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The Navier-Stokes Equations

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Outline

- Introduction: Conservation Principle
- Derivation by Control Volume
    - Convective Terms
    - Forcing Terms
- Solving the Equations
- Guided Example Problem
- Interactive Example Problem

Introduction: Conservation Principle

- N-S is Newton's second law of motion, for a fluid

$$\boldsymbol{\Sigma \vec{F} = m\vec{a}}$$

- The second law is based on the conservation of momentum

$$\boldsymbol{\Sigma \vec{F} = \frac{D(m\vec{v})}{Dt} = \frac{\partial(m\vec{v})}{\partial t} + \frac{\partial(m\vec{v})}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial(m\vec{v})}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial(m\vec{v})}{\partial z}\frac{\partial z}{\partial t}}$$

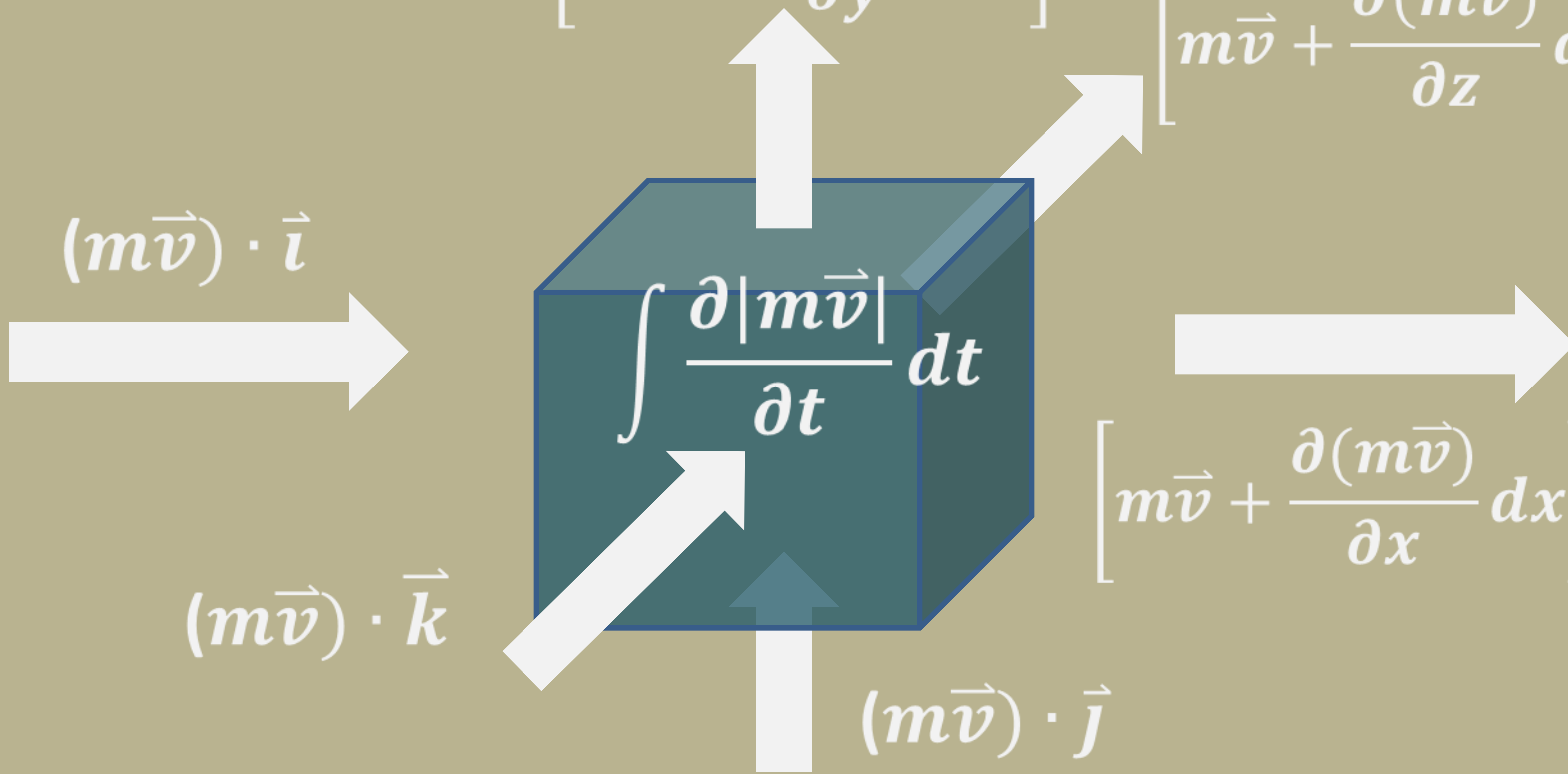
- The spatial derivative terms are a consequence of a Eulerian, rather than Lagrangian, frame of reference

Control Volume

$$\left[m\vec{v} + \frac{\partial(m\vec{v})}{\partial y}dy\right] \cdot \vec{j}$$

$$\left[m\vec{v} + \frac{\partial(m\vec{v})}{\partial z}dz\right] \cdot \vec{k}$$

$$(m\vec{v}) \cdot \vec{i}$$



$$\int \frac{\partial|m\vec{v}|}{\partial t}dt$$

$$\left[m\vec{v} + \frac{\partial(m\vec{v})}{\partial x}dx\right] \cdot \vec{i}$$

$$(m\vec{v}) \cdot \vec{k}$$

$$(m\vec{v}) \cdot \vec{j}$$

Derivation of Convective Terms

- **Momentum change by convection**

$$\int \frac{\partial|m\vec{v}|}{\partial t}dt + \left[\frac{\partial(m\vec{v})}{\partial x}dx\right]\cdot\vec{i} + \left[\frac{\partial(m\vec{v})}{\partial y}dy\right]\cdot\vec{j} + \left[\frac{\partial(m\vec{v})}{\partial z}dz\right]\cdot\vec{k}$$

- **Momentum change per time by convection**

$$\frac{\partial}{\partial t}\left\{\int \frac{\partial|m\vec{v}|}{\partial t}dt + \left[\frac{\partial(m\vec{v})}{\partial x}dx\right]\cdot\vec{i} + \left[\frac{\partial(m\vec{v})}{\partial y}dy\right]\cdot\vec{j} + \left[\frac{\partial(m\vec{v})}{\partial z}dz\right]\cdot\vec{k}\right\}$$

Derivation of Convective Terms

- **Rewritten, these convection terms of the momentum change per time are…**

$$\frac{\partial|m\vec{v}|}{\partial t}+\left[\frac{\partial(m\vec{v})}{\partial x}\frac{dx}{dt}\right]\cdot\vec{\imath}+\left[\frac{\partial(m\vec{v})}{\partial y}\frac{dy}{dt}\right]\cdot\vec{\jmath}+\left[\frac{\partial(m\vec{v})}{\partial z}\frac{dz}{dt}\right]\cdot\vec{k}$$

- **For an infinitesimal volume, dxdydz, with uniform density, the convection terms are written as**

$$\rho\left\{\frac{\partial|\vec{v}|}{\partial t}+\left[\frac{\partial\vec{v}}{\partial x}\frac{dx}{dt}\right]\cdot\vec{\imath}+\left[\frac{\partial\vec{v}}{\partial y}\frac{dy}{dt}\right]\cdot\vec{\jmath}+\left[\frac{\partial\vec{v}}{\partial z}\frac{dz}{dt}\right]\cdot\vec{k}\right\}dxdydz$$

Derivation of Convective Terms

- **The dx/dt, dy/dt, and dz/dt are just the velocity components in the x,y, and z directions, so the momentum change per time per unit volume is…**

$$\rho\left\{\frac{\partial|\vec{v}|}{\partial t}+\left[\frac{\partial\vec{v}}{\partial x}v_x\right]\cdot\vec{\imath}+\left[\frac{\partial\vec{v}}{\partial y}v_y\right]\cdot\vec{\jmath}+\left[\frac{\partial\vec{v}}{\partial z}v_z\right]\cdot\vec{k}\right\}$$

- **Physically, these terms are the momentum transfer by convection, coherent motion of the fluid. Mathematically, these terms represent the "$m\vec{a}$" in "$\Sigma\vec{F}=m\vec{a}$"**

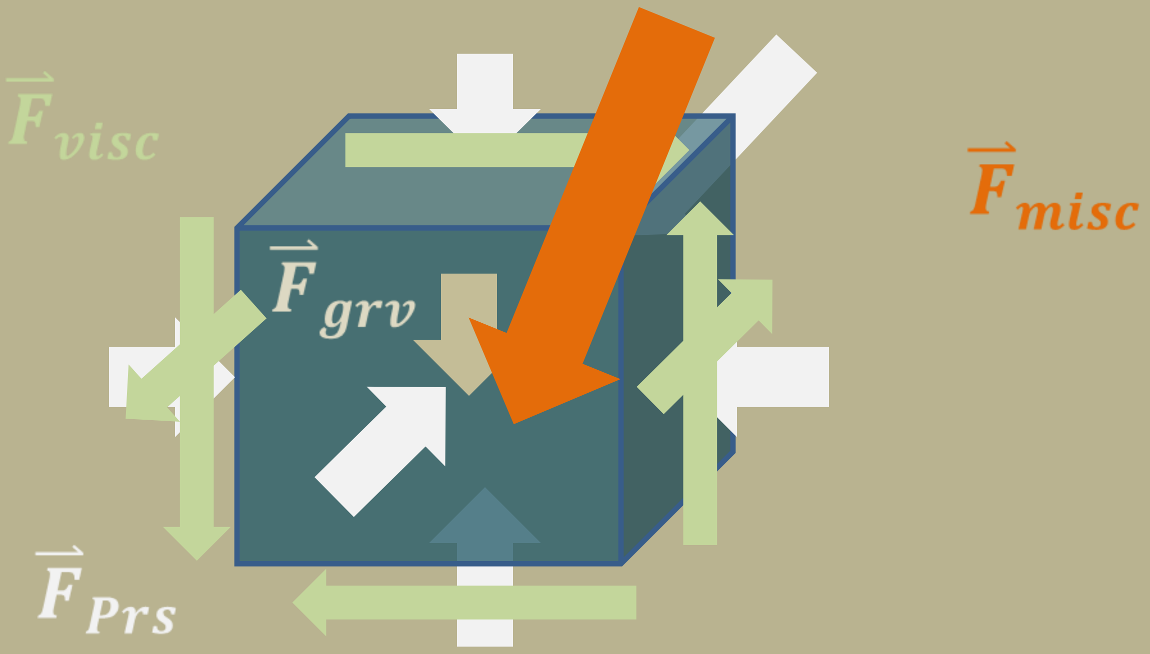
Derivation of Forcing Terms

- **The other half of N-S is the refined definition of the sum of the forces, $\Sigma\vec{F}$, acting on a fluid in order to produce the acceleration.**

- **Many different forces may be imposed on a fluid just as they are imposed on a solid body, for example by a machine and depend on the particular problem at hand**

- **There are some forces, e.g. gravity, that are always present in every situation, so they are spelled out**

Control Volume



Derivation of Forcing Terms: Gravity

- **There are three forces that are always acting on a fluid: gravity, pressure, and viscosity in addition to miscellaneous forces**

$$\vec{F}_{grv} + \vec{F}_{Prs} + \vec{F}_{visc} + \vec{F}_{misc} = \rho\left\{\frac{\partial|\vec{v}|}{\partial t} + \left[v_x\frac{\partial\vec{v}}{\partial x}\right]\cdot\vec{i} + \left[v_y\frac{\partial\vec{v}}{\partial y}\right]\cdot\vec{j} + \left[v_z\frac{\partial\vec{v}}{\partial z}\right]\cdot\vec{k}\right\}dxdydz$$

- **The force of gravity is the same as it is in solid mechanics, $m\vec{g}$, or $\rho\vec{g}dxdydz$**

Derivation of Forcing Terms: Gravity

$$\therefore\ \rho\vec{g}dxdydz + \vec{F}_{Prs} + \vec{F}_{visc} + \vec{F}_{misc} = \rho\left\{\frac{\partial|\vec{v}|}{\partial t} + \left[v_x\frac{\partial\vec{v}}{\partial x}\right]\cdot\vec{i} + \left[v_y\frac{\partial\vec{v}}{\partial y}\right]\cdot\vec{j} + \left[v_z\frac{\partial\vec{v}}{\partial z}\right]\cdot\vec{k}\right\}dxdydz$$

- **Gravity is a body force, which acts on the center of mass in one uniform direction**
- **Just as in solid mechanics, one can apply "$\Sigma\vec{F} = m\vec{a}$" in different directions**

Derivation of Forcing Terms: Gravity

$$\rho g_x + \sum F_x = \rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right)$$

$$\rho g_y + \sum F_y = \rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right)$$

$$\rho g_z + \sum F_z = \rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right)$$

- **Depending on your choice of orienting the x-y-z axiis, some terms may drop out**

Derivation of Forcing Terms: Pressure

- **Pressure is a surface stress always acting normal and inward to the surface of a fluid control volume. It is in a way analogous to the normal force of solid mechanics**

- **The force due to pressure, P, is defined as**

$$\vec{F}_{Prs} = -\nabla P * dV = -\nabla P dxdydz$$

- **The gradient is applied in the direction of the forces examined; for example for sum of the forces in x-direction, the gradient of P is** $^{\partial P}/_{\partial x}$

Derivation of Forcing Terms: Pressure

$$\rho g_x - \frac{\partial P}{\partial x} + \sum F_x = \rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right)$$

$$\rho g_y - \frac{\partial P}{\partial y} + \sum F_y = \rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right)$$

$$\rho g_z - \frac{\partial P}{\partial z} + \sum F_z = \rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right)$$

- **These changes in pressure may be directly caused by external sources, e.g. pumps, or a consequence of other forces, e.g. gravity causing hydrostatic pressure**

Derivation of Forcing Terms: Viscosity

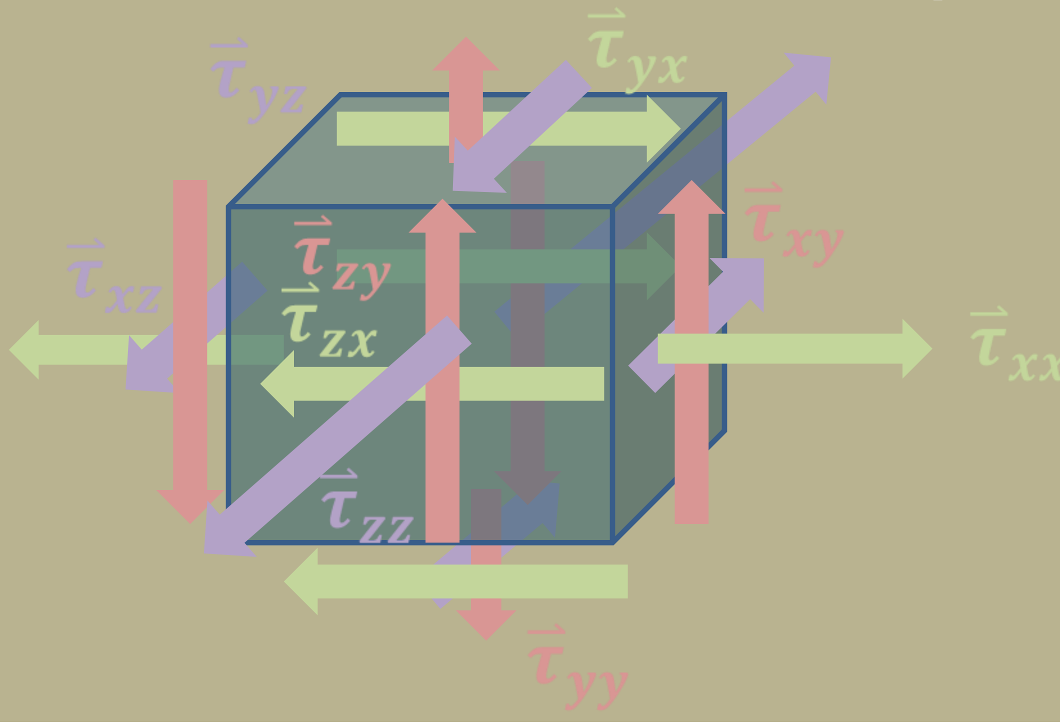
- **The final general force active on a fluid is viscosity, a shear stress, i.e. a stress acting parallel to a surface. It is analogous to the friction force of solid mechanics.**

- **Just as with the pressure, a normal stress, the force due to viscosity, a shear stress, is defined according to a gradient:**

$$\vec{F}_{visc} = \nabla\tau * dV = \nabla\tau dxdydz$$

Derivation of Forcing Terms: Viscosity

- **Unlike pressure, which is has one force couple per direction, shear stress has three force couples in each direction**



Derivation of Forcing Terms: Viscosity

$$\rho g_x - \frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \sum F_x = \rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right)$$

$$\rho g_y - \frac{\partial P}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \sum F_y = \rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right)$$

Derivation of Forcing Terms: Viscosity

$$\rho g_z - \frac{\partial P}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \sum F_z$$
$$= \rho\left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}\right)$$

- **For a Newtonian fluid, the shear stress is proportional to shear strain rate, similar to Hooke's law.**
- **Because a fluid by definition cannot support a shear stress, i.e. Young's modulus is zero, shear stresses create infinite strains, but finite strain rates.**

Derivation of Forcing Terms: Viscosity

$$\tau_{xy} = \tau_{yx} = \mu(\dot{\epsilon}_{xy} + \dot{\epsilon}_{yx}) = \mu\left(\frac{\partial}{\partial t}\frac{\partial y}{\partial x} + \frac{\partial}{\partial t}\frac{\partial x}{\partial y}\right) = \mu\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu(\dot{\epsilon}_{yz} + \dot{\epsilon}_{zy}) = \mu\left(\frac{\partial}{\partial t}\frac{\partial z}{\partial y} + \frac{\partial}{\partial t}\frac{\partial y}{\partial z}\right) = \mu\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right)$$

$$\tau_{xz} = \tau_{zx} = \mu(\dot{\epsilon}_{xz} + \dot{\epsilon}_{zx}) = \mu\left(\frac{\partial}{\partial t}\frac{\partial z}{\partial x} + \frac{\partial}{\partial t}\frac{\partial x}{\partial z}\right) = \mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)$$

$$\tau_{xx} = -\frac{2}{3}\mu\nabla\cdot\vec{v} + 2\mu\frac{\partial v_x}{\partial x}$$

$$\tau_{yy} = -\frac{2}{3}\mu\nabla\cdot\vec{v} + 2\mu\frac{\partial v_y}{\partial y} \qquad \tau_{zz} = -\frac{2}{3}\mu\nabla\cdot\vec{v} + 2\mu\frac{\partial v_z}{\partial z}$$

Derivation of Forcing Terms: Viscosity

- **Substituting these stress-strain rate relations into the momentum rate equations…**

$$\rho g_x - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) = \rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right)$$

$$\rho g_y - \frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) = \rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right)$$

$$\rho g_z - \frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) = \rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right)$$

Derivation

- In summary, the Navier-Stokes equations are the sum of the gravitational force, pressure force, and viscous forces are equal to the mass times acceleration:

$$\vec{F}_{grv} + \vec{F}_{Prs} + \vec{F}_{visc} = m\vec{a}$$

- The three forces are analogous to the forces of gravity, normal, and friction. The last two forces are reaction forces, reacting to the motion. However pressure may be prescribed by machinery.

Solving the Equations

- **How the fluid moves is determined by the initial and boundary conditions; the equations remain the same**

- **Depending on the problem, some terms may be considered to be negligible or zero, and they drop out**

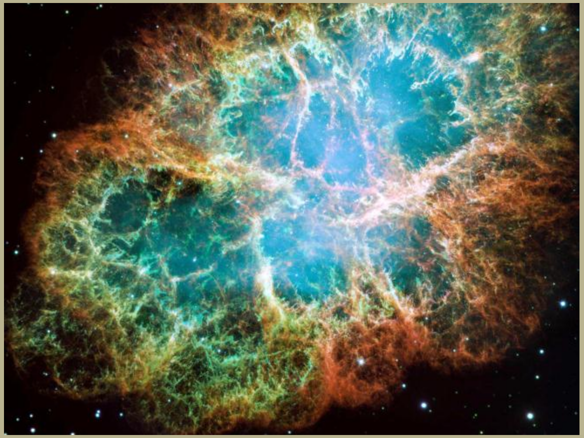
- **In addition to the constraints, the continuity equation (conservation of mass) is frequently required as well. If heat transfer is occuring, the N-S equations may be coupled to the First Law of Thermodynamics (conservation of energy)**

Solving the Equations

- Solving the equations is very difficult except for simple problems. Mathematicians have yet to prove general solutions exist, and is considered the sixth most important unsolved problem in all of math!

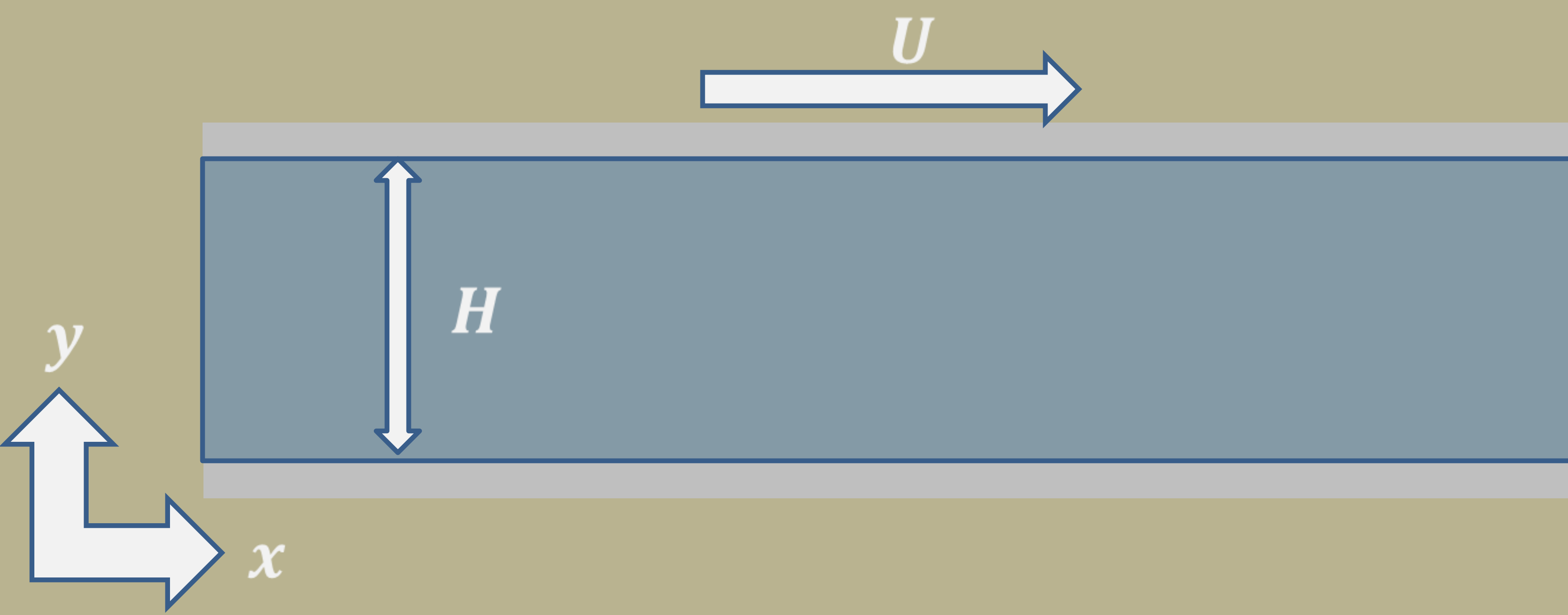
Solving the Equations

- **In addition, the phenomenon of turbulence, caused by the convective terms, is considered the last unsolved problem of classical mechanics. We know more about quantum particles and supernova than we do about the swirling of creamer in a steaming cup of coffee!**



Example Problems: Couette Flow

❖ **Set up the equations and boundary conditions to solve for the following problem at steady state and fully developed:**



Example Problems: Couette Flow

- Two steady-state dimensional problem, therefore only invoke N-S in x and y

$$\rho g_x - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \cancel{\frac{\partial^2 v_x}{\partial z^2}}\right) = \rho\left(\cancel{\frac{\partial v_x}{\partial t}} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + \cancel{v_z\frac{\partial v_x}{\partial z}}\right)$$

$$\rho g_y - \frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \cancel{\frac{\partial^2 v_y}{\partial z^2}}\right) = \rho\left(\cancel{\frac{\partial v_y}{\partial t}} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + \cancel{v_z\frac{\partial v_y}{\partial z}}\right)$$

- First, eliminate all the t and z components

Example Problems: Couette Flow

- **Next, eliminate Vy terms due to impermeability of walls; fluid cannot pass throw the walls so it cannot flow in the y-direction**

$$\rho \cancel{g_x} - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right) = \rho\left(v_x \frac{\partial v_x}{\partial x} + \cancel{v_y} \frac{\partial v_x}{\partial y}\right)$$

$$\rho g_y - \frac{\partial P}{\partial y} + \mu\left(\cancel{\frac{\partial^2 v_y}{\partial x^2}} + \cancel{\frac{\partial^2 v_y}{\partial y^2}}\right) = \rho\left(v_x \cancel{\frac{\partial v_y}{\partial x}} + \cancel{v_y} \cancel{\frac{\partial v_y}{\partial y}}\right)$$

- **Then eliminate gravity in the x-direction, because it is all in the y-direction**

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Example Problems: Couette Flow

- **Now if the flow is fully developed, the velocity in the direction of the flow does not change.**

$$-\frac{\partial P}{\partial x} + \mu\left(\cancel{\frac{\partial^2 v_x}{\partial x^2}} + \frac{\partial^2 v_x}{\partial y^2}\right) = \rho\left(v_x \cancel{\frac{\partial v_x}{\partial x}}\right)$$

$$\rho g_y = \frac{\partial P}{\partial y}$$

Example Problems: Couette Flow

- **Now if the flow is fully developed, the velocity in the direction of the flow does not change.**

$$\mu \frac{\partial^2 v_x}{\partial y^2} = \cancel{\frac{\partial P}{\partial x}}$$

$$\rho g_y y + C(x) = P(x, y)$$

- **If we assume the pressure to be predominately hydrostatic, i.e. gravity is the major cause of pressure, then P(x,y) ~P(y)**

Example Problems: Couette Flow

$$\mu \frac{\partial^2 v_x}{\partial y^2} = 0$$

$$\frac{\partial v_x}{\partial y} = C_1$$

$$v_x = C_1 y + C_2$$

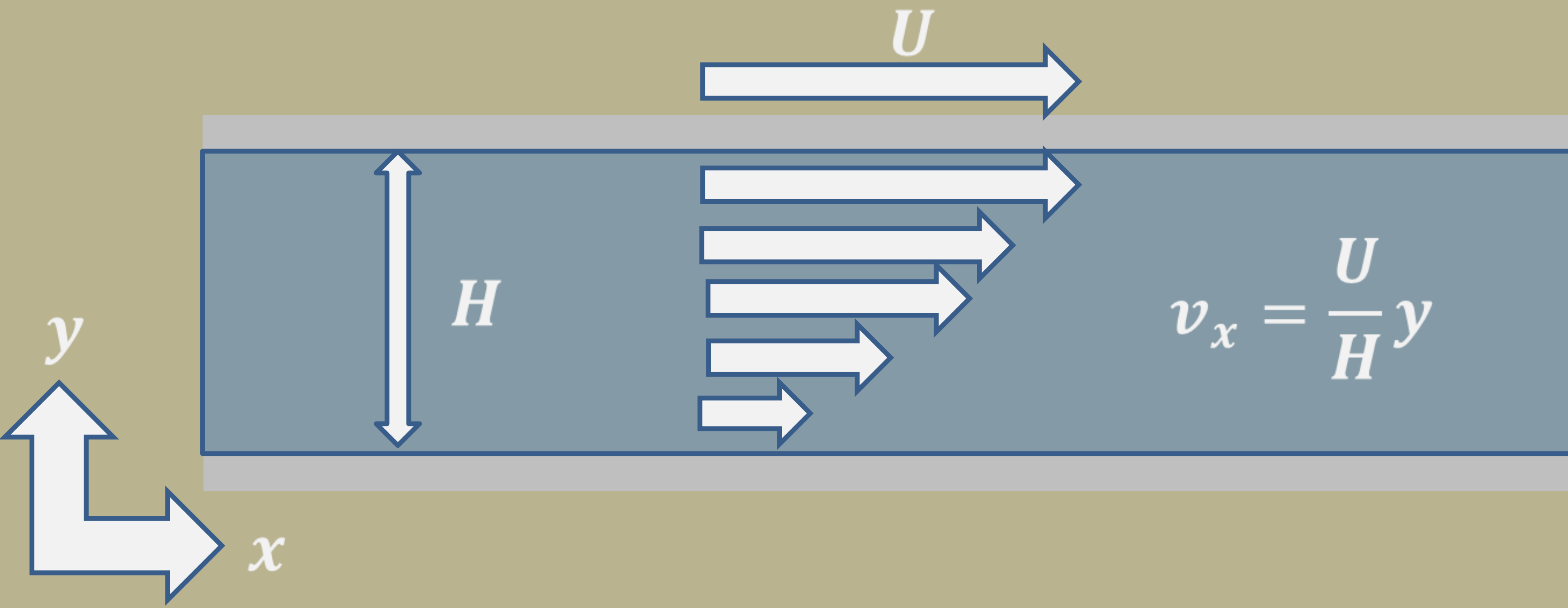
The boundary conditions are

$v_x = 0$ at y=0

$v_x = U$ at y=H

Example Problems: Couette Flow

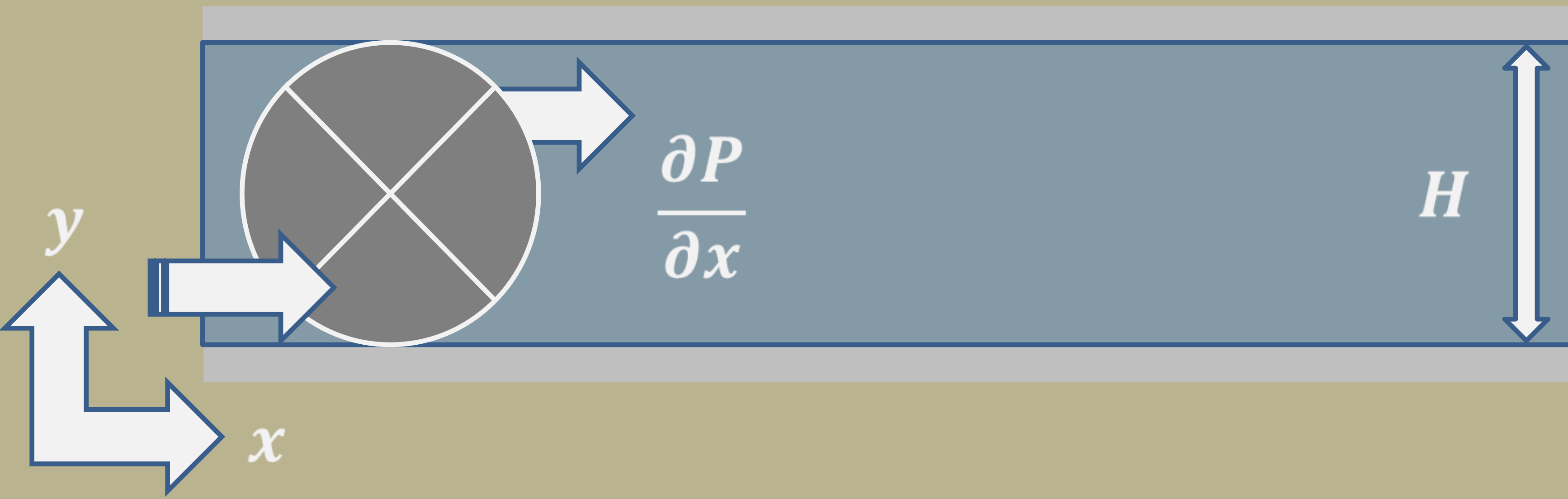
- **Based on the simplified N-S equations, what physical phenomenon is responsible for the velocity profile?**



$$v_x = \frac{U}{H} y$$

Example Problems: Poiseuille Flow

- **Consider the last problem, but without the moving wall and with a pump providing a pressure gradient dP/dx**



Example Problems: Couette Flow

- **The equations reduce to:**

$$\cancel{\rho g_x} - \frac{\partial P}{\partial x} + \mu\left(\cancel{\frac{\partial^2 v_x}{\partial x^2}} + \frac{\partial^2 v_x}{\partial y^2} + \cancel{\frac{\partial^2 v_x}{\partial z^2}}\right) = \rho\left(\cancel{\frac{\partial v_x}{\partial t}} + v_x \cancel{\frac{\partial v_x}{\partial x}} + \cancel{v_y} \frac{\partial v_x}{\partial y} + \cancel{v_z \frac{\partial v_x}{\partial z}}\right)$$

$$\rho g_y - \frac{\partial P}{\partial y} + \mu\left(\cancel{\frac{\partial^2 v_y}{\partial x^2}} + \cancel{\frac{\partial^2 v_y}{\partial y^2}} + \cancel{\frac{\partial^2 v_y}{\partial z^2}}\right) = \rho\left(\cancel{\frac{\partial v_y}{\partial t}} + v_x \cancel{\frac{\partial v_y}{\partial x}} + v_y \cancel{\frac{\partial v_y}{\partial y}} + v_z \cancel{\frac{\partial v_y}{\partial z}}\right)$$

- **Now dP/dx is a known value and presumably driving the flow**

Example Problems: Couette Flow

$$\mu \frac{\partial^2 v_x}{\partial y^2} = \frac{\partial P}{\partial x}$$

$$\frac{\partial v_x}{\partial y} = \frac{1}{\mu} \frac{\partial P}{\partial x} y + C_1$$

$$v_x = \frac{1}{2\mu} \frac{\partial P}{\partial x} y^2 + C_1 y + C_2$$

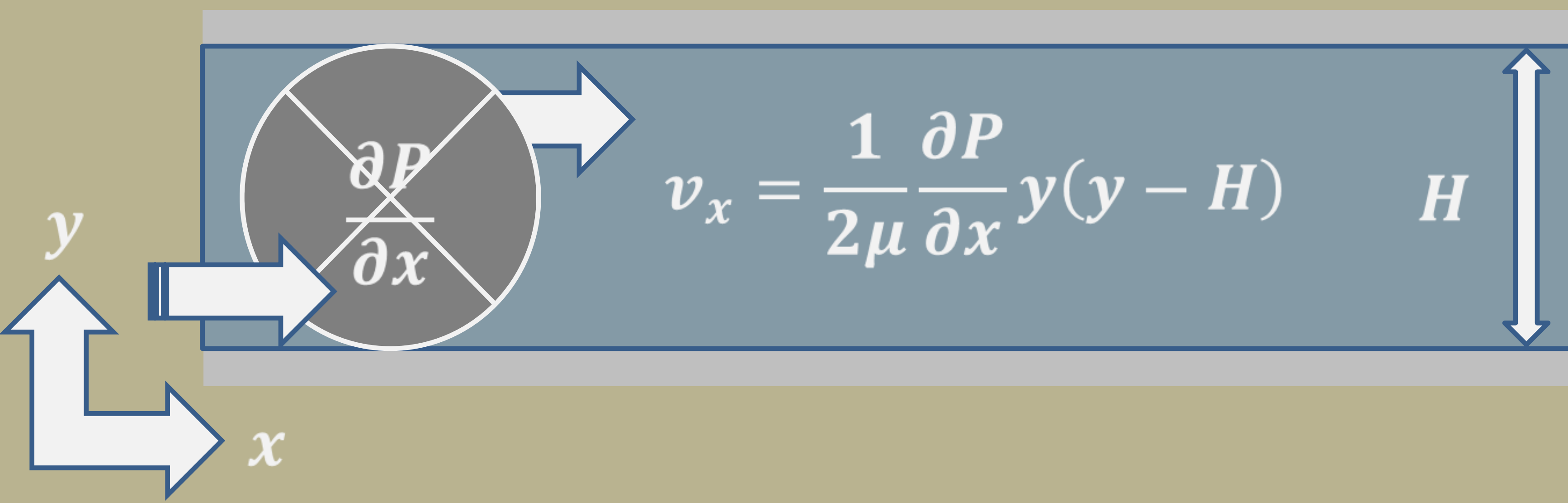
The boundary conditions are

$v_x = 0$ at y=0

$v_x = 0$ at y=H

Example Problems: Poiseuille Flow

- **Now in addition to the viscosity forces, pressure is driving the flow**



Questions?