4 Sobolev spaces

Review. Lax-Milgram lemma

Bilinear form is a mapping $a(\cdot,\cdot)$ defined on $H\times H$ such that for each fixed $v\in H$, the mappings $a(v,\cdot)$ and $a(\cdot,v)$ are linear.

Theorem (Lax-Milgram lemma)

Assume that H is a Hilbert space and the bilinear form $a: H\times H\to\mathbb{R}$ satisfies

$$\begin{aligned} &\text{(continuity)} & |a(u,v)| &\le \alpha\|u\|_H\,\|v\|_H & &\forall u,v\in H \quad (\alpha>0) & (1)\\ &\text{(coercivity)} & \beta\|u\|^2 &\le a(u,u) & &\forall u\in H \quad (\beta>0) & (2)\end{aligned}$$

Then for each bounded linear functional $F: H\to\mathbb{R}$ there exists a unique element $u\in H$ such that

$$B(u,v)=F(v) \qquad \forall v\in H.$$

Proof. We give only a brief proof.

- We define the mapping $A: H\to H$ as follows: for any $u\in H$ there is a unique $w\in H$ so that $a(u,v)=(w,v)$ holds for all $v\in H$ (by Riesz theorem). Then we put $Au=w$.
- The mapping A is linear and bounded, satisfying $\|Au\|_H\le\alpha\|u\|_H$.
- Moreover, from

  $$\beta\|u\|_H^2\le a(u,u)=(Au,u)\le\|Au\|_H\,\|u\|_H$$

  we find that $\beta\|u\|_H\le\|Au\|_H$ holds and therefore A is one-to-one and its range $R(A)$ is closed in H.
- We show that $R(A)=H$. Since $R(A)$ is closed, if $R(A)\ne H$ then there exists a nonzero element $w\in R(A)^\perp$. But then $\beta\|w\|_H^2\le a(w,w)=(Aw,w)=0$, which is a contradiction.
- By Riesz theorem there is a unique w such that

  $$F(v)=(w,v) \qquad \forall v\in H$$

  Because $R(A)=H$ and A is one-to-one, there is exactly one $u\in H$ so that $Au=w$ holds. Hence,

  $$a(u,v)=(Au,v)=(w,v)=F(v) \qquad \forall v\in H$$

- uniqueness: if there are two elements $u_1,u_2\in H$ fulfilling the theorem, then $a(u_1-u_2,v)=F(v)-F(v)=0$. Putting $v=u_1-u_2$ and using the coercivity, we find $u_1=u_2$.

Lax-Milgram lemma actually tells us that $\sqrt{a(\cdot,\cdot)}$ is a norm on H equivalent to $\|\cdot\|_H$.

First, we would like to study, using the Lax-Milgram lemma, the existence of solution to the simplest magnetic field problem, i.e., in the case when $\nu \equiv 1$:

$$\begin{aligned} -\Delta u &= f \qquad \text{for } x \in \Omega, & (3)\\ u(x) &= 0 \qquad \text{on } \partial\Omega. & (4) \end{aligned}$$

Here is the basic idea (for $\Omega = (0, 1)$, for simplicity):

- multiply the equation by a test function φ vanishing at boundary (i.e., $\varphi(0) = \varphi(1) = 0$), integrate over Ω and use by-parts integration:

$$-u'' = f \quad \Rightarrow \quad \int_0^1 -u''\varphi\,dx = \int_0^1 f\varphi\,dx \quad \Rightarrow \quad \int_0^1 u'\varphi'\,dx = \int_0^1 f\varphi\,dx$$

- define the bilinear form a and linear functional F as

$$a(u,\varphi) = \int_0^1 u'\varphi'\,dx, \qquad u, \varphi \in V$$

$$F(\varphi) = \int_0^1 f\varphi\,dx, \qquad \varphi \in V$$

- use the Lax-Milgram lemma to show the existence of solution to $a(u,\varphi) = F(\varphi)\ \ \forall \varphi \in V$.

However, there is an important question how to choose the space V above. From the assumptions of Lax-Milgram lemma, we require that V is a Hilbert space and that it is big enough so that the solution u belongs to it. We could take for V the space of continuously differentiable functions but then it is not a Hilbert space with the usual norm, and, moreover, the solution may not exist for some reasonable f because this space is too small. It turns out that the best choice is a Sobolev space, so before stating the exact proof of existence, we review basic facts about Sobolev spaces.

Review. Functions from $C_0^\infty(\Omega)$.

> These are functions which are infinitely many times differentiable (this fact is denoted by the index ∞) and which are compactly supported inside Ω. Support of a function is the closure of the set of all points where the function is not zero. Here we require that the support is compact in Ω, which means that the function is different from zero only on a bounded subset of Ω and it vanishes on a neighbourhood of the boundary of Ω (that is why the index 0 is used).

Sobolev spaces, loosely said, are spaces of functions which have weak derivatives belonging to some L^p space. We shall first review the notion of weak derivative and then give a more precise definition of Sobolev spaces.

Review. Weak derivative.

The definition of weak derivative is motivated by the following integration by parts formula which is valid for $u \in C^1(\Omega)$ and a test function $\varphi \in C_0^\infty(\Omega)$:

$$\int_\Omega u\varphi_{x_i}\, dx = -\int_\Omega u_{x_i}\varphi, \qquad i = 1, \dots, n$$

There are no boundary terms because φ has compact support in Ω and thus vanishes near $\partial\Omega$, so the boundary integral is zero. Now, the left-hand side of the above equation is defined even for functions u from $L^1(\Omega)$ and hence we can weaken the notion of derivative in the following manner:

Definition. Suppose $u, v \in L^1(\Omega)$. We say that v is the *first weak partial derivative* of u with respect to x_i (written $v = u_{x_i}$) provided

$$\int_\Omega u\varphi_{x_i}\, dx = -\int_\Omega v\varphi\, dx \qquad \text{for all test functions } \varphi \in C_0^\infty(\Omega).$$

(If there does not exist such a function $v \in L^1(\Omega)$, then u does not possess a weak partial derivative.)

Lemma. The weak derivative, if it exists, is uniquely defined up to a set of measure zero.

Example. Let $n = 1$, $\Omega = (0, 2)$ and

$$u(x) = \begin{cases} x & \text{if } 0 < x \le 1 \\ 1 & \text{if } 1 \le x < 2 \end{cases}$$

Then the weak derivative of u is

$$v(x) = \begin{cases} 1 & \text{if } 0 < x \le 1 \\ 0 & \text{if } 1 < x < 2 \end{cases}$$

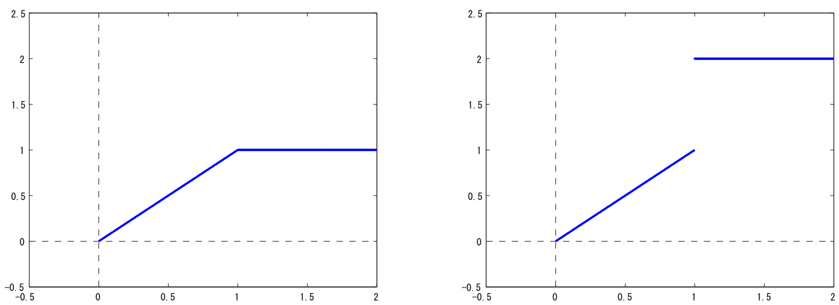
To prove it, we calculate

$$\int_0^2 u\varphi'\, dx = \int_0^1 x\varphi'\, dx + \int_1^2 \varphi'\, dx = -\int_0^1 \varphi\, dx + \varphi(1) - \varphi(1) = -\int_0^2 v\varphi\, dx.$$

Example. We define u by

$$u(x) = \begin{cases} x & \text{if } 0 < x \le 1 \\ 2 & \text{if } 1 \le x < 2 \end{cases}$$

Then u' does not exist in the weak sense.



Now we are ready to define Sobolev spaces $W^{1,p}(\Omega)$. We will consider only domains $\Omega \subset \mathbb{R}^n$ that are bounded and have Lipschitz continuous boundary, which is sufficiently wide class of domains for practical problems.

Review. Sobolev spaces.

> **Definition.** The *Sobolev space* $W^{1,p}(\Omega)$ consists of all locally summable functions $u : \Omega \to \mathbb{R}$ such that u_{x_i} exist for $n = 1, \dots, n$ in the weak sense and u, u_{x_i} belong to $L^p(\Omega)$.
> If $p = 2$ we usually write $H^1(\Omega) = W^{1,2}(\Omega)$.
>
> The norm in $W^{1,p}(\Omega)$ is defined as follows:
>
> $$\|u\|_{W^{1,p}(\Omega)} = \begin{cases} \left(\int_\Omega (|u|^p + \sum_{i=1}^n |u_{x_i}|^p)\, dx\right)^{1/p} & (1 \le \infty) \\ \operatorname{ess\,sup}_\Omega |u| + \sum_{i=1}^n \operatorname{ess\,sup}_\Omega |u_{x_i}| & (p = \infty) \end{cases}$$
>
> **Lemma.** For each $1 \le p \le \infty$ the Sobolev space $W^{1,p}(\Omega)$ is a Banach space.
>
> **Rellich's theorem.** Let Ω be a domain with lipschitz boundary. Then the identity mapping from $H^1(\Omega)$ to $L^2(\Omega)$ is compact, i.e., each bounded sequence in $H^1(\Omega)$ contains a subsequence converging in $L^2(\Omega)$.

Example.

- The space $H^1(\Omega)$ is a Hilbert space with the inner product
$$(f, g) = \int_\Omega (fg + \nabla f \cdot \nabla g)\, dx.$$
- It holds
$$\|v\|^2_{H^1(\Omega)} = \|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}.$$
- From the above relation we have $H^1(\Omega) \subset L^2(\Omega)$.
- In one-dimensional case, H^1-functions are Hölder continuous with exponent $\frac{1}{2}$, which can be shown using Hölder's inequality (we set $\Omega = (0, 1)$):
$$|v(z) - v(y)| = \Big|\int_y^z v'(x)\, dx\Big| \le |z - y|^{1/2}\Big(\int_y^z |v'(x)|^2\, dx\Big)^{1/2} \le \|v\|_{H^1(0,1)}|z - y|^{1/2}, \qquad y, z \in (0, 1).$$

Example. Let $\Omega = B(0, 1)$ be the unit ball in $\mathbb{R}^3$. Then the function

$$u(x) = |x|^{-1/4} \qquad (x \ne 0)$$

belongs to $H^1(\Omega)$. (Although it is not even continuous!)

To see this, we compute

$$u_{x_i}(x) = \frac{-x_i}{4|x|^{9/4}}, \qquad x \ne 0, \tag{5}$$

which implies

$$|\nabla u(x)| = \frac{1}{4|x|^{5/4}}, \qquad x \ne 0.$$

We show that the weak derivative exists by cutting out a small ball of radius ϵ around the origin and sending $\epsilon \to 0$. By Green's theorem

$$\int_{\Omega - B(0,\epsilon)} u\varphi_{x_i}\, dx = -\int_{\Omega - B(0,\epsilon)} u_{x_i}\varphi\, dx + \int_{\partial B(0,\epsilon)} u\varphi n_i\, dS,$$

where $n = (n_1, n_2, n_3)$ denotes the inward normal on $B(0, \epsilon)$, i.e. , $n_i = -x_i/|x|$. Now, the boundary term becomes small when ϵ is small:

$$\left| \int_{\partial B(0,\epsilon)} u\varphi n_i \, dS \right| \leq \max_{\Omega} |\varphi| \int_{\partial B(0,\epsilon)} \epsilon^{-1/4} \, dS \leq C\epsilon^{2-1/4} \to 0, \qquad \text{as } \epsilon \to 0.$$

Thus the function defined by (5) is the weak derivative of u. Moreover,

$$\begin{aligned} \int_{\Omega} |\nabla u|^2 \, dx &= \frac{1}{16} \int_{\Omega} |x|^{-5/2} \, dx = \frac{1}{16} \int_0^1 \int_{\partial B(0,\rho)} \rho^{-5/2} \, dS \, d\rho \\ &= \frac{1}{16} \int_0^1 4\pi\rho^2 \rho^{-5/2} \, d\rho = \frac{\pi}{4} \int_0^1 \rho^{-1/2} \, d\rho = \frac{\pi}{2} \end{aligned}$$

In a similar way we can compute $\int_{\Omega} u^2 \, dx = 8\pi/5$.

Review. Traces.

By $W_0^{1,p}(\Omega)$ we denote the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p}(\Omega)$. This means that if $u \in W_0^{1,p}(\Omega)$, then there exists a sequence of functions $u_m \in C_0^{\infty}(\Omega)$ such that $u_m \to u$ in $W^{1,p}(\Omega)$. It is customary to write $H_0^1(\Omega) = W_0^{1,2}(\Omega)$.

$H_0^1(\Omega)$ is interpreted as functions from $H^1(\Omega)$ that are zero on the boundary of Ω. This is not correct statement because functions from $H^1(\Omega)$ are defined only up to a set of measure zero and boundary has measure zero.
However, it can be made precise using the notion of trace operator.

Theorem on traces. For each lipschitz domain Ω there exists exactly one continuous linear operator $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$, such that

$$\gamma v = v\Big|_{\partial\Omega} \qquad \forall v \in C^{\infty}(\bar{\Omega}).$$

This theorem enables us to define the space $H_0^1(\Omega)$ as

$$H_0^1(\Omega) = \{v \in H^1(\Omega) \,;\ \ \gamma v = 0 \ \text{ on } \partial\Omega\}.$$

Friedrichs inequality. Let Ω be a domain with lipschitz boundary. Then there is a constant C_F such that

$$\|v\|_{H^1(\Omega)} \leq C_F |v|_{H^1(\Omega)} \qquad \forall v \in H_0^1(\Omega).$$

Here $|\cdot|_{H^1(\Omega)}$ is the seminorm defined as

$$|v|_{H^1(\Omega)} = \Big(\int_{\Omega} |\nabla v|^2 \Big)^{1/2}.$$

The fact that v has zero trace on the boundary is important!
This inequality says that the norm and seminorm on $H^1(\Omega)$ are equivalent norms for functions from $H_0^1(\Omega)$:

$$c\|v\|_{H^1(\Omega)} \leq |v|_{H^1(\Omega)} \leq \|v\|_{H^1(\Omega)} \qquad \forall v \in H_0^1(\Omega).$$