Seminar Notes Alifian

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1 3D Linear Elasticity

$$\begin{aligned}
\Omega &\subset \mathbb{R}^d (d = 2, 3) \\
u = \Omega &\rightarrow \mathbb{R}^2 (\text{small displacement}) \\
x &\mapsto u(x)
\end{aligned}$$

1.1 Strain Tensor

$$\begin{aligned}
e[u] &= (e_{ij}[u]) \in \mathbb{R}^{dxd}_{sym} \\
e[u] &:= \frac{1}{2}(\bigtriangledown^T u + (\bigtriangledown^T u)^T)
\end{aligned} \tag{1}$$

1.2 Stress Tensor

$$\sigma[u] = (\sigma ij[u]) \in \mathbb{R}^{dxd}_{sym} \tag{2}$$

Based on Hook's Law, stress tensor must have equality with strain so that

$$\begin{aligned}
\sigma &= \mathbf{C}e \\
\text{with } \mathbf{C} &= \mathbf{C}_{ijkl} (\text{is a 4th order elasticity tensor}) \\
\sigma ij &= \mathbf{C} ijkl e_{kl} \\
\mathbf{C}_{ijkl} &= \mathbf{C}_{ijlk} = \mathbf{C}_{klij} (\text{symmetry}) \\
\mathbf{C}_{ijkl}\xi_{ij}\xi_{kl} &\geq \mathbf{C}_* |\xi|^2
\end{aligned}$$

1.3 Boundary Value Problem

$$\begin{cases}
-\partial_i \sigma_{ij}[u] & = f_j(x), x \in \Omega \\
u & = g(x), x \in \Gamma_D \\
\sigma[u]_\nu & = q(x), x \in \Gamma_N
\end{cases} \tag{3}$$

1.4 Equilibrium Equations of Force in Ω and on Γ_N

1.4.1 Strain Energy Density

$$\omega[u](x) := \frac{1}{2}\sigma[u] : e[u] \tag{4}$$

Solving using Sobolev Space in Isotropic Case, equation 4 becomes

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

with λ, μ called Lame Constant

$$\delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$

$$\begin{aligned} \sigma[u] &= (\sigma_{ij}[u]) \\ \sigma_{ij}[u] &= c_{ijkl}e_{kl}[u] \\ &= \lambda(\delta_k u_k)\delta_{ij} + \mu(\delta_i u_j + \delta_j u_i) \\ &= \lambda(\text{div } u)I + 2\mu e[u] \\ \omega[u] &= \frac{1}{2}(\lambda(\text{div } u)I + 2\mu e[u]) : e[u] \\ \omega[u] &= \frac{1}{2}(\lambda(\text{div } u)^2 + \mu|e[u]|^2 \end{aligned}$$

Remark 1. *Positivity of* $\boldsymbol{C}$

$$\begin{aligned} (\boldsymbol{C}\xi) : \xi &\geq \boldsymbol{C}_*|\xi|^2(\forall\xi \in \mathbb{R}^{dxd}_{sym}) \\ (\boldsymbol{C}\xi) : \xi &= \lambda|tr\ \xi|^2 + 2\mu|\xi|^2 \end{aligned}$$

If $\lambda \geq 0, \mu > 0$*, then* $C_* = 2\mu$

$$\xi = (\xi_{ij}), |\xi|^2 = \xi_{ij}\xi_{ij} = \sum_{i=1...d}^{d}\sum_{j=1...d}^{d}|\xi_{ij}|^2$$

1.5 Elasticity Problem

$$\begin{cases} -\text{div } \sigma[u] &= f(x) \text{ in } \Omega \subset \mathbb{R}^d \\ u &= g(x) \text{ on } \Gamma_D \\ \sigma[u]v &= q(x) \text{ on } \Gamma_N \end{cases} \tag{5}$$

1.6 Crack Problem

$$\begin{cases} -\text{div } \sigma[u] &= f(x) \text{ in } \Omega \setminus \Sigma \subset \mathbb{R}^d \\ u &= g(x) \text{ on } \Gamma_D \\ \sigma[u]v &= q(x) \text{ on } \Gamma_N \\ \sigma[u]v &= 0 \text{ on } \Sigma^+ \cup \Sigma^- \end{cases} \tag{6}$$

1.7 Lebesque Measurable Theory

$$L^p(\Omega) := \left\{ v : \Omega \to \mathbb{R} | \begin{cases} v = \text{Lebesque measurable} \\ \int_\Omega |v(x)|^p dx < \infty \end{cases} \right\} \tag{7}$$

Remark 2. *for $u, v \in \mathbb{L}^p(\Omega)$, if $\exists N \subset \Omega$ such that $\begin{cases} u(x) = v(x)(x \in \Omega \setminus N) \\ \mathcal{L}^d(N) = 0, \end{cases}$ then we identify u and v, $\mathcal{L}^d(N) = 0 \Leftrightarrow$ volume of $N = 0$ for simplicity, we also can say that $u(x) = v(x)$ for a.e. $x \in \Omega$*

for example

$$\begin{aligned} & v : \mathbb{R} \to \mathbb{R} \\ & v(x) = \begin{cases} 1, x \in \mathbb{Q} \\ 0, x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \\ & \int_{\mathbb{R}} v dx = 0, \mathcal{L}^1(\mathbb{Q}) = 0 \\ & v(x) = 0 \text{ on } \mathbb{R} \setminus \mathbb{Q}, \text{ or we can say } v = 0 \text{ a.e. in } \mathbb{R} \end{aligned} \tag{8}$$

1.8 Sobolev Space

$$\mathbb{W}^{1,p}(\Omega) := \left\{ v \in \mathbb{L}^p(\Omega) \frac{\partial v}{\partial x_j}_{(j=1\ldots d)} \in \mathbb{L}^p(\Omega) \right\} \tag{9}$$

such $\frac{\partial v}{\partial x_j}$ we called it distribution sence.
example of Sobolev Space is as follow:

$$v \in \mathbb{L}^p(\Omega) \text{ if } \exists \omega_j \in \mathbb{L}(\Omega)$$

such that

$$\int_\Omega v \frac{\partial \varphi}{\partial x_j} dx = -\int_\Omega \omega_j \varphi dx (\forall \varphi \in \mathbb{C}_0^\infty(\Omega))$$

$$\Rightarrow \frac{\partial \varphi}{\partial x_j} = \omega_j \text{ in distribution sence}$$

for

$$v \in \mathbb{C}^1(\Omega), \frac{\partial v}{\partial x_j}(x) = \omega_j(x)$$

$$\Updownarrow$$

$$\int_\Omega \omega_j \varphi dx = -\int_\Omega v \frac{\partial \varphi}{\partial x_j} dx (\forall \varphi \in \mathbb{C}_0^\infty(\Omega))$$

In particular,

$$\mathbb{H}^1(\Omega) := \mathbb{W}^{1,2}(\Omega), \bigtriangledown u = \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_d} \end{pmatrix}$$

inner product

$$(u, v)_{\mathbb{H}^1(\Omega)} := \int_\Omega uv\ dx + \int_\Omega \bigtriangledown u \cdot \bigtriangledown v\ dx$$

norm

$$||u||_{\mathbb{H}^1(\Omega)} := \sqrt{(u, v)_{\mathbb{H}^1(\Omega)}} = \sqrt{\int_\Omega |u|^2 dx + \int_\Omega | \bigtriangledown u|^2 dx}$$

$\mathbb{H}^1(\Omega)$ is complete ($\mathbb{H}^1(\Omega)$ is a Hilbert Space)

$$(u,v)_{\mathbb{L}^2(\Omega)} = \int_\Omega uvdx$$

1.9 Incomplete Hilbert Space

$\mathbb{V}:$ a vector space in $\mathbb{R}$

$$\begin{cases} u,v & \in \mathbb{V} \Rightarrow \alpha u + \beta v \in \mathbb{V} \\ \alpha, \beta & \in \mathbb{R} \end{cases}$$

If $(\cdot,\cdot) : \mathbb{V}\times\mathbb{V} \to \mathbb{R}$ satisfies

$$\begin{cases} (u\cdot v) \geq 0 \text{ and } (u,u)=0 \Leftrightarrow u = 0_v \in \mathbb{V} \\ (u,v) = (v,u) \\ (\alpha u + \beta v, \omega) = \alpha(u,\omega) + \beta(v,\omega) \end{cases}$$

then we call $[\mathbb{V}\times\mathbb{V}]$ pre Hilbert space or incomplete Hilbert Space.

1.10 Property of $\mathbb{L}^2(\Omega)$

For $v \in \mathbb{C}^1(\Omega)$,

$$\frac{\partial v}{\partial x_j}(x) = w_j(x)$$

$$\Updownarrow$$

$$\int_\Omega w_j \varphi dx = -\int \Omega v \frac{\partial \varphi}{\partial x_j} dx \ \left(\forall \varphi \in \mathbb{C}_0^\infty(\Omega)\right)$$

$(u,v)_{\mathbb{L}^2(\Omega)} = \int_\Omega uvdx$

$$\Rightarrow \left| \int_\Omega uvdx \right| \leq \int_\Omega |u||v|dx \leq ||u||_{\mathbb{L}^2(\Omega)} ||v||_{\mathbb{L}^2(\Omega)}$$

$u, v \in \mathbb{H}^1(\Omega)$

$$\Rightarrow \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j} \in \mathbb{L}^2(\Omega)$$

$$\left| \int_\Omega \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j} dx \right| \leq \int_\Omega \left| \frac{\partial u}{\partial x_j} \right| \left| \frac{\partial v}{\partial x_j} \right| dx \leq \left| \left| \frac{\partial u}{\partial x_j} \right| \right|_{\mathbb{L}^2(\Omega)} \left| \left| \frac{\partial v}{\partial x_j} \right| \right|_{\mathbb{L}^2(\Omega)}$$

$$\nabla u \cdot \nabla v = \sum_{j=1}^{d} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j}$$

$$\begin{aligned}
\left| \int_\Omega \nabla u \cdot \nabla v dx \right| &= \left| \int \sum_{j=1}^{d} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j} dx \right| \leq \sum_{j=1}^{d} \int \left| \frac{\partial u}{\partial x_j} \right| \left| \frac{\partial v}{\partial x_j} \right| dx \\
&\leq \sum_{j=1}^{d} \left| \left| \frac{\partial u}{\partial x_j} \right| \right|_{\mathbb{L}^2(\Omega)} \left| \left| \frac{\partial v}{\partial x_j} \right| \right|_{\mathbb{L}^2(\Omega)} \\
&\leq \sqrt{\sum_{j=1}^{d} \left| \left| \frac{\partial u}{\partial x_j} \right| \right|_{\mathbb{L}^2(\Omega)}} \sqrt{\sum_{j=1}^{d} \left| \left| \frac{\partial v}{\partial x_j} \right| \right|_{\mathbb{L}^2(\Omega)}} \\
&= \sqrt{\sum_{j=1}^{d} \int_\Omega \left| \frac{\partial u}{\partial x_j} \right|^2 dx} \sqrt{\sum_{j=1}^{d} \int_\Omega \left| \frac{\partial v}{\partial x_j} \right|^2 dx} \\
&= \sqrt{\int_\Omega \left(\sum_{j=1}^{d} \left| \frac{\partial u}{\partial x_j} \right|^2 \right) dx} \sqrt{\int_\Omega \left(\sum_{j=1}^{d} \left| \frac{\partial v}{\partial x_j} \right|^2 \right) dx} \\
&= \sqrt{\int_\Omega \left| \nabla u \right|^2 dx} \sqrt{\int_\Omega \left| \nabla v \right|^2 dx}
\end{aligned}$$

$$\therefore \left| \int_\Omega \nabla u \cdot \nabla v dx \right| \leq \sqrt{\int_\Omega \left| \nabla u \right|^2 dx} \sqrt{\int_\Omega \left| \nabla v \right|^2 dx} \tag{10}$$

1.11 Energy (Revisited)

$$E(u) := \frac{1}{2} \int_\Omega \sigma[u] : e[u] dx - \int_\Omega f \cdot u dx - \int_{\Gamma_N} q \cdot u ds \tag{11}$$

with u is a vector of the elasticity problem define by:

$$\begin{aligned}
u \in \mathbb{H}^1(\Omega : \mathbb{R}^d) &:= \{u : \Omega \to \mathbb{R}^d | u = (u_i, \cdots, u_d), u_i \in \mathbb{H}^1(\Omega)\} \\
&\Rightarrow E(u) < \infty
\end{aligned}$$

u : become solution $\Leftrightarrow u = argmin_{v \in \mathbb{H}^1(\Omega : \mathbb{R}^d)} E(v)$ such a technique we call it variational principle.

1.12 Variational Principle

Let's consider a Poisson Equation Problem:

$$\Omega \subset \mathbb{R}^d \begin{cases} -\triangle u & = f(x) \in \Omega \\ u & = g(x) \text{ on } \Gamma_D \ f \in L^2(\Omega), g \in H^1(\Omega), q \in L^2(\Gamma_N) \\ \frac{\partial u}{\partial v} & = q(x) \text{ on } \Gamma_N \end{cases} \tag{12}$$

Remark 3.

$$v \in H^1(\Omega) \Rightarrow \exists v|_\Gamma \in L^2(\Gamma)$$

we choose v on L^2 because it will has value on the boundary

1.12.1 Definition of Weak Solution

$$u \in H^1(\Omega) \text{ s.t. } \begin{cases} \int_\Omega \triangle u \cdot \triangle v dx = \int_\Omega f v dx + \int_{\Gamma_N} q v ds \\ \left(\forall v \in V := v \in H^1(\Omega) \middle| v|_{\Gamma_D} \right) \\ v|_{\Gamma_D} = g|_{\Gamma_D} (v - g \in V) \end{cases}$$

$(v - g \in V)$ mean $(v \in V + g := v + g | v \in V)$ with V is an affine space.

1.12.2 Definition of Strong Solution

$u \in H^2(\Omega)$ and u satisfies (12)

Remark 4.

$$H^2(\Omega) := \{u \in L^2(\Omega) \frac{\partial u}{\partial x_j}, \frac{\partial^2 u}{\partial x_i x_j} \in L^2(\Omega)\}$$

$$u \in H^2(\Omega) \Rightarrow \frac{\partial u}{\partial x_j} \in H^1(\Omega)$$

$$\frac{\partial u}{\partial}$$