1 18-04-18

This research is collaboration with Prof. Yoneda from University of Tokyo.
Main equation discussed is **Navier-Stokes equation** that usually discusses in fluid, for example air.

1.1 Navier-Stokes Equation

1.1.1 General Problem

For dimension $d = 2, 3, ...$ (usually 2 or 3) and $T > 0$, we want to find

$$(u, p) : \Omega \times (0, T) \to \mathbb{R}^d \times \mathbb{R}$$

where u is unknown velocity and p is unknown pressure such that

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \triangle u + \nabla p = f & \text{in } \Omega \times (0, T) \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u = u^0 & \text{in } \Omega, \text{ at } t = 0 \end{cases} \tag{1}$$

where $f : \Omega \times (0, T) \to \mathbb{R}^d$ and $u^0 : \Omega \to \mathbb{R}^d$ are given functions, $\nu > 0$ is a viscosity.

From equation (1) we can see that $\dfrac{\partial u}{\partial t} + (u \cdot \nabla)u$ is the **convection part** that explain the movement of fluid. This part, contain **nonlinear term** $(u \cdot \nabla)u$. We can also see, that $\nu \triangle u$ (similar to Heat equation) is the **diffusion part**. In the second equations, $\nabla \cdot u = 0$ explained the **incompressible condition** of fluid.

Incompressible condition :

$$\nabla \cdot u = div\ u = 0 \Leftrightarrow \text{ fluid is incompressible}$$

means that the total amount of body does not change. By

$$0 = \int_V \nabla \cdot u\ dx = \int_{\partial V} u \cdot n\ ds$$

means that the energy that comes in and comes out is same and the normal component of velocity is $0 = \int_{\partial V} u \cdot n\ ds$ where n is the normal vector works on boundary.

Convection effect :
[**simple explanation**] Let $\phi^0(x), c > 0$ is given. Consider $\phi(x, t) = \phi^0(x - ct)$, that represent the movement of function without changing the shape.

at $t = 0$ we have $\phi(x, 0) = \phi^0(x)$; at at $t = 1$ we have $\phi(x, 1) = \phi^0(x - c)$; at $t = 2$ we have $\phi(x, 2) = \phi^0(x - 2c)$ as shown above.

If we differentiate ϕ over x and t, then we obtain

$$\begin{cases} \dfrac{\partial \phi}{\partial t}(x, t) = \phi^0{'}(x - ct)\ (-c) = -c\ \phi^0{'}(x - ct); \text{ the initial function} \\ \dfrac{\partial \phi}{\partial x}(x, t) = \phi^0{'}(x - ct); \text{ moves to right with velocity c} \end{cases} .$$

From the above relation, we get

$$\frac{\partial \phi}{\partial t} + c\,\frac{\partial \phi}{\partial x} = 0.$$

If we consider velocity, $c \leftarrow u$ then $\dfrac{\partial \phi}{\partial t} + u\,\dfrac{\partial \phi}{\partial x} = 0$. Next, for **multidimensional convection equation**, $\dfrac{\partial}{\partial x} \leftarrow \nabla$, $\dfrac{\partial \phi}{\partial t} + (u \cdot \nabla)\phi = 0$. In Navier-Stokes equations, if we $\phi \leftarrow u_i$, then the first two term in first equation of equation (1).

[explanation] Let $u : \Omega \times (0, T) \to \mathbb{R}^d$ is given.

$$\frac{\partial \phi}{\partial t}(x,t) + [(u \cdot \nabla)\phi](x,t) = 0,\ \ (x,t) \in \Omega \times (0,T).$$

Let us consider the position of a fluid particle that satisfy

$$\begin{cases} X\prime(t) & = u(X(t),t),\ \forall t \\ X(t\star) & = x \end{cases}.$$

Calculate

$$\begin{aligned} \frac{d}{dt}[\phi(X(t),t)] &= (\nabla\phi)(X(t),t) \cdot X\prime(t) + \frac{\partial \phi}{\partial t}(X(t),t) \\ &= [(u \cdot \nabla)\phi](X(t),t) + \frac{\partial \phi}{\partial t}(X(t),t) \\ &= \left[\frac{\partial \phi}{\partial t} + (u \cdot \nabla)\phi\right](X(t),t). \end{aligned}$$

If we set $t = t\star$, then

$$\frac{d}{dt}[\phi(X(t),t)]_{|t=t\star} = \left[\frac{\partial \phi}{\partial t} + (u \cdot \nabla)\phi\right](x,t\star) = 0$$

or means that the function value does not change if it is changes by the velocity u, or called **characterictic line trajectory of particle**.

1.1.2 3D Problem

For $d = 3$, then $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ such that for $(i = 1,2,3)$ we have

$$\frac{\partial u_i}{\partial t} + (u \cdot \nabla)u_i - \nu \triangle u_i + [\nabla p]_i = f_i$$

where

$$\begin{aligned} (u \cdot \nabla)u_i &= \Big(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{bmatrix}\Big)u_i \\ &= (u_1\partial_1 + u_2\partial_2 + u_3\partial_3)u_i \\ &= u_1\frac{\partial u_i}{x_1} + u_2\frac{\partial u_i}{x_2} + u_3\frac{\partial u_i}{x_3} \end{aligned}$$

and

$$\triangle u_i = \frac{\partial^2 u_i}{\partial x_1^2} + \frac{\partial^2 u_i}{\partial x_2^2} + \frac{\partial^2 u_i}{\partial x_3^2}.$$

Note :we have u_1, u_2, u_3, p as four unknown functions and four equations (as first equation defined for three u and second equation), then we could find the solution.

1.2 Research Topic

We will study about axisymmetric flow (example : air). Consider sylindrical domain for first. We do two simulation, first : with the initial velocity with velocity concentration is in the center of axis, second : we include swirl, like tornado type velocity.

In this research it is proved that *if there is blow up, then there is swirl.* But has not proved that there is some blow-up phenomena ($\exists(x\star, t\star), t\star < \infty$ such that $\lim\limits_{(x,t)\to(x\star,t\star)} |u(x,t)| = \infty$) by Navier-Stokes.

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2.1 Weak Formulation

The time dependent Navier-Stokes equation strong formulation is shown as equation (1). We want to find $\{(u,p)(t) \in V \times Q; t \in (0.T)\}$ such that for $t \in (0,T)$

$$\begin{cases} (\frac{\partial u}{\partial t} + (u \cdot \nabla)u, v) + a(u,v) + b(v,p) + b(u,q) = (f,v) & ,\forall (v,q) \in V \times Q \\ u = u^0 & ,t = 0 \end{cases} \tag{2}$$

where

$$\begin{aligned} a(u,v) &= \nu \int_\Omega \nabla u : \nabla v \, dx \\ b(v,q) &= -\int_\Omega (\nabla \cdot v) q \, dx \\ V &= H_0^1(\Omega, \mathbb{R}^d) = H_0^1(\Omega)^d \\ Q &= \{q \in L^2(\Omega); \int_\Omega q \, dx = 0\}. \end{aligned}$$

The term $\int_\Omega q \, dx = 0$ is good for the uniqueness of the pressure. Because if (u,p) is solution, then $(u, p+c)$ for any constant c is also solution.

Notation often used. There are some notation that omit the sum or using index to simplify the writing.

$$A : B = \sum_{i,j=1}^{d} A_{ij} B_{ij} = tr(AB^T) = A_{ij} B_{ij}$$

Einstein's convection

$$\nabla \cdot u = \sum_{i=1}^{d} \frac{\partial u_i}{\partial x_i} = \frac{\partial u_i}{\partial x_i}.$$

$$\frac{\partial u_i}{\partial x_j} = u_{i,j}$$

The parenthesis we used usually defined the L^2 norm

$$(\triangle u, v)_{L^2(\Omega, \mathbb{R}^d)} \text{ or } (\triangle u, v)_{L^2(\Omega, \mathbb{R}^d \times \mathbb{R}^d)}$$

Gauss-Green Theorem. $\int_\Omega f_{,i} g \, dx = \int_{\partial\Omega} f g n_i ds - \int_\Omega f g_{,i} dx$ for $i = 1, \ldots, d$.

To obtain (2), we can multiple equation (1) with $v \in V$ and integrate over Ω.

$$\int_\Omega (\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \triangle u + \nabla p) v \, dx = \int_\Omega f v \, dx$$

using the parenthesis notation we obtain,

$$(\frac{\partial u}{\partial t} + (u \cdot \nabla)u, v) + (-\nu \triangle u, v) + (\nabla p, v) = (f, v)$$

Using index, the equation become,

$$(\frac{\partial u_i}{\partial t} + (u_i \cdot \nabla)u_i, v_i) + (-\nu \triangle u_i, v_i) + (\nabla p_i, v_i) = (f_i, v_i)$$

Using Gauss-Green theorem and integration by parts, with $v = 0$ on $\partial\Omega \times (0,T)$,

$$(\frac{\partial u_i}{\partial t} + (u_i \cdot \nabla)u_i, v_i) + (-\nu)(u_{i,jj}, v_i) + (p_{i,i}, v_i) = (f_i, v_i)$$

$$(\frac{\partial u_i}{\partial t} + (u_i \cdot \nabla)u_i, v_i) + \nu(u_{i,j}, v_{i,j}) - (v_{i,i}, p_i) = (f_i, v_i)$$

Then we can conclude that the weak form of the first equation in (1) is

$$(\frac{\partial u}{\partial t} + (u \cdot \nabla)u, v) + a(u,v) + b(v,p) = (f,v) \tag{3}$$

Then the second equation of (1) weak form can be obtained by multiply it with $q \in Q$. Because it is equal to 0, then we can mutiply it by (-1) such that $\forall q \in Q$

$$-(u_{i,i}, q_i) = b(u, q) = 0. \tag{4}$$

Add equation (3) and (4) we obtain the first equation in (2). Then for the third equation in (1) is already included in V which $u = 0$ on $\partial\Omega$.

If it is smooth enough, then the solution of weak form is equivalent to the strong form. Of course, the solution of strong form always fit for the weak form. But what is with the opposite ? if $v = 0$, then we obtain the (4), and if the $q = 0$ we obtain (3). For example if we can take any $q \in Q$, then because $b(u, q) = 0$ then $\nabla \cdot u$ must be 0.

2.2 FEM

$V_h \subset V$ can be vector valued piecewise linear functions or piecewise polynomial with degree two ($dimV_h < \infty$). Same for $Q_h \subset Q$, with ($dimQ_h < \infty$). For $\Delta t > 0$ and $N_T = \lfloor \frac{T}{\Delta t} \rfloor$. We approximate $u_h^n \approx u(\circ, n\Delta t)$ and $p_h^n \approx p(\circ, n\Delta t)$. Now lets try to solve the nonlinear part with schemes below.

2.2.1 Scheme 0

Find $\{(u_h^n, p_h^n) \in V_h \times Q_h \; ; n = 1, \dots, N_T\}$ such that

$$(\frac{u_h^n - u_h^{n-1}}{\Delta t}, v) + \big((u_h^{n-1} \cdot \nabla)u_h^n, v_h\big) + a(u_h^n, v_h) + b(v_h, p_h^n) + b(u_h^n, q_h) = (f^n, v_h) \tag{5}$$

with $u_h^0 \in V_h$: approximation of u^0 is given.

If u is smooth then

$$(u^n \cdot \nabla)u^n \approx (u^{n-1} \cdot \nabla)u^n + O(\Delta t)$$

such that the nonlinear part become linear, because we always know the previous u^{n-1} or given.

2.2.2 Scheme 1 (modification of Scheme 0)

$$(\frac{u_h^n - u_h^{n-1}}{\Delta t}, v_h) + \frac{1}{2}\Big(\big((u_h^{n-1} \cdot \nabla)u_h^n, v_h\big) - \big((u_h^{n-1} \cdot \nabla)v_h, u_h^n\big)\Big) + a(u_h^n, v_h) + b(v_h, p_h^n) + b(u_h^n, q_h) = (f^n, v_h) \tag{6}$$

Why we use it ?

$$\begin{aligned}
\big((u \cdot \nabla)u, v\big) &= \int_\Omega u_j u_{i,j} v_i \; dx \\
&= \int_\Omega u_{i,j}(u_j v_i) \; dx \text{ (using Gauss-Green and remember } u, v = 0 \text{ in } \partial\Omega) \\
&= \int_\Omega u_i (u_j v_i)_{,j} \; dx \\
&= -\int_\Omega u_i(u_{j,j} v_i + u_j v_{i,j}) \; dx \text{ (by } \nabla \cdot u = u_{i,i} = 0) \\
&= -\int_\Omega u_i u_j v_{i,j} \; dx \\
&= -\int_\Omega (u \cdot \nabla)v u \; dx \\
&= -\big((u \cdot \nabla)v, u\big)
\end{aligned}$$

Using equality above

$$\big((u \cdot \nabla)u, v\big) = \frac{1}{2}\big((u \cdot \nabla)u, v\big) + \frac{1}{2}\big((u \cdot \nabla)u, v\big) = \frac{1}{2}\big((u \cdot \nabla)u, v\big) - \frac{1}{2}\big((u \cdot \nabla)v, u\big).$$

Because it is linear scheme, then if we $(v_h, q_h) \leftarrow (u_h^n, -p_h^n)$ then the term $\frac{1}{2}\Big(\big((u_h^{n-1} \cdot \nabla)u_h^n, v_h\big) - \big((u_h^{n-1} \cdot \nabla)v_h, u_h^n\big)\Big)$ and $b(v_h, p_h^n) + b(u_h^n, q_h)$ vanishes. The term left is

$$(\frac{u_h^n - u_h^{n-1}}{\Delta t}, u_h^n) + a(u_h^n, u_h^n) = (f^n, u_h^n)$$

Since (6) is a system of linear equations, we need to show that $(u_h^n, p_h^n) = 0$ if $f^n = 0$ and $u_h^{n-1} = 0$. Then we can show it by show that $detA \neq 0$ for system $Au = b$ with $u = [\dots, u_h^n, \dots, p_h^n, \dots]$ and $b = [\dots, f^n, \dots, 0, \dots]$.

If we substitute $f^n = 0$ and $u_h^{n-1} = 0$, we obtain $\frac{1}{\Delta t}(u_h^n, u_h^n) + a(u_h^n, u_h^n) = \frac{1}{\Delta t}\|u_h^n\|_{L^2}^2 + \nu\|\nabla u_h^n\|_{L^2}^2 = 0$