1 10-04-18

1.1 Algebraic axioms for real numbers

Two binary operations, $+$ addition and $\cdot$ multiplication on $\mathbb{R}$ are defined and have the following propoerties for all $x, y, z \in \mathbb{R}$:

1. $x + (y + z) = (x + y) + z$. Associative law for addition.
2. $\exists 0$ such that $x + 0 = 0 + x = x$. Existence of additive identity.
3. There exist an element $-x \in \mathbb{R}$ such that $x + (-x) = (-x) + x = 0$. Existence of additive inverse.
4. $x + y = y + x$. Commutative law for addition.
5. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$. Associative law for multiplication.
6. $\exists 1 \neq 0$ such that $x \cdot 1 = 1 \cdot x = x$. Existence of multiplicative identity.
7. If $x \neq 0$, then there exist an element $x^{-1} \in \mathbb{R}$ such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$. Existence of multiplicative inverse.
8. $x \cdot y = y \cdot x$. Commutative law for multiplication.
9. $x \cdot (y + z) = x \cdot y + x \cdot z$. Distributive law.

In the language of algebra, axioms above state that $\mathbb{R}$ with addition and multiplication is a **field**.

1.2 The order axioms for real number

A binary relation $\leq$ on $\mathbb{R}$ is defined and satisfies the following properties for all $x, y, z \in \mathbb{R}$.

1. $x \leq x$. Reflexivity.
2. If $x \leq y,\ y \leq x$ then $x = y$. Antisymmetry.
3. If $x \leq y,\ y \leq z$ then $x \leq z$. Transitivity.
4. Either $x \leq y$ or $y \leq x$. Totality.
5. If $x \leq y$, then $x + y \leq y + z$
6. If $0 \leq x$ and $0 \leq y$, then $0 \leq x \cdot y$.

2 17-04-18

2.1 Real Number

$\mathbb{Q} = \{\frac{n}{m} | n, m \in \mathbb{Z}, m \neq 0\}$. We have $p, q \in \mathbb{Q}$, then

$$p + q = \frac{n}{m} + \frac{k}{l} = \frac{kn + ml}{mk};\ pq = \frac{nl}{mk};\ p \geq q \Leftrightarrow p - q \geq 0$$

For $+, \times, \geq$ satisfy A1-A15.

Remark 1. *$\mathbb{Q}$ is incomplete in the following sense. There is no $r \in \mathbb{Q}$ such that $r^2 = 2$. Remember Phytagoras theorem, $a^2 + b^2 = c^2$. Pict : $\because$ if $c \in \mathbb{Q}$, then $c = \frac{n}{m}$ $(n, m \in \mathbb{Z}, m \neq 0)$. We may assume that either m or n is odd.*

$$c^2 = 2 \rightarrow (\frac{n}{m})^2 = 2 \rightarrow n^2 = 2m^2$$

case 1 : *n is odd $\Rightarrow$ odd = even (impossible)*
case 2 : *n is even $\Rightarrow$ m is odd (from assumtion) $\Rightarrow$ n^2 can be devided by 4 but $2m^2$ can not devided by 4 (contradiction)*

Question : How to fill the gap of $\mathbb{Q}$?
Answer : Idea of Weirstrass (supreme axioms)

Definition 1. *$A \subset \mathbb{R}$.*

- *A is bounded from above $\Leftrightarrow \exists b \in \mathbb{R}$ such that $a \leq b$ $(\forall a \in A)$. such b is called upper bound of A.*
- *A is bounded from below $\Rightarrow \exists b\prime \in \mathbb{R}$ such that $a \geq b\prime$ $(\forall a \in A)$. Such $b\prime$ is called lower bound of A*
- *$\alpha = supA$*
  $\Leftrightarrow$ the minimum of the set of upper bound
  $\Leftrightarrow$ 1. α is an upper bound of A ; 2. if b is an upper bound of A, then $\alpha \leq b$.
- *$\beta = infA \Leftrightarrow$ the maximum of the set of lower bounds of A.*

Remark 2. *$supA(infA)$ is uniquely determined if it exist.*
For example, $sup\mathbb{Q}(inf\mathbb{Q})$ does not exist. $\because$ $\mathbb{Q}$ is not bounded from above (below)

Remark 3. *Every nonempty subset of $\mathbb{R}$ which is bounded from above (below) has a supremum (infimum) in $\mathbb{R}$*

2.2 Real sequence

Definition 2. *For $x \in \mathbb{R}$, $|x| = \begin{cases} x & , x \geq 0 \\ -x & , x \leq 0 \end{cases}$*

Remark 4.
- $|x| \geq 0,\ |x| = 0 \Leftrightarrow x = 0$
- $|xy| = |x||y|$
- $|x + y| \leq |x| + |y|$ *(triangle inequality)*

An infinite sequence of $\mathbb{R} \Leftrightarrow a : \mathbb{N} \rightarrow \mathbb{R}$ usually we write $a_n = a(n),\ n \in \mathbb{N}$ or $\{a_n\}_{n\in\mathbb{N}}$ or $a_1, a_2, \ldots$.
Question : Limiting behavior of a_n as n increases ?
Answer : $a_n \rightarrow l,\ n \rightarrow \infty \Leftrightarrow$ as n become larger and larger, the value a_n become arbitrarily close to l.

Definition 3. *$\epsilon - N$* ***definition of the limit****. $\{a_n\}$ converges to $l \in \mathbb{R} \Leftrightarrow \forall\epsilon > 0, \exists N \in \mathbb{N}$ such that $|a_n - l| < \epsilon, \forall n \geq N$. We write $\lim_{n\to\infty} a_n = l$.*

Definition 4.
- *$a_n \rightarrow +\infty \Leftrightarrow \forall M > 0$. $\exists N \in \mathbb{N}$ such that $a_n > M$ $(\forall n \geq N)$*
- *$a_n \rightarrow -\infty \Leftrightarrow \forall M > 0$. $\exists N \in \mathbb{N}$ such that $a_n < -M$ $(\forall n \geq N)$*

Remark 5. *A convergent sequence has a unique limit.*

$$\because \qquad \epsilon = \frac{1}{2}|l - l\prime| > 0$$
$$\exists N \in \mathbb{N} \text{ such that } |a_n - l| < \epsilon,\ (\forall n \geq N)$$
$$\exists N\prime \in \mathbb{N} \text{ such that } |a_n - l\prime| < \epsilon,\ (\forall n \geq N\prime)$$

Set $\tilde{N} = max\{N, N\prime\} \in \mathbb{N}$. For $n \geq \tilde{N} \Rightarrow |a_n - l| < \epsilon,\ |a_n - l\prime| < \epsilon$ is impossible.