

# 1 Unix Command Terminal

## 1.1 5 October 2017

- **mk dir** : make a file in directory.
- **ls** : file list in directory
- **cd** : change directory
- **pwd** : parent working directory
- **cp -r** : copy
- **rm** : remove
- **open ...** : open ... file
- **.** : here
- **cd ..** : back to previous

## 2 Gnuplot

### 2.1 16 October 2017

#### 2.1.1 Plot from terminal

```
{gnuplot
set terminal x11 (for 2D and 3D)
plotsin(x)
plotsin(x),cos(x)
set hidden3d
splotsin(x)*sin(y)
set xrange[-5:5]
set yrange[-5:5]
q quit
```

#### 2.1.2 Plot from file

We have file **plot.dat** or **.txt** contains list point of triangular format for 2D as shown below:

$x_1^{(i)}$	$x_2^{(i)}$	$u^{(i)}$
$x_1^{(j)}$	$x_2^{(j)}$	$u^{(j)}$
$x_1^{(k)}$	$x_2^{(k)}$	$u^{(k)}$
$x_1^{(i)}$	$x_2^{(i)}$	$u^{(i)}$

(leave blank)

There are some "blank" on line. In 1D case, we will have two lines of points and 1 blank line. Then, for 2D case, we have four lines of points and two blank (anw: I think it still work for one blank thought).

```
{splot 'plot.dat' u 1:2:3 w l palette
```

where **u** means using, **1:2:3** means the column we wish to plot, **w** means with, **l** means line, and **palette** means color.

Other command that maybe used is,

```
{set pm3d map
is used for mapping 3D to 2D. Other is,
{set size ratio -1
then de data will be integers.
```

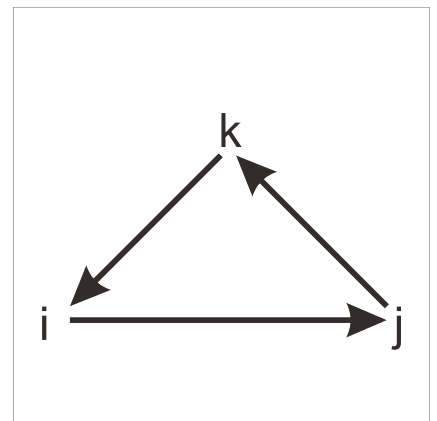


Figure 1: One element with point  $i, j, k$

### 3 FreeFEM++

#### 3.1 16 October 2017

Example 1 :

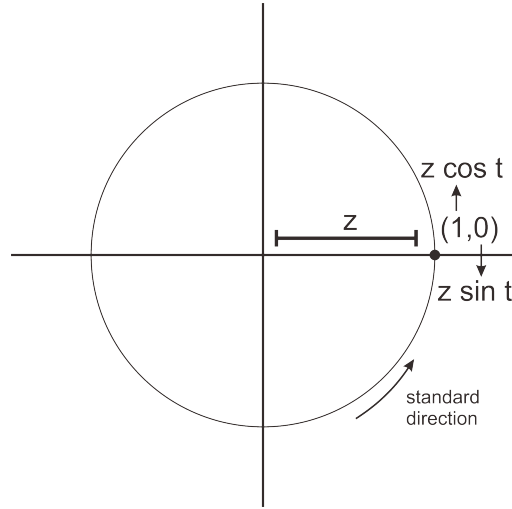
$$\begin{cases} \text{int } n = 50; \\ \text{real } x0 = 0.0, \ y0 = 0.0, \ Lx = 1.0, \ Ly = 1.0, \ z = 1; \\ \text{border } a1(t = 0, 2 * pi) \{ x = z * \cos(t); \ y = z * \sin(t); \} \end{cases}$$


Figure 2: Example 1

Example 2:

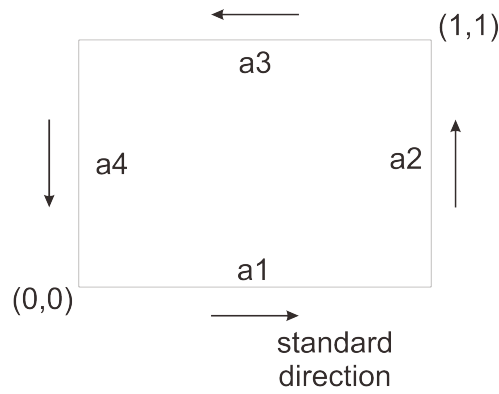
$$\begin{cases} \text{border } a1(t = 0, 1) \{ x = t; y = 0; \} \\ \text{border } a2(t = 0, 1) \{ x = 1; y = t; \} \end{cases}$$


Figure 3: Example 2

Here are image of how to choose the domain.

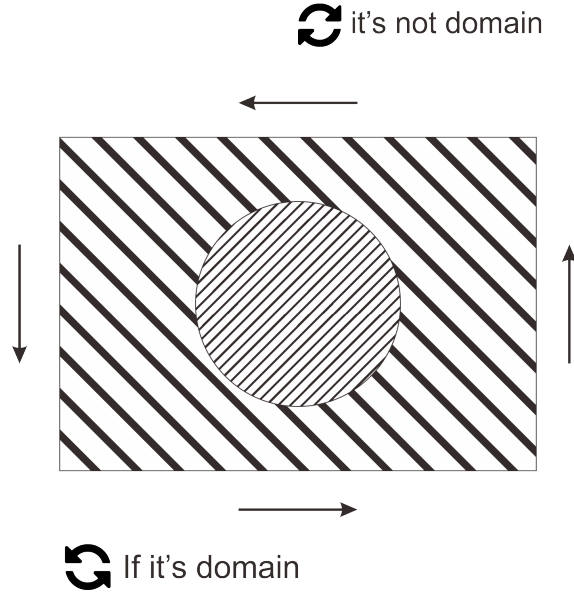


Figure 4: Choose the domain

From file : **membrane.edp**. Find  $\phi : \Omega \rightarrow \mathbb{R}$  such that

$$\begin{cases} -\Delta \phi = f(= 1) & \text{in } \Omega \\ \phi = z(= x_1) & \text{on } \Gamma_1 \\ \frac{\partial \phi}{\partial n} = 0 & \text{on } \Gamma_2 \end{cases} \quad (1)$$

where  $f : \Omega \rightarrow \mathbb{R}$ , given  $f(x) = 1$  and  $z : \Gamma_1 \rightarrow \mathbb{R}$ , given  $z(x) = x_1$ . The equation (1) is called **strong form**, because it contains second derivative.

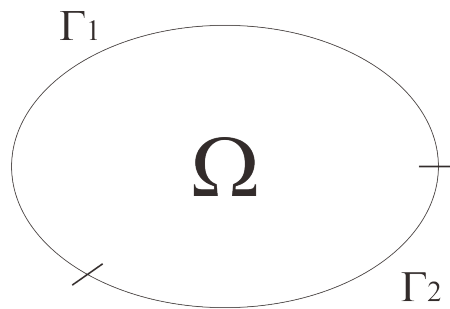


Figure 5:

We multiple equation (1)'s first line by smooth test function  $w$  and integrate over  $\Omega$  such that  $\forall w(w|_{\Gamma_1} = 0)$

$$\int_{\Omega} -\Delta \phi(x)w(x) \, dx = \int_{\Omega} f(x)w(x) \, dx$$

using integration by parts,

$$-\int_{\partial\Omega} \frac{\partial \phi}{\partial n}(x)w(x) \, dx + \int_{\Omega} \nabla \phi(x) \cdot \nabla w(x) \, dx = \int_{\Omega} f(x)w(x) \, dx$$

then we can divide the boundary such that

$$-\int_{\Gamma_1} \frac{\partial \phi}{\partial n}(x) w(x) dx - \int_{\Gamma_2} \frac{\partial \phi}{\partial n}(x) w(x) dx + \int_{\Omega} \nabla \phi(x) \cdot \nabla w(x) dx = \int_{\Omega} f(x) w(x) dx$$

Because on  $\Gamma_1$ , smooth function  $w(x)$  value is equal to 0. And on  $\Gamma_2$  by equation (1) line 3 we obtain that  $\frac{\partial \phi}{\partial n}(x)$  is equal to 0. Then, we conclude that

$$\int_{\Omega} \nabla \phi(x) \cdot \nabla w(x) dx - \int_{\Omega} f(x) w(x) dx = 0 \quad (2)$$

is the **weak form**. Because it only contain first derivative.

## 3.2 6 November 2017

### 3.2.1 Gauss-Green Formula

In **2D Case**, we have  $f, g : \Omega \rightarrow \mathbb{R}$

$$\int_{\Omega} \frac{\partial f}{\partial x_i}(x) g(x) dx = \int_{\partial \Omega} f(x) g(x) n_i ds - \int_{\Omega} f(x) \frac{\partial g}{\partial x_i}(x) dx, \quad (i = 1, 2) \quad (3)$$

In **1D Case**,

$$\int_a^b f(x) g(x) dx = [f(x) g(x)]_{x=a}^b - \int_a^b f(x) g'(x) dx \quad (4)$$

where

$$\begin{aligned} \int_{\partial \Omega} f(x) g(x) n(x) dx &= f(a) g(a) n(a) + f(b) g(b) n(b) \\ &= [f(x) g(x)]_{x=a}^b \end{aligned}$$

### 3.2.2 Strong form

Consider problem to find  $u : \Omega \rightarrow \mathbb{R}$  with **strong form** such as

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g_0 & \text{on } \Gamma_0 \text{ Dirichlet (or essential) B.C.} \\ \frac{\partial u}{\partial n} = g_1 & \text{on } \Gamma_1 \text{ Neumann (or natural) B.C.} \end{cases} \quad (5)$$

where  $f : \Omega \rightarrow \mathbb{R}$ ,  $g_0 : \Gamma_0 \rightarrow \mathbb{R}$ , and  $g_1 : \Gamma_1 \rightarrow \mathbb{R}$  is given.

$$\begin{aligned} \Omega &\subset \mathbb{R}^2 \\ n &: \partial \Omega \rightarrow \mathbb{R}^2 \\ x &\mapsto n(x) \end{aligned}$$

We have notation  $\Delta = \sum_{i=1}^2 \frac{\partial^2}{\partial x_i^2}$  such that  $-\Delta u(x) = -(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2})u(x)$ . So we have

$$\begin{aligned} -\Delta u(x) &= f(x) \\ \Leftrightarrow -(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2})u(x) &= f(x). \end{aligned}$$

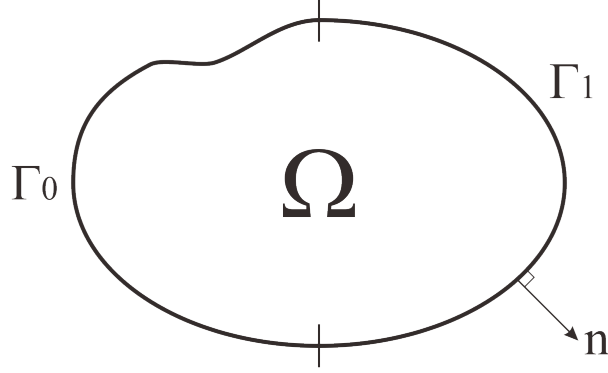


Figure 6:

For all smooth test function  $v(x)$ , where  $v|_{\Gamma_0} = 0$

$$\begin{aligned}
\int_{\Omega} (-\Delta u)(x) v(x) dx &= \int_{\Omega} \left( -\frac{\partial^2 u}{\partial x_1^2}(x) v(x) - \frac{\partial^2 u}{\partial x_2^2}(x) v(x) \right) dx \\
&= -\int_{\Omega} \frac{\partial^2 u}{\partial x_1^2}(x) v(x) dx - \int_{\Omega} \frac{\partial^2 u}{\partial x_2^2}(x) v(x) dx \\
&= -\left( \int_{\partial\Omega} \frac{\partial u}{\partial x_1}(x) v(x) n_i ds - \int_{\Omega} \frac{\partial u}{\partial x_1}(x) \frac{\partial v}{\partial x_1}(x) dx \right) \\
&\quad -\left( \int_{\partial\Omega} \frac{\partial u}{\partial x_2}(x) v(x) n_i ds - \int_{\Omega} \frac{\partial u}{\partial x_2}(x) \frac{\partial v}{\partial x_2}(x) dx \right) \\
&= \int_{\Omega} \left( \frac{\partial u}{\partial x_1}(x) \frac{\partial v}{\partial x_1}(x) + \frac{\partial u}{\partial x_2}(x) \frac{\partial v}{\partial x_2}(x) \right) dx + \int_{\Gamma_0} \left( \frac{\partial u}{\partial x_1}(x) v(x) n_i + \frac{\partial u}{\partial x_2}(x) v(x) n_i \right) ds \\
&\quad + \int_{\Gamma_1} \left( \frac{\partial u}{\partial x_1}(x) v(x) n_i + \frac{\partial u}{\partial x_2}(x) v(x) n_i \right) ds \\
&= \int_{\Omega} \nabla u(x) \cdot \nabla v(x) dx - \int_{\Gamma_1} (\nabla u(x) \cdot n(x)) v(x) ds \\
&= \int_{\Omega} \nabla u(x) \cdot \nabla v(x) dx - \int_{\Gamma_1} \frac{\partial u}{\partial n}(x) v(x) ds \\
&= \int_{\Omega} \nabla u(x) \cdot \nabla v(x) dx - \int_{\Gamma_1} g_1(x) v(x) ds \\
&= \int_{\Omega} f(x) v(x) dx
\end{aligned}$$

such that

$$\int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega} f v dx + \int_{\Gamma_1} g_1 v ds$$

**Note :** Reason why  $\frac{\partial u}{\partial n}(x) = (\nabla u) \cdot n$  In 1D case,

$$u'(x) = \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h}$$

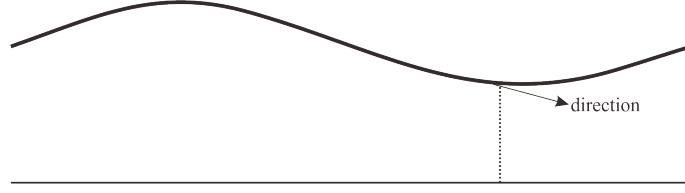


Figure 7:

In 2D case,

$$\begin{aligned}
\frac{\partial u}{\partial n}(x) &= \lim_{h \rightarrow 0} \frac{u(x + hn) - u(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} [u(x_1 + hn_1, x_2 + hn_2) - u(x_1, x_2)] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} [u(x_1 + hn_1, x_2 + hn_2) - u(x_1, x_2 + hn_2) + u(x_1, x_2 + hn_2) - u(x_1, x_2)] \\
&= \lim_{h \rightarrow 0} \frac{u(x_1 + hn_1, x_2 + hn_2) - u(x_1, x_2 + hn_2) + u(x_1, x_2 + hn_2) - u(x_1, x_2)}{hn_1} n_1 \\
&\quad \lim_{h \rightarrow 0} \frac{u(x_1 + hn_2, x_2 + hn_2) - u(x_1, x_2 + hn_2) + u(x_1, x_2 + hn_2) - u(x_1, x_2)}{hn_2} n_2 \\
&= \frac{\partial u}{\partial x_1}(x_1, x_2) n_1 + \frac{\partial u}{\partial x_2}(x_1, x_2) n_2 \\
&= (\nabla u) \cdot n
\end{aligned}$$

### 3.2.3 Weak form

We want to find  $u \in V(g_0)$  such that

$$a(u, v) = l(v), \quad \forall v \in V$$

where

$$V(g_0) \equiv \{v \in H^1(\Omega); v|_{\Gamma_0} = g_0\}, \quad V \equiv V(0).$$

There are some notation we need to know beforehand,

$$L^2(\Omega) \equiv \{v : \Omega \rightarrow \mathbb{R}; \int_{\Omega} v^2(x) dx < \infty\}.$$

For examples,

$$\begin{aligned}
\Omega &= (1, \infty) \\
f(x) &= \frac{1}{x} \in L^2(1, \infty); \quad \int_1^{\infty} f^2(x) dx = [-x^{-1}]_1^{\infty} = 1 \\
f(x) &= \frac{1}{\sqrt{x}} \text{ not } \in L^2(1, \infty); \quad \int_1^{\infty} dx = [\log x]_1^{\infty} = \infty
\end{aligned}$$

Furthermore,  $L^2(\Omega)$  is a Hilbert space or complete space with inner product.

$$H^1(\Omega) \equiv \{v \in L^2(\Omega); \frac{\partial v}{\partial x_i} \in L^2(\Omega), \quad i = 1, 2.\}$$

Inner product is defined by

$$(f, g) \equiv \int_{\Omega} f(x)g(x)dx.$$

Back to the problem, we have bilinear form  $a(u, v)$  and linear form  $l(v)$  as shown below.

$$\begin{aligned} a(u, v) &= \int_{\Omega} \nabla u \cdot \nabla v \, dx \\ l(v) &= \int_{\Omega} f v \, dx + \int_{\Gamma_1} g_1 v \, ds. \end{aligned}$$

$l(v)$  is called linear form, because it holds that

$$l(\alpha v + \beta w) = \alpha l(v) + \beta l(w).$$

Then  $a(u, v)$  is called bilinear form because if  $u$  is fixed, then  $v$  is linear form respect to  $u$ , and vice versa.

### 3.2.4 Discretization

We approach value of smooth function  $u(x)$  by piecewise linear function  $u_h(x)$  as

$$u(x) \approx u_h(x) \equiv \sum_{i=1}^{N_p} c_i \varphi_i(x)$$

where  $N_p$  is total number of nodal points.

For case as shown by picture below,

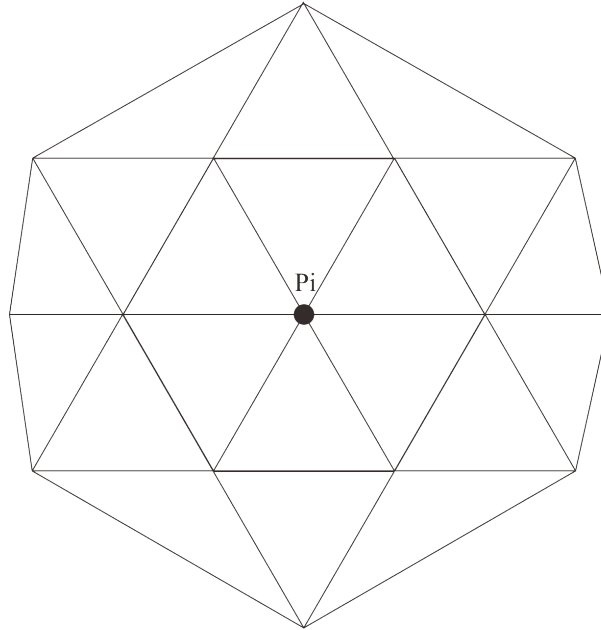


Figure 8:

we choose basis function

$$\begin{aligned} \varphi &: \Omega \rightarrow \mathbb{R} \\ \varphi_i(P_j) &= \begin{cases} 1 & , i = j, \\ 0 & , i \neq j. \end{cases} \end{aligned}$$

Then, in each triangle,

$$\varphi_i(x) = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2.$$

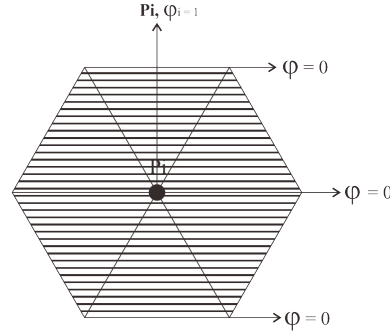


Figure 9:

Then, the problem becomes we want to find  $u_h \in V_h(g_0)$  such that

$$a(u_h, v_h) = l(v_h), \quad \forall v_h \in V_h$$

where

$$V_h(g_0) \equiv \{v_h \in V(g_0); v_h(x) = \sum_{i=1}^{Np} c_i \varphi_i(x), c_i \in \mathbb{R}, \varphi_i \text{ is basis function}\}, V_h \equiv V_h(0).$$

### 3.2.5 Problem

For simplicity, assume  $\Omega = (0, 1)^2$  and  $g_0 = 0$ . We want to find  $u \in V \equiv H_0^1(\Omega)$  which the value on boundary is 0.  $\{P_i\}_{i=1}^{Np}$  is set of nodal points. For example,

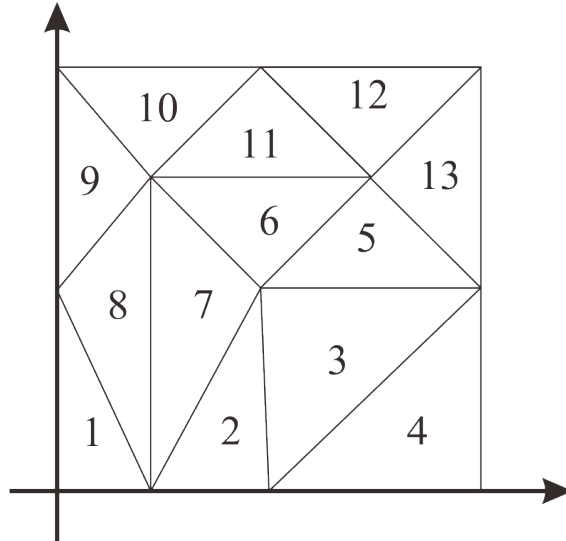


Figure 10:

with nodal point  $Np = 11$  and elements  $Ne = 13$ . The domain  $\tau_h = \{K_k\}_{k=1}^{13}$  or divided into 13 triangle area with point  $\{P_i\}_{i=1}^{11}$ .



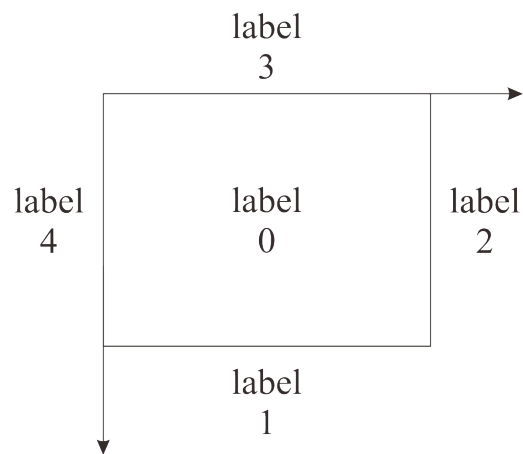


Figure 11:

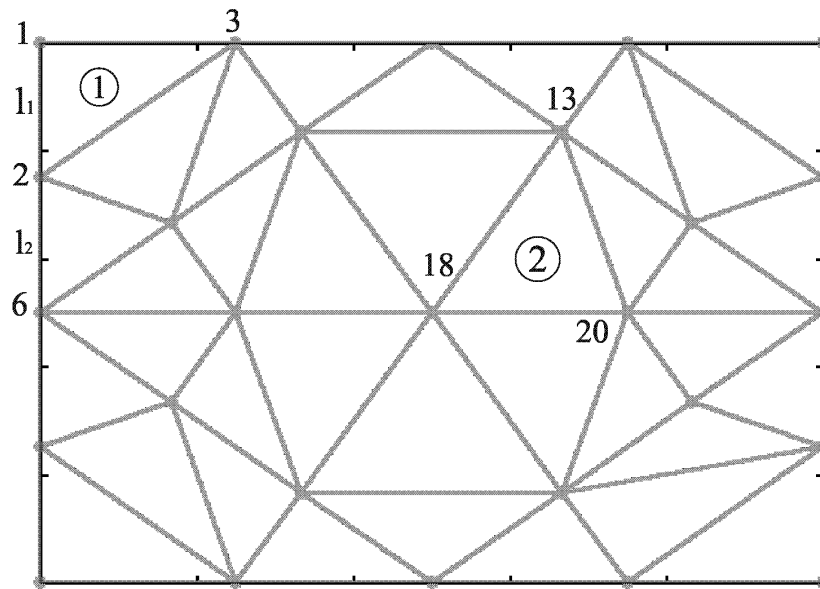


Figure 12:

### 3.3 4 December 2017

#### 3.3.1 Continuous (Partial Differential Equation)

To show that there is exist unique solution  $u$ , we can use the Remark below.

**Remark 1.**  $\exists! u = \underset{v \in V(g)}{\operatorname{argmin}} \left( \frac{1}{2} a(v, v) - l(v) \right) = \underset{w \in V(g)}{\operatorname{argmin}} J(w)$

minimizer or *argmin* is value of  $x$  where the function  $f(x)$  is minimum. For example, for function  $f(x) = 1 + x^2$ ,  $\underset{x \in \mathbb{R}}{\operatorname{min}} f(x) = f(0) = 1$ . Then, the minimizer,  $\underset{x \in \mathbb{R}}{\operatorname{argmin}} f(x) = 0$ .

Using this Remark, the Proposition below is given with proof.

**Proposition 1.** For  $J(v) := \frac{1}{2} a(v, v) - l(v)$ ,  $u = \underset{v \in V(g)}{\operatorname{argmin}} J(v) \iff a(u, v) = l(v)$

**Proof:**

$(\Rightarrow)$  if  $u = \underset{v \in V(g)}{\operatorname{argmin}} J$  then  $u + tv \in V(g)$ ,  $\forall t \in \mathbb{R}, \forall v \in V = H_0^1(\Omega)$ . Since it is on boundary  $\Gamma$ , then  $g = u = u + tv$ .

$$\begin{aligned} J(u) &\leq J(w) \quad , \forall w \in V(g), w = u + tv \in V(g) \\ J(u) &\leq J(u + tv) \quad , \forall t \in \mathbb{R}, \forall v \in V \end{aligned}$$

Then

$$\begin{aligned} J(u + tv) &= \frac{1}{2} a(u + tv, u + tv) - l(u + tv) \\ &= \frac{1}{2} a(u, u) + ta(u, v) + \frac{t^2}{2} a(v, v) - l(u) - tl(v) \\ &= \frac{t^2}{2} a(v, v) + t(a(u, v) - l(v)) + J(u) \\ &=: \varphi(t) \end{aligned}$$

Because  $\varphi(t)$  is in quadratic form, then its minimum obtained at  $t = 0$ . So that  $\varphi = 0$  such that  $a(u, v) - l(v) = 0$ .  $(\Leftarrow) \forall t \in \mathbb{R}, \forall v \in V$  we have

$$J(u, tv) = J(u) + \frac{t^2}{2} a(v, v) \geq J(u).$$

$\forall w \in V(g)$ , we set  $v := w - u \in V$ ,  $t := 1$ ,  $w = u + tv$

$$J(w) = J(u + tv) \geq J(u)$$

### 3.3.2 Discrete (Finite Element Method)

Here introduced some notation,

$$\begin{aligned} X_h &\subset X \text{ (usually } \dim X_h < \infty) \\ V_h &= X_h \cap V \\ g_h &\in X_h \text{ (approximation of } g) \\ V_h(g_h) &= \{v_h \in X_h; v_h - g_h \in V_h\}. \end{aligned}$$

Then the weak form is approximated with

$$\begin{cases} a(u_h, v_h) = l(v_h), \forall v_h \in V_h \\ u_h \in V_h(g_h) \end{cases} \iff u_h = \underset{w_h \in V_h(g_h)}{\operatorname{argmin}} J(w_h)$$

Using Finite Element Method,

$$\begin{aligned} X_h &= \{v_h \in C^0(\bar{\Omega}); v_h|_K \text{ is linear}\} \\ V_h &= X_h \cap H_0^1(\Omega), \end{aligned}$$

or we could write

$$\begin{aligned} X_h &= \langle \varphi_1, \dots, \varphi_{Np} \rangle \\ &= \left\{ \sum_{i=1}^{Np} c_i \varphi_i ; c_i \in \mathbb{R} \right\} \end{aligned}$$

where  $\{\varphi_i\}_{i=1}^{Np}$  become a basis of the vector space  $X_h$ . For nodal points  $\{P_i\}_{i=1}^{Nfp}$  and  $\varphi_i \in X_n$ ;  $\varphi_i(P_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ .

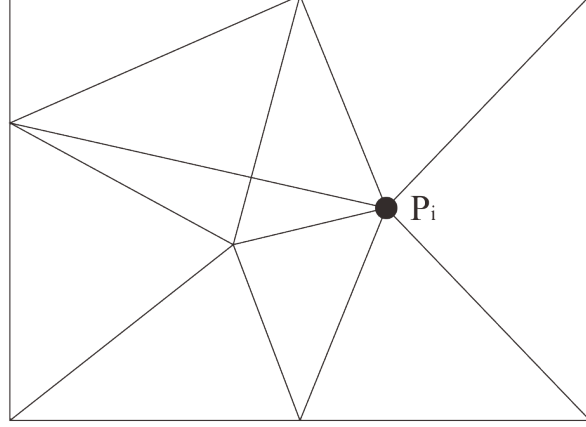


Figure 13:

For  $x = (x_1, x_2) \in \mathbb{R}^2$  and  $\forall v_h \in X_h$ , we will have

$$\begin{aligned} v_h(\cdot) &= \sum_{i=1}^{Np} v_h(P_i) \varphi_i(\cdot) \in X_h \\ w_h &:= \sum_{i=1}^{Np} v_h(P_i) \varphi_i \in X_h \end{aligned}$$

such that

$$\begin{aligned} w_h(P_j) &= \sum_{i=1}^{Np} v_h(P_i) \varphi_i(P_j) \\ &= \sum_{i=1}^{Np} v_h(P_i) \delta_{ij} \\ &= v_h(P_j) \end{aligned}$$

Now, we consider basis of  $V_h$  for

$$\begin{aligned} \Omega \cap \Gamma &= \emptyset \\ \{\varphi_i; P_i \in \Omega\} &\subset \{P_i\}_{i=1}^{Np}. \end{aligned}$$

For simplicity, we assume  $\{\varphi_i; P_i \in \Omega\} = \{P_i\}_{i=1}^N$  for  $(N < Np)$ , such that  $\{P_i\}_{i=1}^N \subset \Omega$  and  $\{P_i\}_{i=N+1}^{Np} \subset \Gamma$ .

Let  $V_h = \langle \varphi_1, \dots, \varphi_N \rangle$ , then

$$a(u_h, v_h) = l(v_h), \quad (\forall v_h \in V_h) \Leftrightarrow a(u_h, \varphi_i) = l(\varphi_i), \quad (i = 1, \dots, N).$$

If we choose  $v_h = \varphi_i \in V$ , then  $\forall v_h \in V_h$  with  $c_i = v_h(P_i)$  and  $v_h = \sum_{i=1}^N c_i \varphi_i$ ,

$$\begin{aligned}
a(u_h, v_h) &= a(u_h, \sum_{i=1}^N c_i \varphi_i) \\
&= \sum_{i=1}^N c_i a(u_h, \varphi_i) \\
&= \sum_{i=1}^N c_i l(\varphi_i) \\
&= l(\sum_{i=1}^N c_i \varphi_i) \\
&= l(v_h)
\end{aligned}$$

We set  $u_j := u_h(P_j)$  for  $j = 1, \dots, Np$  such that at the boundary  $P_j \in \Gamma$  or for  $j = N+1, \dots, Np$

$$u_j = g_j = g_h(P_j), \quad u_h \in V_h(g_h)$$

with  $u_1, \dots, u_N$  is unknown.

Then, we can conclude that

$$\begin{cases} a(u_h, \varphi_i) = l(\varphi_i), & (i = 1, \dots, N) \\ u_h = \sum_{j=1}^N u_j \varphi_j + \sum_{j=N+1}^{Np} g_j \varphi_j, & (u_h \in V_h(g_h)) \end{cases}$$

For simplicity, we set notation  $a_{ij} := a(\varphi_i, \varphi_j) = a(\varphi_j, \varphi_i)$  such that for  $i = 1, \dots, N$ ,

$$\sum_{j=1}^N a_{ij} u_j + \sum_{j=N+1}^{Np} a_{ij} g_j = l(\varphi_i)$$

As conclusion,

$$a(u_h, v_h) = l(v_h), \quad (\forall v_h \in V_h) \Leftrightarrow \mathbf{A} \mathbf{u} = \mathbf{b}$$

where

$$\begin{aligned}
\mathbf{A} &:= (a_{ij}) \in \mathbb{R}_{\text{sym}}^{N \times N} \\
\mathbf{u} &:= \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} \\
\mathbf{b} &:= \left( l(u_i) - \sum_{j=N+1}^{Np} a_{ij} g_j \right), \text{ for } i = 1, \dots, N
\end{aligned}$$

### 3.4 25 Desember 2017

#### 3.4.1 Calculate matrix $\mathbf{A}$

$$\begin{aligned}
A_{ij} &= a(\varphi_j, \varphi_i) \\
&= \int_{\Omega} \nabla \varphi_j(x) \cdot \nabla \varphi_i(x) \, dx \\
&= \sum_{k=1}^{Ne} \int_{K_k} \nabla \varphi_j(x) \cdot \nabla \varphi_i(x) \, dx \\
&= \sum_{k=1}^{Ne} \int_{K_k} 1 \, dx \, (\nabla \varphi_j(x) \cdot \nabla \varphi_i(x)).
\end{aligned}$$

To calculate it, usually we need  $N^3$  computation for code like

```

> for i = 1, ..., Np {
>   for j = 1, ..., Np {
>     for k = 1, ..., Ne {
>       Aij = Aij + ∫Kk ∇φj(x) · ∇φi(x) dx
>     }
>   }

```

```

>     }
>   }
>   But, we can simplify it into  $N^2$  computation using local matrix and global matrix as shown below.
>   for  $k = 1, \dots, Ne$  {
>     (local matrix)
>      $A_{11}^k = \int_{K_k} \nabla \varphi_1(x) \cdot \varphi_1(x) \, dx$ 
>      $A_{12}^k = \int_{K_k} \nabla \varphi_2(x) \cdot \varphi_1(x) \, dx$ 
>
>      $\vdots$ 
>      $A_{33}^k = \int_{K_k} \nabla \varphi_3(x) \cdot \varphi_3(x) \, dx$ 
>     (global matrix)
>      $A_{44} = A_{44} + A_{11}^k$ 
>      $A_{47} = A_{47} + A_{12}^k$ 
>
>      $\vdots$ 
>      $A_{73} = A_{73} + A_{23}^k$ 
>
>      $\vdots$ 
>      $A_{33} = A_{33} + A_{33}^k$ 
>   }

```

### 3.4.2 Calculate vector B

$$\begin{aligned}
b_i &= \int_{\Omega} f(x) \varphi_i(x) \, dx - \sum_{j=1}^{Np} g_h(P_j) \int_{\Omega} \nabla \varphi_j(x) \cdot \nabla \varphi_i(x) \, dx \\
&= \sum_{k=1}^{Ne} \int_{K_k} f(x) \varphi_i(x) \, dx - \sum_{j=1}^{Np} g_h(P_j) \sum_{k=1}^{Ne} \int_{K_k} \nabla \varphi_j(x) \cdot \nabla \varphi_i(x) \, dx
\end{aligned}$$

Consider  $g_h(P_j) = 0$ , with  $f_h(x) = \sum_{j=1}^{Np} f(P_j) \varphi_j(x)$  then for each element,

$$\begin{aligned}
b_i &= b_i + \int_{K_k} f(x) \varphi_i(x) \, dx \\
&= b_i + \int_{K_k} \left( f(P_1) \varphi_1(x) + f(P_2) \varphi_2(x) + f(P_3) \varphi_3(x) \right) \varphi_i(x) \, dx \\
&= b_i + \int_{K_k} f(P_1) \varphi_1(x) \cdot \varphi_i(x) + f(P_2) \varphi_2(x) \cdot \varphi_i(x) + f(P_3) \varphi_3(x) \cdot \varphi_i(x) \, dx \\
&= b_i + f(P_1) \int_{K_k} \varphi_1(x) \cdot \varphi_i(x) \, dx + f(P_2) \int_{K_k} \varphi_2(x) \cdot \varphi_i(x) \, dx + \\
&\quad f(P_3) \int_{K_k} \varphi_3(x) \cdot \varphi_i(x) \, dx
\end{aligned}$$

with

$$\int_{K_k} \varphi_i(x) \varphi_j(x) \, dx = \begin{cases} \frac{\text{meas}(K_k)}{6}, & \text{for } i = j \\ \frac{\text{meas}(K_k)}{12}, & \text{for } i \neq j \end{cases}$$

(way to calculate in program)

### 3.4.3 Other calculation

For any points  $P_i(x_1, x_2)$  in triangle  $K$ ,

$$\varphi_i(x) = c_0 + c_1 x_1 + c_2 x_2, \quad c_j \in \mathbb{R}$$

such that

$$\nabla \varphi_i(x) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

To compute the triangle area  $\int_{K_k} 1 \, dx$ ,

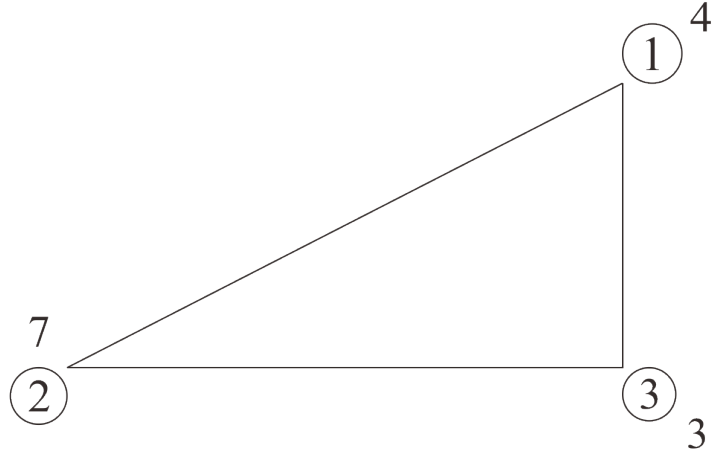


Figure 14:

### 3.5 5 January 2018 (Error Estimate)

#### 3.5.1 Norm

After find the solution  $u(x)$ , we should check if the solution we found is close enough to the exact solution. For numerical solution  $u_h$ , we check for  $h = \frac{1}{N}$  where  $N$  or divider of each boundary side is set to  $N = 4, 8, 16, 32$ .

In this problem we set  $f(x) = -\nabla u(x) = 2\pi^2(\sin(x\pi)\sin(y\pi))$  with exact solution  $u_e(x) = \sin(x\pi)\sin(y\pi)$ . We want to calculate  $\|u_h - \Pi_h u_e\|_X$ ,  $s \ X = L^2(\Omega), H_0^1(\Omega)$ . Here,  $\Pi_h$  is function that mapping continuous function  $C(\bar{\Omega})$  into piecewise linear finite element space.

$$\begin{aligned} \Pi_h : \ C(\bar{\Omega}) &\rightarrow P1 - FEsp \\ v &\mapsto \Pi_h v \end{aligned}$$

such that  $(\Pi_h v)(P) := v(P)$ ,  $P$  is nodes.

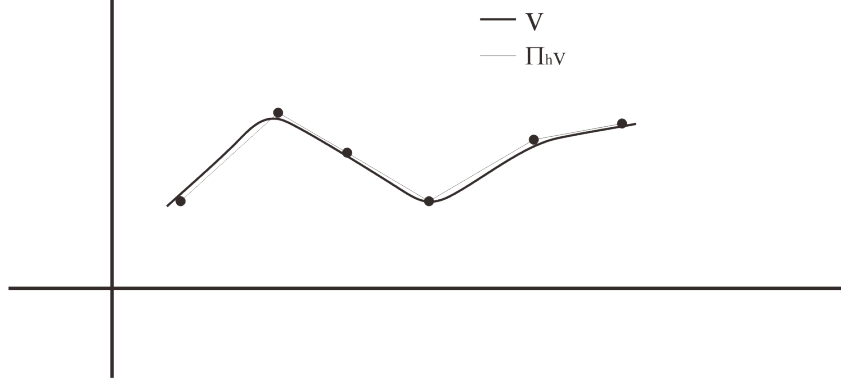


Figure 15:

### 3.5.2 $L^2(\Omega)$ norm

$$\begin{aligned}
\|v_h\|_{L^2(\Omega)} &= \sqrt{\int_{\Omega} (v_h(x))^2 dx} \\
&= \sqrt{\sum_K \int_K (v_h(x))^2 dx} \\
&= \sqrt{\sum_K \int_K (v_i \varphi_i^{(K)}(x) + v_j \varphi_j^{(K)}(x) + v_k \varphi_k^{(K)}(x))^2 dx} \\
&= \sqrt{\sum_K \int_K \left( (v_i \quad v_j \quad v_k) \begin{pmatrix} \varphi_i^{(K)}(x) \\ \varphi_j^{(K)}(x) \\ \varphi_k^{(K)}(x) \end{pmatrix} \right)^2 dx} \\
&= \sqrt{\sum_K \int_K (v_i \quad v_j \quad v_k) \begin{pmatrix} \varphi_i^{(K)}(x) \\ \varphi_j^{(K)}(x) \\ \varphi_k^{(K)}(x) \end{pmatrix} \begin{pmatrix} \varphi_i^{(K)}(x) & \varphi_j^{(K)}(x) & \varphi_k^{(K)}(x) \end{pmatrix} \begin{pmatrix} v_i \\ v_j \\ v_k \end{pmatrix} dx} \\
&= \sqrt{\sum_K (v_i \quad v_j \quad v_k) \int_K \begin{pmatrix} \varphi_i^{(K)} \cdot \varphi_i^{(K)} & \varphi_i^{(K)} \cdot \varphi_j^{(K)} & \varphi_i^{(K)} \cdot \varphi_k^{(K)} \\ \varphi_j^{(K)} \cdot \varphi_i^{(K)} & \varphi_j^{(K)} \cdot \varphi_j^{(K)} & \varphi_j^{(K)} \cdot \varphi_k^{(K)} \\ \varphi_k^{(K)} \cdot \varphi_i^{(K)} & \varphi_k^{(K)} \cdot \varphi_j^{(K)} & \varphi_k^{(K)} \cdot \varphi_k^{(K)} \end{pmatrix} dx \begin{pmatrix} v_i \\ v_j \\ v_k \end{pmatrix}} \\
&= \sqrt{\sum_K (v_i \quad v_j \quad v_k) \frac{|K|}{12} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} v_i \\ v_j \\ v_k \end{pmatrix}} \\
&= \sqrt{\sum_K \frac{|K|}{12} v^T \cdot M \cdot v}
\end{aligned}$$

with

$$v = \begin{pmatrix} v_i \\ v_j \\ v_k \end{pmatrix} \quad M = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

In our problem, we need to find  $\|v_h\|_{L^2(\Omega)} = \|u_h - \Pi_h u_e\|_{L^2(\Omega)}$ . Then we substitute

$$v_h(P) = u_h(P) - u_e(P)$$

for every point  $P$ .

### 3.5.3 $H_0^1(\Omega)$ norm

$$\begin{aligned} \|v_h\|_{H_0^1(\Omega)} &= \sqrt{\int_{\Omega} \nabla v_h(x) \cdot \nabla v_h(x) \, dx} \\ &= \sqrt{\int_{\Omega} |\nabla v_h(x)|^2 \, dx} \\ &= \sqrt{\sum_K |\nabla v_h|_K(x)|^2 |K|} \end{aligned}$$

with  $\nabla v_h|_K(x)$  does not depend on  $x$

$$\nabla v_h|_K(x) = v_i \begin{pmatrix} c_1^{(i)} \\ c_2^{(i)} \end{pmatrix} + v_j \begin{pmatrix} c_1^{(j)} \\ c_2^{(j)} \end{pmatrix} + v_k \begin{pmatrix} c_1^{(i)} \\ c_2^{(i)} \end{pmatrix}.$$

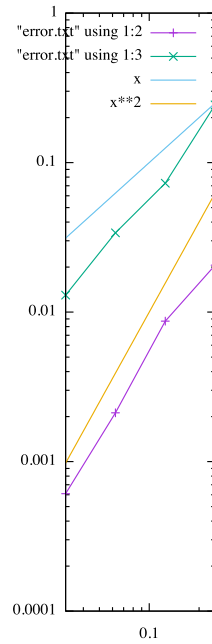


Figure 16:

## 4 13 February 2018

### 4.1 Heat Equation

#### 4.1.1 Problem

Find  $u : \Omega \times (0, T) \rightarrow \mathbb{R}$  such that

$$\begin{cases} u_t - \nu \Delta u = f & \text{in } \Omega \times (0, T) \\ u = g & \text{on } \partial\Omega \times (0, T) \\ u = u^0 & \text{in } \Omega \text{ at } t = 0 \end{cases}$$

where  $f, g, u^0$  is given.



#### 4.1.2 Weak Form

We want to find  $\{u(t) \in V(g(t)); t \in (0, T)\}$  with  $V := H_0^1(\Omega)$ ,  $g : \partial\Omega \rightarrow \mathbb{R}$  smooth function, and  $V(g) := \{v + g \in H^1(\Omega); v \in V\}$  such that

$$\begin{cases} (u_t(t), v) + a(u(t), v) = (f(t), v) & , \forall v \in V \\ u(0) = u^0 & \in V(g(0)) \end{cases}$$

where  $a(u, v) := \nu \int_{\Omega} \nabla u \cdot \nabla v \, dx$

#### 4.1.3 Finite Element Method

For  $n = 1, \dots, N_T$ ,  $\Delta t > 0$ ,  $N_T := \lfloor \frac{T}{\Delta t} \rfloor \in \mathbb{N}$ , we looking for  $\{u_h^n \in V_h(g^n)\}$  such that

$$\begin{cases} \left( \frac{u_h^n - u_h^{n-1}}{\Delta t}, v_h \right) + a(u_h^n, v_h) = (f^n, v_h) & , \forall v_h \in V_h \\ u_h^0 = \Pi_h u^0 & \in V_h(g^0) \end{cases}$$

where

$$\begin{aligned} V_h \subset V & : \quad P1 - FEspace \\ V_h(g) & := \{v_h + g \in V(g); v_h \in V_h\} \\ \Pi_h & : \quad C(\bar{\Omega}) \rightarrow X_h \\ X_h \subset H^1(\Omega) & : \quad P1 - FEspace \\ (V_h \subset X_h) & \Rightarrow V_h \subset H_0^1(\Omega), \quad X_h \subset H^1(\Omega) \\ (\Pi_h u^0)(P) & = u^0(P), \quad P : \text{nodal point} \end{aligned}$$

Then the problem become

$$\begin{aligned} \frac{1}{\Delta t} (u_h^n, v_h) + a(u_h^n, v_h) & = \frac{1}{\Delta t} (u_h^{n-1}, v_h) + (f^n, v_h), \quad \forall v_h \in V_h \\ \left[ \frac{1}{\Delta t} M_{N_p \times N_p} + A_{N_p \times N_p} \right] [U_{N_p \times 1}] & = b_{N_p \times 1}. \end{aligned}$$

#### 4.1.4 Flowchart

The algorithm used to solve heat equation problem is shown as flowchart in Figure 17 and 18

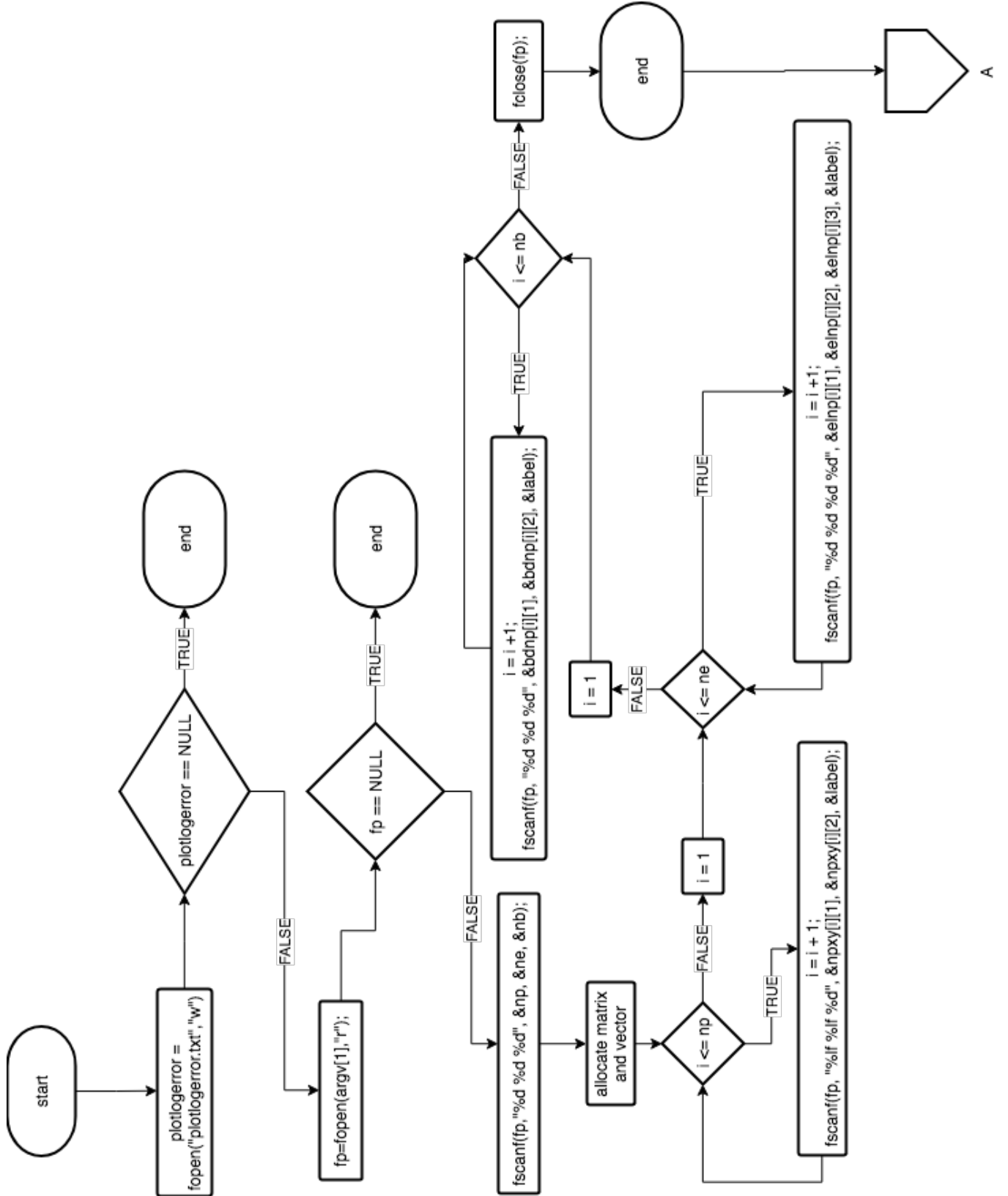


Figure 17:

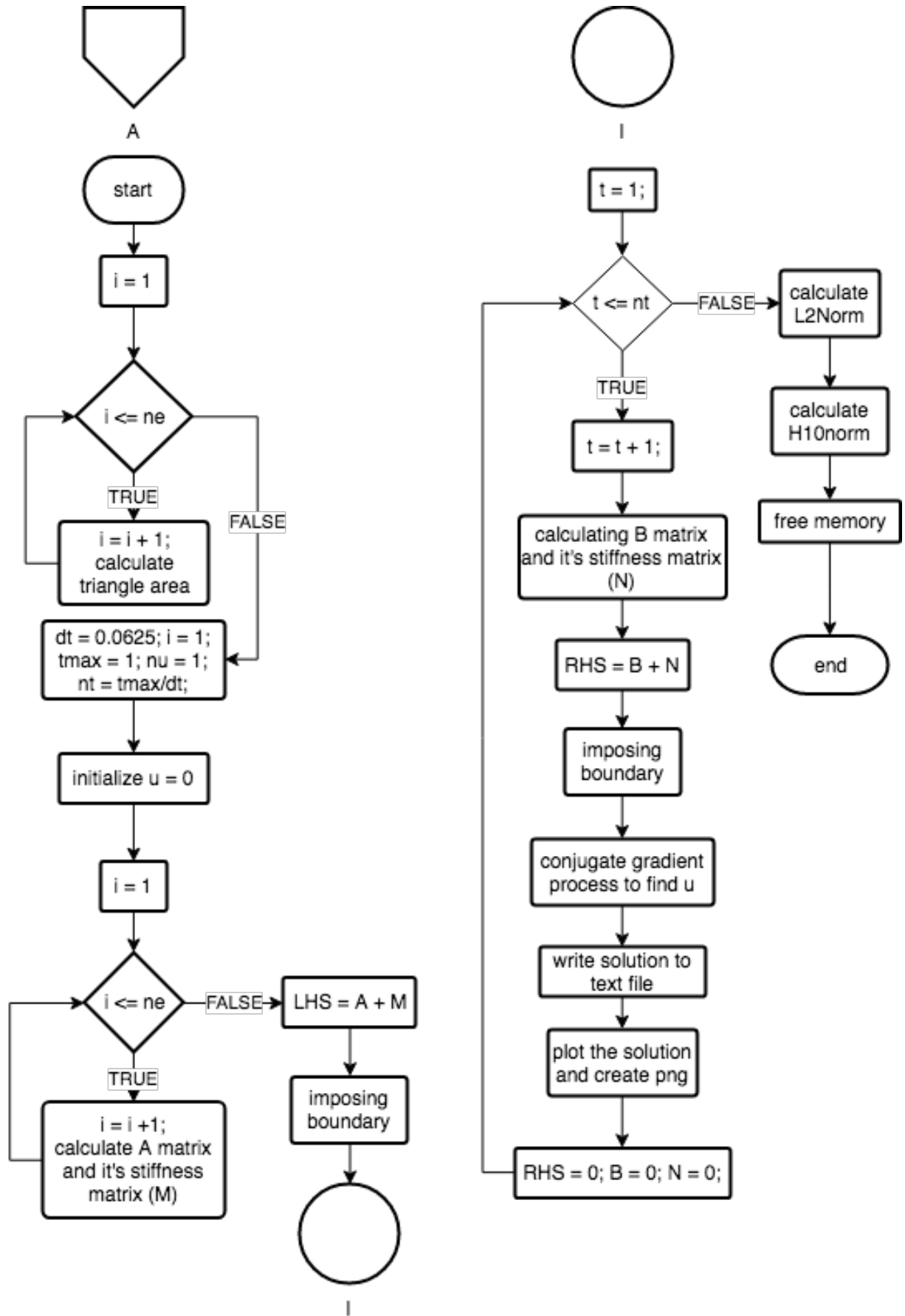


Figure 18:

#### 4.1.5 Numerical experiment

We consider problem

$$\begin{cases} u_t - \nu \Delta u = (1 + 2\nu\pi^2 t) \sin(\pi x_1) \sin(\pi x_2) & , \Omega \times (0, T) \\ u = 0 & , \partial\Omega \times (0, T) \\ u = 0 & , \Omega \text{ at } t = 0 \end{cases}$$

with exact solution

$$u(x_1, x_2, t) = t \sin(\pi x_1) \sin(\pi x_2)$$