla)u_i \approx \frac{u_i^n - u_i^{n-1}(X_1(u^{n-1}, dt))}{dt} + O(dt)$$

Second order in time / Adam-Bashforth Method

$$\frac{\partial u_i}{\partial t} + (u \cdot \nabla)u_i \approx$$

$$\frac{3u_i^n - 4u_i^{n-1}(X_1(\tilde{u}^{n-1}, dt)) + u_i^{n-2}(X_1(\tilde{u}^{n-1}, 2dt))}{2\ dt} + O(dt^2)$$

where

$$X_1(u^{n-1}, dt)(x) = x - u^{n-1}(x)\ dt$$

$$\tilde{u}_i^{n-1} = 2u_i^{n-1} - u_i^{n-2}$$

with stabilization term

With $\delta > 0$ and h as mesh size

$$C_i(p, q) = \delta \sum_k h_k^2 (\nabla p, \nabla q)_k$$

Error estimate

L^2

$$\|u_h^n - u^n\|_{\ell\infty(L^2)} = max \; \|u_h^n - u^n\|_{L^2}$$

with $O(h^2)$.

H^1

$$\|u_h^n - u^n\|_{\ell\infty(H^1)} = max \sqrt{\|u_h^n - u^n\|_{L^2(}^2 + \|\nabla(u_h^n - u^n)\|_{L^2}^2}$$

with $O(h)$

Exact solution

$$
\begin{aligned}
u &= (u_1, u_2, u_3) \\
u_1 &= -\cos(x_1)\sin(x_2)\cos(x_3)e^{-2t} \\
u_2 &= \sin(x_1)\cos(x_2)\cos(x_3)e^{-2t} \\
u_3 &= 0 \\
p &= \frac{-1}{4}e^{-4t}(\cos(2x_1) + \cos(2x_2) + \cos(2x_3))
\end{aligned}
$$

such that equation (1) is satisfied with $f = (f_1, f_2, f_3)$. With
$f_1 = -\cos(x_1)\sin(x_2)\cos(x_3)e^{-2t}$,
$f_2 = -\sin(x_1)\cos(x_2)\cos(x_3)e^{-2t}$, and
$f_3 = -(\frac{1}{4})e^{-4t}\sin(2x_3)(2\cos(2x_3) + 1)$

Error Estimate H^1

With $c = \frac{\sqrt{2}}{4}$, we choose $dt = c\sqrt{h} = \frac{c}{\sqrt{n}}$ such that

$$O(dt^2) + O(h) = O(h) + O(h) = O(h)$$

Error Estimate L^2

With $c = \frac{\sqrt{2}}{4}$, we choose $dt = c\sqrt{h} = \frac{c}{\sqrt{n}}$ such that

$$O(dt^2) + O(h^2) = O(h) + O(h^2) = O(h)$$

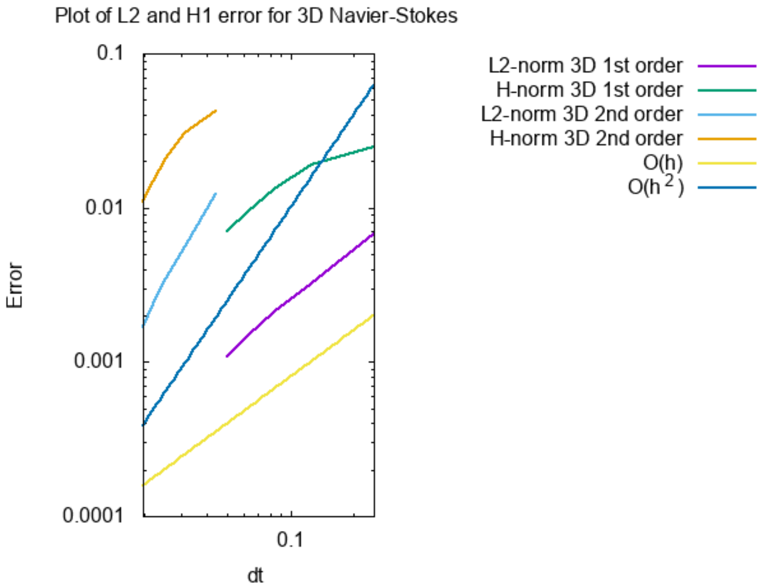

Plot of L2 and H1 error for 3D Navier-Stokes
0.1
0.01
0.001
0.0001
Error
0.1
dt
L2-norm 3D 1st order
H-norm 3D 1st order
L2-norm 3D 2nd order
H-norm 3D 2nd order
O(h)
O(h 2)

Figure:

Using the first order in time for the first iteration, and then second order in time for the rest, we obtain :

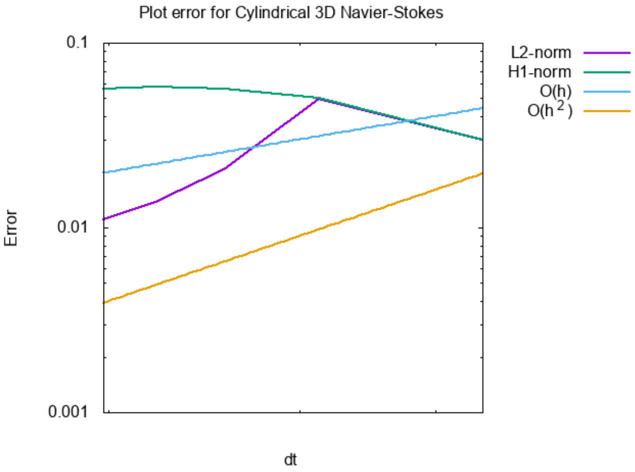


Figure:

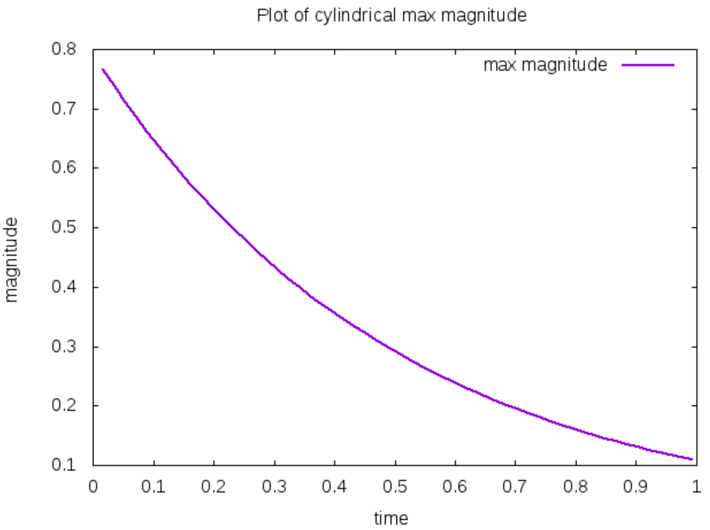

Plot of cylindrical max magnitude
max magnitude
magnitude
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
time

Figure:

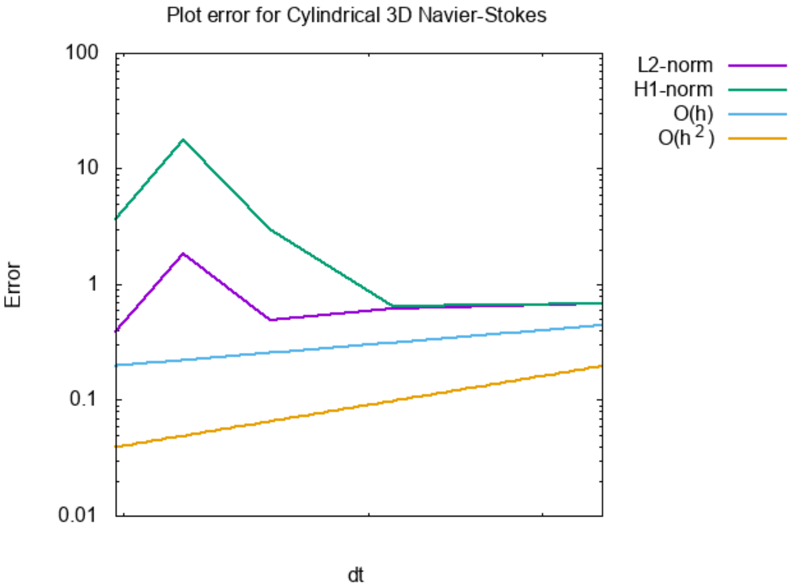
Other trial on exact

Exact solution

with $c = 8\sqrt{3}/27\pi$

$$
\begin{aligned}
u1 &= c\sin(\pi x)\sin^2(\pi y)\sin^2(\pi z)\sin(\pi(y+z+t)) \\
u2 &= c\sin^2(\pi x)\sin(\pi y)\sin^2(\pi z)\sin(\pi(x+z+t)) \\
u1 &= c\sin^2(\pi x)\sin^2(\pi y)\sin(\pi z)\sin(\pi(x+y+t)) \\
p &= \sin(\pi(x+y+z+t))
\end{aligned}
$$

Such that, for the cylindrical domain, we obtain the error



Tornado simulation on cylindrical domain

Domain and initial condition

Taking $a = 1/8, \epsilon_i = 1, \beta_i = 1$ $(i = 1, \ldots, 6)$, with domain $\Omega = \{x = (x, y, z) \in \mathbb{R}^3; -a \leq z \leq 4a,\ \sqrt{x^2 + y^2} < 1\}$ and $u = 0$ on boundary.

$$\begin{cases} \psi(a, \epsilon, \sigma) & = (a^2 + \epsilon)^\sigma \\ u_z & = \psi(r, \epsilon_1, -\beta_1)\psi(z, \epsilon_2, -\beta_2) \\ \rho & = \psi(r, \epsilon_3, -\beta_3)\psi(z, \epsilon_4, \beta_4) \\ u_0 & = \psi(r, \epsilon_5, -\beta_5)\psi(z, \epsilon_6, -\beta_6) \qquad \text{(with swirl)} \\ u_0 & = 0 \qquad\qquad\qquad\qquad\qquad\quad \text{(no swirl)} \\ u_r & = sign(z)\rho u_z \end{cases} \tag{2}$$

Tornado simulation on cylindrical and curved cylindrical domain

Exact solution

$$
\begin{aligned}
u &= (u_1, u_2, u_3) \\
u_1 &= -\cos(x_1)\sin(x_2)e^{-2t} \\
u_2 &= \sin(x_1)\cos(x_2)e^{-2t} \\
u_3 &= 0 \\
p &= \frac{-1}{4}e^{-4t}(\cos(2x_1) + \cos(2x_2))
\end{aligned}
$$

such that equation (1) is satisfied with $f = (f_1, f_2, f_3)$. With $f_1 = 0$, $f_2 = 0$, and $f_3 = 0$. Taking $\nu = 2/3$ which satisfy the strong formulation

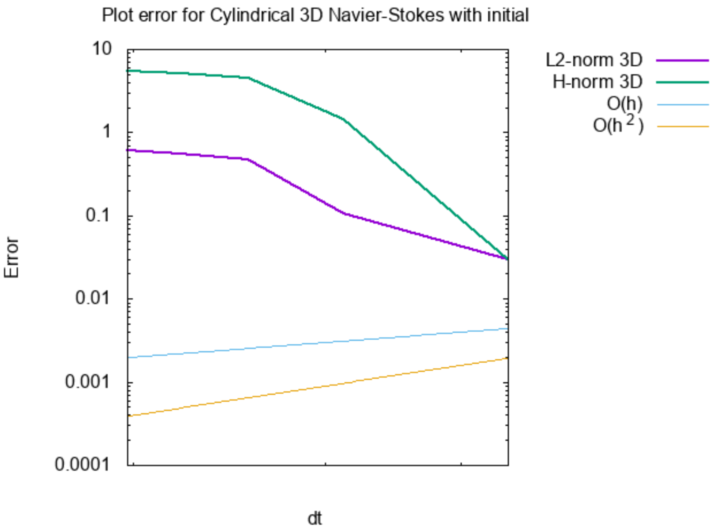

Plot error for Cylindrical 3D Navier-Stokes with initial
10
1
0.1
0.01
0.001
0.0001
Error
dt
L2-norm 3D
H-norm 3D
O(h)
O(h 2)

Figure:

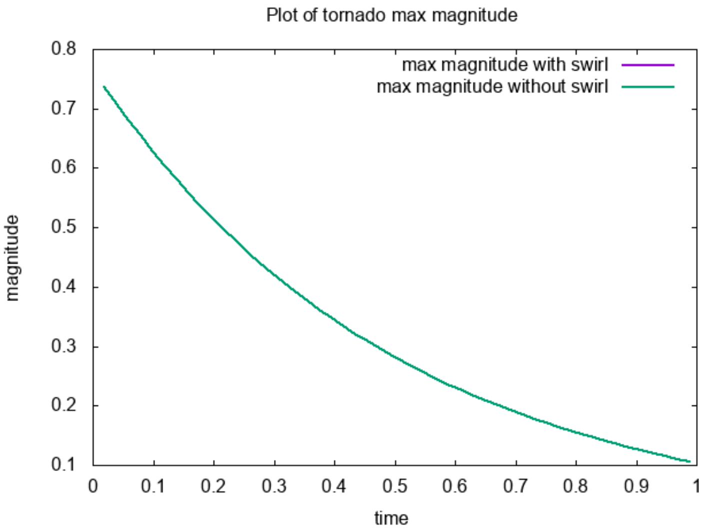
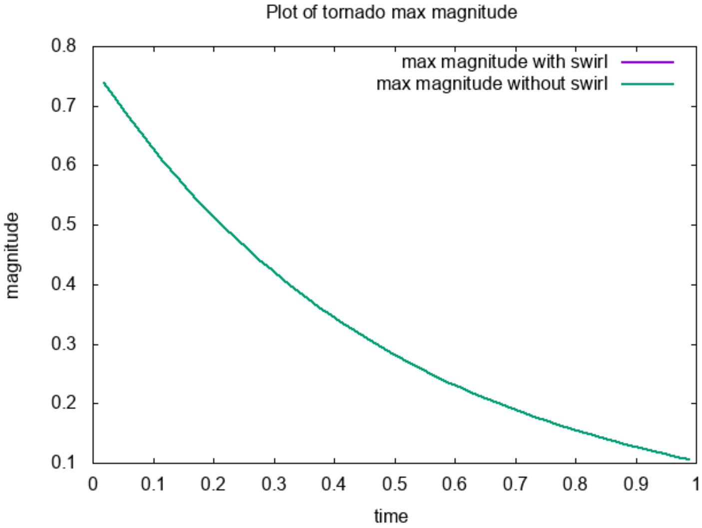


Figure: Max v of tornado simulation every time step

Tornado simulation on curved cylindrical domain

Using FreeFEM++ (applying Kazunori's ideas)



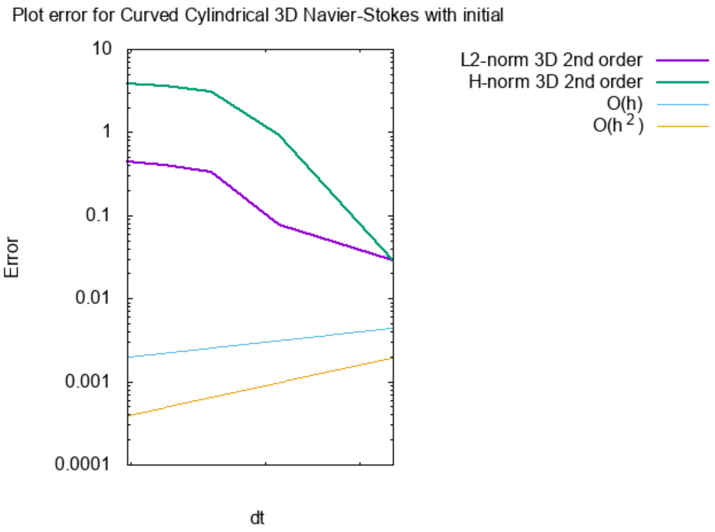

Plot error for Curved Cylindrical 3D Navier-Stokes with initial
10
1
0.1
0.01
0.001
0.0001
Error
dt
L2-norm 3D 2nd order
H-norm 3D 2nd order
O(h)
O(h 2)

Figure:

Labelling the domain cylinder

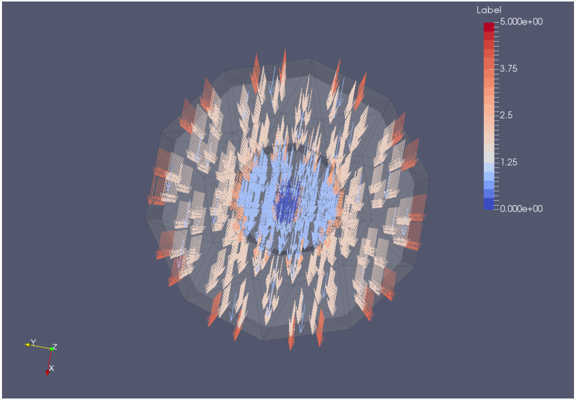


Figure: Labelling

Which is in the program is written as
rup=[0,1,1,1,2,1], rdown=[0,1,1,1,2,1], rmid=[1,4,2,3,3,3];

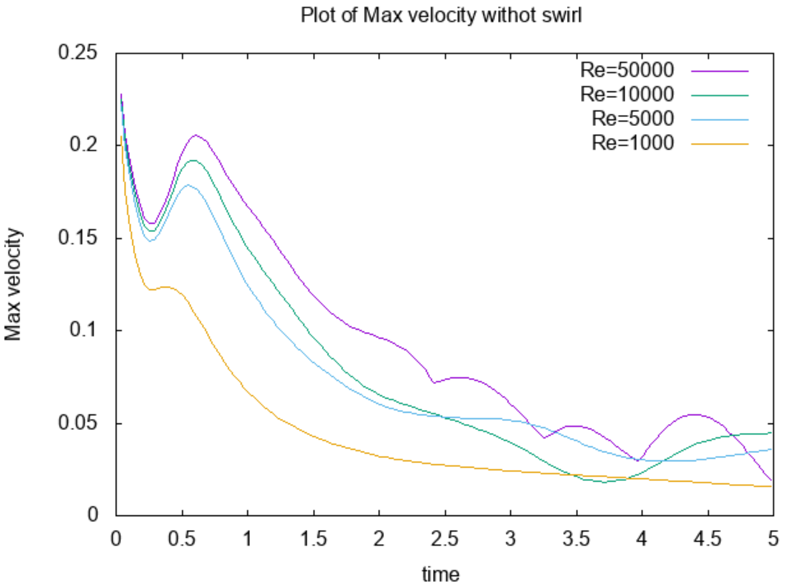
Simulation with initial and Reynolds number

Navier-Stokes problem

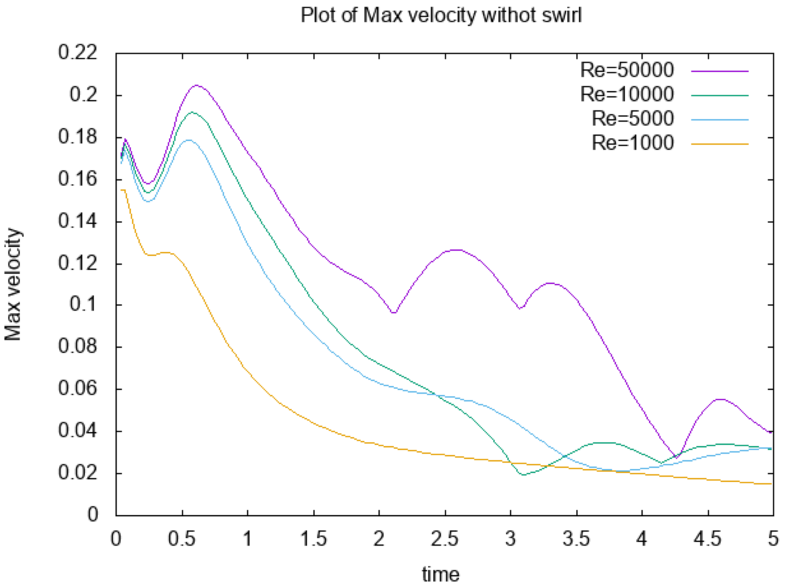
$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \nu \triangle u + \nabla p & = 0 \\ u\big|_{t=0} & = u_0 \\ u\big|_{\partial\Omega} & = 0 \\ \nabla \cdot u & = 0 \end{cases}$$

with the initial in (2). We do the simulation for $\nu = \dfrac{1}{Re}$, where $Re = 50000, 10000, 5000, 1000$. With $T = 5$, $h = 1/n = 1/24$ and $dt = c\sqrt{h}$ with $c = \sqrt{2}/8$.

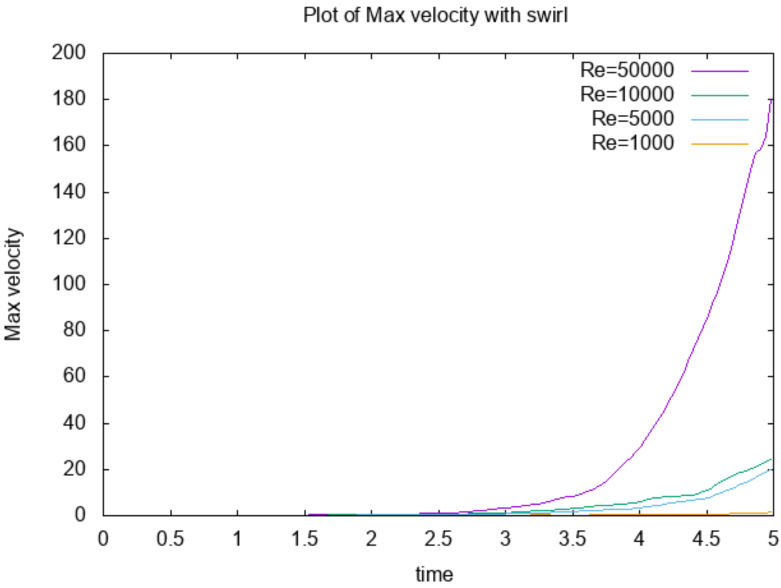
Cylindrical domain with swirl



Cylindrical domain without swirl



Curved cylindrical domain with swirl



Curved cylindrical domain without swirl