1 18-04-18

This research is collaboration with Prof. Yoneda from University of Tokyo.
Main equation discussed is **Navier-Stokes equation** that usually discusses in fluid, for example air.

1.1 Navier-Stokes Equation

1.1.1 General Problem

For dimension $d = 2, 3, ...$ (usually 2 or 3) and $T > 0$, we want to find

$$(u, p) : \Omega \times (0, T) \to \mathbb{R}^d \times \mathbb{R}$$

where u is unknown velocity and p is unknown pressure such that

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \triangle u + \nabla p = f & \text{in } \Omega \times (0, T) \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u = u^0 & \text{in } \Omega, \text{ at } t = 0 \end{cases} \tag{1}$$

where $f : \Omega \times (0, T) \to \mathbb{R}^d$ and $u^0 : \Omega \to \mathbb{R}^d$ are given functions, $\nu > 0$ is a viscosity.

From equation (1) we can see that $\dfrac{\partial u}{\partial t} + (u \cdot \nabla)u$ is the **convection part** that explain the movement of fluid. This part, contain **nonlinear term** $(u \cdot \nabla)u$. We can also see, that $\nu \triangle u$ (similar to Heat equation) is the **diffusion part**. In the second equations, $\nabla \cdot u = 0$ explained the **incompressible condition** of fluid.

Incompressible condition :

$$\nabla \cdot u = div\ u = 0 \Leftrightarrow \text{ fluid is incompressible}$$

means that the total amount of body does not change. By

$$0 = \int_V \nabla \cdot u\ dx = \int_{\partial V} u \cdot n\ ds$$

means that the energy that comes in and comes out is same and the normal component of velocity is $0 = \int_{\partial V} u \cdot n\ ds$ where n is the normal vector works on boundary.

Convection effect :
[**simple explanation**] Let $\phi^0(x), c > 0$ is given. Consider $\phi(x,t) = \phi^0(x - ct)$, that represent the movement of function without changing the shape.

at $t = 0$ we have $\phi(x, 0) = \phi^0(x)$; at at $t = 1$ we have $\phi(x, 1) = \phi^0(x - c)$; at $t = 2$ we have $\phi(x, 2) = \phi^0(x - 2c)$ as shown above.

If we differentiate ϕ over x and t, then we obtain

$$\begin{cases} \dfrac{\partial \phi}{\partial t}(x,t) = \phi^0\prime(x - ct)\ (-c) = -c\ \phi^0\prime(x - ct); \text{ the initial function} \\ \dfrac{\partial \phi}{\partial x}(x,t) = \phi^0\prime(x - ct); \text{ moves to right with velocity c} \end{cases} .$$

From the above relation, we get

$$\frac{\partial \phi}{\partial t} + c\,\frac{\partial \phi}{\partial x} = 0.$$

If we consider velocity, $c \leftarrow u$ then $\dfrac{\partial \phi}{\partial t} + u\,\dfrac{\partial \phi}{\partial x} = 0$. Next, for **multidimensional convection equation**, $\dfrac{\partial}{\partial x} \leftarrow \nabla$, $\dfrac{\partial \phi}{\partial t} + (u \cdot \nabla)\phi = 0$. In Navier-Stokes equations, if we $\phi \leftarrow u_i$, then the first two term in first equation of equation (1).

[explanation] Let $u : \Omega \times (0, T) \to \mathbb{R}^d$ is given.

$$\frac{\partial \phi}{\partial t}(x,t) + [(u \cdot \nabla)\phi](x,t) = 0,\ \ (x,t) \in \Omega \times (0,T).$$

Let us consider the position of a fluid particle that satisfy

$$\begin{cases} X\prime(t) & = u(X(t),t),\ \ \forall t \\ X(t\star) & = x \end{cases}.$$

Calculate

$$\begin{aligned} \frac{d}{dt}[\phi(X(t),t)] &= (\nabla\phi)(X(t),t) \cdot X\prime(t) + \frac{\partial \phi}{\partial t}(X(t),t) \\ &= [(u \cdot \nabla)\phi](X(t),t) + \frac{\partial \phi}{\partial t}(X(t),t) \\ &= \left[\frac{\partial \phi}{\partial t} + (u \cdot \nabla)\phi\right](X(t),t). \end{aligned}$$

If we set $t = t\star$, then

$$\frac{d}{dt}[\phi(X(t),t)]_{|t=t\star} = \left[\frac{\partial \phi}{\partial t} + (u \cdot \nabla)\phi\right](x,t\star) = 0$$

or means that the function value does not change if it is changes by the velocity u, or called **characterictic line trajectory of particle**.

1.1.2 3D Problem

For $d = 3$, then $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ such that for $(i = 1, 2, 3)$ we have

$$\frac{\partial u_i}{\partial t} + (u \cdot \nabla)u_i - \nu \triangle u_i + [\nabla p]_i = f_i$$

where

$$\begin{aligned} (u \cdot \nabla)u_i &= \left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{bmatrix}\right)u_i \\ &= (u_1\partial_1 + u_2\partial_2 + u_3\partial_3)u_i \\ &= u_1\frac{\partial u_i}{x_1} + u_2\frac{\partial u_i}{x_2} + u_3\frac{\partial u_i}{x_3} \end{aligned}$$

and

$$\triangle u_i = \frac{\partial^2 u_i}{\partial x_1^2} + \frac{\partial^2 u_i}{\partial x_2^2} + \frac{\partial^2 u_i}{\partial x_3^2}.$$

Note :we have u_1, u_2, u_3, p as four unknown functions and four equations (as first equation defined for three u and second equation), then we could find the solution.

1.2 Research Topic

We will study about axisymmetric flow (example : air). Consider sylindrical domain for first. We do two simulation, first : with the initial velocity with velocity concentration is in the center of axis, second : we include swirl, like tornado type velocity.

In this research it is proved that *if there is blow up, then there is swirl.* But has not proved that there is some blow-up phenomena ($\exists(x\star, t\star), t\star < \infty$ such that $\lim\limits_{(x,t)\to(x\star,t\star)} |u(x,t)| = \infty$) by Navier-Stokes.