Assignment 6
Topics of Mathematical Science

Alifian Mahardhika Maulana

July 30, 2018

1. Solve :

$$z_0 \in \mathbb{C}, \quad \int_{|\varrho - z_0|=R} \frac{d\varrho}{(\varrho - z_0)^n} = \begin{cases} 2\pi i & (n=1) \\ 0 & (otherwise) \end{cases} \tag{1}$$

Answer:

By Cauchy's integral theorem,
Let $U = \varrho : |\varrho - z_0| < R$ be a simply connected open subset of C and f a function which is holomorphic on U and continuous on $\overline{U}$. Let $\gamma(t) = e^{it}, t \in [0, 2\pi]$ be a loop in $\overline{U}$, then the path integral,

$$\oint_\gamma \frac{1}{\varrho} dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}} dt = \int_0^{2\pi} idt = 2\pi i \tag{2}$$

. We define,

$$f(\varrho) = \frac{d\varrho}{(\varrho - z_0)^n}, \quad \varrho = z_0 + Re^{it},$$

then we can rewrite (1) as follows:

$$\int_0^{2\pi} \frac{1}{(Re^{it})^n} iRe^{it} dt = \frac{i}{R^{n-1}} \int_0^{2\pi} e^{i(1-n)t} dt \tag{3}$$

which according to (1), the integral on the right handside equal to $2\pi i$ if $n = 1$, therefore,

$$\frac{i}{R^{n-1}} \int_0^{2\pi} e^{i(1-n)t} dt = \frac{i}{R^{1-1}} 2\pi i = -2\pi$$

2. Prove the Residue Theorem
Let D be a domain with f is holomorphic around ∂D define by:

$$f(z) = \frac{a_{(n-1)}}{(z - z_0)^{(n-1)}} + \cdots + \frac{a_1}{(z - z_0)} + H(z)$$

then in it's interior, f has a pole except at point z_0. Choosing D_i small enough, and take the integral around ∂D for f, we have:

$$\int_{\partial D} f(z) dz = \int_{\partial D_i} \frac{a_i}{(z - z_0)^i} + \cdots + \frac{a_1}{(z - z_0)} + H(z) dz$$

compute the integration term by term, we get,

$$\int_{\partial D_i} \frac{a_1}{(z - z_0)} = 2\pi i a_1 = 2\pi i \; res(f, z_0)$$

if it's interior except at finite number in interior of C, then,

$$\int_C f(z) dz = 2\pi i (res(f, z_1) + \cdots + res(f, z_n))$$