2 Equation for stationary magnetic field (continuation)

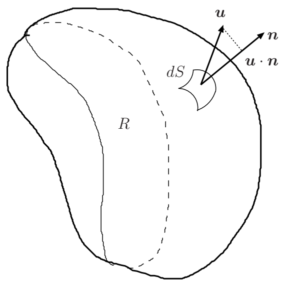
We will soon need some basic theorems from vector analysis. Here is a brief overview.

Review. About Green's theorem.

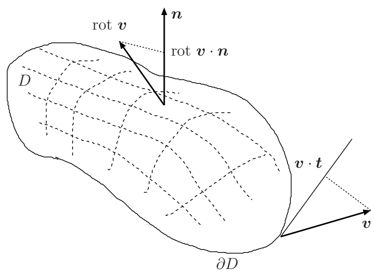
Gauss's divergence theorem

$$\iiint_R \operatorname{div} \boldsymbol{u}\, dV = \iint_S \boldsymbol{u} \cdot \boldsymbol{n}\, dS \tag{1}$$

Physical meaning: the total divergence of a vector field inside a closed domain R is equal to the total of the flux through the boundary of the domain.



Gauss's theorem



Stokes' theorem

Stokes' theorem

$$\iint_D (\operatorname{rot} \boldsymbol{v}) \cdot \boldsymbol{n}\, dS = \int_{\partial D} \boldsymbol{v} \cdot d\boldsymbol{s} \tag{2}$$

Physical meaning: the total circulation inside a closed domain D is equal to the line integral along the boundary ∂D of the domain.

Green's theorem

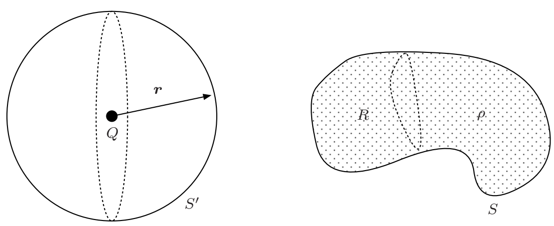
$$\iint_\Omega \varphi \Delta \psi\, dx\, dy = -\iint_\Omega \nabla \psi \cdot \nabla \varphi\, dx\, dy + \oint_{\partial \Omega} \varphi \frac{\partial \psi}{\partial n}\, ds \tag{3}$$

This theorem is obtained from Gauss's theorem by choosing $\boldsymbol{u} = \varphi \nabla \psi$.

If we choose $\boldsymbol{u} = \varphi a$ in Gauss's theorem, we get

$$\iint_\Omega \operatorname{div}\, a\, \varphi\, dx\, dy + \iint_\Omega a \cdot \nabla \varphi\, dx\, dy = \oint_{\partial \Omega} \varphi (a \cdot n)\, ds \tag{4}$$

Ex. Application of divergence theorem: derivation of Gauss's law.



The electric flux density D at any point on a spherical surface S' of radius r centered at an isolated point charge Q is given by:

$$D = \frac{Q}{4\pi r^2}\boldsymbol{r}.$$

Therefore we have

$$\int_{S'} D \cdot n\, dS = \frac{Q}{4\pi r^2} 4\pi r^2 = Q.$$

This is also true for any arbitrary surface S enclosing the charge. By superposition, we can extend this to a system of point charges and to continuous charge distribution. Then the above relation is generalized to

$$\int_S D \cdot n\, dS = \int_R \rho\, dx,$$

where ρ is charge density and R is the region enclosed by the surface S.

Applying the divergence theorem we obtain

$$\int_S D \cdot n\, dS = \int_R \operatorname{div} D\, dx.$$

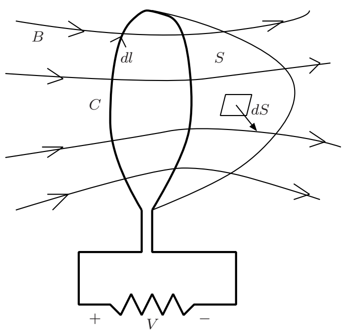
Therefore,

$$\int_R \operatorname{div} D\, dx = \int_R \rho\, dx,$$

and since this relation holds for arbitrary volume R, we arrive at Gauss's law

$$\operatorname{div} D = \rho.$$

Ex. Application of Stokes' theorem: derivation of Faraday's law.



Faraday's circuital law says that the induced electromotive force in any closed circuit C is equal to the time rate of change of the magnetic flux ψ through the circuit.

$$V = -\frac{\partial \psi}{\partial t}.$$

The total magnetic flux linking the closed loop C can be written as

$$\psi = \int_S B \cdot n\, dS,$$

where S is a surface enclosed by contour C and B is magnetic induction. Induced voltage V can be expressed as

$$V = \int_C E \cdot d\boldsymbol{s},$$

where E is the induced electric field. Then we can write Faraday 's law as follows:

$$\int_C E \cdot d\boldsymbol{s} = -\frac{\partial}{\partial t}\int_S B \cdot n\, dS.$$

Applying Stokes' theorem we obtain

$$\int_C E \cdot d\boldsymbol{s} = \int_S \operatorname{rot} E \cdot n \, dS.$$

Therefore,

$$\int_S \operatorname{rot} E \cdot n \, dS = -\int_S \frac{\partial B}{\partial t} \cdot n \, dS,$$

and since this holds for arbitrary surface S, we arrive at Faraday's law

$$\operatorname{rot} E = -\frac{\partial B}{\partial t}.$$

2.2 Equation for nonlinear stationary magnetic field

$$\begin{aligned} \operatorname{rot} H &= J, && (5) \\ \operatorname{div} B &= 0, && (6) \\ B &= \mu H. && (7) \end{aligned}$$

We shall use the second system for magnetic quantities to compute the magnetic field in the nonlinear environment of various electromagnetic machines, such as electromagnets and transformers including their magnetic circuits or shielding, synchronous and non-synchronous rotors, induction coils, magnetic heads of tape-recorders, magnetic lens of electrone microscopes, magnetic confinement devices in tokamaks, linear accelerators etc.

We shall consider only cross-sections of electromagnetic machines, i.e., we will consider the equations in a two-dimensional domain $\Omega \subset \mathbb{R}^2$.

Equation (7)

The relation between magnetic intensity and magnetic induction (7) is usually nonlinear (see Figure below), of the type

$$H(x) = \nu(x, \|B(x)\|^2) B(x), \qquad \text{a.e. } x \in \Omega.$$

Here $\|\cdot\|$ is the euclidian norm and the function ν is called magnetic reluctivity. It can be expressed as follows:

$$\nu(x, \eta) = \begin{cases} \nu_1(\eta) & \text{for } x \in \Omega_1 = \text{ ferromagnetic materials} \\ \nu_0 & \text{for } x \in \Omega_0 = \text{ other materials (copper wires, insulators, air, etc.)} \end{cases}$$

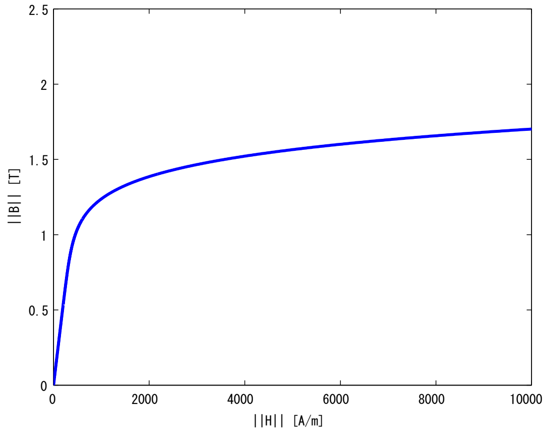
Here, $\nu_0 = 1/\mu_0$, where $\mu_0 = 4\pi \times 10^{-7}$ Tm/A is the permeability of vacuum, and ν_1 is a nondecreasing function satisfying

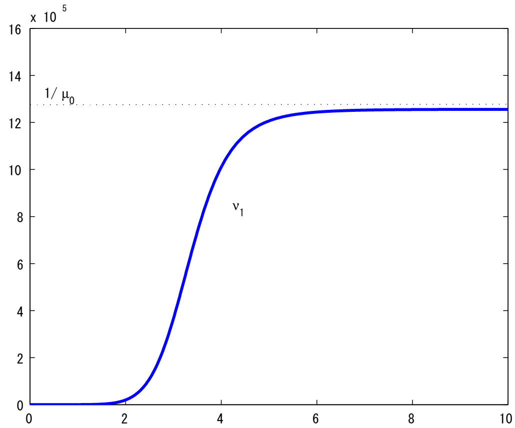
$$\begin{aligned} &C_0 \le \nu_1(\eta) \le C_1, && C_0, C_1 > 0, \\ &|\vartheta \nu_1'(\eta)| \le C_2, && \eta \ge \vartheta \ge 0, \ C_2 > 0. \end{aligned}$$

For example, the function

$$\nu_1(\eta) = \frac{1}{\mu_0} \left(\alpha + (1 - \alpha) \frac{\eta^8}{\eta^8 + \beta} \right),$$

depicted below, satisfies the conditions. This function for $\alpha = 0.0003$ and $\beta = 16000$ describes the reluctivity of stator plates in an electromotor.





Remark. For anisotropic materials the reluctivity becomes a matrix function (e.g., superoriented plates in large transformers).

Equation (6)

Equation (6) says that there exists the stream-function $u \in H^1(\Omega)$, which is determined uniquely up to a constant and which satisfies

$$B = \operatorname{curl} u,$$

where curl $u = (\partial u/\partial x_2, -\partial u/\partial x_1)$. Function u is called scalar magnetic potential, and we see that

$$\|B(x)\| = \|\operatorname{curl} u(x)\| = \|\nabla u(x)\|,$$

and therefore,

$$\nu\big(x, \|B(x)\|^2\big) = \nu\big(x, \|\nabla u(x)\|^2\big), \qquad \text{a.e. } x \in \Omega.$$

Equation (5)

Inserting the above expression into (5), we find

$$J = \operatorname{rot} H = \operatorname{rot}\big[\nu(\cdot, \|B\|^2)B\big] = \operatorname{rot}\big[\nu(\cdot, \|\operatorname{curl} u\|^2)\operatorname{curl} u\big] = -\operatorname{div}\big[\nu(\cdot, \|\nabla u\|^2)\nabla u\big].$$

Boundary condition

If we assume that outside the domain Ω there is a perfect conductor, then the normal component of magnetic induction vanishes on the boundary $\partial\Omega$ and we have the boundary condition

$$B \cdot n = 0.$$

Because $B = \operatorname{curl} u$, this yields

$$0 = B \cdot n = \operatorname{curl} u \cdot n = \nabla u \cdot t = \frac{\partial u}{\partial t},$$

where n is the unit normal vector and t is the unit tangent vector to $\partial\Omega$. This means that u is constant on the boundary and we set for simplicity $u = 0$ on $\partial\Omega$.

The above considerations yield the following model problem (we write f instead of J):

Model problem 1.

$$-\operatorname{div}\Big(\nu(x, \|\nabla u(x)\|^2)\nabla u(x)\Big) = f(x) \qquad \text{for } x \in \Omega \tag{8}$$

$$u(x) = 0 \qquad \text{on } \partial\Omega \tag{9}$$

Ex. We write our model equation in detail for the case of one-dimensional and two-dimensional domains Ω.

For dimension one we get

$$-\frac{d}{dx_1}\Big[\nu\big(x_1, (\frac{du}{dx_1}(x_1))^2\big)\frac{du}{dx_1}(x_1)\Big] = f(x_1).$$

For dimension two it becomes

$$\begin{aligned} &-\frac{\partial}{\partial x_1}\Big[\nu\Big(x_1, x_2, (\frac{\partial u}{\partial x_1}(x_1,x_2))^2 + (\frac{\partial u}{\partial x_2}(x_1,x_2))^2\Big)\frac{\partial u}{\partial x_1}(x_1,x_2)\Big] \\ &-\frac{\partial}{\partial x_2}\Big[\nu\Big(x_1, x_2, (\frac{\partial u}{\partial x_1}(x_1,x_2))^2 + (\frac{\partial u}{\partial x_2}(x_1,x_2))^2\Big)\frac{\partial u}{\partial x_2}(x_1,x_2)\Big] = f(x_1,x_2) \end{aligned}$$

Ex. What happens if we take the constitutive equation $B = \mu H$, where μ is a constant?

Then the function ν is a constant function equal to $1/\mu$ and we have

$$\operatorname{div}\big[\nu(x, \|\nabla u(x)\|^2)\nabla u\big] = \tfrac{1}{\mu}\operatorname{div}\nabla u = \tfrac{1}{\mu}\Big(\frac{\partial}{\partial x_1}\frac{\partial u}{\partial x_1} + \frac{\partial}{\partial x_2}\frac{\partial u}{\partial x_2}\Big) = \tfrac{1}{\mu}\Big(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\Big) = \tfrac{1}{\mu}\Delta u$$

Hence, we get the well-known Poisson equation $\Delta u = \mu f$ which appears in many situations (potential fields, fluid dynamics etc.).