Analysis Ia Report

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May 7, 2018

Problem 1. *Suppose that for every $n \in \mathbb{N}$ we have:*

$$b_n \leq a_n \leq c_n$$

Let

$$\lim_{n\to\infty} b_n = l = \lim_{n\to\infty} c_n$$

given $\epsilon > 0$, then it follows from the convergence of b_n and c_n to l that there exists a natural number N such that if $n \geq N$ then:

$$\begin{matrix} |b_n - l| < \epsilon \\ -\epsilon < b_n - l < \epsilon \end{matrix} \quad \textit{and} \quad \begin{matrix} |c_n - l| < \epsilon \\ -\epsilon < c_n - l < \epsilon \end{matrix}$$

Since the hypothesis implies that

$$\begin{aligned} b_n - l \leq& a_n - l \leq c_n - l \\ -\epsilon < b_n - l \leq& a_n - l \leq c_n - l < \epsilon \end{aligned}$$

it follows that

$$-\epsilon < a_n - l < \epsilon$$

for all $n \geq K$. Since $\epsilon > 0$ is arbitrary, this implies that

$$\lim_{n\to\infty} a_n = l$$

Problem 2. *Prove if a sequence of real numbers converges, then it is bounded and it is a Cauchy sequence.*

1. *If a sequence of real numbers converges, then it is bounded.*

   Proof. Suppose x_n be a sequence converges to x and let $\epsilon := 1$. Then there exist a natural number $K = K(1)$ such that $|x_n - x| < 1$ for all $n \geq K$. Then if we apply Triangle Inequality with $n \geq K$ we obtain

   $$|x_n| = |x_n - x + x| \leq |x_n - x| + |x| < 1 + |x|$$

   put

   $$M := \sup\{|x_1|, |x_2|, \cdots, |x_{K-1}|, 1 + |x|\},$$

   then it follows that $|x_n| \leq M$ for all $n \in \mathbb{N}$. □

2. If a sequence of real numbers converges, then it is a Cauchy sequence.

Proof. Suppose x_n be a sequence converges to x and let $\epsilon := \frac{\epsilon}{2}$, then there exist a natural number $K = K(1)$ such that $|x_n - x| < \frac{\epsilon}{2}$ for all $n \geq K$.
Let x_m be a sequence converges to x and let $\epsilon := \frac{\epsilon}{2}$, then there exist a natural number $K = K(1)$ such that $|x_m - x| < \frac{\epsilon}{2}$ for all $m \geq K$.
Applying Triangular Inequality to substraction of x_n and x_m, we obtain

$$
\begin{aligned}
|x_n - x_m| &= |x_n - x + x - x_m| \leq |x_n - x| + |x_m - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= |x_n - x + x - x_m| \leq |x_n - x| + |x_m - x| < \epsilon
\end{aligned}
$$

then it follows that $|x_n - x_m| < \epsilon$. □