Nonlinear PDEs

8th lecture

Our plan

- Review the concept of Galerkin approximation and learn about its realization for linear problems.
- Explain by way of example the idea of finite element method (FEM) for one-dimensional linear problem.
- Learn the basic general concepts of the finite element method.
- Study the application of the FEM to nonlinear problems.
- Find about the convergence properties of the FEM.

Galerkin approximation

$$\begin{aligned} -\Delta u(x) &= f(x) \qquad x \in \Omega \\ u(x) &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

Weak solution: a function $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx \qquad \forall \varphi \in H_0^1(\Omega)$$

H^1 Hilbert space $\Rightarrow$ has a countable basis $\{w_i\}_{i=1}^{\infty}$

$$X_N = \left\{ v \in H^1(\Omega);\ v(x) = \sum_{i=1}^{N} \alpha_i w_i(x),\ \ \alpha_i \in \mathbb{R} \right\}.$$

$$V_N = \left\{ v \in H^1(\Omega);\ \ v(x) = \sum_{i=1}^{N} \alpha_i w_i(x), \quad v(x) = 0 \text{ on } \ \partial\Omega \right\}.$$

Galerkin approximation:

$$\Big(\ A(u_N, \varphi) = L(\varphi) \quad \forall \varphi \in V_N\ \Big) \Leftrightarrow \Big(\ A(u_N, w_j) = L(w_j) \quad \forall j \le N\ \Big)$$

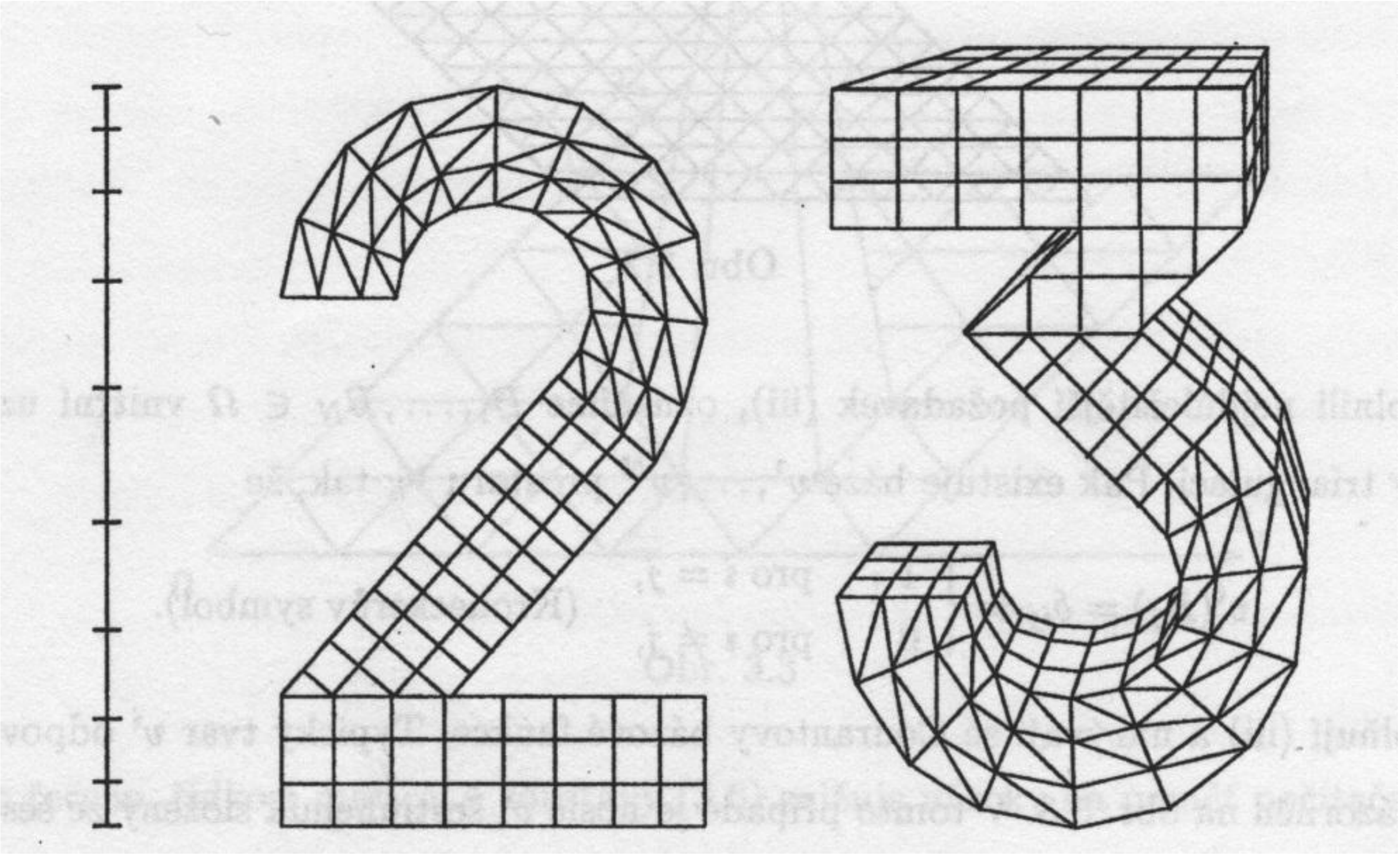
$$u_N(x) = \sum_{i=1}^{N} \alpha_i w_i(x) \quad \Rightarrow \quad A(\sum_{i=1}^{N} \alpha_i w_i, w_j) = L(w_j) \quad j \le N$$

Basic steps of FEM

Sparse matrix is achieved in three steps:

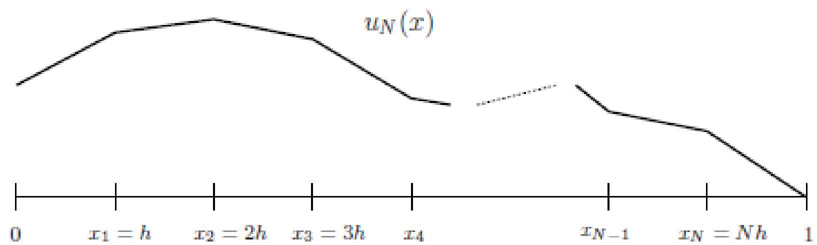
- We create a **triangulation** of the closure of the considered domain into simple closed subdomains. These subdomains are called **elements**.
- The space V_N is chosen so that each function from V_N has a simple form (usually polynomial) on each element. This space is called the **finite element space.**
- We select the basis $w_1, \ldots, w_N$ of the space V_N so that the basis functions w_i have small support (usually, only a few elements).

Triangulations



Example: $u \in H_0^1(0,1): \quad \int_0^1 u'\varphi'\,dx = 2\int_0^1 \varphi\,dx \qquad \forall\varphi \in H_0^1(0,1)$

1. The interval $(0,1)$ is partitioned into say $N+1$ subintervals of length $h = 1/(N+1)$. Let us denote the partition nodes by $x_0 = 0, x_1 = h, x_2 = 2h, \ldots, x_N = Nh, x_{N+1} = 1$.



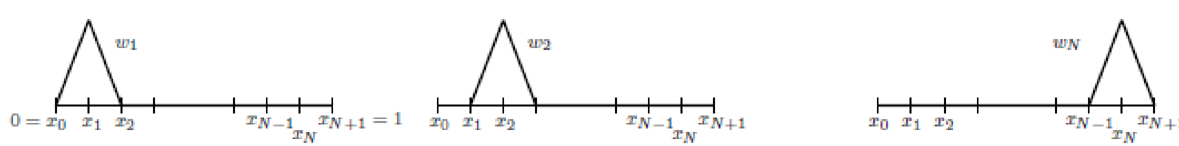
2. We set

$$
\begin{aligned}
X_N &= \{v \in H^1(0,1);\ \ v \text{ is continuous and piecewise linear on the partition } \{x_i\}_{i=0}^{N+1}\} \\
V_N &= \{v \in X_N;\ \ v(0) = v(1) = 0\}
\end{aligned}
$$

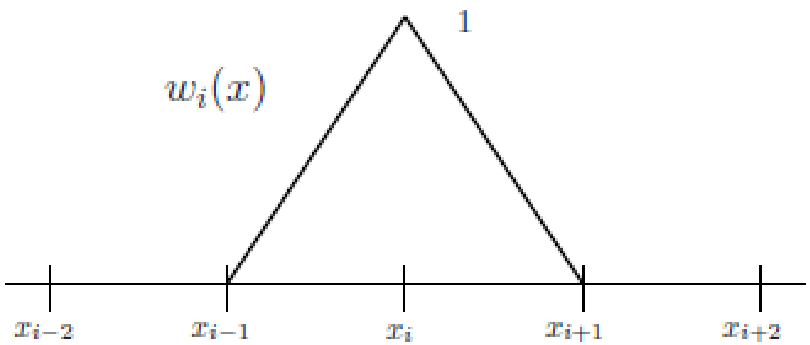
The approximate Galerkin problem reads: find $u_N \in V_N$ satisfying

$$\int_0^1 u_N'\varphi'\,dx = 2\int_0^1 \varphi\,dx \qquad \forall\varphi \in V_N. \tag{7}$$

3. The main idea of FEM is to use the following basis $\{w_i\}_{i=1}^N$ for V_N:



Basis functions



$$w_i(x) = \begin{cases} 0 & x \le x_{i-1} \\ \frac{x - x_i}{h} & x \in [x_{i-1}, x_i] \\ \frac{-x + x_{i+1}}{h} & x \in [x_i, x_{i+1}] \\ 0 & x \ge x_{i+1} \end{cases}$$

This means that w_i is a piecewise linear function fulfilling

$$w_i(x_j) = \begin{cases} 0 & (i \neq j) \\ 1 & (i = j) \end{cases}$$

Such functions are called **Courant basis functions.**

Then if $v \in V_N$ it holds

$$v(x) = \sum_{i=1}^{N} v(x_i) w_i(x).$$