Nonlinear PDEs

4th lecture

Lax-Milgram lemma

Bilinear form is a mapping $a(\cdot,\cdot)$ defined on $H \times H$ such that for each fixed $v \in H$, the mappings $a(v,\cdot)$ and $a(\cdot,v)$ are linear.

Theorem (Lax-Milgram lemma) Assume that the bilinear form $a : H \times H \to \mathbb{R}$ satisfies

$$\begin{array}{lrcll} \text{(continuity)} & |a(u,v)| & \leq & \alpha \|u\|_H \, \|v\|_H & \forall u, v \in H \quad (\alpha > 0) \qquad (1) \\ \text{(coercivity)} & \beta \|u\|^2 & \leq & a(u,u) & \forall u \in H \quad (\beta > 0) \qquad (2) \end{array}$$

Then for each bounded linear functional $F : H \to \mathbb{R}$ there exists a unique element $u \in H$ such that

$$B(u,v) = F(v) \qquad \forall v \in H.$$

Weak derivative

The definition of weak derivative is motivated by the following integration by parts formula which is valid for $u \in C^1(\Omega)$ and a test function $\varphi \in C_0^\infty(\Omega)$:

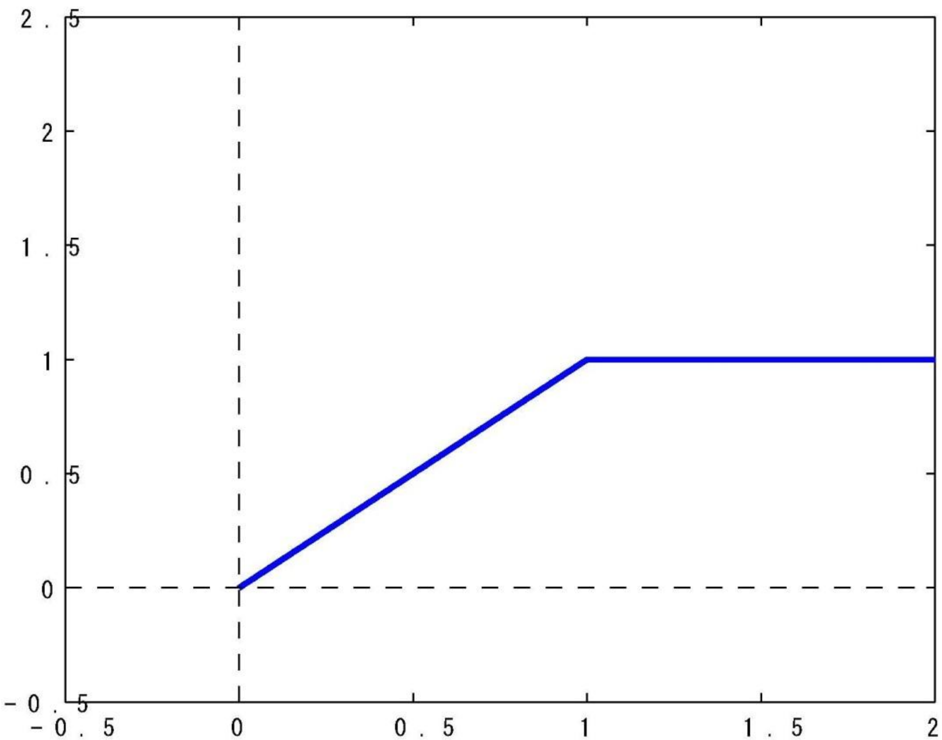
$$\int_\Omega u\varphi_{x_i}\, dx = -\int_\Omega u_{x_i}\varphi, \qquad i = 1, \ldots, n$$

Definition. Suppose $u, v \in L^1(\Omega)$. We say that v is the *first weak partial derivative* of u with respect to x_i (written $v = u_{x_i}$) provided

$$\int_\Omega u\varphi_{x_i}\, dx = -\int_\Omega v\varphi\, dx \qquad \text{for all test functions } \varphi \in C_0^\infty(\Omega).$$

Lemma. The weak derivative, if it exists, is uniquely defined up to a set of measure zero.

Example 1

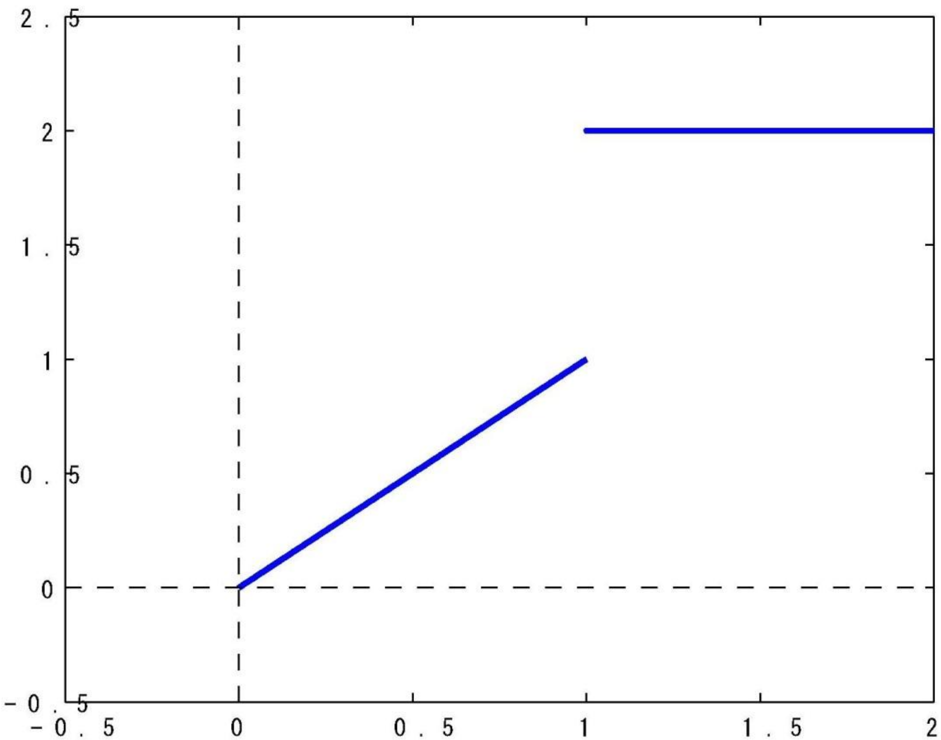


$$u(x) = \begin{cases} x & \text{if } 0 < x \leq 1 \\ 1 & \text{if } 1 \leq x < 2 \end{cases}$$

The weak derivative is

$$v(x) = \begin{cases} 1 & \text{if } 0 < x \leq 1 \\ 0 & \text{if } 1 < x < 2 \end{cases}$$

Example 2



$$u(x) = \begin{cases} x & \text{if } 0 < x \leq 1 \\ 2 & \text{if } 1 \leq x < 2 \end{cases}$$

The weak derivative does not exist.

Sobolev spaces

Definition. The *Sobolev space* $W^{1,p}(\Omega)$ consists of all locally summable functions $u : \Omega \to \mathbb{R}$ such that u_{x_i} exist for $n = 1, \ldots, n$ in the weak sense and u, u_{x_i} belong to $L^p(\Omega)$.

If $p = 2$ we usually write $H^1(\Omega) = W^{1,2}(\Omega)$.

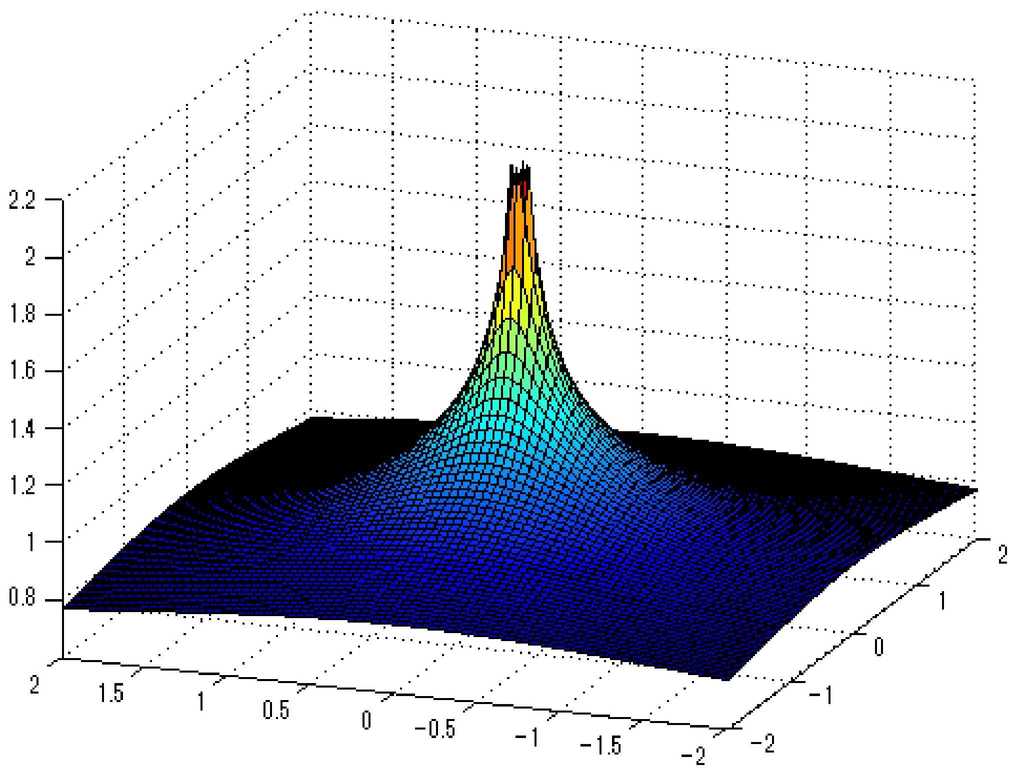
The norm in $W^{1,p}(\Omega)$ is defined as follows:

$$\|u\|_{W^{1,p}(\Omega)} = \begin{cases} \left(\int_\Omega (|u|^p + \sum_{i=1}^n |u_{x_i}|^p)\, dx\right)^{1/p} & (1 \leq \infty) \\ \text{ess sup}_\Omega\, |u| + \sum_{i=1}^n \text{ess sup}_\Omega\, |u_{x_i}| & (p = \infty) \end{cases}$$

Lemma. For each $1 \leq p \leq \infty$ the Sobolev space $W^{1,p}(\Omega)$ is a Banach space.

Rellich's theorem. Let Ω be a domain with lipschitz boundary. Then the identity mapping from $H^1(\Omega)$ to $L^2(\Omega)$ is compact, i.e., each bounded sequence in $H^1(\Omega)$ contains a subsequence converging in $L^2(\Omega)$.

Example



$$u(x) = |x|^{-1/4}$$

- is not continuous
- but belongs to H^1

Traces

By $W_0^{1,p}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. This means that if $u \in W_0^{1,p}(\Omega)$, then there exists a sequence of functions $u_m \in C_0^\infty(\Omega)$ such that $u_m \to u$ in $W^{1,p}(\Omega)$. It is customary to write $H_0^1(\Omega) = W_0^{1,2}(\Omega)$.

$H_0^1(\Omega)$ is interpreted as functions from $H^1(\Omega)$ that are zero on the boundary of Ω. This is not correct statement because functions from $H^1(\Omega)$ are defined only up to a set of measure zero and boundary has measure zero.
However, it can be made precise using the notion of trace operator.

Theorem on traces. For each lipschitz domain Ω there exists exactly one continuous linear operator $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$, such that

$$\gamma v = v\Big|_{\partial\Omega} \qquad \forall v \in C^\infty(\bar{\Omega}).$$

This theorem enables us to define the space $H_0^1(\Omega)$ as

$$H_0^1(\Omega) = \{v \in H^1(\Omega) \ ; \ \ \gamma v = 0 \ \text{ on } \partial\Omega\}.$$

Friedrichs' inequality

Friedrichs inequality. Let Ω be a domain with lipschitz boundary. Then there is a constant C_F such that

$$\|v\|_{H^1(\Omega)} \leq C_F |v|_{H^1(\Omega)} \qquad \forall v \in H_0^1(\Omega).$$

Here $|\cdot|_{H^1(\Omega)}$ is the seminorm defined as

$$|v|_{H^1(\Omega)} = \Big(\int_\Omega |\nabla v|^2 \Big)^{1/2}.$$

The fact that v has zero trace on the boundary is important!

This inequality says that the norm and seminorm on $H^1(\Omega)$ are equivalent norms for functions from $H_0^1(\Omega)$:

$$c\|v\|_{H^1(\Omega)} \leq |v|_{H^1(\Omega)} \leq \|v\|_{H^1(\Omega)} \qquad \forall v \in H_0^1(\Omega).$$