

1 Unix Command Terminal

1.1 5 October 2017

- **mk dir** : make a file in directory.
- **ls** : file list in directory
- **cd** : change directory
- **pwd** : parent working directory
- **cp -r** : copy
- **rm** : remove
- **open ...** : open ... file
- **.** : here
- **cd ..** : back to previous

2 Gnuplot

2.1 16 October 2017

2.1.1 Plot from terminal

```
{gnuplot
set terminal x11 (for 2D and 3D)
plotsin(x)
plotsin(x),cos(x)
set hidden3d
splotsin(x)*sin(y)
set xrange[-5:5]
set yrange[-5:5]
q quit
```

2.1.2 Plot from file

We have file **plot.dat** or **.txt** contains list point of triangular format for 2D as shown below:

$x_1^{(i)}$	$x_2^{(i)}$	$u^{(i)}$
$x_1^{(j)}$	$x_2^{(j)}$	$u^{(j)}$
$x_1^{(k)}$	$x_2^{(k)}$	$u^{(k)}$
$x_1^{(i)}$	$x_2^{(i)}$	$u^{(i)}$

(leave blank)

There are some "blank" on line. In 1D case, we will have two lines of points and 1 blank line. Then, for 2D case, we have four lines of points and two blank (anw: I think it still work for one blank thought).

```
{splot 'plot.dat' u 1:2:3 w l palette
```

where **u** means using, **1:2:3** means the column we wish to plot, **w** means with, **l** means line, and **palette** means color.

Other command that maybe used is,

```
{set pm3d map
is used for mapping 3D to 2D. Other is,
{set size ratio -1
then de data will be integers.
```

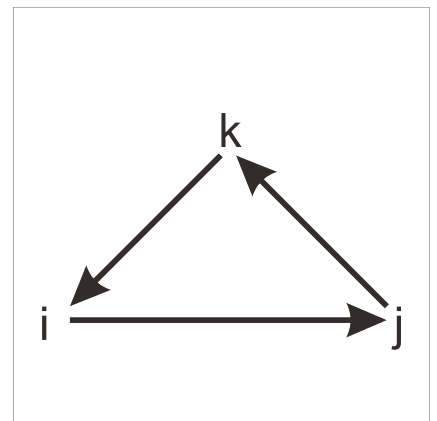


Figure 1: One element with point i, j, k

3 FreeFEM++

3.1 16 October 2017

Example 1 :

$$\begin{cases} \text{int } n = 50; \\ \text{real } x0 = 0.0, \ y0 = 0.0, \ Lx = 1.0, \ Ly = 1.0, \ z = 1; \\ \text{border } a1(t = 0, 2 * \pi) \{ x = z * \cos(t); \ y = z * \sin(t); \} \end{cases}$$

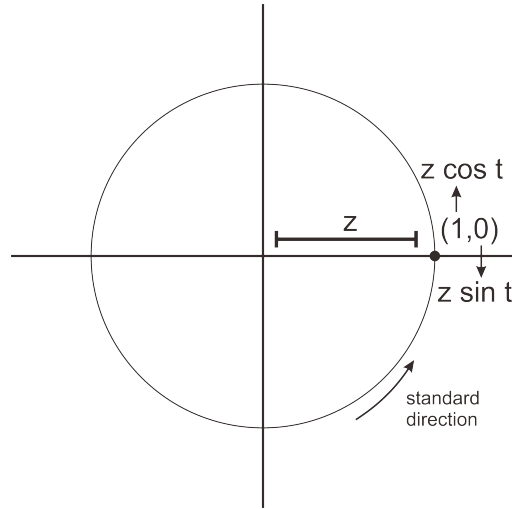


Figure 2: Example 1

Example 2:

$$\begin{cases} \text{border } a1(t = 0, 1) \{ x = t; y = 0; \} \\ \text{border } a2(t = 0, 1) \{ x = 1; y = t; \} \end{cases}$$

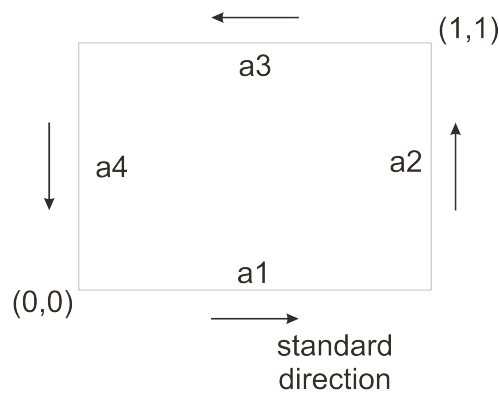


Figure 3: Example 2

Here are image of how to choose the domain.

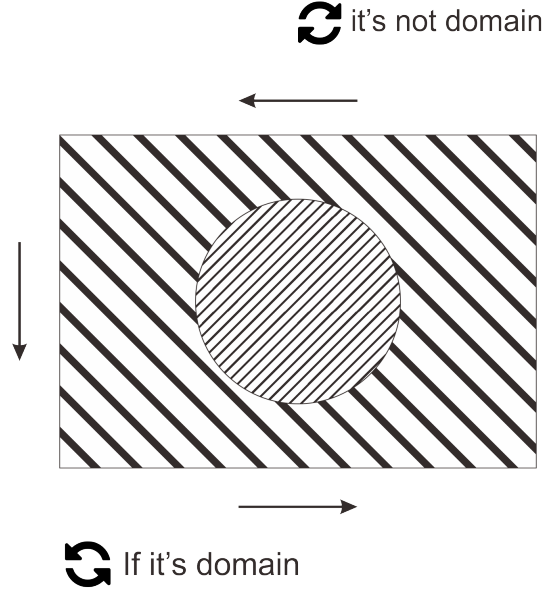


Figure 4: Choose the domain

From file : **membrane.edp**. Find $\phi : \Omega \rightarrow \mathbb{R}$ such that

$$\begin{cases} -\Delta \phi = f(= 1) & \text{in } \Omega \\ \phi = z(= x_1) & \text{on } \Gamma_1 \\ \frac{\partial \phi}{\partial n} = 0 & \text{on } \Gamma_2 \end{cases} \quad (1)$$

where $f : \Omega \rightarrow \mathbb{R}$, given $f(x) = 1$ and $z : \Gamma_1 \rightarrow \mathbb{R}$, given $z(x) = x_1$. The equation (1) is called **strong form**, because it contains second derivative.

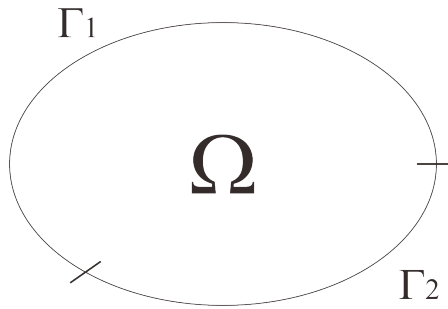


Figure 5

We multiple equation (1)'s first line by smooth test function w and integrate over Ω such that $\forall w(w|_{\Gamma_1} = 0)$

$$\int_{\Omega} -\Delta \phi(x)w(x) \, dx = \int_{\Omega} f(x)w(x) \, dx$$

using integration by parts,

$$-\int_{\partial\Omega} \frac{\partial \phi}{\partial n}(x)w(x) \, dx + \int_{\Omega} \nabla \phi(x) \cdot \nabla w(x) \, dx = \int_{\Omega} f(x)w(x) \, dx$$

then we can divide the boundary such that

$$-\int_{\Gamma_1} \frac{\partial \phi}{\partial n}(x) w(x) dx - \int_{\Gamma_2} \frac{\partial \phi}{\partial n}(x) w(x) dx + \int_{\Omega} \nabla \phi(x) \cdot \nabla w(x) dx = \int_{\Omega} f(x) w(x) dx$$

Because on Γ_1 , smooth function $w(x)$ value is equal to 0. And on Γ_2 by equation (1) line 3 we obtain that $\frac{\partial \phi}{\partial n}(x)$ is equal to 0. Then, we conclude that

$$\int_{\Omega} \nabla \phi(x) \cdot \nabla w(x) dx - \int_{\Omega} f(x) w(x) dx = 0 \quad (2)$$

is the **weak form**. Because it only contain first derivative.

3.2 6 November 2017

3.2.1 Gauss-Green Formula

In **2D Case**, we have $f, g : \Omega \rightarrow \mathbb{R}$

$$\int_{\Omega} \frac{\partial f}{\partial x_i}(x) g(x) dx = \int_{\partial \Omega} f(x) g(x) n_i ds - \int_{\Omega} f(x) \frac{\partial g}{\partial x_i}(x) dx, \quad (i = 1, 2) \quad (3)$$

In **1D Case**,

$$\int_a^b f(x) g(x) dx = [f(x) g(x)]_{x=a}^b - \int_a^b f(x) g'(x) dx \quad (4)$$

where

$$\begin{aligned} \int_{\partial \Omega} f(x) g(x) n(x) dx &= f(a) g(a) n(a) + f(b) g(b) n(b) \\ &= [f(x) g(x)]_{x=a}^b \end{aligned}$$

3.2.2 Strong form

Consider problem to find $u : \Omega \rightarrow \mathbb{R}$ with **strong form** such as

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g_0 & \text{on } \Gamma_0 \text{ Dirichlet (or essential) B.C.} \\ \frac{\partial u}{\partial n} = g_1 & \text{on } \Gamma_1 \text{ Neumann (or natural) B.C.} \end{cases} \quad (5)$$

where $f : \Omega \rightarrow \mathbb{R}$, $g_0 : \Gamma_0 \rightarrow \mathbb{R}$, and $g_1 : \Gamma_1 \rightarrow \mathbb{R}$ is given.

$$\begin{aligned} \Omega &\subset \mathbb{R}^2 \\ n &: \partial \Omega \rightarrow \mathbb{R}^2 \\ x &\mapsto n(x) \end{aligned}$$

We have notation $\Delta = \sum_{i=1}^2 \frac{\partial^2}{\partial x_i^2}$ such that $-\Delta u(x) = -(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2})u(x)$. So we have

$$\begin{aligned} -\Delta u(x) &= f(x) \\ \Leftrightarrow -(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2})u(x) &= f(x). \end{aligned}$$

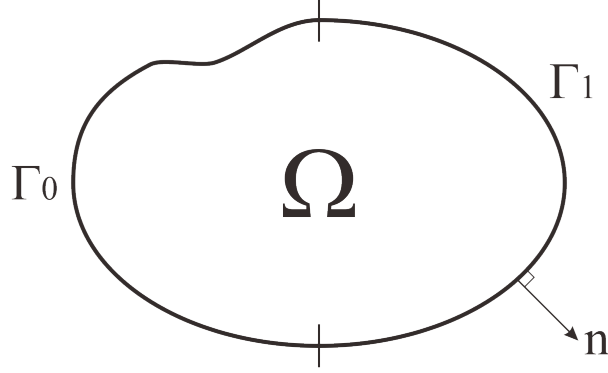


Figure 6

For all smooth test function $v(x)$, where $v|_{\Gamma_0} = 0$

$$\begin{aligned}
\int_{\Omega} (-\Delta u)(x) v(x) dx &= \int_{\Omega} \left(-\frac{\partial^2 u}{\partial x_1^2}(x) v(x) - \frac{\partial^2 u}{\partial x_2^2}(x) v(x) \right) dx \\
&= - \int_{\Omega} \frac{\partial^2 u}{\partial x_1^2}(x) v(x) dx - \int_{\Omega} \frac{\partial^2 u}{\partial x_2^2}(x) v(x) dx \\
&= - \left(\int_{\partial\Omega} \frac{\partial u}{\partial x_1}(x) v(x) n_i ds - \int_{\Omega} \frac{\partial u}{\partial x_1}(x) \frac{\partial v}{\partial x_1}(x) dx \right) \\
&\quad - \left(\int_{\partial\Omega} \frac{\partial u}{\partial x_2}(x) v(x) n_i ds - \int_{\Omega} \frac{\partial u}{\partial x_2}(x) \frac{\partial v}{\partial x_2}(x) dx \right) \\
&= \int_{\Omega} \left(\frac{\partial u}{\partial x_1}(x) \frac{\partial v}{\partial x_1}(x) + \frac{\partial u}{\partial x_2}(x) \frac{\partial v}{\partial x_2}(x) \right) dx + \int_{\Gamma_0} \left(\frac{\partial u}{\partial x_1}(x) v(x) n_i + \frac{\partial u}{\partial x_2}(x) v(x) n_i \right) ds \\
&\quad + \int_{\Gamma_1} \left(\frac{\partial u}{\partial x_1}(x) v(x) n_i + \frac{\partial u}{\partial x_2}(x) v(x) n_i \right) ds \\
&= \int_{\Omega} \nabla u(x) \cdot \nabla v(x) dx - \int_{\Gamma_1} (\nabla u(x) \cdot n(x)) v(x) ds \\
&= \int_{\Omega} \nabla u(x) \cdot \nabla v(x) dx - \int_{\Gamma_1} \frac{\partial u}{\partial n}(x) v(x) ds \\
&= \int_{\Omega} \nabla u(x) \cdot \nabla v(x) dx - \int_{\Gamma_1} g_1(x) v(x) ds \\
&= \int_{\Omega} f(x) v(x) dx
\end{aligned}$$

such that

$$\int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega} f v dx + \int_{\Gamma_1} g_1 v ds$$

Note : Reason why $\frac{\partial u}{\partial n}(x) = (\nabla u) \cdot n$ In 1D case,

$$u'(x) = \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h}$$

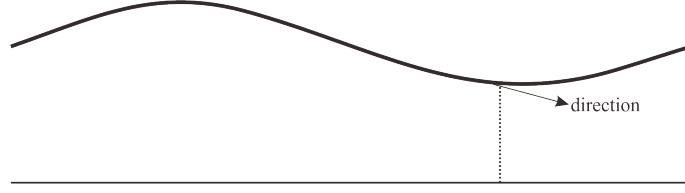


Figure 7

In 2D case,

$$\begin{aligned}
\frac{\partial u}{\partial n}(x) &= \lim_{h \rightarrow 0} \frac{u(x + hn) - u(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} [u(x_1 + hn_1, x_2 + hn_2) - u(x_1, x_2)] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} [u(x_1 + hn_1, x_2 + hn_2) - u(x_1, x_2 + hn_2) + u(x_1, x_2 + hn_2) - u(x_1, x_2)] \\
&= \lim_{h \rightarrow 0} \frac{u(x_1 + hn_1, x_2 + hn_2) - u(x_1, x_2 + hn_2) + u(x_1, x_2 + hn_2) - u(x_1, x_2)}{hn_1} n_1 \\
&\quad \lim_{h \rightarrow 0} \frac{u(x_1 + hn_2, x_2 + hn_2) - u(x_1, x_2 + hn_2) + u(x_1, x_2 + hn_2) - u(x_1, x_2)}{hn_2} n_2 \\
&= \frac{\partial u}{\partial x_1}(x_1, x_2) n_1 + \frac{\partial u}{\partial x_2}(x_1, x_2) n_2 \\
&= (\nabla u) \cdot n
\end{aligned}$$

3.2.3 Weak form

We want to find $u \in V(g_0)$ such that

$$a(u, v) = l(v), \quad \forall v \in V$$

where

$$V(g_0) \equiv \{v \in H^1(\Omega); v|_{\Gamma_0} = g_0\}, \quad V \equiv V(0).$$

There are some notation we need to know beforehand,

$$L^2(\Omega) \equiv \{v : \Omega \rightarrow \mathbb{R}; \int_{\Omega} v^2(x) dx < \infty\}.$$

For examples,

$$\begin{aligned}
\Omega &= (1, \infty) \\
f(x) &= \frac{1}{x} \in L^2(1, \infty); \quad \int_1^{\infty} f^2(x) dx = [-x^{-1}]_1^{\infty} = 1 \\
f(x) &= \frac{1}{\sqrt{x}} \text{ not } \in L^2(1, \infty); \quad \int_1^{\infty} dx = [\log x]_1^{\infty} = \infty
\end{aligned}$$

Furthermore, $L^2(\Omega)$ is a Hilbert space or complete space with inner product.

$$H^1(\Omega) \equiv \{v \in L^2(\Omega); \frac{\partial v}{\partial x_i} \in L^2(\Omega), \quad i = 1, 2.\}$$

Inner product is defined by

$$(f, g) \equiv \int_{\Omega} f(x)g(x)dx.$$

Back to the problem, we have bilinear form $a(u, v)$ and linear form $l(v)$ as shown below.

$$\begin{aligned} a(u, v) &= \int_{\Omega} \nabla u \cdot \nabla v \, dx \\ l(v) &= \int_{\Omega} f v \, dx + \int_{\Gamma_1} g_1 v \, ds. \end{aligned}$$

$l(v)$ is called linear form, because it holds that

$$l(\alpha v + \beta w) = \alpha l(v) + \beta l(w).$$

Then $a(u, v)$ is called bilinear form because if u is fixed, then v is linear form respect to u , and vice versa.

3.2.4 Discretization

We approach value of smooth function $u(x)$ by piecewise linear function $u_h(x)$ as

$$u(x) \approx u_h(x) \equiv \sum_{i=1}^{Np} c_i \varphi_i(x)$$

where Np is total number of nodal points.

For case as shown by picture below,

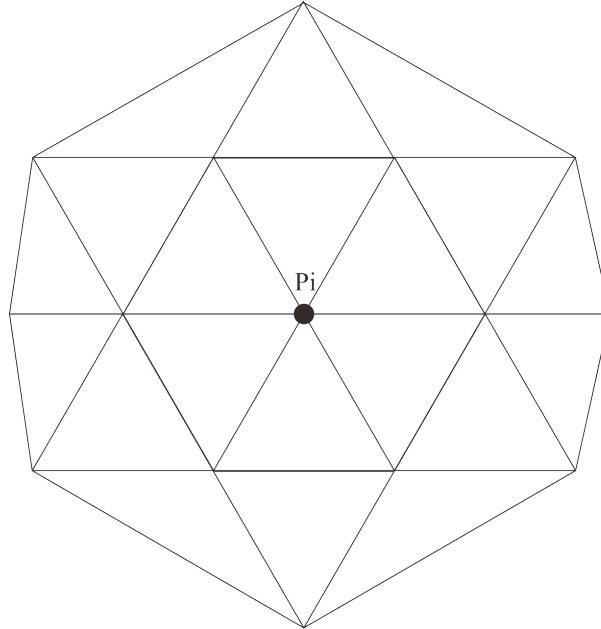


Figure 8

we choose basis function

$$\begin{aligned} \varphi &: \Omega \rightarrow \mathbb{R} \\ \varphi_i(P_j) &= \begin{cases} 1 & , i = j, \\ 0 & , i \neq j. \end{cases} \end{aligned}$$

Then, in each triangle,

$$\varphi_i(x) = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2.$$

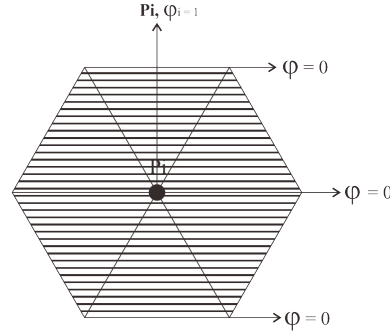


Figure 9

Then, the problem becomes we want to find $u_h \in V_h(g_0)$ such that

$$a(u_h, v_h) = l(v_h), \quad \forall v_h \in V_h$$

where

$$V_h(g_0) \equiv \{v_h \in V(g_0); v_h(x) = \sum_{i=1}^{Np} c_i \varphi_i(x), c_i \in \mathbb{R}, \varphi_i \text{ is basis function}\}, V_h \equiv V_h(0).$$

3.2.5 Problem

For simplicity, assume $\Omega = (0, 1)^2$ and $g_0 = 0$. We want to find $u \in V \equiv H_0^1(\Omega)$ which the value on boundary is 0. $\{P_i\}_{i=1}^{Np}$ is set of nodal points. For example,

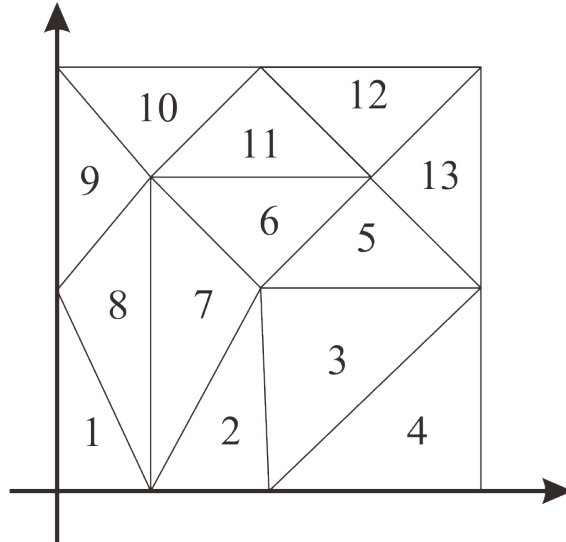


Figure 10

with nodal point $Np = 11$ and elements $Ne = 13$. The domain $\tau_h = \{K_k\}_{k=1}^{13}$ or divided into 13 triangle area with point $\{P_i\}_{i=1}^{11}$.

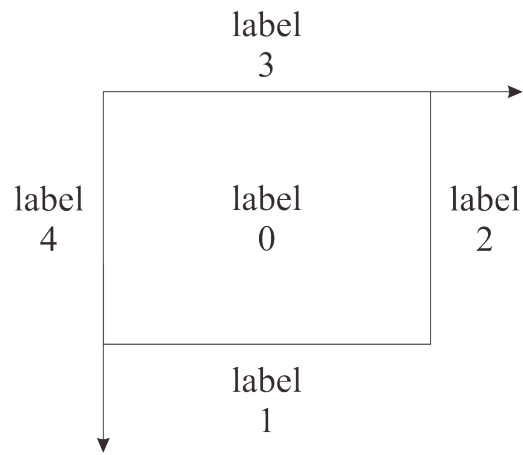


Figure 11

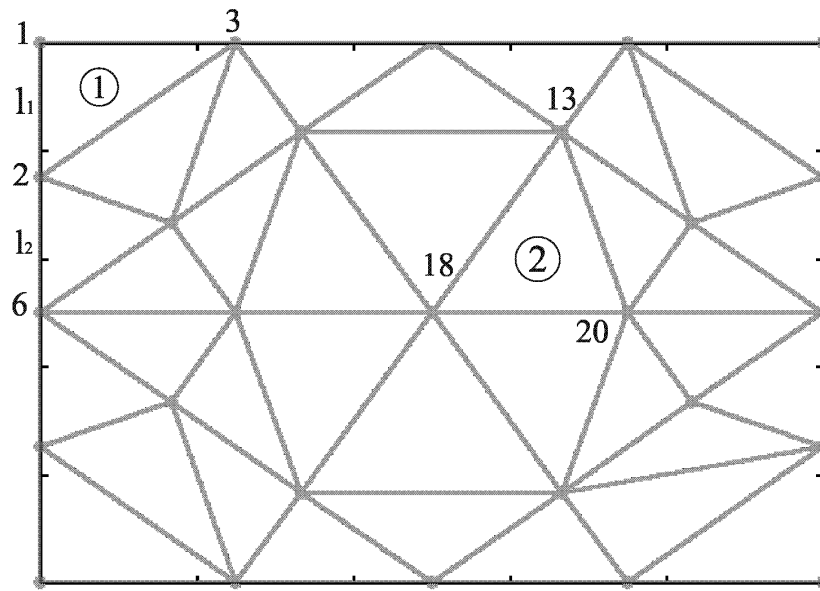


Figure 12

3.3 4 December 2017

3.3.1 Continuous (Partial Differential Equation)

To show that there is exist unique solution u , we can use the Remark below.

Remark 1. $\exists! u = \underset{v \in V(g)}{\operatorname{argmin}} \left(\frac{1}{2} a(v, v) - l(v) \right) = \underset{w \in V(g)}{\operatorname{argmin}} J(w)$

minimizer or *argmin* is value of x where the function $f(x)$ is minimum. For example, for function $f(x) = 1 + x^2$, $\underset{x \in \mathbb{R}}{\operatorname{min}} f(x) = f(0) = 1$. Then, the minimizer, $\underset{x \in \mathbb{R}}{\operatorname{argmin}} f(x) = 0$.

Using this Remark, the Proposition below is given with proof.

Proposition 1. For $J(v) := \frac{1}{2} a(v, v) - l(v)$, $u = \underset{v \in V(g)}{\operatorname{argmin}} J(v) \iff a(u, v) = l(v)$

Proof:

(\Rightarrow) if $u = \underset{v \in V(g)}{\operatorname{argmin}} J$ then $u + tv \in V(g)$, $\forall t \in \mathbb{R}, \forall v \in V = H_0^1(\Omega)$. Since it is on boundary Γ , then $g = u = u + tv$.

$$\begin{aligned} J(u) &\leq J(w) \quad , \forall w \in V(g), w = u + tv \in V(g) \\ J(u) &\leq J(u + tv) \quad , \forall t \in \mathbb{R}, \forall v \in V \end{aligned}$$

Then

$$\begin{aligned} J(u + tv) &= \frac{1}{2} a(u + tv, u + tv) - l(u + tv) \\ &= \frac{1}{2} a(u, u) + ta(u, v) + \frac{t^2}{2} a(v, v) - l(u) - tl(v) \\ &= \frac{t^2}{2} a(v, v) + t(a(u, v) - l(v)) + J(u) \\ &=: \varphi(t) \end{aligned}$$

Because $\varphi(t)$ is in quadratic form, then its minimum obtained at $t = 0$. So that $\varphi = 0$ such that $a(u, v) - l(v) = 0$. $(\Leftarrow) \forall t \in \mathbb{R}, \forall v \in V$ we have

$$J(u, tv) = J(u) + \frac{t^2}{2} a(v, v) \geq J(u).$$

$\forall w \in V(g)$, we set $v := w - u \in V$, $t := 1$, $w = u + tv$

$$J(w) = J(u + tv) \geq J(u)$$

3.3.2 Discrete (Finite Element Method)

Here introduced some notation,

$$\begin{aligned} X_h &\subset X \text{ (usually } \dim X_h < \infty) \\ V_h &= X_h \cap V \\ g_h &\in X_h \text{ (approximation of } g) \\ V_h(g_h) &= \{v_h \in X_h; v_h - g_h \in V_h\}. \end{aligned}$$

Then the weak form is approximated with

$$\begin{cases} a(u_h, v_h) = l(v_h), \forall v_h \in V_h \\ u_h \in V_h(g_h) \end{cases} \iff u_h = \underset{w_h \in V_h(g_h)}{\operatorname{argmin}} J(w_h)$$

Using Finite Element Method,

$$\begin{aligned} X_h &= \{v_h \in C^0(\bar{\Omega}); v_h|_K \text{ is linear}\} \\ V_h &= X_h \cap H_0^1(\Omega), \end{aligned}$$

or we could write

$$\begin{aligned} X_h &= \langle \varphi_1, \dots, \varphi_{Np} \rangle \\ &= \left\{ \sum_{i=1}^{Np} c_i \varphi_i ; c_i \in \mathbb{R} \right\} \end{aligned}$$

where $\{\varphi_i\}_{i=1}^{Np}$ become a basis of the vector space X_h . For nodal points $\{P_i\}_{i=1}^{Nfp}$ and $\varphi_i \in X_n$; $\varphi_i(P_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

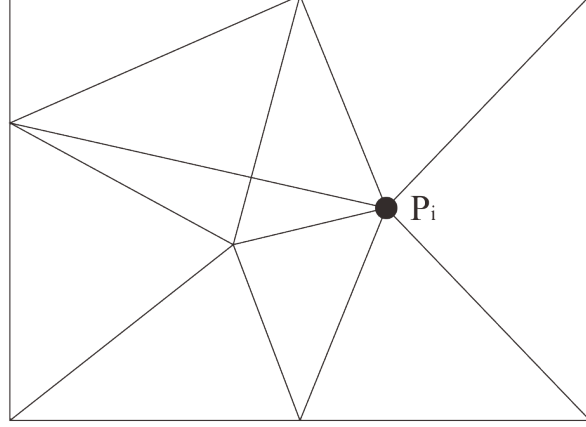


Figure 13

For $x = (x_1, x_2) \in \mathbb{R}^2$ and $\forall v_h \in X_h$, we will have

$$\begin{aligned} v_h(\cdot) &= \sum_{i=1}^{Np} v_h(P_i) \varphi_i(\cdot) \in X_h \\ w_h &:= \sum_{i=1}^{Np} v_h(P_i) \varphi_i \in X_h \end{aligned}$$

such that

$$\begin{aligned} w_h(P_j) &= \sum_{i=1}^{Np} v_h(P_i) \varphi_i(P_j) \\ &= \sum_{i=1}^{Np} v_h(P_i) \delta_{ij} \\ &= v_h(P_j) \end{aligned}$$

Now, we consider basis of V_h for

$$\begin{aligned} \Omega \cap \Gamma &= \emptyset \\ \{\varphi_i; P_i \in \Omega\} &\subset \{P_i\}_{i=1}^{Np}. \end{aligned}$$

For simplicity, we assume $\{\varphi_i; P_i \in \Omega\} = \{P_i\}_{i=1}^N$ for $(N < Np)$, such that $\{P_i\}_{i=1}^N \subset \Omega$ and $\{P_i\}_{i=N+1}^{Np} \subset \Gamma$.

Let $V_h = \langle \varphi_1, \dots, \varphi_N \rangle$, then

$$a(u_h, v_h) = l(v_h), \quad (\forall v_h \in V_h) \Leftrightarrow a(u_h, \varphi_i) = l(\varphi_i), \quad (i = 1, \dots, N).$$

If we choose $v_h = \varphi_i \in V$, then $\forall v_h \in V_h$ with $c_i = v_h(P_i)$ and $v_h = \sum_{i=1}^N c_i \varphi_i$,

$$\begin{aligned}
a(u_h, v_h) &= a(u_h, \sum_{i=1}^N c_i \varphi_i) \\
&= \sum_{i=1}^N c_i a(u_h, \varphi_i) \\
&= \sum_{i=1}^N c_i l(\varphi_i) \\
&= l(\sum_{i=1}^N c_i \varphi_i) \\
&= l(v_h)
\end{aligned}$$

We set $u_j := u_h(P_j)$ for $j = 1, \dots, Np$ such that at the boundary $P_j \in \Gamma$ or for $j = N+1, \dots, Np$

$$u_j = g_j = g_h(P_j), \quad u_h \in V_h(g_h)$$

with u_1, \dots, u_N is unknown.

Then, we can conclude that

$$\begin{cases} a(u_h, \varphi_i) = l(\varphi_i), & (i = 1, \dots, N) \\ u_h = \sum_{j=1}^N u_j \varphi_j + \sum_{j=N+1}^{Np} g_j \varphi_j, & (u_h \in V_h(g_h)) \end{cases} .$$

For simplicity, we set notation $a_{ij} := a(\varphi_i, \varphi_j) = a(\varphi_j, \varphi_i)$ such that for $i = 1, \dots, N$,

$$\sum_{j=1}^N a_{ij} u_j + \sum_{j=N+1}^{Np} a_{ij} g_j = l(\varphi_i)$$

As conclusion,

$$a(u_h, v_h) = l(v_h), \quad (\forall v_h \in V_h) \Leftrightarrow \mathbf{A} \mathbf{u} = \mathbf{b}$$

where

$$\begin{aligned}
\mathbf{A} &:= (a_{ij}) \in \mathbb{R}_{\text{sym}}^{N \times N} \\
\mathbf{u} &:= \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} \\
\mathbf{b} &:= \left(l(u_i) - \sum_{j=N+1}^{Np} a_{ij} g_j \right), \text{ for } i = 1, \dots, N
\end{aligned}$$

3.4 25 Desember 2017

3.4.1 Calculate matrix \mathbf{A}

$$\begin{aligned}
A_{ij} &= a(\varphi_j, \varphi_i) \\
&= \int_{\Omega} \nabla \varphi_j(x) \cdot \nabla \varphi_i(x) \, dx \\
&= \sum_{k=1}^{Ne} \int_{K_k} \nabla \varphi_j(x) \cdot \nabla \varphi_i(x) \, dx \\
&= \sum_{k=1}^{Ne} \int_{K_k} 1 \, dx \, (\nabla \varphi_j(x) \cdot \nabla \varphi_i(x)).
\end{aligned}$$

To calculate it, usually we need N^3 computation for code like

```

> for i = 1, ..., Np {
>   for j = 1, ..., Np {
>     for k = 1, ..., Ne {
>       Aij = Aij + ∫Kk ∇φj(x) · ∇φi(x) dx
>     }
>   }

```

```

>     }
> }
> But, we can simplify it into  $N^2$  computation using local matrix and global matrix as shown below.
> for  $k = 1, \dots, Ne$  {
>     (local matrix)
>      $A_{11}^k = \int_{K_k} \nabla \varphi_1(x) \cdot \varphi_1(x) \, dx$ 
>      $A_{12}^k = \int_{K_k} \nabla \varphi_2(x) \cdot \varphi_1(x) \, dx$ 
>
>          $\vdots$ 
>      $A_{33}^k = \int_{K_k} \nabla \varphi_3(x) \cdot \varphi_3(x) \, dx$ 
>     (global matrix)
>      $A_{44} = A_{44} + A_{11}^k$ 
>      $A_{47} = A_{47} + A_{12}^k$ 
>
>          $\vdots$ 
>      $A_{73} = A_{73} + A_{23}^k$ 
>
>          $\vdots$ 
>      $A_{33} = A_{33} + A_{33}^k$ 
> }

```

3.4.2 Calculate vector B

$$\begin{aligned}
 b_i &= \int_{\Omega} f(x) \varphi_i(x) \, dx - \sum_{j=1}^{Np} g_h(P_j) \int_{\Omega} \nabla \varphi_j(x) \cdot \nabla \varphi_i(x) \, dx \\
 &= \sum_{k=1}^{Ne} \int_{K_k} f(x) \varphi_i(x) \, dx - \sum_{j=1}^{Np} g_h(P_j) \sum_{k=1}^{Ne} \int_{K_k} \nabla \varphi_j(x) \cdot \nabla \varphi_i(x) \, dx
 \end{aligned}$$

Consider $g_h(P_j) = 0$, with $f_h(x) = \sum_{j=1}^{Np} f(P_j) \varphi_j(x)$ then for each element,

$$\begin{aligned}
 b_i &= b_i + \int_{K_k} f(x) \varphi_i(x) \, dx \\
 &= b_i + \int_{K_k} \left(f(P_1) \varphi_1(x) + f(P_2) \varphi_2(x) + f(P_3) \varphi_3(x) \right) \varphi_i(x) \, dx \\
 &= b_i + \int_{K_k} f(P_1) \varphi_1(x) \cdot \varphi_i(x) + f(P_2) \varphi_2(x) \cdot \varphi_i(x) + f(P_3) \varphi_3(x) \cdot \varphi_i(x) \, dx \\
 &= b_i + f(P_1) \int_{K_k} \varphi_1(x) \cdot \varphi_i(x) \, dx + f(P_2) \int_{K_k} \varphi_2(x) \cdot \varphi_i(x) \, dx + \\
 &\quad f(P_3) \int_{K_k} \varphi_3(x) \cdot \varphi_i(x) \, dx
 \end{aligned}$$

with

$$\int_{K_k} \varphi_i(x) \varphi_j(x) \, dx = \begin{cases} \frac{\text{meas}(K_k)}{6}, & \text{for } i = j \\ \frac{\text{meas}(K_k)}{12}, & \text{for } i \neq j \end{cases}$$

(way to calculate in program)

3.4.3 Other calculation

For any points $P_i(x_1, x_2)$ in triangle K ,

$$\varphi_i(x) = c_0 + c_1 x_1 + c_2 x_2, \quad c_j \in \mathbb{R}$$

such that

$$\nabla \varphi_i(x) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

To compute the triangle area $\int_{K_k} 1 \, dx$,

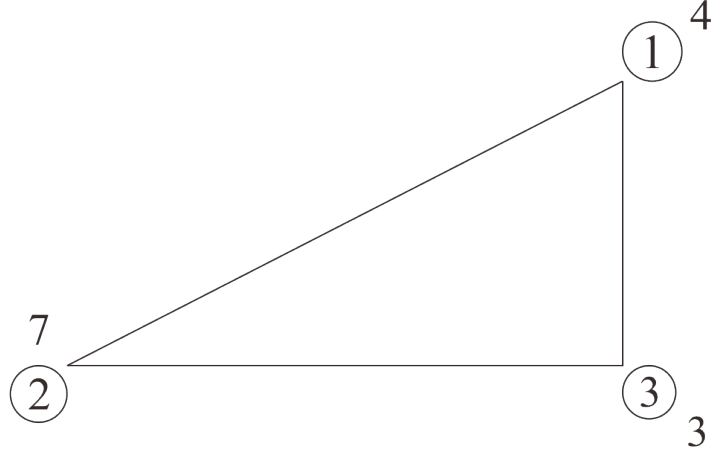


Figure 14

3.5 5 January 2018 (Error Estimate)

3.5.1 Norm

After find the solution $u(x)$, we should check if the solution we found is close enough to the exact solution. For numerical solution u_h , we check for $h = \frac{1}{N}$ where N or divider of each boundary side is set to $N = 4, 8, 16, 32$.

In this problem we set $f(x) = -\nabla u(x) = 2\pi^2(\sin(x\pi)\sin(y\pi))$ with exact solution $u_e(x) = \sin(x\pi)\sin(y\pi)$. We want to calculate $\|u_h - \Pi_h u_e\|_X$, $s \ X = L^2(\Omega), H_0^1(\Omega)$. Here, Π_h is function that mapping continuous function $C(\bar{\Omega})$ into piecewise linear finite element space.

$$\begin{aligned} \Pi_h : \ C(\bar{\Omega}) &\rightarrow P1 - FEsp \\ v &\mapsto \Pi_h v \end{aligned}$$

such that $(\Pi_h v)(P) := v(P)$, P is nodes.

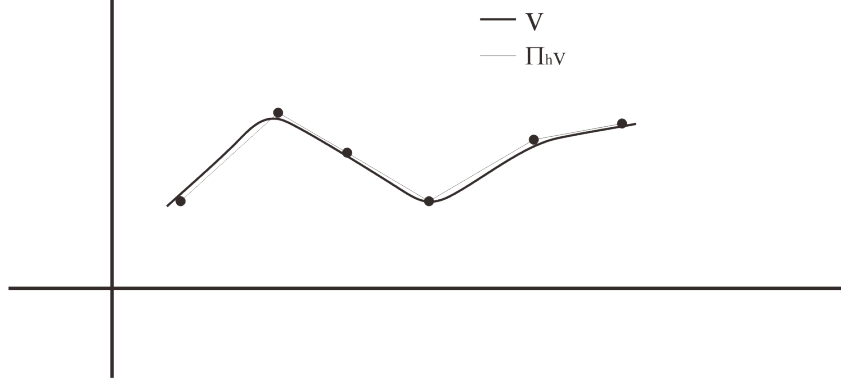


Figure 15

3.5.2 $L^2(\Omega)$ norm

$$\begin{aligned}
\|v_h\|_{L^2(\Omega)} &= \sqrt{\int_{\Omega} (v_h(x))^2 dx} \\
&= \sqrt{\sum_K \int_K (v_h(x))^2 dx} \\
&= \sqrt{\sum_K \int_K (v_i \varphi_i^{(K)}(x) + v_j \varphi_j^{(K)}(x) + v_k \varphi_k^{(K)}(x))^2 dx} \\
&= \sqrt{\sum_K \int_K \left((v_i \quad v_j \quad v_k) \begin{pmatrix} \varphi_i^{(K)}(x) \\ \varphi_j^{(K)}(x) \\ \varphi_k^{(K)}(x) \end{pmatrix} \right)^2 dx} \\
&= \sqrt{\sum_K \int_K (v_i \quad v_j \quad v_k) \begin{pmatrix} \varphi_i^{(K)}(x) \\ \varphi_j^{(K)}(x) \\ \varphi_k^{(K)}(x) \end{pmatrix} \begin{pmatrix} \varphi_i^{(K)}(x) & \varphi_j^{(K)}(x) & \varphi_k^{(K)}(x) \end{pmatrix} \begin{pmatrix} v_i \\ v_j \\ v_k \end{pmatrix} dx} \\
&= \sqrt{\sum_K (v_i \quad v_j \quad v_k) \int_K \begin{pmatrix} \varphi_i^{(K)} \cdot \varphi_i^{(K)} & \varphi_i^{(K)} \cdot \varphi_j^{(K)} & \varphi_i^{(K)} \cdot \varphi_k^{(K)} \\ \varphi_j^{(K)} \cdot \varphi_i^{(K)} & \varphi_j^{(K)} \cdot \varphi_j^{(K)} & \varphi_j^{(K)} \cdot \varphi_k^{(K)} \\ \varphi_k^{(K)} \cdot \varphi_i^{(K)} & \varphi_k^{(K)} \cdot \varphi_j^{(K)} & \varphi_k^{(K)} \cdot \varphi_k^{(K)} \end{pmatrix} dx \begin{pmatrix} v_i \\ v_j \\ v_k \end{pmatrix}} \\
&= \sqrt{\sum_K (v_i \quad v_j \quad v_k) \frac{|K|}{12} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} v_i \\ v_j \\ v_k \end{pmatrix}} \\
&= \sqrt{\sum_K \frac{|K|}{12} v^T \cdot M \cdot v}
\end{aligned}$$

with

$$v = \begin{pmatrix} v_i \\ v_j \\ v_k \end{pmatrix} \quad M = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

In our problem, we need to find $\|v_h\|_{L^2(\Omega)} = \|u_h - \Pi_h u_e\|_{L^2(\Omega)}$. Then we substitute

$$v_h(P) = u_h(P) - u_e(P)$$

for every point P .

3.5.3 $H_0^1(\Omega)$ norm

$$\begin{aligned} \|v_h\|_{H_0^1(\Omega)} &= \sqrt{\int_{\Omega} \nabla v_h(x) \cdot \nabla v_h(x) \, dx} \\ &= \sqrt{\int_{\Omega} |\nabla v_h(x)|^2 \, dx} \\ &= \sqrt{\sum_K |\nabla v_h|_K(x)|^2 |K|} \end{aligned}$$

with $\nabla v_h|_K(x)$ does not depend on x

$$\nabla v_h|_K(x) = v_i \begin{pmatrix} c_1^{(i)} \\ c_1^{(i)} \\ c_2 \end{pmatrix} + v_j \begin{pmatrix} c_1^{(j)} \\ c_1^{(j)} \\ c_2 \end{pmatrix} + v_k \begin{pmatrix} c_1^{(i)} \\ c_1^{(i)} \\ c_2 \end{pmatrix}.$$

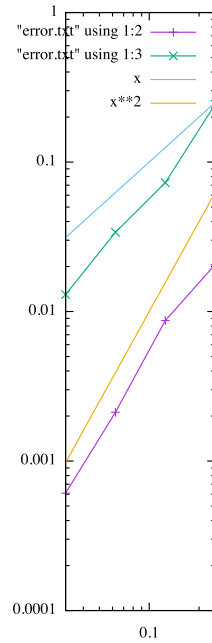


Figure 16

4 13 February 2018

4.1 Heat Equation

4.1.1 Problem

Find $u : \Omega \times (0, T) \rightarrow \mathbb{R}$ such that

$$\begin{cases} u_t - \nu \Delta u = f & \text{in } \Omega \times (0, T) \\ u = g & \text{on } \partial\Omega \times (0, T) \\ u = u^0 & \text{in } \Omega \text{ at } t = 0 \end{cases}$$

where f, g, u^0 is given.

4.1.2 Weak Form

We want to find $\{u(t) \in V(g(t)); t \in (0, T)\}$ with $V := H_0^1(\Omega)$, $g : \partial\Omega \rightarrow \mathbb{R}$ smooth function, and $V(g) := \{v + g \in H^1(\Omega); v \in V\}$ such that

$$\begin{cases} (u_t(t), v) + a(u(t), v) = (f(t), v) & , \forall v \in V \\ u(0) = u^0 & \in V(g(0)) \end{cases}$$

where $a(u, v) := \nu \int_{\Omega} \nabla u \cdot \nabla v \, dx$

4.1.3 Finite Element Method

For $n = 1, \dots, N_T$, $\Delta t > 0$, $N_T := \lfloor \frac{T}{\Delta t} \rfloor \in \mathbb{N}$, we looking for $\{u_h^n \in V_h(g^n)\}$ such that

$$\begin{cases} \left(\frac{u_h^n - u_h^{n-1}}{\Delta t}, v_h \right) + a(u_h^n, v_h) = (f^n, v_h) & , \forall v_h \in V_h \\ u_h^0 = \Pi_h u^0 & \in V_h(g^0) \end{cases}$$

where

$$\begin{aligned} V_h \subset V & : \quad P1 - FEspace \\ V_h(g) & := \{v_h + g \in V(g); v_h \in V_h\} \\ \Pi_h & : \quad C(\bar{\Omega}) \rightarrow X_h \\ X_h \subset H^1(\Omega) & : \quad P1 - FEspace \\ (V_h \subset X_h) & \Rightarrow V_h \subset H_0^1(\Omega), \quad X_h \subset H^1(\Omega) \\ (\Pi_h u^0)(P) & = u^0(P), \quad P : \text{nodal point} \end{aligned}$$

Then the problem become

$$\begin{aligned} \frac{1}{\Delta t} (u_h^n, v_h) + a(u_h^n, v_h) & = \frac{1}{\Delta t} (u_h^{n-1}, v_h) + (f^n, v_h), \quad \forall v_h \in V_h \\ \left[\frac{1}{\Delta t} M_{N_p \times N_p} + A_{N_p \times N_p} \right] [U_{N_p \times 1}] & = b_{N_p \times 1}. \end{aligned}$$

4.1.4 Flowchart

The algorithm used to solve heat equation problem is shown as flowchart in Figure 17 and 18

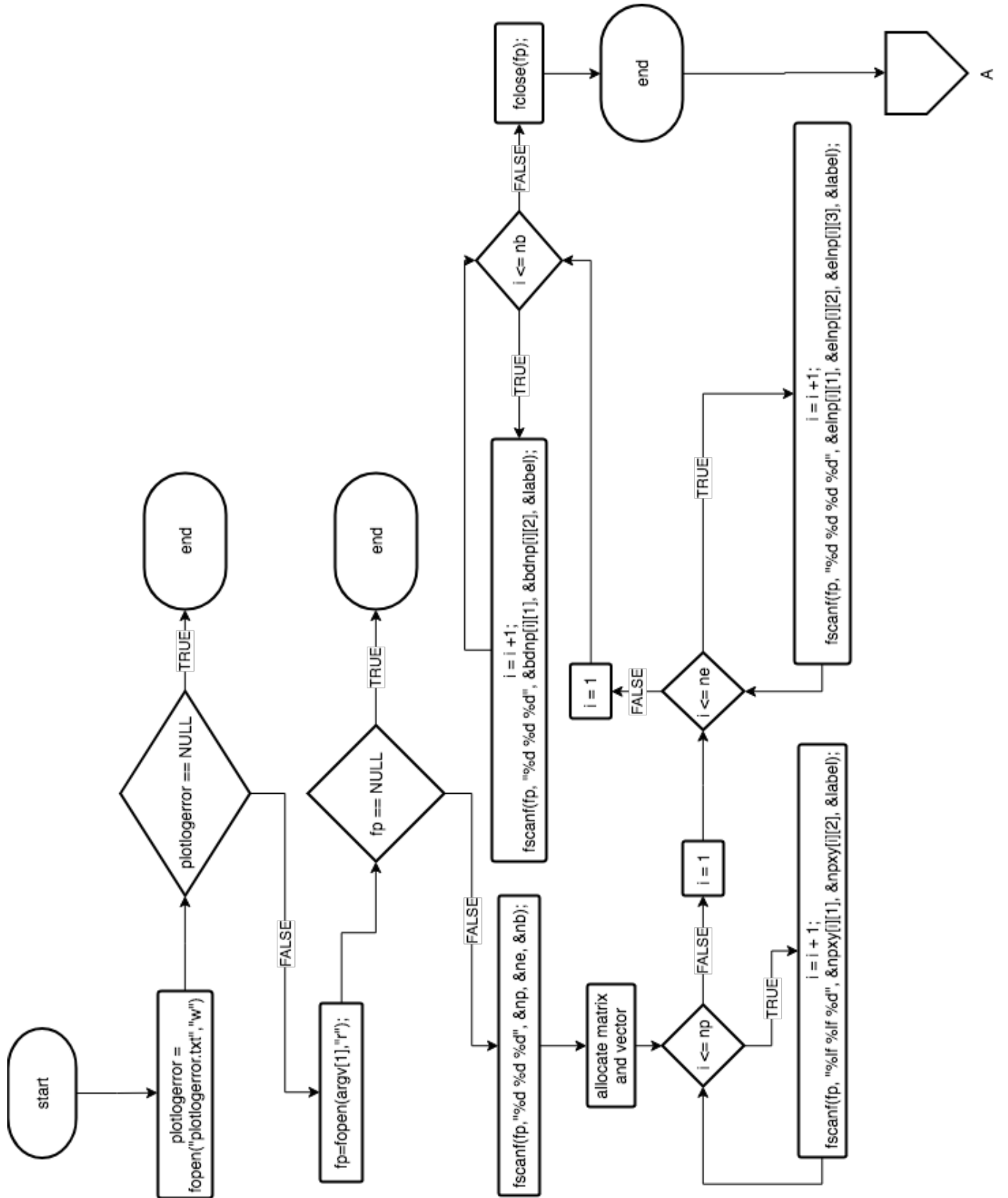


Figure 17

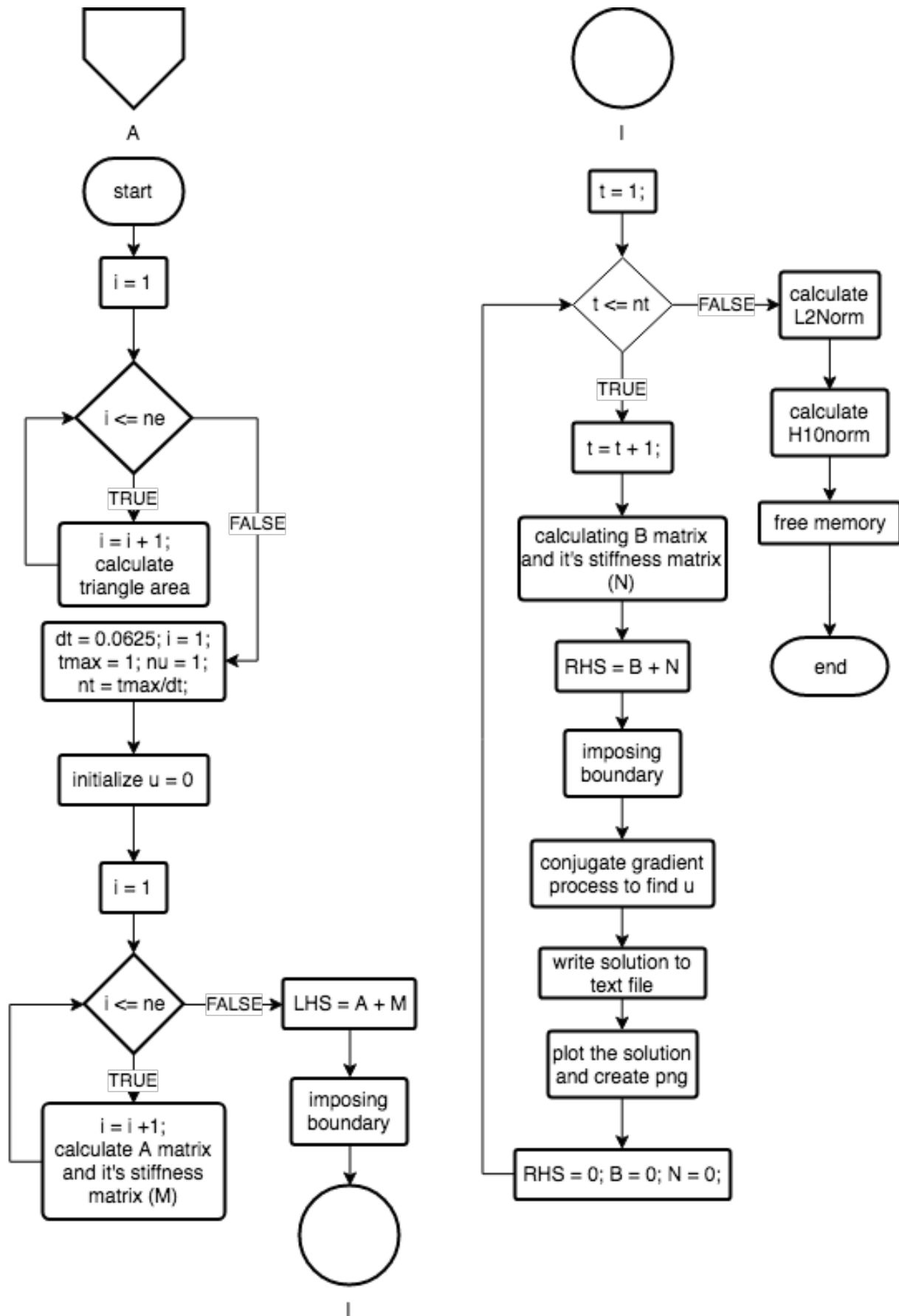


Figure 18

4.1.5 Numerical experiment

We consider problem

$$\begin{cases} u_t - \nu \Delta u = (1 + 2\nu\pi^2 t) \sin(\pi x_1) \sin(\pi x_2) & , \Omega \times (0, T) \\ u = 0 & , \partial\Omega \times (0, T) \\ u = 0 & , \Omega \text{ at } t = 0 \end{cases}$$

with exact solution

$$u(x_1, x_2, t) = t \sin(\pi x_1) \sin(\pi x_2).$$

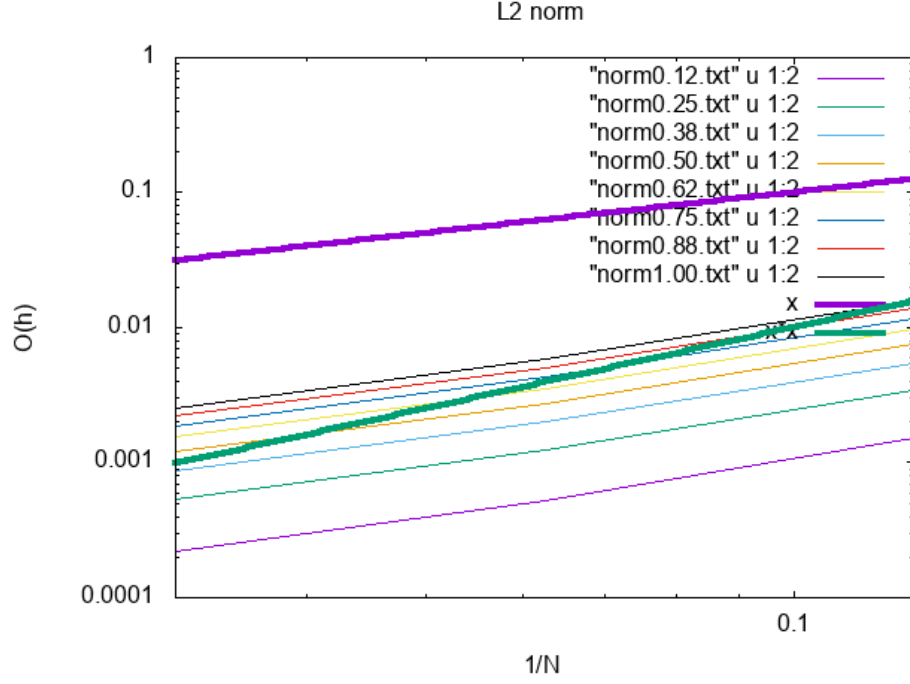


Figure 19

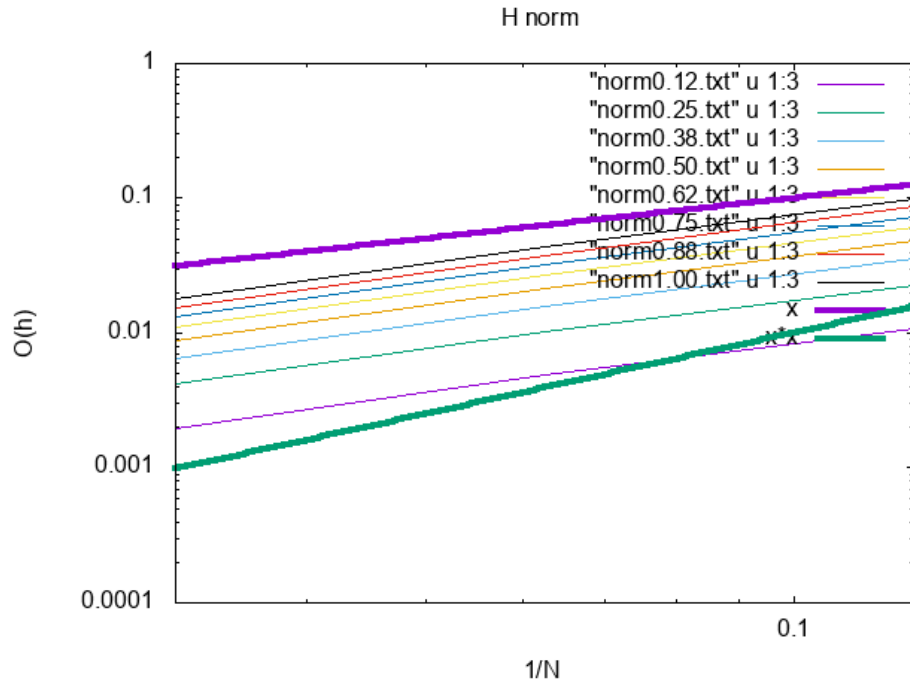
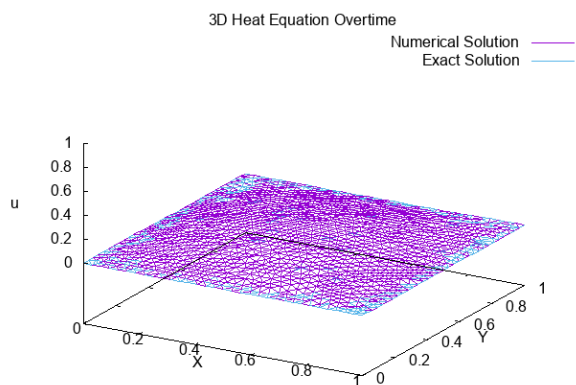
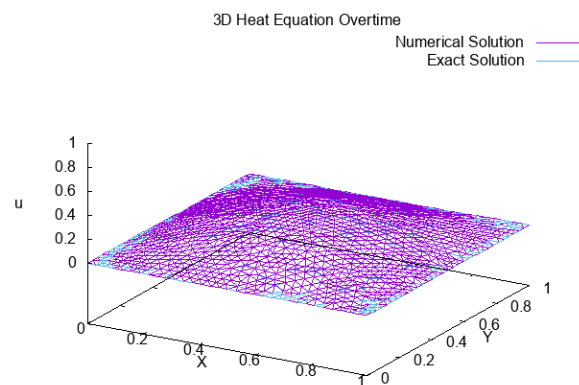


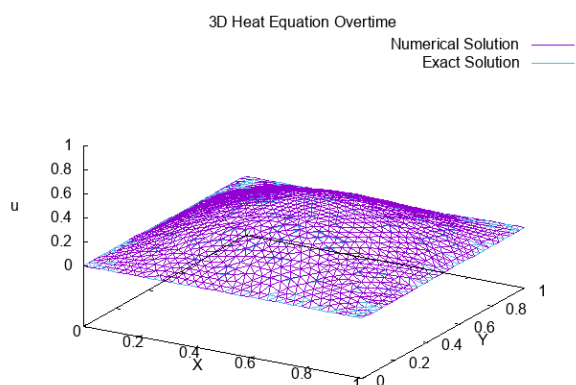
Figure 20



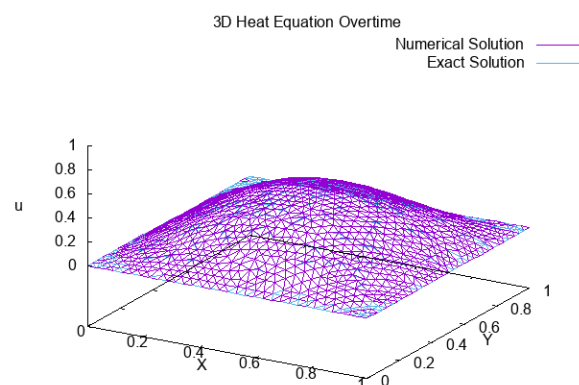
(a) $4 \, dt$



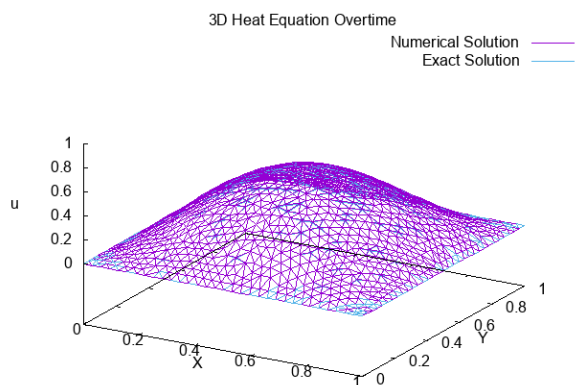
(b) $8 \, dt$



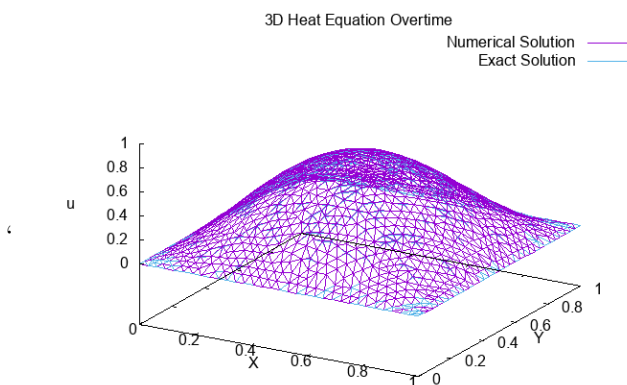
(c) $12 \, dt$



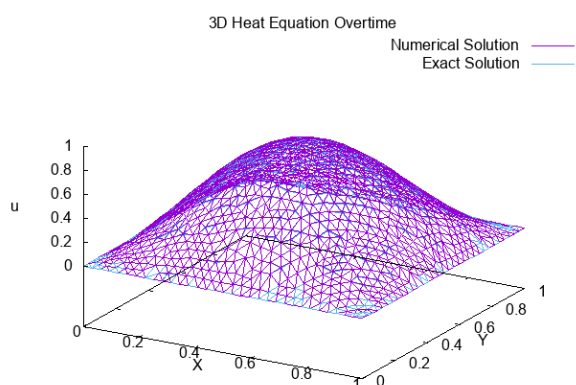
(d) $16 \, dt$



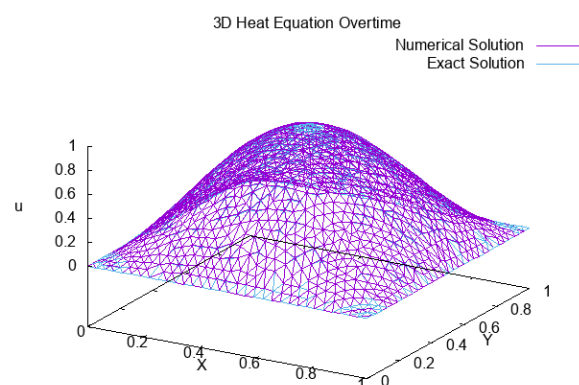
(e) $20 \, dt$



(f) $24 \, dt$



(g) $28 \, dt$



(h) $32 \, dt$