# Topics in Computational Science Report
# Norbert Pozar Class

Alifian Mahardhika Maulana

June 15, 2018

1. Suppose that $u$ is a twice continuously differentiable positive solution of the porous medium equation:

$$u_t - \triangle(u^m) = 0, \quad x \in \mathbb{R}^n, t > 0 \tag{1}$$

In the pressure form:

$$p_t - (m-1)p\triangle p - |\nabla p|^2 = 0 \tag{2}$$

Then, we define:

$$p(x,t) := \frac{m}{m-1} u^{m-1}(x,t) \tag{3}$$

Show that (3) is a solution of (1) in (2) form.

**Answer:**
First we compute:

$$p_t = \frac{\partial}{\partial t} p(x,t), \quad \triangle p = \tfrac{\partial^2}{\partial x^2} p(x,t) = \tfrac{\partial}{\partial x} \nabla p \tag{4}$$

$$\nabla p = \frac{\partial}{\partial x} p(x,t), \qquad \triangle(u^m) = \tfrac{\partial^2}{\partial x^2} u^m \tag{5}$$

Applied Chain Rule to (4),(5), we get:

$$\begin{aligned} p_t &= \frac{\partial}{\partial t} p(x,t) \\ &= \frac{\partial p}{\partial u}\frac{\partial u}{\partial t} \\ &= \frac{m}{(m-1)}(m-1)u^{(m-2)}\frac{\partial}{\partial t}u \\ &= mu^{(m-2)}u_t \end{aligned} \tag{6}$$

$$\begin{aligned} \nabla p &= \frac{\partial}{\partial x} p(x,t) \\ &= \frac{\partial p}{\partial u}\frac{\partial u}{\partial x} \\ &= \frac{m}{(m-1)}(m-1)u^{(m-2)}\frac{\partial}{\partial x}u \\ &= mu^{(m-2)}\nabla u \end{aligned} \tag{7}$$

$$\begin{aligned} \triangle p &= \frac{\partial}{\partial x}\nabla p \\ &= \frac{\partial}{\partial x}\big(mu^{(m-2)}\nabla u\big) \\ &= \big(m(m-2)u^{(m-3)}\nabla u\big)\nabla u + mu^{(m-2)}\triangle u \\ &= m(m-2)u^{(m-3)}|\nabla u|^2 + mu^{(m-2)}\triangle u \end{aligned} \tag{8}$$

$$\begin{aligned} \triangle(u^m) &= \frac{\partial^2}{\partial x^2}u^m \\ &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}u^m\right) \\ &= \frac{\partial}{\partial x}\left(mu^{(m-1)}\nabla u\right) \\ &= \big(m(m-1)u^{(m-2)}\nabla u\big)\nabla u + (mu^{(m-1)})\triangle u \\ &= m(m-1)u^{(m-2)}|\nabla u|^2 + mu^{(m-1)} \end{aligned} \tag{9}$$

Substitute (6), (7) $\rightarrow$ (2), we get:

$$mu^{(m-2)}u_t - mu^{(m-1)}\triangle p - m^2u^{2(m-2)}|\nabla u|^2 = 0 \tag{10}$$

Divide (10) with $mu^{(m-2)}$, we get:

$$u_t - u\triangle p - mu^{(m-2)}|\nabla u|^2 = 0 \tag{11}$$

Substitute (8) $\rightarrow$ (11), we get:

$$
\begin{aligned}
u_t - u\Bigg(m(m-2)u^{(m-3)}|\nabla u|^2 + mu^{(m-2)}\triangle u\Bigg) - mu^{(m-2)}|\nabla u|^2 = 0 \\
u_t - \Bigg(m(m-2)u^{(m-2)}|\nabla u|^2 + mu^{(m-1)}\triangle u\Bigg) - mu^{(m-2)}|\nabla u|^2 = 0 \\
u_t - mu^{(m-2)}|\nabla u|^2\Bigg((m-2)+1\Bigg) - mu^{(m-1)}\triangle u = 0 \\
u_t - m(m-1)u^{(m-2)}|\nabla u|^2 - mu^{(m-1)}\triangle u = 0
\end{aligned}
\tag{12}
$$

Substitute (9) $\rightarrow$ (12),

$$\therefore u_t - \triangle(u^m) = 0 \tag{13}$$

2. For $n \in \mathbb{N}$ and constants $m > 1, C > 0, \alpha > 0, \beta > 0$. We define function $u : \mathbb{R}x(0,\infty) \to \mathbb{R}$ as

$$u(x,t) = t^{-\alpha}\Bigg(max\Bigg(C - \frac{\beta(m-1)}{2m}\frac{|x^2|}{t^{2\beta}}, 0\Bigg)\Bigg)^{\frac{1}{m-1}} \qquad x \in \mathbb{R}^n, t > 0, \tag{14}$$

with $|x| := (\sum_{i=1}^{n} x_i^2)^{1/2}$

**Answer:**

(a) We want to find $\alpha$ and $\beta$ in terms of $m$ and $n$ so that $u$ is a solution of (14) in the set

$$(x,t) : x \in \mathbb{R}^n, t > 0, u(x,t) > 0$$

Because of the set of $u(x,t) > 0$, so we choose $u(x,t)$ as:

$$u(x,t) = t^{-\alpha}\Bigg(C - \frac{\beta(m-1)}{2m}\frac{|x^2|}{t^{2\beta}}\Bigg)^{\frac{1}{m-1}} \tag{15}$$

Then we substitute (15) $\to$ (3), we get:

$$p(x,t) = \frac{m}{(m-1)}t^{-\alpha(m-1)}\Bigg(C - \frac{\beta(m-1)}{2m}\frac{|x|^2}{t^{2\beta}}\Bigg) \tag{16}$$

After that, we compute $p_t, \nabla p,$ and $\triangle p$ of (16).

$$p_t = \frac{\partial}{\partial t}p(x,t) = -\alpha m\Bigg(C - \frac{\beta(m-1)}{2m}\frac{|x|^2}{t^{2\beta}}\Bigg)t^{(-\alpha(m-1)-1)} + \beta^2|x|^2 t^{(-\alpha(m-1)-2\beta-1)} \tag{17}$$

$$\nabla p = \frac{\partial}{\partial x}p(x,t) = -\beta x t^{(-\alpha(m-1)-2\beta)} \tag{18}$$

$$\triangle p = -\beta n t^{(-\alpha(m-1)-2\beta)} \tag{19}$$

Then, we substitute (17), (18), and (19) to (2), we get:

$$\begin{aligned} 0 =& -\alpha m\Bigg(C - \frac{\beta(m-1)}{2m}\frac{|x|^2}{t^{2\beta}}\Bigg)t^{(-\alpha(m-1)-1)} + \beta^2|x|^2 t^{(-\alpha(m-1)-2\beta-1)} \\ &+ (m-1)p\beta n t^{(-\alpha(m-1)-2\beta)} - \beta^2|x|^2 t^{2(-\alpha(m-1)-2\beta-1)} \\ 0 =& p(x,t)(m-1)\Bigg(-\frac{\alpha}{t} + \beta n t^{(-\alpha(m-1)-2\beta)}\Bigg) + \beta^2|x|^2\Bigg(t^{(-\alpha(m-1)-2\beta-1)} + t^{2(-\alpha(m-1)-2\beta)}\Bigg) \end{aligned} \tag{20}$$

From here, we recall the set is $u(x,t) > 0, m > 1,$ and $\beta > 0$ which implied $p(x,t) > 0$, therefore (20) holds for:

$$-\frac{\alpha}{t} + \beta n t^{(-\alpha(m-1)-2\beta)} = 0$$

Then,

$$\beta n t^{(-\alpha(m-1)-2\beta)} = \alpha t^{-1} \tag{21}$$

To satisfies (21),

$$\alpha = \frac{1}{(m-1)+2}, \qquad \beta = \frac{1}{n(m-1)+2} \tag{22}$$

(b) For given $t > 0$, the set $\Omega(t) := x \in \mathbb{R}^n : u(x,t) > 0$ an $n$-dimensional ball. We want to find its radius $r = r(t)$ and $\lim_{t\to 0+} r(t)$ and $\lim_{t\to\infty} r(t)$.
In the set $\Omega$, for $\alpha > 0$,

$$u(x,t) > 0 \iff C - \frac{\beta(m-1)}{2m}\frac{|x^2|}{t^{2\beta}} \geq 0$$

with $x$ as its radius in domain $\Omega$, hence,

$$\begin{aligned} C - \frac{\beta(m-1)}{2m}\frac{|r^2|}{t^{2\beta}} &= 0 \\ \frac{\beta(m-1)}{2m}\frac{|r^2|}{t^{2\beta}} &= C \\ |r^2| &= \frac{C2mt}{\beta(m-1)} \\ \therefore r(t) &= \Bigg(\frac{2mtC}{\beta(m-1)}\Bigg)^{1/2} \end{aligned} \tag{23}$$

Then we compute the limit:

$$\lim_{t\to 0+} r(t) = 0, \qquad \lim_{t\to\infty} r(t)\left(\frac{2mtC}{\beta(m-1)}\right)^{1/2} = \infty \tag{24}$$

(c) With $\alpha$ and $\beta$ from (22), $n=2$, we define:

$$M(t) := \int_{\mathbb{R}^n} u(x,t)dx$$

in polar coordinates,

$$M(t) = \int_{\mathbb{R}^2} u(x,t)dxdy = \iint ru(x,t)drd\theta$$

then, we compute $M(t)$ for $t>0$,

$$\int_0^{2\pi}\int_0^{\infty} ru(x,t)drd\theta = \int_0^{2\pi}\int_0^{r(t)} ru(x,t)drd\theta + \int_0^{2\pi}\int_{r(t)}^{\infty} ru(x,t)drd\theta \tag{25}$$

for $r > r(t)$, we know from (15), $u(x,t)$ tend to 0, hence (25) becomes:

$$\begin{aligned}\int_0^{2\pi}\int_0^{\infty} ru(x,t)drd\theta &= \int_0^{2\pi}\int_0^{r(t)} ru(x,t)drd\theta \\ &= \int_0^{2\pi}\int_0^{r(t)} t^{-\alpha} r\left(C - \frac{\beta(m-1)}{2m}\frac{r^2}{t^{2\beta}}\right)drd\theta \\ &= \int_0^{2\pi} t^{-\alpha}\left(\frac{C}{2}r^2 - \frac{\beta(m-1)}{2mt^{2\beta}}\frac{r^4}{4}\right)\Bigg|_0^{r(t)} d\theta \\ &= \int_0^{2\pi} t^{-\alpha}\left(\frac{Cr^2(t)}{2} - \frac{\beta(m-1)r^4(t)}{8mt^{2\beta}}\right)d\theta\end{aligned} \tag{26}$$

From here, we substitute $r(t)$ defined in (23) to (26), we get:

$$\begin{aligned}\int_0^{2\pi}\int_0^{\infty} ru(x,t)drd\theta &= \int_0^{2\pi} t^{-\alpha}\left(\frac{C2mt^{\beta}C}{2\beta(m-1)} - \frac{\beta(m-1)4m^2t^{4\beta C^2}}{8mt^{2\beta}\beta^2(m-1)^2}\right)d\theta \\ &= \int_0^{2\pi} t^{-\alpha}\frac{mt^{2\beta}C^2}{2\beta(m-1)}d\theta \\ &= \frac{mt^{2\beta}C^2}{2\beta(m-1)}2\pi\end{aligned} \tag{27}$$

from here, we substitute $\alpha$ and $\beta$ from (22) to (27), we get:

$$\therefore M(t) = \int_0^{2\pi}\int_0^{\infty} ru(x,t)drd\theta = \frac{2m^2C^2\pi}{m-1} \tag{28}$$

From (28) we know $M(t)$ on $t>0$ is constant over time, therefore the mass $M(t)$ conserved.