Nonlinear PDEs

Introduction

Where do PDEs stand

Physical reality

a phenomenon that we want to understand or control

↓

Mathematical model: PDE

extracting the physical essence: derive relations between quantities

↓

Discrete model

discretize the mathematical model into a finite-dimensional one

↓

Numerical results

solve the discrete model using computer

Errors

Physical reality

a phenomenon that we want to understand or control

$\downarrow e_0$

Mathematical model: PDE

extracting the physical essence: derive relations between quantities

$\downarrow$

Discrete model

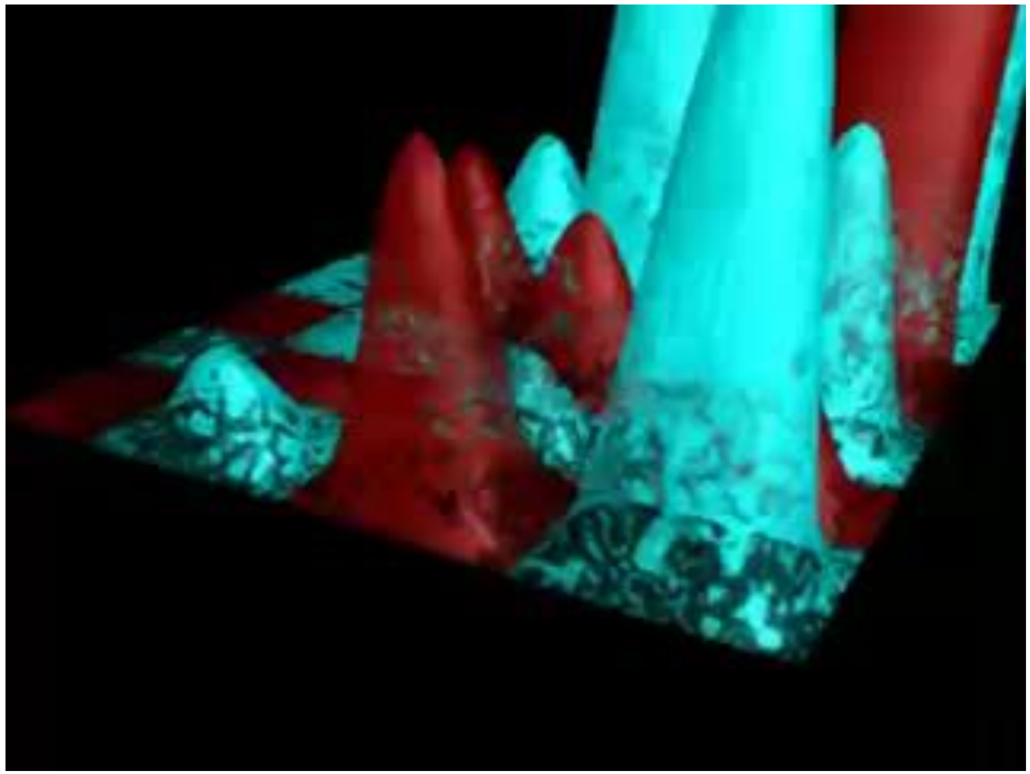
discretize the mathematical model into a finite-dimensional one

$\downarrow$

Numerical results

solve the discrete model using computer

Quantum physics • wave function



(Youtube)

$iu_t + \Delta u = 0$ ... Schrödinger equation

Error e_0

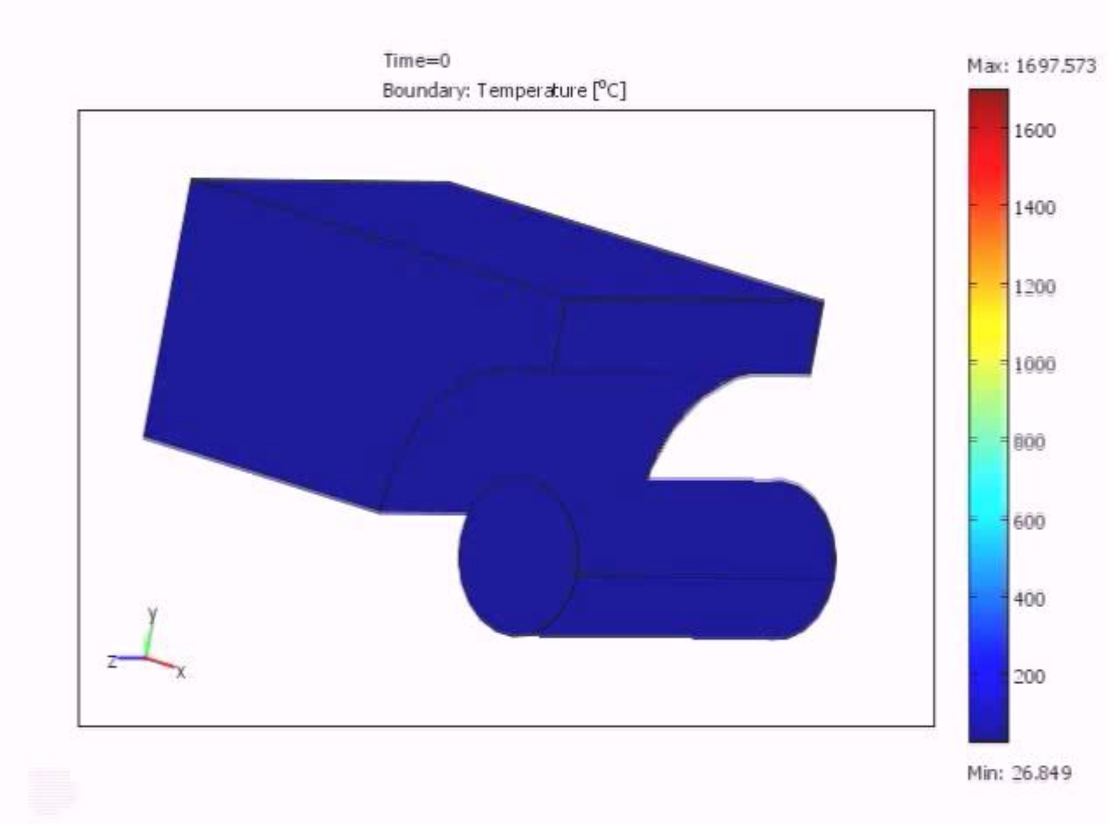
- Heisenberg uncertainty principle

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}$$

- Born's interpretation of wavefunction:
  - as probability distribution

- Example: throwing stone

Approximation by continuum



$$\frac{\partial u}{\partial t} = \Delta u + f(u)$$ ... heat equation

Errors

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e_1

Discrete model

discretize the mathematical model into a finite-dimensional one

Numerical results

solve the discrete model using computer

Error e_1

- Examples:
  - error of finite element method
  - error of numerical quadrature
  - error of approximation of curved boundary by a polygon
  - error of approximation of nonlinearities
  - error of approximation of initial or boundary conditions

Errors

Physical reality

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extracting the physical essence: derive relations between quantities

Discrete model

discretize the mathematical model into a finite-dimensional one

e_2

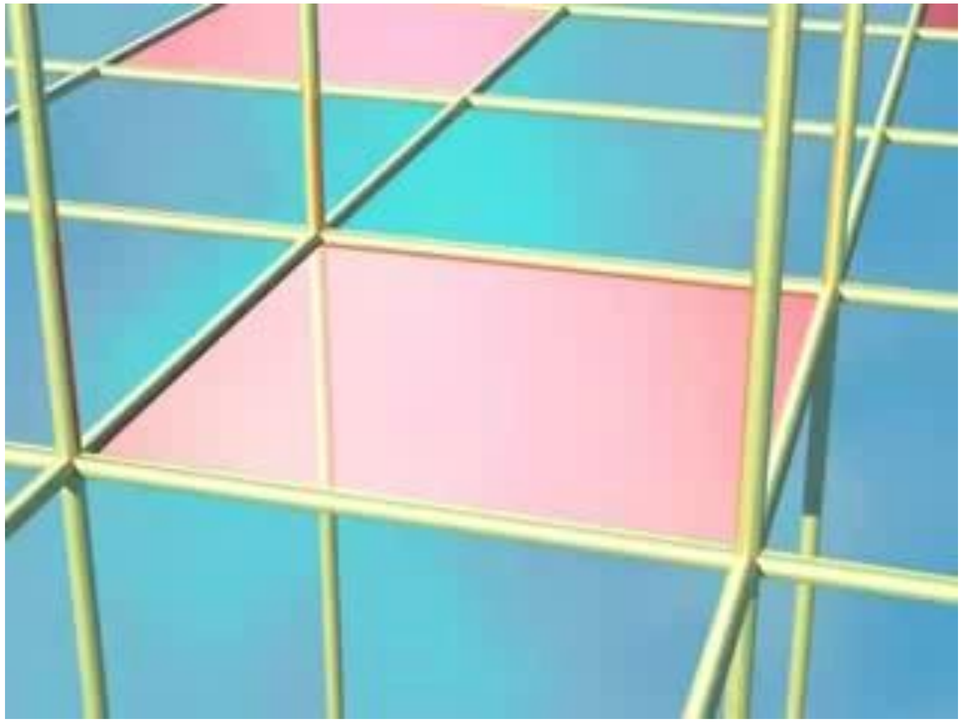
Numerical results

solve the discrete model using computer

Error e_2

- Rounding error
  - machine epsilon (floating point arithmetic)
- Error of iteration

Sound/light waves, string vibration…



(Youtube)

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + f(u)$$ … wave equation

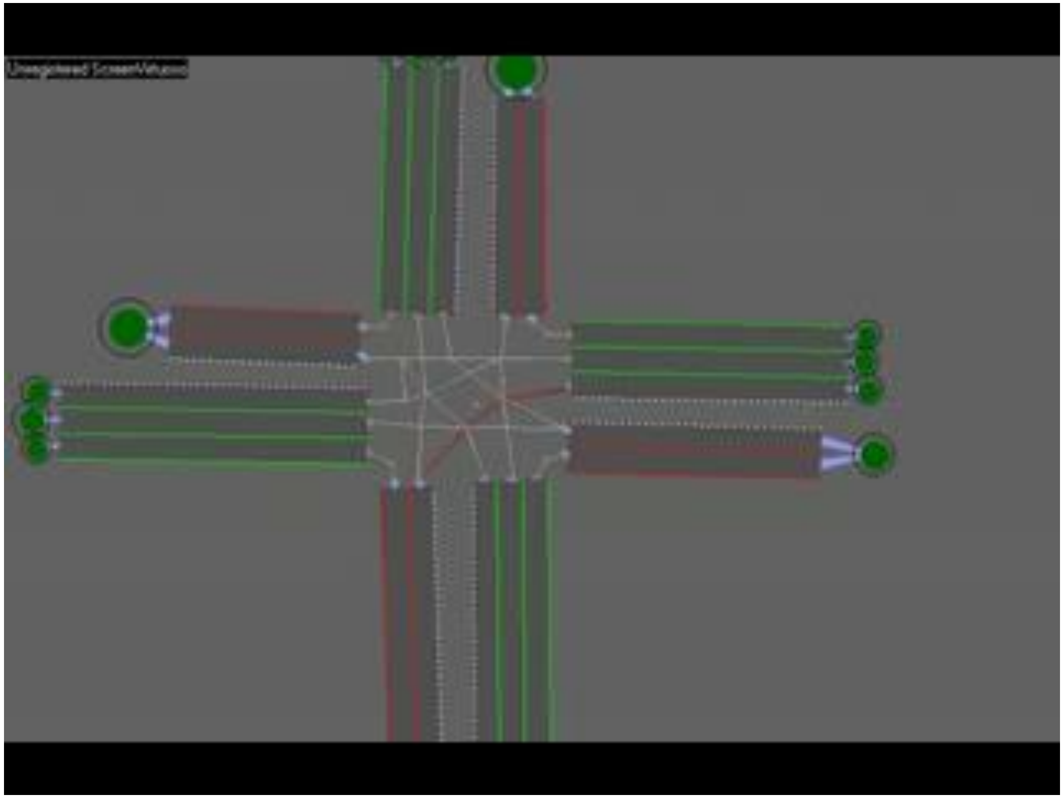
Soap film

(Youtube)

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$$

... minimal surface equation

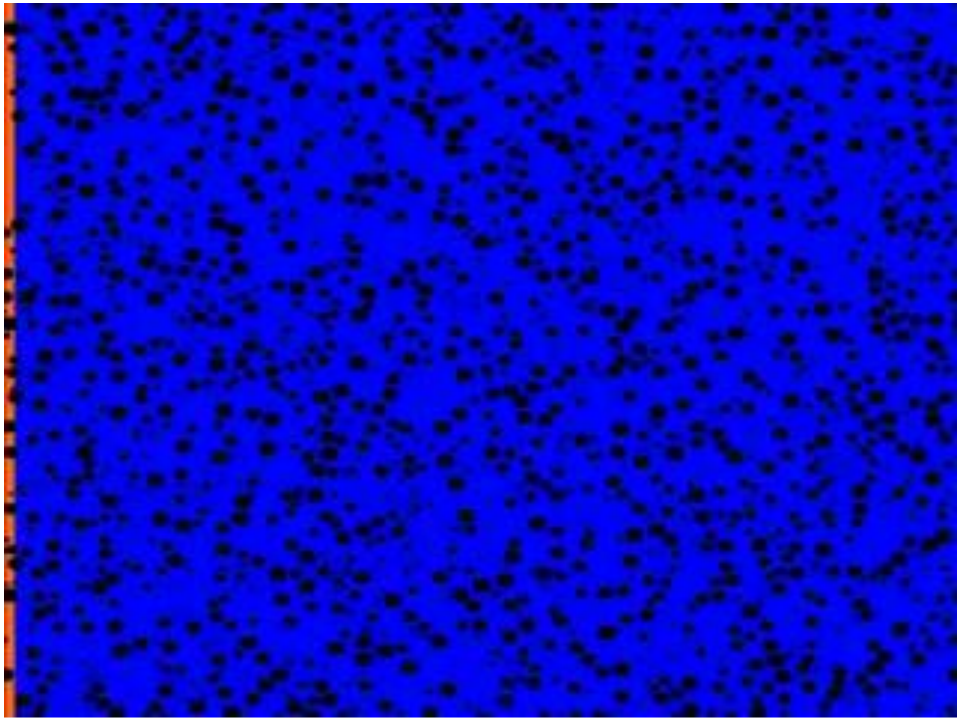
Traffic flow



(Youtube)

$$u_t + uu_x = 0 \qquad \ldots \text{ Burgers' equation}$$

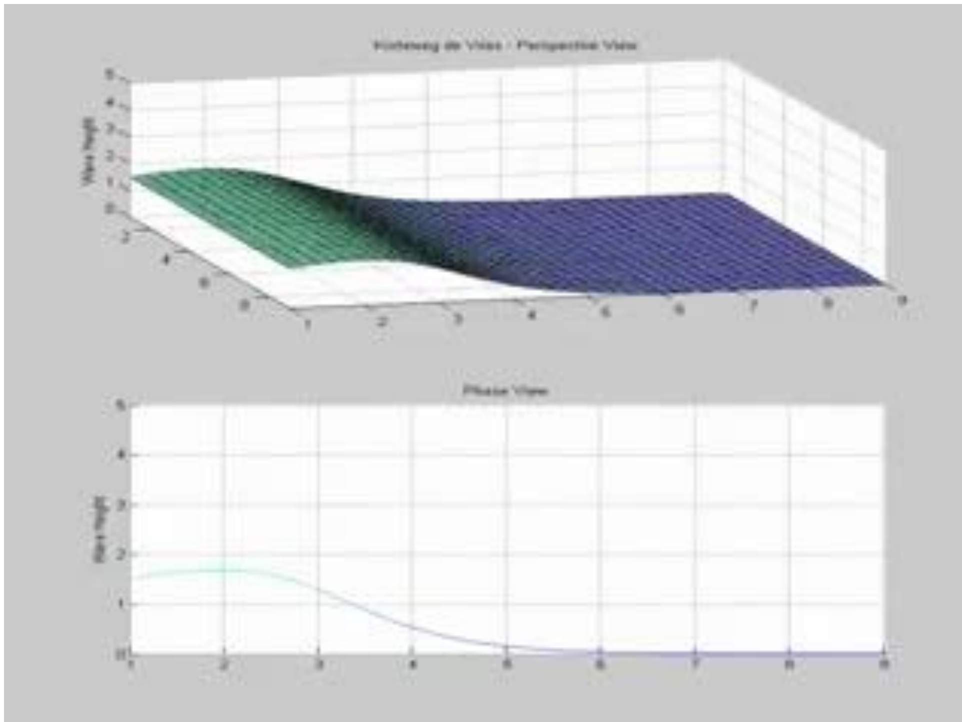
Porous medium



(Youtube)

$$u_t - \Delta(u^\gamma) = 0$$ … porous media equation

Tsunami



(Youtube)

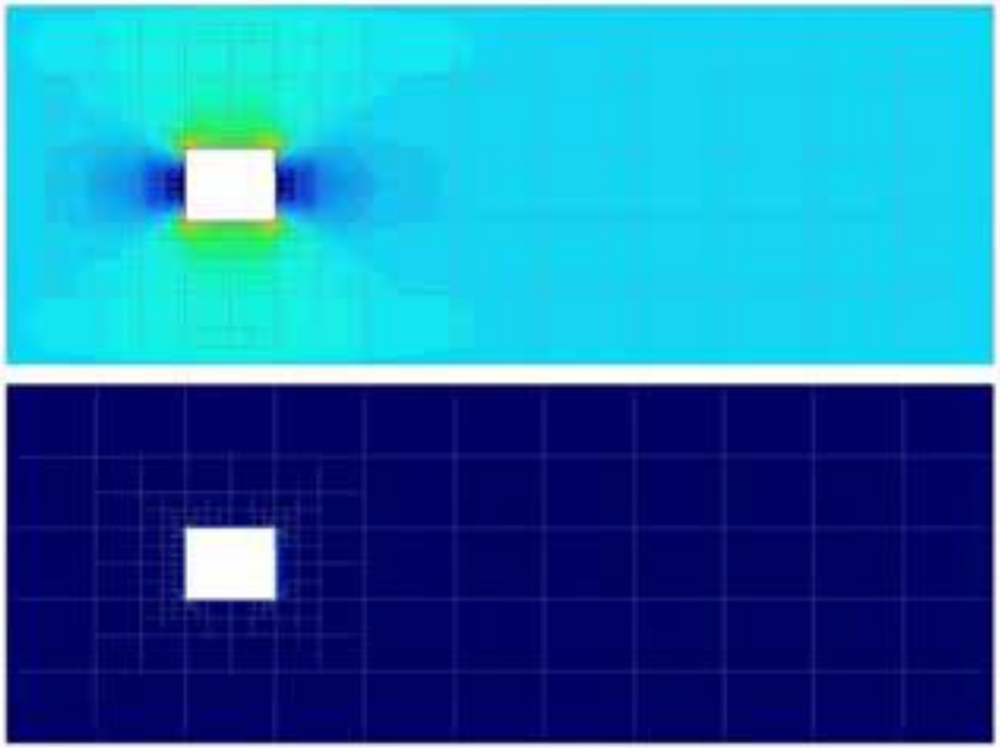
$$u_t + uu_x + u_{xxx} = 0$$ … Korteweg-de Vries equation

Chemical reactions

(Youtube)

$$u_t - \Delta u = f(u)$$ ... Gray-Scott system

Fluid flow



(Youtube)

$$\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \Delta \boldsymbol{u} = -\nabla p$$

$$\operatorname{div} \boldsymbol{u} = 0$$

... Navier-Stokes equations

Superconductor

(Youtube)

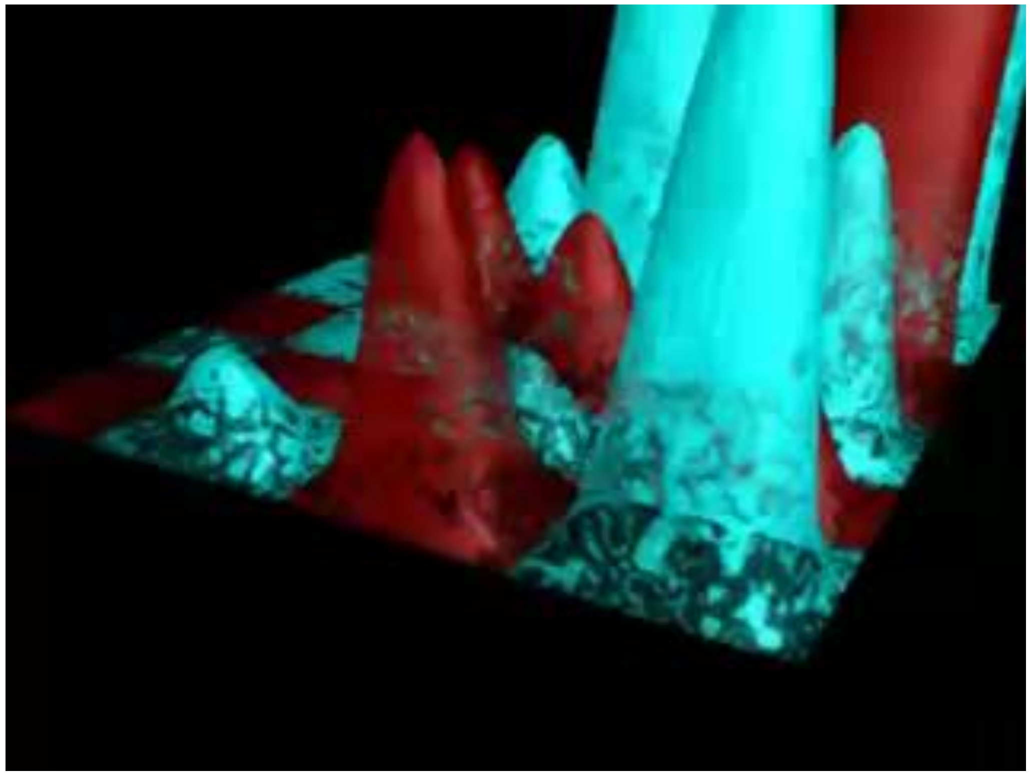
$$iu_t + pu_{xx} + q|u|^2u = i\gamma u$$ ... Ginzburg-Landau equation

Black hole formation

(Youtube)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = \frac{8\pi G}{c^4} T_{\mu\nu}$$ … Einstein's field equations

Quantum physics • wave function



(Youtube)

$$iu_t + \Delta u = 0$$ … Schrödinger equation

Contents of this lecture

- In this lecture we will study the cases of
  - nonlinear stationary magnetic field
  - nonlinear problem of heat radiation
- and focus on
  - weak formulation of the model equation
  - the proof of existence of solution to these problems
  - constructing a practical method for numerical computation of these problems
  - examining the convergence and the speed of convergence of the numerical method

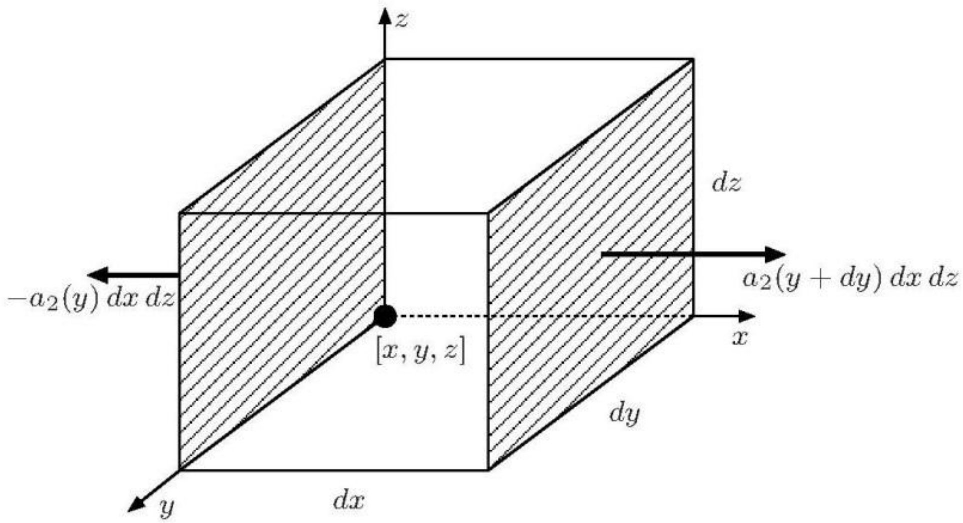
Maxwell's equations

$$\begin{aligned}
\operatorname{rot} H &= J + \frac{\partial D}{\partial t}, && \text{(Ampère's law)} \\
\operatorname{rot} E &= -\frac{\partial B}{\partial t}, && \text{(Faraday's law)} \\
\operatorname{div} D &= \rho, && \text{(Gauss's law)} \\
\operatorname{div} B &= 0, && \text{(Gauss's law for electromagnetism)} \\
D &= \varepsilon E, && \text{(constitutive relation)} \\
B &= \mu H. && \text{(constitutive relation)}
\end{aligned}$$

H	...	magnetic field intensity [A/m]
E	...	electric field intensity [V/m]
D	...	electric induction (or flux density) [C/m^2]
B	...	magnetic induction (or flux density) [Wb/m^2]
J	...	current density [A/m^2]
ρ	...	charge density [C/m^3]
μ	...	permeability
ε	...	permittivity .

Divergence

$$\text{div } a = \sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i} \qquad \left(\text{div } a = \text{div } (a_1, a_2) = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} \quad \text{for } \text{ 2-dim case}\right)$$



Flux in y-direction:

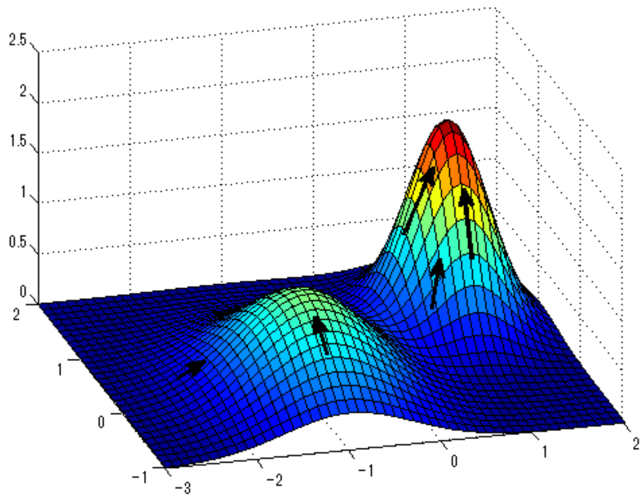
$$a_2(y + dy)\, dx\, dz - a_2(y)\, dx\, dz = \left(a_2(y) + \frac{\partial a_2}{\partial y} dy\right) dx\, dz - a_2(y)\, dx\, dz = \frac{\partial a_2}{\partial y}\, dx\, dy\, dz$$

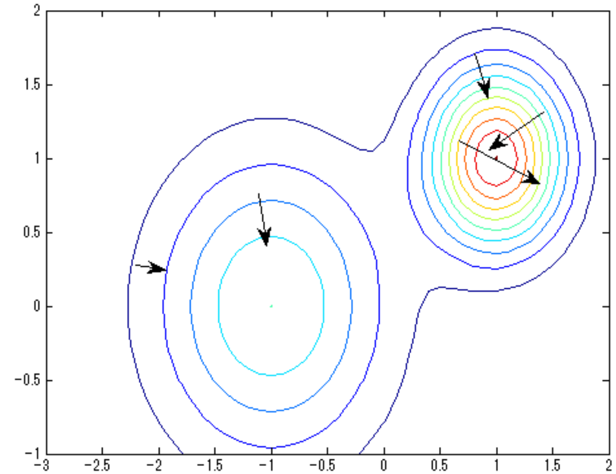
Similarly in x,z directions ⇨ total flux $\left(\frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}\right) dx\, dy\, dz$

Divergence = the extent to which the vector field flow behaves like a source or a sink at a given point – how much more is exiting an infinitesimal region of space than entering it.

Gradient

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right)^T \qquad \left(\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right)^T \quad \text{for 2-dim case}\right)$$



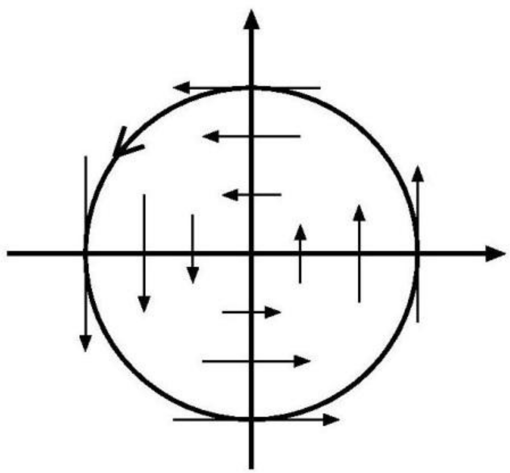


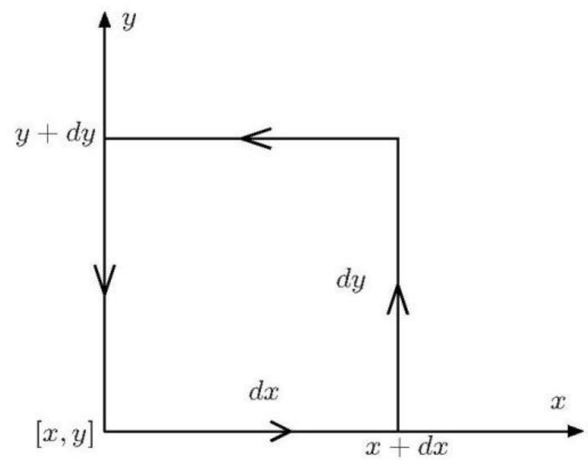
- **Rate of change** of u w.r.t. distance in a particular direction is the projection of the gradient onto that direction.
- Gradient points in the direction of **greatest change** of u and has the magnitude equal to the rate of change of u w.r.t distance in that direction.
- Gradient is everywhere **normal to the contour** lines.

Rotation

$$\text{rot } v = \text{rot } (v_1, v_2, v_3) = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)^T = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$\text{rot } (-y, x, 0) = (0, 0, 2)$





$$a_1(y)\,dx + a_2(x+dx)\,dy - a_1(y+dy)\,dx - a_2(x)\,dy = a_1(y)\,dx + \left(a_2(x) + \frac{\partial a_2}{\partial x}\,dx\right)dy - \left(a_1(y) + \frac{\partial a_1}{\partial y}\,dy\right)dx - a_2(x)\,dy$$

$$= \left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right)dx\,dy$$

Circulation of a vector around a closed curve is the line integral along this curve.
Rotation of a field represents the vorticity, or circulation per unit area, of the field.