3 Review of facts from functional analysis

In order to define the weak solution, we shall uses the theory of vector spaces, and thus we review this topic briefly, focusing especially on Banach, Hilbert and Sobolev spaces.

Review. Normed linear vector spaces

Linear vector space is a set V having the following properties:

- if $v, w \in V$ are arbitrary elements and α is a real (or complex) number, then $v + w$ and αv also belong to V
- the above operations satisfy the rules:

$$
\begin{aligned}
& v + w = w + v \\
& v + (w + z) = (v + w) + z \\
& \alpha(v + w) = \alpha v + \alpha w \\
& (\alpha + \beta)v = \alpha v + \beta v \\
& \alpha(\beta v) = (\alpha\beta)v \\
& 1 \cdot v = v \\
& v + w = v + z \quad \Rightarrow \quad w = z
\end{aligned}
$$

Norm on V is a non-negative real function $\|\cdot\|_V$ fulfilling the conditions:

$$
\begin{aligned}
& \|v + w\|_V \leq \|v\|_V + \|w\|_V \qquad \text{(triangle inequality)} \\
& \|\alpha v\|_V = |\alpha| \; \|v\|_V \\
& \|v\|_V \neq 0 \;\text{ for }\; v \neq 0
\end{aligned}
$$

Normed linear vector space is a linear vector space equipped with a norm.

Ex. Continuous functions on the interval $[0, 1]$ (denoted as $C([0, 1])$) form a linear vector space and the following function is a norm on $C([0, 1])$:

$$\|f\|_{C([0,1])} = \max_{x\in[0,1]} |f(x)|$$

Proof. It is clear that if f, g are continuous functions and $\alpha \in \mathbb{R}$, then $f + g$ and αf are also continuous functions. The rules for operations are obviously satisfied.

We check the conditions for the norm:

- First condition (triangle inequality):

$$
\begin{aligned}
\|f + g\|_{C([0,1])} = \max_{x\in[0,1]} |f(x) + g(x)| \leq \max_{x\in[0,1]} \big(|f(x)| + |g(x)|\big) \quad & \leq \quad \max_{x\in[0,1]} |f(x)| + \max_{x\in[0,1]} |g(x)| \\
& = \quad \|f\|_{C([0,1])} + \|g\|_{C([0,1])}
\end{aligned}
$$

- Second condition

$$\|\alpha f\|_{C([0,1])} = \max_{x\in[0,1]} |\alpha f(x)| = \max_{x\in[0,1]} |\alpha| \cdot |f(x)| = |\alpha| \max_{x\in[0,1]} |f(x)| = |\alpha| \|f\|_{C([0,1])}$$

- Third condition If $f = 0$ then $\|f\|_{C([0,1])} = 0$ is immediate from the definition.

Review. Banach spaces

Definition (Banach space)

1. A sequence $\{v_k\}_{k=1}^{\infty} \subset V$ is called a **Cauchy sequence** provided for each $\epsilon > 0$ there exists $N > 0$ such that
$$\|v_k - v_l\|_V < \epsilon \qquad \text{for all } \ k, l \geq N.$$
2. V is **complete** if each Cauchy sequence in V converges. That is, whenever $\{v_k\}_{k=1}^{\infty}$ is a Cauchy sequence, there exists $v \in V$ such that $\{v_k\}$ converges to v.
3. A **Banach space** V is a complete, normed linear space.

Ex. The space $C([0,1])$ with the above defined norm $\|\cdot\|_{C([0,1])}$ is a Banach space.

Proof. (Idea)

- First we show that if $\{f_n\}$ is a Cauchy sequence then $\{f_n(x)\}$ converges to some value for each $x \in [0,1]$ (o.e., pointwise). We denote the limit value as $f(x)$.
- It remains to show that the obtained function f belongs to $C([0,1])$, i.e., that it is continuous. This follows from the theorem on uniform convergence of continuous functions.

Review. About L^p spaces.

For $1 \leq p \leq \infty$, open subset $\Omega \subset \mathbb{R}^n$ and a measurable function $f : \Omega \to \mathbb{R}$ we define
$$\|f\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |f|^p \, dx\right)^{1/p} & \text{if } 1 \leq p < \infty \\ \operatorname{ess\,sup}_\Omega |f| & \text{if } p = \infty \end{cases}$$
Then $L^p(\Omega)$ is the linear space:
$$L^p(\Omega) = \{f : \Omega \to \mathbb{R} \mid \ \ \|f\|_{L^p(\Omega)} < \infty\}$$
$L^p(\Omega)$ is a Banach space, provided we identify functions which agree almost everywhere.

Lemma. (Hölder's inequality)

Let $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then if $u \in L^p(\Omega), v \in L^q(\Omega)$, we have
$$\int_\Omega |uv| \, dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}$$

Ex.

- If the domain Ω is bounded, then every continuous function f belongs to $L^p(\Omega)$ for any $p > 0$:
$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p \, dx\right)^{1/p} \leq \left(\max_{x\in\Omega} |f|^p \int_\Omega 1 \, dx\right)^{1/p} = \max_{x\in\Omega} |f| \cdot |\Omega|^{1/p}.$$

- If the domain Ω is not bounded, then a continuous function f may not belong to $L^p(\Omega)$ unless it decays fast enough at infinity. For example, the function $f(x) = x^{-1}$ belongs to $L^2(1,\infty)$ but does not belong to $L^1(1,\infty)$:

$$\|x^{-1}\|_{L^2(1,\infty)} = \Big(\int_1^\infty \frac{1}{x^2}\,dx\Big)^{1/2} = \Big[-\frac{1}{x}\Big]_1^\infty = 1$$
$$\|x^{-1}\|_{L^1(1,\infty)} = \int_1^\infty \frac{1}{x}\,dx = [\log|x|]_1^\infty = \infty$$

- There are functions that blow up to infinity but still belong to some L^p-space. For example, the function $f(x) = 1/\sqrt{x}$ belongs to $L^1(0,1)$ because

$$\|1/\sqrt{x}\|_{L^2(0,1)} = \int_0^1 \frac{1}{\sqrt{x}}\,dx = \big[2\sqrt{x}\big]_0^1 = 2.$$

Review. Hilbert spaces

> **Hilbert space** is a Banach space endowed with an inner product.
> Definition (inner product) Let H be a real linear space. A mapping $(\ ,\ ) : H \times H \to \mathbb{R}$ is called an **inner product** if
> (i) $(u,v) = (v,u)\ \ \forall u,v \in H$
> (ii) the mapping $u \mapsto (u,v)$ is linear for each $v \in H$
> (iii) $(u,u) \geq 0$ for all $u \in H$
> (iv) $(u,u) = 0$ if and only if $u = 0$
> If $(\ ,\ )$ is an inner product, the associated norm is
>
> $$\|u\|_H = (u,u)^{1/2}, \qquad u \in H.$$
>
> The **Cauchy-Schwarz inequality** states
>
> $$|(u,v)| \leq \|u\|_H\ \|v\|_H, \qquad u,v \in H.$$

Ex. The space $L^2(\Omega)$ is a Hilbert space with the inner product

$$(f,g) = \int_\Omega fg\,dx.$$

Review. Bounded linear functionals

> Definition (bounded linear operator) A mapping $A : X \to Y$ (X, Y are normed spaces) is a **linear operator** provided
>
> $$A(\lambda u + \mu v) = \lambda A(u) + \mu A(v) \qquad \forall u, v \in X, \ \ \forall \lambda, \mu \in \mathbb{R}$$
>
> A linear operator $A : X \to Y$ is **bounded** if
>
> $$\|A\| = \sup_{\|u\|_X \le 1} \|A(u)\|_Y = \sup_{u \in X} \frac{\|A(u)\|_Y}{\|u\|_X} \ < \ \infty$$
>
> If $Y = \mathbb{R}$, then we call the operator $A : X \to \mathbb{R}$ a **functional**. We write X^* to denote the collection of all bounded linear functionals on X. X^* is the **dual space** of X.
>
> Theorem (Riesz representation theorem) Let H be a real Hilbert space. Then for each $F \in H^*$ there exists a unique element $f \in H$ such that
>
> $$F(v) = (f, v) \qquad \forall v \in H$$
>
> The mapping $F \mapsto f$ is a linear isomorphism of H^* onto H.

Ex. The operator $A : L^2(0,1) \to \mathbb{R}$ defined by

$$A(u) = \int_0^1 x u(x)\, dx, \qquad u \in L^2(0,1)$$

is a bounded linear functional.

Proof.

- Because the operator maps L^2-functions on real numbers, it is a functional.
- It is linear because

$$A(\lambda u + \mu v) = \int_0^1 x(\lambda u(x) + \mu v(x))\, dx = \lambda \int_0^1 x u(x)\, dx + \mu \int_0^1 x v(x)\, dx = \lambda A(u) + \mu A(v)$$

- It is bounded because

$$\|A\| = \sup_{u \in L^2(0,1)} \frac{|A(u)|}{\|u\|_{L^2(0,1)}} \le \frac{\left|\int_0^1 x u(x)\right|}{\left(\int_0^1 u^2(x)\right)^{1/2}} \le \frac{\left(\int_0^1 x^2\right)^{1/2} \left(\int_0^1 u^2(x)\right)^{1/2}}{\left(\int_0^1 u^2(x)\right)^{1/2}} = \left(\int_0^1 x^2\right)^{1/2} = \frac{1}{\sqrt{3}} < \infty.$$

  We have used here Hölder's inequality. Moreover if we set $u(x) = x$, we find that $|A(u)|/\|u\|_{L^2(0,1)} = 1/\sqrt{3}$ and, therefore, the norm of the operator is equal to $\|A\| = 1/\sqrt{3}$.