Progress Report

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Strong Form

We want to find

$$(u, p) : \Omega \times (0, T) \to \mathbb{R}^d \times \mathbb{R}$$

where u is unknown velocity and p is unknown pressure such that

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \triangle u + \nabla p = f & \text{in } \Omega \times (0, T) \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u = u^0 & \text{in } \Omega, \text{ at } t = 0 \end{cases} \tag{1}$$

where $f : \Omega \times (0, T) \to \mathbb{R}^d$ and $u^0 : \Omega \to \mathbb{R}^d$ are given functions, $\nu > 0$ is a viscosity.

Weak Form

The weak formulation for equation (1) is shown below. We want to find $\{(u,p)(t) \in V \times Q; t \in (0.T)\}$ such that for $t \in (0,T)$

$$\begin{cases} (\frac{\partial u}{\partial t} + (u \cdot \nabla)u, v) + a(u,v) + b(v,p) + b(u,q) = (f,v) & , \\ & \forall (v,q) \in V \times Q \\ u = u^0, & t = 0 \end{cases}$$

$$\begin{aligned} a(u,v) &= \nu \int_\Omega \nabla u : \nabla v \; dx \\ b(v,q) &= -\int_\Omega (\nabla \cdot v) q \; dx \\ V &= H_0^1(\Omega, \mathbb{R}^d) = H_0^1(\Omega)^d \\ Q &= \{q \in L^2(\Omega); \int_\Omega q \; dx = 0\}. \end{aligned}$$

3D Discretization

First order in time

Before applying to FreeFEM++, we need to discritize $\dfrac{\partial u_i}{\partial t} + (u \cdot \nabla)u_i$ part, where dt as time increment.

$$\frac{\partial u_i}{\partial t} + (u \cdot \nabla)u_i \approx \frac{u_i^n - u_i^{n-1}(X_1(u^{n-1}, dt))}{dt} + O(dt + h)$$

Second order in time / Adam-Bashforth Method

$$\frac{\partial u_i}{\partial t} + (u \cdot \nabla)u_i \approx$$
$$\frac{3u_i^n - 4u_i^{n-1}(X_1(\tilde{u}^{n-1}, dt)) + u_i^{n-2}(X_1(\tilde{u}^{n-1}, 2dt))}{2\ dt} + O(dt^2 + h^2)$$

where

$$X_1(u^{n-1}, dt)(x) = x - u^{n-1}(x)\ dt$$

$$\tilde{u}_i^{n-1} = 2u_i^{n-1} - u_i^{n-2}$$

with stabilization term

With $\delta > 0$ and h as mesh size

$$C_i(p, q) = \delta \sum_k h_k^2 (\nabla p, \nabla q)_k$$

Cylindrical domain simulation

Exact solution

$$
\begin{aligned}
u &= (u_1, u_2, u_3) \\
u_1 &= -\cos(x_1)\sin(x_2)\cos(x_3)e^{-2t} \\
u_2 &= -\sin(x_1)\cos(x_2)\cos(x_3)e^{-2t} \\
u_3 &= 0 \\
p &= \frac{1}{4}e^{-4t}(\cos(2x_1) + \cos(2x_2) + \cos(2x_3))
\end{aligned}
$$

such that equation (1) is satisfied with $f = (f_1, f_2, f_3)$. With
$f_1 = -\cos(x_1)\sin(x_2)\cos(x_3)e^{-2t}$,
$f_2 = -\sin(x_1)\cos(x_2)\cos(x_3)e^{-2t}$, and
$f_3 = -(\frac{1}{4})e^{-4t}\sin(2x_3)(2\cos(2x_3) + 1)$

Tornado simulation

Domain and initial condition

Taking $a = 1/8, \epsilon_i = 1, \beta_i = 1$ $(i = 1, \dots, 6)$, with domain $\Omega = \{x = (x, y, z) \in \mathbb{R}^3; -a \leq z \leq 4a,\ \sqrt{x^2 + y^2} < 1\}$ and $u = 0$ on boundary.

$$\begin{aligned}
\psi(a, \epsilon, \sigma) &= (a^2 + \epsilon)^\sigma \\
u_z &= \psi(r, \epsilon_1, -\beta_1)\psi(z, \epsilon_2, -\beta_2) \\
\rho &= \psi(r, \epsilon_3, -\beta_3)\psi(z, \epsilon_4, \beta_4) \\
u_0 &= \psi(r, \epsilon_5, -\beta_5)\psi(z, \epsilon_6, -\beta_6) && \text{(with swirl)} \\
u_0 &= 0 && \text{(no swirl)} \\
u_r &= sign(z)\rho u_z
\end{aligned}$$

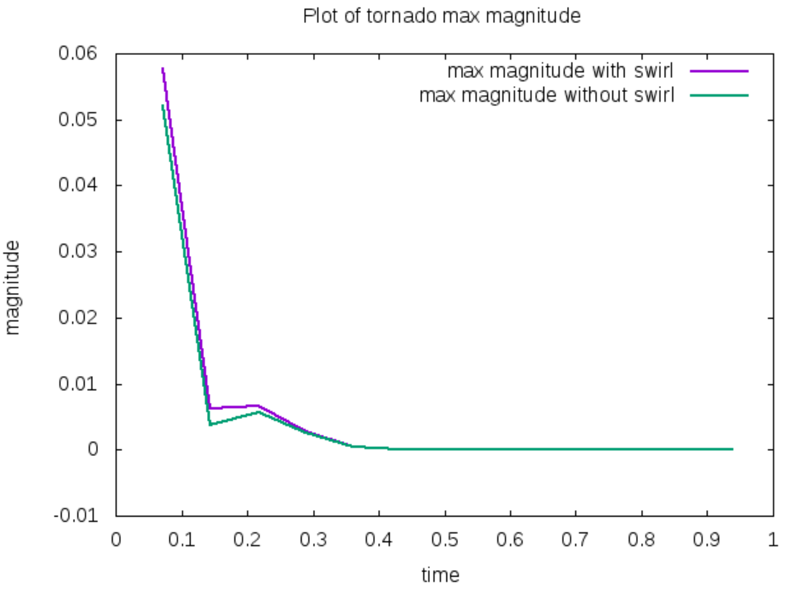

Figure: Max v of tornado simulation every time step

Error estimate

L^2

$$\|u_h^n - u^n\|_{L^\infty(L^2)} = max\ \|u_h^n - u^n\|_{L^2}$$

H_1

$$\|u_h^n - u^n\|_{L^\infty(H^1)} = max\sqrt{\|u_{h_1}^n - u_1^n\|_{L^2(\Omega)}^2 + \|\nabla(u_{h_1}^n - u_1^n)\|_{L^2(\Omega)}^2}$$