Chapter 6

The Time-Dependent Navier–Stokes Equations – Laminar Flows

Remark 6.1. Motivation. The time-dependent Navier–Stokes equations (1.24) were derived in Chapter 1 as a model for describing the behavior of incompressible fluids. From the point of view of numerical simulations, one has to distinguish between laminar and turbulent flows. It does not exist an exact definition of these terms. From the point of view of simulations, a flow is considered to be laminar, if on reasonable grids all flow structures can be represented or resolved. In this case, it is possible to simulate the flow with standard discretization techniques in space, like the Galerkin finite element method.

In addition to the discretization in space, the simulation of time-dependent flows requires a discretization of the temporal derivative of the velocity. There are a number of so-called time stepping schemes available which lead to a variety of different algorithms for the numerical simulation of the time-dependent Navier–Stokes equations. □

6.1 The Continous Equations.

Remark 6.2. The incompressible Navier–Stokes equations. This chapter considers the incompressible Navier–Stokes equations given by

$$\begin{aligned} \partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } (0,T] \times \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 \quad \text{in } [0,T] \times \Omega, \\ \boldsymbol{u}(0,\cdot) &= \boldsymbol{u}_0 \text{ in } \Omega, \end{aligned} \tag{6.1}$$

where $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, is a domain. In this chapter, only the case will be considered that Ω is a bounded domain with sufficently smooth boundary Γ. In addition, Ω does not change in time. For simplicity of presentation, homogeneous Dirichlet boundary conditions

$$\boldsymbol{u} = \mathbf{0} \text{ in } [0,T] \times \Gamma \tag{6.2}$$

are assumed. □

Definition 6.3. Classical solution of the Navier–Stokes equations. reference missing A pair $(\boldsymbol{u}, p)$ is called classical solution of the Navier–Stokes equations (6.1), (6.2) if:

- $(\boldsymbol{u}, p)$ satisfies the Navier–Stokes equations (6.1) with the boundary conditions (6.2),
- $\boldsymbol{u}$ and p are infinitely often differentiable with respect to space and time.

□

Remark 6.4. On classical solutions. A necessary condition for the existence of a classical solution is that all data of the problem are sufficently smooth. □

Remark 6.5. On the weak equation for the velocity. In the case of the time-dependent incompressible Navier–Stokes equations, one restricts the analysis in the first step to an appropriate divergence-free subspace and studies the existence and uniqueness of an appropriate velocity. In the second step, the existence of a corresponding pressure is studied. One can find different forms of a weak formulation for the velocity in the literature. □

Remark 6.6. First form of a weak equation for the velocity. One considers, as for the stationary equations, test functions from V_{div}, multiplies the momentum equation in (6.1) with these test functions, applies integration by parts in space and obtains an ordinary differential equation for the velocity

$$\begin{gathered}\frac{d}{dt}(\boldsymbol{u}, \boldsymbol{v}) + (\nu \nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + n(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{V',V} \ \forall\, \boldsymbol{v} \in V_{\text{div}}, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0.\end{gathered} \tag{6.3}$$

Here, the derivative with respect to time and the initial condition has to be understood in a weak sense. This form of the weak equation can be found, e.g., in (Girault and Raviart, 1979, p. 158), (Temam, 1984, pp. 280). □

Definition 6.7. Weak or variational velocity solution of the Navier–Stokes equations, (Girault and Raviart, 1979, p. 158), (Temam, 1984, pp. 280). Let $\boldsymbol{f} \in L^2(0,T;V')$ and $\boldsymbol{u}_0 \in L^2_{\text{div}}(\Omega)$. Then $\boldsymbol{u} \in L^2(0,T;V_{\text{div}})$ is called weak or variational velocity solution of the Navier–Stokes equations if $\boldsymbol{u}$ satisfies (6.3) in $(C_0^\infty((0,T)))'$. □

Remark 6.8. Second form of a weak equation for the velocity. To obtain this form, one integrates in addition with respect to time and applies integration by parts. One can consider this approach also in the way that the momentum equation of (6.1) is multiplied with test functions depending on time and space, integrated in the time-space domain, and then integration by parts is applied.

Usually, one uses smooth test functions for this purpose. The extension of the statements to test function in appropriate Lebesgue and Sobolev spaces is then based on the density of the smooth functions in Lebesgue and Sobolev spaces, like stated in Theorem A.38. To be concrete, one uses test functions from the space $C_0^\infty\left([0,T),C_{0,\mathrm{div}}^\infty(\Omega)\right)$. Since these functions are in $C_0^\infty\left([0,T)\right)$ check 0 with respect to time, they vanish at the final time. Applying now the approach for deriving a weak formulation, one obtains

$$\int_0^T \Big[-\left(\boldsymbol{u},\partial_t\boldsymbol{\phi}\right) + \nu\left(\nabla\boldsymbol{u},\nabla\boldsymbol{\phi}\right) + \left(\left(\boldsymbol{u}\cdot\nabla\right)\boldsymbol{u},\boldsymbol{\phi}\right)\Big](\tau)\ d\tau$$
$$= \int_0^T \langle \boldsymbol{f},\boldsymbol{\phi}\rangle_{V',V}(\tau)\ d\tau + \left(\boldsymbol{u}_0,\boldsymbol{\phi}\left(0,\cdot\right)\right) \quad \forall\,\boldsymbol{\phi}\in C_0^\infty\left([0,T),C_{0,\mathrm{div}}^\infty(\Omega)\right) \tag{6.4}$$

Note that the temporal derivative is applied to the test function and not to the velocity. There is no contribution from the integration by parts in time at the final time because the test functions vanish at this time. A weak formulation of form (6.4) can be found, e.g., in Galdi (2000); Amann (2000), and (Sohr, 2001, p. 263). some words about standard and integral form of ODEs □

Definition 6.9. Weak or variational solution of the Navier–Stokes equations, Galdi (2000), (Sohr, 2001, p. 263). Let $\boldsymbol{f}\in L^2\left(0,T;V'\right)$ and $\boldsymbol{u}_0\in L^2_{\mathrm{div}}(\Omega)$. A function $\boldsymbol{u}$ is called weak or variational solution of the Navier–Stokes equations if

- $\boldsymbol{u}$ satisfies (6.4),
- $\boldsymbol{u}$ has the following regularity

$$\boldsymbol{u}\in L^2\left(0,T;V_{\mathrm{div}}\right)\cap L^\infty\left(0,T;L^2_{\mathrm{div}}\left(\Omega\right)\right). \tag{6.5}$$

□

Remark 6.10. General idea for proving existence of a weak solution. The general idea to prove the existence of a weak solution is as follows:

1. Consider a sequence of simpler problems than (6.4) which in an appropriate limit converges to (6.4).
2. Show that each of the simpler problems has a unique solution.
3. Show that a subsequence of the sequence of the unique solutions converges to a weak solution of the Navier–Stokes equations.

□

Remark 6.11. On the definition of simpler problems. There are different proposals for defining simpler problems in the literature.

- *Leray's regularization approach.* The first results using this approach where obtained by Leray:
  - ◦ 1933: $\Omega=\mathbb{R}^2$, Leray (1933),

∘ 1934: Ω is a fixed oval in $\mathbb{R}^2$, Leray (1934),
∘ 1934: $\Omega = \mathbb{R}^3$, Leray (1934).

The most remarkable one is the last paper. There, the simplified equations have the convective term

$$(\boldsymbol{u}_\varepsilon \cdot \nabla)\, \boldsymbol{u},$$

instead of $(\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u}$ in the Navier–Stokes equations (6.1), where $\boldsymbol{u}_\varepsilon$ is an average of $\boldsymbol{u}$ in space with averaging radius ε

$$\boldsymbol{u}_\varepsilon(\boldsymbol{x}) := \frac{1}{\varepsilon^3} \int_\Omega \lambda \left(\frac{\|\boldsymbol{x} - \boldsymbol{y}\|_2^2}{\varepsilon^2} \right) \boldsymbol{u}(\boldsymbol{y})\ d\boldsymbol{y}, \quad \lambda(s) = a e^{\frac{1}{s-1}},\ s \in (0,1).$$

what is a? The use of the convection field $\boldsymbol{u}_\varepsilon$ corresponds to a regularization of the Navier–Stokes equations. In this approach, the limit $\varepsilon \to 0$ is studied.

- *Semidiscretization in space, Galerkin method.* In Hopf (1951), a different type of simpler problems to be considered in the first step of the general approach was introduced. In this method, (6.3) or (6.4) is considered in finite-dimensional spaces with dimension n, which is the so-called Galerkin method. That means, the equation has the same form as (6.3) or (6.4) but the test functions are from a finite-dimensional space and the solution is sought in the same space. Then, $n \to \infty$ is studied. The weak solution of the Navier–Stokes equations obtained with this approach is called weak solution in the sense of Leray–Hopf.
- *Semidiscretization in time.* It is also possible to define the simpler problem with a discretization in time, see (Temam, 1984, Chap. III.4). In the limit, one passes from the discrete times to the continuous time.
- *The semi group method.* Another approach for proving the existence of a weak solution was developed in Sohr (1983), see also Sohr (2001). Similarly to Leray's approach, the simpler problems rely on a regularisation of the convective term, which is however more complicated than in Leray's approach.

Below, one approach using the Galerkin method will be presented. □

Remark 6.12. Starting point of the Galekin method: the Navier–Stokes equations in a finite-dimensional subspace. The first step of the Galerkin method consists in considering the weak form of the Navier–Stokes equations (6.3) or (6.4) in a finite-dimensional space. It can be shown, see (Galdi, 2000, Lemma 2.3), that there is a basis of $\{\boldsymbol{v}_l\}_{l=1}^\infty$ of $C_{0,\mathrm{div}}^\infty(\Omega)$ where the basis functions are orthonormal with respect to the inner product of $L^2(\Omega)$. Consider now the finite-dimensional subspace

$$V_{\mathrm{div}}^n = \operatorname{span}\{\boldsymbol{v}_l^n\}_{l=1}^n \subset C_{0,\mathrm{div}}^\infty(\Omega)\,.$$

In this subspace, the Galerkin method applied to (6.3) reads as follows: Find $\boldsymbol{u}^n \in V_{\mathrm{div}}^n$ such that

$$(\partial_t \boldsymbol{u}^n, \boldsymbol{v}^n) + (\nu \nabla \boldsymbol{u}^n, \nabla \boldsymbol{v}^n) + n\,(\boldsymbol{u}^n, \boldsymbol{u}^n, \boldsymbol{v}^n) = \langle \boldsymbol{f}, \boldsymbol{v}^n \rangle_{V',V} \quad \forall\, \boldsymbol{v}^n \in V_{\mathrm{div}}^n, \tag{6.6}$$

and $\boldsymbol{u}^n(0) = \boldsymbol{u}_0^n$, where $\boldsymbol{u}_0^n$ is the $L^2(\Omega)$ orthogonal projection of $\boldsymbol{u}_0$ into V_{div}^n. It is clear that (6.6) is satisfied if this equation is satisfied for all basis functions. Using the representation

$$\boldsymbol{u}^n\,(t, \boldsymbol{x}) = \sum_{l=1}^{n} \alpha_l^n(t) \boldsymbol{v}_l^n\,(\boldsymbol{x})\,, \tag{6.7}$$

one obtains from (6.6) the following system of ordinary differential equations

$$\frac{d\alpha_l^n}{dt} + \sum_{j=1}^{n} a_{lj} \alpha_j^n + \sum_{j,k=1}^{n} n_{ljk} \alpha_j^n \alpha_k^n = f_l, \quad l = 1, \ldots, n, \tag{6.8}$$

$$\alpha_l^n(0) = u_{0l},, \quad l = 1, \ldots, n \tag{6.9}$$

with

$$a_{lj} = \left(\nu \nabla \boldsymbol{v}_j^n, \nabla \boldsymbol{v}_l^n\right), \quad n_{ljk} = \left(\left(\boldsymbol{v}_j^n \cdot \nabla\right) \boldsymbol{v}_k^n, \boldsymbol{v}_l^n\right) = n\left(\boldsymbol{v}_j^n, \boldsymbol{v}_k^n, \boldsymbol{v}_l^n\right),$$
$$f_l = \langle \boldsymbol{f}, \boldsymbol{v}_l^n \rangle_{V',V}, \quad u_{0l} = (\boldsymbol{u}_0, \boldsymbol{v}_l^n)\,.$$

The orthonormality of the basis functions is not essential but only simplifies the presentation. For non-orthonormal basis functions, the Grammian matrix (mass matrix) of V_{div}^n would appear at the first term of (6.8). This matrix is non-singular, thus multiplying with its inverse gives the same first term as in (6.8) and it leads to some changes in the other terms, e.g., see (Temam, 1984, p. 283). □

Lemma 6.13. Unique solvability of the problem in the finite-dimensional space. *Let $\Omega \in \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded Lipschitz domain. Let the regularity assumptions on $\boldsymbol{f}$ and $\boldsymbol{u}_0$ from Definitions 6.9 and 6.7 be satisfied. Then system* (6.7) – (6.9) *has a unique solution which is absolutely continuous in* $[0, T]$. *There hold the a priori estimates*

$$\sup_{t \in [0,T]} \|\boldsymbol{u}^n\,(t)\|_{L^2(\Omega)}^2 \leq \|\boldsymbol{u}_0\|_{L^2(\Omega)}^2 + \frac{1}{\nu} \|\boldsymbol{f}\|_{L^2(0,T;V')}^2 \tag{6.10}$$

and

$$\|\boldsymbol{u}^n\,(T)\|_{L^2(\Omega)}^2 + \nu \|\nabla \boldsymbol{u}^n\|_{L^2(0,T;L^2(\Omega))}^2 \leq \|\boldsymbol{u}_0\|_{L^2(\Omega)}^2 + \frac{1}{\nu} \|\boldsymbol{f}\|_{L^2(0,T;V')}^2\,, \tag{6.11}$$

which are both uniformly with respect to n. Hence

$$\boldsymbol{u}^n \in L^2\,(0, T; V_{\mathrm{div}}) \cap L^\infty\left(0, T; L_{\mathrm{div}}^2\,(\Omega)\right).$$

Proof. The proof of the lemma will be based on the application of the theorem of Carathéodory, see Theorem A.62. Thus, one has to show a Lipschitz condition for the right-hand side of

$$\frac{d\alpha_l^n}{dt}(t) = F(\alpha_l^n), \quad t \in (0,T], \tag{6.12}$$

with $F \in L^2(0,T)$. If F would be continuous in $[0,T]$, one could apply the famous theorem of Peano, but for the given regularity, the theorem of Carathéodory has to be used.

The functions α_l^n appear linearly and quadratically on the right-hand side of (6.12). Hence, the Lipschitz condition is satisfied, since linear and quadratic functions are Lipschitz continuous. Hence, the local existence and uniqueness of an absolutely continuous solution $\boldsymbol{u}^n(t,\boldsymbol{x})$ in some maximal interval $[0,t_n]$ with $0 < t_n \le T$ can be concluded from the theorem of Carathéodory. If $t_n < T$, then $\boldsymbol{u}^n(t)$ blows up as $t \to t_n$.

Next, the a priori estimates (6.10) and (6.11) will be proved which show that this situation cannot happen and therefore $t_n = T$.

Taking the solution $\boldsymbol{u}^n(t,\boldsymbol{x})$, which is now known to exist, as test function in (6.6) for some arbitrary $t \in (0,T)$, using the product rule for the first term

$$\begin{aligned}\frac{d}{dt}\|\boldsymbol{u}^n(t)\|_{L^2(\Omega)}^2 &= \frac{d}{dt}(\boldsymbol{u}^n(t),\boldsymbol{u}^n(t)) = (\partial_t\boldsymbol{u}^n(t),\boldsymbol{u}^n(t)) + (\boldsymbol{u}^n(t),\partial_t\boldsymbol{u}_n(t))\\ &= 2(\partial_t\boldsymbol{u}^n(t),\boldsymbol{u}^n(t)),\end{aligned} \tag{6.13}$$

the skew-symmetry (5.15) of the convective term, the estimate of the dual pairing, and Young's inequality yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}^n(t)\|_{L^2(\Omega)}^2 + \nu\|\nabla\boldsymbol{u}^n(t)\|_{L^2(\Omega)}^2 &= \langle \boldsymbol{f}(t),\boldsymbol{u}^n(t)\rangle_{V',V} \le \|\boldsymbol{f}(t)\|_{V'}\|\nabla\boldsymbol{u}^n(t)\|_{L^2(\Omega)}\\ &\le \frac{1}{2\nu}\|\boldsymbol{f}(t)\|_{V'}^2 + \frac{\nu}{2}\|\nabla\boldsymbol{u}^n(t)\|_{L^2(\Omega)}^2,\end{aligned} \tag{6.14}$$

which gives

$$\frac{d}{dt}\|\boldsymbol{u}^n(t)\|_{L^2(\Omega)}^2 + \nu\|\nabla\boldsymbol{u}^n(t)\|_{L^2(\Omega)}^2 \le \frac{1}{\nu}\|\boldsymbol{f}(t)\|_{V'}^2. \tag{6.15}$$

Integrating this equation in some time interval $[0,t]$ with arbitrary $t \le T$ and applying the estimate for the $L^2(\Omega)$ projection (C.21) leads to

$$\|\boldsymbol{u}^n(t)\|_{L^2(\Omega)}^2 \le \|\boldsymbol{u}_0^n\|_{L^2(\Omega)}^2 + \frac{1}{\nu}\int_0^t \|\boldsymbol{f}(\tau)\|_{V'}^2 \, d\tau \le \|\boldsymbol{u}_0\|_{L^2(\Omega)}^2 + \frac{1}{\nu}\|\boldsymbol{f}\|_{L^2(0,t;V')}^2.$$

Taking the supremum in $[0,T]$ gives the a priori estimate (6.10) and since clearly $\boldsymbol{u}^n \in L^2_{\mathrm{div}}(\Omega)$, it follows that $\boldsymbol{u}^n \in L^\infty\left(0,T;L^2_{\mathrm{div}}(\Omega)\right)$.

Integrating now (6.15) in $[0,T]$ leads in the same way as used for deriving the first estimate to

$$\|\boldsymbol{u}^n(T)\|_{L^2(\Omega)}^2 + \nu\|\nabla\boldsymbol{u}^n\|_{L^2(0,T;L^2(\Omega))}^2 \le \|\boldsymbol{u}_0\|_{L^2(\Omega)}^2 + \frac{1}{\nu}\|\boldsymbol{f}\|_{L^2(0,T;V')}^2,$$

which is the a priori estimate (6.11). Since $\boldsymbol{u}^n \in V_{\mathrm{div}}$, one obtains in particular that $\boldsymbol{u}^n \in L^2(0,T;V_{\mathrm{div}})$. ■

Corollary 6.14. Weak convergence. *There is a subsequence* $\{\boldsymbol{u}^{n_l}\}_{l=1}^\infty$ *of* $\{\boldsymbol{u}^n\}_{n=1}^\infty$ *and a function* $\boldsymbol{u} \in L^2(0,T;V_{\mathrm{div}}) \cap L^\infty\left(0,T;L^2_{\mathrm{div}}(\Omega)\right)$ *such that*

$$\begin{aligned}\boldsymbol{u}^{n_l} &\overset{*}{\rightharpoonup} \boldsymbol{u} \quad \text{in } L^\infty\left(0,T;L^2_{\mathrm{div}}(\Omega)\right),\\ \boldsymbol{u}^{n_l} &\rightharpoonup \boldsymbol{u} \quad \text{in } L^2(0,T;V_{\mathrm{div}}).\end{aligned}$$

as $l \to \infty$.

Proof. It is known from Lemma 6.13 that the sequence $\{\boldsymbol{u}^n\}_{n=1}^{\infty}$ is bounded uniformly in $L^2(0,T;V_{\text{div}})$. This space is a Hilbert space, thus in particular a reflexive Banach space, such that the existence of a weakly convergence subsequence to some element $\boldsymbol{u}_1 \in L^2(0,T;V_{\text{div}})$ follows, see Remark A.49. The limit is unique.

The space $L^1\left(0,T;L^2_{\text{div}}(\Omega)\right)$ is a separable Banach space and its dual space is $L^\infty\left(0,T;L^2_{\text{div}}(\Omega)\right)$. From Lemma 6.13 it is known that $\{\boldsymbol{u}^n\}_{n=1}^{\infty}$ is bounded uniformly in the dual space, such that the existence of a weakly* convergent subsequence to some function $\boldsymbol{u}_2 \in L^\infty\left(0,T;L^2_{\text{div}}(\Omega)\right)$, follows, see Remark A.49. Also this limit is unique.

Since

$$\{\boldsymbol{u}^n\}_{n=1}^{\infty} \in L^2(0,T;V_{\text{div}}) \cap L^\infty\left(0,T;L^2_{\text{div}}(\Omega)\right)$$

and both spaces are complete, it follows that

$$\boldsymbol{u}_1, \boldsymbol{u}_2 \in L^2(0,T;V_{\text{div}}) \cap L^\infty\left(0,T;L^2_{\text{div}}(\Omega)\right).$$

The uniqueness of the limits gives finally that $\boldsymbol{u}_1 = \boldsymbol{u}_2 = \boldsymbol{u}$. ■

Remark 6.15. Consequences of the weak convergence, convergence of the linear terms. For simplicity of notation, the subsequence will be denoted again by $\{\boldsymbol{u}^n\}_{n=1}^{\infty}$.

Consider the solution of the Galerkin problem (6.6), take an arbitrary $\boldsymbol{v} \in V_{\text{div}}$, and let $\phi \in C_0^\infty((0,T))$. One obtains with integration by parts

$$\int_0^T (\partial_t \boldsymbol{u}^n(t), \boldsymbol{v})\, \phi(t)\ dt = -\int_0^T (\boldsymbol{u}^n(t), \boldsymbol{v})\, \frac{d}{dt}\phi(t)\ dt.$$

Since $\boldsymbol{u}^n \stackrel{*}{\rightharpoonup} \boldsymbol{u}$ in $L^\infty\left(0,T;L^2_{\text{div}}(\Omega)\right)$ and $\frac{d}{dt}\phi \in L^1(0,T)$, one gets

$$\begin{aligned}\lim_{n\to\infty} -\int_0^T (\boldsymbol{u}^n(t), \boldsymbol{v})\, \frac{d}{dt}\phi(t)\ dt &= \int_0^T (\boldsymbol{u}(t), \boldsymbol{v})\, \frac{d}{dt}\phi(t)\ dt \\ &= \int_0^T (\partial_t \boldsymbol{u}(t), \boldsymbol{v})\, \phi(t)\ dt\end{aligned}$$

for all $\boldsymbol{v} \in V_{\text{div}}$. Hence, $\boldsymbol{u}$ satisfies the weak form of the first term of (6.3).

For the viscous term, it follows from $\boldsymbol{u}^n \rightharpoonup \boldsymbol{u}$ in $L^2(0,T;V_{\text{div}})$ and $\phi \in L^2(0,T)$ that for all $\boldsymbol{v} \in V_{\text{div}}$ and $\phi \in C_0^\infty([0,T])$

$$\lim_{n\to\infty} \int_0^T (\nu \nabla \boldsymbol{u}^n, \nabla \boldsymbol{v})\, \phi(t)\ dt = \int_0^T (\nu \nabla \boldsymbol{u}, \nabla \boldsymbol{v})\, \phi(t)\ dt, \qquad (6.16)$$

which is the weak form of the viscous term in (6.3).

In order to show that $\boldsymbol{u}$ is a weak solution of the Navier–Stokes equations, one has to show the convergence of the nonlinear convective term

$$n\left(\boldsymbol{u}^h, \boldsymbol{u}^n, \boldsymbol{v}\right) \to n\left(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}\right).$$

There are different ways to prove this property, e.g., using a Fourier transform as in (Temam, 1984, pp. 285) or a special Friedrichs inequality as in Galdi

(2000). Below, the approach used in (Girault and Raviart, 1979, pp. 161) will be sketched. □

Lemma 6.16. Estimate of the difference of nonlinear convective terms. *Let* $\boldsymbol{u}, \overline{\boldsymbol{u}} \in L^2\left(0,T;V_{\mathrm{div}}\right) \cap L^\infty\left(0,T;L^2_{\mathrm{div}}\left(\Omega\right)\right)$, $\boldsymbol{v} \in V_{\mathrm{div}}$ *and* $r \in [1,4/3)$, *then*

$$\begin{aligned}
&\int_0^T \left|n(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) - n(\overline{\boldsymbol{u}},\overline{\boldsymbol{u}},\boldsymbol{v})\right|^r \ dt \qquad (6.17)\\
&\le C\left\|\nabla \boldsymbol{v}\right\|^r_{L^2(\Omega)} \left(\left\|\boldsymbol{u}\right\|^{r/4}_{L^\infty(0,T;L^2(\Omega))} + \left\|\overline{\boldsymbol{u}}\right\|^{r/4}_{L^\infty(0,T;L^2(\Omega))}\right)\\
&\quad \times \left\|\boldsymbol{u} - \overline{\boldsymbol{u}}\right\|^{r/4}_{L^{r/(4-3r)}(0,T;L^2(\Omega))} \left(\left\|\nabla \boldsymbol{u}\right\|^{3r/2}_{L^2(0,T;L^2(\Omega))} + \left\|\nabla \overline{\boldsymbol{u}}\right\|^{3r/2}_{L^2(0,T;L^2(\Omega))}\right).
\end{aligned}$$

Proof. By adding and subtracting the same term, applying the triangle inequality, and using the skew-symmetry of the nonlinear convective term (5.12), one obtains

$$\begin{aligned}
\left|n(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) - n(\overline{\boldsymbol{u}},\overline{\boldsymbol{u}},\boldsymbol{v})\right| &= \left|n(\boldsymbol{u}-\overline{\boldsymbol{u}},\boldsymbol{u},\boldsymbol{v}) - n(\overline{\boldsymbol{u}},\overline{\boldsymbol{u}}-\boldsymbol{u},\boldsymbol{v})\right|\\
&\le \left|n(\boldsymbol{u}-\overline{\boldsymbol{u}},\boldsymbol{u},\boldsymbol{v})\right| + \left|n(\overline{\boldsymbol{u}},\overline{\boldsymbol{u}}-\boldsymbol{u},\boldsymbol{v})\right|\\
&= \left|n(\boldsymbol{u}-\overline{\boldsymbol{u}},\boldsymbol{v},\boldsymbol{u})\right| + \left|n(\overline{\boldsymbol{u}},\boldsymbol{v},\overline{\boldsymbol{u}}-\boldsymbol{u})\right|.
\end{aligned}$$

The goal of the proof is to obtain an estimated where the norm of the difference $\boldsymbol{u} - \overline{\boldsymbol{u}}$ is as weak as possible.

The next steps of the estimate consist in applying the generalized Hölder inequality (5.19) with $p = r = 1/4$ and $q = 2$, using the Sobolev embedding (A.12) with $m = 3/4$, $p = 2$, $q = 4$, and using the interpolation estimate for Sobolev spaces (A.10)

$$\begin{aligned}
&\left|n(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) - n(\overline{\boldsymbol{u}},\overline{\boldsymbol{u}},\boldsymbol{v})\right|\\
&\le \left\|\boldsymbol{u}-\overline{\boldsymbol{u}}\right\|_{L^4(\Omega)} \left\|\nabla \boldsymbol{v}\right\|_{L^2(\Omega)} \left\|\boldsymbol{u}\right\|_{L^4(\Omega)} + \left\|\overline{\boldsymbol{u}}\right\|_{L^4(\Omega)} \left\|\nabla \boldsymbol{v}\right\|_{L^2(\Omega)} \left\|\overline{\boldsymbol{u}}-\boldsymbol{u}\right\|_{L^4(\Omega)}\\
&= \left\|\nabla \boldsymbol{v}\right\|_{L^2(\Omega)} \left\|\boldsymbol{u}-\overline{\boldsymbol{u}}\right\|_{L^4(\Omega)} \left(\left\|\boldsymbol{u}\right\|_{L^4(\Omega)} + \left\|\overline{\boldsymbol{u}}\right\|_{L^4(\Omega)}\right)\\
&\le C\left\|\nabla \boldsymbol{v}\right\|_{L^2(\Omega)} \left\|\boldsymbol{u}-\overline{\boldsymbol{u}}\right\|_{H^{3/4}(\Omega)} \left(\left\|\boldsymbol{u}\right\|_{H^{3/4}(\Omega)} + \left\|\overline{\boldsymbol{u}}\right\|_{H^{3/4}(\Omega)}\right)\\
&\le C\left\|\nabla \boldsymbol{v}\right\|_{L^2(\Omega)} \left\|\boldsymbol{u}-\overline{\boldsymbol{u}}\right\|^{1/4}_{L^2(\Omega)} \left\|\nabla\left(\boldsymbol{u}-\overline{\boldsymbol{u}}\right)\right\|^{3/4}_{L^2(\Omega)}\\
&\quad \times \left(\left\|\boldsymbol{u}\right\|^{1/4}_{L^2(\Omega)} \left\|\nabla \boldsymbol{u}\right\|^{3/4}_{L^2(\Omega)} + \left\|\overline{\boldsymbol{u}}\right\|^{1/4}_{L^2(\Omega)} \left\|\nabla \overline{\boldsymbol{u}}\right\|^{3/4}_{L^2(\Omega)}\right).
\end{aligned}$$

Note that in the application of the generalized Hölder inequality, the maximal regularity for $\boldsymbol{v}$ was used and the other two terms are treated in the same way. Note also that the application of Poincaré's inequality would lead to a strong norm for the difference $\boldsymbol{u} - \overline{\boldsymbol{u}}$.

Let $r \in [1,4/3)$, then it follows that

$$\begin{aligned}
&\int_0^T \left|n(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) - n(\overline{\boldsymbol{u}},\overline{\boldsymbol{u}},\boldsymbol{v})\right|^r \ dt\\
&\le C\left\|\nabla \boldsymbol{v}\right\|^r_{L^2(\Omega)} \int_0^T \Big[\left\|\boldsymbol{u}-\overline{\boldsymbol{u}}\right\|^{r/4}_{L^2(\Omega)} \left\|\nabla\left(\boldsymbol{u}-\overline{\boldsymbol{u}}\right)\right\|^{3r/4}_{L^2(\Omega)}\\
&\quad \times \left(\left\|\boldsymbol{u}\right\|^{1/4}_{L^2(\Omega)} \left\|\nabla \boldsymbol{u}\right\|^{3/4}_{L^2(\Omega)} + \left\|\overline{\boldsymbol{u}}\right\|^{1/4}_{L^2(\Omega)} \left\|\nabla \overline{\boldsymbol{u}}\right\|^{3/4}_{L^2(\Omega)}\right)^r \Big] \ dt. \qquad (6.18)
\end{aligned}$$

Taking the supremum in $(0,T)$ of two terms, using

$$(ab+cd) \le (a+c)(b+d), \quad a,b,c,d \ge 0,$$

and using $(a+b)^n \le C\left(a^n+b^n\right)$ for $a,b\ge 0$, $n\ge 1$, gives for the integral

$$\begin{aligned}
&\int_0^T \Big[\|\boldsymbol{u}-\overline{\boldsymbol{u}}\|_{L^2(\Omega)}^{r/4} \|\nabla(\boldsymbol{u}-\overline{\boldsymbol{u}})\|_{L^2(\Omega)}^{3r/4} \\
&\quad\times\left(\|\boldsymbol{u}\|_{L^2(\Omega)}^{1/4}\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}^{3/4}+\|\overline{\boldsymbol{u}}\|_{L^2(\Omega)}^{1/4}\|\nabla\overline{\boldsymbol{u}}\|_{L^2(\Omega)}^{3/4}\right)^r\Big]\,dt \\
&\le \int_0^T \Big[\|\boldsymbol{u}-\overline{\boldsymbol{u}}\|_{L^2(\Omega)}^{r/4} \|\nabla(\boldsymbol{u}-\overline{\boldsymbol{u}})\|_{L^2(\Omega)}^{3r/4} \\
&\quad\times\left(\|\boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega))}^{1/4}\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}^{3/4}+\|\overline{\boldsymbol{u}}\|_{L^\infty(0,T;L^2(\Omega))}^{1/4}\|\nabla\overline{\boldsymbol{u}}\|_{L^2(\Omega)}^{3/4}\right)^r\Big]\,dt \\
&\le \int_0^T \Big[\|\boldsymbol{u}-\overline{\boldsymbol{u}}\|_{L^2(\Omega)}^{r/4} \|\nabla(\boldsymbol{u}-\overline{\boldsymbol{u}})\|_{L^2(\Omega)}^{3r/4}\left(\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}^{3/4}+\|\nabla\overline{\boldsymbol{u}}\|_{L^2(\Omega)}^{3/4}\right)^r \\
&\quad\times\left(\|\boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega))}^{1/4}+\|\overline{\boldsymbol{u}}\|_{L^\infty(0,T;L^2(\Omega))}^{1/4}\right)^r\Big]\,dt \\
&\le C\left(\|\boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega))}^{r/4}+\|\overline{\boldsymbol{u}}\|_{L^\infty(0,T;L^2(\Omega))}^{r/4}\right) \qquad (6.19)\\
&\quad\times\int_0^T \Big[\|\boldsymbol{u}-\overline{\boldsymbol{u}}\|_{L^2(\Omega)}^{r/4} \|\nabla(\boldsymbol{u}-\overline{\boldsymbol{u}})\|_{L^2(\Omega)}^{3r/4}\left(\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}^{3r/4}+\|\nabla\overline{\boldsymbol{u}}\|_{L^2(\Omega)}^{3r/4}\right)\Big]\,dt.
\end{aligned}$$

Concentrating again on the integral, using the triangle inequality (in the case of Lebesgue spaces also called Minkowski's inequality),

$$(a+b)^q \le a^q+b^q \quad a,b\ge 0, q\in(0,1), \quad\Longleftrightarrow\quad c^p+d^p\le(c+d)^p, \quad c,d\ge 0, p>1,$$

and Young's inequality (A.4) yields

$$\begin{aligned}
&\int_0^T \Big[\|\boldsymbol{u}-\overline{\boldsymbol{u}}\|_{L^2(\Omega)}^{r/4} \|\nabla(\boldsymbol{u}-\overline{\boldsymbol{u}})\|_{L^2(\Omega)}^{3r/4}\left(\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}^{3r/4}+\|\nabla\overline{\boldsymbol{u}}\|_{L^2(\Omega)}^{3r/4}\right)\Big]\,dt \\
&\le \int_0^T \Big[\|\boldsymbol{u}-\overline{\boldsymbol{u}}\|_{L^2(\Omega)}^{r/4}\left(\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}^{3r/4}+\|\nabla\overline{\boldsymbol{u}}\|_{L^2(\Omega)}^{3r/4}\right)\left(\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}^{3r/4}+\|\nabla\overline{\boldsymbol{u}}\|_{L^2(\Omega)}^{3r/4}\right)\Big]\,dt \\
&\le C\int_0^T \|\boldsymbol{u}-\overline{\boldsymbol{u}}\|_{L^2(\Omega)}^{r/4}\left(\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}^{3r/2}+\|\nabla\overline{\boldsymbol{u}}\|_{L^2(\Omega)}^{3r/2}\right)\,dt. \qquad (6.20)
\end{aligned}$$

Note that the integral is well defined for $r\le 4/3$ since the first term is in $L^\infty(0,T)$ by assumption and the norms with the gradient are in $L^2(0,T)$.

Next, Hölder's inequality (A.7) will be applied. Since $\boldsymbol{u},\overline{\boldsymbol{u}}\in L^2\left(0,T;V_{\mathrm{div}}\right)$, the terms in the parentheses should get the power 2. Therefore, one chooses in (A.7) the values $p=4/(3r)>1$, $q=4/(4-3r)$ and one obtains

$$\begin{aligned}
&\int_0^T \Big[\|\boldsymbol{u}-\overline{\boldsymbol{u}}\|_{L^2(\Omega)}^{r/4} \|\nabla(\boldsymbol{u}-\overline{\boldsymbol{u}})\|_{L^2(\Omega)}^{3r/4}\left(\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}^{3r/4}+\|\nabla\overline{\boldsymbol{u}}\|_{L^2(\Omega)}^{3r/4}\right)\Big]\,dt \\
&\le\left(\int_0^T \|\boldsymbol{u}-\overline{\boldsymbol{u}}\|_{L^2(\Omega)}^{r/(4-3r)}\,dt\right)^{(4-3r)/4} \\
&\quad\times\left[\left(\int_0^T\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}^2\,dt\right)^{3r/4}+\left(\int_0^T\|\nabla\overline{\boldsymbol{u}}\|_{L^2(\Omega)}^2\,dt\right)^{3r/4}\right].
\end{aligned}$$

Inserting this estimate into (6.18), (6.19), and (6.20) gives estimate (6.17). ■

Lemma 6.17. Convergence of the nonlinear convective term. *It holds for all $\boldsymbol{v}\in V_{\mathrm{div}}$ and $\phi\in C_0^\infty([0,T])$*

$$\lim_{n\to\infty} \int_0^T n\left(\boldsymbol{u}^n(t), \boldsymbol{u}^n(t), \boldsymbol{v}\right) \phi(t)\ dt = \int_0^T n\left(\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{v}\right) \phi(t)\ dt \qquad (6.21)$$

in $L^2\left(0,T;V_{\mathrm{div}}\right)$.

Proof. Only a sketch of the proof will be presented.

In the first step, one proves that

$$\boldsymbol{u}^n \in H^\sigma\left(0,T;L^2_{\mathrm{div}}(\Omega)\right) \quad \text{with } \sigma \in \left(0, \frac{1}{4}\right).$$

The norm in this space is defined by

$$\|\boldsymbol{u}^n\|_{H^\sigma\left(0,T;L^2_{\mathrm{div}}(\Omega)\right)} = \int_0^T \int_0^T |t-s|^{-(1+2\sigma)} \|\boldsymbol{u}^n(t) - \boldsymbol{u}^n(s)\|_{L^2(\Omega)}\ dsdt.$$

The proof of this property is quite lengthy and it is refered to the literature, e.g., (Girault and Raviart, 1979, pp. ??). insert details

Next, it is known (generalized theorem of Lions–Aubin, (Girault and Raviart, 1979, pp. 153)) that

$$L^2\left(0,T;V_{\mathrm{div}}\right) \cap H^\sigma\left(0,T;L^2_{\mathrm{div}}(\Omega)\right)$$

is compactly embedded check into $L^2\left(0,T;L^2_{\mathrm{div}}(\Omega)\right)$. Since $\{\boldsymbol{u}^n\}_{n=1}^\infty$ is bounded in $L^2\left(0,T;V_{\mathrm{div}}\right) \cap H^\sigma\left(0,T;L^2_{\mathrm{div}}(\Omega)\right)$, it follows from the compactness of the embedding that there is a subsequence which is a Cauchy sequence in $L^2\left(0,T;L^2_{\mathrm{div}}(\Omega)\right)$. Since $L^2\left(0,T;L^2_{\mathrm{div}}(\Omega)\right)$ is complete, this subsequence converges strongly to some $\boldsymbol{u} \in L^2\left(0,T;L^2_{\mathrm{div}}(\Omega)\right)$. Because of the uniqueness of the limit and because this limit is contained in $L^2\left(0,T;V_{\mathrm{div}}\right)$, it has to be the same limit as the limit from Corollary 6.14.

Taking $\overline{\boldsymbol{u}} = \boldsymbol{u}^n$ and $r = 8/7$ in (6.17) gives the term

$$\|\boldsymbol{u} - \boldsymbol{u}^n\|^{r/4}_{L^{r/(4-3r)}(0,T;L^2(\Omega))} = \|\boldsymbol{u} - \boldsymbol{u}^n\|^{2/7}_{L^2(0,T;L^2(\Omega))},$$

which converges to zero as $n \to \infty$. Thus, one gets from (6.17) that

$$\lim_{n\to\infty} \int_0^T |n(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) - n\left(\boldsymbol{u}^n,\boldsymbol{u}^n,\boldsymbol{v}\right)|^{8/7}\ dt = 0.$$

Since this expression converges for the power 8/7, it converges also for the power one, such that one obtains

$$\lim_{n\to\infty} \int_0^T |n\left(\boldsymbol{u}^n,\boldsymbol{u}^n,\boldsymbol{v}\right) - n\left(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}\right))|\ dt = 0.$$

Finally, it follows that

$$\begin{aligned}&\lim_{n\to\infty} \left| \int_0^T \left(n\left(\boldsymbol{u}^n,\boldsymbol{u}^n,\boldsymbol{v}\right) - n\left(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}\right)\right)\phi(t)\ dt\right| \\ &\le \|\phi\|_{L^\infty((0,T))} \lim_{n\to\infty} \int_0^T |n\left(\boldsymbol{u}^n,\boldsymbol{u}^n,\boldsymbol{v}\right) - n\left(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}\right))|\ dt = 0,\end{aligned}$$

which is the statement of the lemma. ■

Lemma 6.18. Satisfaction of the initial condition. *The limit* $\boldsymbol{u}$ *from Corollary 6.14 and Lemma 6.17 satisfies the initial condition.*

Proof. Let $\boldsymbol{v}^n \in V^n_{\mathrm{div}}$ be arbitrary, then the fundamental theorem of calculus and the product rule yields

$$-\left(\boldsymbol{u}(0),\boldsymbol{v}^n\right) = \left(\boldsymbol{u}(t),\boldsymbol{v}^n\right)\frac{T-t}{T}\Big|_{t=0}^{T} = \int_0^T \frac{d}{dt}\left(\left(\boldsymbol{u}(t),\boldsymbol{v}^n\right)\frac{T-t}{T}\right)\,dt$$
$$= \int_0^T \left(\frac{d}{dt}\left(\boldsymbol{u},\boldsymbol{v}^n\right)\right)\frac{T-t}{T}\,dt - \frac{1}{T}\int_0^T \left(\boldsymbol{u},\boldsymbol{v}^n\right)\,dt.$$

Inserting for the time derivative the equation (6.3) and replacing the term with the exterior force by (6.6) gives

$$\begin{aligned}
&-\left(\boldsymbol{u}(0),\boldsymbol{v}^n\right)\\
&= \int_0^T \left(\langle \boldsymbol{f},\boldsymbol{v}^n\rangle_{V',V} - (\nu\nabla\boldsymbol{u},\nabla\boldsymbol{v}^n) - n\left(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}^n\right)\right)\frac{T-t}{T}\,dt - \frac{1}{T}\int_0^T \left(\boldsymbol{u},\boldsymbol{v}^n\right)\,dt\\
&= \int_0^T \left((\nu\nabla\boldsymbol{u}^n,\nabla\boldsymbol{v}^n) - (\nu\nabla\boldsymbol{u},\nabla\boldsymbol{v}^n)\right)\frac{T-t}{T}\,dt\\
&\quad+ \int_0^T \left(n\left(\boldsymbol{u}^n,\boldsymbol{u}^n,\boldsymbol{v}^n\right) - n\left(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}^n\right)\right)\frac{T-t}{T}\,dt\\
&\quad+ \int_0^T \left(\partial_t\boldsymbol{u}^n,\boldsymbol{v}^n\right)\frac{T-t}{T}\,dt - \frac{1}{T}\int_0^T \left(\boldsymbol{u},\boldsymbol{v}^n\right)\,dt
\end{aligned}$$

Now, one considers $n\to\infty$. From (6.16) and (6.21), it follows that the first two terms vanish. One obtains, using integration by parts in time and $\boldsymbol{u}^n \overset{*}{\rightharpoonup} \boldsymbol{u}$ in $L^\infty\left(0,T;L^2_{\mathrm{div}}(\Omega)\right)$

$$\begin{aligned}
&-\left(\boldsymbol{u}(0),\boldsymbol{v}\right)\\
&= \lim_{n\to\infty} -\left(\boldsymbol{u}(0),\boldsymbol{v}^n\right)\\
&= \lim_{n\to\infty}\int_0^T \left(\partial_t\boldsymbol{u}^n,\boldsymbol{v}^n\right)\frac{T-t}{T}\,dt - \frac{1}{T}\int_0^T \left(\boldsymbol{u},\boldsymbol{v}^n\right)\,dt\\
&= \lim_{n\to\infty}\left(\boldsymbol{u}^n,\boldsymbol{v}^n\right)\frac{T-t}{T}\Big|_{t=0}^{T} + \frac{1}{T}\lim_{n\to\infty}\left(\int_0^T \left(\boldsymbol{u}^n,\boldsymbol{v}^n\right)\,dt - \int_0^T \left(\boldsymbol{u},\boldsymbol{v}^n\right)\,dt\right)\\
&= -\lim_{n\to\infty}\left(\boldsymbol{u}^n(0),\boldsymbol{v}^n\right)\\
&= -\lim_{n\to\infty}\left(\boldsymbol{u}_0^n,\boldsymbol{v}^n\right).
\end{aligned}$$

Hence, the limit function $\boldsymbol{u}$ satisfies the initial condition. ■

Lemma 6.19. Regularity of the time derivative. *Let* $\boldsymbol{u}\in L^2\left(0,T;V_{\mathrm{div}}\right)\cap L^\infty\left(0,T;L^2_{\mathrm{div}}(\Omega)\right)$ *be the Leray–Hopf weak solution of* (6.3)*, then*

$$\partial_t\boldsymbol{u} \in \begin{cases} L^2\left(0,T;V'\right) & \text{if } d=2,\\ L^{4/3}\left(0,T;V'\right) & \text{if } d=3.\end{cases} \tag{6.22}$$

Proof. The proof can be found in (Girault and Raviart, 1979, pp. 158) and it is based on statements proved on (Girault and Raviart, 1979, p. 155) and (Girault and Raviart, 1979, p. 157). work out ■

Theorem 6.20. Existence of a weak solution in the sense of Leray–Hopf. *Let* $\boldsymbol{f}\in L^2\left(0,T;V'\right)$ *and* $\boldsymbol{u}_0\in L^2_{\mathrm{div}}(\Omega)$*, then there exists a weak solution*

$$\boldsymbol{u}\in L^2\left(0,T;V_{\mathrm{div}}\right)\cap L^\infty\left(0,T;L^2_{\mathrm{div}}(\Omega)\right)$$

of (6.3) *or* (6.4) *in the sense of Leray–Hopf.*

Proof. From Remark 6.15 and Lemma 6.17 it follows that there is a subsequence of $\{\boldsymbol{u}^n\}_{n=1}^{\infty}$ whose limit $\boldsymbol{u}$ satisfies the weak equation. In Lemma 6.18 it was proved that also the initial condition is satisfied from $\boldsymbol{u}$. ∎

Definition 6.21. Energy inequality. A weak solution $\boldsymbol{u}$ is said to satisfy the energy inequality if

$$\begin{aligned} &\|\boldsymbol{u}(t)\|_{L^2(\Omega)}^2 + 2\nu \int_0^t \|\nabla \boldsymbol{u}(\tau)\|_{L^2(\Omega)}^2 \ d\tau \\ &\le \|\boldsymbol{u}(0)\|_{L^2(\Omega)}^2 + 2\int_0^t \langle \boldsymbol{f}, \boldsymbol{u}\rangle_{V'V}(\tau)\ d\tau \end{aligned} \tag{6.23}$$

for all $t \in [0,T]$. The first term on the left-hand side is (twice of) the kinetic energy and the second term the energy dissipation due to the viscosity of the fluid. □

Lemma 6.22. Energy inequality, stability of the solution. *Let $d = 2$ and let $\boldsymbol{u}$ be any weak solution of (6.3) or (6.4). Then, this solution satisfies an energy equality. In addition, the following stability bound for the velocity holds*

$$\|\boldsymbol{u}(t)\|_{L^2(\Omega)}^2 + \nu \|\nabla \boldsymbol{u}\|_{L^2(0,t;L^2(\Omega))}^2 \le \|\boldsymbol{u}(0)\|_{L^2(\Omega)}^2 + \frac{1}{\nu}\|\boldsymbol{f}\|_{L^2(0,t;H^{-1}(\Omega))}^2 \tag{6.24}$$

for all $t \in [0,T]$.

If $d = 3$ and $\boldsymbol{u}$ is the weak solution of (6.3) or (6.4) in the sense of Leray–Hopf, then this solution satisfies the energy inequality (6.23) and a stability estimate of form (6.24).

Proof. The proof is different for two and three dimensions.

Two-dimensionial case. In this case, one can apply the usual approach, i.e., one can take the weak solution $\boldsymbol{u}$ as a test function. One obtains from (6.3), (6.13), and Lemma 5.9,

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|_{L^2(\Omega)}^2 + \nu\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}^2 = \langle \boldsymbol{f}, \boldsymbol{u}\rangle_{V',V}.$$

Integrating in $(0,t)$ yields

$$\frac{1}{2}\|\boldsymbol{u}(t)\|_{L^2(\Omega)}^2 + \nu\|\nabla\boldsymbol{u}\|_{L^2(0,t;L^2(\Omega))}^2 = \frac{1}{2}\|\boldsymbol{u}(0)\|_{L^2(\Omega)}^2 + \int_0^t \langle \boldsymbol{f}, \boldsymbol{u}\rangle_{V',V}(\tau)\ d\tau,$$

which is just the energy equality. Then, using the inequality for the dual pairing and Young's inequality gives

$$\begin{aligned} &\frac{1}{2}\|\boldsymbol{u}(t)\|_{L^2(\Omega)}^2 + \nu\|\nabla\boldsymbol{u}\|_{L^2(0,t;L^2(\Omega))}^2 \\ &\le \frac{1}{2}\|\boldsymbol{u}(0)\|_{L^2(\Omega)}^2 + \frac{1}{2\nu}\|\boldsymbol{f}\|_{L^2(0,t;V')}^2 + \frac{\nu}{2}\|\nabla\boldsymbol{u}\|_{L^2(0,t;L^2(\Omega))}^2, \end{aligned}$$

which leads directly to (6.24).

Three-dimensionial case. The difficulty in three dimensions with the usual approach is that the term

$$\int_0^t (\partial_t \boldsymbol{u}, \boldsymbol{u})(\tau)\ d\tau$$

might not be well defined, since $\partial_t \boldsymbol{u} \in L^{4/3}(0,t;V')$, see (6.22), and not in $L^2(0,t;V')$. Thus, with respect to time, the term in the integral might not be in $L^1(0,t)$ such that the integral might not be well defined.

Starting point for proving the energy inequality is the first line of (6.14), i.e.,

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}^n(t)\|^2_{L^2(\Omega)} + \nu\|\nabla\boldsymbol{u}^n(t)\|^2_{L^2(\Omega)} = \langle \boldsymbol{f}(t), \boldsymbol{u}^n(t)\rangle_{V',V}.$$

Integration in $(0,t)$ gives

$$\|\boldsymbol{u}^n(t)\|^2_{L^2(\Omega)} + 2\nu\|\nabla\boldsymbol{u}^n\|^2_{L^2(0,t;L^2(\Omega))} = \|\boldsymbol{u}^n(0)\|^2_{L^2(\Omega)} + 2\int_0^t \langle \boldsymbol{f}, \boldsymbol{u}^n\rangle_{V',V}(\tau)\ d\tau. \quad (6.25)$$

Since there is a subsequence of $\{\boldsymbol{u}^n\}_{n=1}^\infty$, which will be denoted for simplicity with the same symbol, which converges weakly to $\boldsymbol{u}$ in $L^2(0,T;V_{\mathrm{div}})$ and weakly* to $\boldsymbol{u}$ in $L^\infty\left(0,T;L^2_{\mathrm{div}}\right)$, see Corallary 6.14, it is known reference that

$$\begin{aligned}\|\boldsymbol{u}\|_{L^\infty(0,t;L^2(\Omega))} &\le \liminf_{n\to\infty}\|\boldsymbol{u}^n\|_{L^\infty(0,t;L^2(\Omega))},\\ \|\boldsymbol{u}\|_{L^2(0,t;V)} &\le \liminf_{n\to\infty}\|\boldsymbol{u}^n\|_{L^2(0,t;V)}\end{aligned}$$

for $t\in[0,T]$. Taking $n\to\infty$ in (6.25), using these estimates, noting that $\|\boldsymbol{u}(t)\|_{L^2(\Omega)} \le \|\boldsymbol{u}\|_{L^\infty(0,t;L^2(\Omega))}$, using that the limit satisfies the initial condition in the sense of $L^2(\Omega)$, see Lemma 6.18, and using the weak convergence in $L^2(0,t;V_{\mathrm{div}})$ leads to

$$\begin{aligned}&\|\boldsymbol{u}(t)\|^2_{L^2(\Omega)} + 2\nu\|\nabla\boldsymbol{u}\|^2_{L^2(0,t;L^2(\Omega))}\\ &\le \lim_{n\to\infty}\|\boldsymbol{u}^n(0)\|^2_{L^2(\Omega)} + 2\lim_{n\to\infty}\int_0^t\langle \boldsymbol{f}, \boldsymbol{u}^n\rangle_{V',V}(\tau)\ d\tau\\ &= \|\boldsymbol{u}(0)\|^2_{L^2(\Omega)} + 2\int_0^t \langle \boldsymbol{f}, \boldsymbol{u}\rangle_{V',V}(\tau)\ d\tau,\end{aligned}$$

which is just the energy inequality (6.23). Estimate (6.24) is now proved analogously to the two-dimensional case by using the estimate for the dual pairing and Korn's inequality. ■

Remark 6.23. Uniqueness of the weak solution. So far, the uniqueness of the weak solution cannot be proved for the three-dimensional case. Below, a few of the known results are sketched. □

Theorem 6.24. Serrin's condition, Serrin (1963). *Let $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ be two weak solutions corresponding to the same data $\boldsymbol{u}_0$ and $\boldsymbol{f}$. Assume that $\boldsymbol{u}_1$ satisfies the energy inequality* (6.23) *and that*

$$\boldsymbol{u}_2 \in L^r(0,T;L^s(\Omega)) \text{ for some } r,s \text{ with } \frac{d}{s}+\frac{2}{r}=1,\ s\in(d,\infty]. \quad (6.26)$$

Then $\boldsymbol{u}_1 = \boldsymbol{u}_2$ almost everywhere in $(0,T)\times\Omega$.

Proof. reference ■

Remark 6.25. Weak solution in the sense of Leray–Hopf. Since the weak solution in the sense of Leray–Hopf satisfies the energy inequality (6.23), see Lemma 6.22, one can take this solution as $\boldsymbol{u}_1$. For $\boldsymbol{u}_2$ it is known by the definition of weak solution that

$$\boldsymbol{u}_2 \in L^2\left(0,T;V_{\text{div}}\right) \cap L^\infty\left(0,T;L^2_{\text{div}}\left(\Omega\right)\right).$$

Now, one can check if $\boldsymbol{u}_2$ satisfies (6.26) □

Remark 6.26. 2D Navier–Stokes equations. Consider the case $d=2$ and let $\boldsymbol{u}$ be an arbitrary weak solution of the Navier–Stokes equations. Then, one has by the Sobolev embedding $H^{1/2}\left(\Omega\right) \to L^4\left(\Omega\right)$, see (A.12), the interpolation theorem between Sobolev spaces (A.10), and Poincaré's inequality (A.9)

$$\left\|\boldsymbol{u}\right\|^4_{L^4(\Omega)} \le C\left\|\boldsymbol{u}\right\|^4_{H^{1/2}(\Omega)} \le C\left\|\boldsymbol{u}\right\|^2_{L^2(\Omega)}\left\|\boldsymbol{u}\right\|^2_{H^1(\Omega)} \le C\left\|\boldsymbol{u}\right\|^2_{L^2(\Omega)}\left\|\nabla\boldsymbol{u}\right\|^2_{L^2(\Omega)}.$$

It follows that

$$\begin{aligned}\int_0^t \left\|\boldsymbol{u}\left(\tau\right)\right\|^4_{L^4(\Omega)}\ d\tau &\le C\int_0^t \left\|\boldsymbol{u}\left(\tau\right)\right\|^2_{L^2(\Omega)}\left\|\nabla\boldsymbol{u}\left(\tau\right)\right\|^2_{L^2(\Omega)}\,d\tau \\ &\le C\left\|\boldsymbol{u}\right\|^2_{L^\infty((0,T);L^2(\Omega))}\int_0^t \left\|\nabla\boldsymbol{u}\left(\tau\right)\right\|^2_{L^2(\Omega)}\,d\tau \\ &= C\left\|\boldsymbol{u}\right\|^2_{L^\infty((0,T);L^2(\Omega))}\left\|\boldsymbol{u}\right\|^2_{L^2((0,T);V)} < \infty.\end{aligned}$$

Hence, if $\boldsymbol{u}$ is an arbitrary weak solution of the Navier–Stokes equations for $d=2$, then $\boldsymbol{u} \in L^4\left(\left(0,T\right);L^4\left(\Omega\right)\right)$ and Serrin's condition (6.26) is fulfilled with $r=s=4$. Consequently, the weak solution of the Navier–Stokes equations is unique in two dimensions. Different ways for constructing a weak solution lead to the same solution. □

Remark 6.27. 3D Navier–Stokes equations. In three dimensions, one has by Sobolev embeddings (A.12) and the interpolation theorem for Sobolev spaces (A.10)

$$\left\|\boldsymbol{u}\right\|_{L^s(\Omega)} \le C\left\|\boldsymbol{u}\right\|^{(6-s)/(2s)}_{L^2(\Omega)}\left\|\nabla\boldsymbol{u}\right\|^{3(s-2)/(2s)}_{L^2(\Omega)}, \quad s \in [2,6].$$

It follows that $\boldsymbol{u} \in L^r\left(0,T;L^s\left(\Omega\right)\right)$ if

$$\int_0^t \left\|\boldsymbol{u}\left(\tau\right)\right\|^{r(6-s)/(2s)}_{L^2(\Omega)}\left\|\nabla\boldsymbol{u}\left(\tau\right)\right\|^{3r(s-2)/(2s)}_{L^2(\Omega)}\ d\tau < \infty$$

for almost all $t \in [0,T]$. By Definition 6.9 of a weak solution, the power of the $L^2\left(\Omega\right)$ norm of the gradient of $\boldsymbol{u}$ is not known to be larger than 2. This gives

$$2 \ge \frac{3r\left(s-2\right)}{2s} \quad\Longleftrightarrow\quad \frac{3}{s}+\frac{2}{r} \ge \frac{3}{2} \quad\Longleftrightarrow\quad \frac{d}{s}+\frac{2}{r} \ge \frac{3}{2}.$$

That means, Serrin's condition (6.26) is not fulfilled for $d = 3$ and the uniqueness cannot be proved with Theorem 6.24.

Attempts to construct counter examples for the uniqueness of the weak solution failed so far, too.

Summary. The uniqueness of a weak solution can be proved up to now only for the two-dimensional Navier–Stokes equations. In three dimensions, one cannot exclude so far that weak solutions constructed with different methods, see Remark 6.11, are not the same. □

Remark 6.28. Connection to the classical solution, miscellaneous.

- Theorem 6.24 states that the weak solution is unique even for the three-dimensional Navier–Stokes equations if it satisfies additional regularity assumptions (Serrin's condition). But it is not known for $d = 3$ if every weak solution possesses such additional regularity. The situation for the classical solution is correspondingly.
  Classical solution in three dimensions: Its existence cannot yet be proved. But if a classical solution exists, it is unique.
- The mathematical reason of the gap in the theory is the dependence of the Sobolev embeddings on the dimension of the domain. From the physical point of view even the question arises if the Navier–Stokes equations are the correct model for three-dimensional (turbulent) flows.
- The answer of the question of *uniqueness of the weak solution in 3d* or *existence of a classical solution in 3d* is one of the major mathematical challenges of this century. There is a prize of one million dollar for those scientists who answer these questions, see Fefferman Fefferman (2000).
- After having established properties for the velocity solution, one can introduce the pressure and study its properties, like its regularity.
- There are much more results on the existence and uniqueness of solutions of the Navier–Stokes equations, e.g., for small data and unbounded domains.

□

6.2 Temporal Discretizations Leading to Coupled Problems

6.2.1 *θ*-Schemes as Discretization in Time

Remark 6.29. Principal approach in the application of θ-schemes. θ-schemes use the following strategy for the full discretization and linearization of (6.1):

1. *Semi discretization of* (6.1) *in time.* The semi discretization in time leads in each discrete time step to a nonlinear system of equations of saddle point type.

2. *Variational formulation and linearization.* The nonlinear system of equations is reformulated as variational problem and the nonlinear variational problem is linearized.
3. *Discretization of the linear systems in space.* The linear system of equations arising in each step of the nonlinear iteration is discretized by a finite element discretization using, e.g., an inf-sup stable pair of finite element spaces.

This approach, which applies first the discretization in time and then in space, is also called method of Rothe or horizontal method of lines. The other way, discretizing first in space to get an ordinary differential equation and then in time, is called (vertical) method of lines. The individual steps in this strategy are described in detail in this section. check what happens if different order is applied? Hunsdorfer, Verwer? □

Remark 6.30. General θ-scheme for the Navier–Stokes equations. Let Δt_{n+1} be the current time step from t_n to t_{n+1}, i.e. $\Delta t_{n+1} = t_{n+1} - t_n$. Quantities at time level t_k, with $t_k \in [t_n, t_{n+1}]$, are denoted by a subscript k. To describe the time stepping scheme the incompressible Navier–Stokes equations (6.1), a general time step of the form

$$\begin{aligned} \boldsymbol{u}_{k+1} + \theta_1 \Delta t_{n+1} \big[-\nu \Delta \boldsymbol{u}_{k+1} + (\boldsymbol{u}_{k+1} \cdot \nabla)\, \boldsymbol{u}_{k+1} \big] + \Delta t_{k+1} \nabla p_{k+1} & \\ = \boldsymbol{u}_k - \theta_2 \Delta t_{n+1} \big[-\nu \Delta \boldsymbol{u}_k + (\boldsymbol{u}_k \cdot \nabla)\, \boldsymbol{u}_k \big] + \theta_3 \Delta t_{n+1} \boldsymbol{f}_k & \\ + \theta_4 \Delta t_{n+1} \boldsymbol{f}_{k+1}, & \qquad (6.27) \\ \nabla \cdot \boldsymbol{u}_{k+1} = 0, & \end{aligned}$$

with the parameters $\theta_1, \ldots, \theta_4$, is introduced. Note the different indices k and n, which are necessary to describe the fractional-step θ-scheme. For one-step θ-schemes, it is $n = k$. The time step (6.27) allows the implementation of a number of time stepping schemes by one single formula and the choice between the schemes by setting four parameters. □

Example 6.31. Explicit and implicit Euler scheme, Crank–Nicolson scheme. Three well known one-step θ-schemes, the forward and backward Euler scheme and the Crank–Nicolson scheme, are obtained by appropriate choices of these parameters, see Table 6.1.

Table 6.1 One-step θ-schemes.

	θ_1	θ_2	θ_3	θ_4	t_k	t_{k+1}	Δt_{k+1}	order
forward Euler scheme	0	1	1	0	t_n	t_{n+1}	Δt_{n+1}	1
backward Euler scheme (BWE)	1	0	0	1	t_n	t_{n+1}	Δt_{n+1}	1
Crank–Nicolson scheme (CN)	0.5	0.5	0.5	0.5	t_n	t_{n+1}	Δt_{n+1}	2

A few remarks concerning these schemes should be given already here. More detailed comments will be provided in Remark 6.39, after the presentation of a numerical example.

- Forward or explicit Euler scheme. The appearance of the viscous term, which has the same form as a diffusive term, leads to a stiff ordinary differential equation with respect to time, see (Hairer and Wanner, 2010, pp. 6). From the numerical analysis of ordinary differential equations it is known that explicit schemes have to be used with very small time steps for stiff problems to obtain stable simulations. For the Navier–Stokes equations, the time step has to be usually so small that the simulations become very inefficient. For this reason, the forward Euler scheme is not recommended for the discretization of the incompressible Navier–Stokes equations.
- Backward or implicit Euler scheme. This first order scheme is quite popular. However, it is known from the literature, e.g., from John et al. (2006), and it will be demonstrated in Example 6.38, that the use of the backward Euler scheme in combination with higher order discretizations in space might lead to rather inaccurate results, compared with the results computed with higher order temporal discretizations.
- Crank–Nicolson scheme, trapezoidal rule. The Crank–Nicolson scheme is a second order scheme whose use is very popular. However, meanwhile situations are known where this scheme is not sufficiently stable and should not be applied, e.g., if adaptively refined grids are used which change in time, see Besier and Wollner (2012). check

□

remark on CFL

Example 6.32. Fractional-step θ-scheme. The fractional-step θ-scheme, developed in Bristeau et al. (1987), is obtained by three steps of form (6.27). There exist two variants of this scheme. The two variants, FS0 and FS1, are presented in Table 6.2, where

$$\theta = 1 - \frac{\sqrt{2}}{2}, \quad \tilde{\theta} = 1 - 2\theta, \quad \tau = \frac{\tilde{\theta}}{1-\theta}, \quad \eta = 1 - \tau.$$

A fractional-step θ-scheme is a clever combination of three first order one-step schemes to achieve a strongly A-stable second order scheme.

FS1 requires the evaluation of $\boldsymbol{f}$ only at the times t_n and $t_{n+1} - \theta\Delta t_{n+1}$ whereas FS0 needs the evaluation of $\boldsymbol{f}$ in addition at $t_n + \theta\Delta t_{n+1}$ and at t_{n+1}. Both variants are second order schemes but FS1 does not integrate second order polynomials (with respect to t) exactly. However, most of other fundamental properties, like stability, are the same for both variants. Results for the two–dimensional Navier–Stokes equations, John et al. (2006), show that FS0 is often considerable more accurate than FS1. Thus, if the evaluation of the right-hand side is not very expensive, FS0 should be preferred. □

Table 6.2 The two variants of the fractional-step θ-schemes

	θ_1	θ_2	θ_3	θ_4	t_k	t_{k+1}	Δt_{k+1}	order
FS0	$\tau\theta$	$\eta\theta$	$\eta\theta$	$\tau\theta$	t_n	$t_n + \theta\Delta t_{n+1}$	$\theta\Delta t_{n+1}$	
	$\eta\tilde{\theta}$	$\tau\tilde{\theta}$	$\tau\tilde{\theta}$	$\eta\tilde{\theta}$	$t_n + \theta\Delta t_{n+1}$	$t_{n+1} - \theta\Delta t_{n+1}$	$\tilde{\theta}\Delta t_{n+1}$	2
	$\tau\theta$	$\eta\theta$	$\eta\theta$	$\tau\theta$	$t_{n+1} - \theta\Delta t_{n+1}$	t_{n+1}	$\theta\Delta t_{n+1}$	
FS1	$\tau\theta$	$\eta\theta$	θ	0	t_n	$t_n + \theta\Delta t_{n+1}$	$\theta\Delta t_{n+1}$	
	$\eta\tilde{\theta}$	$\tau\tilde{\theta}$	0	$\tilde{\theta}$	$t_n + \theta\Delta t_{n+1}$	$t_{n+1} - \theta\Delta t_{n+1}$	$\tilde{\theta}\Delta t_{n+1}$	2
	$\tau\theta$	$\eta\theta$	θ	0	$t_{n+1} - \theta\Delta t_{n+1}$	t_{n+1}	$\theta\Delta t_{n+1}$	

Remark 6.33. On the pressure, the pressure at the initial time. In scheme (6.27), the pressure term is scaled in a different way than the viscous and the nonlinear convective term if $\theta_1 \neq 1$. The derivation of (6.27) can be considered to consist of two steps. Starting point is the variational form of the Navier–Stokes equations in V_{div}, where no pressure appears, and the θ–scheme is applied. In the second step, the variational form is extended to $V \times Q$, where one needs a Lagrange multiplicator for the incompressibility constraint for the unknown velocity $\boldsymbol{u}_{k+1}$. The term $\Delta t_{k+1}\nabla p_{k+1}$, depending only on the time t_{k+1} is this Lagrange multiplicator, leading to the problem: Find $(\boldsymbol{u}_{k+1}, p_{k+1}) \in V \times Q$ such that for all $(\boldsymbol{v}, q) \in V \times Q$

$$
\begin{aligned}
(\boldsymbol{u}_{k+1}, \boldsymbol{v}) + \theta_1 \Delta t_{n+1}\big[\, & (\nu\nabla \boldsymbol{u}_{k+1}, \nabla \boldsymbol{v}) + ((\boldsymbol{u}_{k+1}\cdot\nabla)\,\boldsymbol{u}_{k+1}, \boldsymbol{v})\,\big] \\
& -\Delta t_{k+1}\,(\nabla\cdot\boldsymbol{v}, p_{k+1}) \\
& = (\boldsymbol{u}_k, \boldsymbol{v}) - \theta_2 \Delta t_{n+1}\big[\,(\nu\nabla \boldsymbol{u}_k, \nabla \boldsymbol{v}) + ((\boldsymbol{u}_k\cdot\nabla)\,\boldsymbol{u}_k, \boldsymbol{v})\,\big] \\
& \quad +\theta_3 \Delta t_{n+1}\langle \boldsymbol{f}_k, \boldsymbol{v}\rangle_{V',V} + \theta_4 \Delta t_{n+1}\langle \boldsymbol{f}_{k+1}, \boldsymbol{v}\rangle_{V',V}, \qquad (6.28)\\
(\nabla\cdot\boldsymbol{u}_{k+1}, q) = 0. &
\end{aligned}
$$

Then, (6.27) is just the corresponding strong form of this equation.

It is also possible to apply to θ-scheme to the complete Navier–Stokes equations which leads to the variational form: Find $(\boldsymbol{u}_{k+1}, p_{k+1}) \in V \times Q$ for all $(\boldsymbol{v}, q) \in V \times Q$ such that

$$
\begin{aligned}
(\boldsymbol{u}_{k+1}, \boldsymbol{v}) + \theta_1 \Delta t_{n+1}\big[\, & (\nu\nabla \boldsymbol{u}_{k+1}, \nabla \boldsymbol{v}) + ((\boldsymbol{u}_{k+1}\cdot\nabla)\,\boldsymbol{u}_{k+1}, \boldsymbol{v}) \\
& - (\nabla\cdot\boldsymbol{v}, p_{k+1})\,\big] \\
& = (\boldsymbol{u}_k, \boldsymbol{v}) - \theta_2 \Delta t_{n+1}\big[\,(\nu\nabla \boldsymbol{u}_k, \nabla \boldsymbol{v}) + ((\boldsymbol{u}_k\cdot\nabla)\,\boldsymbol{u}_k, \boldsymbol{v}) \\
& \quad - (\nabla\cdot\boldsymbol{v}, p_k)\,\big] \\
& \quad +\theta_3 \Delta t_{n+1}\langle \boldsymbol{f}_k, \boldsymbol{v}\rangle_{V',V} + \theta_4 \Delta t_{n+1}\langle \boldsymbol{f}_{k+1}, \boldsymbol{v}\rangle_{V',V}, \qquad (6.29)\\
(\nabla\cdot\boldsymbol{u}_{k+1}, q) = 0. &
\end{aligned}
$$

In this approach, the pressure at the previous time appears in the scheme.

For (6.29), it follows that the pressure at the initial time is needed in the first time step. This pressure is not part of the definition of the problem. Thus, one has to construct an appropriate pressure. A possible way, which is the most common one, starts by assuming that all functions in the Navier–Stokes equations (6.1) are sufficiently smooth. Then, the divergence operator is applied to the momentum equation. Using that $\boldsymbol{u}$ is divergence-free, such that

$$
\begin{aligned}
\nabla \cdot \partial_t \boldsymbol{u} &= \partial_t \left(\nabla \cdot \boldsymbol{u}\right) = 0, \\
\nabla \cdot \Delta \boldsymbol{u} &= \partial_x \left(\partial_{xx} u_1 + \partial_{yy} u_1 + \partial_{zz} u_1\right) + \partial_y \left(\partial_{xx} u_2 + \partial_{yy} u_2 + \partial_{zz} u_2\right) \\
&\quad + \partial_z \left(\partial_{xx} u_3 + \partial_{yy} u_3 + \partial_{zz} u_3\right) \\
&= \partial_{xx} \left(\partial_x u_1 + \partial_y u_2 + \partial_z u_3\right) + \partial_{xx} \left(\partial_x u_1 + \partial_y u_2 + \partial_z u_3\right) \\
&\quad + \partial_{xx} \left(\partial_x u_1 + \partial_y u_2 + \partial_z u_3\right) \\
&= 0,
\end{aligned}
$$

one obtains

$$
- \nabla \cdot \nabla p = -\Delta p = -\nabla \cdot \boldsymbol{f} + \nabla \cdot \left((\boldsymbol{u} \cdot \nabla))\boldsymbol{u}\right). \tag{6.30}
$$

This equation is a Poisson equation for the pressure, a so-called pressure Poisson equation. To get a well-posed problem, it has to be equipped with appropriate boundary conditions. To this end, one considers the restriction of the momentum equation in (6.1) to the boundary Γ, multiplies this restriction with the outward pointing unit normal vector $\boldsymbol{n}$, and uses that $\boldsymbol{u}$ vanishes at the boundary for all times, which yields

$$
\nabla p \cdot \boldsymbol{n} = (\boldsymbol{f} + \nu \Delta \boldsymbol{u}) \cdot \boldsymbol{n} \quad \text{on } \Gamma. \tag{6.31}
$$

Applying this approach at the initial time gives the following pressure Poisson problem with Neumann boundary conditions for the initial pressure p_0

$$
\begin{aligned}
-\Delta p_0 &= -\nabla \cdot \boldsymbol{f}(0,\cdot) + \nabla \cdot \left(\left(\boldsymbol{u}_0 \cdot \nabla\right) \boldsymbol{u}_0\right) && \text{in } \Omega, \\
\frac{\partial p_0}{\partial \boldsymbol{n}} &= \left(\boldsymbol{f}(0,\cdot) + \nu \Delta \boldsymbol{u}_0\right) \cdot \boldsymbol{n} && \text{on } \Gamma, \\
\int_\Omega p_0(\boldsymbol{x}) \; d\boldsymbol{x} &= 0.
\end{aligned}
$$

In practice, the initial pressure might not be a difficulty at all. Often, even an initial condition for the velocity is not known, e.g., see Example D.10. Then, one starts the simulation with some initial condition and let it run until the flow is developed. Of course, in the so generated developed flow field, one has velocity and pressure. Then, a time instance of the developed flow field is saved, both velocity and pressure, and the saved flow field is used as initial condition for all further simulations.

Both forms (6.28) and (6.29) are used. references To the best of our knowledge, a comprehensive numerical comparison cannot be found so far in the literature. But it is believed that one obtains very similar results with both formulations. In the numerical examples presented here, always the form (6.28) was used. □

Remark 6.34. Variational formulation. The solution of (6.27) will be approximated by a finite element method, where the basis of the finite element method is a variational formulation of (6.27). The derivation of the variational problem is done in the usual way by multiplying the equations in (6.27) with test functions, integrating on Ω and applying integration by parts, which gives (6.28). Of course, any other form of the convective term discussed in Section 5.1.2 can be used. For brevity, the approach will be described only for the convective form of the convective term. □

Remark 6.35. Linearization of the variational form. The nonlinear system (6.28) is solved iteratively starting with an initial guess $\left(\boldsymbol{u}_{k+1}^0, p_{k+1}^0\right)$. The nonlinear convective term has to be linearized, where the same linearizations as for the steady-state Navier–Stokes equations can be applied, see Section 5.3.

Given a known velocity field $\boldsymbol{u}_{k+1}^{(m)}$, The fixed point iteration or Picard iteration uses the approximation

$$\left(\boldsymbol{u}_{k+1}^{(m+1)} \cdot \nabla\right) \boldsymbol{u}_{k+1}^{(m+1)} \approx \left(\boldsymbol{u}_{k+1}^{(m)} \cdot \nabla\right) \boldsymbol{u}_{k+1}^{(m+1)}.$$

Then, the fixed point iteration for solving (6.28) has the following form: Given $\left(\boldsymbol{u}_{k+1}^{(m)}, p_{k+1}^{(m)}\right)$, the iterate $\left(\boldsymbol{u}_{k+1}^{(m+1)}, p_{k+1}^{(m+1)}\right)$ is computed by solving

$$\begin{aligned}
\left(\boldsymbol{u}_{k+1}^{(m+1)}, \boldsymbol{v}\right) &+ \theta_1 \Delta t_{n+1} \left[\left(\nu \nabla \boldsymbol{u}_{k+1}^{(m+1)}, \nabla \boldsymbol{v}\right) + \left(\left(\boldsymbol{u}_{k+1}^{(m)} \cdot \nabla\right) \boldsymbol{u}_{k+1}^{(m+1)}, \boldsymbol{v}\right)\right] \\
&- \Delta t_{k+1} \left(\nabla \cdot \boldsymbol{v}, p_{k+1}^{(m+1)}\right) \\
&= (\boldsymbol{u}_k, \boldsymbol{v}) + \theta_3 \Delta t_{n+1} \langle \boldsymbol{f}_k, \boldsymbol{v}\rangle_{V',V} + \theta_4 \Delta t_{n+1} \langle \boldsymbol{f}_{k+1}, \boldsymbol{v}\rangle_{V',V} \qquad (6.32) \\
&- \theta_2 \Delta t_{n+1} \left[(\nu \nabla \boldsymbol{u}_k, \nabla \boldsymbol{v}) + ((\boldsymbol{u}_k \cdot \nabla) \boldsymbol{u}_k, \boldsymbol{v}) \right] \\
0 &= \left(\nabla \cdot \boldsymbol{u}_{k+1}^{(m+1)}, q\right), \quad \forall\ (\boldsymbol{v}, q) \in V \times Q,
\end{aligned}$$

$m = 0, 1, 2, \ldots$. The equations (6.32) are of Oseen type, see (4.2). The right-hand side of this equation does not change during the iteration.

For Newton's method, the nonlinear convective term is linearized as follows

$$\begin{aligned}
&\left(\boldsymbol{u}_{k+1}^{(m+1)} \cdot \nabla\right) \boldsymbol{u}_{k+1}^{(m+1)} \\
&\approx \left(\boldsymbol{u}_{k+1}^{(m)} \cdot \nabla\right) \boldsymbol{u}_{k+1}^{(m+1)} + \left(\boldsymbol{u}_{k+1}^{(m+1)} \cdot \nabla\right) \boldsymbol{u}_{k+1}^{(m)} - \left(\boldsymbol{u}_{k+1}^{(m)} \cdot \nabla\right) \boldsymbol{u}_{k+1}^{(m)},
\end{aligned}$$

such that the iteration is of the form: Given $\left(\boldsymbol{u}_{k+1}^{(m)}, p_{k+1}^{(m)}\right)$, the iterate $\left(\boldsymbol{u}_{k+1}^{(m+1)}, p_{k+1}^{(m+1)}\right)$ is computed by solving

$$
\begin{aligned}
&\left(\boldsymbol{u}_{k+1}^{(m+1)}, \boldsymbol{v}\right) + \theta_1 \Delta t_{n+1} \Big[\left(\nu \nabla \boldsymbol{u}_{k+1}^{(m+1)}, \nabla \boldsymbol{v}\right) + \left(\left(\boldsymbol{u}_{k+1}^{(m)} \cdot \nabla\right) \boldsymbol{u}_{k+1}^{(m+1)}, \boldsymbol{v}\right) \\
&\qquad + \left(\left(\boldsymbol{u}_{k+1}^{(m+1)} \cdot \nabla\right) \boldsymbol{u}_{k+1}^{(m)}, \boldsymbol{v}\right) \Big] - \Delta t_{k+1} \left(\nabla \cdot \boldsymbol{v}, p_{k+1}^{(m+1)}\right) \\
&\quad = (\boldsymbol{u}_k, \boldsymbol{v}) + \theta_3 \Delta t_{n+1} \langle \boldsymbol{f}_k, \boldsymbol{v}\rangle_{V',V} + \theta_4 \Delta t_{n+1} \langle \boldsymbol{f}_{k+1}, \boldsymbol{v}\rangle_{V',V} \\
&\qquad + \theta_1 \Delta t_{n+1} \left(\left(\boldsymbol{u}_{k+1}^{(m)} \cdot \nabla\right) \boldsymbol{u}_{k+1}^{(m)}, \boldsymbol{v}\right) \\
&\qquad - \theta_2 \Delta t_{n+1} \big[(\nu \nabla \boldsymbol{u}_k, \nabla \boldsymbol{v}) + ((\boldsymbol{u}_k \cdot \nabla) \boldsymbol{u}_k, \boldsymbol{v}) \big] \\
&\quad 0 = \left(\nabla \cdot \boldsymbol{u}_{k+1}^{(m+1)}, q\right), \quad \forall\ (\boldsymbol{v}, q) \in V \times Q,
\end{aligned}
\tag{6.33}
$$

$m = 0, 1, 2, \ldots$.

The initial guess can be chosen to be the solution of the previous time step $(\boldsymbol{u}_{k+1}^0, p_{k+1}^0) = (\boldsymbol{u}_k, p_k)$ or some extrapolation from more than one previous times.

The numerical approximation of the Navier–Stokes equations (6.1) using the approach described in this section requires the repeated solution of linear saddle point problems of form (6.32) or (6.33) in each discrete time. □

Remark 6.36. Comparison of fixed point iteration and Newton's method. For the comparison of the fixed point iteration and Newton's method, the same statements as given for the stationary Navier–Stokes equations in Section 5.3, e.g., in Remarks 5.37 and 5.39 apply.

Using an approach which does not solve the linear systems of equations exactly, the numerical studies at a three-dimensional problem in John (2006) showed that the Picard iteration is considerably more efficient then Newton's method in this situation. It will be discussed in ?? that for problems in three dimensions usually iterative solvers for the linear systems of equations should be applied and that it is not efficient to solve the linear systems of equations with high accuracy in this case. □

Remark 6.37. Semi-implicit methods, IMEX schemes. A popular approach is the use of semi-implicit schemes, so called IMEX (implicitexplicit) schemes, which avoid the solution of a nonlinear problem at each discrete time. In these schemes, the nonlinear term $(\boldsymbol{u}_{k+1} \cdot \nabla) \boldsymbol{u}_{k+1}$, e.g., appearing in (6.27), is replaced by $(\boldsymbol{u}_{\text{prev}} \cdot \nabla) \boldsymbol{u}_{k+1}$, where $\boldsymbol{u}_{\text{prev}}$ can be obtained from already computed solutions. The simplest way consists in using $\boldsymbol{u}_{\text{prev}} = \boldsymbol{u}_k$, but often a linear extrapolation of the previous two time steps is used, i.e.,

$$
\boldsymbol{u}_{\text{prev}} = \frac{\Delta t_{k+1}}{\Delta t_k} (\boldsymbol{u}_k - \boldsymbol{u}_{k-1}) + \boldsymbol{u}_k.
$$

In Caiazzo et al. (2014), a comparison of the fully implicit Crank–Nicolson scheme (6.27), i.e., $\theta_1 = \ldots = \theta_4 = 0.5$, and the IMEX Crank–Nicolson scheme with $(\boldsymbol{u}_k \cdot \nabla) \boldsymbol{u}_{k+1}$ for Example D.10 can be found. It could be observed that the results obtained with the IMEX Crank–Nicolson scheme are notable less accurate in comparison with the fully nonlinear scheme.

If not noted otherwise, then the fully implicit scheme was used in all examples presented here. A discussion on the effect of using different stopping criteria on the accuracy of the results can be found in Example 6.38. □

Example 6.38. Accuracy studies at Example D.11. This examples considers the two-dimensional flow around a cylinder defined in Example D.11. At the outlet, the do-nothing boundary conditions (D.26) were applied.

Results are presented for three implicit θ-schemes: the backward Euler scheme, the Crank–Nicolson schmeme, and the fractional-step θ-scheme. Note that in this problem $\boldsymbol{f} = \mathbf{0}$ for all times such that FS0 and FS1 are identical. With respect to the spatial discretization, results for the P_2/P_1 finite element on triangular grids and the Q_2/P_1^{disc} finite element on quadrilateral grids are shown. The grids from Figure 5.5 were used as initial grids (level 0) in the simulations. Table 6.3 gives information about the number of degrees of freedom for different refinement levels.

Table 6.3 Example 6.38. Number of degrees of freedom in space (including Dirichlet nodes).

	P_2/P_1			Q_2/P_1^{disc}		
level	velocity	pressure	all	velocity	pressure	all
3	25 408	3248	28 656	27 232	9984	37 216
4	100 480	12 704	113 184	107 712	39 936	147 648
5	399 616	50 240	449 856	428 416	159 744	588 160

The nonlinear system in each discrete time was solved until the Euclidean norm of the residual vector was less than 10^{-10}. For the evaluation of the drag and lift coefficient with (**??**) and (**??**), the same approach was applied as described in Example 5.29. The temporal derivative in (**??**) and (**??**) was approximated by a backward difference formula

$$\partial_t \boldsymbol{u}_{n+1}^h \approx \frac{\boldsymbol{u}_{n+1}^h - \boldsymbol{u}_n^h}{\Delta t_{n+1}}.$$

Figures 6.1 – 6.6 present results obtained for the Q_2/P_1^{disc} finite element discretization. It can be observed that the use of the backward Euler scheme leads by far to the most inaccurate results among all time stepping schemes. In the considered example, the temporal and the spatial error are both of importance. Using the backward Euler scheme, the temporal error dominates the error in space. Considerably smaller time steps would be necessary to reach a similar error level as obtained with the second order time stepping schemes. The temporal evolution of the lift coefficient, Figure 6.3 shows that even the vortex shedding is not correctly predicted with the backward Euler scheme with time steps larger than or equal to $\Delta t = 0.005$. Altogether, this example demonstrates the qualitative difference of using first and second order time stepping schemes, in combination with higher order discretizations in space, in problems where the error in space does not dominate.

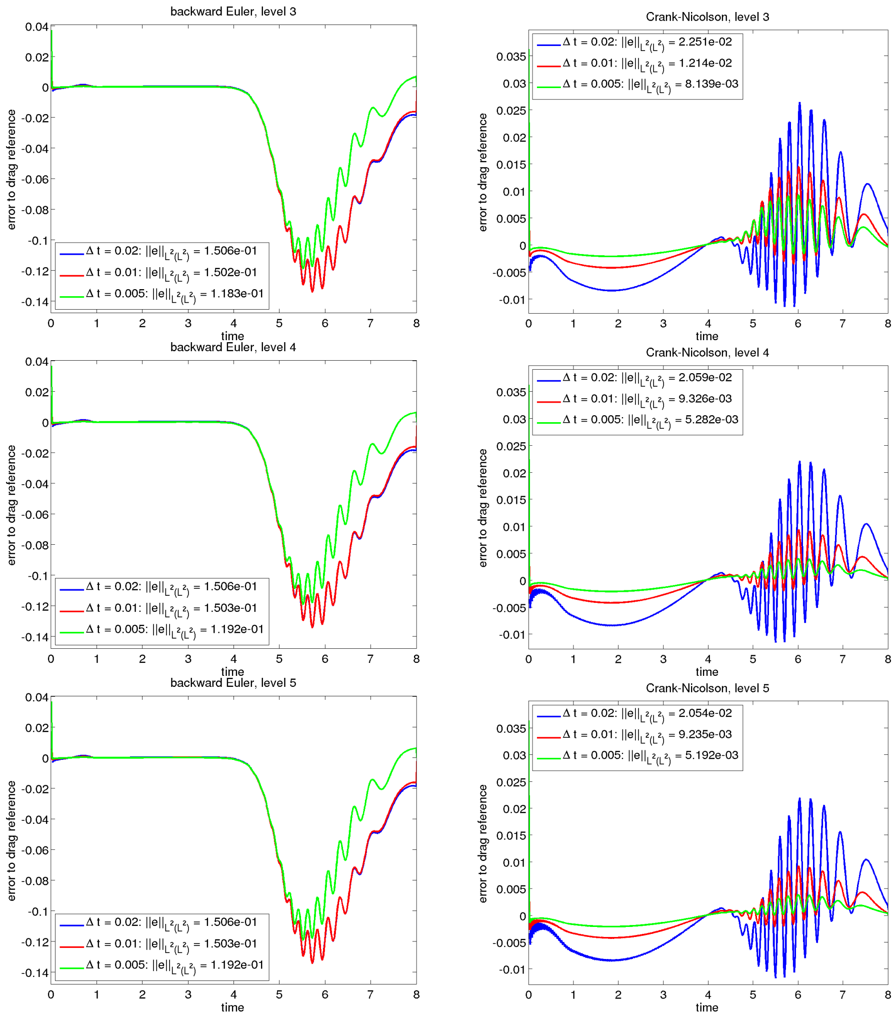

Fig. 6.1 Example 6.38. Error to the reference curve for the drag coefficient from Figure D.6, Q_2/P_1^{disc}, levels 3 – 5 (top to bottom), left: backward Euler scheme, right: Crank–Nicolson, both for different lengths of the time step.

Also for the second order schemes, the error reductions with decreasing length of the time step can be clearly observed. Results with the second order time stepping schemes can be compared in Figures 6.2, 6.5, and 6.6. Despite the same order of the Crank–Nicolson scheme and the fractional-step θ-scheme, it can be observed that the coefficients computed with the

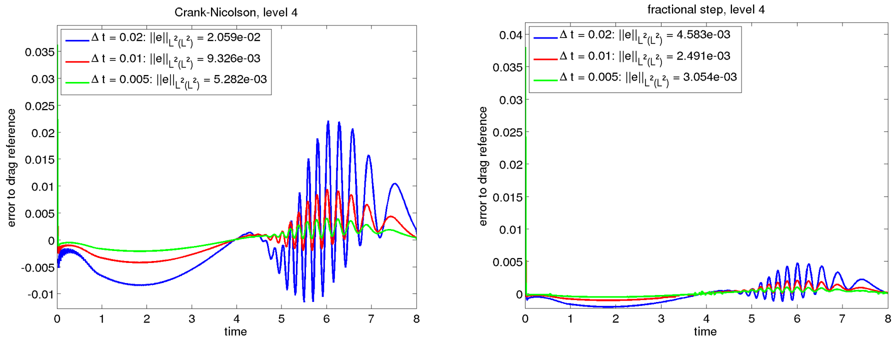

Fig. 6.2 Example 6.38. Error to the reference curve for the drag coefficient from Figure D.6, Q_2/P_1^{disc}, left: Crank–Nicolson scheme, right: fractional-step θ-scheme. Note the different scales in the pictures.new pics

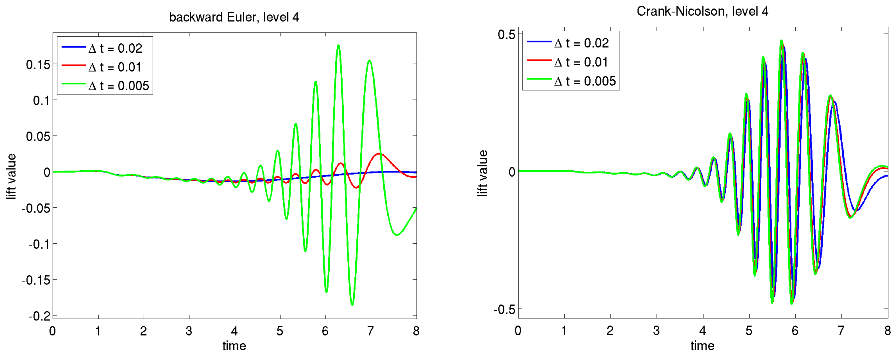

Fig. 6.3 Example 6.38. Temporal evolution of the lift coefficient, Q_2/P_1^{disc}, left: backward Euler scheme, right: Crank–Nicolson scheme. Note the different scales in both pictures.

fractional-step θ-scheme are considerably more accurate. However, it will be discussed in Remark 6.39 that the numerical costs (computing times) for simulating the flow with the fractional-step θ-scheme are usually two to three times higher than the costs for the Crank–Nicolson scheme.

Some representative results for the P_2/P_1 finite element discretization are presented in Figures 6.7 – 6.9. The assessment of these results leads to the same conclusions as they were drawn for the Q_2/P_1^{disc} finite element discretization.

The choice of the stopping criterion for solving the nonlinear problems in each discrete time is a delictate issue. If the stopping criterion is rather hard, the computation of the solution in the discrete times might become time-consuming and the overall computing time might become large. But one can expect to get accurate results. On the other hand, using a soft stopping criterion might speed up the simulations considerably. However, a rather inaccurate solution might be computed in some discrete times such that the

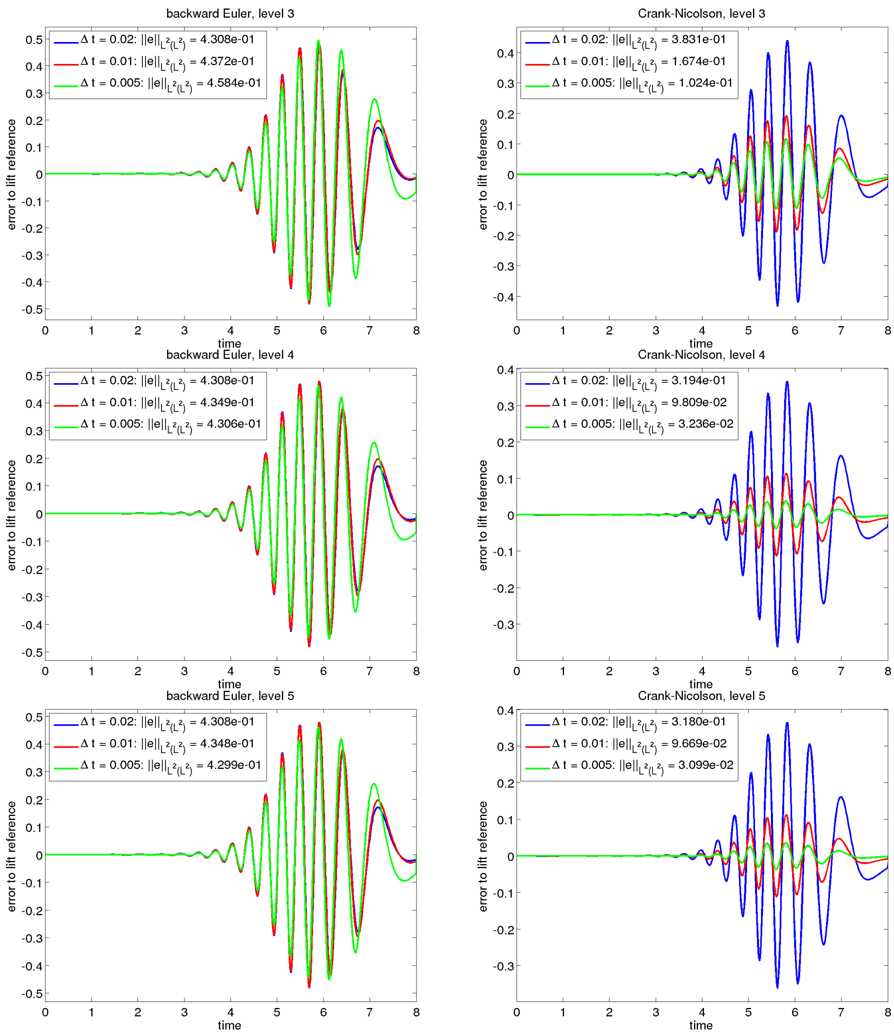

Fig. 6.4 Example 6.38. Error to the reference curve for the lift coefficient from Figure D.6, Q_2/P_1^{disc}, levels 3 – 5 (top to bottom), left: backward Euler scheme, right: Crank–Nicolson, both for different lengths of the time step.

overall error increases noteably. This effect is demonstrated in Figure 6.10. It can be seen that there is a notable gain in accuracy if as stopping criterion an Euclidean norm of the residual vector lower than or equal to 10^{-8} is used instead of 10^{-6}. It was found that the computing time for the tolerance 10^{-8} is around 2.5 times longer than for 10^{-6}. Decreasing the tolerance to 10^{-10} does not posses much impact on the accuracy compared with the tolerance

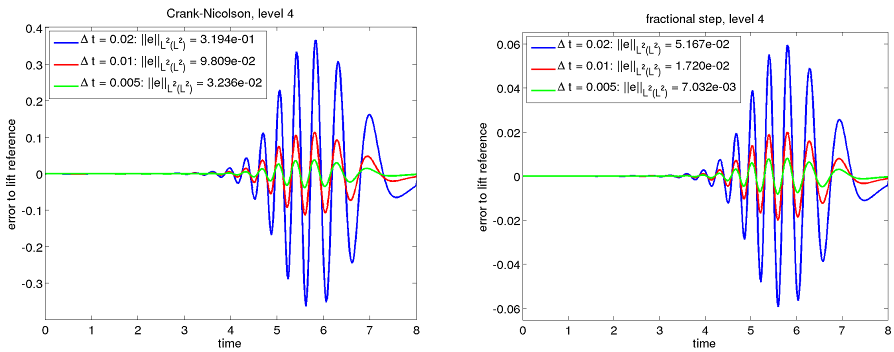

Fig. 6.5 Example 6.38. Error to the reference curve for the lift coefficient from Figure D.6, Q_2/P_1^{disc}, left: Crank–Nicolson scheme, right: fractional-step θ-scheme. Note the different scales in the pictures.

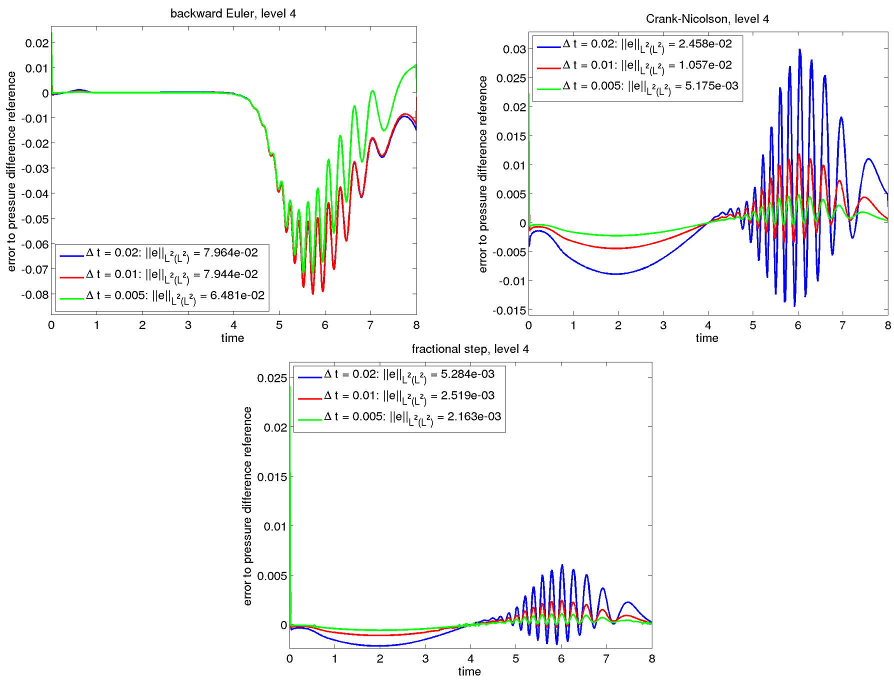

Fig. 6.6 Example 6.38. Error to the reference curve for the pressure difference between the front and the back of the cylinder from Figure D.6, Q_2/P_1^{disc}, backward Euler scheme, Crank–Nicolson scheme, fractional-step θ-scheme (left to right, top to bottom).

10^{-8}, however the computing time increased once more by a factor of around 2.8.

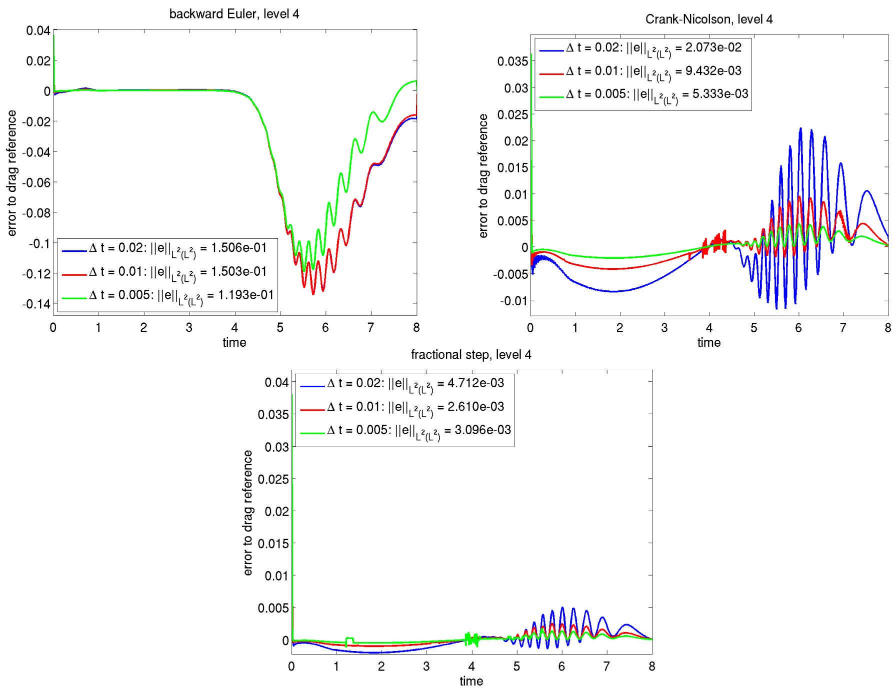

Fig. 6.7 Example 6.38. Error to the reference curve for the drag coefficient from Figure D.6, P_2/P_1, backward Euler scheme, Crank–Nicolson scheme, fractional-step θ-scheme (left to right, top to bottom).

What does it mean 'hard' and 'soft' stopping criterion depends on the concrete example and on the used methods. □

Remark 6.39. Numerical experience with implicit θ-schemes for the Navier–Stokes equations. The Crank–Nicolson and the fractional-step θ-scheme are already well tested and compared for the Navier–Stokes equations, see Emmrich (2001) for an overview. The Crank–Nicolson scheme is A-stable whereas the fractional-step θ-scheme is even strongly A-stable. That means, the Crank–Nicolson scheme may lead to numerical oscillations in problems with rough initial data or boundary conditions. These oscillations are damped out only if sufficiently small time steps are used. Compared with the fractional-step θ-scheme, a smaller time step might be necessary for the Crank–Nicolson scheme to ensure robustness. In numerical tests for the two–dimensional Navier–Stokes equations, John et al. (2006), the results obtained with FS0 were often somewhat more accurate than the results computed with CN. However, the computing time of FS0 was in general twice to three times the computing time of CN. check John and Rang (2010) □

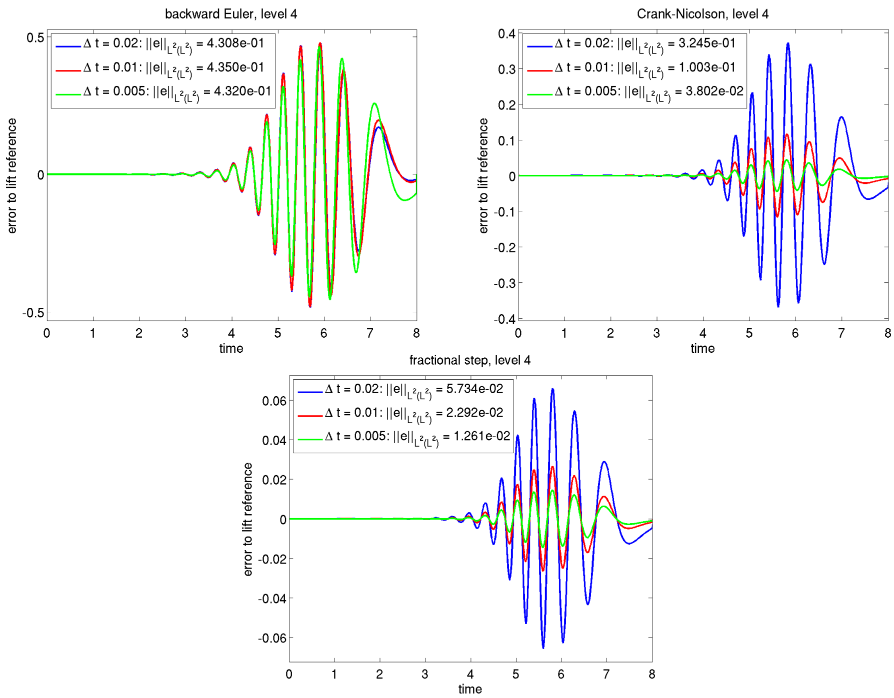

Fig. 6.8 Example 6.38. Error to the reference curve for the lift coefficient from Figure D.6, P_2/P_1, backward Euler scheme, Crank–Nicolson scheme, fractional-step θ-scheme (left to right, top to bottom).

Remark 6.40. Adpative time step control. There are well understood techniques for an adaptive time step control in the numerical simulation of ordinary differential equations. These techniques rely on comparing two solutions obtained with methods of different order and on procedures, so-called controllers, which propose the length of the next time step. If possible, so-called embedded schemes are used, as an inexpensive approach, for computing a solution with one order less than for the original time stepping scheme. However, there are no embedded schemes for the class of θ-schemes such that other approaches for an adaptive time step control are necessary.

The approach proposed in Turek (1999) compares the results of the fractional-step θ-scheme and the Crank–Nicolson scheme. These schemes have a different constant in the leading term of their error expansions. This difference, together with the difference of the results obtained with both schemes, can be used to estimate an appropriate length of the time step. The main drawback is the high computational effort of this approach. The step with the Crank-Nicolson scheme is used only to determine the size of the next time step. The costs of this step are not neglibible such that the adaptive time step control increases the costs per time step notably. A more heuristic

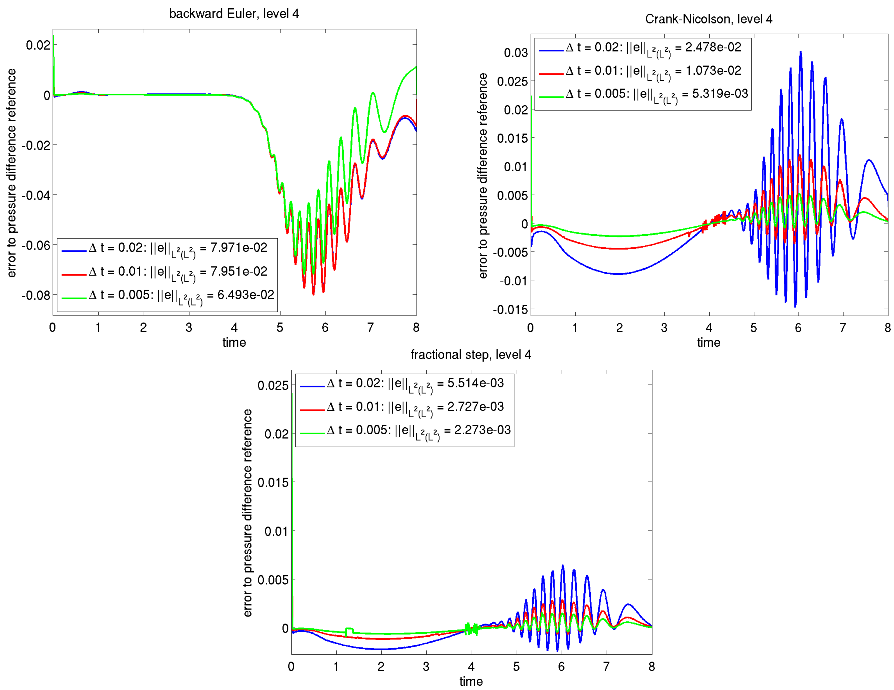

Fig. 6.9 Example 6.38. Error to the reference curve for the pressure difference between the front and the back of the cylinder from Figure D.6, P_2/P_1, backward Euler scheme, Crank–Nicolson scheme, fractional-step θ-scheme (left to right, top to bottom).

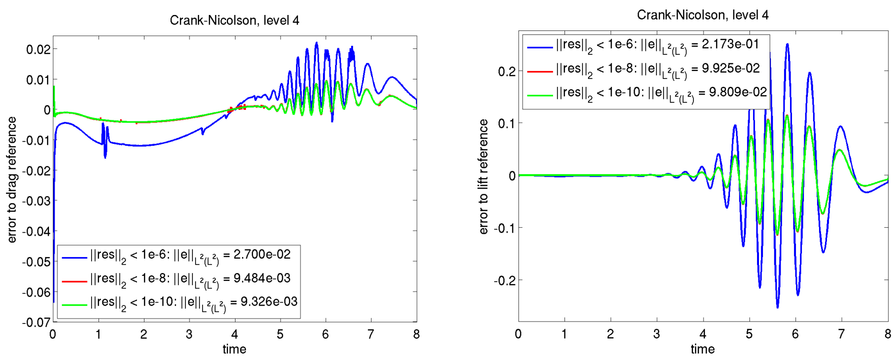

Fig. 6.10 Example 6.38. Dependency of the accuracy of the results on the stopping criterion for the Picard iteration in each discrete time, Q_2/P_1^{disc}, Crank–Nicolson scheme, $\Delta t = 0.01$.

approach monitors just the change of the computed solution in some norm and the length of the time step is varied due to this change, e.g., see Berrone and Marro (2009). □

PSPG Implementation

6.2.2 Other Schemes

BDF2, higher order, adaptive time step control

6.3 Finite Element Error Analysis

6.3.1 The Time-Continuous Case

Remark 6.41. Motivation. The error analysis in the time-continuous case does not consider a discretization of the temporal derivative in the momentum equation in (6.1). Thus, one concentrates on the errors that are introduced by the spatial discretization and the dependency of the constants in the estimates on the viscosity ν. □

Remark 6.42. The weak formulation. In the finite element error analysis, a weak formulation of the time-dependent Navier–Stokes equations is considered that does not apply an integration by parts with respect to the temporal derivative. Consider (6.1) with the boundary condition (6.2). The weak formulation reads as follows: Find $\boldsymbol{u} : [0,T] \to V$ and $p : (0,T] \to Q$ such that

$$(\partial_t \boldsymbol{u}, \boldsymbol{v}) + (\nu \nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + n(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}, p) + (\nabla \cdot \boldsymbol{u}, q) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{V',V} \quad (6.34)$$

for all $(\boldsymbol{v}, q) \in V \times Q$ and $\boldsymbol{u}(0, \boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}) \in L^2_{\mathrm{div}}(\Omega)$. In (6.34), the nonlinear term $n(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})$ can be choosen as any of the terms that are introduced in Remark 5.7.

By definition, the regularity (6.5) is known. The satisfaction of the energy inequality (6.23) and the stability estimate (6.24) were proved in Lemma 6.22. For the finite element error analysis it will turn out that the regularity (6.5) is not sufficient. Higher regularity assumptions are used, see Theorem 6.46. □

Remark 6.43. Time-continuous Galerkin finite element formulation. This section considers inf-sup stable and conforming finite element spaces. Let $V^h \subset V$ and $Q^h \subset Q$, then the time-continuous Galerkin finite element formulation reads as follows: Find $\boldsymbol{u}^h : [0,T] \to V^h$ and $p^h : (0,T] \to Q^h$ such that

$$\begin{aligned} \left(\partial_t \boldsymbol{u}^h, \boldsymbol{v}^h\right) + \left(\nu \nabla \boldsymbol{u}^h, \nabla \boldsymbol{v}^h\right) + n\left(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h\right) \\ - \left(\nabla \cdot \boldsymbol{v}^h, p^h\right) + \left(\nabla \cdot \boldsymbol{u}^h, q^h\right) = \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle_{V',V} \end{aligned} \quad (6.35)$$

for all $(\boldsymbol{v}^h, q^h) \in V^h \times Q^h$ and $\boldsymbol{u}^h(0,\boldsymbol{x}) = \boldsymbol{u}_0^h(\boldsymbol{x}) \in V^h$, where $\boldsymbol{u}_0^h(\boldsymbol{x})$ is an approximation of $\boldsymbol{u}_0(\boldsymbol{x})$, for instance an appropriate interpolation (if the initial condition is sufficiently smooth) or a projection. For the domain, the usual assumptions have to be made: Ω should be a bounded domain with polyhedral and Lipschitz continuous boundary.

Like in the stability analysis of the steady-state equations, it is important that the nonlinear term vanishes if its second and third argument are identical, see Lemma 5.9. Thus, in (6.35), the nonlinear term in the stability estimates might be $n_{\text{skew}}\left(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h\right)$, $n_{\text{rot}}\left(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h\right)$, or $n_{\text{div}}\left(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h\right)$.

In contrast to the finite element error analysis of the steady-state Navier–Stokes equations, the $L^2(\Omega)$ norm of the velocity plays an important role in the error analysis of the time-dependent problem. This fact is reflected by the use of an estimate of the convective term that involves the $L^2(\Omega)$ norm of the first argument, see Lemma 6.45. This estimate is only true for the convective and the skew-symmetric form. Since the results of the stability analysis are applied in the finite element error analysis, this analysis will be carried out for $n_{\text{skew}}(\cdot,\cdot,\cdot)$ only.

In practical simulations, $n_{\text{conv}}(\cdot,\cdot,\cdot)$ is often used. □

Lemma 6.44. Existence, uniqueness, and stability of the finite element solution. *Let $\boldsymbol{u}_0^h \in V_{\text{div}}^h$ and $\boldsymbol{f} \in L^2(0,t;V')$, then the finite element problem* (6.35) *has a unique solution $\left(\boldsymbol{u}^h, p^h\right) \in V^h \times Q^h$. If the nonlinear terms $n_{\text{skew}}\left(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h\right)$, $n_{\text{rot}}\left(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h\right)$, or $n_{\text{div}}\left(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h\right)$ are used, then it holds for all $t \in (0,T]$ that*

$$\left\|\boldsymbol{u}^h(t)\right\|_{L^2(\Omega)}^2 + \nu\left\|\nabla\boldsymbol{u}^h\right\|_{L^2(0,t;L^2(\Omega))}^2 \le \left\|\boldsymbol{u}_0^h\right\|_{L^2(\Omega)}^2 + \frac{1}{\nu}\left\|\boldsymbol{f}\right\|_{L^2(0,t;V')}^2. \quad (6.36)$$

Proof. Consider first the Galerkin finite element method in V_{div}^h: Find $\boldsymbol{u}^h \in V_{\text{div}}^h$ such that

$$\left(\partial_t\boldsymbol{u}^h, \boldsymbol{v}^h\right) + \left(\nu\nabla\boldsymbol{u}^h, \nabla\boldsymbol{v}^h\right) + n\left(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{v}^h\right) = \langle\boldsymbol{f}, \boldsymbol{v}^h\rangle_{V',V} \quad \forall\, \boldsymbol{v}^h \in V_{\text{div}}^h.$$

This problem is a problem in a finite-dimensional space, exactly as problem (6.6) or equivalently (6.7) – (6.9). Now, the existence and uniqueness of a velocity solution can be proved with the same arguments as in the proof of Lemma 6.13. The exsitence and uniquness of a corresponding pressure follows from the discrete inf-sup condition (2.45).

Estimate (6.36) is the analog of estimate (6.11). Note that in the derivation of (6.11) the skew-symmetry of the nonlinear term was used, which is given for finite element spaces usually only for the forms of the convective term mentioned in the formulation of the lemma. In contrast to (6.11), the estimate of the projection for the initial velocity is not yet applied in (6.36). ■

Lemma 6.45. Estimate of the convective term. *Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$, then it holds*

$$n(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \le C\left\|\boldsymbol{u}\right\|_{L^2(\Omega)}^{1-s}\left\|\nabla\boldsymbol{u}\right\|_{L^2(\Omega)}^{s}\left\|\nabla\boldsymbol{v}\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{w}\right\|_{L^2(\Omega)} \quad (6.37)$$

with arbitrary $s \in (0,1]$ if $d = 2$ and $s \in [1/2,1]$ if $d = 3$. In (6.37) *the trinlinear forms if $n_{\text{conv}}(\cdot,\cdot,\cdot)$ and $n_{\text{skew}}(\cdot,\cdot,\cdot)$ can be used.*

Proof. Consider the convective form of the nonlinear term and $d = 2$. Then, one obtains with Hölders inequality (5.19), the Sobolev imbedding (A.16), Poincaré's inequality (A.9), and the Sobolev imbedding (A.12) (with $p = 2, q = 2 + \varepsilon, m = \varepsilon/(2 + \varepsilon) = s$ for arbitrary $\varepsilon > 0$)

$$\begin{aligned}((\boldsymbol{u} \cdot \nabla)\,\boldsymbol{v}, \boldsymbol{w}) &\le \|\boldsymbol{u}\|_{L^p(\Omega)} \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \|\boldsymbol{w}\|_{L^q(\Omega)} \quad \text{with } p^{-1} + q^{-1} = 1/2\\ &\le C \|\boldsymbol{u}\|_{L^{2+\varepsilon}(\Omega)} \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \|\nabla \boldsymbol{w}\|_{L^2(\Omega)}\\ &\le C \|\boldsymbol{u}\|_{H^s(\Omega)} \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \|\nabla \boldsymbol{w}\|_{L^2(\Omega)}\,.\end{aligned}$$

Finally, the interpolation estimate (A.10) is applied to the first factor on the right-hand side, followed by Poincaré's inequality, such that

$$((\boldsymbol{u} \cdot \nabla)\,\boldsymbol{v}, \boldsymbol{w}) \le C \|\boldsymbol{u}\|_{L^2(\Omega)}^{1-s} \|\nabla \boldsymbol{u}\|_{L^2(\Omega)}^{s} \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \|\nabla \boldsymbol{w}\|_{L^2(\Omega)}$$

for arbitrary small positive s.

For $d = 3$, the way to prove the estimate is similar, it will be presented in detail only for $s = 1/2$. After having applied Hölder's inequality, the Sobolev imbedding (A.17), the Sobolev imbedding (A.12) with $p = 2, q = 3, m = 1/2$, the interpolation estimate (A.10), and Poincaré's inequality, one obtains

$$\begin{aligned}((\boldsymbol{u} \cdot \nabla)\,\boldsymbol{v}, \boldsymbol{w}) &\le \|\boldsymbol{u}\|_{L^3(\Omega)} \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \|\boldsymbol{w}\|_{L^6(\Omega)}\\ &\le C \|\boldsymbol{u}\|_{L^3(\Omega)} \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \|\nabla \boldsymbol{w}\|_{L^2(\Omega)}\\ &\le C \|\boldsymbol{u}\|_{H^{1/2}(\Omega)} \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \|\nabla \boldsymbol{w}\|_{L^2(\Omega)}\\ &\le C \|\boldsymbol{u}\|_{L^2(\Omega)}^{1/2} \|\nabla \boldsymbol{u}\|_{L^2(\Omega)}^{1/2} \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \|\nabla \boldsymbol{w}\|_{L^2(\Omega)}\,.\end{aligned}$$

The estimate for the skew-symmetric term follows from twice applying the estimate of the convective term. ■

Theorem 6.46. Finite element error estimate for the time-continuous Galerkin finite element method. *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded domain with polyhedral and Lipschitz continuous boundary, let $\boldsymbol{f} \in L^2(0, T; V')$, $\boldsymbol{u}_0 \in L^2_{\text{div}}(\Omega)$, and $\boldsymbol{u}_0^h \in V^h_{\text{div}}$. In addition, the following regularities are assumed*

$$\partial_t \boldsymbol{u} \in L^2(0, T; V')\,, \quad \nabla \boldsymbol{u} \in L^4\big(0, T; L^2(\Omega)\big)\,, \quad p \in L^4\big(0, T; L^2(\Omega)\big)\,. \tag{6.38}$$

Then the following error estimate holds for the convective term $n_{\text{skew}}(\cdot, \cdot, \cdot)$ and for all $t \in (0, T]$

$$
\begin{aligned}
& \left\| \left(\boldsymbol{u}-\boldsymbol{u}^h\right)(t)\right\|_{L^2(\Omega)}^2 + \nu \left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(0,t;L^2(\Omega))}^2 \\
& \le C\Bigg\{ \left\|\left(\boldsymbol{u}-I_{\mathrm{St}}^h\boldsymbol{u}\right)(t)\right\|_{L^2(\Omega)}^2 + \nu\left\|\nabla\left(\boldsymbol{u}-I_{\mathrm{St}}^h\boldsymbol{u}\right)\right\|_{L^2(0,t;L^2(\Omega))}^2 \\
& + \exp\left(\frac{C}{\nu^3}\|\nabla\boldsymbol{u}\|_{L^4(0,t;L^2(\Omega))}^4\right)\Bigg[\left\|\boldsymbol{u}_0^h - I_{\mathrm{St}}^h\boldsymbol{u}(0)\right\|_{L^2(\Omega)}^2 \\
& +\nu\left\|\nabla\left(\boldsymbol{u}-I_{\mathrm{St}}^h\boldsymbol{u}\right)\right\|_{L^2(0,t;L^2(\Omega))}^2 + \nu^{-1}\Bigg(\left\|\partial_t\left(\boldsymbol{u}-I_{\mathrm{St}}^h\boldsymbol{u}\right)\right\|_{L^2(0,t;V')}^2 \\
& + \left\|\nabla\left(\boldsymbol{u}-I_{\mathrm{St}}^h\boldsymbol{u}\right)\right\|_{L^4(0,t;L^2(\Omega))}^2 \|\nabla\boldsymbol{u}\|_{L^4(0,t;L^2(\Omega))}^2 \\
& + \inf_{q^h\in L^2(0,t;Q^h)}\left\|p-q^h\right\|_{L^2(0,t;L^2(\Omega))}^2\Bigg) \\
& + \frac{1}{\nu^{3/2}}\left(\left\|\boldsymbol{u}_0^h\right\|_{L^2(\Omega)}^2 + \frac{1}{\nu}\|\boldsymbol{f}\|_{L^2(0,t;V')}^2\right)\left\|\nabla\left(\boldsymbol{u}-I_{\mathrm{St}}^h\boldsymbol{u}\right)\right\|_{L^4(0,t;L^2(\Omega))}^2\Bigg]\Bigg\},
\end{aligned}
\tag{6.39}
$$

where $I_{\mathrm{St}}^h\boldsymbol{u}(t)$ *is the Stokes projection at time* t*, see* (3.49)*, for which*

$$
\partial_t I_{\mathrm{St}}^h\boldsymbol{u} \in L^2\left(0,T;V'\right) \tag{6.40}
$$

is assumed.

Proof. The proof proceeds in several steps:

1. derivation of an error equation and splitting of the error,
2. estimate all terms on the right hand-side of the error equation,
3. application of Gronwall's lemma,
4. application of the triangle inequality.

1. Derivation of an error equation and splitting of the error. An error equation is usually derived by subtracting the weak form and the finite element formulation for test functions that can be applied in both equations. As usual, the error is decomposed into an interpolation error and a finite element remainder

$$
\boldsymbol{e}(t) = \boldsymbol{u}(t) - \boldsymbol{u}^h(t) = \left(\boldsymbol{u}(t) - I_{\mathrm{St}}^h\boldsymbol{u}(t)\right) + \left(I_{\mathrm{St}}^h\boldsymbol{u}(t) - \boldsymbol{u}^h(t)\right) = \boldsymbol{\eta}(t) - \boldsymbol{\phi}^h(t).
$$

From the estimate (3.50) and the regularity assumptions (6.38) it follows that

$$
\nabla I_{\mathrm{St}}^h\boldsymbol{u} \in L^4\left(0,T;L^2(\Omega)\right). \tag{6.41}
$$

For simplicity of notation, the argument will be neglected in the following. Now, (6.34) and (6.35) are considered for test functions $\boldsymbol{v}^h \in V_{\mathrm{div}}^h \subset V^h \subset V$ and $q^h \in Q^h \subset Q$. Subtracting both equations and using that $\boldsymbol{u}^h$ is discretely divergence-free leads to

$$
\left(\partial_t\boldsymbol{e},\boldsymbol{v}^h\right) + \left(\nu\nabla\boldsymbol{e},\nabla\boldsymbol{v}^h\right) + n_{\mathrm{skew}}\left(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}^h\right) - n_{\mathrm{skew}}\left(\boldsymbol{u}^h,\boldsymbol{u}^h,\boldsymbol{v}^h\right) - \left(\nabla\cdot\boldsymbol{v}^h,p\right) = 0
$$

for all $\boldsymbol{v}^h \in V_{\mathrm{div}}^h$. Since $\boldsymbol{v}^h$ is discretely divergence-free, it is $\left(\nabla\cdot\boldsymbol{v}^h,q^h\right) = 0$ for all $q^h \in Q^h$ and this term can be added to this equation. Rearranging terms yields

$$
\begin{aligned}
& \left(\partial_t\boldsymbol{\phi}^h,\boldsymbol{v}^h\right) + \left(\nu\nabla\boldsymbol{\phi}^h,\nabla\boldsymbol{v}^h\right) \\
& = \left(\partial_t\boldsymbol{\eta},\boldsymbol{v}^h\right) + \left(\nu\nabla\boldsymbol{\eta},\nabla\boldsymbol{v}^h\right) + n_{\mathrm{skew}}\left(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}^h\right) - n_{\mathrm{skew}}\left(\boldsymbol{u}^h,\boldsymbol{u}^h,\boldsymbol{v}^h\right) - \left(\nabla\cdot\boldsymbol{v}^h,p-q^h\right)
\end{aligned}
$$

for all $\left(\boldsymbol{v}^h, q^h\right) \in V_{\text{div}}^h \times Q^h$. Since $I_{\text{St}}^h \boldsymbol{u}$ is as Stokes projection discretely divergence-free and $\boldsymbol{u}^h$ is as solution of (6.35) discretely divergence-free, it follows that $\boldsymbol{\phi}^h$ is also discretely divergence-free. Hence, one can choose $\boldsymbol{v}^h = \boldsymbol{\phi}^h$ and obtains the error equation

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 + \nu\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 \\
&= \left(\partial_t\boldsymbol{\eta}, \boldsymbol{\phi}^h\right) + \left(\nu\nabla\boldsymbol{\eta}, \nabla\boldsymbol{\phi}^h\right) + n_{\text{skew}}\left(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\phi}^h\right) - n_{\text{skew}}\left(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{\phi}^h\right) \\
&\quad - \left(\nabla\cdot\boldsymbol{\phi}^h, p - q^h\right)
\end{aligned} \tag{6.42}$$

for all $q^h \in Q^h$.

2. Estimate all terms on the right hand-side of the error equation (6.42). The goal of these estimates consists in absorbing in the left-hand side of (6.42) as many terms as possible that contain $\boldsymbol{\phi}^h$ on the right-hand side. There is not much choice for doing this job, in principle only the second term on the left-hand side can be used for this purpose.

The linear terms in (6.42) are estimated with the usual tools: dual pairing or Cauchy–Schwarz inequality (A.8), and Young's inequality (A.4). One obtains

$$\left|\left(\partial_t\boldsymbol{\eta}, \boldsymbol{\phi}^h\right)\right| \le \left\|\partial_t\boldsymbol{\eta}\right\|_{V'}\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \le \frac{2}{\nu}\left\|\partial_t\boldsymbol{\eta}\right\|_{V'}^2 + \frac{\nu}{8}\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2$$

and

$$\left|\left(\nu\nabla\boldsymbol{\eta}, \nabla\boldsymbol{\phi}^h\right)\right| \le \nu\left\|\nabla\boldsymbol{\eta}\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \le 2\nu\left\|\nabla\boldsymbol{\eta}\right\|_{L^2(\Omega)}^2 + \frac{\nu}{8}\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2.$$

For the third linear term, one applies in addition (2.39) to get

$$\begin{aligned}
\left|\left(\nabla\cdot\boldsymbol{\phi}^h, p - q^h\right)\right| &\le \left\|\nabla\cdot\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}\left\|p - q^h\right\|_{L^2(\Omega)} \\
&\le \left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}\left\|p - q^h\right\|_{L^2(\Omega)} \\
&\le \frac{2}{\nu}\left\|p - q^h\right\|_{L^2(\Omega)}^2 + \frac{\nu}{8}\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2.
\end{aligned}$$

The nonlinear terms are decomposed as in (5.43). One obtains

$$\begin{aligned}
&\left|n_{\text{skew}}\left(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\phi}^h\right) - n_{\text{skew}}\left(\boldsymbol{u}^h, \boldsymbol{u}^h, \boldsymbol{\phi}^h\right)\right| \\
&\le \left|n_{\text{skew}}\left(\boldsymbol{\eta}, \boldsymbol{u}, \boldsymbol{\phi}^h\right)\right| + \left|n_{\text{skew}}\left(\boldsymbol{\phi}^h, \boldsymbol{u}, \boldsymbol{\phi}^h\right)\right| + \left|n_{\text{skew}}\left(\boldsymbol{u}^h, \boldsymbol{\eta}, \boldsymbol{\phi}^h\right)\right|.
\end{aligned}$$

Applying (6.37) with $s = 1/2$ and Young's inequality gives

$$\begin{aligned}
\left|n_{\text{skew}}\left(\boldsymbol{\eta}, \boldsymbol{u}, \boldsymbol{\phi}^h\right)\right| &\le C\left\|\boldsymbol{\eta}\right\|_{L^2(\Omega)}^{1/2}\left\|\nabla\boldsymbol{\eta}\right\|_{L^2(\Omega)}^{1/2}\left\|\nabla\boldsymbol{u}\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \\
&\le \frac{C}{\nu}\left\|\boldsymbol{\eta}\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{\eta}\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{u}\right\|_{L^2(\Omega)}^2 + \frac{\nu}{8}\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2,
\end{aligned}$$

$$\begin{aligned}
\left|n_{\text{skew}}\left(\boldsymbol{\phi}^h, \boldsymbol{u}, \boldsymbol{\phi}^h\right)\right| &\le \left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^{1/2}\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^{1/2}\left\|\nabla\boldsymbol{u}\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \\
&\le \frac{C}{\nu^3}\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2\left\|\nabla\boldsymbol{u}\right\|_{L^2(\Omega)}^4 + \frac{\nu}{8}\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2,
\end{aligned}$$

and

$$\begin{aligned}
\left|n_{\text{skew}}\left(\boldsymbol{u}^h, \boldsymbol{\eta}, \boldsymbol{\phi}^h\right)\right| &\le C\left\|\boldsymbol{u}^h\right\|_{L^2(\Omega)}^{1/2}\left\|\nabla\boldsymbol{u}^h\right\|_{L^2(\Omega)}^{1/2}\left\|\nabla\boldsymbol{\eta}\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \\
&\le \frac{C}{\nu}\left\|\boldsymbol{u}^h\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{u}^h\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{\eta}\right\|_{L^2(\Omega)}^2 + \frac{\nu}{8}\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2.
\end{aligned}$$

Inserting all estimates into (6.42) gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 + \frac{\nu}{4}\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 \\
&\le \frac{2}{\nu}\left\|\partial_t\boldsymbol{\eta}\right\|_{V'}^2 + 2\nu\left\|\nabla\boldsymbol{\eta}\right\|_{L^2(\Omega)}^2 + \frac{2}{\nu}\left\|p-q^h\right\|_{L^2(\Omega)}^2 \\
&\quad + \frac{C}{\nu}\Big(\left\|\boldsymbol{\eta}\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{\eta}\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{u}\right\|_{L^2(\Omega)}^2 + \left\|\boldsymbol{u}^h\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{u}^h\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{\eta}\right\|_{L^2(\Omega)}^2\Big) \\
&\quad + \frac{C}{\nu^3}\left\|\nabla\boldsymbol{u}\right\|_{L^2(\Omega)}^4\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2
\end{aligned}
$$

for all $q^h \in Q^h$.

3. Application of Gronwall's lemma E. Next, this estimate is integrated in $(0,t)$ leading to

$$
\begin{aligned}
&\left\|\boldsymbol{\phi}^h(t)\right\|_{L^2(\Omega)}^2 + \frac{\nu}{2}\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(0,t;L^2(\Omega))}^2 \\
&\le \frac{2}{\nu}\left\|\partial_t\boldsymbol{\eta}\right\|_{L^2(0,t;V')}^2 + 2\nu\left\|\nabla\boldsymbol{\eta}\right\|_{L^2(0,t;L^2(\Omega))}^2 + \frac{2}{\nu}\left\|p-q^h\right\|_{L^2(0,t;L^2(\Omega))}^2 \\
&\quad + \frac{C}{\nu}\int_0^t \left\|\boldsymbol{\eta}\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{\eta}\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{u}\right\|_{L^2(\Omega)}^2\,d\tau \\
&\quad + \frac{C}{\nu}\int_0^t \left\|\boldsymbol{u}^h\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{u}^h\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{\eta}\right\|_{L^2(\Omega)}^2\;d\tau \\
&\quad + \frac{C}{\nu^3}\int_0^t \left\|\nabla\boldsymbol{u}\right\|_{L^2(\Omega)}^4\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2\;d\tau
\end{aligned}
\tag{6.43}
$$

for all $q^h \in Q^h$. Before Gronwall's lemma can be applied to estimate (6.43), it has to be checked if the assumptions for its application are satisfied. All terms on the right-hand side of (6.43) are non-negative. From the regularity assumptions (6.38) and (6.40), it follows immediately that the first three terms on the right-hand side of (6.43) are well defined. For the fourth term, it follows with Poincaré's inequality (A.9) and the Cauchy–Schwarz inequality that

$$
\begin{aligned}
&\int_0^t \left\|\boldsymbol{\eta}\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{\eta}\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{u}\right\|_{L^2(\Omega)}^2\;d\tau \\
&\le C\int_0^t \left\|\nabla\boldsymbol{\eta}\right\|_{L^2(\Omega)}^2\left\|\nabla\boldsymbol{u}\right\|_{L^2(\Omega)}^2\;d\tau \\
&\le C\int_0^t \left(\left\|\nabla\boldsymbol{\eta}\right\|_{L^2(\Omega)}^4\;d\tau\right)^{1/2}\left(\int_0^t \left\|\nabla\boldsymbol{u}\right\|_{L^2(\Omega)}^4\;d\tau\right)^{1/2} \\
&= C\left\|\nabla\boldsymbol{\eta}\right\|_{L^4(0,t;L^2(\Omega))}^2\left\|\nabla\boldsymbol{u}\right\|_{L^4(0,t;L^2(\Omega))}^2 < \infty,
\end{aligned}
$$

because of the regularity assumptions (6.38) and of (6.41). For the fifth term on the right-hand side of (6.43) one obtains with the stability estimate (6.36) and the Cauchy–Schwarz inequality

$$
\begin{aligned}
&\int_0^t \left\|\boldsymbol{u}^h\right\|_{L^2(\Omega)} \left\|\nabla \boldsymbol{u}^h\right\|_{L^2(\Omega)} \|\nabla \boldsymbol{\eta}\|_{L^2(\Omega)}^2 \, d\tau \\
&\le \left\|\boldsymbol{u}^h\right\|_{L^\infty(0,t;L^2(\Omega))} \int_0^t \left\|\nabla \boldsymbol{u}^h\right\|_{L^2(\Omega)} \|\nabla \boldsymbol{\eta}\|_{L^2(\Omega)}^2 \, d\tau \\
&\le \left\|\boldsymbol{u}^h\right\|_{L^\infty(0,t;L^2(\Omega))} \left\|\nabla \boldsymbol{u}^h\right\|_{L^2(0,t;L^2(\Omega))} \|\nabla \boldsymbol{\eta}\|_{L^4(0,t;L^2(\Omega))}^2 \\
&\le \frac{C}{\nu^{1/2}} \left(\left\|\boldsymbol{u}_0^h\right\|_{L^2(\Omega)}^2 + \frac{1}{\nu} \|\boldsymbol{f}\|_{L^2(0,t;V')}^2 \right) \|\nabla \boldsymbol{\eta}\|_{L^4(0,t;L^2(\Omega))}^2 < \infty
\end{aligned}
$$

as a consequence of the regularity assumptions (6.38) and of (6.41). From (6.38) it follows that

$$
\int_0^t \|\nabla \boldsymbol{u}\|_{L^2(\Omega)}^4 \, d\tau < \infty.
$$

Hence, all conditions for the application of Gronwall's lemma are satisfied and one obtains for all $t \in (0,T]$ and all $q^h \in Q^h$

$$
\begin{aligned}
&\left\|\boldsymbol{\phi}^h(t)\right\|_{L^2(\Omega)}^2 + \nu \left\|\nabla \boldsymbol{\phi}^h\right\|_{L^2(0,t;L^2(\Omega))}^2 \\
&\le C \exp\left(\frac{C}{\nu^3} \|\nabla \boldsymbol{u}\|_{L^4(0,t;L^2(\Omega))}^4 \right) \\
&\quad \times \Bigg[\left\|\boldsymbol{\phi}^h(0)\right\|_{L^2(\Omega)}^2 + \nu \|\nabla \boldsymbol{\eta}\|_{L^2(0,t;L^2(\Omega))}^2 + \nu^{-1} \Big(\|\partial_t \boldsymbol{\eta}\|_{L^2(0,t;V')}^2 \\
&\quad + \|\nabla \boldsymbol{\eta}\|_{L^4(0,t;L^2(\Omega))}^2 \|\nabla \boldsymbol{u}\|_{L^4(0,t;L^2(\Omega))}^2 + \left\|p - q^h\right\|_{L^2(0,t;L^2(\Omega))}^2 \Big) \\
&\quad + \nu^{-3/2} \left(\left\|\boldsymbol{u}_0^h\right\|_{L^2(\Omega)}^2 + \frac{1}{\nu} \|\boldsymbol{f}\|_{L^2(0,t;V')}^2 \right) \|\nabla \boldsymbol{\eta}\|_{L^4(0,t;L^2(\Omega))}^2 \Bigg].
\end{aligned}
\tag{6.44}
$$

4. Application of the triangle inequality. The triangle inequality gives

$$
\begin{aligned}
&\|\boldsymbol{e}(t)\|_{L^2(\Omega)}^2 + \nu \|\nabla \boldsymbol{e}\|_{L^2(0,t;L^2(\Omega))}^2 \\
&\le 2 \left(\|\boldsymbol{\eta}(t)\|_{L^2(\Omega)}^2 + \nu \|\nabla \boldsymbol{\eta}\|_{L^2(0,t;L^2(\Omega))}^2 + \left\|\boldsymbol{\phi}^h(t)\right\|_{L^2(\Omega)}^2 + \nu \left\|\nabla \boldsymbol{\phi}^h\right\|_{L^2(0,t;L^2(\Omega))}^2 \right).
\end{aligned}
$$

Inserting (6.44) concludes the proof. ■

Remark 6.47. On error estimate (6.39). Estimate (6.39) has the typical form of a finite element error estimate. On the right-hand side one finds interpolation errors, data of the problem, and norms of the solution of the continuous problem. The term

$$
\left\|\boldsymbol{u}_0^h - I_{\mathrm{St}}^h \boldsymbol{u}(0)\right\|_{L^2(\Omega)} \le \left\|\boldsymbol{u}_0 - I_{\mathrm{St}}^h \boldsymbol{u}(0)\right\|_{L^2(\Omega)} + \left\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\right\|_{L^2(\Omega)}
$$

states that the initial condition has to be approximated sufficiently accurately. The dominating interpolation errors are

$$
\left\|\nabla \left(\boldsymbol{u} - I_{\mathrm{St}}^h \boldsymbol{u}\right)\right\|_{L^2(0,t;L^2(\Omega))}, \quad \inf_{q^h \in L^2(0,t;Q^h)} \left\|p - q^h\right\|_{L^2(0,t;L^2(\Omega))}^2,
$$

where the error for the Stokes projection is estimated in (3.52). □

Remark 6.48. On the proof of Theorem 6.46. To use the decomposition (5.42) instead of (5.43) gives the term $\left\|\nabla \boldsymbol{u}^h\right\|_{L^4(0,t;L^2(\Omega))}$. However, the boundedness of this term is not known and it does not follow from the stability estimates for the finite element solution. A higher regularity, which can be assumed for the solution of the continuous problem, cannot simply assumed for the finite element solution. □

6.3.2 The Fully Discrete Case

Remark 6.49. Analysis of temporal discretizations applied to the Navier–Stokes equations. There are a number of investigations of the time discretizations introduced above applied to the Navier–Stokes equations, see (Gresho and Sani, 2000, Section 3.16) or (Emmrich, 2001, Section 4.1) for a survey of the present state of art. The Crank–Nicolson scheme was studied in Temam (1977), Heywood and Rannacher (1990), and Bause (1997) at the already spatially discretized Navier–Stokes equations (with a finite element method). One can prove, under a number of assumptions on the smoothness of the data, that the error between the time discrete and the time-continuous finite element solution in $L^2(\Omega)$ behaves like $(\Delta t)^2$ for the equidistant time step Δt. The fractional-step θ-scheme was investigated analytically in Klouček and Rys (1994) and Müller-Urbaniak (1993). A second order error estimate similar to the Crank–Nicolson scheme was proved in Müller-Urbaniak (1993). □

todo
implementation

6.4 Approaches Decoupling Velocity and Pressure – Projection Methods

Remark 6.50. Motivation and general ideas. The motivation for the construction of projection methods was the wish to obtain simple schemes for simulating the time-dependent Navier–Stokes equations (6.1) which require only the solution of scalar linear systems of equations and not of nonlinear saddle point problems.

To this end, the Navier–Stokes equations (6.1) are decoupled such that separate equations for velocity and pressure are obtained. Let an approximation of the time derivative by a q-step scheme be given, for simplicity of presentation with an equidistant time step,

$$\partial_t \boldsymbol{u}\,(t_{n+1}) \approx \frac{1}{\Delta t}\left(\tau_q \boldsymbol{u}_{n+1} + \sum_{j=0}^{q-1} \tau_{q-1-j}\boldsymbol{u}_{n-j}\right), \quad \sum_{j=0}^{q} \tau_j = 0. \tag{6.45}$$

The equation for an intermediate velocity comes from the momentum balance of the Navier–Stokes equations. Given $\hat{p} \in L^2(\Omega)$ or $\nabla\hat{p}$ and the initial condition $\tilde{\boldsymbol{u}}(0) = \boldsymbol{u}(0) = \boldsymbol{u}_0$, it has the form

$$\begin{aligned} \frac{1}{\Delta t}\left(\tau_q \tilde{\boldsymbol{u}}_{n+1} + \sum_{j=0}^{q-1} \tau_{q-1-j}\boldsymbol{u}_{n-j}\right) & \\ -\nu\Delta\tilde{\boldsymbol{u}}_{n+1} + (\tilde{\boldsymbol{u}}_n\cdot\nabla)\tilde{\boldsymbol{u}}_{n+1} &= \boldsymbol{f}_{n+1} - \nabla\hat{p} \text{ in } \Omega, \\ \tilde{\boldsymbol{u}}_{n+1} &= \boldsymbol{0} \qquad\qquad \text{on } \Gamma. \end{aligned} \tag{6.46}$$

The use of $\tilde{\boldsymbol{u}}_n$ as convection field will be discussed below in Remark 6.62. The intermediate velocity field $\tilde{\boldsymbol{u}}$ will be in general not divergence-free. For this reason, a correction step

$$\begin{aligned} \frac{1}{\Delta t}\left(\tau_q \boldsymbol{u}_{n+1} - \tau_q\tilde{\boldsymbol{u}}_{n+1}\right) + \nabla\varphi\left(\tilde{\boldsymbol{u}}_{n+1}\right) + \nabla p &= \nabla\hat{p} \text{ in } \Omega, \\ \nabla\cdot\boldsymbol{u}_{n+1} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{u}_{n+1}\cdot\boldsymbol{n} &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{6.47}$$

will be performed, where $\varphi(\cdot)$ is a given function with values in $L^2(\Omega)$. In Lemma 6.51 it will become clear that (6.47) describes the $L^2(\Omega)$ projection of $\tilde{\boldsymbol{u}}_{n+1}$ onto a velocity $\boldsymbol{u}_{n+1}$ in an appropriate divergence-free space. The examples given below will illustrate that (6.47) can be transformed into an equation for the pressure.

Note that (6.47) is only a first order partial differential equation with respect to space for $\boldsymbol{u}_{n+1}$. Hence, to define a well-posed problem, it is not possible to use the same boundary condition for $\boldsymbol{u}_{n+1}$ as for the Navier–Stokes equations, which are a second order partial differential equation with respect to space.

Thus, projection methods compute two velocity fields: $\tilde{\boldsymbol{u}}_{n+1}$ satisfies the boundary conditions but it is not divergence free whereas $\boldsymbol{u}_{n+1}$ is divergence free but it does not satisfy the boundary conditions given in the Navier–Stokes equations.

Adding (6.46) and (6.47), it is obvious that the separation of the equations introduces a splitting error, whose size depends on the term $\varphi\left(\tilde{\boldsymbol{u}}_{n+1}\right)$. To obtain only linear systems of equations, the nonlinear term in (6.46) has to be treated semi-implicitly or even explicitly. Here, the semi-implicit variant will be presented.

A survey on projection methods for incompressible flow problems can be found in Guermond et al. (2006). Although projection schemes aim to facilitate the numerical solution of the incompressible Navier–Stokes equations, their analysis turns out to be more involved than the analysis for coupled schemes. Here, only the most important results concerning their convergence will be mentioned. For their proofs, it is referred to the literature. □

Lemma 6.51. Projection property of (6.47)**.** *The velocity computed by* (6.47) *is the* $L^2(\Omega)$ *projection of* $\tilde{\boldsymbol{u}}_{n+1}$ *into* $H_{\mathrm{div}}(\Omega)$.

Proof. Let $\boldsymbol{v} \in H_{\mathrm{div}}(\Omega)$ be an arbitrary function. Multiplication of (6.47) with $\boldsymbol{v}$ and applying integration by parts gives

$$\begin{aligned} 0 &= \frac{\tau_q}{\Delta t}\left(\boldsymbol{u}_{n+1} - \tilde{\boldsymbol{u}}_{n+1}, \boldsymbol{v}\right) + \left(\nabla\varphi\left(\tilde{\boldsymbol{u}}_{n+1}\right) + \nabla\left(p - \hat{p}\right), \boldsymbol{v}\right) \\ &= \frac{\tau_q}{\Delta t}\left(\boldsymbol{u}_{n+1} - \tilde{\boldsymbol{u}}_{n+1}, \boldsymbol{v}\right) + \int_\Gamma \left(\varphi\left(\tilde{\boldsymbol{u}}_{n+1}\right) + p - \hat{p}\right)\boldsymbol{v}\cdot\boldsymbol{n}\; ds - \left(\varphi\left(\tilde{\boldsymbol{u}}_{n+1}\right) + p - \hat{p}, \nabla\cdot\boldsymbol{v}\right) \\ &= \frac{\tau_q}{\Delta t}\left(\boldsymbol{u}_{n+1} - \tilde{\boldsymbol{u}}_{n+1}, \boldsymbol{v}\right), \end{aligned}$$

since $\boldsymbol{v}\cdot\boldsymbol{n}$ vanishes on Γ, $\varphi\left(\tilde{\boldsymbol{u}}_{n+1}\right) + p - \hat{p} \in L^2(\Omega)$, and $\nabla\cdot\boldsymbol{v} = 0$ in $L^2(\Omega)$. It follows that

$$\left(\boldsymbol{u}_{n+1} - \tilde{\boldsymbol{u}}_{n+1}, \boldsymbol{v}\right) = 0 \quad \forall\, \boldsymbol{v} \in H_{\mathrm{div}}(\Omega),\ n = 0, 1, 2, \ldots.$$

This relation is just the definition of the $L^2(\Omega)$ projection into $H_{\mathrm{div}}(\Omega)$. ■

Example 6.52. The non-incremental pressure-correction scheme. This scheme is the simplest pressure-correction scheme, proposed in Chorin (1968) and in Témam (1969). check It is given by

$$\hat{p} = 0 \quad \text{in (6.46)}, \qquad \varphi\left(\tilde{\boldsymbol{u}}_{n+1}\right) = 0 \quad \text{in (6.47)}.$$

It will turn out that the magnitude of the splitting error of this scheme is such that the use of a first order time stepping scheme is sufficient.

The non-incremental pressure-correction scheme with backward Euler time stepping ($q = 1$, $\tau_1 = 1$, $\tau_0 = -1$ in (6.45)) has the following form: Given $\boldsymbol{u}_0$, compute $(\tilde{\boldsymbol{u}}_{n+1}, \boldsymbol{u}_{n+1}, p_{n+1})$ by solving

$$\begin{aligned} \tilde{\boldsymbol{u}}_{n+1} + \Delta t_{n+1}\left(-\nu\Delta\tilde{\boldsymbol{u}}_{n+1} + (\tilde{\boldsymbol{u}}_n\cdot\nabla)\tilde{\boldsymbol{u}}_{n+1}\right) &= \boldsymbol{u}_n + \Delta t_{n+1}\boldsymbol{f}_{n+1} && \text{in } \Omega, \\ \tilde{\boldsymbol{u}}_{n+1} &= \boldsymbol{0} && \text{on } \Gamma, \end{aligned} \tag{6.48}$$

and

$$\begin{aligned} \boldsymbol{u}_{n+1} + \Delta t_{n+1}\nabla p_{n+1} &= \tilde{\boldsymbol{u}}_{n+1} && \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u}_{n+1} &= 0 && \text{in } \Omega, \\ \boldsymbol{u}_{n+1}\cdot\boldsymbol{n} &= 0 && \text{on } \Gamma, \end{aligned} \tag{6.49}$$

for $n = 1, 2, 3, \ldots$.

To obtain a scalar equation for the pressure from (6.49), the (negative of the) divergence of the first equation of (6.49) is considered. This approach gives, using also the second equation,

$$-\nabla\cdot\nabla p_{n+1} = -\Delta p_{n+1} = -\frac{1}{\Delta t_{n+1}}\nabla\cdot\tilde{\boldsymbol{u}}_{n+1}. \tag{6.50}$$

This equation is just a Poisson equation for the pressure, which still has to be equipped with boundary conditions. From (6.49) and from the boundary conditions for $\tilde{\boldsymbol{u}}_{n+1}$ and $\boldsymbol{u}_{n+1}$, one obtains

$$\nabla p_{n+1} \cdot \boldsymbol{n} = -\frac{1}{\Delta t_{n+1}} \left(\boldsymbol{u}_{n+1} - \tilde{\boldsymbol{u}}_{n+1}\right) \cdot \boldsymbol{n} = 0. \tag{6.51}$$

Thus, homogeneous Neumann boundary conditions are applied.

Instead of a semi-implicit treatment of the convective term in (6.48), also a fully implicit or an explicit discretization is possible. □

Theorem 6.53. Error estimate for the non-incremental pressure-correction scheme. *Let the solution* $(\boldsymbol{u}, p)$ *of* (6.1) *be sufficiently smooth. Consider* (6.48) *and* (6.49) *with an equidistant time step and denote the solution of* (6.49) *by* $(\overline{\boldsymbol{u}}, \overline{p})$. *Let* Ω *be a domain with Lipschitz-continuous boundary, then the solution of* (6.48) *and* (6.49) *satisfies*

$$\|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{l^\infty(0,T;H^1(\Omega))} + \|p - \overline{p}\|_{l^\infty(0,T;L^2(\Omega))} \le C\left(\boldsymbol{u}, p, T\right) \Delta t^{1/2}. \tag{6.52}$$

If Ω *has the elliptic regularity property, see Remark 3.27, it holds in addition*

$$\|\boldsymbol{u} - \overline{\boldsymbol{u}}\|_{l^\infty(0,T;L^2(\Omega))} + \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{l^\infty(0,T;L^2(\Omega))} \le C\left(\boldsymbol{u}, p, T\right) \Delta t.$$

Proof. On can find proofs of these statements in Prohl (1997), Rannacher (1992), and Shen (1992). check if all proofs are for Navier–Stokes ■

Remark 6.54. On the non-incremental pressure-correction scheme.

- For the numerical solution, (6.48) and (6.50) has to be discretized in space. Since it is known that the solution of a Poisson problem with homogeneous Neumann boundary conditions is unique up to an additive constant, this method does not need an inf-sup condition for obtaining a unique pressure. Hence, arbitrary finite element spaces can be used for velocity and pressure. But spurious oscillations may appear if the time step becomes too small, see the discussion in Guermond et al. (2006). more details
- The artificial Neumann boundary condition for the pressure (6.51) induces a numerical boundary layer that prevents the scheme to obtain a first order convergence in (6.52), see Rannacher (1992).
- It can be shown that the scheme has a splitting error of order $\mathcal{O}\left(\Delta t\right)$ which cannot be reduced. Hence, using a higher order time stepping scheme does not lead to improvements of the order of convergence.

□

Example 6.55. The standard incremental pressure-correction scheme. The goal of this scheme is to increase the order of convergence compared with the non-incremental pressure-correction scheme by using a better approximation for $\hat{p}$. A natural choice is the pressure from the previous discrete time

$$\hat{p} = p_n \quad \text{in (6.46)}, \quad \varphi\left(\tilde{\boldsymbol{u}}_{n+1}\right) = 0 \quad \text{in (6.47)}.$$

Since the scheme should perform better than first order in space, it should be combined with a higher order temporal discretization than the backward Euler scheme. Popular second order discretizations are the Crank–Nicolson scheme, see Example 6.31, and the backward difference formula 2 (BDF2) scheme. The standard incremental pressure-correction scheme became popular by van Kan (1986).

The standard incremental pressure-correction scheme with BDF2 ($q = 2$, $\tau_2 = 3/2$, $\tau_1 = -2$, $\tau_0 = 1/2$ in (6.45)) and equidistant time step Δt has the form: Given $(\boldsymbol{u}_0, p_0)$ and $(\boldsymbol{u}_1, p_1)$, compute $(\tilde{\boldsymbol{u}}_{n+1}, \boldsymbol{u}_{n+1}, p_{n+1})$ by solving

$$\begin{aligned} 3\tilde{\boldsymbol{u}}_{n+1} &+ 2\Delta t \left(-\nu \Delta \tilde{\boldsymbol{u}}_{n+1} + (\tilde{\boldsymbol{u}}_n \cdot \nabla)\tilde{\boldsymbol{u}}_{n+1}\right) \\ &= 4\boldsymbol{u}_n - \boldsymbol{u}_{n-1} + 2\Delta t \left(\boldsymbol{f}_{n+1} - \nabla p_n\right) \quad \text{in } \Omega, && (6.53) \\ \tilde{\boldsymbol{u}}_{n+1} &= \boldsymbol{0} \quad \text{on } \Gamma, && (6.54) \end{aligned}$$

and

$$\left.\begin{aligned} 3\boldsymbol{u}_{n+1} + 2\Delta t \nabla \left(p_{n+1} - p_n\right) &= 3\tilde{\boldsymbol{u}}_{n+1} && \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}_{n+1} &= 0 && \text{in } \Omega, \\ \boldsymbol{u}_{n+1} \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma, \end{aligned}\right. \qquad (6.55)$$

for $n = 2, 3, 4 \ldots$. By applying the negative of the divergence to (6.55), one obtains the following equation for the update of the pressure

$$-\Delta \left(p_{n+1} - p_n\right) = -\frac{3}{2\Delta t} \nabla \cdot \tilde{\boldsymbol{u}}_{n+1} \quad \text{in } \Omega. \qquad (6.56)$$

From $\boldsymbol{u}_{n+1} \cdot \boldsymbol{n} = 0$ on the boundary and the no-slip boundary condition of $\tilde{\boldsymbol{u}}_{n+1}$, it follows that the boundary condition for (6.56) is obtained as follows

$$\nabla \left(p_{n+1} - p_n\right) \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma. \qquad (6.57)$$

□

Remark 6.56. The initial step. Besides the initial velocity field $\boldsymbol{u}_0$, the standard incremental pressure-correction scheme with BDF2 requires the computation of quantities $p_0 = p(0, \boldsymbol{x})$ and $(\boldsymbol{u}_1, p_1)$. An initial pressure can be computed as described in Remark 6.33. Then, $(\boldsymbol{u}_1, p_1)$ is computed in principle with the same scheme as the incremental pressure-correction scheme, only a first order, one-step discretization in time is applied

$$\begin{aligned} \tilde{\boldsymbol{u}}_1 + \Delta t \left(-\nu \Delta \tilde{\boldsymbol{u}}_1 + (\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{u}_1\right) &= \boldsymbol{u}_0 + \Delta t \left(\boldsymbol{f}_1 - \nabla p_0\right) && \text{in } \Omega, \\ \tilde{\boldsymbol{u}}_1 &= \boldsymbol{0} && \text{on } \Gamma, \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{u}_1 + \Delta t \nabla \left(p_1 - p_0\right) &= \tilde{\boldsymbol{u}}_1 && \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}_1 &= 0 && \text{in } \Omega, \\ \boldsymbol{u}_1 \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma. \end{aligned}$$

For this initialization of the method, one can prove the following error estimates

$$\|\boldsymbol{u}(\Delta t) - \tilde{\boldsymbol{u}}_1\|_{L^2(\Omega)} \le C\Delta t^2, \tag{6.58}$$

$$\|\boldsymbol{u}(\Delta t) - \tilde{\boldsymbol{u}}_1\|_{H^1(\Omega)} \le C\Delta t^{3/2}, \tag{6.59}$$

$$\|p(\Delta t) - p_1\|_{H^1(\Omega)} \le C\Delta t. \tag{6.60}$$

□

Theorem 6.57. Error estimate for the standard incremental pressure-correction scheme. *Let the solution* $(\boldsymbol{u}, p)$ *of* (6.1) *be sufficiently smooth. Consider* (6.53) *and* (6.55) *with an equidistant time step and denote the numerical solution of* (6.55) *by* $(\overline{\boldsymbol{u}}, \overline{p})$. *Let the first step be performed such that* (6.58) – (6.60) *holds. Let* Ω *be a domain with Lipschitz-continuous boundary, then the error estimate*

$$\|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{l^\infty(0,T;H^1(\Omega))} + \|p - \overline{p}\|_{l^\infty(0,T;L^2(\Omega))} \le C\left(\boldsymbol{u}, p, T\right)\Delta t$$

holds. If Ω *has the elliptic regularity property, see Remark 3.27, it holds in addition*

$$\|\boldsymbol{u} - \overline{\boldsymbol{u}}\|_{l^\infty(0,T;L^2(\Omega))} + \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{l^2(0,T;L^2(\Omega))} \le C\left(\boldsymbol{u}, p, T\right)\Delta t^2.$$

Proof. The proof can be found in Guermond (1999). ■

Remark 6.58. On the standard incremental pressure-correction scheme.
- A similar error estimate for the Crank–Nicolson time stepping scheme was proved in Shen (1996).
- The boundary condition (6.57) implies

$$\nabla p_{n+1} \cdot \boldsymbol{n} = \nabla p_n \cdot \boldsymbol{n} = \ldots = \nabla p_0 \cdot \boldsymbol{n} \text{ on } \Gamma.$$

  This boundary condition is an unphysical boundary conditions that leads to numerical boundary layers.
- The scheme has an irreducible splitting error of order $\mathcal{O}\left(\Delta t^2\right)$. Hence, the use of a higher order discretization in time does not improve the order of convergence.

□

Example 6.59. The rotational incremental pressure-correction scheme. The goal of this scheme consists in overcoming the difficulties caused by the artificial boundary conditions of the pressure of the standard incremental pressure-correction scheme. It is obtained by choosing

$$\hat{p} = p_n \quad \text{in (6.46)}, \quad \varphi\left(\tilde{\boldsymbol{u}}_{n+1}\right) = \nu \nabla \cdot \tilde{\boldsymbol{u}}_{n+1} \quad \text{in (6.47)}.$$

Hence, the velocity step of this scheme is identical to (6.53) and the projection step has the form

$$\begin{aligned} 3\boldsymbol{u}_{n+1} + 2\Delta t \nabla (p_{n+1} - p_n) &= 3\tilde{\boldsymbol{u}}_{n+1} - 2\nu \Delta t \nabla (\nabla \cdot \tilde{\boldsymbol{u}}_{n+1}) && \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}_{n+1} &= 0 && \text{in } \Omega, \\ \boldsymbol{u}_{n+1} \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma. \end{aligned} \tag{6.61}$$

The problem for the pressure becomes, using $\nabla \cdot \nabla = \Delta$,

$$-\Delta (p_{n+1} - p_n) = -\frac{3}{2\Delta t} \nabla \cdot \tilde{\boldsymbol{u}}_{n+1} + \nu \Delta (\nabla \cdot \tilde{\boldsymbol{u}}_{n+1}) \tag{6.62}$$

or

$$-\Delta \tilde{p}_n = -\frac{3}{2\Delta t} \nabla \cdot \tilde{\boldsymbol{u}}_{n+1} \quad \textit{with} \quad \tilde{p}_n = p_{n+1} - p_n + \nu \nabla \cdot \tilde{\boldsymbol{u}}_{n+1}. \tag{6.63}$$

The boundary conditions for (6.62) and (6.63) will be discussed in Remark 6.60, see (6.65) for the result. Using (6.63), the update for the pressure has to be computed by subtracting $\nu \nabla \cdot \tilde{\boldsymbol{u}}_{n+1}$ from $\tilde{p}_n$. This scheme was proposed in Timmermans et al. (1996).

If standard finite elements are used for velocity and pressure with a continuous pressure space, one has to modify (6.63) since the term $\nu \nabla \cdot \tilde{\boldsymbol{u}}_{n+1}$ is usually discontinuous. □

Remark 6.60. On the rotational incremental pressure-correction scheme. Inserting (6.61) into (6.53) yields

$$\begin{aligned} 3\boldsymbol{u}_{n+1} + 2\Delta t \Big(\nu \nabla (\nabla \cdot \tilde{\boldsymbol{u}}_{n+1}) - \nu \Delta \tilde{\boldsymbol{u}}_{n+1} + (\tilde{\boldsymbol{u}}_n \cdot \nabla) \tilde{\boldsymbol{u}}_{n+1} + \nabla p_{n+1} \Big) \\ = 4\boldsymbol{u}_n - \boldsymbol{u}_{n-1} + 2\Delta t \boldsymbol{f}_{n+1}. \end{aligned}$$

Using now $\nabla (\nabla \cdot \boldsymbol{v}) - \Delta \boldsymbol{v} = \nabla \times \nabla \times \boldsymbol{v}$ for all sufficiently smooth vector fields $\boldsymbol{v}$ leads to

$$\begin{aligned} 3\boldsymbol{u}_{n+1} + 2\Delta t \Big(\nu \nabla \times \nabla \times \tilde{\boldsymbol{u}}_{n+1} + (\tilde{\boldsymbol{u}}_n \cdot \nabla) \tilde{\boldsymbol{u}}_{n+1} + \nabla p_{n+1} \Big) \\ = 4\boldsymbol{u}_n - \boldsymbol{u}_{n-1} + 2\Delta t \boldsymbol{f}_{n+1}. \end{aligned}$$

From (6.61), it follows with $\nabla \times \nabla \boldsymbol{v} = \boldsymbol{0}$ that $\nabla \times \tilde{\boldsymbol{u}}_{n+1} = \nabla \times \boldsymbol{u}_{n+1}$ such that an equation for $(\boldsymbol{u}_{n+1}, p_{n+1})$ is obtained

$$\begin{aligned} & 3\boldsymbol{u}_{n+1} + 2\Delta t (\nu \nabla \times \nabla \times \boldsymbol{u}_{n+1} + (\tilde{\boldsymbol{u}}_n \cdot \nabla) \tilde{\boldsymbol{u}}_{n+1} + \nabla p_{n+1}) \\ & \qquad = 4\boldsymbol{u}_n - \boldsymbol{u}_{n-1} + 2\Delta t \boldsymbol{f}_{n+1} && \text{in } \Omega, \\ & \nabla \cdot \boldsymbol{u}_{n+1} = 0 && \text{in } \Omega, \\ & \boldsymbol{u}_{n+1} \cdot \boldsymbol{n} = 0 && \text{on } \Gamma. \end{aligned} \tag{6.64}$$

Because of form (6.64), the scheme is called rotational incremental pressure correction scheme. From (6.64), also the boundary condition for the pressure can be derived by using the boundary conditions for $\tilde{\boldsymbol{u}}_{n+1}$, $\boldsymbol{u}_{n+1}$, $\boldsymbol{u}_n$, and

$\boldsymbol{u}_{n-1}$

$$\nabla p_{n+1} \cdot \boldsymbol{n} = \left(\boldsymbol{f}_{n+1} - \nu \nabla \times \nabla \times \boldsymbol{u}_{n+1}\right) \cdot \boldsymbol{n} \text{ on } \Gamma. \tag{6.65}$$

This condition is a consistent boundary condition because it is also obtained by restricting the momentum balance of the Navier–Stokes equations to the boundary and multiplying it with $\boldsymbol{n}$. Denoting with $(\boldsymbol{u}_{n+1}, p_{n+1})$ the solution of the Navier–Stokes equations and using $-\Delta \boldsymbol{v} = \nabla \times \nabla \times \boldsymbol{v} - \nabla(\nabla \cdot \boldsymbol{v})$ yields for the momentum balance, discretized with BDF2,

$$\begin{aligned} 3\boldsymbol{u}_{n+1} + 2\Delta t\Big(\nu \nabla \times \nabla \times \boldsymbol{u}_{n+1} - \nu \nabla(\nabla \cdot \boldsymbol{u}_{n+1}) + (\boldsymbol{u}_{n+1} \cdot \nabla)\boldsymbol{u}_{n+1} \\ + \nabla p_{n+1}\Big) = 4\boldsymbol{u}_n - \boldsymbol{u}_{n-1} + 2\Delta t \boldsymbol{f}_{n+1}. \end{aligned}$$

Since the velocity field in the Navier–Stokes equations is divergence-free and it vanishes on the boundary, (6.65) is obtained. Thus, (6.65) does not come from some method, but it can be derived from the Navier–Stokes equations. □

Theorem 6.61. Error estimate for the rotational incremental pressure-correction scheme. *Let the solution* $(\boldsymbol{u}, p)$ *of* (6.1) *be sufficiently smooth. Consider* (6.53) *and* (6.61) *with an equidistant time step and denote the numerical solution by* $(\overline{\boldsymbol{u}}, \overline{p})$. *Let the first step be performed such that* (6.58) – (6.60) *holds. Let* Ω *be a domain with Lipschitz-continuous boundary, then the error estimate*

$$\begin{aligned} \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{l^2(0,T;H^1(\Omega))} + \|\boldsymbol{u} - \overline{\boldsymbol{u}}\|_{l^2(0,T;H^1(\Omega))} \\ + \|p - \overline{p}\|_{l^2(0,T;L^2(\Omega))} \le C(\boldsymbol{u}, p, T)\, \Delta t^{3/2} \end{aligned}$$

holds. If Ω *has the elliptic regularity property, see Remark 3.27, it holds in addition*

$$\|\boldsymbol{u} - \overline{\boldsymbol{u}}\|_{l^2(0,T;L^2(\Omega))} + \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{l^2(0,T;L^2(\Omega))} \le C(\boldsymbol{u}, p, T)\, \Delta t^2.$$

Proof. The proof can be found in Guermond and Shen (2004). ■

Remark 6.62. On $\boldsymbol{u}_{n+1}$ *and* $\tilde{\boldsymbol{u}}_{n+1}$. In implementing and using the pressure-correction schemes, one has two velocity fields in the computation: $\boldsymbol{u}_{n+1}$ and $\tilde{\boldsymbol{u}}_{n+1}$. The field $\boldsymbol{u}_{n+1}$ is (discretely) divergence-free but it does not satisfy the no-slip boundary condition. Only the normal component of $\boldsymbol{u}_{n+1}$ is required to vanish at the boundary, but not the tangential component(s). On the contrary, the boundary condition is satisfied from $\tilde{\boldsymbol{u}}_{n+1}$, but this field is not (discretely) divergence-free. Thus, none of the fields is perfect.

From the point of view of convergence orders, one obtains the same results for both, $\boldsymbol{u}_{n+1}$ and $\tilde{\boldsymbol{u}}_{n+1}$, see Theorems 6.53, 6.57, and 6.61.

From the point of view of implementation, $\boldsymbol{u}_{n+1}$ has another disadvantage. After having computed the pressure by solving the Poisson equation (6.50), (6.56), or (6.62) using standard continuous finite elements, then $\boldsymbol{u}_{n+1}$ can be recovered by (6.49), (6.55), or (6.61). This recovery requires the computation of the gradient of the computed pressure, which will give in general a discontinuous finite element function. Consequently, $\boldsymbol{u}_{n+1}$ will be a discontinuous finite element function, too.

Note that $\boldsymbol{u}_{n+1}$ is not needed in computing the pressure p_{n+1}. It can be also removed from the equation which accounts for the momentum balance in all schemes. The idea consists in solving the equation for the projection for $\boldsymbol{u}_{n+1}$ and in inserting the result to the equation for the momentum balance. For instance, for the non-incremental pressure-correction scheme, one obtains in the first step from (6.49)

$$\boldsymbol{u}_k = \tilde{\boldsymbol{u}}_k - \Delta t_k p_k, \quad k = 1, 2, \ldots .$$

Inserting this expression for $k = n$ into (6.48) gives an equation without $\boldsymbol{u}_n$

$$\tilde{\boldsymbol{u}}_{n+1} + \Delta t_{n+1} \left(-\nu \Delta \tilde{\boldsymbol{u}}_{n+1} + (\tilde{\boldsymbol{u}}_n \cdot \nabla)\tilde{\boldsymbol{u}}_{n+1}\right) = \tilde{\boldsymbol{u}}_n - \Delta t_n p_n + \Delta t_{n+1} \boldsymbol{f}_{n+1}.$$

It was shown in Guermond and Quartapelle (1998) that the use of $\tilde{\boldsymbol{u}}_n$ as convection field, as it was already done in this section, does not spoil the overall error.

Altogether, $\boldsymbol{u}_{n+1}$ can be completely eliminated from the solution process. In the literature, e.g., in Guermond et al. (2006), it is advised to use only $\tilde{\boldsymbol{u}}_{n+1}$ in the implementation. □

Remark 6.63. Some numerical experience with pressure-correction schemes. In Caiazzo et al. (2014), the standard incremental pressure-correction scheme, see Example 6.55, was compared with the standard fully implicit Galerkin finite element method (**??**) and the IMEX Galerkin method, see Remark 6.37 at Example D.10. All schemes were used with the Crank–Nicolson method, see Example 6.31. It could be observed taht the accuracy of the results obtained with the standard incremental pressure-correction scheme was by far worst among the studied schemes. check more literature □