Nonlinear PDEs

7th lecture

Our plan

$$
\begin{aligned}
-\operatorname{div}\left(\nu(x, \|\nabla u(x)\|^2)\nabla u(x)\right) &= f(x) \qquad && \text{for } x \in \Omega \\
u(x) &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

We shall solve this problem in the following steps:

- Write the problem in the operator form $A(u) = f$, where $A : H \to H$ is a nonlinear operator expressing the left-hand side of the equation.
- Proof an existence theorem for this abstract operator equation (using monotone operator theory).
- Check that the assumptions of the existence theorem for our problem are satisfied.

Theorem

Theorem. Let $A : H \to H$ be strongly monotone with respect to H (with constant η) and let A satisfy Lipschitz condition (with constant L). Then for each $f \in H$ there exists a unique solution u of the problem

$$Au = f \qquad \text{in } H.$$

A is **strongly monotone** if there exists $\eta > 0$ such that

$$\langle Au - Av, u - v \rangle \geq \eta \|u - v\|_H^2 \qquad \forall u, v \in H.$$

A satisfies **Lipschitz condition** with constant L if

$$\|Au - Av\|_H \leq L \|u - v\|_H \qquad \forall u, v \in H.$$

Nonlinear operator

Find $u \in H_0^1(\Omega)$ satisfying

weak solution

$$\int_\Omega \nu(x, |\nabla u(x)|^2) \nabla u(x) \cdot \nabla \varphi(x)\, dx = \int_\Omega f(x)\varphi(x)\, dx$$

for all test functions $\varphi \in H_0^1(\Omega)$.

Define the operator A by

$$\langle Au, \varphi \rangle = \int_\Omega \nu(x, |\nabla u(x)|^2) \nabla u(x) \cdot \nabla \varphi(x)\, dx$$

$u,\ \varphi \in H_0^1(\Omega)$

$\langle \cdot, \cdot \rangle$ is the inner product in $H^1(\Omega)$

Coefficient

The function ν is given by

$$\nu(x,\eta) = \begin{cases} \nu_1(\eta) & \text{for } x \in \Omega_1 = \text{ ferromagnetic materials} \\ \nu_0 & \text{for } x \in \Omega_0 = \text{ other materials (copper wires, insulators, air, etc.)} \end{cases}$$

where $\nu_0 = 1/\mu_0$ with $\mu_0 = 4\pi \times 10^{-7}$ Tm/A, the permeability of vacuum, and ν_1 is a nondecreasing function satisfying

$$\begin{aligned} &C_0 \le \nu_1(\eta) \le C_1, \qquad C_0, C_1 > 0, \\ &|\vartheta \nu_1'(\eta)| \le C_2, \qquad \eta \ge \vartheta \ge 0,\ C_2 > 0. \end{aligned}$$

Galerkin solution

Definition Let $V_h \subset V$ be a nonempty finite-dimensional subspace of V. Then a function $u_h \in V_h$ is called a **Galerkin approximation** to the solution of $Au = f$ if

$$\langle Au_h, \varphi_h \rangle = \langle f, \varphi_h \rangle \qquad \forall \varphi_h \in V_h$$

Theorem (Error of Galerkin approximations)

Let $A : V \to V$ be a strongly monotone and Lipschitz continuous operator and $V_h \neq \emptyset$ be a finite-dimensional subspace of V. Then there exists a constant independent on V_h such that

$$\|u - u_h\|_V \leq C \inf_{v_h \in V_h} \|u - v_h\|_V,$$

where u is the solution to $Au = f$ and u_h is the Galerkin approximation.

Method of successive approximations

$$a(w; u, \varphi) = \int_\Omega \nu(x, |\nabla w(x)|^2) \nabla u(x) \cdot \nabla \varphi(x)\, dx, \qquad u, w, \varphi \in V$$

$$\boxed{a(u_h; u_h, \varphi_h) = \langle A u_h, \varphi_h \rangle = (f, \varphi_h)_0 \qquad \forall \varphi_h \in V_h}$$

1. Choose $w_0 \in V_h$ arbitrarily.

2. Compute $w_k + 1$ for $k = 0, 1, 2, \ldots$ by repeating the solution of the following **linear** problem

$$a(w_k; w_{k+1}, \varphi_h) = (f, \varphi_h)_0 \qquad \forall \varphi_h \in V_h \tag{1}$$

Such a $w_{k+1} \in V_h$ uniquely exists by Lax-Milgram lemma.

Minimization method

Minimize the following non-quadratic convex functional on V_h:

$$J(u) = \frac{1}{2}\int_\Omega \mathcal{N}(x, |\nabla u(x)|^2)\,dx - \int_\Omega fu\,dx$$

Here $\mathcal{N}$ is a primitive function to ν, i.e., $d\mathcal{N}/ds(x,s) = \nu(x,s)$.

There are several methods how to solve the minimization problem numerically:

- Newton's method
- generalized conjugate gradient method
- relaxation method
- method of successive approximations - this amounts to solving the linear problem by minimization, i.e., minimizing

  $$J_k(w) = \frac{1}{2}a(w_k; w, w) - (f, w)_0 - \frac{1}{2}a(w_k; w_k, w_k) + (f, w_k)_0 + J(w_k),$$

  where the last three constant terms do not have any influence on the minimization but are selected so that $J_k(w_k) = J(w_k)$ for all k.

Theorem

Theorem (Main theorem on monotone operators)

Let V be a separable Hilbert space and let $A : V \to V$ be monotone, demi-continuous and coercive operator. Then the set of solutions to $Au = f$ is nonempty, convex and closed for each $f \in V$.

Separable space and coercivity

Separable space: V is separable if it contains a countable dense subset. The space $H^1(\Omega)$ is separable. An example of a countable dense subset is the set of all polynomials with rational coefficients.

Coercive operator: $A : V \to V$ is called coercive if

$$\lim_{\|v\|_V \to \infty} \frac{\langle Av, v \rangle}{\|v\|_V} = \infty.$$

It is clear that a strongly monotone operator is coercive.

Demi-continuity and weak convergence

Demi-continuous operator : $A : V \to V$ is demi-continuous if

$$v_k \to v \quad \Rightarrow \quad Av_k \rightharpoonup Av$$

Here, the arrow $\rightharpoonup$ means **weak convergence** defined as follows:

$$v_k \rightharpoonup v \quad \Leftrightarrow \quad F(v_k) \to F(v) \quad \forall F \in V^*$$

V^* is the dual space of V, i.e., the space of continuous linear functionals on V.

1. Strong convergence implies weak convergence but the converse is not true unless the space is finite-dimensional.

2. Uniform boundedness principle: If $\{v_k\} \subset V$ is a sequence such that $F(v_k)$ is bounded for every $F \in V^*$, then $\{\|v_k\|_V\}$ is also bounded.

3. The following two statements are equivalent

   (i) v_k converges weakly to v in V

   (ii) $\{\|v_k\|_V\}$ is bounded and $\lim_{k\to\infty}\langle v_k, f\rangle = \langle v, f\rangle$ for all f from a set which is dense in V.

Riesz theorem

A mapping $A : X \to Y$ (X, Y are normed spaces) is a **linear operator** provided

$$A(\lambda u + \mu v) = \lambda A(u) + \mu A(v) \qquad \forall u, v \in X, \;\; \forall \lambda, \mu \in \mathbb{R}$$

A linear operator $A : X \to Y$ is **bounded** if

$$\|A\| = \sup_{\|u\|_X \leq 1} \|A(u)\|_Y = \sup_{u \in X} \frac{\|A(u)\|_Y}{\|u\|_X} < \infty$$

If $Y = \mathbb{R}$, then we call the operator $A : X \to \mathbb{R}$ a **functional**.

X^* = collection of all bounded linear functionals on X = **dual space** of X

Theorem (Riesz representation theorem) Let H be a real Hilbert space. Then for each $F \in H^*$ there exists a unique element $f \in H$ such that

$$F(v) = (f, v) \qquad \forall v \in H$$

The mapping $F \mapsto f$ is a linear isomorphism of H^* onto H.