Weierstrass Institute for
Applied Analysis and Stochastics

Finite Element Methods for the Simulation of Incompressible Flows

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Outline of the Lectures

1. The Navier–Stokes Equations as Model for Incompressible Flows
2. Function Spaces For Linear Saddle Point Problems
3. The Stokes Equations
4. The Oseen Equations
5. The Stationary Navier–Stokes Equations
6. The Time-Dependent Navier–Stokes Equations – Laminar Flows

1 A Model for Incompressible Flows

- conservation laws
  - conservation of linear momentum
  - conservation of mass
- flow variables
  - $\rho(t, \mathbf{x})$: density $[kg/m^3]$
  - $\mathbf{v}(t, \mathbf{x})$: velocity $[m/s]$
  - $P(t, \mathbf{x})$: pressure $[N/m^2]$

  assumed to be sufficiently smooth in
- $\Omega \subset \mathbb{R}^3$
- $[0, T]$

1 Conservation of Mass

- change of fluid in arbitrary volume V

$$-\frac{\partial}{\partial t}\underbrace{\int_V \rho \, d\mathbf{x}}_{\text{mass}} = \underbrace{\int_{\partial V} \rho \mathbf{v}\cdot\mathbf{n} \, ds}_{\text{transport through bdry}} = \int_V \nabla\cdot(\rho\mathbf{v}) \, d\mathbf{x}$$

- V arbitrary $\Longrightarrow$ continuity equation

$$\rho_t + \nabla\cdot(\rho\mathbf{v}) = 0$$

- incompressibility ($\rho = \text{const}$)

$$\nabla\cdot\mathbf{v} = 0$$

1 Newton's Second Law of Motion

- Newton's second law of motion

$$\text{net force} = \text{mass} \times \text{acceleration}$$

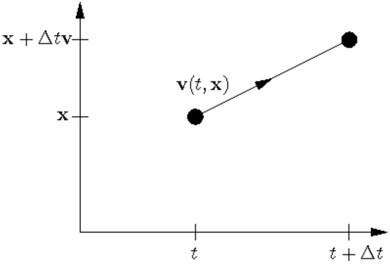
1 Newton's Second Law of Motion

- Newton's second law of motion

net force = mass $\times$ acceleration

- acceleration: using first order Taylor series expansion in time (board)

$$\frac{d\mathbf{v}}{dt}(t,\mathbf{x}) = \partial_t \mathbf{v}(t,\mathbf{x}) + \left(\mathbf{v}(t,\mathbf{x}) \cdot \nabla\right) \mathbf{v}(t,\mathbf{x})$$



movement of a particle

1 Newton's Second Law of Motion

- acting forces on an arbitrary volume V:

  sum of external (body) forces

  - gravity

  and internal (molecular) forces

  - pressure
  - viscous drag that a 'fluid element' exerts on the 'adjacent element'
  - contact forces: act only on surface of 'fluid element'

$$\int_V \mathbf{F}(t,\mathbf{x})\, d\mathbf{x} + \int_{\partial V} \mathbf{t}(t,\mathbf{s})\, d\mathbf{s}$$

  $\mathbf{t}$ $[N/m^2]$ – Cauchy stress vector

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$$\int_V \mathbf{F}(t,\mathbf{x})\,d\mathbf{x} + \int_{\partial V} \mathbf{t}(t,\mathbf{s})\,d\mathbf{s}$$

  $\mathbf{t}\ [N/m^2]$ – Cauchy stress vector

- principle of Cauchy: internal contact forces depend (geometrically) only on the orientation of the surface

$$\mathbf{t} = \mathbf{t}(\mathbf{n})$$

  $\mathbf{n}$ – unit normal vector of the surface pointing outwards of V

1 Newton's Second Law of Motion

- it can be shown: conservation of linear momentum results in linear dependency on $\mathbf{n}$

$$\mathbf{t} = \mathbb{S}\mathbf{n}$$

$\mathbb{S}(t,\mathbf{x})$ $[N/m^2]$ – stress tensor, dimension 3×3

- divergence theorem

$$\int_{\partial V} \mathbf{t}(t,\mathbf{s})\, d\mathbf{s} = \int_V \nabla \cdot \mathbb{S}(t,\mathbf{x})\, d\mathbf{x}$$

- momentum equation

$$\rho\left(\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = \nabla\cdot\mathbb{S} + \mathbf{F} \quad \forall\, t \in (0,T],\ \mathbf{x}\in\Omega$$

- model for the stress tensor
  - torque

$$\mathbf{M_0} = \int_V \mathbf{r} \times \mathbf{F}\, d\mathbf{x} + \int_{\partial V} \mathbf{r} \times (\mathbb{S}\mathbf{n})\ d\mathbf{s} \quad [N\, m]$$

    at equilibrium is zero $\Longrightarrow$ symmetry $\mathbb{S} = \mathbb{S}^T$
  - decomposition

$$\mathbb{S} = \mathbb{V} + P\mathbb{I}$$

    $\mathbb{V}$ $[N/m^2]$ – viscous stress tensor
  - pressure P acts only normal to the surface, directed into V

$$-\int_{\partial V} P\mathbf{n}\, d\mathbf{s} = -\int_V \nabla P\, d\mathbf{x} = -\int_V \nabla \cdot (P\mathbb{I})\, d\mathbf{x}$$

1 Newton's Second Law of Motion

- model for the stress tensor (cont.)
  - viscous stress tensor
    - friction between fluid particles can only occur if the particles move with different velocities
    - viscous stress tensor depends on gradient of velocity
    - because of symmetry: on symmetric part of the gradient: velocity deformation tensor

$$\mathbb{D}(\mathbf{v}) = \frac{\nabla \mathbf{v} + (\nabla \mathbf{v})^T}{2}$$

    - velocity not too large: dependency is linear (Newtonian fluids)

$$\mathbb{V} = 2\mu\mathbb{D}(\mathbf{v}) + \left(\zeta - \frac{2\mu}{3}\right)(\nabla \cdot \mathbf{v})\,\mathbb{I}$$

$\mu\ [kg/(m\ s)]$ – dynamic or shear viscosity
$\zeta\ [kg/(m\ s)]$ – second order viscosity

1 Navier–Stokes Equations

- general Navier–Stokes equations

$$
\begin{aligned}
\rho\left(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) & & & \\
-2\nabla\cdot(\mu\mathbb{D}(\mathbf{v})) - \nabla\cdot\left(\left(\zeta - \frac{2\mu}{3}\right)\nabla\cdot\mathbf{v}\mathbb{I}\right) + \nabla P &= \mathbf{F} & & \text{in } (0,T]\times\Omega, \\
\rho_t + \nabla\cdot(\rho\mathbf{v}) &= 0 & & \text{in } (0,T]\times\Omega
\end{aligned}
$$

1 Navier–Stokes Equations

- general Navier–Stokes equations

$$\begin{aligned} \rho\left(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) & & & \\ -2\nabla\cdot\left(\mu\mathbb{D}(\mathbf{v})\right) - \nabla\cdot\left(\left(\zeta - \tfrac{2\mu}{3}\right)\nabla\cdot\mathbf{v}\mathbb{I}\right) + \nabla P &= \mathbf{F} & & \text{in } (0,T]\times\Omega, \\ \rho_t + \nabla\cdot(\rho\mathbf{v}) &= 0 & & \text{in } (0,T]\times\Omega \end{aligned}$$

- incompressible flows: incompressible Navier–Stokes equations

$$\begin{aligned} \partial_t \mathbf{v} - 2\nu\nabla\cdot\mathbb{D}(\mathbf{v}) + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla\frac{P}{\rho_0} &= \frac{\mathbf{F}}{\rho_0} & & \text{in } (0,T]\times\Omega, \\ \nabla\cdot\mathbf{v} &= 0 & & \text{in } (0,T]\times\Omega \end{aligned}$$

1 Navier–Stokes Equations

- Claude Louis Marie Henri Navier (1785 – 1836)
  George Gabriel Stokes (1819 – 1903)

1 Dimensionless Incompressible Navier–Stokes Equations

- dimensionless equations needed for (numerical) analysis and numerical simulations
- reference quantities of flow problem
  - $L\ [m]$ – a characteristic length scale
  - $U\ [m/s]$ – a characteristic velocity scale
  - $T^*\ [s]$ – a characteristic time scale
- transform of variables

$$\mathbf{x} = \frac{\mathbf{x}'}{L}, \quad \mathbf{u} = \frac{\mathbf{v}}{U}, \quad t = \frac{t'}{T^*}$$

- rescaling

$$\begin{array}{rcll} \dfrac{L}{UT^*}\partial_t \mathbf{u} - \dfrac{2\nu}{UL}\nabla\cdot\mathbb{D}(\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\dfrac{P}{\rho_0 U^2} & = & \dfrac{L}{\rho_0 U^2}\mathbf{F} & \text{in } (0,T]\times\Omega, \\ \nabla\cdot\mathbf{u} & = & 0 & \text{in } (0,T]\times\Omega, \end{array}$$

1 Dimensionless Incompressible Navier–Stokes Equations

- defining

$$p = \frac{P}{\rho_0 U^2}, \quad Re = \frac{UL}{\nu}, \quad St = \frac{L}{UT^*}, \quad \mathbf{f} = \frac{L}{\rho_0 U^2}\mathbf{F}$$

  p – new pressure
  Re – Reynolds number
  St – Strouhal number
  $\mathbf{f}$ – new right hand side

- result

$$\begin{aligned} St\,\partial_t \mathbf{u} - \frac{2}{Re}\nabla\cdot\mathbb{D}(\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } (0,T]\times\Omega, \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } (0,T]\times\Omega \end{aligned}$$

- generally $T^* = L/U \implies St = 1$

1 Dimensionless Incompressible Navier–Stokes Equations

- dimensionless Navier–Stokes equations
  - conservation of linear momentum
  - conservation of mass

$$
\begin{array}{rcll}
\mathbf{u}_t - 2Re^{-1}\nabla\cdot\mathbb{D}(\mathbf{u}) + \nabla\cdot(\mathbf{u}\mathbf{u}^T) + \nabla p & = & \mathbf{f} & \text{in } (0,T]\times\Omega \\
\nabla\cdot\mathbf{u} & = & 0 & \text{in } [0,T]\times\Omega \\
\mathbf{u}(0,\mathbf{x}) & = & \mathbf{u}_0 & \text{in } \Omega
\end{array}
$$

\+ boundary conditions

- given:
  - $\Omega \subset \mathbb{R}^d, d \in \{2,3\}$: domain
  - T: final time
  - $\mathbf{u}_0$: initial velocity
  - boundary conditions
- to compute:
  - velocity $\mathbf{u}$, with

$$\mathbb{D}(\mathbf{u}) = \frac{\nabla\mathbf{u} + \nabla\mathbf{u}^T}{2},$$

velocity deformation tensor

  - pressure p

- parameter: Reynolds number Re

1 The Reynolds Number

- Reynolds number

$$Re = \frac{LU}{\nu} = \frac{\text{convective forces}}{\text{viscous forces}}$$

Osborne Reynolds (1842 – 1912)

- rough classification of flows:
  - Re small: steady-state flow field (if data do not depend on time)
  - Re larger: laminar time-dependent flow field
  - Re very large: turbulent flows

1 Dimensionless Incompressible Navier–Stokes Equations

- simplified form (for mathematics)

$$\begin{aligned} \partial_t \mathbf{u} - 2\nu\nabla\cdot\mathbb{D}(\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } (0,T]\times\Omega, \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } (0,T]\times\Omega \end{aligned}$$

$\nu = Re^{-1}$ – dimensionless viscosity

1 Dimensionless Incompressible Navier–Stokes Equations

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  $\nu = Re^{-1}$ – dimensionless viscosity

- alternative expression of viscous term (due to $\nabla\cdot\mathbf{u} = 0$)

$$2\nabla\cdot\mathbb{D}(\mathbf{u}) = \Delta\mathbf{u}$$

- alternative expression of convective term (due to $\nabla\cdot\mathbf{u} = 0$)

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\cdot(\mathbf{u}\mathbf{u}^T)$$

- special cases
  - steady-state Navier–Stokes equations: stationary flow fields

$$
\begin{aligned}
-\nu\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p &= \mathbf{f} \quad \text{in } \Omega\\
\nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega
\end{aligned}
$$

- special cases
  - steady-state Navier–Stokes equations: stationary flow fields

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\nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega
\end{aligned}
$$

  - Oseen equations: convection field known (only for analysis)

$$
\begin{aligned}
-\nu\Delta\mathbf{u} + (\mathbf{u}_0\cdot\nabla)\mathbf{u} + \nabla p + c\mathbf{u} &= \mathbf{f} \quad \text{in } \Omega \\
\nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega
\end{aligned}
$$

- special cases
  - steady-state Navier–Stokes equations: stationary flow fields

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  - Oseen equations: convection field known (only for analysis)

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  - Stokes equations: no convection

$$\begin{aligned} -\Delta\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega \end{aligned}$$

- boundary conditions
  - Dirichlet boundary conditions (inflows)

$$\mathbf{u}(t,\mathbf{x}) = \mathbf{g}(t,\mathbf{x}) \text{ in } (0,T] \times \Gamma_{\text{diri}} \subset \Gamma$$

$\mathbf{g}(t,\mathbf{x}) = \mathbf{0}$ – no slip boundary condition (walls)

$$\mathbf{u}(t,\mathbf{x}) = \mathbf{0} \iff \mathbf{u}(t,\mathbf{x}) \cdot \mathbf{n} = 0,\ \mathbf{u}(t,\mathbf{x}) \cdot \mathbf{t}_1 = 0,\ \ \mathbf{u}(t,\mathbf{x}) \cdot \mathbf{t}_2 = 0$$

no penetration, no slip

- boundary conditions
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    no penetration, no slip
  - free slip boundary condition (e.g. symmetry planes)

$$\begin{aligned} \mathbf{u} \cdot \mathbf{n} &= g \quad \text{in } (0,T] \times \Gamma_{\text{slip}} \subset \Gamma, \\ \mathbf{n}^T \mathbb{S} \mathbf{t}_k &= 0 \quad \text{in } (0,T] \times \Gamma_{\text{slip}}, \quad 1 \le k \le d-1 \end{aligned}$$

1 Incompressible Navier–Stokes Equations

- boundary conditions (cont.)
  - do-nothing boundary conditions (outflow)

$$\mathbb{S}\mathbf{n} = \mathbf{0} \ \text{ in } \ (0,T] \times \Gamma_{\text{outf}} \subset \Gamma$$

- boundary conditions (cont.)
  - do-nothing boundary conditions (outflow)

$$\mathbb{S}\mathbf{n} = \mathbf{0} \;\text{ in }\; (0,T] \times \Gamma_{\text{outf}} \subset \Gamma$$

  - periodic boundary conditions (only for analysis, $\Omega = (0,l)^d$)

$$\mathbf{u}(t, \mathbf{x} + l\mathbf{e}_i) = \mathbf{u}(t, \mathbf{x}) \quad \forall\, (t, \mathbf{x}) \in (0,T] \times \Gamma$$

- difficulties for mathematical analysis and numerical simulations
  - coupling of velocity and pressure
  - nonlinearity of the convective term
  - the convective term dominates the viscous term, i.e. ν is small

2 Linear Saddle Point Problems

- motivation
  - iterative solution of Navier–Stokes equations leads to linear system of equations
  - linear system have special form: saddle point problem
  - sufficient and necessary condition on unique solvability needed
  - can be derived in abstract form, see [1]

[1] Girault, Raviart: *Finite Element Methods for Navier-Stokes Equations* 1986

2 Linear Saddle Point Problems

- spaces: V, Q – real Hilbert spaces
- bilinear forms:

$$a(\cdot,\cdot)\ :\ V \times V \to \mathbb{R}, \quad b(\cdot,\cdot)\ :\ V \times Q \to \mathbb{R}$$

- linear problem: Find $(u,p) \in V \times Q$ such that for given $(f,r) \in V' \times Q'$

$$\begin{aligned} a(u,v) + b(v,p) &= \langle f, v\rangle_{V',V} \quad \forall\, v \in V, \\ b(u,q) &= \langle r, q\rangle_{Q',Q} \quad \forall\, q \in Q \end{aligned}$$

- conditions on the spaces and bilinear forms necessary

2 Linear Saddle Point Problems

- associated linear operators

$$A \in \mathscr{L}\left(V, V'\right) \quad \text{defined by} \quad \langle Au, v\rangle_{V',V} = a(u,v) \quad \forall\, u, v \in V$$
$$B \in \mathscr{L}\left(V, Q'\right) \quad \text{defined by} \quad \langle Bu, q\rangle_{Q',Q} = b(u,q) \quad \forall\, u \in V,\ \forall\, q \in Q$$

- dual operator: $B' \in \mathscr{L}(Q, V')$ defined by

$$\left\langle B'q, v\right\rangle_{V',V} = \langle Bv, q\rangle_{Q',Q} = b(v,q) \quad \forall\, v \in V,\ \forall\, q \in Q$$

- linear problem in operator form: Find $(u,p) \in V \times Q$ such that

$$\begin{array}{cccccl} Au & +B'p & = & f & \text{in } V' \\ Bu & & = & r & \text{in } Q' \end{array}$$

- spaces
  - $V_0 := V(0) = \ker(B), \quad V = V_0^\perp \oplus V_0$
  - $\tilde{V}' = \{\phi \in V' \; : \; \langle \phi, v \rangle_{V',V} = 0 \;\; \forall\, v \in V_0\} \subset V'$
- inf-sup condition: The three following properties are equivalent:

  i) There exists a constant $\beta_{\text{is}} > 0$ such that

$$\inf_{q\in Q} \sup_{v\in V} \frac{b(v,q)}{\|v\|_V \, \|q\|_Q} \geq \beta_{\text{is}}.$$

  ii) The operator B' is an isomorphism from Q onto $\tilde{V}'$ and

$$\left\|B'q\right\|_{V'} \geq \beta_{\text{is}} \, \|q\|_Q \quad \forall\, q \in Q.$$

  iii) The operator B is an isomorphism from $V_0^\perp$ onto Q' and

$$\|Bv\|_{Q'} \geq \beta_{\text{is}} \, \|v\|_V \quad \forall\, v \in V_0^\perp.$$

2 The Inf-Sup Condition – Bilinear Form $b(\cdot,\cdot)$

- independently derived in [1,2]: Babuška–Brezzi condition
- sometimes: Ladyzhenskaya–Babuška–Brezzi condition, LBB condition
- it follows:

$$V(r) = \{v \in V \ : \ Bv = r\}$$

is not empty for all $r \in Q'$

[1] Babuška: Numer. Math. 20, 179–192, 1973

[2] Brezzi: Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge 8, 129–151, 1974

2 Unique Solution of Linear Saddle Point Problem

- sufficient and necessary conditions for unique solution of saddle point problem can be formulated with projection operator, see literature
- sufficient conditions
  - $a(\cdot,\cdot)$ is V_0-elliptic, i.e., there is a constant $\alpha > 0$ such that

$$a(v,v) \geq \alpha \|v\|_V^2 \quad \forall\, v \in V_0$$

  - $b(\cdot,\cdot)$ satisfies inf-sup condition

- for simplicity: Dirichlet boundary conditions on whole boundary
- velocity space

$$V = H_0^1(\Omega) = \left\{ \mathbf{v} \ : \ \mathbf{v} \in H^1(\Omega) \text{ with } \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega \right\}$$

with

$$(\mathbf{v}, \mathbf{w}) = \int_\Omega (\nabla \mathbf{v} \cdot \nabla \mathbf{w})(\mathbf{x})\, d\mathbf{x}, \quad \|\mathbf{v}\|_V := \|\nabla \mathbf{v}\|_{L^2(\Omega)}$$

dual space: $V' = H^{-1}(\Omega)$

- for simplicity: Dirichlet boundary conditions on whole boundary
- velocity space

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dual space: $V' = H^{-1}(\Omega)$

- pressure space

$$Q = L_0^2(\Omega) = \left\{ q \,:\, q \in L^2(\Omega) \text{ with } \int_\Omega q(\mathbf{x})\, d\mathbf{x} = 0 \right\}$$

with

$$(q, r) = \int_\Omega (qr)(\mathbf{x})\, d\mathbf{x}, \quad \|q\|_Q = \|q\|_{L^2(\Omega)}$$

- dual space: $Q' = Q$

- bilinear form for coupling velocity and pressure

$$b(\mathbf{v},q) = -\int_\Omega q\nabla\cdot\mathbf{v}\,d\mathbf{x} = -(\nabla\cdot\mathbf{v},q) \quad \mathbf{v}\in V,\ q\in Q$$

- bilinear form for coupling velocity and pressure

$$b(\mathbf{v},q) = -\int_\Omega q\nabla\cdot\mathbf{v}\,d\mathbf{x} = -(\nabla\cdot\mathbf{v},q) \quad \mathbf{v}\in V,\ q\in Q$$

- divergence operator

$$\text{div} \,:\, V \to \mathrm{range}(\text{div}), \quad \mathbf{v}\mapsto\nabla\cdot\mathbf{v}$$

- it can be shown: $\mathrm{range}(\text{div}) = Q'$
- associated linear operator: negative divergence operator

$$B\in\mathscr{L}(V,Q'), \quad B = -\text{div}$$

- dual operator: gradient operator

$$\text{grad} \ : \ Q \to \mathrm{range}(\text{grad}), \quad q \mapsto \nabla q$$

with

$$B' \in \mathscr{L}(Q, V'), \quad B' = \text{grad}$$

2 Continuous Incompressible Flow Problems

- dual operator: gradient operator

$$\text{grad} \;:\; Q \to \mathrm{range}(\text{grad}), \quad q \mapsto \nabla q$$

with

$$B' \in \mathscr{L}(Q, V'), \quad B' = \text{grad}$$

- kernel of B: space of weakly divergence-free functions

$$V_0 = V_{\mathrm{div}} = \{\mathbf{v} \in V \;:\; (\nabla \cdot \mathbf{v}, q) = 0 \; \forall \, q \in Q\}$$

- estimating divergence by gradient

$$\|\nabla \cdot \mathbf{v}\|_{L^2(\Omega)} \le \sqrt{d}\, \|\nabla \mathbf{v}\|_{L^2(\Omega)} \quad \forall\, \mathbf{v} \in H^1(\Omega)$$

  - proof: board
  - estimate is sharp

- estimating divergence by gradient

$$\|\nabla \cdot \mathbf{v}\|_{L^2(\Omega)} \le \sqrt{d}\,\|\nabla \mathbf{v}\|_{L^2(\Omega)} \quad \forall\, \mathbf{v} \in H^1(\Omega)$$

  - proof: board
  - estimate is sharp
- boundedness and continuity of $b(\cdot,\cdot)$

$$|b(\mathbf{v},q)| \le \sqrt{d}\,\|\mathbf{v}\|_V\,\|q\|_Q$$

  - proof: board

- one can show: div is an isomorphism from $V_{\text{div}}^{\perp}$ onto Q
- corollary: each pressure is the divergence of a velocity field:
  for each $q \in Q$ there is a unique $\mathbf{v} \in V_{\text{div}}^{\perp} \subset V$ such that

$$\nabla \cdot \mathbf{v} = q \quad \text{and} \quad \|q\|_Q \le \sqrt{d}\,\|\mathbf{v}\|_V\,, \quad \|\mathbf{v}\|_V \le C\,\|q\|_Q$$

  with C independent of $\mathbf{v}$ and q
  - proof: board

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  with C independent of $\mathbf{v}$ and q
  - proof: board
- V and Q fulfill the inf-sup condition, i.e. there is a $\beta_{\mathrm{is}} > 0$ such that

$$\inf_{q\in Q}\sup_{\mathbf{v}\in V}\frac{(\nabla\cdot\mathbf{v},q)}{\|\mathbf{v}\|_V} \ge \beta_{\mathrm{is}}$$

  - proof: board

2 Finite Element Spaces

- finite element spaces
  - V^h – finite element velocity space
  - Q^h – finite element pressure space
  - V^h/Q^h – pair
- conforming finite element spaces: $V^h \subset V$ and $Q^h \subset Q$

2 Finite Element Spaces

- finite element spaces
  - V^h – finite element velocity space
  - Q^h – finite element pressure space
  - V^h/Q^h – pair
- conforming finite element spaces: $V^h \subset V$ and $Q^h \subset Q$
- bilinear form $b^h \ : \ V^h \times Q^h \to \mathbb{R}$

$$b^h\left(\mathbf{v}^h, q^h\right) := -\sum_{K\in\mathscr{T}^h}\left(\nabla\cdot\mathbf{v}^h, q^h\right)_K$$

  - $\mathscr{T}^h$ – triangulation of Ω
  - $K \in \mathscr{T}^h$ – mesh cells
  - norm in V^h

$$\left\|\mathbf{v}^h\right\|_{V^h} = \sum_{K\in\mathscr{T}^h}\left(\nabla\mathbf{v}^h, \nabla\mathbf{v}^h\right)_K$$

2 Finite Element Spaces

- space of discretely divergence-free functions

$$V_{\text{div}}^h = \left\{ \mathbf{v}^h \in V^h \;:\; b^h\left(\mathbf{v}^h, q^h\right) = 0 \;\forall\, q^h \in Q^h \right\}$$

- generally $V_{\text{div}}^h \not\subset V_{\text{div}}$
  - finite element velocities not weakly or pointwise divergence-free
  - conservation of mass violated

2 Finite Element Spaces

- space of discretely divergence-free functions

$$V_{\mathrm{div}}^h = \left\{ \mathbf{v}^h \in V^h \; : \; b^h\left(\mathbf{v}^h, q^h\right) = 0 \; \forall \, q^h \in Q^h \right\}$$

- generally $V_{\mathrm{div}}^h \not\subset V_{\mathrm{div}}$
  - finite element velocities not weakly or pointwise divergence-free
  - conservation of mass violated
- discrete inf-sup condition

$$\inf_{q^h \in Q^h} \sup_{\mathbf{v}^h \in V^h} \frac{b^h\left(\mathbf{v}^h, q^h\right)}{\left\|\mathbf{v}^h\right\|_{V^h} \left\|q^h\right\|_{L^2(\Omega)}} \geq \beta_{\mathrm{is}}^h > 0$$

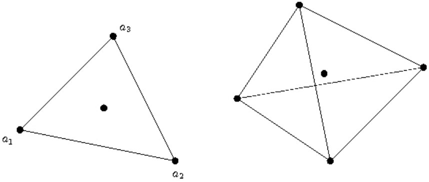
  - not inherited from inf-sup condition fulfilled by V and Q
  - discussion: board

- Interpolation estimate for V_{div}^h. Let $\mathbf{v} \in V_{\mathrm{div}}$ and let the discrete inf-sup condition hold. Then

$$\inf_{\mathbf{v}^h \in V_{\mathrm{div}}^h} \left\| \nabla \left(\mathbf{v} - \mathbf{v}^h \right) \right\|_{L^2(\Omega)} \leq \left(1 + \frac{\sqrt{d}}{\beta_{\mathrm{is}}^h} \right) \inf_{\mathbf{w}^h \in V^h} \left\| \nabla \left(\mathbf{v} - \mathbf{w}^h \right) \right\|_{L^2(\Omega)}$$

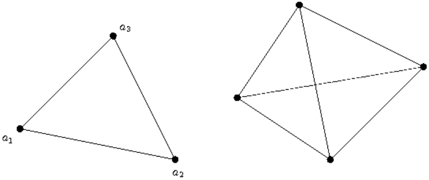
  - proof: board

- piecewise constant finite elements P_0, (Q_0)



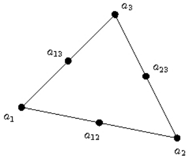
one degree of freedom (d.o.f.) per mesh cell

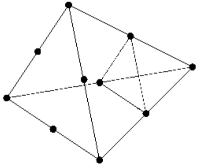
- continuous piecewise linear finite elements P_1



d d.o.f. per mesh cell

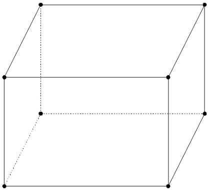
- continuous piecewise quadratic finite elements P_2





  $(d+1)(d+2)/2$ d.o.f. per mesh cell

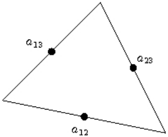
- continuous piecewise bilinear finite elements Q_1

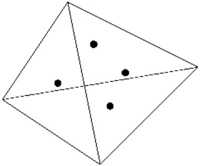


  2^d d.o.f. per mesh cell

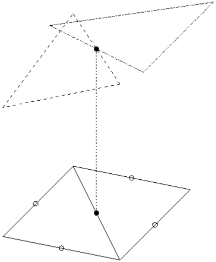
- and so on for continuous finite elements of higher order

- nonconforming linear finite elements P_1^{nc}, Crouzeix, Raviart (1973)





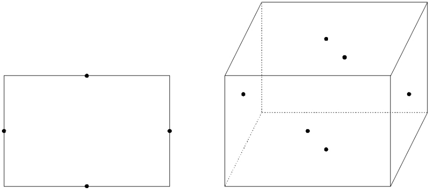
continuous only in barycenters of faces



$d+1$ d.o.f. per mesh cell

2 Finite Elements

- rotated bilinear finite element Q_1^{rot}, Rannacher, Turek (1992)



  - continuous only in barycenters of faces
  - $2d$ d.o.f. per mesh cell

- discontinuous linear finite element P_1^{disc}
  - defined by integral nodal functionals
    e.g. $\varphi^h \in P_1^{\text{disc}}$ if φ^h is linear on a mesh cell K (2d) and

$$\int_K \varphi^h(\mathbf{x})\, d\mathbf{x} = 0,\ \int_K x\varphi^h(\mathbf{x})\, d\mathbf{x} = 1,\ \int_K y\varphi^h(\mathbf{x})\, d\mathbf{x} = 0$$

  - $d+1$ d.o.f. per mesh cell

- criterion for violation of discrete inf-sup condition: there is non-trivial $q^h \in Q^h$ such that

$$b^h\left(\mathbf{v}^h, q^h\right) = 0 \quad \forall\, \mathbf{v}^h \in V^h$$

$\Longrightarrow$

$$\sup_{\mathbf{v}^h \in V^h} \frac{b^h\left(\mathbf{v}^h, q^h\right)}{\left\|\mathbf{v}^h\right\|_{V^h}} = 0$$

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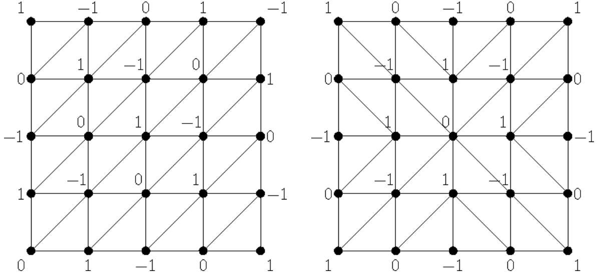
$\Longrightarrow$

$$\sup_{\mathbf{v}^h \in V^h} \frac{b^h\left(\mathbf{v}^h, q^h\right)}{\|\mathbf{v}^h\|_{V^h}} = 0$$

- P_1/P_1 pair of finite element spaces violates discrete inf-sup condition
  - counter example: checkerboard instability, board



2 Finite Element Spaces

- other pairs which violated discrete inf-sup condition
  - P_1/P_0
  - Q_1/Q_0
  - P_k/P_k, $k \geq 1$
  - Q_k/Q_k, $k \geq 1$
  - $P_k/P_{k-1}^{\mathrm{disc}}$, $k \geq 2$, on a special macro cell
- summary:
  - many easy to implement pairs violate discrete inf-sup condition
  - different finite element spaces for velocity and pressure necessary

2 Finite Element Spaces

- pairs which fulfill discrete inf-sup condition
  - P_k/P_{k-1}, Q_k/Q_{k-1}: Taylor–Hood finite elements [1]
    - proofs: 2D, $k = 2$ [2]
  - $Q_k/Q_{k-1}^{\text{disc}}$
  - $P_k/P_{k-1}^{\text{disc}}$, $k \geq d$, on very special meshes (Scott–Vogelius element)
  - P_1^{bubble}/P_1, mini element
  - $P_k^{\text{bubble}}/P_{k-1}^{\text{disc}}$ [3]
  - P_1^{nc}/P_0, Crouzeix–Raviart element [4]
  - Q_1^{rot}/Q_0, Rannacher–Turek element [5]
  - $\vdots$

[1] Taylor, Hood: Comput. Fluids 1, 73-100, 1973

[2] Verfürth: RAIRO Anal. Numér. 18, 175–182, 1984

[3] Bernardi, Raugel: Math. Comp. 44, 71–79, 1985

[4] Crouzeix, Raviart: RAIRO. Anal. Numér. 7, 33–76, 1973

[5] Rannacher, Turek: Numer. Meth. Part. Diff. Equ. 8, 97–111, 1992

- techniques for proving the discrete inf-sup condition
  - construction of Fortin operator [1]
  - using projection to piecewise constant pressure [2]
  - macroelement technique [3]
  - survey in [4]

[1] Fortin: RAIRO Anal. Numér. 11, 341–354, 1977

[2] Brezzi, Bathe: Comput. Methods Appl. Mech. Engrg. 82, 27–57, 1990

[3] Stenberg: Math. Comput. 32, 9–23, 1984

[4] Boffi, Brezzi, Fortin: Lecture Notes in Mathematics 1939, Springer, 45–100, 2008

3 The Stokes Equations

- continuous equation

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega \end{aligned} \tag{1}$$

  for simplicity: homogeneous Dirichlet boundary conditions
- difficulty: coupling of velocity and pressure
- properties
  - linear
  - form

$$\begin{aligned} -\nu \Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega \end{aligned}$$

    becomes (1) by rescaling with new pressure, right hand side

- weak form: Find $(\mathbf{u}, p) \in H_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\begin{array}{rcll} (\nabla \mathbf{u}, \nabla \mathbf{v}) - (\nabla \cdot \mathbf{v}, p) & = & \langle \mathbf{f}, \mathbf{v} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} & \forall\, \mathbf{v} \in H_0^1(\Omega), \\ -(\nabla \cdot \mathbf{u}, q) & = & 0 & \forall\, q \in L_0^2(\Omega) \end{array}$$

- casting into abstract framework
  - spaces

$$V = H_0^1(\Omega),\ \|\cdot\|_V = |\cdot|_{H^1(\Omega)}, \quad Q = L_0^2(\Omega),\ \|\cdot\|_Q = \|\cdot\|_{L^2(\Omega)}$$

  - bilinear forms

$$a(\mathbf{u}, \mathbf{v}) = (\nabla \mathbf{u}, \nabla \mathbf{v}), \quad b(\mathbf{v}, q) = -(\nabla \cdot \mathbf{v}, q)$$

3 The Stokes Equations

- weak form: Find $(\mathbf{u}, p) \in H_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\begin{array}{rcll} (\nabla \mathbf{u}, \nabla \mathbf{v}) - (\nabla \cdot \mathbf{v}, p) & = & \langle \mathbf{f}, \mathbf{v} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} & \forall\, \mathbf{v} \in H_0^1(\Omega), \\ -(\nabla \cdot \mathbf{u}, q) & = & 0 & \forall\, q \in L_0^2(\Omega) \end{array}$$

- casting into abstract framework
  - spaces

$$V = H_0^1(\Omega),\ \|\cdot\|_V = |\cdot|_{H^1(\Omega)}, \quad Q = L_0^2(\Omega),\ \|\cdot\|_Q = \|\cdot\|_{L^2(\Omega)}$$

  - bilinear forms

$$a(\mathbf{u}, \mathbf{v}) = (\nabla \mathbf{u}, \nabla \mathbf{v}), \quad b(\mathbf{v}, q) = -(\nabla \cdot \mathbf{v}, q)$$

- equivalent formulation: Find $(\mathbf{u}, p) \in V \times Q$ such that

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) - b(\mathbf{u}, q) = \langle \mathbf{f}, \mathbf{v} \rangle_{V', V} \quad \forall\, (\mathbf{v}, q) \in V \times Q$$

- V_{div} – space of weakly divergence-free functions
- associated problem: Find $\mathbf{u} \in V_{\mathrm{div}}$ such that

$$(\nabla \mathbf{u}, \nabla \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_{V',V} \quad \forall\, \mathbf{v} \in V_{\mathrm{div}}$$

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$$(\nabla \mathbf{u}, \nabla \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_{V',V} \quad \forall\, \mathbf{v} \in V_{\mathrm{div}}$$

- existence and uniqueness of solution
  - $a(\cdot,\cdot)$ is V_{div}-elliptic

$$a(\mathbf{v}, \mathbf{v}) = |\mathbf{v}|^2_{H^1(\Omega)} \quad \forall\, \mathbf{v} \in V \supset V_{\mathrm{div}}$$

  - $b(\cdot,\cdot)$ satisfies inf-sup condition

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  - $b(\cdot,\cdot)$ satisfies inf-sup condition
- stability of solution

$$\|\nabla \mathbf{u}\|_{L^2(\Omega)} \le \|\mathbf{f}\|_{H^{-1}(\Omega)}, \quad \|p\|_{L^2(\Omega)} \le \frac{2}{\beta_{\text{is}}} \|\mathbf{f}\|_{H^{-1}(\Omega)}$$

  - proof and discussion: board

- finite element problem: Find $(\mathbf{u}^h, p^h) \in V^h \times Q^h$ such that

$$\begin{array}{rcll} a^h\left(\mathbf{u}^h, \mathbf{v}^h\right) + b^h\left(\mathbf{v}^h, p^h\right) & = & \left(\mathbf{f}, \mathbf{v}^h\right) & \forall\, \mathbf{v}^h \in V^h, \\ b^h\left(\mathbf{u}^h, q^h\right) & = & 0 & \forall\, q^h \in Q^h \end{array}$$

with

$$a^h\left(\mathbf{v}^h, \mathbf{w}^h\right) = \sum_{K \in \mathscr{T}^h} \left(\nabla \mathbf{v}^h, \nabla \mathbf{w}^h\right)_K, \quad b^h\left(\mathbf{v}^h, q^h\right) = -\sum_{K \in \mathscr{T}^h} \left(\nabla \cdot \mathbf{v}^h, q^h\right)_K$$

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$$a^h\left(\mathbf{v}^h, \mathbf{w}^h\right) = \sum_{K\in\mathscr{T}^h} \left(\nabla \mathbf{v}^h, \nabla \mathbf{w}^h\right)_K, \quad b^h\left(\mathbf{v}^h, q^h\right) = -\sum_{K\in\mathscr{T}^h} \left(\nabla\cdot \mathbf{v}^h, q^h\right)_K$$

- only conforming inf-sup stable finite element spaces
  - $V^h \subset V$ and $Q^h \subset Q$
  - $$\inf_{q^h\in Q^h} \sup_{\mathbf{v}^h\in V^h} \frac{b^h\left(\mathbf{v}^h, q^h\right)}{\|\mathbf{v}^h\|_{V^h} \|q^h\|_{L^2(\Omega)}} \geq \beta_{\mathrm{is}}^h > 0$$

- existence and uniqueness of a solution
  - same proof as for continuous problem

- existence and uniqueness of a solution
  - same proof as for continuous problem
- stability

$$\left\|\nabla \mathbf{u}^h\right\|_{L^2(\Omega)} \le \|\mathbf{f}\|_{H^{-1}(\Omega)}, \quad \left\|p^h\right\|_{L^2(\Omega)} \le \frac{2}{\beta_{\mathrm{is}}^h} \|\mathbf{f}\|_{H^{-1}(\Omega)}$$

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  - same proof as for continuous problem
- goal of finite element error analysis: estimate error by interpolation errors
  - interpolation errors depend only on finite element spaces, not on problem
  - estimates for interpolation error are known

3 Finite Element Methods for the Stokes Equations

- existence and uniqueness of a solution
  - same proof as for continuous problem
- stability

$$\left\|\nabla \mathbf{u}^h\right\|_{L^2(\Omega)} \le \|\mathbf{f}\|_{H^{-1}(\Omega)}, \quad \left\|p^h\right\|_{L^2(\Omega)} \le \frac{2}{\beta_{\text{is}}^h} \|\mathbf{f}\|_{H^{-1}(\Omega)}$$

  - same proof as for continuous problem
- goal of finite element error analysis: estimate error by interpolation errors
  - interpolation errors depend only on finite element spaces, not on problem
  - estimates for interpolation error are known
- reduction to a problem on the space of discretely divergence-free functions

$$a\left(\mathbf{u}^h, \mathbf{v}^h\right) = \left(\nabla \mathbf{u}^h, \nabla \mathbf{v}^h\right) = \left(\mathbf{f}, \mathbf{v}^h\right) \ \forall\, \mathbf{v}^h \in V_{\text{div}}^h$$

- finite element error estimate for the $L^2(\Omega)$ norm of the gradient of the velocity
  - $\Omega \subset \mathbb{R}^d$, bounded, polyhedral, Lipschitz-continuous boundary

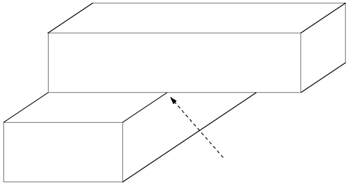
$$
\begin{aligned}
\left\|\nabla(\mathbf{u}-\mathbf{u}^h)\right\|_{L^2(\Omega)} \quad &\leq \quad 2\left(1+\frac{\sqrt{d}}{\beta_{\mathrm{is}}^h}\right)\inf_{\mathbf{v}^h\in V^h}\left\|\nabla(\mathbf{u}-\mathbf{v}^h)\right\|_{L^2(\Omega)} \\
&\quad +\sqrt{d}\inf_{q^h\in Q^h}\left\|p-q^h\right\|_{L^2(\Omega)}
\end{aligned}
$$

  - proof: board

- finite element error estimate for the $L^2(\Omega)$ norm of the gradient of the velocity
  - $\Omega \subset \mathbb{R}^d$, bounded, polyhedral, Lipschitz-continuous boundary

$$\left\|\nabla(\mathbf{u}-\mathbf{u}^h)\right\|_{L^2(\Omega)} \leq 2\left(1+\frac{\sqrt{d}}{\beta_{\mathrm{is}}^h}\right)\inf_{\mathbf{v}^h\in V^h}\left\|\nabla(\mathbf{u}-\mathbf{v}^h)\right\|_{L^2(\Omega)} + \sqrt{d}\inf_{q^h\in Q^h}\left\|p-q^h\right\|_{L^2(\Omega)}$$

  - proof: board
- polyhedral domain in three dimensions which is not Lipschitz-continuous



- finite element error estimate for the $L^2(\Omega)$ norm of the pressure
  - same assumptions as for previous estimate

$$
\begin{aligned}
\left\|p-p^h\right\|_{L^2(\Omega)} \quad \le \quad & \frac{2}{\beta_{\mathrm{is}}^h}\left(1+\frac{\sqrt{d}}{\beta_{\mathrm{is}}^h}\right)\inf_{\mathbf{v}^h\in V^h}\left\|\nabla(\mathbf{u}-\mathbf{v}^h)\right\|_{L^2(\Omega)} \\
& +\left(1+\frac{2\sqrt{d}}{\beta_{\mathrm{is}}^h}\right)\inf_{q^h\in Q^h}\left\|p-q^h\right\|_{L^2(\Omega)}
\end{aligned}
$$

  - proof: board

- error of the velocity in the $L^2(\Omega)$ norm
  - by Poincaré inequality not optimal

$$\left\|\mathbf{u}-\mathbf{u}^h\right\|_{L^2(\Omega)} \leq C\left\|\nabla(\mathbf{u}-\mathbf{u}^h)\right\|_{L^2(\Omega)}$$

- error of the velocity in the $L^2(\Omega)$ norm
  - by Poincaré inequality not optimal

$$\left\|\mathbf{u}-\mathbf{u}^h\right\|_{L^2(\Omega)} \le C \left\|\nabla(\mathbf{u}-\mathbf{u}^h)\right\|_{L^2(\Omega)}$$

- regular dual Stokes problem: For given $\hat{\mathbf{f}} \in L^2(\Omega)$, find $(\phi_{\hat{\mathbf{f}}}, \xi_{\hat{\mathbf{f}}}) \in V \times Q$ such that

$$\begin{aligned} -\Delta\phi_{\hat{\mathbf{f}}} + \nabla\xi_{\hat{\mathbf{f}}} &= \hat{\mathbf{f}} \quad \text{in } \Omega, \\ \nabla\cdot\phi_{\hat{\mathbf{f}}} &= 0 \quad \text{in } \Omega \end{aligned}$$

  - regular if mapping

$$\left(\phi_{\hat{\mathbf{f}}}, \xi_{\hat{\mathbf{f}}}\right) \mapsto -\Delta\phi_{\hat{\mathbf{f}}} + \nabla\xi_{\hat{\mathbf{f}}}$$

    is an isomorphism from $\left(H^2(\Omega)\cap V\right) \times \left(H^1(\Omega)\cap Q\right)$ onto $L^2(\Omega)$
  - Γ of class C^2
  - bounded, convex polygons in two dimensions

- finite element error estimate for the $L^2(\Omega)$ norm of the velocity
  - same assumptions as for previous estimates
  - dual Stokes problem regular with solution $(\phi_{\hat{\mathbf{f}}}, \xi_{\hat{\mathbf{f}}})$

$$
\begin{aligned}
&\left\|\mathbf{u}-\mathbf{u}^h\right\|_{L^2(\Omega)} \\
&\quad\leq \quad \sqrt{d}\left(\left\|\nabla\left(\mathbf{u}-\mathbf{u}^h\right)\right\|_{L^2(\Omega)}+\inf_{q^h\in Q^h}\left\|p-q^h\right\|_{L^2(\Omega)}\right) \\
&\qquad\quad \times \sup_{\hat{\mathbf{f}}\in L^2(\Omega)} \frac{1}{\left\|\hat{\mathbf{f}}\right\|_{L^2(\Omega)}}\left[\left(1+\frac{\sqrt{d}}{\beta_{\mathrm{is}}^h}\right)\inf_{\phi^h\in V^h}\left\|\nabla\left(\phi_{\hat{\mathbf{f}}}-\phi^h\right)\right\|_{L^2(\Omega)}\right. \\
&\qquad\qquad\qquad \left.+\inf_{r^h\in Q^h}\left\|\xi_{\hat{\mathbf{f}}}-r^h\right\|_{L^2(\Omega)}\right]
\end{aligned}
$$

  - proof: board (if time admits)

- finite element error estimates for conforming pairs of finite element spaces
  - same assumptions on domain as for previous estimates
  - solution sufficiently regular
  - h – mesh width of triangulation
  - spaces
    - P_k^{bubble}/P_k, $k = 1$ (mini element),
    - P_k/P_{k-1}, Q_k/Q_{k-1}, $k \geq 2$ (Taylor–Hood element),
    - $P_k^{\text{bubble}}/P_{k-1}^{\text{disc}}$, $Q_k/P_{k-1}^{\text{disc}}$, $k \geq 2$

$$
\begin{aligned}
\left\|\nabla(\mathbf{u}-\mathbf{u}^h)\right\|_{L^2(\Omega)} &\leq Ch^k\left(\|\mathbf{u}\|_{H^{k+1}(\Omega)} + \|p\|_{H^k(\Omega)}\right) \\
\left\|p-p^h\right\|_{L^2(\Omega)} &\leq Ch^k\left(\|\mathbf{u}\|_{H^{k+1}(\Omega)} + \|p\|_{H^k(\Omega)}\right)
\end{aligned}
$$

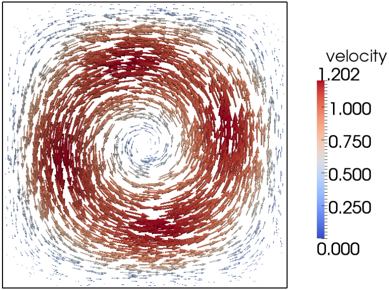
- finite element error estimates for conforming pairs of finite element spaces (cont.)
  - in addition: dual Stokes problem regular

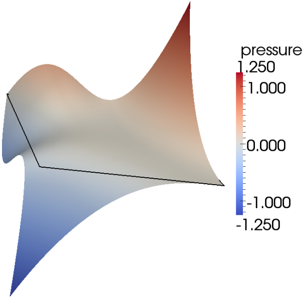
$$\left\|\mathbf{u}-\mathbf{u}^h\right\|_{L^2(\Omega)} \leq C h^{k+1}\left(\|\mathbf{u}\|_{H^{k+1}(\Omega)}+\|p\|_{H^k(\Omega)}\right)$$

  - all C depend on the discrete inf-sup constant β_{is}^h

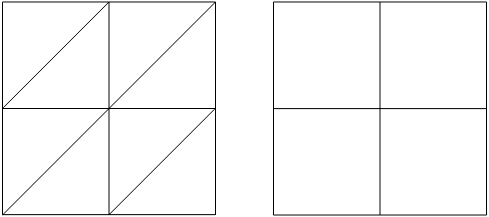
- analytical example which supports the error estimates
- prescribed solution

$$
\begin{aligned}
\mathbf{u} &= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \partial_y \psi \\ -\partial_x \psi \end{pmatrix} = 200 \begin{pmatrix} x^2(1-x)^2 y(1-y)(1-2y) \\ -x(1-x)(1-2x)y^2(1-y)^2 \end{pmatrix} \\
p &= 10\left(\left(x-\frac{1}{2}\right)^3 y^2 + (1-x)^3\left(y-\frac{1}{2}\right)^3\right)
\end{aligned}
$$



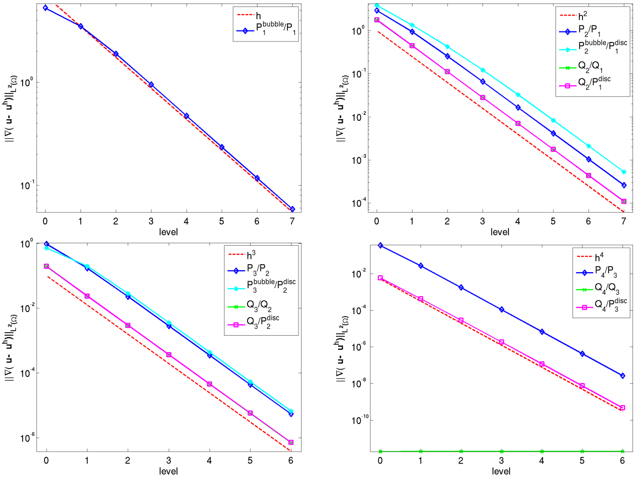


- initial grids (level 0)

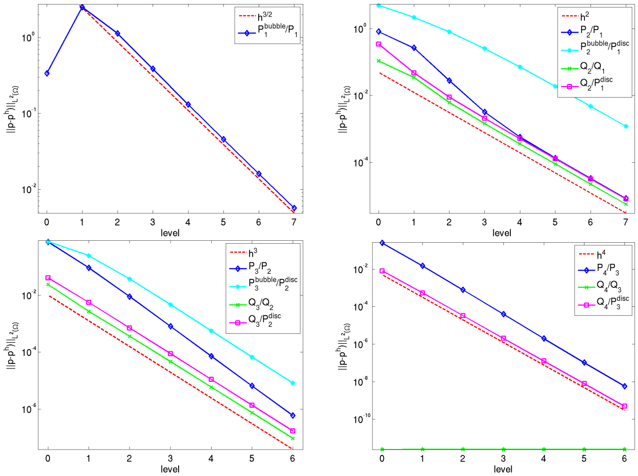


- red refinement

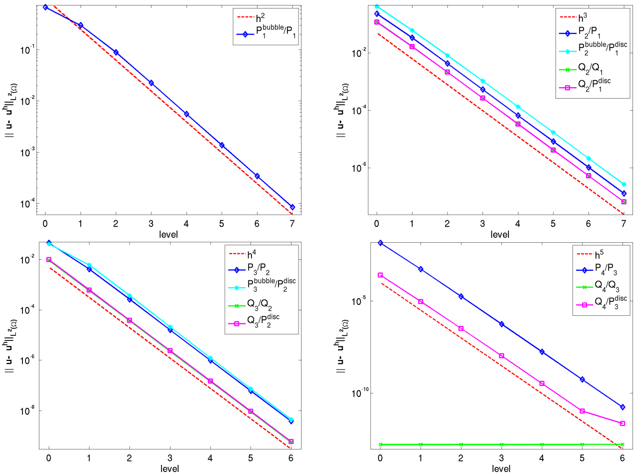
- convergence of the errors $\left\|\nabla(\mathbf{u}-\mathbf{u}^h)\right\|_{L^2(\Omega)}$ for different discretizations with different orders k



- convergence of the errors $\left\|p - p^h\right\|_{L^2(\Omega)}$ for different discretizations with different orders k



- convergence of the errors $\left\|\mathbf{u}-\mathbf{u}^h\right\|_{L^2(\Omega)}$ for different discretizations with different orders k



- implementation
  - vector-valued velocity space

$$
\begin{aligned}
V^h &= \text{span}\{\phi_i^h\}_{i=1}^{3N_v} \\
&= \text{span}\left\{ \left\{ \begin{pmatrix} \phi_i^h \\ 0 \\ 0 \end{pmatrix} \right\}_{i=1}^{N_v} \cup \left\{ \begin{pmatrix} 0 \\ \phi_i^h \\ 0 \end{pmatrix} \right\}_{i=1}^{N_v} \cup \left\{ \begin{pmatrix} 0 \\ 0 \\ \phi_i^h \end{pmatrix} \right\}_{i=1}^{N_v} \right\}
\end{aligned}
$$

  - pressure space

$$
Q^h = \text{span}\{\psi_i^h\}_{i=1}^{N_p}
$$

  - representation of unknown solution

$$
\mathbf{u}^h = \sum_{j=1}^{3N_v} u_j^h \phi_j^h, \quad p^h = \sum_{j=1}^{N_p} p_j^h \psi_j^h
$$

- pressure finite element space
  - standard basis functions not in $L_0^2(\Omega)$
  - it can be shown under mild assumptions that standard basis functions can be used as ansatz and test functions
  - computed pressure with standard basis functions has to be projected into $L_0^2(\Omega)$ at the end

- linear saddle point problem

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix}$$

with

$$\begin{aligned} (A)_{ij} = a_{ij} &= \sum_{K\in\mathscr{T}^h} \left(\nabla\phi_j^h, \nabla\phi_i^h\right)_K, i,j = 1,\ldots,3N_v, \\ (B)_{ij} = b_{ij} &= -\sum_{K\in\mathscr{T}^h} \left(\nabla\cdot\phi_j^h, \psi_i^h\right)_K, i = 1,\ldots,N_p, j = 1,\ldots,3N_v, \\ (\underline{f})_i = f_i &= \sum_{K\in\mathscr{T}^h} \left(\mathbf{f}, \phi_i^h\right)_K, i = 1,\ldots,3N_v \end{aligned}$$

- dimension (3d): $(3N_v + N_p) \times (3N_v + N_p)$

- matrix A
  - symmetric
  - positive definite
  - block-diagonal matrix

$$A = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{11} & 0 \\ 0 & 0 & A_{11} \end{pmatrix}$$

- matrix A
  - symmetric
  - positive definite
  - block-diagonal matrix

$$A = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{11} & 0 \\ 0 & 0 & A_{11} \end{pmatrix}$$

- $\left(\mathbb{D}\left(\mathbf{u}^h\right), \mathbb{D}\left(\mathbf{v}^h\right)\right)$ instead of $\left(\nabla \mathbf{u}^h, \nabla \mathbf{v}^h\right)$
  - equivalent only if $\mathbf{u}^h$ weakly divergence-free
  - generally not given for finite element velocities
  - not longer block-diagonal matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^T & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{pmatrix}$$

4 The Oseen Equations

- continuous equation

$$
\begin{aligned}
-\nu\Delta\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{u} + c\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega,\\
\nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega
\end{aligned}
$$

for simplicity: homogeneous Dirichlet boundary conditions

- difficulties:
  - coupling of velocity and pressure
  - dominating convection
- properties
  - linear

Carl Wilhelm Oseen (1879 – 1944)

- coefficients
  - $\nu > 0$
  - $\mathbf{b} \in W^{1,\infty}(\Omega)$, $\nabla \cdot \mathbf{b} = 0$
  - $c \in L^\infty(\Omega)$, $c(\mathbf{x}) \geq c_0 \geq 0$

- coefficients
  - $\nu > 0$
  - $\mathbf{b} \in W^{1,\infty}(\Omega)$, $\nabla \cdot \mathbf{b} = 0$
  - $c \in L^{\infty}(\Omega)$, $c(\mathbf{x}) \geq c_0 \geq 0$
- scaling of momentum equation: one of these possibilities
  - $\|\mathbf{b}\|_{L^{\infty}(\Omega)} = \mathcal{O}(1)$ if $\nu \leq \|\mathbf{b}\|_{L^{\infty}(\Omega)}$
  - $\nu = \mathcal{O}(1)$ if $\|\mathbf{b}\|_{L^{\infty}(\Omega)} \leq \nu$

4 The Oseen Equations

- coefficients
  - $\nu > 0$
  - $\mathbf{b} \in W^{1,\infty}(\Omega)$, $\nabla \cdot \mathbf{b} = 0$
  - $c \in L^\infty(\Omega)$, $c(\mathbf{x}) \geq c_0 \geq 0$
- scaling of momentum equation: one of these possibilities
  - $\|\mathbf{b}\|_{L^\infty(\Omega)} = \mathcal{O}(1)$ if $\nu \leq \|\mathbf{b}\|_{L^\infty(\Omega)}$
  - $\nu = \mathcal{O}(1)$ if $\|\mathbf{b}\|_{L^\infty(\Omega)} \leq \nu$
- interesting cases
  - ν of moderate size, $c = 0$

    in numerical solution of steady-state Navier–Stokes equations
  - ν of arbitrary size, $c = \mathcal{O}\left((\Delta t)^{-1}\right)$

    in numerical solution of time-dependent Navier–Stokes equations

- weak form

$$\begin{aligned} \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + ((\mathbf{b}\cdot\nabla)\mathbf{u} + c\mathbf{u}, \mathbf{v}) - (\nabla\cdot\mathbf{v}, p) &= \langle \mathbf{f}, \mathbf{v}\rangle_{V',V} \quad \forall\, \mathbf{v} \in V, \\ -(\nabla\cdot\mathbf{u}, q) &= 0 \qquad\qquad \forall\, q \in Q \end{aligned}$$

- bilinear forms

$$\begin{aligned} a \,:\, V\times V \to \mathbb{R}, \qquad a(\mathbf{u},\mathbf{v}) &= \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + ((\mathbf{b}\cdot\nabla)\mathbf{u} + c\mathbf{u}, \mathbf{v}), \\ b \,:\, V\times Q \to \mathbb{R}, \qquad b(\mathbf{v},q) &= -(\nabla\cdot\mathbf{v}, q) \end{aligned}$$

4 The Oseen Equations

- weak form

$$\begin{aligned} \nu(\nabla\mathbf{u},\nabla\mathbf{v})+((\mathbf{b}\cdot\nabla)\mathbf{u}+c\mathbf{u},\mathbf{v})-(\nabla\cdot\mathbf{v},p) &= \langle\mathbf{f},\mathbf{v}\rangle_{V',V} & \forall\,\mathbf{v}\in V,\\ -(\nabla\cdot\mathbf{u},q) &= 0 & \forall\,q\in Q \end{aligned}$$

- bilinear forms

$$\begin{aligned} a &: V\times V\to\mathbb{R}, & a(\mathbf{u},\mathbf{v}) &= \nu(\nabla\mathbf{u},\nabla\mathbf{v})+((\mathbf{b}\cdot\nabla)\mathbf{u}+c\mathbf{u},\mathbf{v}),\\ b &: V\times Q\to\mathbb{R}, & b(\mathbf{v},q) &= -(\nabla\cdot\mathbf{v},q) \end{aligned}$$

- existence and uniqueness of solution
  - proof: board
  - essential condition

$$((\mathbf{b}\cdot\nabla)\mathbf{v},\mathbf{v})=0\quad\forall\,\mathbf{v}\in V$$

can be proved is $\mathbf{b}$ is weakly divergence-free and has zero trace on Γ

- stability of solution
  - dependency of bounds on coefficients is important
  - depending on regularity of data, different estimates possible
    - most general

$$\frac{\nu}{2}\left\|\nabla \mathbf{u}\right\|_{L^2(\Omega)}^2 + \left\|c^{1/2}\mathbf{u}\right\|_{L^2(\Omega)}^2 \le \frac{1}{2\nu}\left\|\mathbf{f}\right\|_{H^{-1}(\Omega)}^2$$

    - $\mathbf{f} \in L^2(\Omega)$ and $c_0 > 0$

$$\nu\left\|\nabla \mathbf{u}\right\|_{L^2(\Omega)}^2 + \frac{1}{2}\left\|c^{1/2}\mathbf{u}\right\|_{L^2(\Omega)}^2 \le \frac{1}{2c_0}\left\|\mathbf{f}\right\|_{L_2(\Omega)}^2$$

  - proof: board

- stability of solution
  - dependency of bounds on coefficients is important
  - depending on regularity of data, different estimates possible
    - most general

$$\frac{\nu}{2}\|\nabla\mathbf{u}\|_{L^2(\Omega)}^2+\left\|c^{1/2}\mathbf{u}\right\|_{L^2(\Omega)}^2\le\frac{1}{2\nu}\|\mathbf{f}\|_{H^{-1}(\Omega)}^2$$

    - $\mathbf{f}\in L^2(\Omega)$ and $c_0>0$

$$\nu\|\nabla\mathbf{u}\|_{L^2(\Omega)}^2+\frac{1}{2}\left\|c^{1/2}\mathbf{u}\right\|_{L^2(\Omega)}^2\le\frac{1}{2c_0}\|\mathbf{f}\|_{L_2(\Omega)}^2$$

  - proof: board
  - estimates for pressure with inf-sup condition
  - discussion: board

- Galerkin finite element method

$$\begin{array}{rcll} a\left(\mathbf{u}^h,\mathbf{v}^h\right)+b\left(\mathbf{v}^h,p^h\right) & = & \left(\mathbf{f},\mathbf{v}^h\right) & \forall\,\mathbf{v}^h\in V^h, \\ b\left(\mathbf{u}^h,q^h\right) & = & 0 & \forall\,q^h\in Q^h \end{array}$$

  - homogeneous Dirichlet boundary conditions
  - conforming, inf-sup stable finite element spaces
- existence, uniqueness, stability like for continuous problem

- finite element error estimate for the $L^2(\Omega)$ norm of the gradient of the velocity
  - $\Omega \subset \mathbb{R}^d$, bounded, polyhedral, Lipschitz-continuous boundary
  - regularity of coefficients like stated above

$$
\begin{aligned}
& \nu^{1/2} \left\| \nabla \left(\mathbf{u} - \mathbf{u}^h \right) \right\|_{L^2(\Omega)} + \left\| c^{1/2} \left(\mathbf{u} - \mathbf{u}^h \right) \right\|_{L^2(\Omega)} \\
\leq \quad & C \left[\left(1 + \frac{1}{\beta_{\mathrm{is}}^h} \right) C_{\mathrm{os}} \inf_{\mathbf{v}^h \in V^h} \left\| \nabla (\mathbf{u} - \mathbf{v}^h) \right\|_{L^2(\Omega)} + \frac{1}{\nu^{1/2}} \inf_{q^h \in Q^h} \left\| p - q^h \right\|_{L^2(\Omega)} \right],
\end{aligned}
$$

where

$$
C_{\mathrm{os}} = \nu^{1/2} + \|c\|_{L^\infty(\Omega)}^{1/2} + \|\mathbf{b}\|_{L^\infty(\Omega)} \min \left\{ \frac{1}{\nu^{1/2}}, \frac{1}{c_0^{1/2}} \right\}
$$

  - C does not depend on coefficients and triangulation, but on Ω (Poincaré–Friedrichs inequality)

- finite element error estimate for the $L^2(\Omega)$ norm of the gradient of the velocity (cont.)
  - proof: principally same as for Stokes equations
  - estimates for convective term

$$
\begin{aligned}
\left|\left((\mathbf{b}\cdot\nabla)\,\eta,\phi^h\right)\right| &= \left|-\left((\mathbf{b}\cdot\nabla)\,\phi^h,\eta\right)\right| \le \|\mathbf{b}\|_{L^\infty(\Omega)} \left\|\nabla\phi^h\right\|_{L^2(\Omega)} \|\eta\|_{L^2(\Omega)} \\
&\le \frac{2}{\nu}\|\mathbf{b}\|_{L^\infty(\Omega)}^2 \|\eta\|_{L^2(\Omega)}^2 + \frac{\nu}{8}\left\|\nabla\phi^h\right\|_{L^2(\Omega)}^2
\end{aligned}
$$

or if $c_0 > 0$

$$
\begin{aligned}
\left|\left((\mathbf{b}\cdot\nabla)\,\eta,\phi^h\right)\right| &\le \|\mathbf{b}\|_{L^\infty(\Omega)} \|\nabla\eta\|_{L^2(\Omega)} \left\|\phi^h\right\|_{L^2(\Omega)} \\
&\le \frac{\|\mathbf{b}\|_{L^\infty(\Omega)}^2 \|\nabla\eta\|_{L^2(\Omega)}^2}{c_0} + \frac{\left\|c^{1/2}\phi^h\right\|_{L^2(\Omega)}^2}{4}
\end{aligned}
$$

- finite element error estimate for the $L^2(\Omega)$ norm of the pressure
  - same assumptions as for previous estimate

$$
\begin{aligned}
\left\| p - p^h \right\|_{L^2(\Omega)} \quad \le \quad & C \left[\frac{1}{\beta_{\mathrm{is}}^h} \left(1 + \frac{1}{\beta_{\mathrm{is}}^h} \right) C_{\mathrm{os}}^2 \inf_{\mathbf{v}^h \in V^h} \left\| \nabla (\mathbf{u} - \mathbf{v}^h) \right\|_{L^2(\Omega)} \right. \\
& \left. + \left(1 + \frac{1}{\beta_{\mathrm{is}}^h} + \frac{1}{\beta_{\mathrm{is}}^h} \frac{C_{\mathrm{os}}}{\nu^{1/2}} \right) \inf_{q^h \in Q^h} \left\| p - q^h \right\|_{L^2(\Omega)} \right],
\end{aligned}
$$

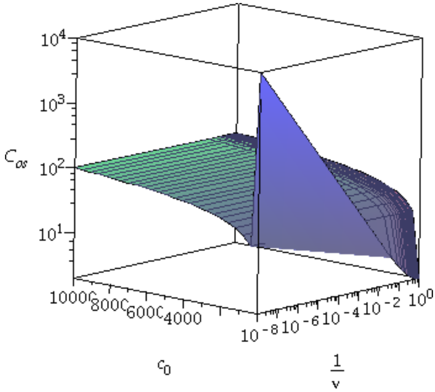
- proof: as for Stokes equations, with discrete inf-sup condition

- finite element error estimates for conforming pairs of finite element spaces
  - same assumptions on domain as for previous estimates
  - solution sufficiently regular
  - h – mesh width of triangulation
  - spaces
    - P_k^{bubble}/P_k, $k=1$ (mini element),
    - P_k/P_{k-1}, Q_k/Q_{k-1}, $k \geq 2$ (Taylor–Hood element),
    - $P_k^{\text{bubble}}/P_{k-1}^{\text{disc}}$, $Q_k/P_{k-1}^{\text{disc}}$,$k \geq 2$

$$
\begin{aligned}
\left\|\nabla(\mathbf{u}-\mathbf{u}^h)\right\|_{L^2(\Omega)} &\leq \frac{C}{\nu^{1/2}} h^k \left(C_{\text{os}} \|\mathbf{u}\|_{H^{k+1}(\Omega)} + \frac{1}{\nu^{1/2}} \|p\|_{H^k(\Omega)} \right), \\
\left\|p-p^h\right\|_{L^2(\Omega)} &\leq C h^k \left(C_{\text{os}}^2 \|\mathbf{u}\|_{H^{k+1}(\Omega)} + \left(1+\frac{C_{\text{os}}}{\nu^{1/2}}\right) \|p\|_{H^k(\Omega)} \right)
\end{aligned}
$$

4 The Oseen Equations – Galerkin FEM

- C_{os} for $\|\mathbf{b}\|_{L^\infty(\Omega)} = 1$



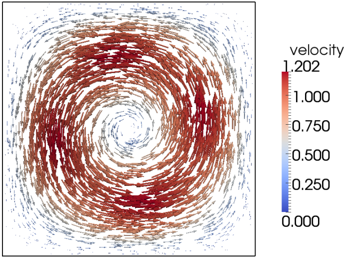
discussion: board

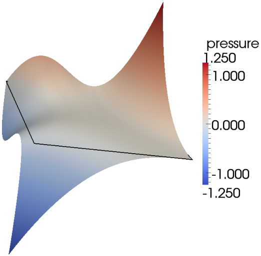
- error bounds not uniform for small ν or small time steps

- analytical example which supports the error estimates
- prescribed solution

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \partial_y \psi \\ -\partial_x \psi \end{pmatrix} = 200 \begin{pmatrix} x^2(1-x)^2 y(1-y)(1-2y) \\ -x(1-x)(1-2x)y^2(1-y)^2 \end{pmatrix}$$

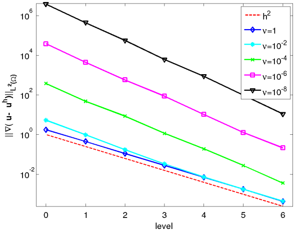
$$p = 10\left(\left(x-\frac{1}{2}\right)^3 y^2 + (1-x)^3\left(y-\frac{1}{2}\right)^3\right)$$

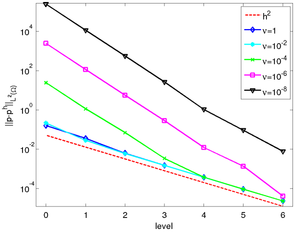




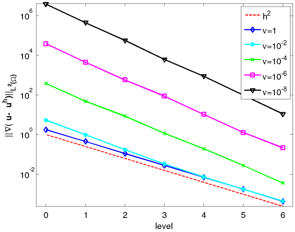
- $\mathbf{b} = \mathbf{u}$

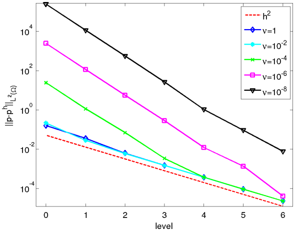
- Q_2/Q_1, convergence of errors for $c = 0$ and different values of ν



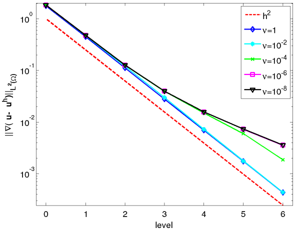


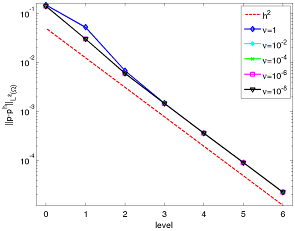
- Q_2/Q_1, convergence of errors for $c = 0$ and different values of ν



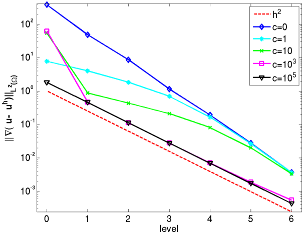


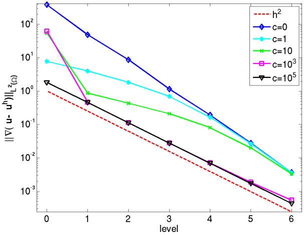
- Q_2/Q_1, convergence of errors for $c = 100$ and different values of ν



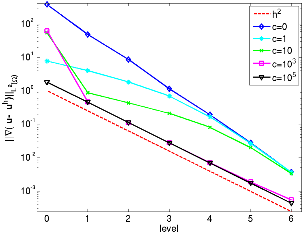


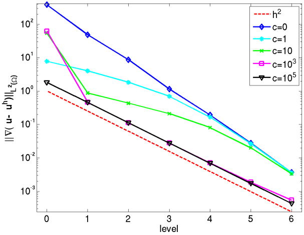
- Q_2/Q_1, convergence of errors for $\nu = 10^{-4}$ and different values of c





- Q_2/Q_1, convergence of errors for $\nu = 10^{-4}$ and different values of c





- summary
  - Galerkin discretization in some cases unstable

- principal idea
- given: linear partial differential equation in strong form

$$A_{\text{str}} u_{\text{str}} = f, \quad f \in L^2(\Omega)$$

- Galerkin discretization

$$a^h\left(u^h, v^h\right) = \left(f, v^h\right) \quad \forall\, v^h \in V^h$$

- needed: modification of strong operator $A^h_{\text{str}} \,:\, V^h \to L^2(\Omega)$
- residual

$$r^h\left(u^h\right) = A^h_{\text{str}} u^h - f \, \in L^2(\Omega)$$

- generally $r^h\left(u^h\right) \neq 0$

- principal idea (cont.)
- consider optimization problem

$$\arg\min_{u^h \in V^h} \left\| r^h\left(u^h\right) \right\|_{L^2(\Omega)}^2 = \arg\min_{u^h \in V^h} \left(r^h\left(u^h\right), r^h\left(u^h\right) \right)$$

- necessary condition for solution (board)

$$\left(r^h\left(u^h\right), A_{\mathrm{str}}^h v^h \right) = 0$$

- principal idea (cont.)
- consider optimization problem

$$\arg\min_{u^h \in V^h} \left\| r^h\left(u^h\right) \right\|_{L^2(\Omega)}^2 = \arg\min_{u^h \in V^h} \left(r^h\left(u^h\right), r^h\left(u^h\right) \right)$$

- necessary condition for solution (board)

$$\left(r^h\left(u^h\right), A_{\mathrm{str}}^h v^h \right) = 0$$

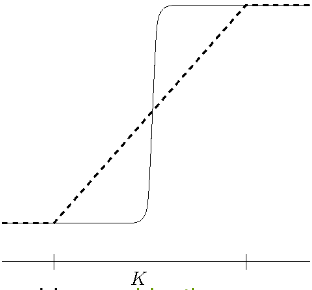
- generalization $\delta(\mathbf{x}) > 0$

$$\arg\min_{u^h \in V^h} \left\| \delta^{1/2} r^h\left(u^h\right) \right\|_{L^2(\Omega)}^2 = \arg\min_{u^h \in V^h} \left(\delta r^h\left(u^h\right), r^h\left(u^h\right) \right)$$

with necessary condition

$$\left(\delta r^h\left(u^h\right), A_{\mathrm{str}}^h v^h \right) = 0$$

- principal idea (cont.)
- minimizing residual alone: not good



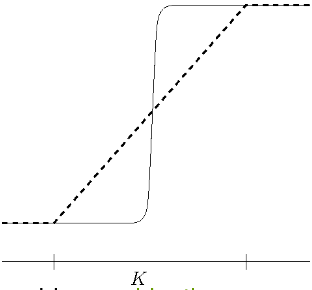
- solid line – function with layer
- dashed line – optimal piecewise linear approximation

- consider combination

$$a^h\left(u^h, v^h\right) + \left(\delta r^h\left(u^h\right), A_{\text{str}}^h v^h\right) = \left(f, v^h\right) \quad \forall\, v^h \in V^h$$

optimal choice of weighting function $\delta(\mathbf{x})$ by numerical analysis

- principal idea (cont.)
- minimizing residual alone: not good



- solid line – function with layer
- dashed line – optimal piecewise linear approximation

- consider combination

$$a^h\left(u^h, v^h\right) + \left(\delta r^h\left(u^h\right), A^h_{\text{str}} v^h\right) = \left(f, v^h\right) \quad \forall\, v^h \in V^h$$

optimal choice of weighting function $\delta(\mathbf{x})$ by numerical analysis

- example: Oseen equations, board

- SUPG/PSPG/grad-div stabilization
- find $\left(\mathbf{u}^h, p^h\right) \in V^h \times Q^h$ such that

$$A_{\mathrm{spg}}\left(\left(\mathbf{u}^h, p^h\right), \left(\mathbf{v}^h, q^h\right)\right) = L_{\mathrm{spg}}\left(\left(\mathbf{v}^h, q^h\right)\right) \quad \forall \left(\mathbf{v}^h, q^h\right) \in V^h \times Q^h,$$

with $A_{\mathrm{spg}} \; : \; \left(V \times \tilde{Q}\right) \times \left(V \times \tilde{Q}\right) \to \mathbb{R}$

$$\begin{aligned} & A_{\mathrm{spg}}\left((\mathbf{u}, p), (\mathbf{v}, q)\right) \\ = \quad & \nu\left(\nabla \mathbf{u}, \nabla \mathbf{v}\right) + \left((\mathbf{b} \cdot \nabla)\mathbf{u} + c\mathbf{u}, \mathbf{v}\right) - \left(\nabla \cdot \mathbf{v}, p\right) + \left(\nabla \cdot \mathbf{u}, q\right) \\ & + \sum_{K \in \mathscr{T}^h} \mu_K \left(\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}\right)_K + \sum_{E \in \mathscr{E}^h} \delta_E \left([|p|]_E, [|q|]_E\right)_E \\ & + \sum_{K \in \mathscr{T}^h} \left(-\nu \Delta \mathbf{u} + (\mathbf{b} \cdot \nabla)\mathbf{u} + c\mathbf{u} + \nabla p, \delta_K^v (\mathbf{b} \cdot \nabla)\mathbf{v} + \delta_K^p \nabla q\right)_K \end{aligned}$$

and $L_{\mathrm{spg}} \; : \; \left(V \times \tilde{Q}\right) \to \mathbb{R}$

$$L_{\mathrm{spg}}\left((\mathbf{v}, q)\right) = (\mathbf{f}, \mathbf{v}) + \sum_{K \in \mathscr{T}^h} \left(\mathbf{f}, \delta_K^v (\mathbf{b} \cdot \nabla)\mathbf{v} + \delta_K^p \nabla q\right)_K$$

4 The Oseen Equations – Residual-Based Stabilizations

- SUPG/PSPG/grad-div stabilization (cont.)
- finite element error analysis in [1]
- $\delta_K = \delta_K^v = \delta_K^p$ for all $K \in \mathscr{T}^h$

$$\delta = \max_{K \in \mathscr{T}^h} \delta_K, \quad \mu = \max_{K \in \mathscr{T}^h} \mu_K$$

[1] Tobiska, Verfürth, SINUM 33, 107–127, 1996

- SUPG/PSPG/grad-div stabilization (cont.)
- finite element error analysis in [1]
- $\delta_K = \delta_K^v = \delta_K^p$ for all $K \in \mathscr{T}^h$

$$\delta = \max_{K \in \mathscr{T}^h} \delta_K, \quad \mu = \max_{K \in \mathscr{T}^h} \mu_K$$

- no saddle point problem because of

$$-\sum_{E \in \mathscr{E}^h} \delta_E \left(\left[\left| p^h \right|\right]_E, \left[\left| q^h \right|\right]_E \right)_E - \sum_{K \in \mathscr{T}^h} \delta_K \left(\nabla p^h, \nabla q^h \right)_K$$

  - analysis for elliptic partial differential equations applicable
  - inf-sup stable spaces not necessary
  - choice of stabilization parameters affected by choice of finite element spaces

[1] Tobiska, Verfürth, SINUM 33, 107–127, 1996

- properties
  - consistency

$$A_{\text{spg}}\left((\mathbf{u},p),\left(\mathbf{v}^h,q^h\right)\right)=L_{\text{spg}}\left(\left(\mathbf{v}^h,q^h\right)\right),\quad\forall\left(\mathbf{v}^h,q^h\right)\in V^h\times Q^h$$

  - Galerkin orthogonality

$$A_{\text{spg}}\left(\left(\mathbf{u}-\mathbf{u}^h,p-p^h\right),\left(\mathbf{v}^h,q^h\right)\right)=0,\quad\forall\left(\mathbf{v}^h,q^h\right)\in V^h\times Q^h$$

- mesh-dependent norm

$$\|(\mathbf{v},q)\|_{\mathrm{spg}} = \left\{ \nu \|\nabla \mathbf{v}\|_{L^2(\Omega)}^2 + \left\| c^{1/2}\mathbf{v} \right\|_{L^2(\Omega)}^2 + \sum_{K\in\mathscr{T}^h} \mu_K \|\nabla\cdot\mathbf{v}\|_{L^2(K)}^2 \right.$$
$$\left. + \sum_{E\in\mathscr{E}^h} \delta_E \|[|q|]_E\|_{L^2(E)}^2 + \sum_{K\in\mathscr{T}^h} \delta_K \|(\mathbf{b}\cdot\nabla)\mathbf{v} + \nabla q\|_{L^2(K)}^2 \right\}^{1/2}$$

  - proof: board
  - additional control on error of
    - divergence
    - pressure jumps
    - streamline derivative + gradient of pressure
  - norm with pressure: later

- existence and uniqueness of a solution
  - assumptions

$$\mu_K \geq 0, \quad 0 < \delta_K \leq \min\left\{ \frac{h_K^2}{3\nu C_{\text{inv}}^2}, \frac{1}{3\left\|c\right\|_{L^\infty(K)}} \right\}$$

$\delta_E > 0$ if $Q^h \not\subset C(\overline{\Omega})$

  - proof: application of Lax–Milgram lemma
    - coercivity (board if time admits), $\forall \left(\mathbf{v}^h, q^h\right) \in V^h \times Q^h$

$$A_{\text{spg}}\left(\left(\mathbf{v}^h, q^h\right), \left(\mathbf{v}^h, q^h\right)\right) \geq \frac{1}{2}\left\|\left(\mathbf{v}^h, q^h\right)\right\|_{\text{spg}}^2$$

    - boundedness, $\forall \left(\mathbf{u}^h, p^h\right), \left(\mathbf{v}^h, q^h\right) \in V^h \times Q^h$

$$A_{\text{spg}}\left(\left(\mathbf{u}^h, p^h\right), \left(\mathbf{v}^h, q^h\right)\right) \leq C\left\|\left(\mathbf{u}^h, p^h\right)\right\|_{\text{spg}}\left\|\left(\mathbf{v}^h, q^h\right)\right\|_{\text{spg}}$$

using: all norms are equivalent in finite-dimensional spaces

- stability

$$\left\| \left(\mathbf{u}^h, p^h \right) \right\|_{\mathrm{spg}}^2 \le \frac{12}{5} \min \left\{ \frac{\|\mathbf{f}\|_{H^{-1}(\Omega)}^2}{\nu}, \frac{\|\mathbf{f}\|_{L_2(\Omega)}^2}{c_0} \right\} + 4 \sum_{K \in \mathscr{T}^h} \delta_K \|\mathbf{f}\|_{L^2(K)}^2$$

  - proof: as usual
  - estimate in stronger norm than for Galerkin finite element method
  - estimate for pressure with inf-sup condition possible

- norm for finite element error estimates

$$\|(\mathbf{v},q)\|_{\text{spg,p}} = \left(\|(\mathbf{v},q)\|_{\text{spg}} + w_{\text{pres}}^{-2} \|q\|_{L^2(\Omega)}^2 \right)^{1/2}$$

with

$$w_{\text{pres}} = \max\left\{1, \nu^{-1/2}, \|c\|_{L^\infty(\Omega)}^{1/2}\right\}$$

for the interesting cases of small ν and large c: small contribution of the pressure

- norm for finite element error estimates

$$\|(\mathbf{v},q)\|_{\mathrm{spg,p}} = \left(\|(\mathbf{v},q)\|_{\mathrm{spg}} + w_{\mathrm{pres}}^{-2}\,\|q\|_{L^2(\Omega)}^2\right)^{1/2}$$

  with

$$w_{\mathrm{pres}} = \max\left\{1, \nu^{-1/2}, \|c\|_{L^\infty(\Omega)}^{1/2}\right\}$$

  for the interesting cases of small ν and large c: small contribution of the pressure

- first step: inf-sup conditions for A_{spg}

$$\inf_{\substack{(\mathbf{v}^h,q^h)\in V^h\times Q^h \\ \|(\mathbf{u}^h,p^h)\|_{\mathrm{spg,p}}=1}} \sup_{\substack{(\mathbf{w}^h,r^h)\in V^h\times Q^h \\ \|(\mathbf{v}^h,q^h)\|_{\mathrm{spg,p}}=1}} A_{\mathrm{spg}}\left(\left(\mathbf{v}^h,q^h\right),\left(\mathbf{w}^h,r^h\right)\right) \ge \beta_{\mathrm{spg}}$$

  - some conditions on stabilization parameters, e.g., $\delta_0 h_K^2 \le \delta_K$
  - proof very technical
  - $\beta_{\mathrm{spg}} = \mathcal{O}(\delta_0)$

- finite element error estimate

$$\begin{aligned}
&\left\|\left(\mathbf{u}-\mathbf{u}^h, p-p^h\right)\right\|_{\text{spg}} + \nu^{1/2}\left\|p-p^h\right\|_{L^2\Omega} \\
&\quad \leq \quad C\left[h^{k}\left(\nu^{1/2}+\frac{\nu\delta^{1/2}}{h}+\frac{h}{\delta^{1/2}}+\delta^{1/2}+\frac{\mu\delta^{1/2}}{h}+\|c\|_{L^\infty(\Omega)}^{1/2} h+\delta\|c\|_{L^\infty(\Omega)} h\right)\right. \\
&\qquad\qquad \left.+h^{l+1}\left(\nu^{1/2}+\frac{\delta^{1/2}}{h}+\frac{1}{\nu^{1/2}}\left(\max\left\{1,\frac{\mu}{\nu}\right\}\right)^{-1/2}\right)\|p\|_{H^{l+1}(\Omega)}\right]
\end{aligned}$$

  - $k \geq 1$, $l \geq 0$
  - C independent of the coefficients of the problem
  - proof: based on inf-sup condition

- optimal asymptotics for stabilization parameters, $\nu < h$ (board)
  - inf-sup stable discretizations with $k = l + 1$

$$\delta = \mathscr{O}\left(h^2\right), \quad \mu = \mathscr{O}\left(1\right) \quad \Longrightarrow \quad \text{order of error reduction: } k$$

- optimal asymptotics for stabilization parameters, $\nu < h$ (board)
  - inf-sup stable discretizations with $k = l + 1$

$$\delta = \mathcal{O}\left(h^2\right), \quad \mu = \mathcal{O}\left(1\right) \quad \Longrightarrow \quad \text{order of error reduction: } k$$

  - equal-order discretizations with $k = l \geq 1$

$$\mathcal{O}\left(\delta\right) = \mathcal{O}\left(\mu\right) = \mathcal{O}\left(h\right) \quad \Longrightarrow \quad \text{order of error reduction: } k + \frac{1}{2}$$

- optimal asymptotics for stabilization parameters, $\nu < h$ (board)
  - inf-sup stable discretizations with $k = l + 1$

$$\delta = \mathcal{O}\left(h^2\right), \quad \mu = \mathcal{O}(1) \quad \Longrightarrow \quad \text{order of error reduction: } k$$

  - equal-order discretizations with $k = l \geq 1$

$$\mathcal{O}(\delta) = \mathcal{O}(\mu) = \mathcal{O}(h) \quad \Longrightarrow \quad \text{order of error reduction: } k + \frac{1}{2}$$

- optimal asymptotics for stabilization parameters, $\nu \geq h$
  - inf-sup stable discretizations with $k = l + 1$

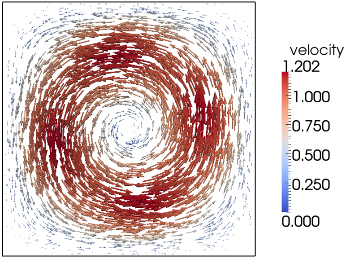
$$\delta = \mathcal{O}\left(h^2\right), \quad \mu = \mathcal{O}(1) \quad \Longrightarrow \quad \text{order of convergence: } k$$

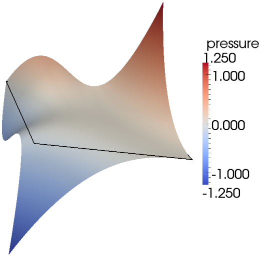
  - equal-order discretizations with $k = l \geq 1$

$$\delta = \mathcal{O}\left(h^2\right), \quad \mu \text{ arbitrary} \quad \Longrightarrow \quad \text{order of convergence: } k$$

- analytical example which supports the error estimates
- prescribed solution

$$
\begin{aligned}
\mathbf{u} &= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \partial_y \psi \\ -\partial_x \psi \end{pmatrix} = 200 \begin{pmatrix} x^2(1-x)^2 y(1-y)(1-2y) \\ -x(1-x)(1-2x)y^2(1-y)^2 \end{pmatrix} \\
p &= 10\left(\left(x-\frac{1}{2}\right)^3 y^2 + (1-x)^3\left(y-\frac{1}{2}\right)^3\right)
\end{aligned}
$$



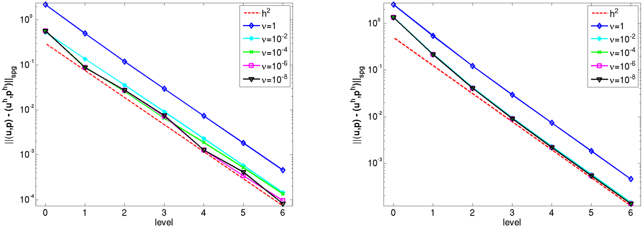


- $\mathbf{b} = \mathbf{u}$

- Q_2/Q_1 finite element
- stabilization parameters (based on numerical simulations from [1])

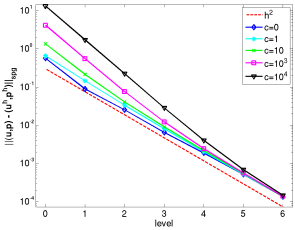
$$\mu_K = 0.2, \quad \delta_K = 0.1 h_K^2$$

- convergence of errors for $c = 0$ and $c = 100$, different values of ν



[1] Matthies, Lube, Röhe, Comput. Methods Appl. Math. 9, 368 – 390, 2009

- Q_2/Q_1, convergence of errors for $\nu = 10^{-4}$ and different values of c

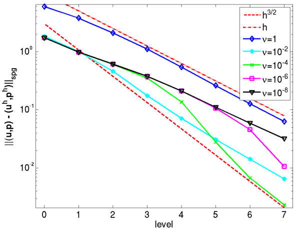


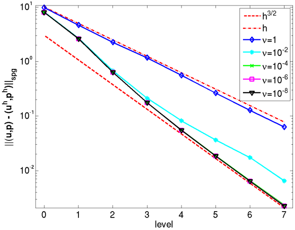
4 The Oseen Equations – Residual-Based Stabilizations

- P_1/P_1 finite element
- stabilization parameters

$$\delta_K = \begin{cases} 0.5 h_K & \text{if } \nu < h_K, \\ 0.5 h_K^2 & \text{else,} \end{cases} \qquad \mu_K = 0.5 h_K$$

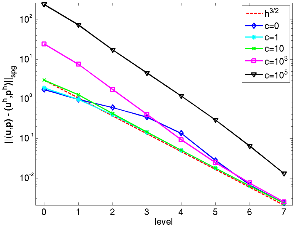
- convergence of errors for $c = 0$ and $c = 100$, different values of ν





4 The Oseen Equations – Residual-Based Stabilizations

- P_1/P_1, convergence of errors for $\nu = 10^{-4}$ and different values of c



- implementation: same approach as for Stokes equations
- grad-div term leads to matrix block

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^T & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{pmatrix} \quad \text{instead of} \quad \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{11} & 0 \\ 0 & 0 & A_{11} \end{pmatrix}$$

- PSPG term introduces pressure-pressure couplings
- SUPG term influences velocity-velocity coupling and the pressure (ansatz) - velocity (test) coupling
- final system

$$\begin{pmatrix} A & D \\ B & C \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{f_p} \end{pmatrix}$$

much more matrix blocks to store than for Galerkin FEM

- Summary and remarks
  - errors $\left\|(\mathbf{u},p)-(\mathbf{u}^h,p^h)\right\|_{\mathrm{spg}}$ independent of ν
  - versions without pressure couplings available
    - only for inf-sup stable pairs of finite elements
    - easier to implement than SUPG/PSPG/grad-div stabilization
  - numerical analysis in [1,2,3]

[1] Tobiska, Verfürth, SINUM 33, 107–127, 1996

[2] Lube, Rapin, M3AS 16, 949–966, 2006

[3] Matthies, Lube, Röhe, Comput. Methods Appl. Math. 9, 368–390, 2009

5 The Stationary Navier–Stokes Equations

- continuous equation

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega \end{aligned}$$

  for simplicity: homogeneous Dirichlet boundary conditions

- difficulties:
  - coupling of velocity and pressure
  - dominating convection
  - nonlinear

- different forms of the convective term

$$\begin{array}{lcl} (\mathbf{u}\cdot\nabla)\mathbf{u} & : & \text{convective form}, \\ \nabla\cdot\left(\mathbf{u}\mathbf{u}^T\right) & : & \text{divergence form}, \\ (\nabla\times\mathbf{u})\times\mathbf{u} & : & \text{rotational form} \end{array}$$

  - convective form and divergence form equivalent if $\nabla\cdot\mathbf{u}=0$ (board, if time permits)
  - convective form and rotational form

$$(\nabla\times\mathbf{u})\times\mathbf{u}+\frac{1}{2}\nabla\left(\mathbf{u}^T\mathbf{u}\right)=(\mathbf{u}\cdot\nabla)\mathbf{u}$$

definition of new pressure (Bernoulli pressure) in rotational form

$$p_{\text{Bern}}=p+\frac{1}{2}\mathbf{u}^T\mathbf{u}$$

- variational form of the steady-state Navier–Stokes equations: Find $(\mathbf{u}, p) \in V \times Q$ such that

$$\begin{aligned} (\nu \nabla \mathbf{u}, \nabla \mathbf{v}) + ((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}) \\ -(\nabla \cdot \mathbf{u}, q) &= 0 \end{aligned}$$

  for all $(\mathbf{v}, q) \in V \times Q$
- equivalent: Find $(\mathbf{u}, p) \in V \times Q$ such that

$$(\nu \nabla \mathbf{u}, \nabla \mathbf{v}) + ((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, p) + (\nabla \cdot \mathbf{u}, q) = (\mathbf{f}, \mathbf{v})$$

  for all $(\mathbf{v}, q) \in V \times Q$

- properties of convective term
  - linear in each component (trilinear)
  - $\mathbf{u}, \mathbf{v}, \mathbf{w} \in H^1(\Omega)$, product rule

$$((\mathbf{u}\cdot\nabla)\mathbf{v}, \mathbf{w}) = \left(\nabla\cdot\left(\mathbf{v}\mathbf{u}^T\right), \mathbf{w}\right) - ((\nabla\cdot\mathbf{u})\mathbf{v}, \mathbf{w})$$

  - $\mathbf{u}, \mathbf{v}, \mathbf{w} \in H^1(\Omega)$, product rule

$$((\mathbf{u}\cdot\nabla)\mathbf{v}, \mathbf{w}) = (\mathbf{u}, \nabla(\mathbf{v}\cdot\mathbf{w})) - ((\mathbf{u}\cdot\nabla)\mathbf{w}, \mathbf{v})$$

- convective terms in the variational formulation
  - convective form
$$n_{\text{conv}}(\mathbf{u},\mathbf{v},\mathbf{w}) = ((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w})$$
  - divergence form
$$n_{\text{div}}(\mathbf{u},\mathbf{v},\mathbf{w}) = n_{\text{conv}}(\mathbf{u},\mathbf{v},\mathbf{w}) + ((\nabla\cdot\mathbf{u})\,\mathbf{v},\mathbf{w})$$
  - rotational form
$$n_{\text{rot}}(\mathbf{u},\mathbf{v},\mathbf{w}) = ((\nabla\times\mathbf{u})\times\mathbf{v},\mathbf{w})$$
    with momentum equation
$$(\nu\nabla\mathbf{u},\nabla\mathbf{v}) + n_{\text{rot}}(\mathbf{u},\mathbf{u},\mathbf{v}) - (\nabla\cdot\mathbf{v}, p_{\text{Bern}}) = (\mathbf{f},\mathbf{v}) \quad \forall\,\mathbf{v}\in V,$$
  - skew-symmetric form (for $\mathbf{u}$ weakly divergence-free, $\mathbf{u}\cdot\mathbf{n}=0$ on Γ, board)
$$n_{\text{skew}}(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{1}{2}\left(n_{\text{conv}}(\mathbf{u},\mathbf{v},\mathbf{w}) - n_{\text{conv}}(\mathbf{u},\mathbf{w},\mathbf{v})\right)$$

- further properties of convective term
- vanishing
  - rotational and skew-symmetric form

$$n_{\mathrm{rot}}(\mathbf{u},\mathbf{v},\mathbf{v}) = n_{\mathrm{skew}}(\mathbf{u},\mathbf{v},\mathbf{v}) = 0$$

  - convective and divergence form: if $\mathbf{u}$ weakly divergence-free and $\mathbf{u}\cdot\mathbf{n} = 0$ on Γ

$$n_{\mathrm{conv}}(\mathbf{u},\mathbf{v},\mathbf{v}) = n_{\mathrm{div}}(\mathbf{u},\mathbf{v},\mathbf{v}) = 0$$

- further properties of convective term
- vanishing
  - rotational and skew-symmetric form

$$n_{\mathrm{rot}}(\mathbf{u},\mathbf{v},\mathbf{v}) = n_{\mathrm{skew}}(\mathbf{u},\mathbf{v},\mathbf{v}) = 0$$

  - convective and divergence form: if $\mathbf{u}$ weakly divergence-free and $\mathbf{u}\cdot\mathbf{n} = 0$ on Γ

$$n_{\mathrm{conv}}(\mathbf{u},\mathbf{v},\mathbf{v}) = n_{\mathrm{div}}(\mathbf{u},\mathbf{v},\mathbf{v}) = 0$$

- estimates: $\mathbf{u},\mathbf{v},\mathbf{w} \in H^1(\Omega)$

$$\begin{aligned} |n_{\mathrm{conv}}(\mathbf{u},\mathbf{v},\mathbf{w})| &\le C\,\|\mathbf{u}\|_{H^1(\Omega)}\,\|\nabla\mathbf{v}\|_{L^2(\Omega)}\,\|\mathbf{w}\|_{H^1(\Omega)}, \\ |n_{\mathrm{skew}}(\mathbf{u},\mathbf{v},\mathbf{w})| &\le C\,\|\mathbf{u}\|_{H^1(\Omega)}\,\|\mathbf{v}\|_{H^1(\Omega)}\,\|\mathbf{w}\|_{H^1(\Omega)} \end{aligned}$$

  - proof: board

- existence and uniqueness of a solution
  - $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, bounded domain with Lipschitz boundary
  - $\mathbf{f} \in H^{-1}(\Omega)$
  - then: existence
- main ideas of the proof
  - equivalent problem in the divergence-free subspace, only velocity
  - consider problem in finite dimensional spaces (Galerkin method)
  - fixed point equation, existence of a solution of the finite dimensional problems: fixed point theorem of Brouwer
  - dimension of the spaces $\to \infty$: show subsequence of the solutions tends to a solution of the problem in the divergence-free subspace
  - existence of the pressure: inf-sup condition

- existence and uniqueness of a solution (cont.)
  - ν sufficiently large, i.e.

$$\|\mathbf{f}\|_{H^{-1}(\Omega)} \sup_{\mathbf{u},\mathbf{v},\mathbf{w}\in V} \frac{((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w})}{\|\nabla\mathbf{u}\|_{L^2(\Omega)}\|\nabla\mathbf{v}\|_{L^2(\Omega)}\|\nabla\mathbf{w}\|_{L^2(\Omega)}} < \nu^2$$

  - then: uniqueness
- main idea of the proof
  - construct a contraction, apply Banach's fixed point theorem

- existence and uniqueness of a solution (cont.)
  - ν sufficiently large, i.e.

$$\|\mathbf{f}\|_{H^{-1}(\Omega)} \sup_{\mathbf{u},\mathbf{v},\mathbf{w}\in V} \frac{((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w})}{\|\nabla\mathbf{u}\|_{L^2(\Omega)}\|\nabla\mathbf{v}\|_{L^2(\Omega)}\|\nabla\mathbf{w}\|_{L^2(\Omega)}} < \nu^2$$

  - then: uniqueness
- main idea of the proof
  - construct a contraction, apply Banach's fixed point theorem
- numerical simulations
  - case of unique solution is of interest
  - steady-state solutions unstable in non-unique case, solve time-dependent solution

- stability

$$
\begin{aligned}
\|\nabla \mathbf{u}\|_{L^2(\Omega)} &\le \frac{1}{\nu}\,\|\mathbf{f}\|_{H^{-1}(\Omega)}\,, \\
\|p\|_{L^2(\Omega)} &\le \frac{1}{\beta_{\mathrm{is}}}\left(2\,\|\mathbf{f}\|_{H^{-1}(\Omega)} + \frac{C}{\nu^2}\,\|\mathbf{f}\|^2_{H^{-1}(\Omega)}\right)
\end{aligned}
$$

  - proof: as usual, using

$$
n(\mathbf{u},\mathbf{u},\mathbf{u}) = 0
$$

- Galerkin finite element method

$$\begin{array}{rcll} \nu\left(\nabla \mathbf{u}^h, \nabla \mathbf{v}^h\right) + n\left(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h\right) - \left(\nabla \cdot \mathbf{v}^h, p^h\right) & = & \left(\mathbf{f}, \mathbf{v}^h\right) & \forall\, \mathbf{v}^h \in V^h, \\ -\left(\nabla \cdot \mathbf{u}^h, q^h\right) & = & 0 & \forall\, q^h \in Q^h, \end{array}$$

- inf-sup stable pair of finite element spaces

- Galerkin finite element method

$$
\begin{aligned}
\nu\left(\nabla \mathbf{u}^h, \nabla \mathbf{v}^h\right) + n\left(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h\right) - \left(\nabla\cdot\mathbf{v}^h, p^h\right) &= \left(\mathbf{f}, \mathbf{v}^h\right) \quad \forall\, \mathbf{v}^h \in V^h, \\
-\left(\nabla\cdot\mathbf{u}^h, q^h\right) &= 0 \qquad\quad\; \forall\, q^h \in Q^h,
\end{aligned}
$$

- inf-sup stable pair of finite element spaces
- finite element error analysis for $n_{\mathrm{skew}}(\cdot,\cdot,\cdot)$

$$
n_{\mathrm{skew}}\left(\mathbf{u}^h, \mathbf{v}^h, \mathbf{v}^h\right) = \frac{1}{2}\left(n_{\mathrm{conv}}\left(\mathbf{u}^h, \mathbf{v}^h, \mathbf{v}^h\right) - n_{\mathrm{conv}}\left(\mathbf{u}^h, \mathbf{v}^h, \mathbf{v}^h\right)\right) = 0
$$

note that in general $\mathbf{u}^h \notin V_{\mathrm{div}} \quad \Longrightarrow$

$$
n_{\mathrm{conv}}\left(\mathbf{u}^h, \mathbf{v}^h, \mathbf{v}^h\right) \neq 0, \quad n_{\mathrm{div}}\left(\mathbf{u}^h, \mathbf{v}^h, \mathbf{v}^h\right) \neq 0
$$

- Galerkin finite element method

$$\begin{aligned} \nu\left(\nabla\mathbf{u}^h,\nabla\mathbf{v}^h\right)+n\left(\mathbf{u}^h,\mathbf{u}^h,\mathbf{v}^h\right)-\left(\nabla\cdot\mathbf{v}^h,p^h\right) &= \left(\mathbf{f},\mathbf{v}^h\right) \quad \forall\ \mathbf{v}^h\in V^h, \\ -\left(\nabla\cdot\mathbf{u}^h,q^h\right) &= 0 \qquad\qquad \forall\ q^h\in Q^h, \end{aligned}$$

- inf-sup stable pair of finite element spaces
- finite element error analysis for $n_{\text{skew}}(\cdot,\cdot,\cdot)$

$$n_{\text{skew}}\left(\mathbf{u}^h,\mathbf{v}^h,\mathbf{v}^h\right)=\frac{1}{2}\left(n_{\text{conv}}\left(\mathbf{u}^h,\mathbf{v}^h,\mathbf{v}^h\right)-n_{\text{conv}}\left(\mathbf{u}^h,\mathbf{v}^h,\mathbf{v}^h\right)\right)=0$$

note that in general $\mathbf{u}^h\notin V_{\text{div}} \quad \Longrightarrow$

$$n_{\text{conv}}\left(\mathbf{u}^h,\mathbf{v}^h,\mathbf{v}^h\right)\neq 0,\quad n_{\text{div}}\left(\mathbf{u}^h,\mathbf{v}^h,\mathbf{v}^h\right)\neq 0$$

- same as for continuous problem:
  - existence, uniqueness
  - stability

- Finite element error estimate for the $L^2(\Omega)$ norm of the gradient of the velocity
  - $\Omega \subset \mathbb{R}^d$ bounded domain with polyhedral boundary
  - $\nu^{-2} \|\mathbf{f}\|_{H^{-1}(\Omega)}$ be sufficiently small such that unique solution
  - inf-sup stable finite element spaces $V^h \times Q^h$

$$
\begin{aligned}
&\left\| \nabla(\mathbf{u} - \mathbf{u}^h) \right\|_{L^2(\Omega)} \\
&\quad \leq \quad C \left(\left(1 + \frac{1}{\nu^2} \|\mathbf{f}\|_{H^{-1}(\Omega)} \right) \left(1 + \frac{1}{\beta_{\text{is}}^h} \right) \inf_{\mathbf{v}^h \in V^h} \left\| \nabla \left(\mathbf{u} - \mathbf{v}^h \right) \right\|_{L^2(\Omega)} \right. \\
&\qquad\qquad \left. + \frac{1}{\nu} \inf_{q^h \in Q^h} \left\| p - q^h \right\|_{L^2(\Omega)} \right)
\end{aligned}
$$

  - proof: main ideas and treatment of nonlinear term: board

- Finite element error estimate for the $L^2(\Omega)$ norm of the pressure

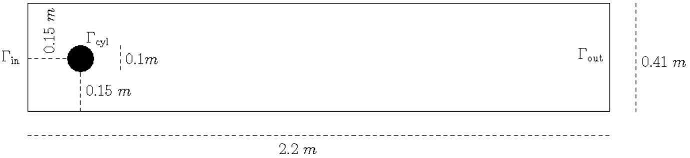
$$
\begin{aligned}
&\left\|p-p^h\right\|_{L^2(\Omega)} \\
&\leq \quad C\frac{\nu}{\beta_{\mathrm{is}}^h}\left(\left(1+\frac{1}{\nu^2}\|\mathbf{f}\|_{H^{-1}(\Omega)}\right)^2\left(1+\frac{1}{\beta_{\mathrm{is}}^h}\right)\inf_{\mathbf{v}^h\in V^h}\left\|\nabla\left(\mathbf{u}-\mathbf{v}^h\right)\right\|_{L^2(\Omega)}\right. \\
&\qquad \left. +C\frac{\nu}{\beta_{\mathrm{is}}^h}\left(1+\frac{1}{\nu^2}\|\mathbf{f}\|_{H^{-1}(\Omega)}\right)\inf_{q^h\in Q^h}\left\|p-q^h\right\|_{L^2(\Omega)}\right)
\end{aligned}
$$

- Finite element error estimate for the $L^2(\Omega)$ norm of the pressure

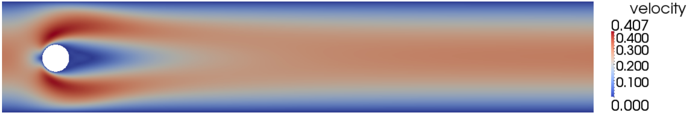
$$
\begin{aligned}
& \left\| p - p^h \right\|_{L^2(\Omega)} \\
\leq \quad & C \frac{\nu}{\beta_{\text{is}}^h} \left(\left(1 + \frac{1}{\nu^2} \|\mathbf{f}\|_{H^{-1}(\Omega)} \right)^2 \left(1 + \frac{1}{\beta_{\text{is}}^h} \right) \inf_{\mathbf{v}^h \in V^h} \left\| \nabla \left(\mathbf{u} - \mathbf{v}^h \right) \right\|_{L^2(\Omega)} \right. \\
& \left. + C \frac{\nu}{\beta_{\text{is}}^h} \left(1 + \frac{1}{\nu^2} \|\mathbf{f}\|_{H^{-1}(\Omega)} \right) \inf_{q^h \in Q^h} \left\| p - q^h \right\|_{L^2(\Omega)} \right)
\end{aligned}
$$

- analytical results can be supported numerically by analytical test examples

- Example: steady-state flow around a cylinder at $Re = 20$
  - domain

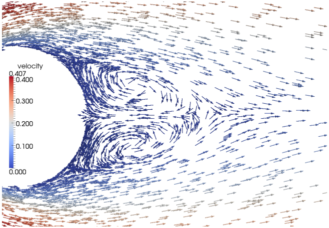


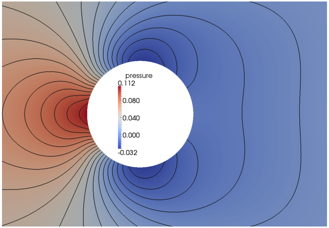
  - velocity



  - pressure

- Example: steady-state flow around a cylinder at $Re = 20$
  - at the cylinder





- important: drag and lift coefficient at the cylinder

$$
\begin{aligned}
c_{\text{drag}} &= \frac{2}{\rho d U_{\text{mean}}^2} \int_{\Gamma_{\text{cyl}}} \left(\mu \frac{\partial \mathbf{v_t}}{\partial \mathbf{n}} n_y - P n_x \right) \, ds, \\
c_{\text{lift}} &= -\frac{2}{\rho d U_{\text{mean}}^2} \int_{\Gamma_{\text{cyl}}} \left(\mu \frac{\partial \mathbf{v_t}}{\partial \mathbf{n}} n_x + P n_y \right) \, ds
\end{aligned}
$$

- important: drag and lift coefficient at the cylinder

$$
\begin{aligned}
c_{\text{drag}} &= \frac{2}{\rho d U_{\text{mean}}^2} \int_{\Gamma_{\text{cyl}}} \left(\mu \frac{\partial \mathbf{v_t}}{\partial \mathbf{n}} n_y - P n_x \right) \, ds, \\
c_{\text{lift}} &= -\frac{2}{\rho d U_{\text{mean}}^2} \int_{\Gamma_{\text{cyl}}} \left(\mu \frac{\partial \mathbf{v_t}}{\partial \mathbf{n}} n_x + P n_y \right) \, ds
\end{aligned}
$$

- reformulation with volume integrals possible, long but elementary derivation, e.g.

$$
c_{\text{drag}} = -\frac{2U^2}{d U_{\text{mean}}^2} \big((\nu \nabla \mathbf{u}, \nabla \mathbf{w}_d) + n(\mathbf{u}, \mathbf{u}, \mathbf{w}_d) - (\nabla \cdot \mathbf{w}_d, p) - (\mathbf{f}, \mathbf{w}_d) \big)
$$

for any function $\mathbf{w}_d \in H^1(\Omega)$ with $\mathbf{w}_d = \mathbf{0}$ on $\Gamma \setminus \Gamma_{\text{cyl}}$ and $\mathbf{w}_d|_{\Gamma_{\text{cyl}}} = (1,0)^T$

- reference values
  - [1] : compiled from simulations of different groups

$$c_{\text{drag,ref}} \in [5.57, 5.59], \quad c_{\text{lift,ref}} \in [0.104, 0.110]$$

  - [2] : do-nothing conditions at outlet

$$c_{\text{drag,ref}} = 5.57953523384, \quad c_{\text{lift,ref}} = 0.010618948146$$

  - [3] : Dirichlet conditions at outlet

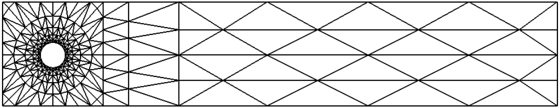
$$c_{\text{drag,ref}} = 5.57953523384, \quad c_{\text{lift,ref}} = 0.010618937712$$

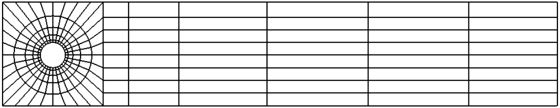
[1] Schäfer, Turek, Notes on Numerical Fluid Mechanics 52, 547–566, 1996

[2] Nabh, PhD thesis, Heidelberg, 1998

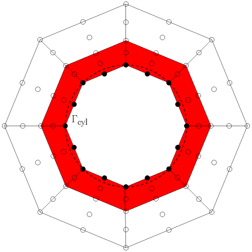
[3] J., Matthies, IJNMF 37, 885–903, 2001

- initial grids

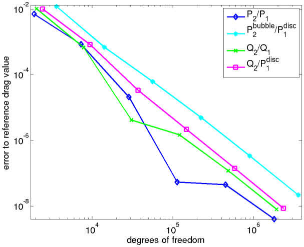


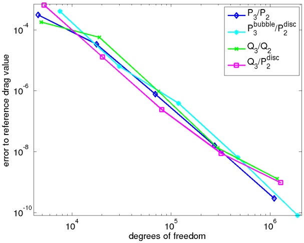


- patch for test function in computation of coefficients, Q_2

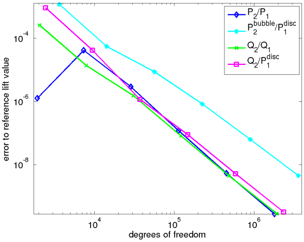


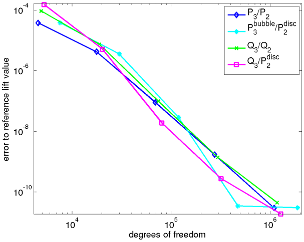
- convective form of convective term
- do-nothing boundary conditions
- convergence of drag coefficient





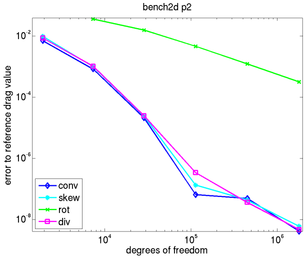
- convergence of lift coefficient

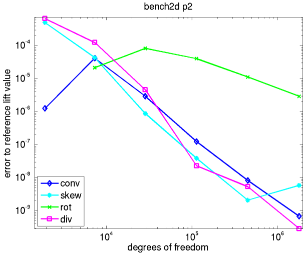




5 The Stationary Navier–Stokes Equations – Galerkin FEM

- preliminary results: different forms of the convective term, P_2/P_1





- rotational form
  - reconstructed pressure has boundary layers, inaccurate results

- schemes for solving the nonlinearity
- fixed point iteration

$$\begin{pmatrix} \mathbf{u}^{(m+1)} \\ p^{(m+1)} \end{pmatrix} = \begin{pmatrix} \mathbf{u}^{(m)} \\ p^{(m)} \end{pmatrix} - \vartheta \mathbf{N}_{\text{lin}}^{-1} \left(\begin{pmatrix} (\mathbf{f}, \mathbf{v}) - \mathbf{N}\left(\mathbf{u}^{(m)}; \mathbf{u}^{(m)}, p^{(m)}\right) \\ 0 \end{pmatrix} \right)$$

with

$$\mathbf{N}\left(\mathbf{w}; \mathbf{u}, p\right) = \begin{pmatrix} a(\mathbf{u}, \mathbf{v}) + n(\mathbf{w}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) \\ b(\mathbf{u}, q) \end{pmatrix}$$

$\mathbf{N}_{\text{lin}}$ – linear operator
$\vartheta \in (0, 1]$ – damping factor

- fixed point iteration
  - linear system to be solved

$$\mathbf{N}_{\text{lin}} \begin{pmatrix} \delta \mathbf{u}^{(m+1)} \\ \delta p^{(m+1)} \end{pmatrix} = \left(\begin{pmatrix} (\mathbf{f}, \mathbf{v}) - \mathbf{N}\left(\mathbf{u}^{(m)}; \mathbf{u}^{(m)}, p^{(m)}\right) \\ 0 \end{pmatrix} \right)$$

  - setting

$$\begin{pmatrix} \delta \mathbf{u}^{(m+1)} \\ \delta p^{(m+1)} \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{u}}^{(m+1)} - \mathbf{u}^{(m)} \\ \tilde{p}^{(m+1)} - p^{(m)} \end{pmatrix},$$

then

$$\mathbf{N}_{\text{lin}} \begin{pmatrix} \tilde{\mathbf{u}}^{(m+1)} \\ \tilde{p}^{(m+1)} \end{pmatrix} = \left(\begin{pmatrix} (\mathbf{f}, \mathbf{v}) - \mathbf{N}\left(\mathbf{u}^{(m)}; \mathbf{u}^{(m)}, p^{(m)}\right) \\ 0 \end{pmatrix} \right) + \mathbf{N}_{\text{lin}} \begin{pmatrix} \mathbf{u}^{(m)} \\ p^{(m)} \end{pmatrix}$$

5 The Stationary Navier–Stokes Equations – Galerkin FEM

- iteration with Stokes equations

$$\mathbf{N}_{\text{lin}} = \mathbf{N}\left(\mathbf{0};\tilde{\mathbf{u}}^{(m+1)},\tilde{p}^{(m+1)}\right)$$

- then

$$\begin{aligned}
&\begin{pmatrix} a(\tilde{\mathbf{u}}^{(m+1)},\mathbf{v}) + b(\mathbf{v},\tilde{p}^{(m+1)}) \\ b(\tilde{\mathbf{u}}^{(m+1)},q) \end{pmatrix} \\
&= \quad \begin{pmatrix} (\mathbf{f},\mathbf{v}) - a(\mathbf{u}^{(m)},\mathbf{v}) - n(\mathbf{u}^{(m)},\mathbf{u}^{(m)},\mathbf{v}) - b(\mathbf{v},\tilde{p}^{(m)}) \\ -b(\tilde{\mathbf{u}}^{(m)},q) \end{pmatrix} \\
&\qquad + \begin{pmatrix} a(\mathbf{u}^{(m)},\mathbf{v}) + b(\mathbf{v},\tilde{p}^{(m)}) \\ b(\tilde{\mathbf{u}}^{(m)},q) \end{pmatrix} = \begin{pmatrix} (\mathbf{f},\mathbf{v}) - n(\mathbf{u}^{(m)},\mathbf{u}^{(m)},\mathbf{v}) \\ 0 \end{pmatrix}
\end{aligned}$$

  - converges only if ν is sufficiently large
  - not recommended

- iteration with Oseen-type equations, Picard iteration

$$\mathbf{N}_{\text{lin}} = \mathbf{N}\left(\mathbf{u}^{(m)}; \tilde{\mathbf{u}}^{(m+1)}, \tilde{p}^{(m+1)}\right)$$

- then

$$\begin{aligned}
&\begin{pmatrix} a(\tilde{\mathbf{u}}^{(m+1)}, \mathbf{v}) + n(\mathbf{u}^{(m)}, \tilde{\mathbf{u}}^{(m+1)}, \mathbf{v}) + b(\mathbf{v}, \tilde{p}^{(m+1)}) \\ b(\tilde{\mathbf{u}}^{(m+1)}, q) \end{pmatrix} \\
&= \begin{pmatrix} (\mathbf{f}, \mathbf{v}) - a(\mathbf{u}^{(m)}, \mathbf{v}) - n(\mathbf{u}^{(m)}, \mathbf{u}^{(m)}, \mathbf{v}) - b(\mathbf{v}, \tilde{p}^{(m)}) \\ -b(\tilde{\mathbf{u}}^{(m)}, q) \end{pmatrix} \\
&\quad + \begin{pmatrix} a(\mathbf{u}^{(m)}, \mathbf{v}) + n(\mathbf{u}^{(m)}, \mathbf{u}^{(m)}, \mathbf{v}) + b(\mathbf{v}, \tilde{p}^{(m)}) \\ b(\tilde{\mathbf{u}}^{(m)}, q) \end{pmatrix} \\
&= \begin{pmatrix} (\mathbf{f}, \mathbf{v}) \\ 0 \end{pmatrix}
\end{aligned}$$

- iteration with Oseen-type equations, Picard iteration (cont.)
- different forms of nonlinear term

$$
\begin{aligned}
n_{\text{conv}}\left(\mathbf{u}^{(m)},\tilde{\mathbf{u}}^{(m+1)},\mathbf{v}\right) &= \left(\left(\mathbf{u}^{(m)}\cdot\nabla\right)\tilde{\mathbf{u}}^{(m+1)},\mathbf{v}\right),\\
n_{\text{div}}\left(\mathbf{u}^{(m)},\tilde{\mathbf{u}}^{(m+1)},\mathbf{v}\right) &= \left(\left(\mathbf{u}^{(m)}\cdot\nabla\right)\tilde{\mathbf{u}}^{(m+1)},\mathbf{v}\right)+\left(\left(\nabla\cdot\mathbf{u}^{(m)}\right)\tilde{\mathbf{u}}^{(m+1)},\mathbf{v}\right),\\
n_{\text{rot}}\left(\mathbf{u}^{(m)},\tilde{\mathbf{u}}^{(m+1)},\mathbf{v}\right) &= \left(\left(\nabla\times\mathbf{u}^{(m)}\right)\times\tilde{\mathbf{u}}^{(m+1)},\mathbf{v}\right)\\
n_{\text{skew}}\left(\mathbf{u}^{(m)},\tilde{\mathbf{u}}^{(m+1)},\mathbf{v}\right) &= \frac{1}{2}\Big[\left(\left(\mathbf{u}^{(m)}\cdot\nabla\right)\tilde{\mathbf{u}}^{(m+1)},\mathbf{v}\right)\\
&\quad -\left(\left(\mathbf{u}^{(m)}\cdot\nabla\right)\mathbf{v},\tilde{\mathbf{u}}^{(m+1)}\right)\Big]
\end{aligned}
$$

  - discussion: board
- widely used

- Newton's method
- linear operator is derivative of the nonlinear operator at the current position

$$\mathbf{N}_{\text{lin}} = D\mathbf{N}\begin{pmatrix} \mathbf{u}^{(m)} \\ p^{(m)} \end{pmatrix}$$

  - with Gâteaux derivative at $(\mathbf{u}, p)^T$

$$\begin{aligned} D\mathbf{N}\begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} &= \lim_{\varepsilon \to 0} \frac{\mathbf{N}(\mathbf{u} + \varepsilon\phi; \mathbf{u} + \varepsilon\phi, p + \varepsilon\psi) - \mathbf{N}(\mathbf{u}; \mathbf{u}, p)}{\varepsilon} \\ &= \mathbf{N}(\phi; \mathbf{u}, p) + \mathbf{N}(\mathbf{u}; \phi, p) + \mathbf{N}(\mathbf{u}, \mathbf{u}, \psi) \end{aligned}$$

  - inserting and collecting terms (board)

$$\begin{pmatrix} a(\tilde{\mathbf{u}}^{(m+1)}, \mathbf{v}) + n(\mathbf{u}^{(m)}, \tilde{\mathbf{u}}^{(m+1)}, \mathbf{v}) + n(\tilde{\mathbf{u}}^{(m+1)}, \mathbf{u}^{(m)}, \mathbf{v}) + b(\mathbf{v}, \tilde{p}^{(m+1)}) \\ b(\tilde{\mathbf{u}}^{(m+1)}, q) \end{pmatrix}$$

$$= \begin{pmatrix} (\mathbf{f}, \mathbf{v}) + n(\mathbf{u}^{(m)}, \mathbf{u}^{(m)}, \mathbf{v}) \\ 0 \end{pmatrix}$$

- Newton's method (cont.)
  - order of convergence of Newton's method expected to be better than of the Picard iteration if
    - the solution $(\mathbf{u}, p)$ is sufficiently smooth
    - the linear systems are solved sufficiently accurately
  - properties of term $n(\tilde{\mathbf{u}}^{(m+1)}, \mathbf{u}^{(m)}, \mathbf{v})$ not clear
  - sometimes used in practice

- implementation
  - same principal approach as for Stokes and Oseen equations
  - inf-sup stable finite elements lead to linear saddle point problems in fixed point iteration

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix}$$

- implementation
  - same principal approach as for Stokes and Oseen equations
  - inf-sup stable finite elements lead to linear saddle point problems in fixed point iteration

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix}$$

  - convective form of convective term
    - Picard iteration

$$A = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{11} & 0 \\ 0 & 0 & A_{11} \end{pmatrix}$$

    - Newton iteration

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

- residual-based (and other) stabilizations possible
  - better: solve time-dependent problem

6 The Time-Dependent NSE – Laminar Flows

- continuous equation

$$\begin{aligned} \partial_t \mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } (0,T]\times\Omega, \\ \nabla\cdot\mathbf{u} &= 0 && \text{in } [0,T]\times\Omega, \\ \mathbf{u}(0,\cdot) &= \mathbf{u}_0 && \text{in } \Omega, \end{aligned}$$

with

$$\mathbf{u} = \mathbf{0} \text{ in } [0,T]\times\Gamma$$

- weak or variational formulation obtained by
  - multiply Navier–Stokes equations with a suitable test function φ
  - integrate on $(0,T)\times\Omega$
  - apply integration by parts
- weak or variational formulation: find $\mathbf{u} \ : \ (0,T] \to H_0^1(\Omega)$ such that

$$
\begin{aligned}
&\int_0^T \Big[-(\mathbf{u},\partial_t\varphi) + Re^{-1}(\nabla\mathbf{u},\nabla\varphi) + ((\mathbf{u}\cdot\nabla)\mathbf{u},\varphi) \Big](\tau)\,d\tau \\
&= \quad \int_0^T (\mathbf{f},\varphi)(\tau)\,d\tau + (\mathbf{u}_0,\varphi(0,\cdot))
\end{aligned}
$$

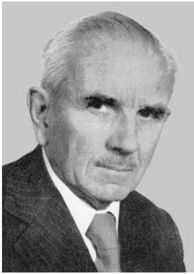
- weak or variational formulation obtained by
  - multiply Navier–Stokes equations with a suitable test function φ
  - integrate on $(0,T)\times\Omega$
  - apply integration by parts
- weak or variational formulation: find $\mathbf{u} \ : \ (0,T] \to H_0^1(\Omega)$ such that

$$\int_0^T \Big[-(\mathbf{u},\partial_t\varphi) + Re^{-1}(\nabla\mathbf{u},\nabla\varphi) + ((\mathbf{u}\cdot\nabla)\mathbf{u},\varphi)\Big](\tau)\,d\tau$$
$$= \int_0^T (\mathbf{f},\varphi)(\tau)\,d\tau + (\mathbf{u}_0,\varphi(0,\cdot))$$

- properties
  - no time derivative with respect to $\mathbf{u}$
  - no second order space derivative with respect to $\mathbf{u}$
  - the pressure vanished because

$$\int_\Omega \nabla p\cdot\varphi\,d\mathbf{x} = (\nabla p,\varphi) = \int_\Gamma p(\mathbf{s})\underbrace{\varphi(\mathbf{s})}_{=\mathbf{0}}\cdot\mathbf{n}(\mathbf{s})\,d\mathbf{s} - (p,\underbrace{\nabla\cdot\varphi}_{=0}) = 0$$

- mathematical analysis
  - 2d: existence and uniqueness of weak solution, Leary (1933), Hopf (1951)
  - 3d: existence of weak solution, Leary (1933), Hopf (1951)
- Jean Leray (1906 – 1998) Eberhard Hopf (1902 – 1983)



Uniqueness of weak solution of 3d Navier–Stokes equations is open problem !

- different form of the variational formulation

$$
\begin{array}{rcll}
(\partial_t \mathbf{u}, \mathbf{v}) + \nu\,(\nabla \mathbf{u}, \nabla \mathbf{v}) + n\,(\mathbf{u}, \mathbf{u}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, p) & = & (\mathbf{f}, \mathbf{v}) & \forall\, \mathbf{v} \in V, \\
-\,(\nabla \cdot \mathbf{u}, q) & = & 0 & \forall\, q \in Q,
\end{array}
$$

and $\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x})$

- different form of the variational formulation

$$\begin{aligned} (\partial_t \mathbf{u}, \mathbf{v}) + \nu (\nabla \mathbf{u}, \nabla \mathbf{v}) + n(\mathbf{u}, \mathbf{u}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}) \quad \forall\, \mathbf{v} \in V, \\ -(\nabla \cdot \mathbf{u}, q) &= 0 \quad \forall\, q \in Q, \end{aligned}$$

  and $\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x})$

- stability of velocity (board)

$$\|\mathbf{u}(T)\|_{L^2(\Omega)}^2 + \nu \|\nabla \mathbf{u}\|_{L^2(0,T;L^2(\Omega))}^2 \le \|\mathbf{u}(0)\|_{L^2(\Omega)}^2 + \frac{1}{\nu} \|\mathbf{f}\|_{L^2(0,T;H^{-1}(\Omega))}^2$$

- implicit θ-schemes as semi discretization in time
  - $\Delta t_{n+1} = t_{n+1} - t_n$
  - subscript k for quantities at time level k

$$
\begin{aligned}
\mathbf{u}_{k+1} + \theta_1 \Delta t_{n+1} \left[-\nu \Delta \mathbf{u}_{k+1} + (\mathbf{u}_{k+1} \cdot \nabla)\, \mathbf{u}_{k+1} \right] + \Delta t_{k+1} \nabla p_{k+1} & \\
= \quad & \mathbf{u}_k - \theta_2 \Delta t_{n+1} \left[-\nu \nabla \cdot \Delta \mathbf{u}_k + (\mathbf{u}_k \cdot \nabla)\, \mathbf{u}_k \right] + \theta_3 \Delta t_{n+1} \mathbf{f}_k \\
& + \theta_4 \Delta t_{n+1} \mathbf{f}_{k+1}, \\
\nabla \cdot \mathbf{u}_{k+1} \quad = \quad & 0,
\end{aligned}
$$

- implicit θ-schemes as semi discretization in time
  - $\Delta t_{n+1} = t_{n+1} - t_n$
  - subscript k for quantities at time level k

$$
\begin{aligned}
\mathbf{u}_{k+1} + \theta_1 \Delta t_{n+1}\left[-\nu \Delta \mathbf{u}_{k+1} + (\mathbf{u}_{k+1}\cdot\nabla)\,\mathbf{u}_{k+1}\right] + \Delta t_{k+1}\nabla p_{k+1} & \\
= \quad & \mathbf{u}_k - \theta_2 \Delta t_{n+1}\left[-\nu \nabla\cdot\Delta \mathbf{u}_k + (\mathbf{u}_k\cdot\nabla)\,\mathbf{u}_k\right] + \theta_3 \Delta t_{n+1}\mathbf{f}_k \\
& + \theta_4 \Delta t_{n+1}\mathbf{f}_{k+1}, \\
\nabla\cdot\mathbf{u}_{k+1} \quad = \quad & 0,
\end{aligned}
$$

- one-step θ-schemes: $n = k$

	θ_1	θ_2	θ_3	θ_4	t_k	t_{k+1}	Δt_{k+1}	order
forward Euler scheme	0	1	1	0	t_n	t_{n+1}	Δt_{n+1}	
backward Euler scheme (BWE)	1	0	0	1	t_n	t_{n+1}	Δt_{n+1}	1
Crank–Nicolson scheme (CN)	0.5	0.5	0.5	0.5	t_n	t_{n+1}	Δt_{n+1}	2

- fractional-step θ-scheme [1]
  - three-step scheme
  - two variants

$$\theta = 1 - \frac{\sqrt{2}}{2}, \quad \tilde{\theta} = 1 - 2\theta, \quad \tau = \frac{\tilde{\theta}}{1-\theta}, \quad \eta = 1 - \tau$$

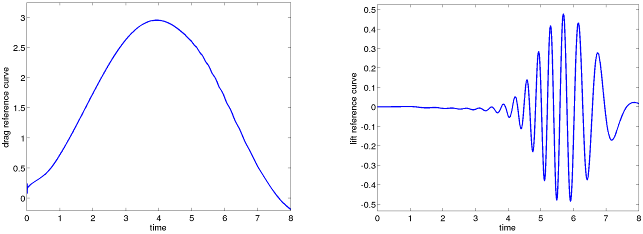
	θ_1	θ_2	θ_3	θ_4	t_k	t_{k+1}	Δt_{k+1}	order
FS0	$\tau\theta$	$\eta\theta$	$\eta\theta$	$\tau\theta$	t_n	$t_n + \theta\Delta t_{n+1}$	$\theta\Delta t_{n+1}$	
	$\eta\tilde{\theta}$	$\tau\tilde{\theta}$	$\tau\tilde{\theta}$	$\eta\tilde{\theta}$	$t_n + \theta\Delta t_{n+1}$	$t_{n+1} - \theta\Delta t_{n+1}$	$\tilde{\theta}\Delta t_{n+1}$	2
	$\tau\theta$	$\eta\theta$	$\eta\theta$	$\tau\theta$	$t_{n+1} - \theta\Delta t_{n+1}$	t_{n+1}	$\theta\Delta t_{n+1}$	
FS1	$\tau\theta$	$\eta\theta$	θ	0	t_n	$t_n + \theta\Delta t_{n+1}$	$\theta\Delta t_{n+1}$	
	$\eta\tilde{\theta}$	$\tau\tilde{\theta}$	0	$\tilde{\theta}$	$t_n + \theta\Delta t_{n+1}$	$t_{n+1} - \theta\Delta t_{n+1}$	$\tilde{\theta}\Delta t_{n+1}$	2
	$\tau\theta$	$\eta\theta$	θ	0	$t_{n+1} - \theta\Delta t_{n+1}$	t_{n+1}	$\theta\Delta t_{n+1}$	

[1] Bristeau, Glowinski, Periaux: Finite elements in physics, North-Holland, 73–187, 1986

- popular approaches: BWE, CN
- stability
  - BWE, FS0, FS1: strongly A-stable
  - CN: A-stable
- FS1 less expensive than FS0 if computation of right hand side costly

- popular approaches: BWE, CN
- stability
  - BWE, FS0, FS1: strongly A-stable
  - CN: A-stable
- FS1 less expensive than FS0 if computation of right hand side costly
- number of papers with finite element error estimates available
  - proofs become long
  - same techniques as for steady-state problems + Gronwall's lemma

- flow around a cylinder
  - reference curves for drag and lift [1]



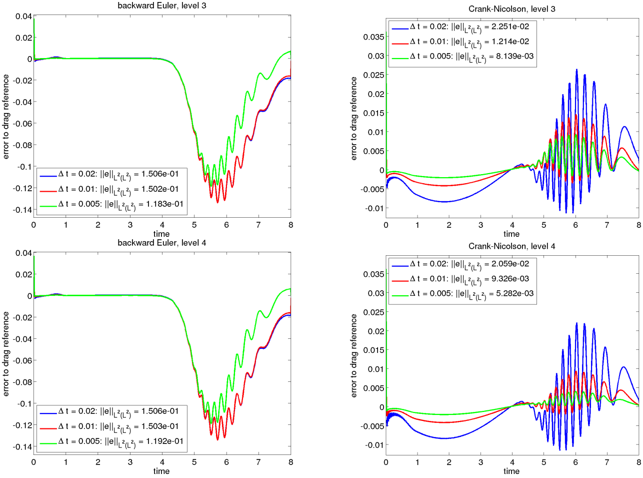
[1] J., Rang, CMAME 199, 514–524, 2010

- refinement in space with Q_2/P_1^{disc}

	P_2/P_1			Q_2/P_1^{disc}		
level	velocity	pressure	all	velocity	pressure	all
3	25 408	3248	28 656	27 232	9984	37 216
4	100 480	12 704	113 184	107 712	39 936	147 648
5	399 616	50 240	449 856	428 416	159 744	588 160

- refinement in time: $\Delta t \in \{0.02, 0.01, 0.005\}$

- error to the reference curve for the drag coefficient



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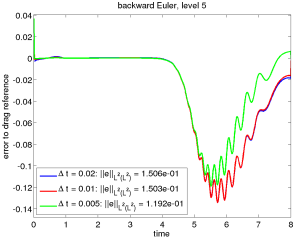
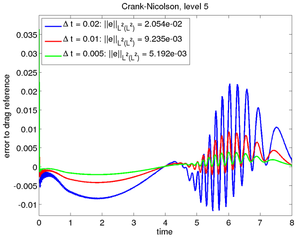
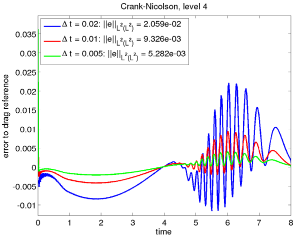
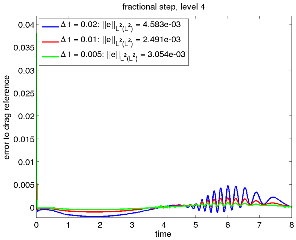
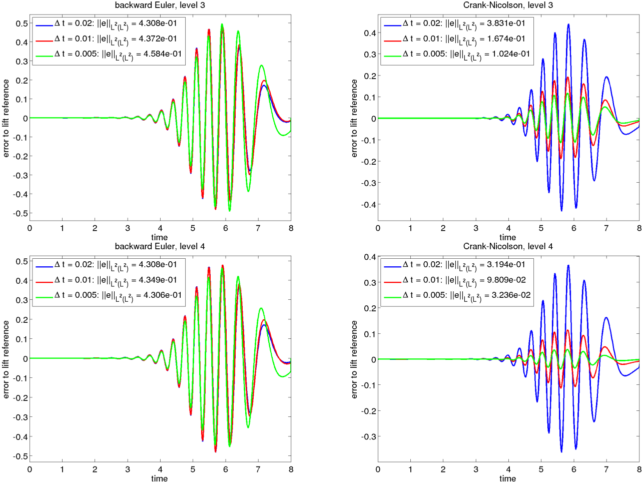




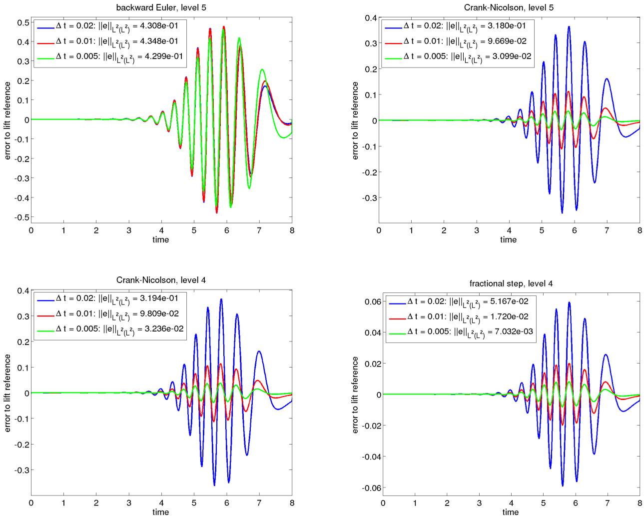




- error to the reference curve for the lift coefficient

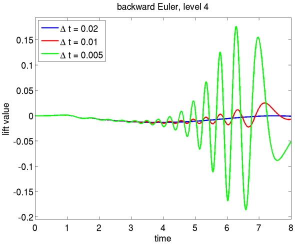


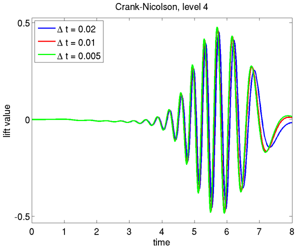
- error to the reference curve for the lift coefficient



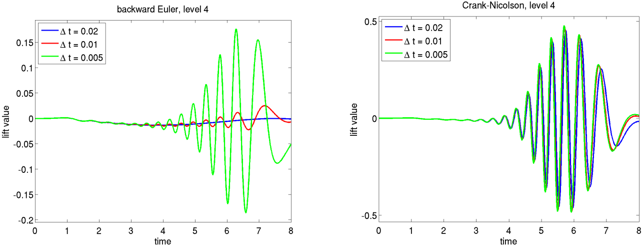
6 The Time-Dependent NSE – Laminar Flows

- temporal evolution of lift coefficient





- temporal evolution of lift coefficient



- BWE much to inaccurate (dissipative)

- projection method
  - motivation: schemes without need to solve (nonlinear) saddle point problems
  - survey in [1]

[1] Guermond, Minev, Shen, CMAME 195, 6011–6045, 2006

- idea: decoupled NSE to obtain separate equations for velocity and pressure
  - approximation of time derivative given (q-step scheme)

$$\partial_t \mathbf{u}(t_{n+1}) \approx \frac{1}{\Delta t}\left(\tau_q \mathbf{u}_{n+1} + \sum_{i=0}^{q-1} \tau_j \mathbf{u}_{n-j}\right), \quad \sum_{i=0}^{q} \tau_j = 0$$

  - equation for intermediate velocity: given $\hat{p}$ or $\nabla \hat{p}$

$$\frac{1}{\Delta t}\left(\tau_q \tilde{\mathbf{u}}_{n+1} + \sum_{i=0}^{q-1} \tau_j \mathbf{u}_{n-j}\right) - \nu \Delta \tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}} = \mathbf{f} - \nabla \hat{p} \quad \text{in } (0,T] \times \Omega$$

  - correction step for divergence-free velocity

$$\begin{aligned} \frac{1}{\Delta t}\left(\tau_q \mathbf{u}_{n+1} - \tau_q \tilde{\mathbf{u}}_{n+1}\right) + \nabla \varphi(\tilde{\mathbf{u}}) + \nabla p &= \nabla \hat{p} \quad \text{in } (0,T] \times \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in } [0,T] \times \Omega \end{aligned}$$

$\varphi(\cdot)$ – given function

- velocity computed in projection step is $L^2(\Omega)$ projection of $\tilde{\mathbf{u}}_{n+1}$ into

$$H_{\text{div}}(\Omega) = \left\{\mathbf{v} \in L^2(\Omega),\ \nabla\cdot\mathbf{v} \in L^2(\Omega)\ :\ \nabla\cdot\mathbf{v} = 0 \text{ and } \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma\right\}$$

- non-incremental pressure-correction scheme
  - $\hat{p} = 0$, $\varphi(\cdot) = 0$
  - proposed in [1,2]
  - with backward Euler
- intermediate velocity

$$\tilde{\mathbf{u}}_{n+1} + \Delta t_{n+1}\left(-\nu\Delta\tilde{\mathbf{u}}_{n+1} + (\tilde{\mathbf{u}}_n\cdot\nabla)\tilde{\mathbf{u}}_{n+1}\right) = \mathbf{u}_n + \Delta t_{n+1}\mathbf{f}_{n+1} \quad \text{in } \Omega$$

with $\tilde{\mathbf{u}}_{n+1} = \mathbf{0}$ on Γ

- projection step

$$\begin{aligned} \mathbf{u}_{n+1} + \Delta t_{n+1}\nabla p_{n+1} &= \tilde{\mathbf{u}}_{n+1} && \text{in } \Omega, \\ \nabla\cdot\mathbf{u}_{n+1} &= 0 && \text{in } \Omega, \\ \mathbf{u}_{n+1}\cdot\mathbf{n} &= 0 && \text{on } \Gamma \end{aligned}$$

[1] Chorin, Math. Comp. 22, 745–762, 1968

[2] Temam, Arch. Rational Mech. Anal. 33, 377–385, 1969

6 The Time-Dependent NSE – Laminar Flows

- non-incremental pressure-correction scheme (cont.)
- taking divergence of projection step

$$\nabla \cdot \nabla p_{n+1} = \Delta p_{n+1} = \frac{1}{\Delta t_{n+1}} \nabla \cdot \tilde{\mathbf{u}}_{n+1}$$

  - Poisson equation for the pressure
  - boundary condition

$$\nabla p_{n+1} \cdot \mathbf{n} = -\frac{1}{\Delta t_{n+1}} \left(\mathbf{u}_{n+1} - \tilde{\mathbf{u}}_{n+1} \right) \cdot \mathbf{n} = 0$$

- non-incremental pressure-correction scheme (cont.)
- taking divergence of projection step

$$\nabla \cdot \nabla p_{n+1} = \Delta p_{n+1} = \frac{1}{\Delta t_{n+1}} \nabla \cdot \tilde{\mathbf{u}}_{n+1}$$

  - Poisson equation for the pressure
  - boundary condition

$$\nabla p_{n+1} \cdot \mathbf{n} = -\frac{1}{\Delta t_{n+1}} \left(\mathbf{u}_{n+1} - \tilde{\mathbf{u}}_{n+1}\right) \cdot \mathbf{n} = 0$$

- error estimates: $(\overline{\mathbf{u}}, \overline{p})$ result of projection step

$$\|p - \overline{p}\|_{l^\infty\left(0,T;L^2(\Omega)\right)} + \|\mathbf{u} - \tilde{\mathbf{u}}\|_{l^\infty\left(0,T;H^1(\Omega)\right)} \le C\left(\mathbf{u}, p, T\right) \Delta t^{1/2}$$

  if in addition domain has regularity property

$$\|\mathbf{u} - \overline{\mathbf{u}}\|_{l^\infty\left(0,T;L^2(\Omega)\right)} + \|\mathbf{u} - \tilde{\mathbf{u}}\|_{l^\infty\left(0,T;L^2(\Omega)\right)} \le C\left(\mathbf{u}, p, T\right) \Delta t$$

- non-incremental pressure-correction scheme (cont.)
  - inf-sup stable finite elements not necessary
  - however, spurious oscillations may appear if the time step becomes too small
  - low orders of convergence
  - splitting error is $\mathcal{O}(\Delta t) \Longrightarrow$ first order time stepping scheme sufficient
  - artificial Neumann boundary condition for the pressure induces a numerical boundary layer

- standard incremental pressure-correction scheme
  - $\hat{p} = p_n,\ \varphi(\cdot) = 0$
  - with BDF2
- intermediate velocity

$$\begin{aligned} 3\tilde{\mathbf{u}}_{n+1} + 2\Delta t \left(-\nu\Delta\tilde{\mathbf{u}}_{n+1} + (\tilde{\mathbf{u}}_n \cdot \nabla)\tilde{\mathbf{u}}_{n+1}\right) & \\ = \quad 4\mathbf{u}_n - \mathbf{u}_{n-1} + 2\Delta t \left(\mathbf{f}_{n+1} - \nabla p_n\right) & \quad \text{in } \Omega, \end{aligned}$$

with $\tilde{\mathbf{u}}_{n+1} = \mathbf{0}$ on Γ

- projection step

$$\begin{aligned} 3\mathbf{u}_{n+1} + 2\Delta t \nabla \left(p_{n+1} - p_n\right) &= 3\tilde{\mathbf{u}}_{n+1} && \text{in } \Omega, \\ \nabla \cdot \mathbf{u}_{n+1} &= 0 && \text{in } \Omega, \\ \mathbf{u}_{n+1} \cdot \mathbf{n} &= 0 && \text{on } \Gamma \end{aligned}$$

- standard incremental pressure-correction scheme (cont.)
- taking divergence of projection step

$$\Delta(p_{n+1} - p_n) = \frac{3}{2\Delta t} \nabla \cdot \tilde{\mathbf{u}}_{n+1} \quad \text{in } \Omega$$

  - Poisson equation for the pressure update
  - boundary condition

$$\nabla(p_{n+1} - p_n) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma$$

- standard incremental pressure-correction scheme (cont.)
- taking divergence of projection step

$$\Delta(p_{n+1}-p_n)=\frac{3}{2\Delta t}\nabla\cdot\tilde{\mathbf{u}}_{n+1}\quad\text{in }\Omega$$

  - Poisson equation for the pressure update
  - boundary condition

$$\nabla(p_{n+1}-p_n)\cdot\mathbf{n}=0\quad\text{on }\Gamma$$

- error estimates, with appropriate initial step, $(\overline{\mathbf{u}},\overline{p})$ result of projection step

$$\|p-\overline{p}\|_{l^\infty(0,T;L^2(\Omega))}+\|\mathbf{u}-\tilde{\mathbf{u}}\|_{l^\infty(0,T;H^1(\Omega))}\le C(\mathbf{u},p,T)\Delta t$$

  if in addition domain has regularity property

$$\|\mathbf{u}-\overline{\mathbf{u}}\|_{l^\infty(0,T;L^2(\Omega))}+\|\mathbf{u}-\tilde{\mathbf{u}}\|_{l^2(0,T;L^2(\Omega))}\le C(\mathbf{u},p,T)\Delta t^2$$

- standard incremental pressure-correction scheme (cont.)
  - similar estimates for Crank–Nicolson scheme
  - splitting error is $\mathscr{O}\left(\Delta t^2\right) \Longrightarrow$ second order time stepping scheme sufficient
  - artificial Neumann boundary condition for the pressure induces a numerical boundary layer

- rotational incremental pressure-correction scheme
  - $\hat{p} = p_n$, $\varphi(\tilde{\mathbf{u}}) = \nu \nabla \cdot \tilde{\mathbf{u}}_{n+1}$
  - with BDF2
- intermediate velocity

$$\begin{aligned} & 3\tilde{\mathbf{u}}_{n+1} + 2\Delta t \left(-\nu \Delta \tilde{\mathbf{u}}_{n+1} + (\tilde{\mathbf{u}}_n \cdot \nabla)\tilde{\mathbf{u}}_{n+1}\right) \\ = \quad & 4\mathbf{u}_n - \mathbf{u}_{n-1} + 2\Delta t \left(\mathbf{f}_{n+1} - \nabla p_n\right) \quad \text{in } \Omega, \end{aligned}$$

with $\tilde{\mathbf{u}}_{n+1} = \mathbf{0}$ on Γ

- projection step

$$\begin{aligned} 3\mathbf{u}_{n+1} + 2\Delta t \nabla \left(p_{n+1} - p_n\right) &= 3\tilde{\mathbf{u}}_{n+1} - 2\nu\Delta t \nabla \left(\nabla \cdot \tilde{\mathbf{u}}_{n+1}\right) && \text{in } \Omega, \\ \nabla \cdot \mathbf{u}_{n+1} &= 0 && \text{in } \Omega, \\ \mathbf{u}_{n+1} \cdot \mathbf{n} &= 0 && \text{on } \Gamma \end{aligned}$$

- rotational incremental pressure-correction scheme (cont.)
- taking divergence of projection step

$$\Delta \tilde{p}_n = \frac{3}{2\Delta t} \nabla \cdot \tilde{\mathbf{u}}_{n+1} \quad with \quad \tilde{p}_n = p_{n+1} - p_n + \nu \nabla \cdot \tilde{\mathbf{u}}_{n+1}$$

  - Poisson equation for the modified pressure
  - boundary condition

$$\nabla p_{n+1} \cdot \mathbf{n} = \left(\mathbf{f}_{n+1} - \nu \nabla \times \nabla \times \mathbf{u}_{n+1} \right) \cdot \mathbf{n} \text{ on } \Gamma$$

- rotational incremental pressure-correction scheme (cont.)
- taking divergence of projection step

$$\Delta \tilde{p}_n = \frac{3}{2\Delta t}\nabla \cdot \tilde{\mathbf{u}}_{n+1} \quad with \quad \tilde{p}_n = p_{n+1} - p_n + \nu \nabla \cdot \tilde{\mathbf{u}}_{n+1}$$

  - Poisson equation for the modified pressure
  - boundary condition

$$\nabla p_{n+1} \cdot \mathbf{n} = \left(\mathbf{f}_{n+1} - \nu \nabla \times \nabla \times \mathbf{u}_{n+1}\right) \cdot \mathbf{n} \text{ on } \Gamma$$

- error estimates, with appropriate initial step, $(\overline{\mathbf{u}}, \overline{p})$ result of projection step

$$\|p - \overline{p}\|_{l^2\left(0,T;L^2(\Omega)\right)} + \|\mathbf{u} - \tilde{\mathbf{u}}\|_{l^2\left(0,T;H^1(\Omega)\right)} + \|\mathbf{u} - \overline{\mathbf{u}}\|_{l^2\left(0,T;H^1(\Omega)\right)} \le C\left(\mathbf{u}, p, T\right)\Delta t^{3/2}$$

  if in addition domain has regularity property

$$\|\mathbf{u} - \overline{\mathbf{u}}\|_{l^2\left(0,T;L^2(\Omega)\right)} + \|\mathbf{u} - \tilde{\mathbf{u}}\|_{l^2\left(0,T;L^2(\Omega)\right)} \le C\left(\mathbf{u}, p, T\right)\Delta t^2$$

- rotational incremental pressure-correction scheme (cont.)
  - equivalent formulation of velocity step

$$
\begin{aligned}
3\mathbf{u}_{n+1} + 2\Delta t \left(\nu \nabla \times \nabla \times \mathbf{u}_{n+1} + (\tilde{\mathbf{u}}_n \cdot \nabla)\tilde{\mathbf{u}}_{n+1} + \nabla p_{n+1}\right) & \\
= \quad & 4\mathbf{u}_n - \mathbf{u}_{n-1} + 2\Delta t \mathbf{f}_{n+1} && \text{in } \Omega, \\
\nabla \cdot \mathbf{u}_{n+1} \quad = \quad & 0 && \text{in } \Omega
\end{aligned}
$$

  - boundary condition for the pressure is consistent, can be derived from the Navier–Stokes equations

- only $\tilde{\mathbf{u}}_{n+1}$ needed in implementation
- first experience with non-incremental and standard incremental scheme: very inaccurate at boundaries (bad drag and lift coefficients)

Thank you for your attention !

http://www.wias-berlin.de/people/john/