Nonlinear PDEs

3rd lecture

Vector space

- Is a set V satisfying
  - linearity (α is a real number)

$$v, w \in V \quad \Rightarrow v + w \in V \;\text{ and }\; \alpha v \in V$$

  - conditions on operations

$$v + w = w + v$$

$$v + (w + z) = (v + w) + z$$

$$\alpha(v + w) = \alpha v + \alpha w$$

$$(\alpha + \beta)v = \alpha v + \beta v$$

$$\alpha(\beta v) = (\alpha\beta)v$$

$$1 \cdot v = v$$

$$v + w = v + z \quad \Rightarrow \quad w = z$$

Norm

- **Norm** on V is a non-negative real function $\|\cdot\|_V$ fulfilling the conditions:

$$\|v+w\|_V \leq \|v\|_V + \|w\|_V \qquad \text{(triangle inequality)}$$

$$\|\alpha v\|_V = |\alpha| \, \|v\|_V$$

$$\|v\|_V \neq 0 \ \text{ for } \ v \neq 0$$

- **Normed linear vector space** is a linear vector space equipped with a norm.

Banach space

- A sequence $\{v_k\}_{k=1}^{\infty} \subset V$ is called a **Cauchy sequence** provided for each $\epsilon > 0$ there exists $N > 0$ such that

$$\|v_k - v_l\|_V < \epsilon \qquad \text{for all } \ k, l \geq N$$

- V is **complete** if each Cauchy sequence in V converges.
  That is, whenever $\{v_k\}_{k=1}^{\infty}$ is a Cauchy sequence, there exists $v \in V$ such that $\{v_k\}_{k=1}^{\infty}$ converges to v.

- A **Banach space** V is a complete, normed linear space.

L^p spaces

For $1 \leq p \leq \infty$, open subset $\Omega \subset \mathbb{R}^n$ and a measurable function $f : \Omega \to \mathbb{R}$ we define

$$\|f\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |f|^p \, dx \right)^{1/p} & \text{if } 1 \leq p < \infty \\ \text{ess sup}_\Omega |f| & \text{if } p = \infty \end{cases}$$

Then $L^p(\Omega)$ is the linear space:

$$L^p(\Omega) = \{ f : \Omega \to \mathbb{R} \mid \ \|f\|_{L^p(\Omega)} < \infty \}$$

$L^p(\Omega)$ is a Banach space, provided we identify functions which agree almost everywhere.

Lemma. (Hölder's inequality)

Let $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then if $u \in L^p(\Omega), v \in L^q(\Omega)$, we have

$$\int_\Omega |uv| \, dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}$$

Hilbert spaces

Let H be a real linear space. A mapping $(\ ,\ ) : H \times H \to \mathbb{R}$ is called an **inner product** if

(i) $(u, v) = (v, u) \ \ \forall u, v \in H$
(ii) the mapping $u \mapsto (u, v)$ is linear for each $v \in H$
(iii) $(u, u) \geq 0$ for all $u \in H$
(iv) $(u, u) = 0$ if and only if $u = 0$

If $(\ ,\ )$ is an inner product, the associated norm is

$$\|u\|_H = (u, u)^{1/2}, \qquad u \in H.$$

The **Cauchy-Schwarz inequality** states

$$|(u, v)| \leq \|u\|_H \, \|v\|_H, \qquad u, v \in H.$$

Hilbert space is a Banach space endowed with an inner product.

Bounded linear operators

A mapping $A : X \to Y$ (X, Y are normed spaces) is a **linear operator** provided

$$A(\lambda u + \mu v) = \lambda A(u) + \mu A(v) \qquad \forall u, v \in X, \ \ \forall \lambda, \mu \in \mathbb{R}$$

A linear operator $A : X \to Y$ is **bounded** if

$$\|A\| = \sup_{\|u\|_X \leq 1} \|A(u)\|_Y = \sup_{u \in X} \frac{\|A(u)\|_Y}{\|u\|_X} < \infty$$

If $Y = \mathbb{R}$, then we call the operator $A : X \to \mathbb{R}$ a **functional.**

X^* = collection of all bounded linear functionals on X = **dual space** of X

<u>Theorem</u> (Riesz representation theorem) Let H be a real Hilbert space. Then for each $F \in H^*$ there exists a unique element $f \in H$ such that

$$F(v) = (f, v) \qquad \forall v \in H$$

The mapping $F \mapsto f$ is a linear isomorphism of H^* onto H.