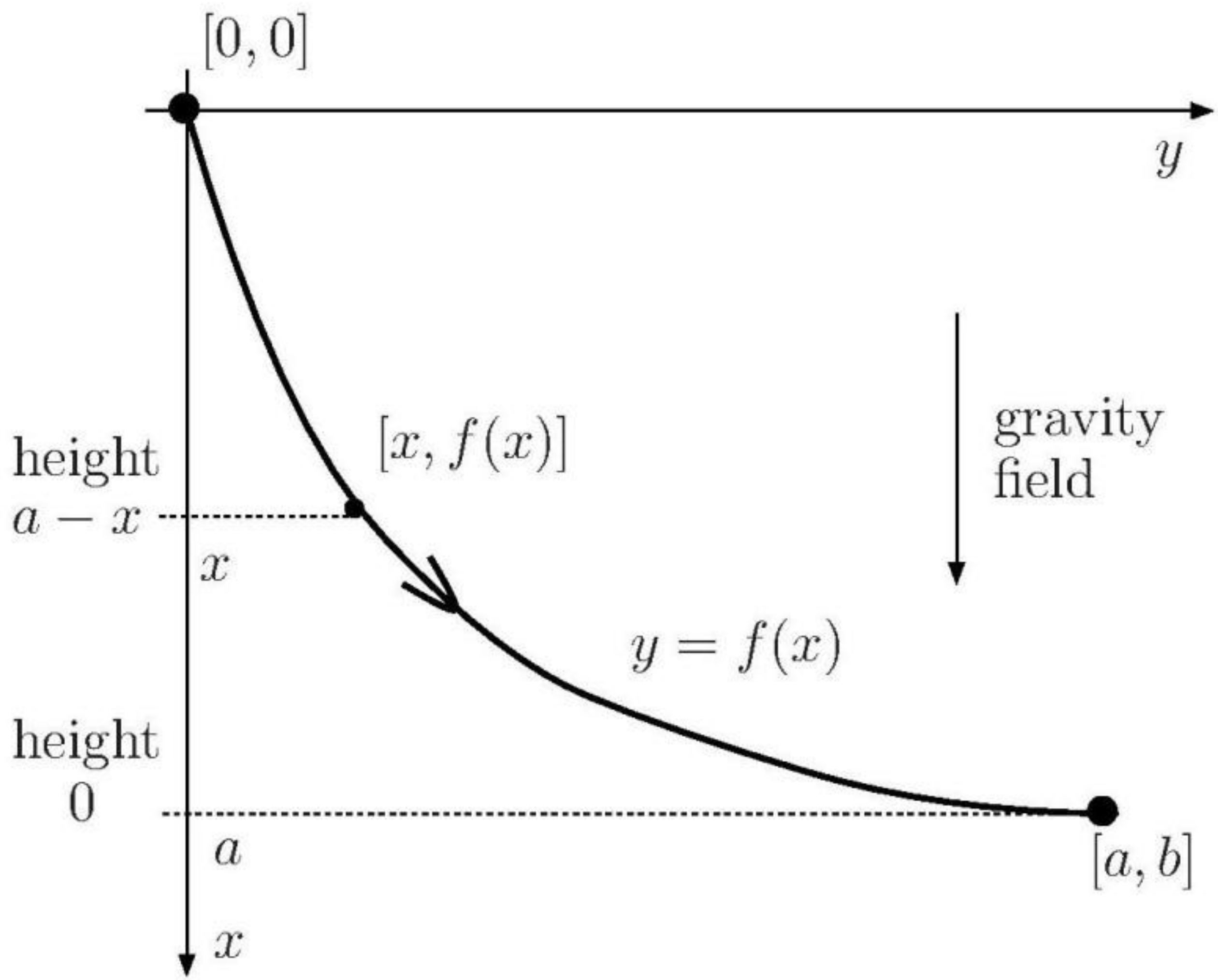
Nonlinear PDEs

12th lecture

Brachystochrone problem



Minimization problem

Find $\tilde{f} \in C^1[0,a]$ so that $\tilde{f}(0) = 0,\ \tilde{f}(a) = b$
yielding the minimum of

$$T(f) = \int_0^a \frac{\sqrt{1 + (f'(x))^2}}{\sqrt{2gx}}\, dx$$

Find $\min_{g \in Y} T(g)$, $\quad T(g) = \int_0^a \frac{\sqrt{1 + (g'(x) + \frac{b}{a})^2}}{\sqrt{x}}\, dx,$

where Y is the linear space

$$Y = \{g \in C^1[0,a],\ \ g(0) = g(a) = 0\}$$

Euler-Lagrange equation

$$\frac{d}{d\varepsilon} T(g + \varepsilon\varphi)|_{\varepsilon=0} = 0$$

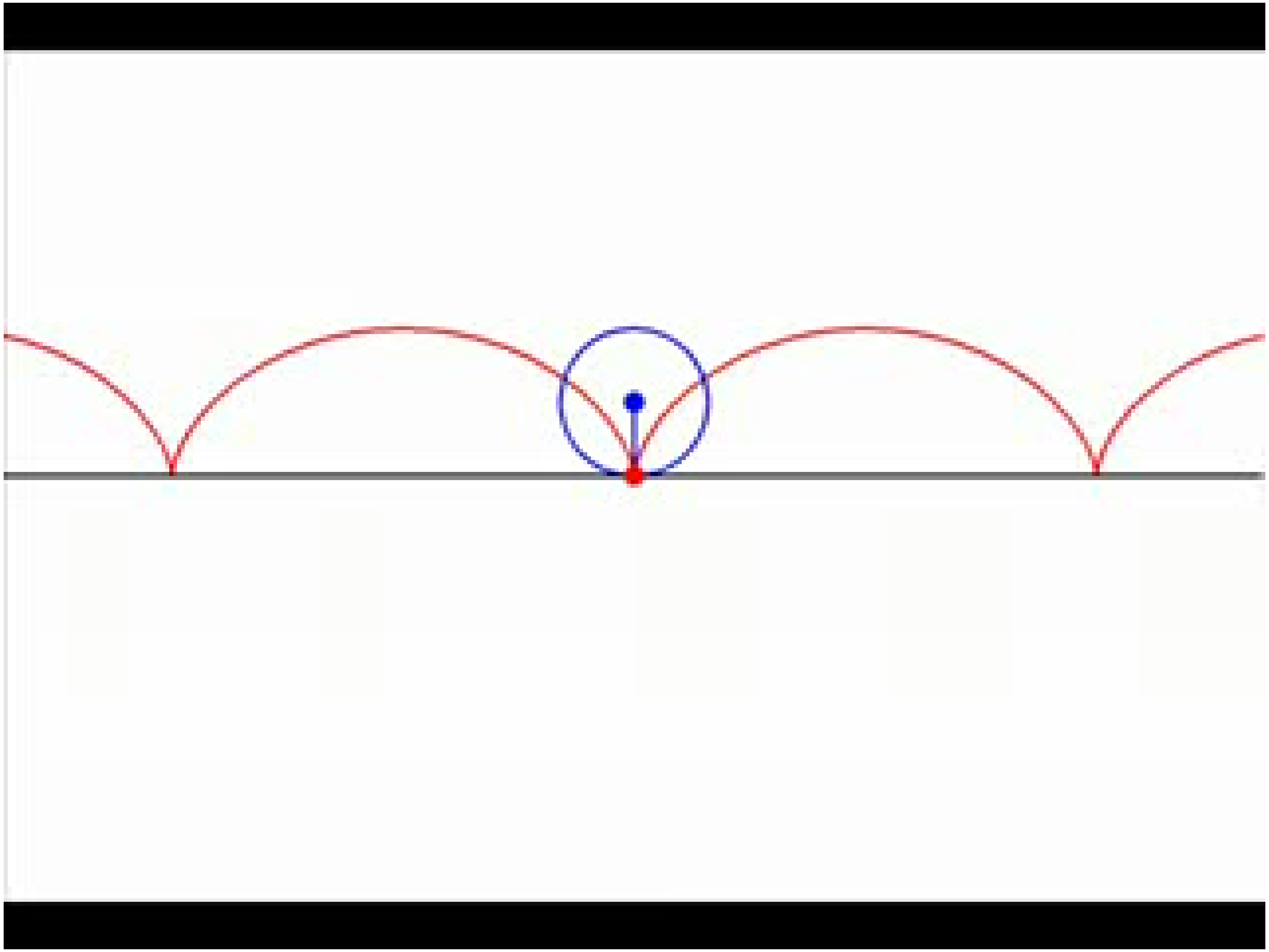
$$\boxed{\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\,\frac{f'(x)}{\sqrt{1+(f'(x))^2}}\right) = 0 \qquad \forall x \in (0,a)}$$

Integration of the equation

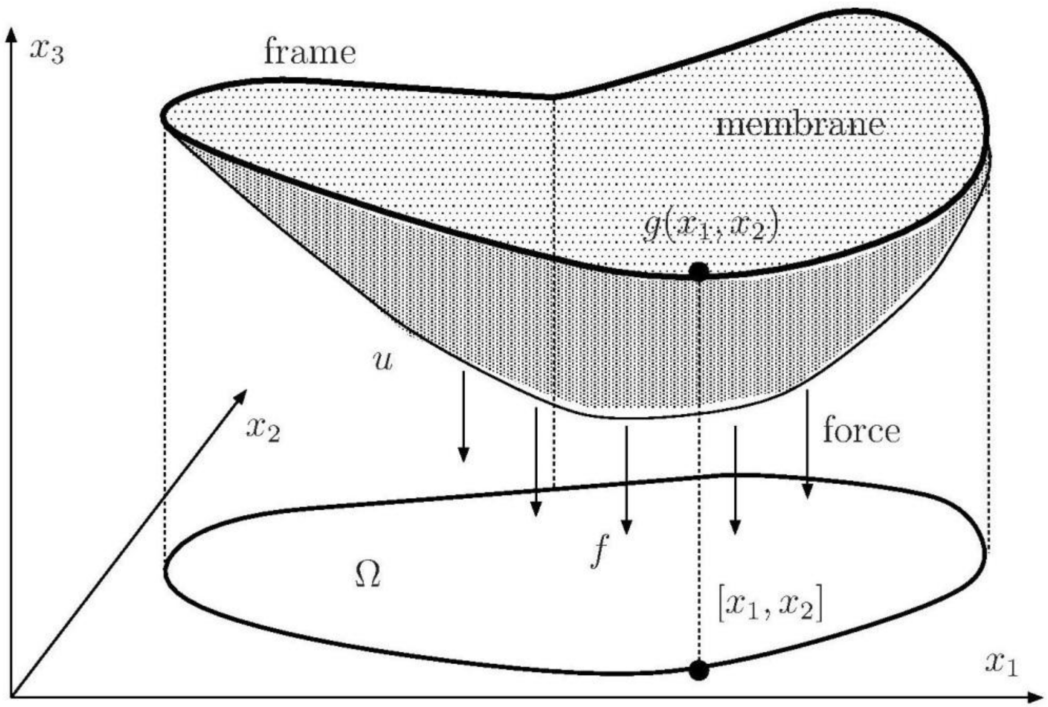
$$\begin{aligned}
\frac{1}{\sqrt{x}}\frac{f'(x)}{\sqrt{1+(f'(x))^2}} &= \sqrt{C}, \qquad C>0 \ \text{ integration constant} \\
\frac{f'(x)}{\sqrt{1+(f'(x))^2}} &= \sqrt{Cx} \\
\frac{(f'(x))^2}{1+(f'(x))^2} &= Cx \\
f'(x) &= \sqrt{\frac{Cx}{1-Cx}} \\
f(x) &= \frac{1}{C}\left[\arctan\sqrt{\frac{Cx}{1-Cx}}-\sqrt{Cx(1-Cx)}\right]+k
\end{aligned}$$

$f(0)=0,\ f(a)=b$ ⇨ determine $k=0$ and C

Cycloid



Membrane problem



Membrane energy:

$$J(u) = \int_\Omega [\frac{1}{2}|\nabla u|^2 - uf]\, dx_1\, dx_2$$

The stationary shape u satisfies

$$J(u) = \min_{w \in \mathcal{A}} J(w), \qquad \mathcal{A} = \{w \in C^2(\bar{\Omega}) \mid w = g \text{ on } \partial\Omega\}$$

Dirichlet's principle

$$u \in C^2(\bar{\Omega}) \text{ is a solution of } \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

$$\Updownarrow$$

$$u \in \mathcal{A} \text{ satisfies } J(u) = \min_{w \in \mathcal{A}} J(w)$$