5 Existence for the linear problem

We shall use the Lax-Milgram lemma to show existence of solution for the simple linear problem.

Review. Lax-Milgram lemma

Theorem (Lax-Milgram lemma)

Assume that H is a Hilbert space and the bilinear form $a : H \times H \to \mathbb{R}$ satisfies

$$\begin{aligned} &\text{(continuity)} & |a(u,v)| &\leq \alpha\|u\|_H \|v\|_H & \forall u,v \in H \quad (\alpha > 0) \quad (1)\\ &\text{(coercivity)} & \beta\|u\|^2 &\leq a(u,u) & \forall u \in H \quad (\beta > 0) \quad (2) \end{aligned}$$

Then for each bounded linear functional $F : H \to \mathbb{R}$ there exists a unique element $u \in H$ such that

$$B(u,v) = F(v) \qquad \forall v \in H.$$

We study the linear problem for magnetic field:

$$\begin{aligned} -\Delta u &= f & \text{for } x \in \Omega, \quad (3)\\ u(x) &= 0 & \text{on } \partial\Omega. \quad (4) \end{aligned}$$

Now we derive the so-called **weak formulation**. There are basically two types of solutions:

- **classical solution**: a function $u \in C^2(\Omega) \cap C(\bar{\Omega})$ fulfilling (3) in Ω and (4) on $\partial\Omega$.
  However, in many practical situations, the real solution is often not continuously differentiable (for example, shock waves in fluid dynamics) and therefore, we have to think of another "weaker" solution.
- **weak solution**: weak formulation is derived by integrating the equation and transferring some derivatives to a test function. In this way, the solution does not have to be so smooth as in the classical case.

1. Multiply the equation by a test function $\varphi \in H_0^1(\Omega)$ and integrate over Ω:
$$-\Delta u = f \qquad \Rightarrow \qquad \int_\Omega -\Delta u \varphi \, dx = \int_\Omega f\varphi \, dx$$
2. Use Green's theorem to transfer one derivative to the test function:
$$\int_\Omega -\Delta u \varphi \, dx = \int_\Omega f\varphi \, dx \qquad \Rightarrow \qquad \int_\Omega \nabla u \cdot \nabla\varphi \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n}\varphi \, dS = \int_\Omega f\varphi \, dx$$
Since φ has zero trace on the boundary $\partial\Omega$, the last integral vanishes and we obtain
$$\int_\Omega \nabla u \cdot \nabla\varphi \, dx = \int_\Omega f\varphi \, dx \qquad \text{for all } \varphi \in H_0^1(\Omega).$$
This equation is called a weak formulation to the original problem (3),(4).

Definition (weak solution)

A **weak solution** to the problem (3),(4) is a function $u \in H_0^1(\Omega)$, which satisfies the relation

$$\int_\Omega \nabla u \cdot \nabla\varphi \, dx = \int_\Omega f\varphi \, dx \qquad \text{for all } \varphi \in H_0^1(\Omega). \tag{5}$$

Note that the boundary condition (4) is satisfied in the sense that the traces of the solution are required to be zero.

What is good about this rearrangement in the equation?

- weak solution needs to have only first derivatives because one derivative was transfered to the test function and only first derivatives of u appear in the weak equation
- u and its first derivative does not have to be continuous, it is enough if they are integrable (it means that they do not have to be smooth at every point but only "on the whole")
- we can use this weak formulation for numerical approximations, such as finite element method (we shall see later how)

3. Define bilinear form a and linear functional F as

$$a(u,\varphi) = \int_\Omega \nabla u \nabla \varphi \, dx, \qquad u,\varphi \in H_0^1(\Omega)$$

$$F(\varphi) = \int_\Omega f \varphi \, dx, \qquad \varphi \in H_0^1(\Omega)$$

and use the Lax-Milgram lemma to show the existence of solution to

$$a(u,\varphi) = F(\varphi) \qquad \forall \varphi \in H_0^1(\Omega)$$

. For the proof, we need just to check that the assumptions of Lax-Milgram lemma are satisfied:

- $H = H_0^1$ is a Hilbert space (the norm is $\|\cdot\|_{H^1(\Omega)}$)
- $a : H \times H \to \mathbb{R}$ is a bilinear form
- a is continuous:

$$|a(u,v)| = \Big| \int_\Omega \nabla u \cdot \nabla v \, dx \Big| \leq \Big(\int_\Omega |\nabla u|^2 \, dx \Big)^{1/2} \Big(\int_\Omega |\nabla v|^2 \, dx \Big)^{1/2} \leq \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$$

  Here we have used Hölder's inequality for $p = q = 2$.
- a is coercive
- F is a bounded linear functional, if we assume that $f \in L^2(\Omega)$

All the assumptions are fulfilled, hence we can apply Lax-Milgram lemma and we arrive at the result

Theorem. If Ω is a domain with lipschitz boundary and $f \in L^2(\Omega)$, then there exists exactly one weak solution to the problem (3),(4), defined in (5).

6 Existence for the nonlinear problem

Linear problems, such as the one examined above, have a unified theory and are relatively easy to implement. However, for nonlinear problems there is no general method for solution and each problem has to be treated separately. Here we shall study the theory of monotone operators and apply it to the nonlinear problem of stationary magnetic field

$$-\mathrm{div}\left(\nu(x, \|\nabla u(x)\|^2)\nabla u(x)\right) = f(x) \qquad \text{for } x \in \Omega \tag{6}$$

$$u(x) = 0 \qquad \text{on } \partial\Omega \tag{7}$$

Later on, we shall write this problem in the operator form

$$Au = f,$$

where $A : H \to H$ is a nonlinear operator expressing the left-hand side of the equation. H is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ (in our case it will be $H^1(\Omega)$ or $H_0^1(\Omega)$). Note that the above equation can be written also as

$$\langle Au, \varphi \rangle = \langle f, \varphi \rangle \qquad \forall \varphi \in H. \tag{8}$$

Proof. It is clear that $Au = f$ implies (8) (we just multiply by a $\varphi \in h$). The reverse implication can be shown by setting $\varphi = Au - f$. Then we obtain $\langle Au - f, Au - f \rangle = 0$ but the left-hand side is the norm $\|Au - f\|_H^2$, so it must hold that $Au - f = 0$.

Definition Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. We say that an operator $A : H \to H$ is

- **monotone** if
$$\langle Au - Av, u - v \rangle \geq 0 \qquad \forall u, v \in H$$
- **strongly monotone** if there exists $\eta > 0$ such that
$$\langle Au - Av, u - v \rangle \geq \eta \|u - v\|_H^2 \qquad \forall u, v \in H.$$

We say that A satisfies **Lipschitz condition** with constant L if

$$\|Au - Av\|_H \leq L \|u - v\|_H \qquad \forall u, v \in H.$$

Examples.

(1) Let $H = \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function. Then
$$(f(x_1) - f(x_2))(x_1 - x_2) \geq 0 \qquad x_1, x_2 \in \mathbb{R},$$
so f represents a monotone operator.

(2) Let $H = \mathbb{R}^N$ and M be symmetric positive definite matrix of size $N \times N$. Then the linear mapping $A : \mathbb{R}^N \to \mathbb{R}^N$ defined by
$$A(x) = Mx$$
is a monotone operator because
$$\langle A(x_1) - A(x_2), x_1 - x_2 \rangle = (x_1 - x_2)^T M (x_1 - x_2) \geq 0 \qquad \forall x_1, x_2 \in \mathbb{R}^N.$$

(3) Let H be any Hilbert space and $a(\cdot, \cdot)$ be a continuous coercive bilinear form. Then we can define an operator A by
$$\langle Au, v \rangle = a(u, v), \qquad u, v \in H.$$
This definition is correct since for a fixed $u \in H$ the right-hand side is a bounded linear functional and, therefore, it follows from Riesz representation theorem that there is a unique element $Au \in H$ satisfying the above relation.

We find that due to the coercivity the operator A is strongly monotone:
$$\begin{aligned} \langle Au - Av, u - v \rangle &= \langle Au, u - v \rangle - \langle Av, u - v \rangle \\ &= a(u, u - v) - a(v, u - v) \\ &= a(u - v, u - v) \geq \beta \|u - v\|_H^2 \qquad u, v \in H. \end{aligned}$$

(4) Let $f \in L^2(0,1)$ and consider the nonlinear problem

$$-u'' + e^u = f \qquad \text{in } (0,1), \tag{9}$$
$$u(0) = u(1) = 0. \tag{10}$$

This kind of problems appears in simulations of semiconductor parts.

The weak formulation reads: Find a $u \in H = H_0^1(0,1)$ satisfying

$$\langle Au, \varphi \rangle = (f, \varphi)_0 \qquad \forall \varphi \in H.$$

Here,

$$\langle Au, \varphi \rangle = (u', \varphi')_0 + (e^u, \varphi)_0$$

and $(\cdot,\cdot)_0$ denotes the inner product in $L^2(0,1)$ and $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $H^1(0,1)$.

Let us check that the operator A is well defined. The term $(e^u, \varphi)_0$ is finite for any $u, \varphi \in H$ because due to the continuity of H^1-functions in one dimension, both u and φ are continuous on $[0,1]$. Therefore, the mapping $\varphi \mapsto (u', \varphi')_0 + (e^u, \varphi)_0$ represents a bounded linear functional on H and the existence of $Au \in H$ follows again from Riesz theorem.

Since the exponential function is increasing, we have

$$(e^u - e^v)(u - v) \geq 0 \qquad \forall u, v \in \mathbb{R}.$$

Using this fact and Friedrichs inequality, there is a constant $\eta > 0$ such that

$$\langle Au - Av, u - v \rangle = (u' - v', u' - v')_0 + (e^u - e^v, u - v)_0 \geq \|u' - v'\|^2_{L^2(0,1)} \geq \eta \|u - v\|^2_{H^1(0,1)} \quad u, v \in H.$$

Hence, the operator A is strongly monotone.

Next time we shall show the following theorem:

Theorem. Let $A : H \to H$ be a strongly monotone operator which satisfies the Lipschitz condition. Then for each $f \in H$ there exists a unique solution to the problem (8).