7 Finite element method for nonlinear elliptic problems

We shall study the finite element method for nonlinear PDEs. The outline of our plan is as follows:

1. Review the concept of Galerkin approximation and learn about its realization for linear problems.
2. Explain by way of example the idea of finite element method (FEM) for one-dimensional linear problem.
3. Learn the basic general concepts of the finite element method.
4. Study the application of the FEM to nonlinear problems.
5. Find about the convergence properties of the FEM.

7.1 Galerkin approximation

Let us consider a simple linear problem

$$-\Delta u(x) = f(x) \qquad x \in \Omega \tag{1}$$

$$u(x) = 0 \qquad \text{on } \partial\Omega \tag{2}$$

We have given the definition of a weak solution as a function $u \in H_0^1(\Omega)$ satisfying the following condition

$$A(u,\varphi) = L(\varphi) \qquad \forall \varphi \in H_0^1(\Omega). \tag{3}$$

Here,

$$\begin{aligned} A(u,\varphi) &= \int_\Omega \nabla u \cdot \nabla \varphi \, dx = (\nabla u, \nabla \varphi)_0 \\ L(\varphi) &= \int_\Omega f \varphi \, dx = (f,\varphi)_0, \end{aligned}$$

where $(\cdot,\cdot)$ is the inner product on $(L^2(\Omega))^r, r = 1, 2, \ldots$, so the identity (3) means

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx \qquad \forall \varphi \in H_0^1(\Omega). \tag{4}$$

Our aim now is to compute the weak solution numerically. In order to do so, we need first to approximate the continuous problem by a finite-dimensional problem so that it can be handled by computers. The basic idea is to replace the infinite-dimensional space $H_0^1(\Omega)$ in the definition of weak solution by its suitable finite-dimensional subspace. That the dimension of $H_0^1(\Omega)$ is infinite means that there does not exist a set of finite number of independent functions that would form a basis of $H_0^1(\Omega)$.

Fortunately, H^1 is a Hilbert space which means that it has a countable basis. There are many choices for the basis but let us consider one of them and denote it by $\{w_i\}_{i=1}^\infty$. To approximate our problem by a finite dimensional one, we restrict our considerations only on the subspace of $H^1(\Omega)$ generated by a finite part of the basis:

$$X_N = \left\{ v \in H^1(\Omega);\ v(x) = \sum_{i=1}^N \alpha_i w_i(x), \quad \alpha_i \in \mathbb{R} \right\}.$$

We define also the finite-dimensional counterpart of the space $H_0^1(\Omega)$:

$$V_N = \left\{ v \in H^1(\Omega);\ \ v(x) = \sum_{i=1}^N \alpha_i w_i(x), \quad v(x) = 0 \text{ on } \ \partial\Omega \right\}.$$

We then define the approximate solution to be a function $u_N \in X_N$ which satisfies

$$A(u_N, \varphi) = L(\varphi) \qquad \forall \varphi \in V_N.$$

This is exactly the definition of **Galerkin approximation** to the solution of (1), (2) that was introduced in the last lecture.

Since A and L are linear in φ and $\varphi \in V_N$ can be written as a linear combination of basis functions, the above is the same as to require

$$A(u_N, w_j) = L(w_j) \qquad \forall j = 1, 2, \ldots, N. \tag{5}$$

If we write u_N in the form

$$u_N(x) = \sum_{i=1}^{N} \alpha_i w_i(x), \tag{6}$$

equation (5) becomes

$$A(\sum_{i=1}^{N} \alpha_i w_i, w_j) = L(w_j) \qquad j = 1, 2, \ldots, N.$$

This is a system of N algebraic equations for the unknown coefficients $\alpha_1, \alpha_2, \ldots, \alpha_N$. If we can solve this system, we obtain the approximate solution from (6). Since A is linear in our case, the system can be rewritten as

$$\sum_{i=1}^{N} \alpha_i A(w_i, w_j) = L(w_j) \qquad j = 1, 2, \ldots, N,$$

where the specific form of the matrix

$$A = \begin{pmatrix} (\nabla w_1, \nabla w_1)_0 & (\nabla w_1, \nabla w_2)_0 & \ldots & (\nabla w_1, \nabla w_N)_0 \\ (\nabla w_2, \nabla w_1)_0 & (\nabla w_2, \nabla w_2)_0 & \ldots & (\nabla w_2, \nabla w_N)_0 \\ \vdots & \vdots & \ddots & \vdots \\ (\nabla w_N, \nabla w_1)_0 & (\nabla w_N, \nabla w_2)_0 & \ldots & (\nabla w_N, \nabla w_N)_0 \end{pmatrix},$$

and we have to solve only a system of linear equations for $\alpha_1, \ldots, \alpha_N$ with the matrix $A = (A(w_i, w_j))_{i,j=1}^N$ called **stiffness matrix** (because in the problems of linear statics where this method was first used this matrix expresses stiffness, i.e., the relation between the applied stress and the resulting strain of a body). It is symmetric and positive definite but generally it is a full matrix (i.e., it does not have many zero elements).

7.2 Example of finite element method

We would like the stiffness matrix to be as close to a diagonal matrix as possible. Similarly, in the nonlinear case we want to choose the basis $\{w_i\}_1^N$ so that the resulting system of equations is as simple as possible.

Finite element method can be characterized as a special case of Galerkin approximation, where the space V_N and its basis functions are chosen so that the stiffness matrix is sparse (i.e., it has only $O(N)$ nonzero elements).

The sparse matrix is achieved in three steps:

1. We create a **triangulation** of the closure of the considered domain $\bar{\Omega}$ into simple closed subdomains. These subdomains are called **elements** and they are usually line segments (in 1D), triangles or quadrilaterals (in 2D), tetrahedra, pentahedra or hexahedra (cuboids) (in 3D), etc.

2. The space V_N is chosen so that each function $v \in V_N$ has a simple form (usually polynomial) on each element. This space is called the **finite element space**.

3. We select the basis $w_1, \dots, w_N$ of the space V_N so that the basis functions w_i have small support (usually, only a few elements).

Example. Let us look at a concrete problem to solve the one-dimensional Poisson equation:

$$\begin{aligned} -u''(x) &= 2 \qquad x \in (0,1) \\ u(0) = u(1) &= 0. \end{aligned}$$

To derive the weak formulation, we multiply the equation by a test function $\varphi \in C_0^\infty(0,1)$ and integrate over $(0,1)$. We get

$$\int_0^1 -u''\varphi\,dx = 2\int_0^1 \varphi\,dx$$

Using integration by part we have

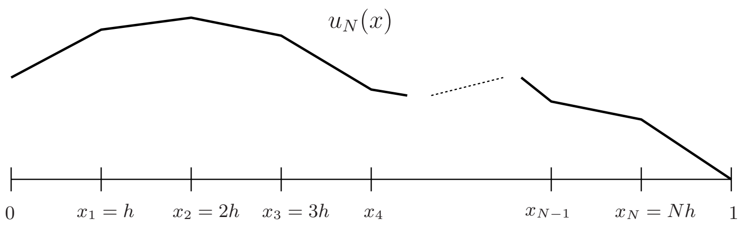
$$\int_0^1 u'\varphi'\,dx = 2\int_0^1 \varphi\,dx$$

The weak solution is thus defined as a function $u \in H_0^1(0,1)$ satisfying

$$\int_0^1 u'\varphi'\,dx = 2\int_0^1 \varphi\,dx \qquad \forall \varphi \in H_0^1(0,1).$$

We find an approximate solution to this problem by finite element method (FEM) according to the above steps.

1. The interval $(0,1)$ is partitioned into say $N+1$ subintervals of length $h = 1/(N+1)$. Let us denote the partition nodes by $x_0 = 0, x_1 = h, x_2 = 2h, \dots, x_N = Nh, x_{N+1} = 1$.



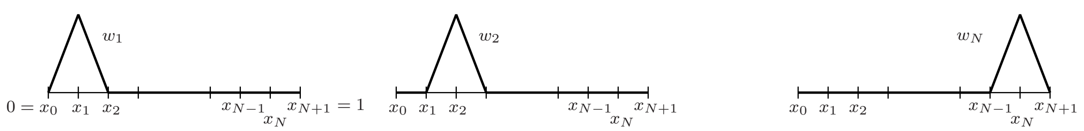
2. We set

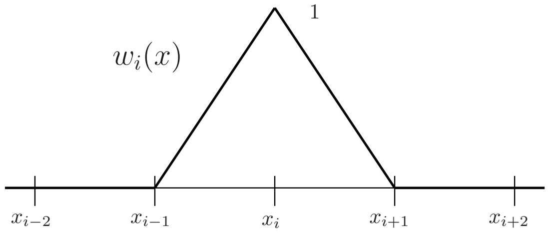
$$\begin{aligned} X_N &= \{v \in H^1(0,1);\ \ v \text{ is continuous and piecewise linear on the partition } \{x_i\}_{i=0}^{N+1}\} \\ V_N &= \{v \in X_N;\ \ v(0) = v(1) = 0\} \end{aligned}$$

The approximate Galerkin problem reads: find $u_N \in V_N$ satisfying

$$\int_0^1 u_N'\varphi'\,dx = 2\int_0^1 \varphi\,dx \qquad \forall \varphi \in V_N. \tag{7}$$

3. The main idea of FEM is to use the following basis $\{w_i\}_{i=1}^N$ for V_N:





$$w_i(x) = \begin{cases} 0 & x \le x_{i-1} \\ \frac{x-x_i}{h} & x \in [x_{i-1}, x_i] \\ \frac{-x+x_{i+1}}{h} & x \in [x_i, x_{i+1}] \\ 0 & x \ge x_{i+1} \end{cases}$$

This means that w_i is a piecewise linear function fulfilling

$$w_i(x_j) = \begin{cases} 0 & (i \neq j) \\ 1 & (i = j) \end{cases} \tag{8}$$

Such functions are called **Courant basis functions**.

Then if $v \in V_N$ it holds

$$v(x) = \sum_{i=1}^{N} v(x_i) w_i(x).$$

It is because the right-hand side is a continuous piecewise linear function (linear combination of such functions) and (8) holds.

We can see that the stiffness matrix A will be sparse because

$$a_{ij} = (w_i', w_j')_0 = \int_\Omega w_i' w_j' \, dx = 0 \qquad \text{if } \ i, j \ \text{ are not neighboring numbers or } \ i = j.$$

Now, since u_N belongs to V_N, we can write it using the basis $\{w_i\}$ of V_N:

$$u_N(x) = \sum_{i=1}^{N} \alpha_i w_i(x).$$

Inserting this form into (7), we have

$$\int_0^1 \Big(\sum_{i=1}^{N} \alpha_i w_i \Big)' \varphi' \, dx = 2 \int_0^1 \varphi \, dx \qquad \forall \varphi \in V_N.$$

This is the same as

$$\int_0^1 \sum_{i=1}^{N} \alpha_i w_i' w_j' \, dx = 2 \int_0^1 w_j \, dx \qquad j = 1, 2, \ldots, N.$$

Denoting

$$a_{ij} = \int_0^1 w_i' w_j' \, dx, \quad A = (a_{ij})_{i,j=1}^N, \quad b_j = 2 \int_0^1 w_j \, dx, \quad \boldsymbol{b} = (b)_{j=1}^N, \quad \boldsymbol{\alpha} = (\alpha_i)_{i=1}^N$$

we have obtained a linear system of equations with stiffness matrix A and right-hand side $\boldsymbol{b}$:

$$\sum_{i=1}^{N} a_{ij} \alpha_i = b_j \quad (j = 1, 2, \ldots, N) \qquad \text{or} \qquad A\boldsymbol{\alpha} = \boldsymbol{b}.$$

For completeness, let us compute the entries of A and $\boldsymbol{b}$. We immediately have

$$b_j = 2 \int_0^1 w_j \, dx = 2h, \qquad j = 1, \ldots, N.$$

As for A, we notice that a_{ij} vanishes if the supports of w_i and w_j do not overlap which is in every case except when $|i-j| \leq 1$. Hence, we have a tridiagonal matrix

$$a_{ij} = \begin{cases} \frac{2}{h} & (i=j) \\ -\frac{1}{h} & (|i-j|=1) \\ 0 & (\text{otherwise}) \end{cases}$$

The concrete form of the system is

$$\frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \vdots & . & . & . & & \\ \vdots & & . & . & . & \\ & & & & & \\ 0 & \dots & 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \\ \\ \vdots \\ \\ \alpha_N \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ \\ \\ \vdots \\ \\ 2 \end{pmatrix}.$$