1 Introduction

Partial differential equations often arise as a result of mathematical **modeling of natural phenomena**. Since the universe is composed of elementary particles which can be described only with a certain probability (due to Heisenberg's principle of uncertainty and Born's interpretation of wavefunction), every equation describing a physical phenomenon must be imprecise. Differential equations are obtained when we approximate the discrete structure of mass by a continuum (for example, we can think of temperature as being defined at all points in an object, even between atoms).

Since each mathematical model is only an approximation, it is important to study the reliability of the model. This usually means to study the **existence and uniqueness of solutions** to the model equation. Often the equation does not have solution in the classical sense, so we reformulate the equation in an integral form and define a **weak solution**.

Only if we know that the solution exists, has it sense to try to calculate it by some **numerical method**. Computers can handle only finite-dimensional problems and therefore, it is necessary to discretize the infinite-dimensional problem represented by the differential equation.

The flow-chart of mathematical modeling and numerical computation is as follows

$$\boxed{\text{physical reality}} \overset{e_0}{\longleftrightarrow} \boxed{\text{mathematical model}} \overset{e_1}{\longleftrightarrow} \boxed{\text{discrete model}} \overset{e_2}{\longleftrightarrow} \boxed{\text{numerical results}}$$

The errors e_0, e_1, e_2 have the following meaning:

- e_0 expresses the fact that no equation describes the reality exactly
- e_1 appears due to discretization of the infinite-dimensional mathematical model into a finite-dimensional model, so that it can be computed using computers, e.g,,
  - error of finite element method
  - error of numerical quadrature
  - error of approximation of curved boundary by a polygon
  - error of approximation of nonlinearities
  - error of approximation of initial or boundary conditions
- e_2 includes errors arising during the computation itself (e.g., rounding error, error of iteration)

In this lecture we will study the cases of

- nonlinear stationary magnetic field
- nonlinear problem of heat radiation

and focus on

- weak formulation of the model equation
- the proof of existence of solution to these problems
- constructing a practical method for numerical computation of these problems
- examining the convergence and the speed of convergence of the numerical method

2 Equation for stationary magnetic field

2.1 Maxwell's equations

Maxwell's equation describe how electric chrages and currents act as sources for the electric and magnetic fields.

- **Gauss's law** and **Gauss's law for magnetism** describe how the fields emanate from charges.
- **Ampère's law** describes how the magnetic field circulates around electric currents and time-varying electric field.
- **Faraday's law** describes how the electric field circulates around time-varying magnetic fields.

$$\begin{aligned}
\operatorname{rot} H &= J + \frac{\partial D}{\partial t}, && \text{(Ampère's law)} \\
\operatorname{rot} E &= -\frac{\partial B}{\partial t}, && \text{(Faraday's law)} \\
\operatorname{div} D &= \rho, && \text{(Gauss's law)} \\
\operatorname{div} B &= 0, && \text{(Gauss's law for electromagnetism)} \\
D &= \varepsilon E, && \text{(constitutive relation)} \\
B &= \mu H. && \text{(constitutive relation)}
\end{aligned}$$

Here, the symbols have the following meaning:

H	...	magnetic field intensity
E	...	electric field intensity
D	...	electric induction
B	...	magnetic induction
J	...	current density
ρ	...	charge density
μ	...	permeability
ε	...	permittivity .

Let us assume that all the quantities are time-independent. Then the above system of equations splits into two independent systems:

$$\begin{aligned}
\operatorname{rot} E &= 0, \\
\operatorname{div} D &= \rho, \\
D &= \varepsilon E,
\end{aligned}$$

and

$$\operatorname{rot} H = J, \tag{1}$$

$$\operatorname{div} B = 0, \tag{2}$$

$$B = \mu H. \tag{3}$$

Note that the first system contains only electric quantities E and D, while the second system contains only magnetic quantities H and B.

Various types of operators appeared in the above explanation and therefore, we review some basic facts from vector analysis.

Review. Basic operators "div", "∇", "rot" and "Δ".

- "div" or "$\nabla\cdot$" of a vector field a is a scalar which describes the rate at which density exits a given region of space (or the rate of change in volume):

$$\text{div } a = \sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i} \qquad \Big(\text{div } a = \text{div } (a_1, a_2) = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} \quad \text{for 2-dim case}\Big)$$

- "∇" or "grad" of a scalar function u is a vector which describes the rate of change in the values of the scalar functions (it points in the direction of its greatest rate of increase):

$$\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n})^T \qquad \Big(\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2})^T \quad \text{for 2-dim case}\Big)$$

- "rot" or "$\nabla\times$" of a vector field v is a vector expressing the quantity and direction of vortices in the field (usually used only in 3-dim case):

$$\text{rot } v = \text{rot } (v_1, v_2, v_3) = (\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2})^T = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

- The operator "rot" is sometimes also called "curl" but here we shall use the name "curl" for scalar functions in two dimensions, defined as

$$\text{curl}\, v = (\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1})^T$$

- "Δ" is called Laplacian and is defined for a scalar u as follows:

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \qquad \Big(\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \quad \text{for 2-dim case}\Big)$$

Ex. Let f be a twice continuously differentiable scalar function defined in a three-dimensional domain. Then it holds that $\text{rot}\,(\nabla f)$ is a zero vector.

Proof.

$$\begin{aligned}
\text{rot}\,(\nabla f) &= \text{rot}\Big[\Big(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\Big)^T\Big] \\
&= \Big(\frac{\partial}{\partial x_2}(\frac{\partial f}{\partial x_3}) - \frac{\partial}{\partial x_3}(\frac{\partial f}{\partial x_2}), \frac{\partial}{\partial x_3}(\frac{\partial f}{\partial x_1}) - \frac{\partial}{\partial x_1}(\frac{\partial f}{\partial x_3}), \frac{\partial}{\partial x_1}(\frac{\partial f}{\partial x_2}) - \frac{\partial}{\partial x_2}(\frac{\partial f}{\partial x_1})\Big)^T \\
&= \Big(\frac{\partial^2 f}{\partial x_2 \partial x_3} - \frac{\partial^2 f}{\partial x_3 \partial x_2}, \frac{\partial^2 f}{\partial x_3 \partial x_1} - \frac{\partial^2 f}{\partial x_1 \partial x_3}, \frac{\partial^2 f}{\partial x_1 \partial x_2} - \frac{\partial^2 f}{\partial x_2 \partial x_1}\Big)^T
\end{aligned}$$

Since the function f is twice continuously differentiable, we can change the order of differentiation and obtain the result.