A stabilized Lagrange-Galerkin scheme for the Navier-Stokes equations and its computation

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Contents

Introduction

- LG (Lagrange-Galerkin) method
  = FEM + the method of characteristics

A basic idea of LG schemes

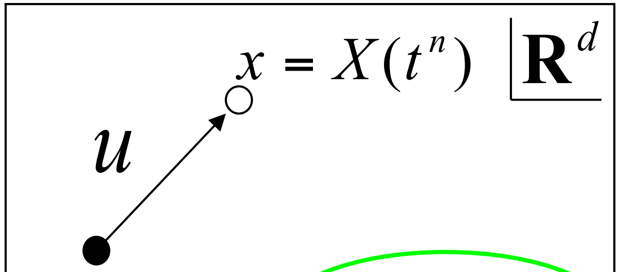
$\Omega \subset \mathbf{R}^d\ (d=2,3),\ u:\Omega\times(0,T)\to\mathbf{R}^d$: given, $\phi:\Omega\times(0,T)\to\mathbf{R}$: unknown, $t^n \equiv n\Delta t$.

Material derivative : $\dfrac{D\phi}{Dt} \equiv \left(\dfrac{\partial}{\partial t} + u\cdot\nabla\right)\phi$

is discretized as follows.

Let $X(\cdot;x,t^n):(0,T)\to\mathbf{R}^d$ be the sol. of the ODE;

$$\begin{cases} X'(t) = u(X,t) \text{ in } (t^{n-1},\, t^n), \\ X(t^n) = x. \end{cases}$$



$$X(t^{n-1}) \approx x - u^{n-1}(x)\Delta t$$

$$\left(\frac{X(t^n) - X(t^{n-1})}{\Delta t} \approx u^{n-1}\big(X(t^n)\big)\right)$$

$$\boxed{\frac{D\phi}{Dt}(X(t^n),t^n) = \frac{d}{dt}\phi\big(X(t),t\big)\Big|_{t=t^n}} \approx \frac{\phi\big(X(t^n),t^n\big) - \phi\big(X(t^{n-1}),t^{n-1}\big)}{\Delta t}$$

$$\approx \frac{\boxed{\phi^n(x)} - \boxed{\phi^{n-1}\big(x - u^{n-1}(x)\Delta t\big)}}{\Delta t} = \frac{\phi^n - \phi^{n-1}\circ X_1(u^{n-1},\Delta t)}{\Delta t}(x)$$

$$A\vec{x} = \vec{b}$$

non-symmetric part goes to RHS vector

$\boxed{\phi \leftarrow u_i}$

LG schemes for flow problems

Accuracy in time	Navier-Stokes Conventional (P2/P1)	Navier-Stokes Stabilized (P1/P1)
First order	Pironneau, NM, 1982 Süli, NM, 1988	N-Tabata, M2AN (to appear)
Second order	Boukir et al., IJNMF, 1997	N-Tabata, a book chapter, Springer (to appear)

Note: Achdou-Guermond, SINUM, 2000, Projection-type.
N-Tabata, JSC, 2015, a stabilized LG scheme for Oseen eqns.

LG schemes for the Navier-Stokes eqns.

The Navier-Stokes equations

Find $(u,p) : \Omega \times (0,T) \to \mathbf{R}^d \times \mathbf{R}$ s.t.

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix}$$

$\mathbf{R}^d \ (d = 2,3)$

$$\boxed{\frac{\partial u}{\partial t} + (u \cdot \nabla) u} - \nabla \cdot \left[2\nu D(u) \right] + \nabla p = f$$

Material derivative

$$\nabla \cdot u = 0$$

$$u(0) = u^0$$

$\Gamma (\equiv \partial \Omega)$

$u = 0$

Ω

ν : viscosity, $\underset{d\times 1}{f}$, $\underset{d\times 1}{u^0}$: given, $D(u) \equiv \frac{1}{2}\left[\nabla u + (\nabla u)^T \right]$.

An LG scheme for NS eqns.

Scheme (P2/P1)

Pironneau, 1982, NM,
Süli, 1988, NM.

Find $\left\{(u_h^n, p_h^n)\right\}_{n=1}^{N_T} \subset V_h \times Q_h$ s.t. for $n = 1, 2, \cdots, N_T$,

$$\left(\frac{u_h^n - u_h^{n-1} \circ X_1(u_h^{n-1}, \Delta t)}{\Delta t}, v_h\right) + 2\nu\Big(D(u_h^n), D(v_h)\Big) - \Big(\nabla \cdot v_h, p_h^n\Big) = (f^n, v_h), \quad \forall v_h \in V_h,$$

$$\approx \frac{Du}{Dt}$$

$$-\Big(\nabla \cdot u_h^n, q_h\Big) = 0, \quad \forall q_h \in Q_h,$$

$V_h \subset H_0^1(\Omega)^d$ and $Q_h \subset L_0^2(\Omega)$: P2 and P1 finite element spaces,

$(\,\cdot\,,\,\cdot\,) : L^2(\Omega)$ inner product,

u_h^0 : an approximation of u^0, $X_1(w, \Delta t)(x) \equiv x - w(x)\Delta t$.

The matrix is symmetric $\begin{pmatrix} A & B^T \\ B & O \end{pmatrix}$ $A = A^T$

A stabilized LG scheme for NS eqns.

Scheme (P1/P1)

N-Tabata, M2AN (to appear)

Find $\left\{ (u_h^n, p_h^n) \right\}_{n=1}^{N_T} \subset V_h \times Q_h$ s.t. for $n = 1, 2, \cdots, N_T$,

$$\left(\frac{u_h^n - u_h^{n-1} \circ X_1(u_h^{n-1}, \Delta t)}{\Delta t}, v_h \right) + 2\nu \Big(D(u_h^n), D(v_h) \Big) - \Big(\nabla \cdot v_h, p_h^n \Big) = (f^n, v_h), \quad \forall v_h \in V_h,$$

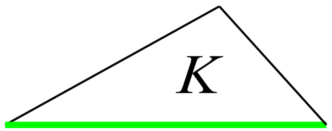
$$\approx \frac{Du}{Dt} \qquad -\Big(\nabla \cdot u_h^n, q_h \Big) - \sum_{K \in \mathcal{T}_h} h_K^2 \Big(\nabla p_h^n, \nabla q_h \Big)_K = 0, \quad \forall q_h \in Q_h,$$

Pressure stabilization (Brezzi-Pitkäranta, 1984)

V_h and Q_h : P1-finite element spaces, $(\,\cdot\,,\,\cdot\,) : L^2(\Omega)$ inner product,

u_h^0 : an approximation of u^0, $X_1(w, \Delta t)(x) \equiv x - w(x)\Delta t$,

$\boxed{h_K}$ = diam(K), $(\,\cdot\,,\,\cdot\,)_K : L^2(K)^d$-inner product.



The matrix is symmetric

$$\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} \qquad A = A^T, \quad C = C^T.$$

Theorem: P2/P1 (Süli, NM, 1988)

$\{\mathcal{T}_h\}_{h\downarrow 0}$: regular family of triangulations with the inverse assumption.

(u,p): $u \in C^0([0,T];W^{1,\infty}) \cap H^2(0,T;L^2) \cap H^1(0,T;V \cap H^3)$, $\ p \in H^1(0,T;Q \cap H^2)$,

$u_h^0 \in V_h$: first component of the Stokes projection of $(u^0, 0)$.

$\Rightarrow$

$\boxed{\bar{D}_{\Delta t} a^n \equiv (a^n - a^{n-1})/\Delta t.}$

$\exists h_0 > 0$ and $c_0 > 0$ indep. of h and Δt s.t. the following hold for any

$$h \in (0, h_0] \text{ and } \Delta t \le c_0 h^{d/4}.$$

$\leftarrow \Delta t = O(h^{d/4})$

(i) $\exists (u_h, p_h) = \left\{(u_h^n, p_h^n)\right\}_{n=1}^{N_T}$: FE sol. of the scheme.

$\leftarrow$ Existence

(ii) $\|u_h\|_{\ell^\infty(L^\infty)} \le \|u\|_{C(L^\infty)} + 1.$

$\leftarrow$ Stability

(iii) $\|u_h - u\|_{\ell^\infty(H^1)},\ \|\bar{D}_{\Delta t} u_h - \partial u/\partial t\|_{\ell^2(L^2)},\ \|p_h - p\|_{\ell^2(L^2)} \le c(\Delta t + h^2).$

$\leftarrow$ Error estimates

(iv) Stokes problem is regular $\Rightarrow \|u_h - u\|_{\ell^\infty(L^2)} \le c(\Delta t + h^3).$

Theorem: P1/P1 (N-Tabata, M2AN, to appear)

$\{\mathcal{T}_h\}_{h\downarrow 0}$: regular family of triangulations with the inverse assumption.

$(u,p):\ u\in C([0,T];W^{1,\infty})\cap H^2(0,T;L^2)\cap H^1(0,T;V\cap H^2),\quad p\in H^1(0,T;Q\cap H^1),$

$u_h^0\in V_h$: first component of the Stokes projection of $(u^0,0)$.

$\Longrightarrow$

$\boxed{\bar{D}_{\Delta t}a^n \equiv (a^n - a^{n-1})/\Delta t.}$

$\exists h_0>0$ and $c_0>0$ indep. of h and Δt s.t. the following hold for any

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(iii) $\|u_h-u\|_{\ell^\infty(H^1)},\ \|\bar{D}_{\Delta t}u_h-\partial u/\partial t\|_{\ell^2(L^2)},\ \|p_h-p\|_{\ell^2(L^2)}\le c(\Delta t+h).$

$\leftarrow$Error estimates

(iv) Stokes problem is regular $\Rightarrow$ $\|u_h-u\|_{\ell^\infty(L^2)}\le c(\Delta t+h^2).$

Equation of errors

$$\mathcal{A}((u,p),(v,q)) \equiv a(u,v)+b(v,p)+b(u,q),$$

$$\mathcal{A}_h((u,p),(v,q)) \equiv \begin{cases} \mathcal{A}((u,p),(v,q)) & \text{(P2/P1 case)}, \\ \mathcal{A}((u,p),(v,q)) - C_h(p,q) & \text{(P1/P1 case)}, \end{cases}$$

where

$$a(u,v) \equiv 2\nu\big(D(u),D(v)\big), \quad b(v,q) \equiv -\big(\nabla\cdot v,q\big), \quad C_h(p,q) \equiv \sum_{K\in\mathcal{T}_h} h_K^2 \left(\nabla p,\nabla q\right)_K.$$

$*$Scheme: $\frac{1}{\Delta t}\Big(u_h^n - u_h^{n-1}\circ X_1(u_h^{n-1},\Delta t),v_h\Big) + \mathcal{A}_h((u_h^n,p_h^n),(v_h,q_h)) = (f^n,v_h), \ \forall(v_h,q_h)\in V_h\times Q_h.$

$$\boxed{X_1(w,\Delta t)(x) \equiv x - w(x)\Delta t}$$

$(\hat{u}_h,\hat{p}_h)\in V_h\times Q_h$: the Stokes projection of (u,p).

$\Leftrightarrow \mathcal{A}_h\big((\hat{u}_h,\hat{p}_h),(v_h,q_h)\big) = \mathcal{A}\big((u,p),(v_h,q_h)\big), \quad \forall(v_h,q_h)\in V_h\times Q_h.$

- Error estimates for the Stokes projection:

(i) For P2/P1: $\|\hat{u}_h - u\|_1, \|\hat{p}_h - p\|_0 \le ch^2,$

(ii) For P1/P1: $\|\hat{u}_h - u\|_1, \|\hat{p}_h - p\|_0 \le ch.$

Equation of errors (cont.)

$*$Scheme: $\frac{1}{\Delta t}\left(u_h^n - u_h^{n-1}\circ X_1(u_h^{n-1},\Delta t), v_h\right) + \mathcal{A}_h((u_h^n, p_h^n),(v_h,q_h)) = (f^n, v_h),\ \forall (v_h,q_h)\in V_h\times Q_h.$

Stokes projection: $\mathcal{A}_h\left((\hat{u}_h^n,\hat{p}_h^n),(v_h,q_h)\right) = \mathcal{A}\left((u^n,p^n),(v_h,q_h)\right),\quad \forall (v_h,q_h)\in V_h\times Q_h.$

Let $(e_h^n,\varepsilon_h^n)\equiv(u_h^n-\hat{u}_h^n, p_h^n-\hat{p}_h^n),\ \ \eta(t)\equiv(u-\hat{u}_h)(t).$

$\boxed{\bar{D}_{\Delta t}a^n \equiv (a^n - a^{n-1})/\Delta t.}$

Equation of the errors: $\left(\bar{D}_{\Delta t}e_h^n, v_h\right) + \mathcal{A}_h\left((e_h^n,\varepsilon_h^n),(v_h,q_h)\right) = \left\langle R_h^n, v_h\right\rangle,\ \ \forall (v_h,q_h)\in V_h\times Q_h,$

where

$$R_h^n \equiv R_{h1}^n + R_{h2}^n + R_{h3}^n + R_{h4}^n,$$

$$R_{h1}^n \equiv \frac{Du^n}{Dt} - \frac{u^n - u^{n-1}\circ X_1(u^{n-1},\Delta t)}{\Delta t},$$

$$R_{h2}^n \equiv \frac{1}{\Delta t}\left\{u^{n-1}\circ X_1(u_h^{n-1},\Delta t) - u^{n-1}\circ X_1(u^{n-1},\Delta t)\right\},$$

$$R_{h3}^n \equiv \frac{1}{\Delta t}\left\{\eta^n - \eta^{n-1}\circ X_1(u_h^{n-1},\Delta t)\right\},$$

$$R_{h4}^n \equiv \frac{1}{\Delta t}\left\{e_h^{n-1} - e_h^{n-1}\circ X_1(u_h^{n-1},\Delta t)\right\}.$$

<u>Lemma</u> (P1/P1)

$$\left\|R_h^n\right\|_0 \le c\left(\left\|u_h^{n-1}\right\|_{0,\infty}+1\right)\left(\Delta t + h + \left\|e_h^{n-1}\right\|_1\right).$$

Proof.

$$\left\|R_{h1}^n\right\|_0 \le c\Delta t,$$

$$\left\|R_{h2}^n\right\| \le c\left(\left\|e_h^{n-1}\right\|_0 + h\right),$$

$$\left\|R_{h3}^n\right\| \le c\left(\left\|u_h^{n-1}\right\|_{0,\infty}+1\right)h,$$

$$\left\|R_{h4}^n\right\| \le c\left\|u_h^{n-1}\right\|_{0,\infty}\left\|e_h^{n-1}\right\|_1.$$

The key point for the error estimates

$*$Scheme: $\frac{1}{\Delta t}\left(u_h^n - u_h^{n-1}\circ X_1(u_h^{n-1},\Delta t), v_h\right) + \mathcal{A}_h((u_h^n, p_h^n),(v_h,q_h)) = (f^n, v_h),\ \forall (v_h,q_h)\in V_h\times Q_h.$

Stokes projection: $\mathcal{A}_h\left((\hat{u}_h^n,\hat{p}_h^n),(v_h,q_h)\right) = \mathcal{A}\left((u^n,p^n),(v_h,q_h)\right),\ \ \forall (v_h,q_h)\in V_h\times Q_h.$

Let $(e_h^n,\varepsilon_h^n)\equiv(u_h^n-\hat{u}_h^n, p_h^n-\hat{p}_h^n),\ \ \eta(t)\equiv(u-\hat{u}_h)(t).$

Equation of the errors: $\left(\bar{D}_{\Delta t}e_h^n, v_h\right) + \mathcal{A}_h\left((e_h^n,\varepsilon_h^n),(v_h,q_h)\right) = \left\langle R_h^n, v_h\right\rangle,\ \forall (v_h,q_h)\in V_h\times Q_h,$

with $\left\|R_h^n\right\|_0 \le c\left(\boxed{\left\|u_h^{n-1}\right\|_{0,\infty}}+1\right)\left(\Delta t + h + \left\|e_h^{n-1}\right\|_1\right).$

Mathematical induction

If $\left\|u_h^{n-1}\right\|_{0,\infty}$ is bounded $\Rightarrow$ $\left\|e_h^n\right\|_1 = \left\|u_h^n-\hat{u}_h^n\right\|_1 \le c\left(\Delta t + h\right)$ by Gronwall's ineq.

$$\Rightarrow \left\|u_h^n\right\|_{0,\infty} \le \left\|\Pi_h u^n\right\|_{0,\infty} + \left\|u_h^n - \Pi_h u^n\right\|_{0,\infty} \le \left\|\Pi_h u^n\right\|_{0,\infty} + ch^{-d/6}\left\|u_h^n - \Pi_h u^n\right\|_1 \le \ldots \le \left\|u\right\|_{C(L^\infty)} + 1$$

($\exists h_0$ and c_0 that the argument holds.)

Π_h : the Lagrange interpolation op.

Key issues

$*$Scheme: $\frac{1}{\Delta t}\left(u_h^n - u_h^{n-1}\circ X_1(u_h^{n-1},\Delta t), v_h\right) + \mathcal{A}_h((u_h^n, p_h^n),(v_h,q_h)) = (f^n, v_h),\ \forall (v_h,q_h)\in V_h\times Q_h.$

$\boxed{X_1(w,\Delta t)(x) \equiv x - w(x)\Delta t}$

- Upwind points: $u\in W_0^{1,\infty}(\Omega),\ \Delta t\|u\|_{1,\infty} < 1 \Rightarrow X_1(u,\Delta t)(\Omega) = \Omega.$
- Inverse inequalities: $(i)\ \|v_h\|_{0,\infty} \le \alpha_0 h^{-d/6}\|v_h\|_1,\quad (ii)\ \|v_h\|_{1,\infty} \le \alpha_1 h^{-d/2}\|v_h\|_1.$
- <u>Lemma</u>

$$a,b\in W_0^{1,\infty}(\Omega),\quad \Delta t\|a\|_{1,\infty},\ \Delta t\|b\|_{1,\infty} \le 1/4$$

$$\Rightarrow$$

$$(i)\quad \frac{1}{2} \le J \equiv \det\left(\frac{\partial X_1(a,\Delta t)}{\partial t}\right) \le \frac{3}{2}.$$

$$(ii)\quad \|g - g\circ X_1(a,\Delta t)\|_0 \le c\Delta t\|a\|_{0,\infty}\|g\|_1,\quad \forall g\in H^1(\Omega)^d,$$

$$\|g\circ X_1(b,\Delta t) - g\circ X_1(a,\Delta t)\|_0 \le c\Delta t\|b-a\|_0\|g\|_{1,\infty},\quad \forall g\in W^{1,\infty}(\Omega)^d.$$

Numerical results to see convergence order

Test problem$(\mathrm{Re} = 10^k,\ k = 1,\dots,4)$

$$\Omega = (0,1)^d, T = 1, \nu = 10^{-k}, k = 1,\dots,4,$$

$(i)\ d = 2:$

$$u = \left(-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1}\right), \quad p(x,t) = \sin\{\pi(x_1 + x_2 + t)\},$$

$$\psi(x,t) = \left(\sqrt{3}/2\pi\right)\sin^2(\pi x_1)\sin^2(\pi x_2)\sin\{\pi(x_1 + x_2 + t)\}.$$

$(ii)\ d = 3:$

$$u = \mathrm{rot}\,\Psi, \quad p(x,t) = \sin\{\pi(x_1 + x_2 + x_3 + t)\},$$

$$\Psi_1 = c\sin(\pi x_1)\sin^2(\pi x_2)\sin^2(\pi x_3)\sin\{\pi(x_2 + x_3 + t)\},$$

$$\Psi_2 = c\sin^2(\pi x_1)\sin(\pi x_2)\sin^2(\pi x_3)\sin\{\pi(x_3 + x_1 + t)\},$$

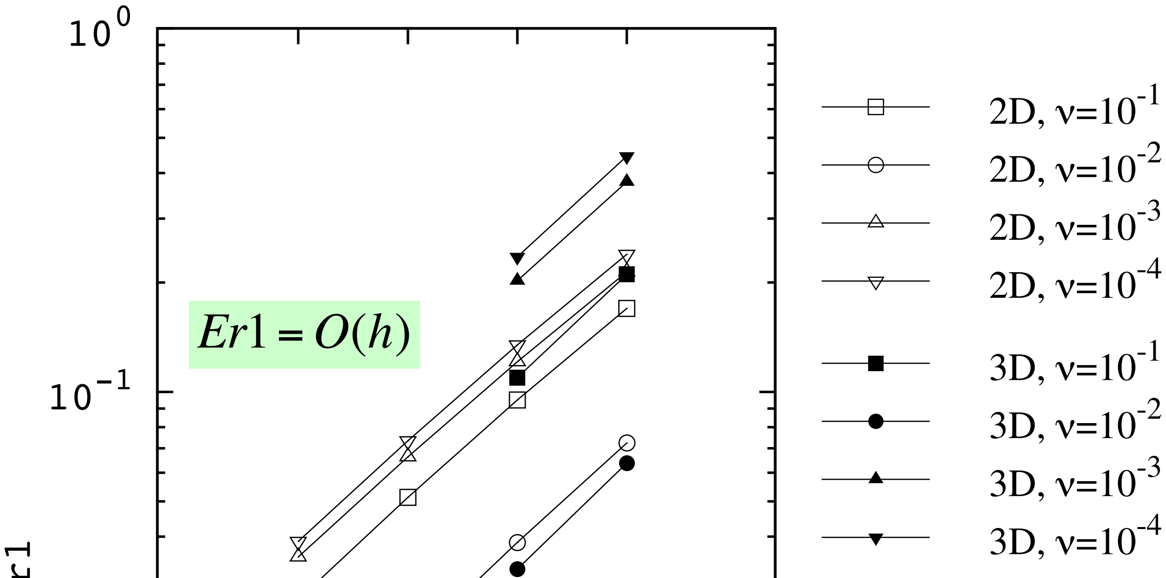
$$\Psi_3 = c\sin^2(\pi x_1)\sin^2(\pi x_2)\sin(\pi x_3)\sin\{\pi(x_1 + x_2 + t)\},$$

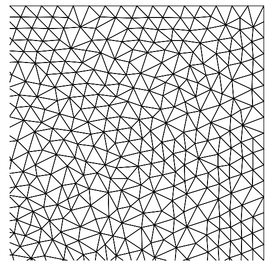
$$c = 8\sqrt{3}/27\pi.$$

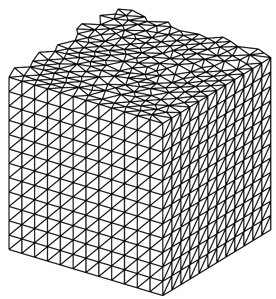
$\delta_0 = 1,\ h = 1/N,\ N = 64, 128, 256, 512.$

$\Delta t = 4h$

$$Er1 \equiv \frac{\|u_h - \Pi_h u\|_{\ell^2(H^1(\Omega))} + \|p_h - \Pi_h p\|_{\ell^2(L^2(\Omega))}}{\|\Pi_h u\|_{\ell^2(H^1(\Omega))} + \|\Pi_h p\|_{\ell^2(L^2(\Omega))}}.$$







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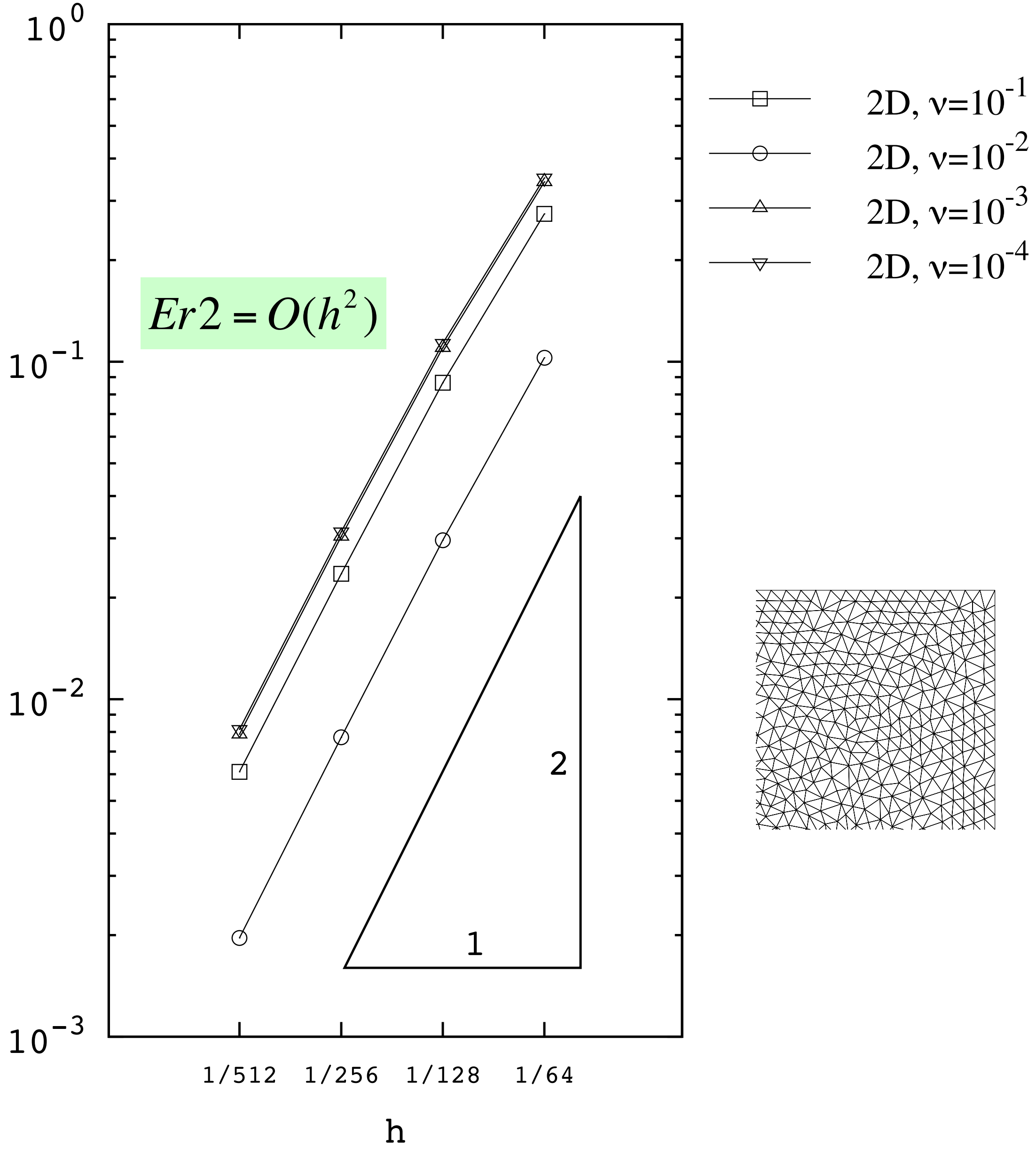
$$\psi(x,t) = \left(\sqrt{3}/2\pi\right)\sin^2(\pi x_1)\sin^2(\pi x_2)\sin\{\pi(x_1 + x_2 + t)\}.$$

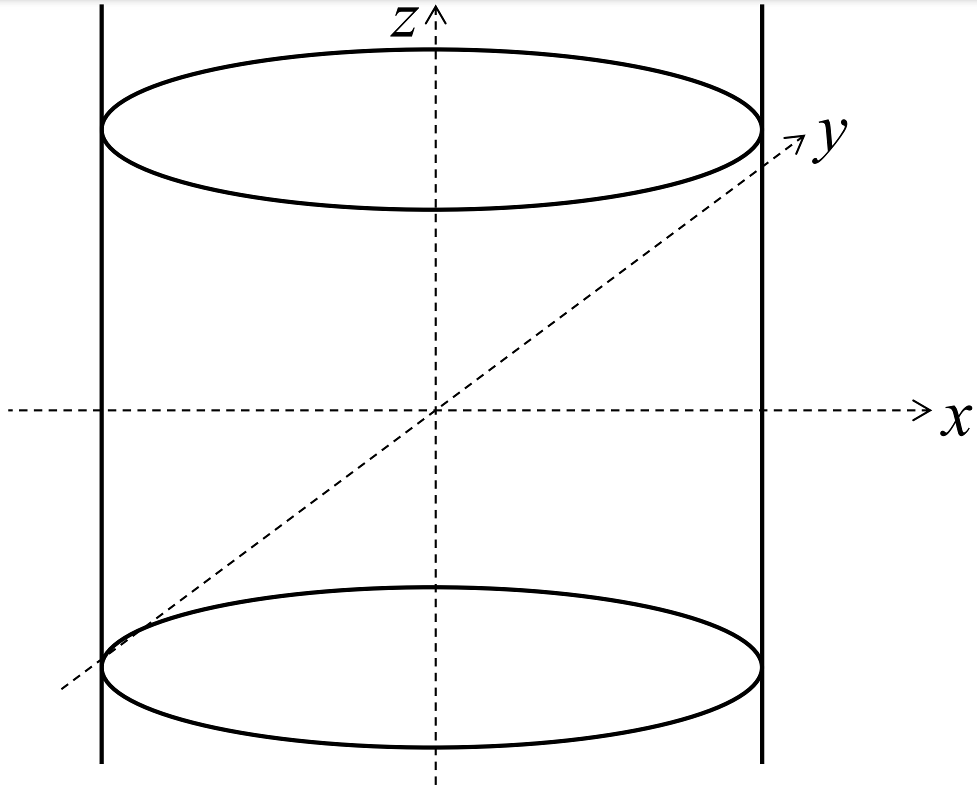
$$h = 1/N,\ N = 64, 128, 256, 512.$$

$$\Delta t = 256h^2$$

$$Er2 \equiv \frac{\|u_h - \Pi_h u\|_{\ell^\infty(L^2(\Omega))}}{\|\Pi_h u\|_{\ell^\infty(L^2(\Omega))}}.$$



Computation of axisymmetric NS flow



Hsu-N-Yoneda, submitted

- Blow-up phenomena: possible only on Z-axis [Caffarelli-Kohn-Nirenberg,1982]

- Suppose that blow-up appears, then, there exists a swirl [Ukhovskii-Yudovich, 1968]

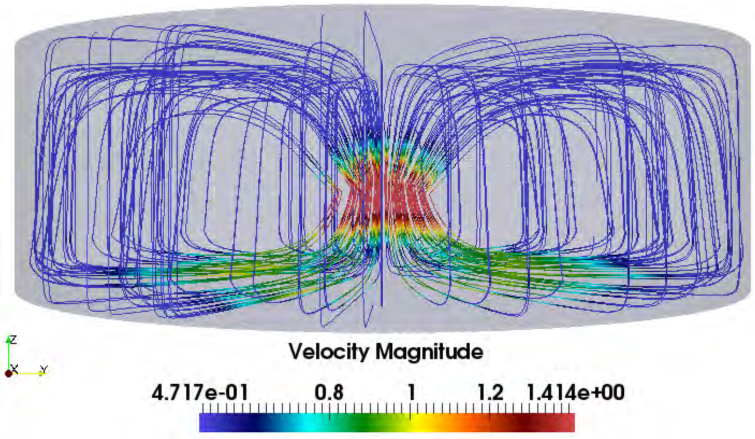
Initial flow

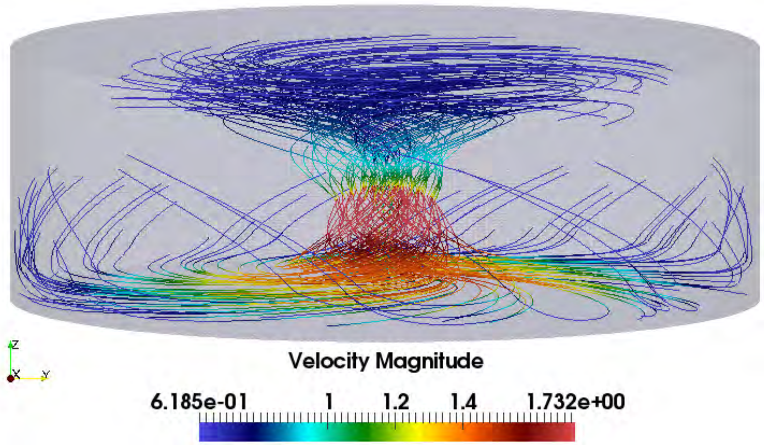
Without swirl
(only concentration)

With swirl

z=1/2

z=-1/8





Numerical results

Without swirl

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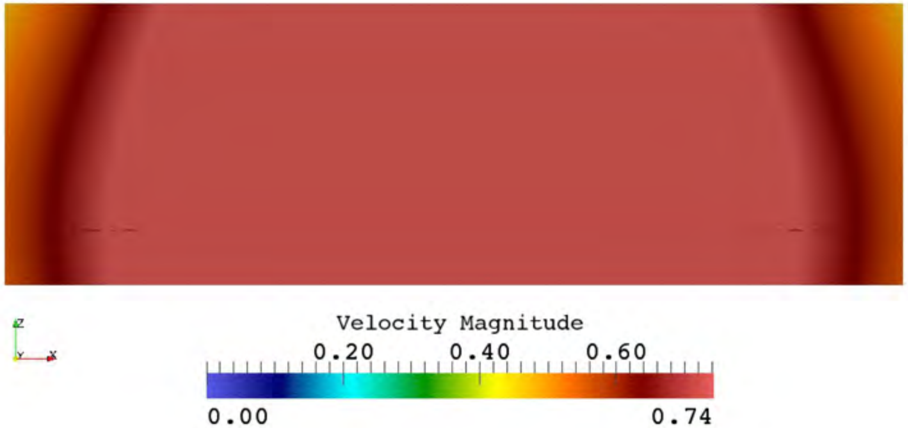
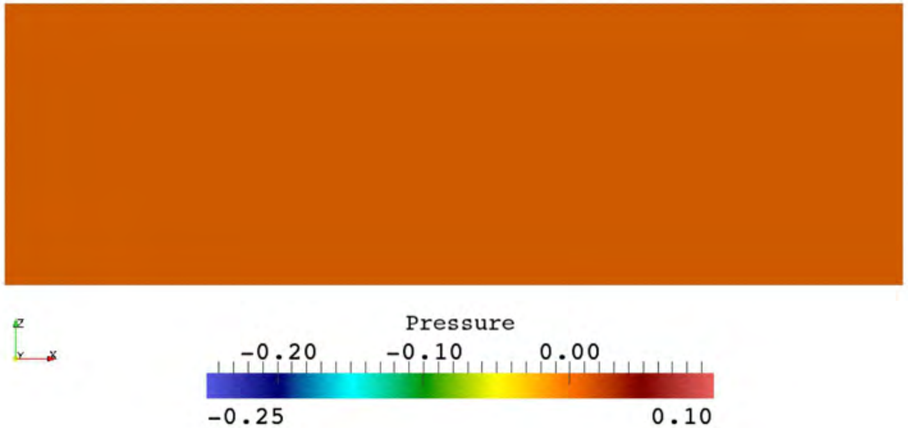
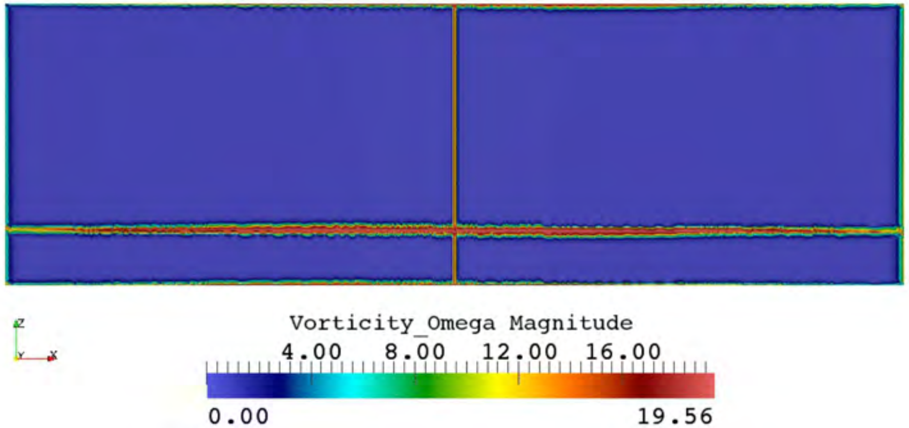


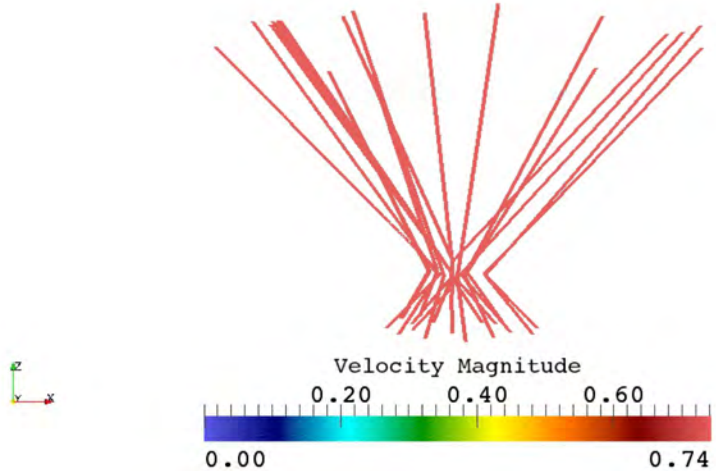
p



vorticity



stream lines

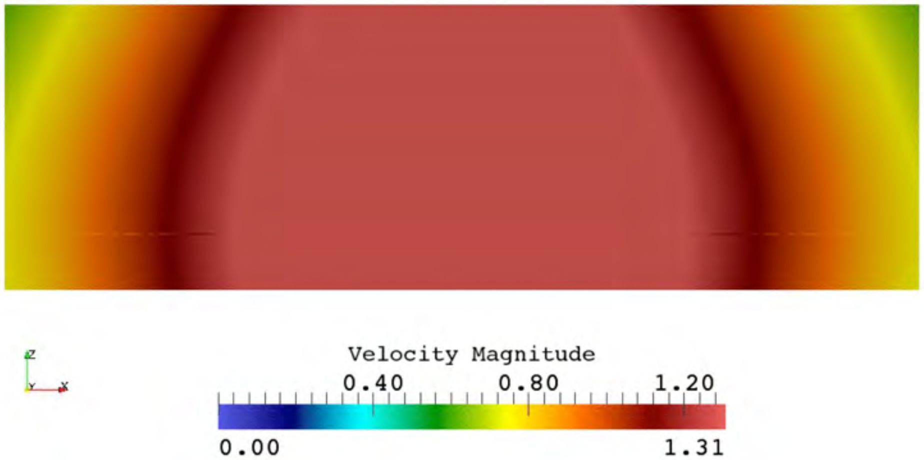


Re=50,000

Numerical results

With swirl

|u|



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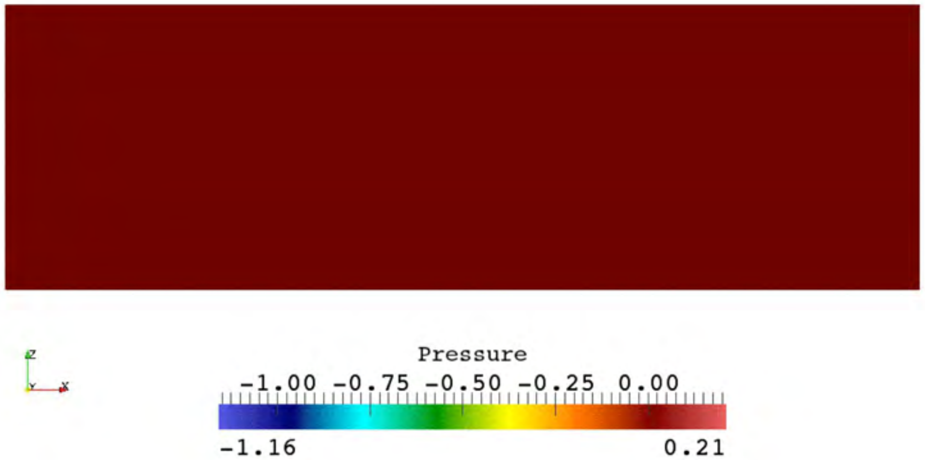
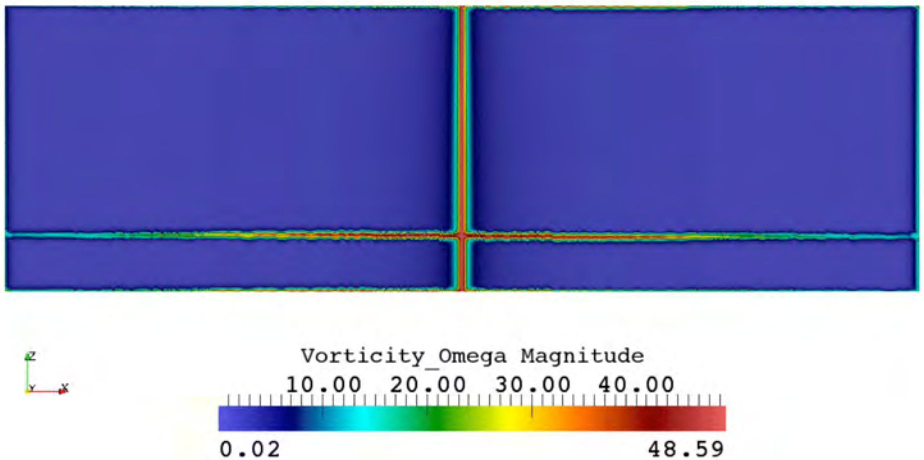
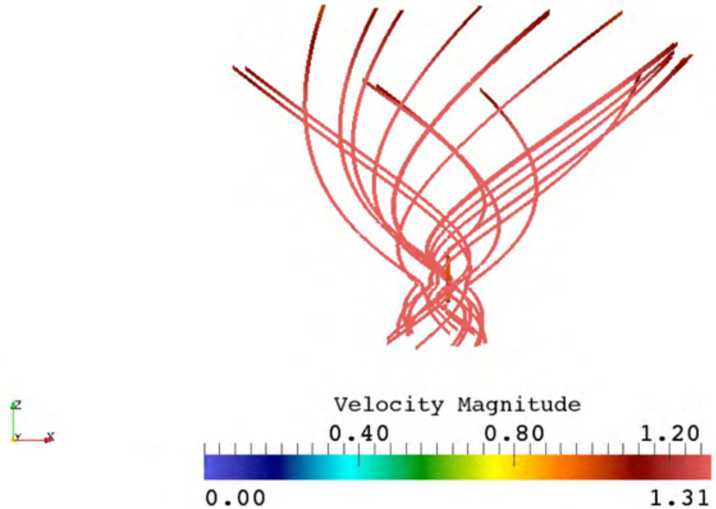


vorticity

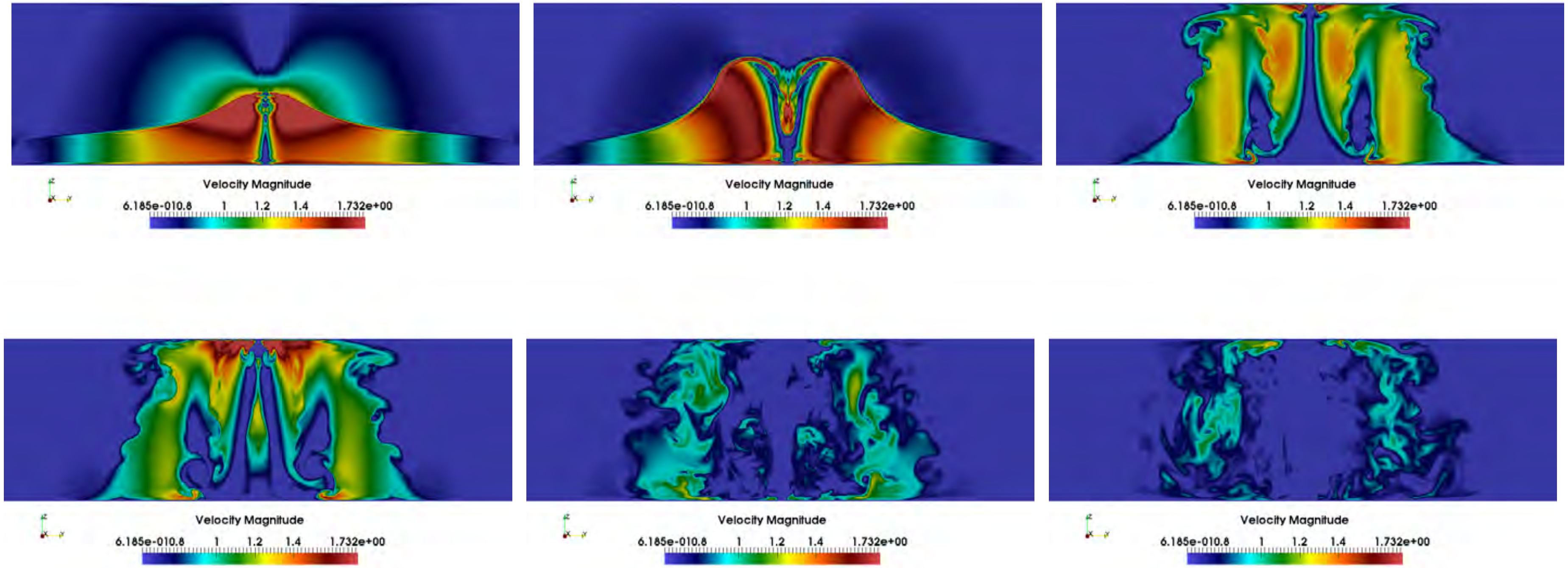


stream lines



Re=50,000

With swirl


Velocity Magnitude
6.185e-010.8 1 1.2 1.4 1.732e+00
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6.185e-010.8 1 1.2 1.4 1.732e+00
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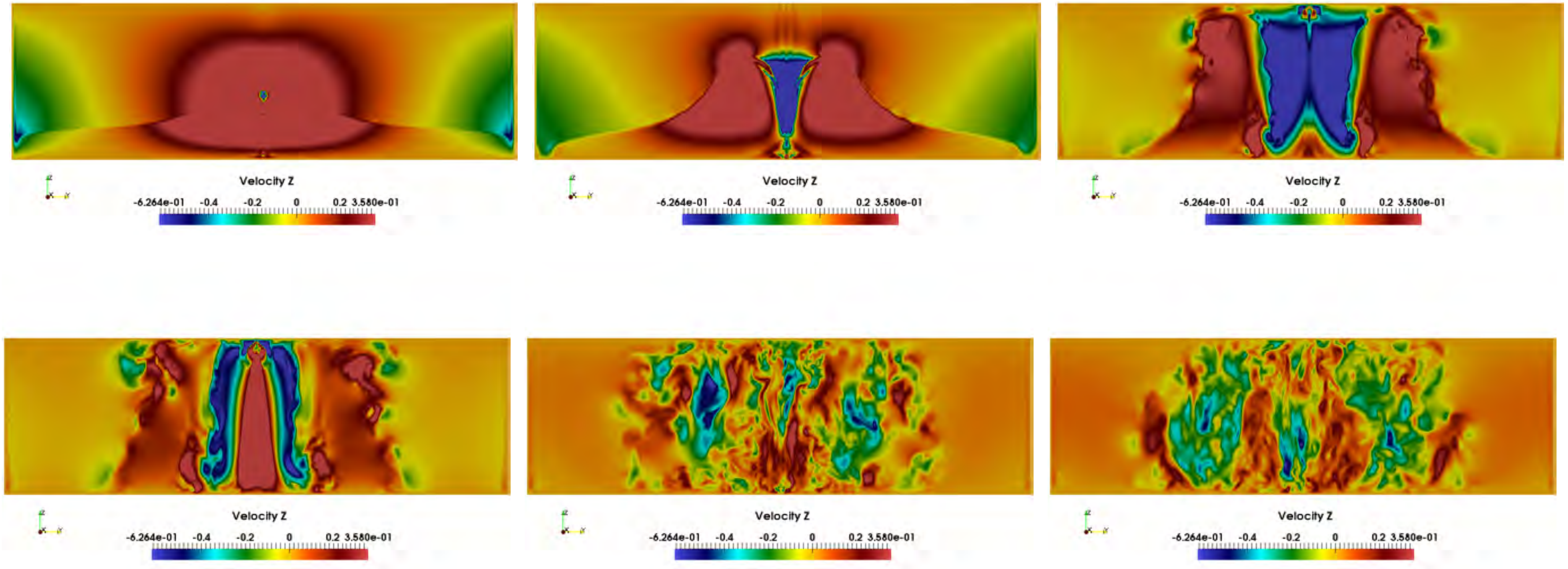
6.185e-010.8 1 1.2 1.4 1.732e+00

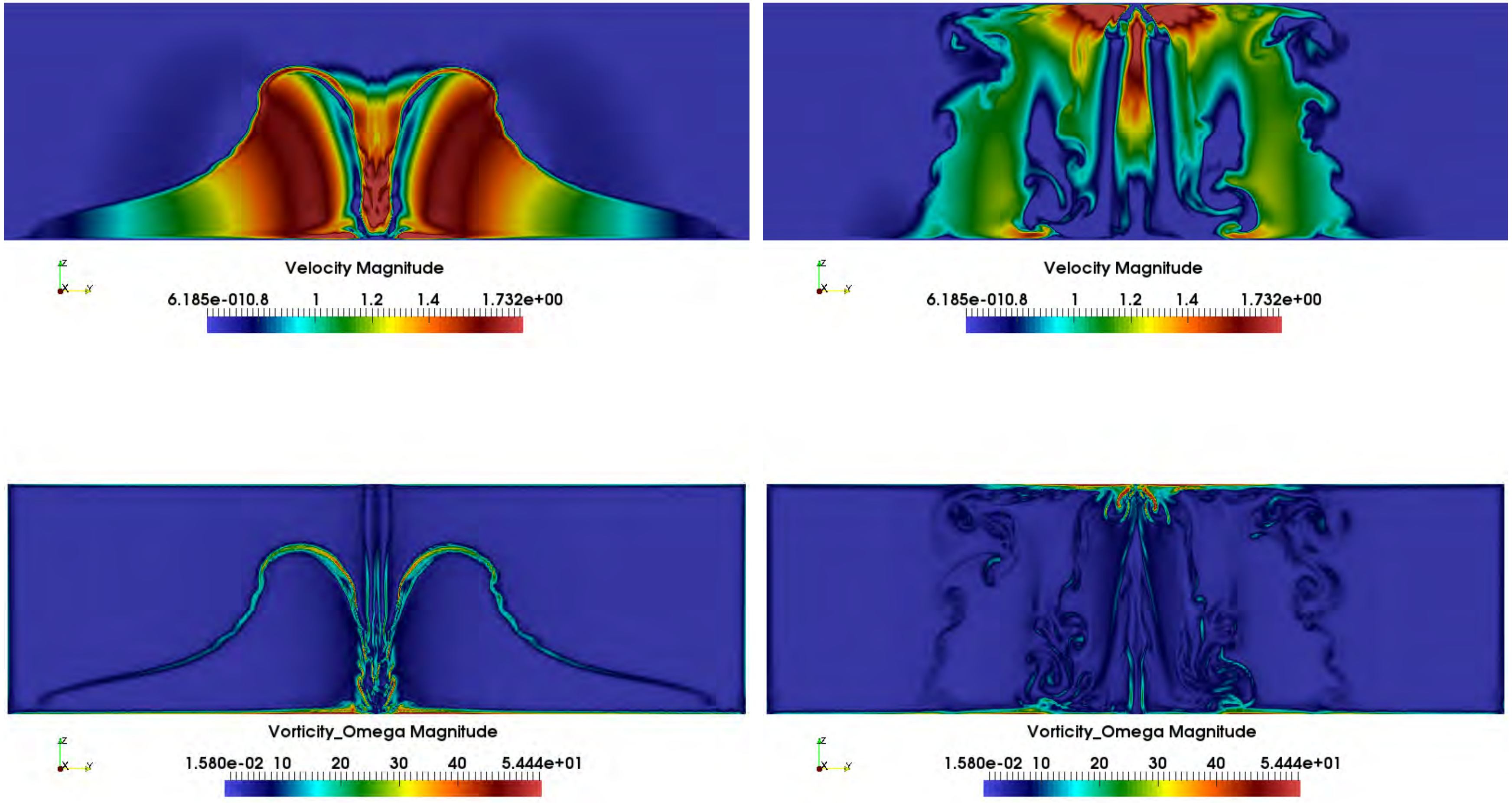
U_Z

With swirl



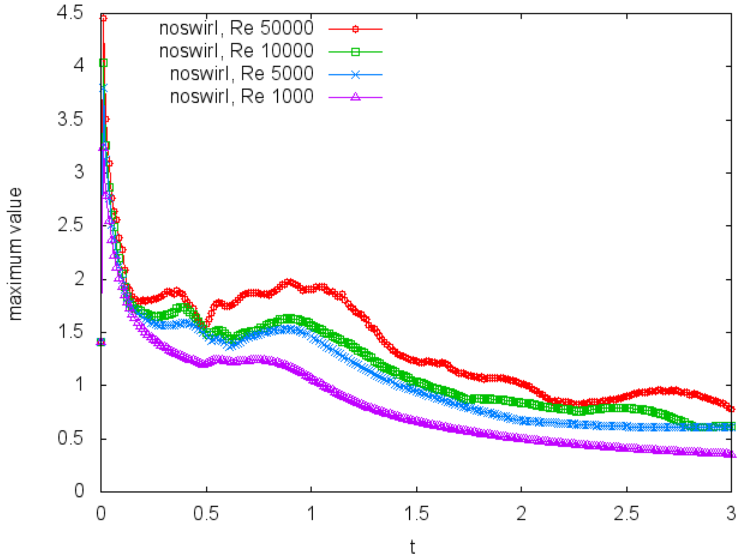
|u| and Vorticity

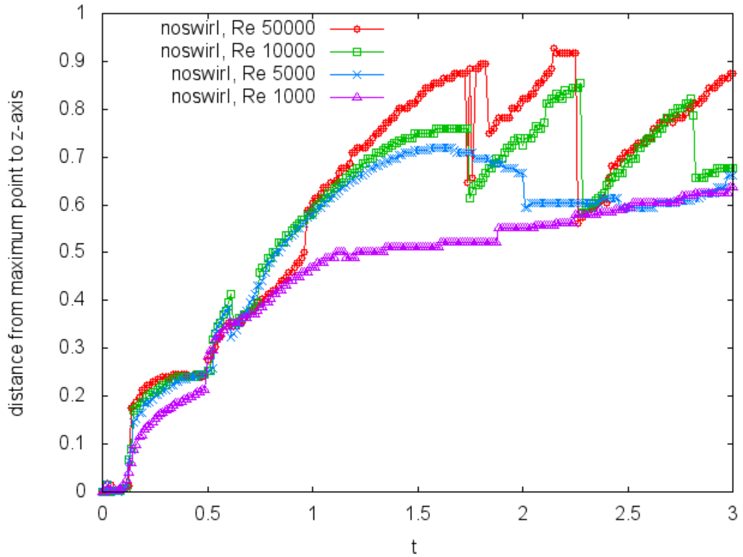
With swirl



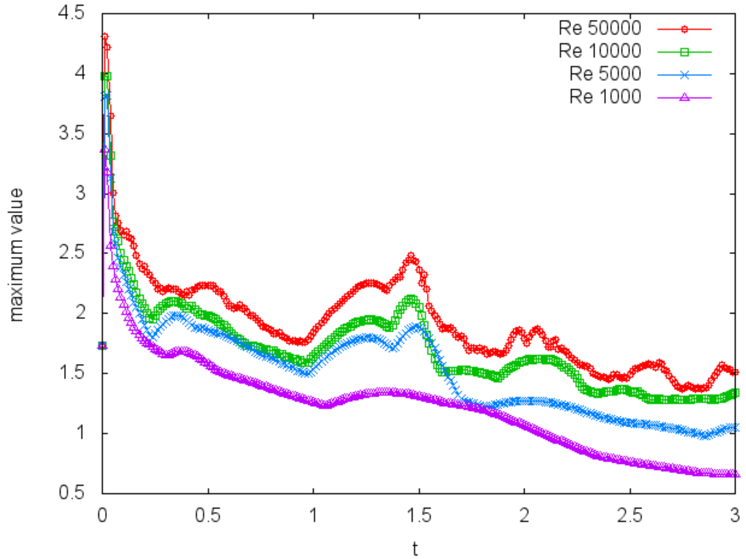
max |u|, and distance from Z-axis to the maximum point

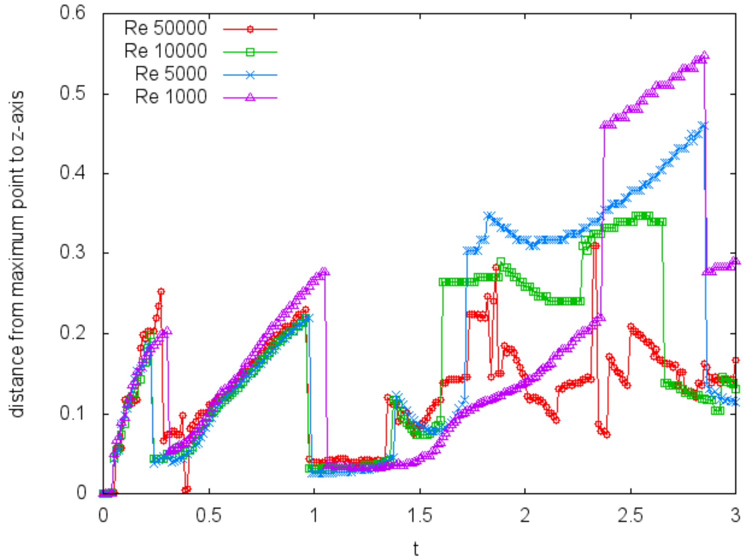
Without swirl





With swirl





Conclusions and Remarks

- We have shown optimal error estimates of a stabilized LG scheme for the Navier-Stokes equations and its computation

Accuracy in time	Navier-Stokes: Conventional (P2/P1)	Navier-Stokes: Stabilized (P1/P1)
First order	Pironneau, NM, 1982 Süli, NM, 1988	N-Tabata, M2AN (to appear)
Second order	Boukir et al., IJNMF, 1997	N-Tabata, a book chapter, Springer (to appear)

- For the second order scheme, we employ the following.

$$\frac{1}{\Delta t}\left(u_h^n - u_h^{n-1}\circ X_1(u_h^{n-1},\Delta t), v_h\right) \to \frac{1}{2\Delta t}\left(3u_h^n - 4u_h^{n-1}\circ X_1(2u_h^{n-1}-u_h^{n-2},\Delta t) + u_h^{n-2}\circ X_1(2u_h^{n-1}-u_h^{n-2},2\Delta t), v_h\right)$$

Theorem: P1/P1 (N-Tabata, a book chapter, Springer, to appear)

$*$Scheme: $\frac{1}{2\Delta t}\left(3u_h^n - 4u_h^{n-1}\circ X_1(u_h^{(n-1)*},\Delta t) + u_h^{n-2}\circ X_1(u_h^{(n-1)*},2\Delta t), v_h\right) + \mathcal{A}_h((u_h^n,p_h^n),(v_h,q_h)) = (f^n, v_h),\ \forall (v_h,q_h)\in V_h\times Q_h.$

$\boxed{u_h^{(n-1)*} \equiv 2u_h^{n-1} - u_h^{n-2} = u_h^n + O(\Delta t^2)}$

$\{\mathcal{T}_h\}_{h\downarrow 0}$: regular family of triangulations with the inverse assumption.

(u,p) : smooth enough,

$u_h^0,\ u_h^1 \in V_h$: "good" approximations of u^0 and u^1, resp.

$\Rightarrow$

$\exists h_0 > 0$ and $c_0 > 0$ indep. of h and Δt s.t. the following hold for any

$$h \in (0, h_0] \text{ and } \Delta t \le c_0 h^{d/6}.$$

$\leftarrow \Delta t = O(h^{d/6})$

(i) $\exists (u_h, p_h) = \left\{(u_h^n, p_h^n)\right\}_{n=2}^{N_T}$:FE sol. of the scheme.

$\leftarrow$Existence

(ii) $\|u_h\|_{\ell^\infty(L^\infty)} \le \|u\|_{C(L^\infty)} + 1.$

$\leftarrow$Stability

(iii) $\|u_h - u\|_{\ell^\infty(H^1)},\ \|p_h - p\|_{\ell^2(L^2)} \le c(\Delta t^2 + h),$

$\leftarrow$Error estimates

(iv) Stokes problem is regular $\Rightarrow \|u_h - u\|_{\ell^\infty(L^2)} \le c(\Delta t^2 + h^2).$

Theorem: P1/P1 (N-Tabata, a book chapter, Springer, to appear)

$*$Scheme: $\frac{1}{2\Delta t}\left(3u_h^n - 4u_h^{n-1}\circ X_1(u_h^{(n-1)*},\Delta t) + u_h^{n-2}\circ X_1(u_h^{(n-1)*},2\Delta t), v_h\right) + \mathcal{A}_h((u_h^n,p_h^n),(v_h,q_h)) = (f^n, v_h),\ \forall (v_h,q_h)\in V_h\times Q_h.$

$\boxed{u_h^{(n-1)*} \equiv 2u_h^{n-1} - u_h^{n-2} = u_h^n + O(\Delta t^2)}$

$\{\mathcal{T}_h\}_{h\downarrow 0}$: regular family of triangulations with the inverse assumption.

(u,p): smooth enough, $u_h^0\in V_h$: first component of the Stokes projection of $(u^0,0)$,

$(u_h^1,p_h^1)\in V_h\times Q_h$: solution of the stabilized LG scheme of first-order in time.

$\Longrightarrow$

$\exists h_0>0$ and $c_0>0$ indep. of h and Δt s.t. the following hold for any

$$h\in(0,h_0] \text{ and } \Delta t\le c_0 h^{d/5}.$$

$\leftarrow \Delta t = O(h^{d/5})$

(i) $\exists (u_h,p_h) = \left\{(u_h^n,p_h^n)\right\}_{n=2}^{N_T}$: FE sol. of the scheme.

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(iii) $\|u_h-u\|_{\ell^2(H^1)},\ \|p_h-p\|_{\ell^2(L^2)} \le c(\Delta t^2+h),$

$$\|u_h-u\|_{\ell^\infty(H^1)} \le c(\Delta t^{3/2}+h).$$

$\leftarrow$Error estimates

(iv) Stokes problem is regular $\Rightarrow \|u_h-u\|_{\ell^\infty(L^2)} \le c(\Delta t^2+h^2).$