# 1 18-04-18

This research is collaboration with Prof. Yoneda from University of Tokyo.
Main equation discussed is **Navier-Stokes equation** that usually discusses in fluid, for example air.

## 1.1 Navier-Stokes Equation

### 1.1.1 General Problem

For dimension $d = 2, 3, ...$ (usually 2 or 3) and $T > 0$, we want to find

$$(u, p) : \Omega \times (0, T) \to \mathbb{R}^d \times \mathbb{R}$$

where $u$ is unknown velocity and $p$ is unknown pressure such that

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \triangle u + \nabla p = f & \text{in } \Omega \times (0, T) \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u = u^0 & \text{in } \Omega, \text{ at } t = 0 \end{cases} \tag{1}$$

where $f : \Omega \times (0, T) \to \mathbb{R}^d$ and $u^0 : \Omega \to \mathbb{R}^d$ are given functions, $\nu > 0$ is a viscosity.

From equation (1) we can see that $\dfrac{\partial u}{\partial t} + (u \cdot \nabla)u$ is the **convection part** that explain the movement of fluid. This part, contain **nonlinear term** $(u \cdot \nabla)u$. We can also see, that $\nu \triangle u$ (similar to Heat equation) is the **diffusion part**. In the second equations, $\nabla \cdot u = 0$ explained the **incompressible condition** of fluid.

**Incompressible condition :**

$$\nabla \cdot u = div\ u = 0 \Leftrightarrow \text{ fluid is incompressible}$$

means that the total amount of body does not change. By

$$0 = \int_V \nabla \cdot u\ dx = \int_{\partial V} u \cdot n\ ds$$

means that the energy that comes in and comes out is same and the normal component of velocity is $0 = \int_{\partial V} u \cdot n\ ds$ where $n$ is the normal vector works on boundary.

**Convection effect :**
**[simple explanation.]** Let $\phi^0(x), c > 0$ is given. Consider $\phi(x, t) = \phi^0(x - ct)$, that represent the movement of function without changing the shape.

at $t = 0$ we have $\phi(x, 0) = \phi^0(x)$ ; at at $t = 1$ we have $\phi(x, 1) = \phi^0(x - c)$ ; at $t = 2$ we have $\phi(x, 2) = \phi^0(x - 2c)$ as shown above.

### 1.1.2 3D Problem

For $d = 3$, then $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ such that for $(i = 1, 2, 3)$ we have

$$\frac{\partial u_i}{\partial t} + (u \cdot \nabla)u_i - \nu \triangle u_i + [\nabla p]_i = f_i$$

where

$$
\begin{aligned}
(u \cdot \nabla) u_i &= \left( \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{bmatrix} \right) u_i \\
&= (u_1 \partial_1 + u_2 \partial_2 + u_3 \partial_3) u_i \\
&= u_1 \frac{\partial u_i}{x_1} + u_2 \frac{\partial u_i}{x_2} + u_3 \frac{\partial u_i}{x_3}
\end{aligned}
$$

and

$$
\triangle u_i = \frac{\partial^2 u_i}{\partial x_1^2} + \frac{\partial^2 u_i}{\partial x_2^2} + \frac{\partial^2 u_i}{\partial x_3^2}.
$$

**Note :**we have $u_1, u_2, u_3, p$ as four unknown functions and four equations (as first equation defined for three $u$ and second equation), then we could find the solution.