Finite Difference Method (FDM) and Finite Element Method (FEM)

Afifah Maya Iknaningrum

May 4, 2018

0.1 Introduction

Partial Differential Equation (PDE) is an equation with two or more variables and its partial derivative. For integer $k \geq 1$ and U is open subset of $\mathbb{R}^n$, Evans, L.C.(2010) define PDE as an expression of the form

$$F(D^k u(x), D^{k-1}u(x), \dots, Du(x), u(x), x) = 0,$$

that is called a k^{th}-*order* partial differential equation, where

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$$

is given and

$$u : U \to \mathbb{R}$$

is the unknown. Some example of PDE is Heat Equations, Wave Equations, Elasticity Equations, Maxwell Equations, Navier-Stokes Equations, etc.

In the study of these equations, there are some PDEs that the analytical solution is not easy to get. To solve it, usually the solution is approached by numerical method. Such as Boundary (Integral) Element Method, Particle Method, Spectral Method, Finite Difference Method (FDM) and Finite Element Method (FEM), etc. In this report, we will discuss about the comparison of FDM and FEM.

To understand the FDM and FEM, we would like to show how these method can be used to solve Poisson Equation for bounded domain $\Omega \subset \mathbb{R}^2$,

$$\begin{cases} - \triangle u = f(x) & , x \in \Omega \\ u = g(x) & , x \in \partial\Omega \end{cases} \tag{1}$$

with $x = (x_1, x_2)$, $u = u(x) = u(x_1, x_2)$, and Laplacian $\triangle = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}$.

0.2 Finite Difference Method

We consider mesh as shown below with mesh size $h > 0$.

The solution of PDE $u(\xi_{ij})$ is approximated by solution of FDM u_{ij} for each point $\xi_{ij} = (ih, jh) \in \mathbb{R}^2$. Using one of FDM, <u>central difference scheme</u>, the solution of Poisson equation in (1) can be obtained.

Let $\Omega = (0,1)\times(0,1)$ and $h = \dfrac{1}{N}$ where N is number of devider of domain. We define $w_h := \{\xi_{ij}|\ \xi_{ij} \in \Omega\}$ and $\gamma_h := \{\xi_{ij}|\ \xi_{ij} \in \partial\Omega\}$. Then (1) can be approximated by

$$\begin{cases} -\dfrac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4\ u_{ij}}{h^2} = f_{ij} & ,(\xi_{ij} \in w_h) \\ u_{ij} = g_{ij} & ,(\xi_{ij} \in \gamma_h) \end{cases} \tag{2}$$

where we assume $f \in C(\bar{\Omega})$ and $g \in C(\partial\Omega)$,

$$\begin{cases} f_{ij} := f(\xi_{ij}) \\ g_{ij} := g(\xi_{ij}) \end{cases}$$

0.3 Finite Element Method

Consider Poisson equation in (1), by variational principle, there exist a unique solution $u = argmin\ E(v)$, for $v \in H_0^1(\Omega)$, $v = 0$ at boundary, where

$$E(v) := \frac{1}{2}\int_\Omega |\nabla v|^2\ dx - \int_\Omega f\ v\ dx$$

with $\nabla v = \left[\dfrac{\partial v}{\partial x_1}\ ,\ \dfrac{\partial v}{\partial x_2}\right]$. There exist weak solution for equation (1) if and only if exist $u = argmin\ E(v)$.

Let Ω be a polygon. We define $\bar{\Omega} = \bigcup_{K\in T_h} K$

with K is closed triangle and T_h is triangular division.

0.4 Comparison

FDM

1. FDM is approximation of the differential operator by finite difference.
2. The function is approximated in grid points.
3. Difficult to apply for not rectangular domain.

FEM

1. The domain is approximated by triangular mesh.
2. Approximate the function space under variational structure.
3. Easy to aply for curved domain

As we can see, the FEM in two dimesional or three dimensional problem is much more powerful than FDM.