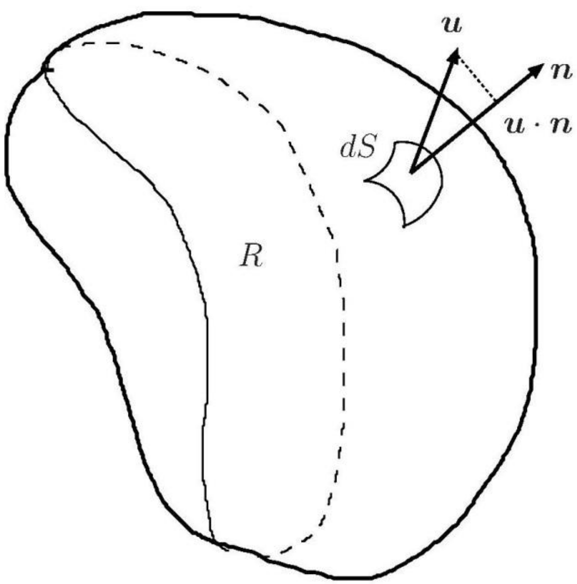
Nonlinear PDEs

2^{nd} lecture

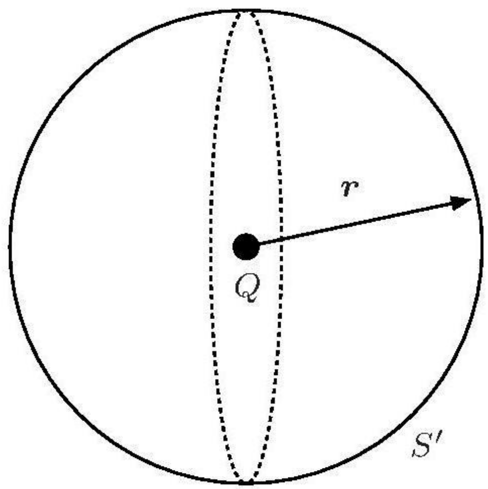
Divergence theorem

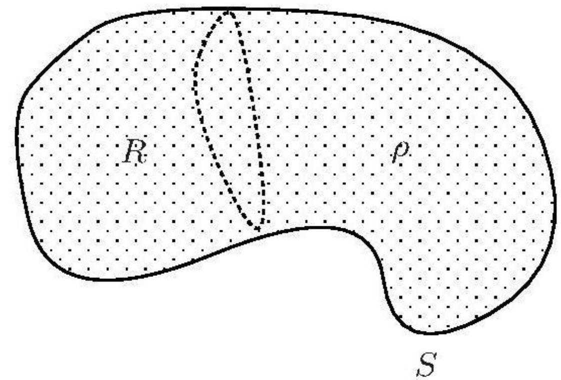


$$\iiint_R \operatorname{div} \boldsymbol{u}\, dV = \iint_S \boldsymbol{u}\cdot\boldsymbol{n}\, dS$$

Physical meaning: the total divergence of a vector field inside a closed domain R is equal to the total of the flux through the boundary of the domain.

Gauss's law





Electric flux density $\boldsymbol{D}$ at any point on a spherical surface S' of radius r centered at an isolated point charge Q is:

$$\boldsymbol{D} = \frac{Q}{4\pi r^2}\boldsymbol{r} \quad [\mathrm{C/m^2}]$$

Integrate over the sphere:

$$\int_{S'} \boldsymbol{D}\cdot d\boldsymbol{S} = \frac{Q}{4\pi r^2}4\pi r^2 = Q$$

Similarly for any domain R with surface S and charge density ρ:

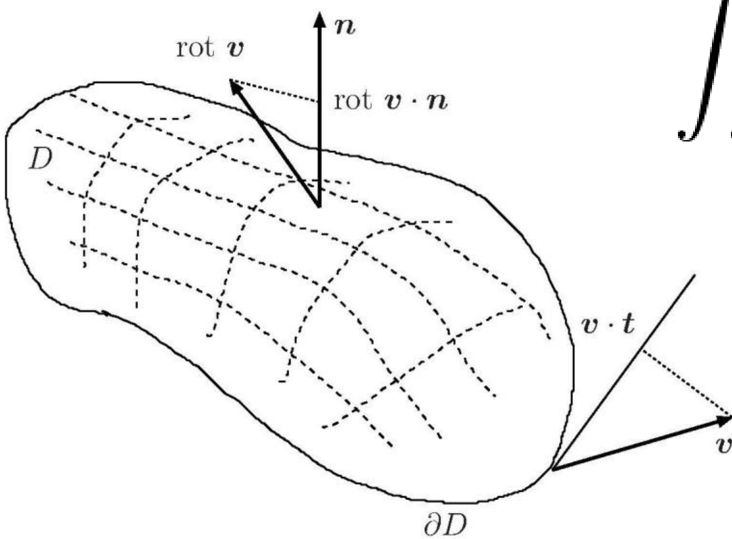
$$\iint_S \boldsymbol{D}\cdot d\boldsymbol{S} = \iiint_R \rho\, dx$$

By **divergence theorem**:

$$\iint_S \boldsymbol{D}\cdot d\boldsymbol{S} = \iiint_R \mathrm{div}\boldsymbol{D}\, dx$$

Gauss's law: $\boxed{\mathrm{div}\,\boldsymbol{D} = \rho}$

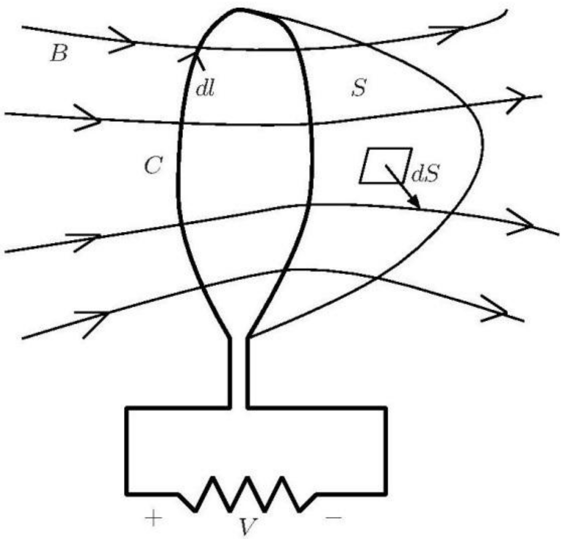
Stokes' theorem



$$\iint_D (\operatorname{rot} \boldsymbol{v}) \cdot \boldsymbol{n} \, dS = \int_{\partial D} \boldsymbol{v} \cdot d\boldsymbol{s}$$

Physical meaning: The total circulation inside a closed domain D is equal to the line integral along the boundary ∂D of the domain.

Faraday's law



Faraday's law: The induced electromotive force in any closed circuit is equal to the time rate of change of the magnetic flux through the circuit.

$$V = -\frac{\partial}{\partial t}\psi$$

$$V = \int_C E \cdot dl$$

$$\psi = \int_S B \cdot dS$$

Therefore, $$\int_C E \cdot dl = -\frac{\partial}{\partial t}\int_S B \cdot dS$$

By **Stokes' theorem**

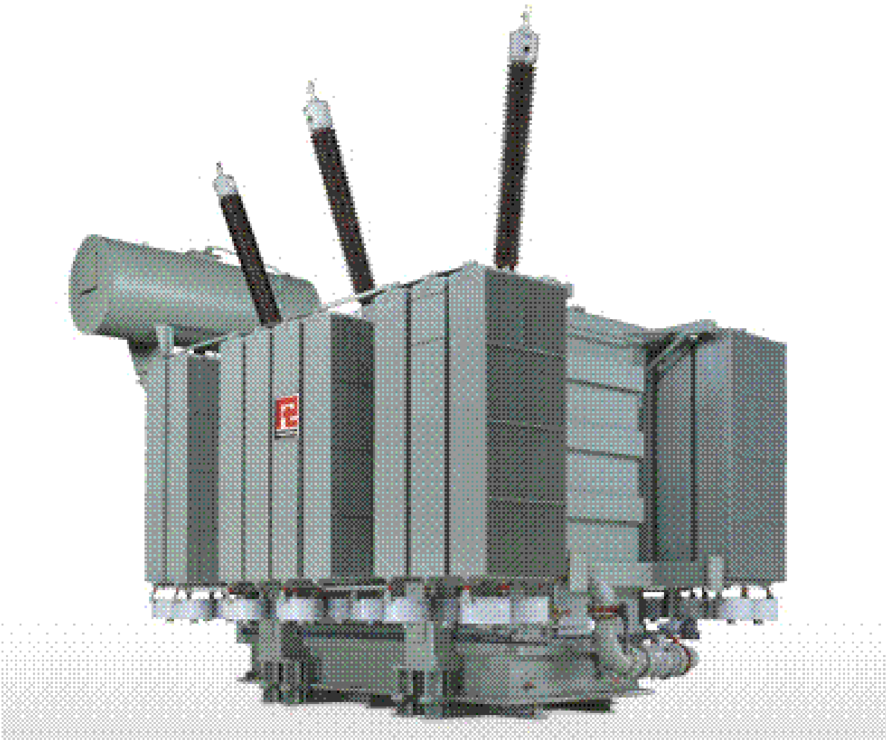
$$\int_C E \cdot dl = \int_S \mathrm{rot}\, E \cdot dS$$

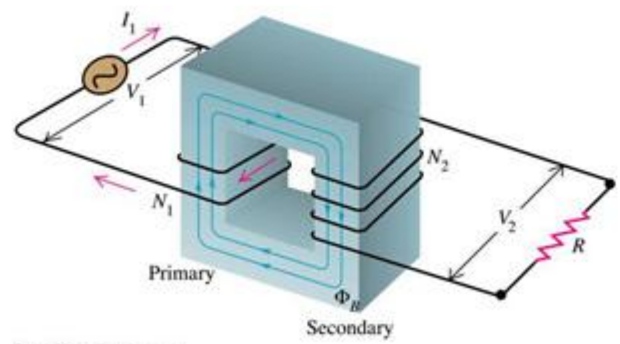
Faraday's law: $$\boxed{\mathrm{rot}\, E = -\frac{\partial B}{\partial t}}$$

Examples of electromagnetic devices

Electromagnets

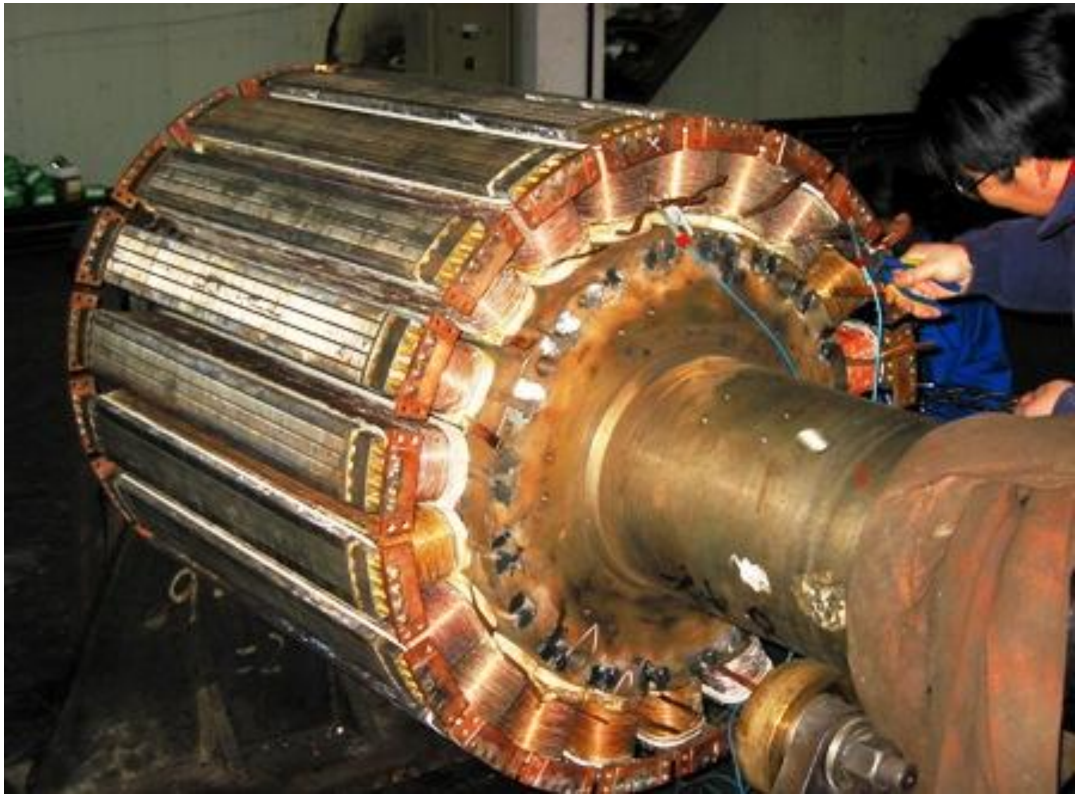
Examples of electromagnetic devices





Transformers

Examples of electromagnetic devices



Synchronous rotors

Examples of electromagnetic devices

Asynchronous motors

Examples of electromagnetic devices

Induction coils

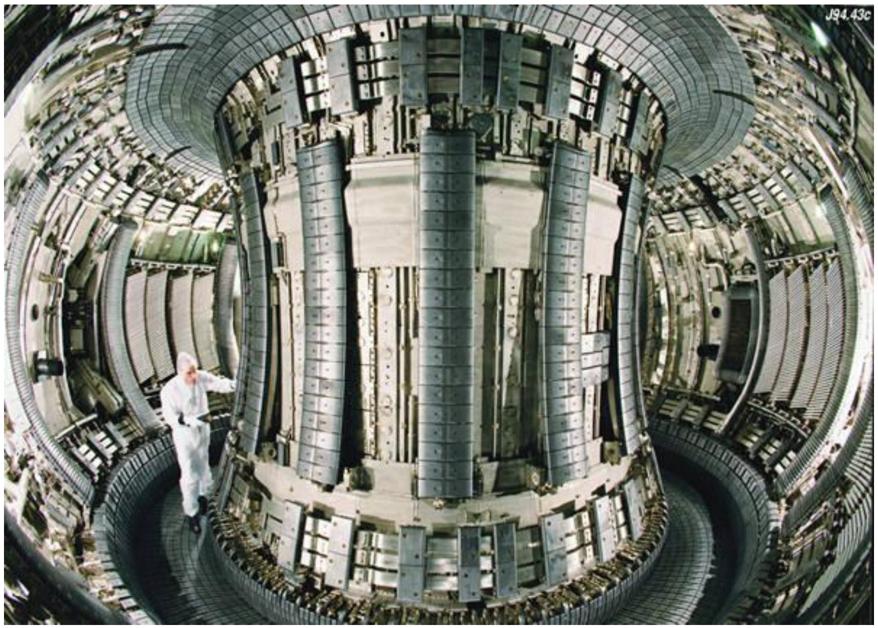
Examples of electromagnetic devices

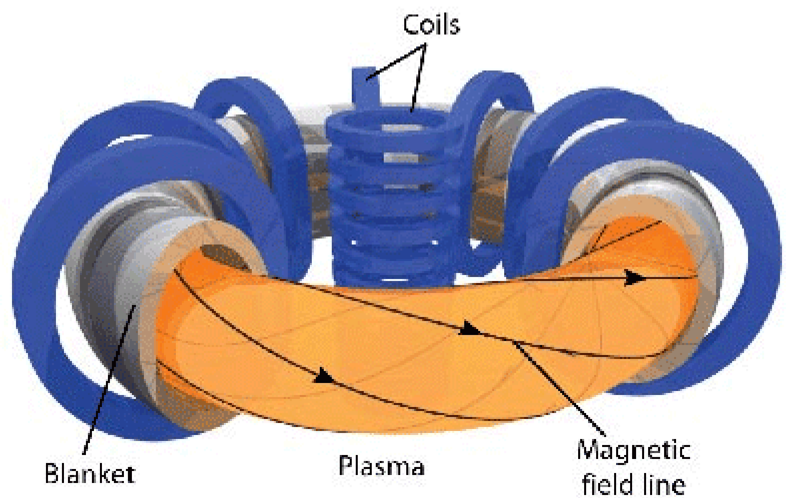
Magnetic heads

Examples of electromagnetic devices

Electron microscopes

Examples of electromagnetic devices





Tokamaks

Examples of electromagnetic devices

Linear accelerators

Maxwell's equations

$$
\begin{aligned}
\operatorname{rot} H &= J + \frac{\partial D}{\partial t}, && \text{(Ampère's law)}\\
\operatorname{rot} E &= -\frac{\partial B}{\partial t}, && \text{(Faraday's law)}\\
\operatorname{div} D &= \rho, && \text{(Gauss's law)}\\
\operatorname{div} B &= 0, && \text{(Gauss's law for electromagnetism)}\\
D &= \varepsilon E, && \text{(constitutive relation)}\\
B &= \mu H. && \text{(constitutive relation)}
\end{aligned}
$$

H	...	magnetic field intensity [A/m]
E	...	electric field intensity [V/m]
D	...	electric induction (or flux density) [C/m^2]
B	...	magnetic induction (or flux density) [Wb/m^2]
J	...	current density [A/m^2]
ρ	...	charge density [C/m^3]
μ	...	permeability
ε	...	permittivity .

Constitutive equation

$$H = \frac{1}{\mu} B \quad \Rightarrow \quad H(x) = \nu(x, \|B(x)\|^2) B(x)$$

$$\nu(x, \eta) = \begin{cases} \nu_1(\eta) & \text{for } x \in \Omega_1 = \text{ ferromagnetic materials} \\ \nu_0 & \text{for } x \in \Omega_0 = \text{ other materials (insulators, air, etc.)} \end{cases}$$