1 Unix Command Terminal

1.1 5 October 2017

- mk dir : make a file in directory.
- ls : file list in directory
- cd : change directory
- pwd : parent working directory
- cp -r : copy
- rm : remove
- open ... : open ... file
- . : here
- cd .. : back to previous

2 Gnuplot

2.1 16 October 2017

2.1.1 Plot from terminal

$$\begin{cases} \text{gnuplot} \\ \text{set terminal x11 (for 2D and 3D)} \\ \text{plot}sin(x) \\ \text{plot}sin(x), cos(x) \\ \text{set hidden3d} \\ \text{splot}sin(x) * sin(y) \\ \text{set xrange}[-5:5] \\ \text{set yrange}[-5:5] \\ \text{q quit} \end{cases}$$

2.1.2 Plot from file

We have file **plot.dat or .txt** contains list point of triangular format for 2D as shown below:

$$\begin{array}{lll} x_1^{(i)} & x_2^{(i)} & u^{(i)} \\ x_1^{(j)} & x_2^{(j)} & u^{(j)} \\ x_1^{(k)} & x_2^{(k)} & u^{(k)} \\ x_1^{(i)} & x_2^{(i)} & u^{(i)} \end{array}$$

(leave blank)

There are some "blank" on line. In 1D case, we will have two lines of points and 1 blank line. Then, for 2D case, we have four llines of points and two blank (anw: I think it still work for one blank thought).

$\{$splot 'plot.dat' u 1:2:3 w l pallete

where **u** means using, **1:2:3** means the column we wish to plot, **w** means with, **l** means line, and **pallete** means color.

Other command that maybe used is,

$\{$set pm3d map

is used for mapping 3D to 2D. Other is,

$\{$set size ratio -1

then de data will be integers.

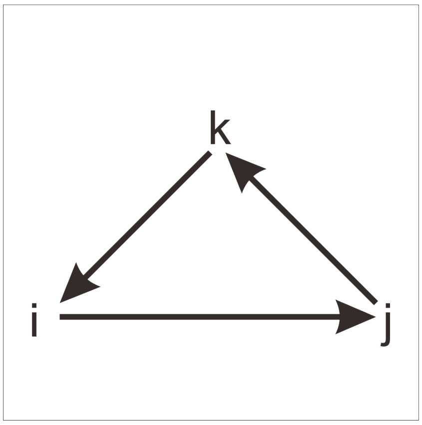


Figure 1: One element with point i, j, k

3 FreeFEM++

3.1 16 October 2017

Example 1 :

$$\begin{cases} \text{int } n = 50; \\ \text{real } x0 = 0.0,\ y0 = 0.0,\ Lx = 1.0,\ Ly = 1.0,\ z = 1; \\ \text{border } a1(t = 0, 2 * pi)\{x = z * cos(t);\ y = z * sin(t);\} \end{cases}$$

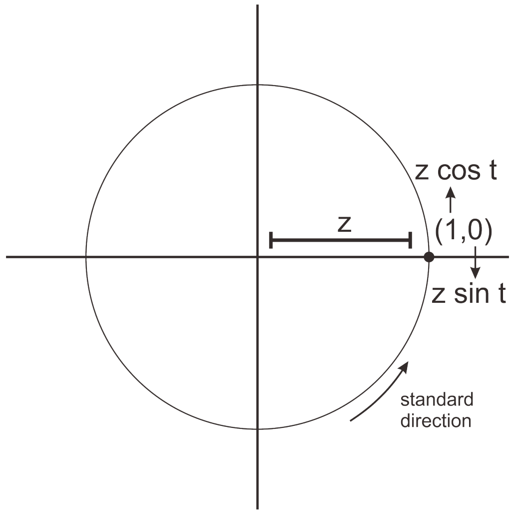


Figure 2: Example 1

Example 2:

$$\begin{cases} \text{border } a1(t = 0, 1)\{x = t; y = 0;\} \\ \text{border } a2(t - 0, 1)\{x = 1; y = t;\} \end{cases}$$

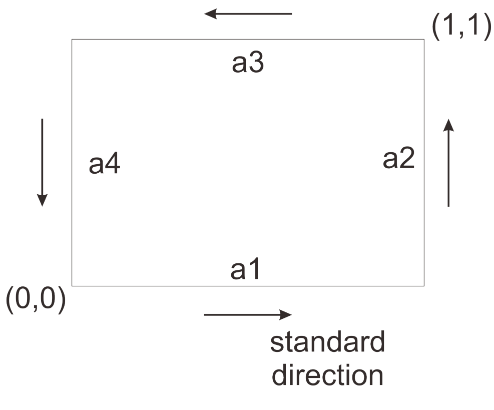


Figure 3: Example 2

Here are image of how to choose the domain.

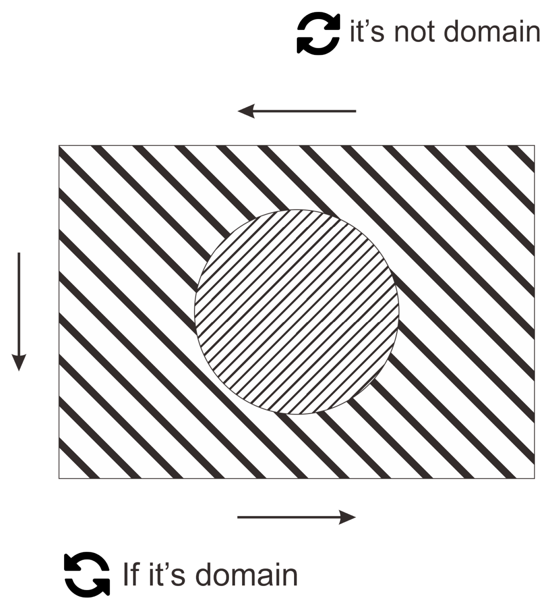


Figure 4: Choose the domain

From file : **membrane.edp**. Find $\phi : \Omega \to \mathbb{R}$ such that

$$\begin{cases} - \triangle \phi = f(=1) & \text{in } \Omega \\ \phi = z(= x_1) & \text{on } \Gamma_1 \\ \dfrac{\partial \phi}{\partial n} = 0 & \text{on } \Gamma_2 \end{cases} \tag{1}$$

where $f : \Omega \to \mathbb{R}$, given $f(x) = 1$ and $z : \Gamma_1 \to \mathbb{R}$, given $z(x) = x_1$. The equation (1) is called **strong form**, because it contains second derivative.

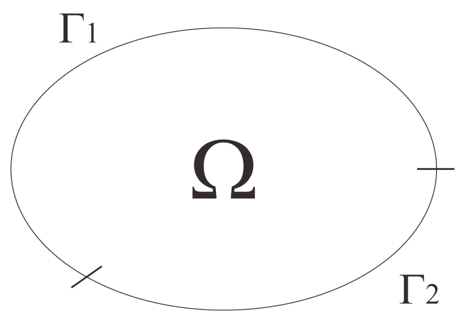


Figure 5

We multiple equation (1)'s first line by smooth test function w and integrate over Ω such that $\forall w(w|_{\Gamma_1} = 0)$

$$\int_\Omega - \triangle \phi(x) w(x)\ dx = \int_\Omega f(x)w(x)\ dx$$

using integration by parts,

$$- \int_{\partial\Omega} \frac{\partial \phi}{\partial n}(x) w(x)\ dx + \int_\Omega \triangledown \phi(x) \cdot \triangledown w(x)\ dx = \int_\Omega f(x) w(x)\ dx$$

then we can devide the boundary such that

$$-\int_{\Gamma_1} \frac{\partial \phi}{\partial n}(x)w(x)\ dx - \int_{\Gamma_2} \frac{\partial \phi}{\partial n}(x)w(x)\ dx + \int_{\Omega} \triangledown \phi(x) \cdot \triangledown w(x)\ dx = \int_{\Omega} f(x)w(x)\ dx$$

Because on Γ_1, smooth function $w(x)$ value is equal to 0. And on Γ_2 by equation (1) line 3 we obtain that $\frac{\partial \phi}{\partial n}(x)$ is equal to 0. Then, we conclude that

$$\int_{\Omega} \triangledown \phi(x) \cdot \triangledown w(x)\ dx - \int_{\Omega} f(x)w(x)\ dx = 0 \tag{2}$$

is the **weak form**. Because it only contain first derivative.

3.2 6 November 2017

3.2.1 Gauss-Green Formula

In **2D Case**, we have $f, g : \Omega \to \mathbb{R}$

$$\int_{\Omega} \frac{\partial f}{\partial x_i}(x)g(x)\ dx = \int_{\partial\Omega} f(x)g(x)n_i\ ds - \int_{\Omega} f(x)\frac{\partial g}{\partial x_i}(x)\ dx,\ (i = 1, 2) \tag{3}$$

In **1D Case**,

$$\int_a^b f(x)g(x)\ dx = [f(x)g(x)]_{x=a}^b - \int_a^b f(x)g'(x)\ dx \tag{4}$$

where

$$\begin{aligned} \int_{\partial\Omega} f(x)g(x)n(x)\ dx &= f(a)g(a)n(a) + f(b)g(b)n(b) \\ &= [f(x)g(x)]_{x=a}^b \end{aligned}$$

3.2.2 Strong form

Consider problem to find $u : \Omega \to \mathbb{R}$ with **strong form** such as

$$\begin{cases} -\triangle u = f & \text{in } \Omega \\ u = g_0 & \text{on } \Gamma_0 \text{ Dirichlet (or essential) B.C.} \\ \frac{\partial u}{\partial n} = g_1 & \text{on} \Gamma_1 \text{ Neumann (or natural) B.C.} \end{cases} \tag{5}$$

where $f : \Omega \to \mathbb{R}$, $g_0 : \Gamma_0 \to \mathbb{R}$, and $g_1 : \Gamma_1 \to \mathbb{R}$ is given.

$$\begin{array}{lll} \Omega & \subset & \mathbb{R}^2 \\ n & : \partial\Omega & \to \mathbb{R}^2 \\ & x & \mapsto n(x) \end{array}$$

We have notation $\triangle = \sum_{i=1}^2 \frac{\partial^2}{\partial x_1^2}$ such that $-\triangle u(x) = -(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2})u(x)$. So we have

$$\begin{aligned} & -\triangle u(x) = f(x) \\ \Leftrightarrow \quad & -(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2})u(x) = f(x). \end{aligned}$$

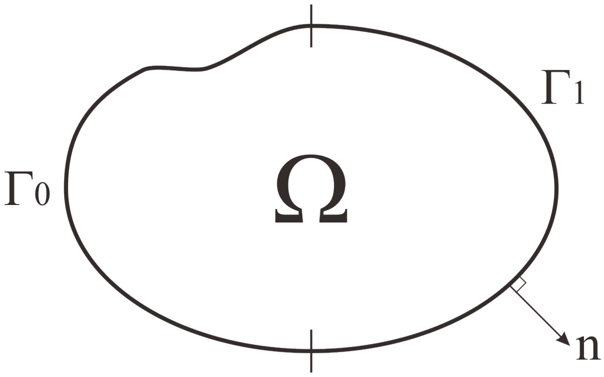


Figure 6

For all smooth test function $v(x)$, where $v|_{\Gamma_0} = 0$

$$
\begin{aligned}
\int_\Omega (-\Delta u)(x)v(x)dx &= \int_\Omega \Big(-\frac{\partial^2 u}{\partial {x_1}^2}(x)v(x) - \frac{\partial^2 u}{\partial {x_2}^2}(x)v(x)\Big)\ dx \\
&= -\int_\Omega \frac{\partial^2 u}{\partial x_1^2}(x)v(x)\ dx - \int_\Omega \frac{\partial^2 u}{\partial x_2^2}(x)v(x)\ dx \\
&= -\Big(\int_{\partial\Omega} \frac{\partial u}{\partial x_1}(x)v(x)n_i\ ds - \int_\Omega \frac{\partial u}{\partial x_1}(x)\frac{\partial v}{\partial x_1}(x)\ dx\Big) \\
&\quad -\Big(\int_{\partial\Omega} \frac{\partial u}{\partial x_2}(x)v(x)n_i\ ds - \int_\Omega \frac{\partial u}{\partial x_2}(x)\frac{\partial v}{\partial x_2}(x)\ dx\Big) \\
&= \int_\Omega \Big(\frac{\partial u}{\partial x_1}(x)\frac{\partial v}{\partial x_1}(x) + \frac{\partial u}{\partial x_2}(x)\frac{\partial v}{\partial x_2}(x)\Big)\ dx + \int_{\Gamma_0} \Big(\frac{\partial u}{\partial x_1}(x)v(x)n_i + \frac{\partial u}{\partial x_2}(x)v(x)n_i\Big)\ ds \\
&\quad + \int_{\Gamma_1} \Big(\frac{\partial u}{\partial x_1}(x)v(x)n_i + \frac{\partial u}{\partial x_2}(x)v(x)n_i\Big)\ ds \\
&= \int_\Omega \nabla u(x)\cdot\nabla v(x)\ dx - \int_{\Gamma_1} (\nabla u(x)\cdot n(x))v(x)\ ds \\
&= \int_\Omega \nabla u(x)\cdot\nabla v(x)\ dx - \int_{\Gamma_1} \frac{\partial u}{\partial n}(x)v(x)\ ds \\
&= \int_\Omega \nabla u(x)\cdot\nabla v(x)\ dx - \int_{\Gamma_1} g_1(x)v(x)\ ds \\
&= \int_\Omega f(x)v(x)\ dx
\end{aligned}
$$

such that

$$\int_\Omega \nabla u\cdot\nabla v\ dx = \int_\Omega fv\ dx + \int_{\Gamma_1} g_1 v\ ds$$

Note : Reason why $\dfrac{\partial u}{\partial n}(x) = (\bigtriangledown u)\cdot n$ In 1D case,

$$u'(x) = \lim_{h\to 0}\frac{u(x+h)-u(x)}{h}$$

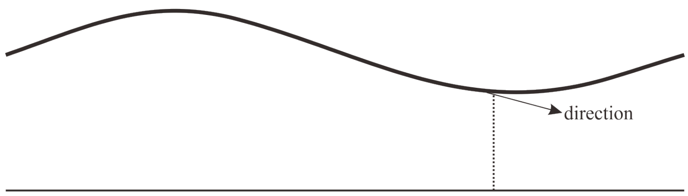


Figure 7

In 2D case,

$$\begin{aligned}
\frac{\partial u}{\partial n}(x) &= \lim_{h\to 0}\frac{u(x+hn)-u(x)}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\big[u(x_1+hn_1,\ x_2+hn_2)-u(x_1,x_2)\big] \\
&= \lim_{h\to 0}\frac{1}{h}\big[u(x_1+hn_1,\ x_2+hn_2)-u(x_1,x_2+hn_2)+u(x_1,x_2+hn_2)-u(x_1,x_2)\big] \\
&= \lim_{h\to 0}\frac{u(x_1+hn_1,\ x_2+hn_2)-u(x_1,x_2+hn_2)+u(x_1,x_2+hn_2)-u(x_1,x_2)}{hn_1}n_1 \\
& \lim_{h\to 0}\frac{u(x_1+hn_2,\ x_2+hn_2)-u(x_1,x_2+hn_2)+u(x_1,x_2+hn_2)-u(x_1,x_2)}{hn_2}n_2 \\
&= \frac{\partial u}{\partial x_1}(x_1,x_2)n_1+\frac{\partial u}{\partial x_2}(x_1,x_2)n_2 \\
&= (\nabla u)\cdot n
\end{aligned}$$

3.2.3 Weak form

We want to find $u \in V(g_0)$ such that

$$a(u,v) = l(v),\ \ \forall v \in V$$

where

$$V(g_0) \equiv \{v \in H^1(\Omega);\ v|_{\Gamma_0} = g_0\},\ V \equiv V(0).$$

There are some notation we need to know beforehand,

$$L^2(\Omega) \equiv \{v : \Omega \to \mathbb{R}; \int_\Omega v^2(x)\ dx < \infty\}.$$

For examples,

$$\begin{aligned}
\Omega &= (1,\infty) \\
f(x) &= \frac{1}{x} \in L^2(1,\infty);\ \int_1^\infty f^2(x)\ dx = [-x^{-1}]_1^\infty = 1 \\
f(x) &= \frac{1}{\sqrt{x}} not \in L^2(1,\infty);\ \int_1^\infty\ \ dx = [logx]_1^\infty = \infty
\end{aligned}$$

Furthermore, $L^2(\Omega)$ is a Hilbert space or complete space with inner product.

$$H^1(\Omega) \equiv \{v \in L^2(\Omega);\ \frac{\partial v}{\partial x_i} \in L^2(\Omega),\ i = 1,2.\}$$

Inner product is defined by

$$(f, g) \equiv \int_\Omega f(x)g(x)dx.$$

Back to the problem, we have bilinear form $a(u, v)$ and linear form $l(v)$ as shown below.

$$\begin{aligned} a(u, v) &= \int_\Omega \nabla u \cdot \nabla v \; dx \\ l(v) &= \int_\Omega f v \; dx + \int_{\Gamma_1} g_1 v \; ds. \end{aligned}$$

$l(v)$ is called linear form, because it holds that

$$l(\alpha v + \beta w) = \alpha l(v) + \beta l(w).$$

Then $a(u, v)$ is called bilinear form because if u is fixed, them v is linear form respect to u, and vice versa.

3.2.4 Discretization

We approach value of smooth function $u(x)$ by piecewise linear function $u_h(x)$ as

$$u(x) \; u_h(x) \equiv \sum_{i=1}^{Np} c_i \varphi_i(x)$$

where Np is total number of nodal points.

For case as shown by picture below,

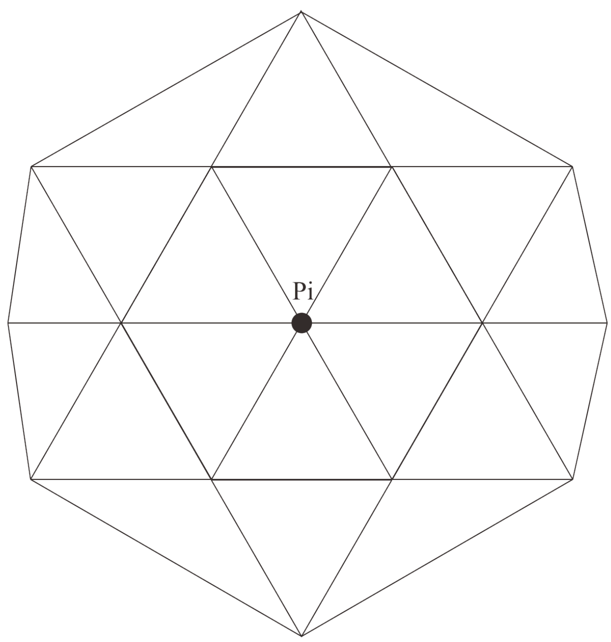


Figure 8

we choose basis function

$$\begin{aligned} \varphi &: \Omega \to \mathbb{R} \\ \varphi_i(P_j) &= \begin{cases} 1 & , i = j, \\ 0 & , i \neq j. \end{cases} \end{aligned}$$

Then, in each triangle,

$$\varphi_i(x) = \alpha_0 + \alpha_1 x_1 + \alpha_1 x_2.$$

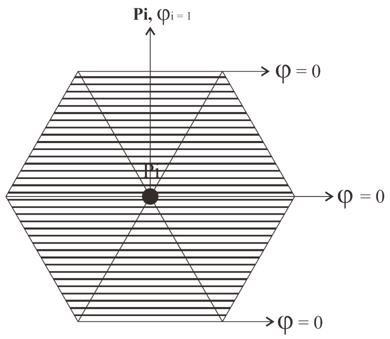


Figure 9

Then, the problem becomes we want to find $u_h \in V_h(g_0)$ such that

$$a(u_h, v_h) = l(v_h), \ \forall v_h \in V_h$$

where

$$V_h(g_0) \equiv \{v_h \in V(g_0); \ v_h(x) = \sum_{i=1}^{Np} c_i \varphi_i(x), \ c_i \in \mathbb{R}, \ \varphi_i \text{ is basis function}\}, \ V_h \equiv V_h(0).$$

3.2.5 Problem

For simplicity, assume $\Omega = (0,1)^2$ and $g_0 = 0$. We want to find $u \in V \equiv H_0^1(\Omega)$ which the value on boundary is 0. $\{P_i\}_{i=1}^{Np}$ is set of nodal points. For example,

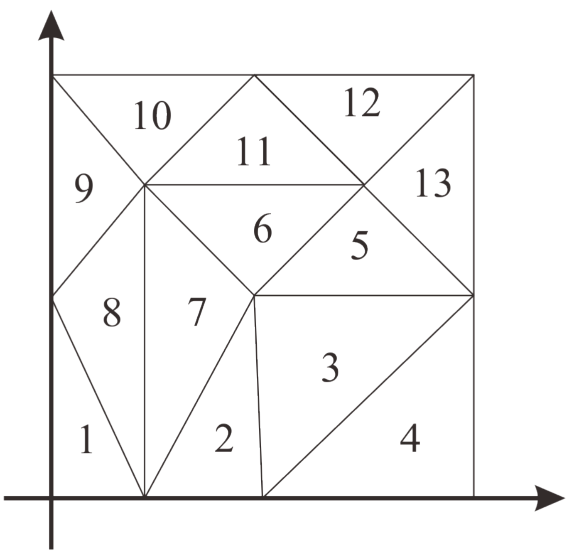


Figure 10

with nodal point $Np = 11$ and elements $Ne = 13$. The domain $\tau_h = \{K_k\}_{k=1}^{13}$ or devided into 13 triangle area with point $\{P_i\}_{i=1}^{11}$.

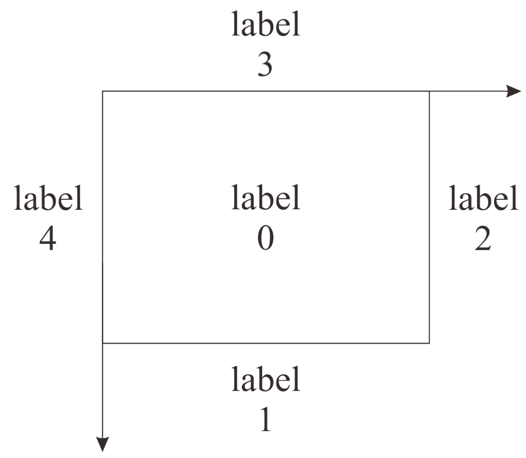


Figure 11

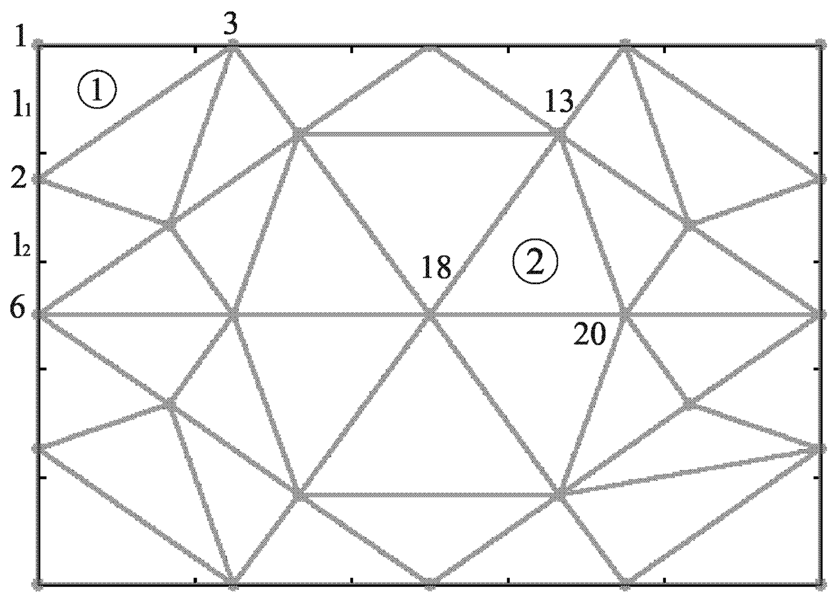


Figure 12

3.3 4 December 2017

3.3.1 Continuous (Partial Differential Equation)

To show that there is exist unique solution u, we can use the Remark below.

Remark 1. $\exists! u = \underset{v\in V(g)}{argmin}\ (\frac{1}{2}a(v,v) - l(v)) = \underset{w\in V(g)}{argmin}\ J(w)$

minimizer or *argmin* is value of x where the function $f(x)$ is minimum. For example, for function $f(x) = 1 + x^2$, $\underset{x\in\mathbb{R}}{min} f(x) = f(0) = 1$. Then, the minimizer, $\underset{x\in\mathbb{R}}{argmin} f(x) = 0$.

Using this Remark, the Proposition below is given with proof.

Proposition 1. *For* $J(v) := \frac{1}{2}a(v,v) - l(v)$, $u = \underset{v\in V(g)}{argmin}\ J(v) \iff a(u,v) = l(v)$

Proof:

($\Rightarrow$) if $u = argmin\ J$ then $u + tv \in V(g),\ \forall t \in \mathbb{R}, \forall v \in V = H_0^1(\Omega)$. Since it is on boundary Γ, then $g = u = u + tv$.

$$\begin{array}{lll} J(u) & \leq J(w) & ,\forall w \in V(g),\ w = u + tv \in V(g) \\ J(u) & \leq J(u+tv) & ,\forall t \in \mathbb{R},\ \forall v \in V \end{array}$$

Then

$$\begin{aligned} J(u+tv) &= \frac{1}{2}a(u+tv, u+tv) - l(u+tv) \\ &= \frac{1}{2}a(u,u) + ta(u,v) + \frac{t^2}{2}a(v,v) - l(u) - tl(v) \\ &= \frac{t^2}{2}a(v,v) + t(a(u,v) - l(v)) + J(u) \\ &=: \varphi(t) \end{aligned}$$

Because $\varphi(t)$ is in quadratic form, then its minimum obtained at $t = 0$. So that $\varphi = 0$ such that $a(u,v) - l(v) = 0$.
($\Leftarrow$) $\forall t \in \mathbb{R}, \forall v \in V$ we have

$$J(u,tv) = J(u) + \frac{t^2}{2}a(v,v) \geq J(u).$$

$\forall w \in V(g)$, we set $v := w - u \in V,\ t := 1,\ w = u + tv$

$$J(w) = J(u+tv) \geq J(u)$$

3.3.2 Discrete (Finite Element Method)

Here introduced some notation,

$$\begin{array}{rcl} X_h & \subset & X \text{ (usually dim } X_h < \infty) \\ V_h & = & X_h \cap V \\ g_h & \in & X_h \text{(approximation of } g) \\ V_h(g_h) & = & \{v_h \in X_h; v_h - g_h \in V_h\}. \end{array}$$

Then the weak form is approximated with

$$\begin{cases} a(u_h, v_h) & = l(v_h),\ \forall v_h \in V_h \\ u_h & \in V_h(g_h) \end{cases} \iff u_h = \underset{w_h\in V_h(g_h)}{argmin}\ J(w_h)$$

Using Finite Element Method,

$$\begin{array}{rcl} X_h & = & \{v_h \in C^0(\overline{\Omega});\ v_h|_K \text{ is linear}\} \\ V_h & = & X_h \cap H_0^1(\Omega), \end{array}$$

or we could write

$$\begin{aligned} X_h &= \langle \varphi_1, \ldots, \varphi_{Np} \rangle \\ &= \{\sum_{i=1}^{Np} c_i\varphi_i\ ;\ c_i \in \mathbb{R}\} \end{aligned}$$

where $\{\varphi_i\}_{i=1}^{Np}$ become a basis of the vector space X_h. For nodal points $\{P_i\}_{i=1}^{Nfp}$ and $\varphi_i \in X_n\ ;\ \varphi_i(P_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

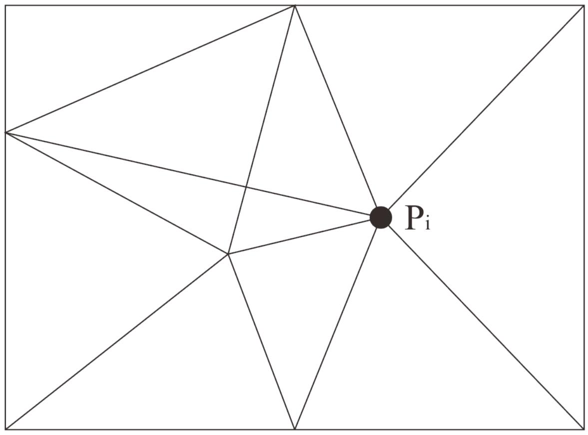


Figure 13

For $x = (x_1, x_2) \in \mathbb{R}^2$ and $\forall v_h \in X_h$, we will have

$$v_h(\cdot) = \sum_{i=1}^{Np} v_h(P_i)\varphi_i(\cdot) \quad \in \quad X_h$$
$$w_h := \sum_{i=1}^{Np} v_h(P_i)\varphi_i \quad \in \quad X_h$$

such that

$$\begin{aligned} w_h(P_j) &= \sum_{i=1}^{Np} v_h(P_i)\varphi_i(P_j) \\ &= \sum_{i=1}^{Np} v_h(P_i)\delta_{ij} \\ &= v_h(P_j) \end{aligned}$$

Now, we consider basis of V_h for

$$\begin{aligned} \Omega \cap \Gamma &= \emptyset \\ \{\varphi_i; P_i \in \Omega\} &\subset \{P_i\}_{i=1}^{Np}. \end{aligned}$$

For simplicity, we assume $\{\varphi_i; P_i \in \Omega\} = \{P_i\}_{i=1}^{N}$ for $(N < Np)$, such that $\{P_i\}_{i=1}^{N} \subset \Omega$ and $\{P_i\}_{i=N+1}^{Np} \subset \Gamma$.

Let $V_h = \langle \varphi_1, \cdots, \varphi_N \rangle$, then

$$a(u_h, v_h) = l(v_h),\ (\forall v_h \in V_h) \ \Leftrightarrow \ a(u_h, \varphi_i) = l(\varphi_i),\ (i = 1, \cdots, N).$$

If we choose $v_h = \varphi_i \in V$, then $\forall v_h \in V_h$ with $c_i = v_h(P_i)$ and $v_h = \sum_{i=1}^{N} c_i\varphi_i$,

$$\begin{aligned}
a(u_h, v_h) &= a(u_h, \sum_{i=1}^{N} c_i\varphi_i) \\
&= \sum_{i=1}^{N} c_i a(u_h, \varphi_i) \\
&= \sum_{i=1}^{N} c_i l(\varphi_i) \\
&= l(\sum_{i=1}^{N} c_i\varphi_i) \\
&= l(v_h)
\end{aligned}$$

We set $u_j := u_h(P_j)$ for $j = 1, \cdots, Np$ such that at the boundary $P_j \in \Gamma$ or for $j = N+1, \cdots, Np$

$$u_j = g_j = g_h(P_j),\ u_h \in V_h(g_h)$$

with $u_1, \cdots, u_N$ is unknown.

Then, we can conclude that

$$\begin{cases} a(u_h, \varphi_i) = l(\varphi_i),\ (i = 1, \cdots, N) \\ u_h = \sum_{j=1}^{N}\ u_j\varphi_j + \sum_{j=N+1}^{Np}\ g_j\varphi_j,\ (u_h \in V_h(g_h)) \end{cases}.$$

For simplicity, we set notation $a_{ij} := a(\varphi_i, \varphi_j) = a(\varphi_j, \varphi_i)$ such that for $i = 1, \cdots, N$,

$$\sum_{j=1}^{N} a_{ij}u_j + \sum_{j=N+1}^{Np} a_{ij}g_j = l(\varphi_i)$$

As conclusion,

$$a(u_h, v_h) = l(v_h),\ (\forall v_h \in V_h) \Leftrightarrow \mathbf{A}\mathbf{u} = \mathbf{b}$$

where

$$\begin{aligned}
A &:= (a_{ij}) \in \mathbb{R}^{N\times N}_{\text{sym}} \\
\mathbf{u} &:= \begin{pmatrix} u_i \\ \vdots \\ u_N \end{pmatrix} \\
\mathbf{b} &:= \Big(l(u_i) - \sum_{j=N+1}^{Np} a_{ij}g_j\Big), \text{ for } i = 1, \cdots, N
\end{aligned}$$

3.4 25 Desember 2017

3.4.1 Calculate matrix A

$$\begin{aligned}
A_{ij} &= a(\varphi_j, \varphi_i) \\
&= \int_\Omega \nabla\varphi_j(x) \cdot \nabla\varphi_i(x)\ dx \\
&= \sum_{k=1}^{Ne} \int_{K_k} \nabla\varphi_j(x) \cdot \nabla\varphi_i(x)\ dx \\
&= \sum_{k=1}^{Ne} \int_{K_k} 1\ dx\ \big(\nabla\varphi_j(x) \cdot \nabla\varphi_i(x)\big).
\end{aligned}$$

To calculate it, usually we need N^3 computation for code like

> for $i = 1, \ldots, Np$ {
> for $j = 1, \ldots, Np$ {
> for $k = 1, \ldots, Ne$ {
> $A_{ij} = A_{ij} + \int_{K_k} \nabla\varphi_j(x) \cdot \nabla\varphi_i(x) dx$
> }

> }
> }

But, we can simplify it into N^2 computation using local matrix and global matrix as shown below.

> for $k = 1, \dots, Ne$ {
> (local matrix)
> $A^k_{11} = \int_{K_k} \nabla\varphi_1(x) \cdot \varphi_1(x)\ dx$
> $A^k_{12} = \int_{K_k} \nabla\varphi_2(x) \cdot \varphi_1(x)\ dx$
> $\vdots$
> $A^k_{33} = \int_{K_k} \nabla\varphi_3(x) \cdot \varphi_3(x)\ dx$
> (global matrix)
> $A_{44} = A_{44} + A^k_{11}$
> $A_{47} = A_{47} + A^k_{12}$
> $\vdots$
> $A_{73} = A_{73} + A^k_{23}$
> $\vdots$
> $A_{33} = A_{33} + A^k_{33}$
> }

3.4.2 Calculate vector B

$$\begin{aligned} b_i &= \int_\Omega f(x)\varphi_i(x)\ dx - \sum_{j=1}^{Np} g_h(P_j) \int_\Omega \nabla\varphi_j(x) \cdot \nabla\varphi_i(x)\ dx \\ &= \sum_{k=1}^{Ne} \int_{K_k} f(x)\varphi_i(x)\ dx - \sum_{j=1}^{Np} g_h(P_j) \sum_{k=1}^{Ne} \int_{K_k} \nabla\varphi_j(x) \cdot \nabla\varphi_i(x)\ dx \end{aligned}$$

Consider $g_h(P_j) = 0$, with $f_h(x) = \sum_{j=1}^{Np} f(P_j)\varphi_j(x)$ then for each element,

$$\begin{aligned} b_i &= b_i + \int_{K_k} f(x)\varphi_i(x)\ dx \\ &= b_i + \int_{K_k} \Big(f(P_1)\varphi_1(x) + f(P_2)\varphi_2(x) + f(P_3)\varphi_3(x)\Big)\varphi_i(x)\ dx \\ &= b_i + \int_{K_k} f(P_1)\varphi_1(x) \cdot \varphi_i(x) + f(P_2)\varphi_2(x) \cdot \varphi_i(x) + f(P_3)\varphi_3(x) \cdot \varphi_i(x)\ dx \\ &= b_i + f(P_1) \int_{K_k} \varphi_1(x) \cdot \varphi_i(x)\ dx + f(P_2) \int_{K_k} \varphi_2(x) \cdot \varphi_i(x)\ dx + \\ &\quad f(P_3) \int_{K_k} \varphi_3(x) \cdot \varphi_i(x)\ dx \end{aligned}$$

with

$$\int_{K_k} \varphi_i(x)\varphi_j(x) = \begin{cases} \dfrac{meas(K_k)}{6}, & for\ i = j \\ \dfrac{meas(K_k)}{12}, & for\ i \neq j \end{cases}$$

(way to calculate in program)

3.4.3 Other calculation

For any points $P_i(x_1, x_2)$ in triangle K,

$$\varphi_i(x) = c_0 + c_1x_1 + c_2x_2,\ c_j \in \mathbb{R}$$

such that

$$\nabla\varphi_i(x) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

To compute the triangle area $\int_{K_k} 1\ dx$,

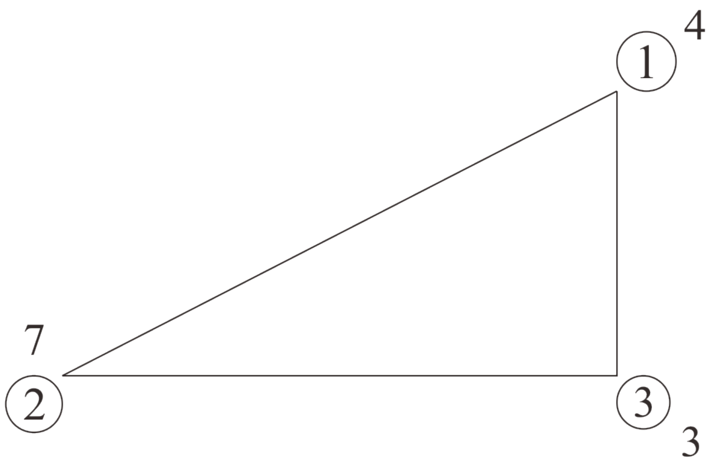


Figure 14

3.5 5 January 2018 (Error Estimate)

3.5.1 Norm

After find the solution $u(x)$, we should check if the solution we found is close enough to the exact solution. For numerical solution u_h, we check for $h = \frac{1}{N}$ where N or devider of each boundary side is set to $N = 4, 8, 16, 32$.

In this problem we set $f(x) = -\nabla u(x) = 2\pi^2(sin(x\pi)sin(y\pi))$ with exact solution $u_e(x) = sin(x\pi)sin(y\pi)$. We want to calculate $\|u_h - \Pi_h u_e\|_X, s\ X = L^2(\Omega), H_0^1(\Omega)$. Here, Π_h is function that mapping continuous function $C(\bar{\Omega})$ into piecewise linear finite element space.

$$
\begin{aligned}
\Pi_h : \quad C(\bar{\Omega}) &\rightarrow P1 - FEsp \\
v &\mapsto \Pi_h v
\end{aligned}
$$

such that $(\Pi_h v)(P) := v(P)$, P is nodes.

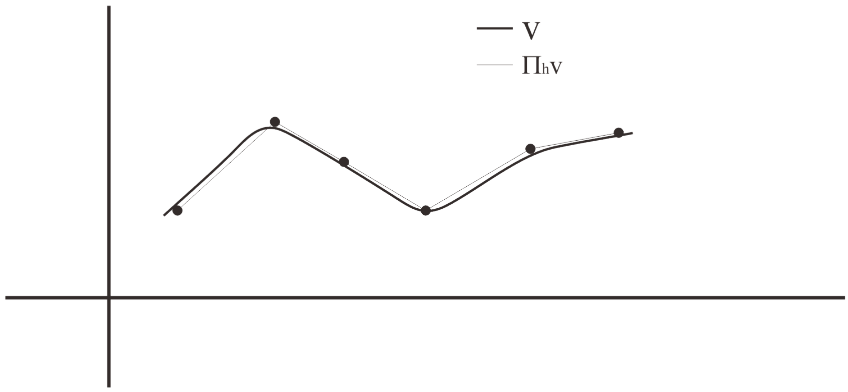


Figure 15

3.5.2 $L^2(\Omega)$ norm

$$
\begin{aligned}
\|v_h\|_{L^2(\Omega)} &= \sqrt{\int_\Omega \big(v_h(x)\big)^2 \; dx} \\
&= \sqrt{\sum_K \int_K \big(v_h(x)\big)^2 \; dx} \\
&= \sqrt{\sum_K \int_K \big(v_i \varphi_i^{(K)}(x) + v_j \varphi_j^{(K)}(x) + v_k \varphi_k^{(K)}(x)\big)^2 \; dx} \\
&= \sqrt{\sum_K \int_K \left(\begin{pmatrix} v_i & v_j & v_k \end{pmatrix} \begin{pmatrix} \varphi_i^{(K)}(x) \\ \varphi_j^{(K)}(x) \\ \varphi_k^{(K)}(x) \end{pmatrix} \right)^2 \; dx} \\
&= \sqrt{\sum_K \int_K \begin{pmatrix} v_i & v_j & v_k \end{pmatrix} \begin{pmatrix} \varphi_i^{(K)}(x) \\ \varphi_j^{(K)}(x) \\ \varphi_k^{(K)}(x) \end{pmatrix} \begin{pmatrix} \varphi_i^{(K)}(x) & \varphi_j^{(K)}(x) & \varphi_k^{(K)}(x) \end{pmatrix} \begin{pmatrix} v_i \\ v_j \\ v_k \end{pmatrix} \; dx} \\
&= \sqrt{\sum_K \begin{pmatrix} v_i & v_j & v_k \end{pmatrix} \int_K \begin{pmatrix} \varphi_i^{(K)} \cdot \varphi_i^{(K)} & \varphi_i^{(K)} \cdot \varphi_j^{(K)} & \varphi_i^{(K)} \cdot \varphi_k^{(K)} \\ \varphi_j^{(K)} \cdot \varphi_i^{(K)} & \varphi_j^{(K)} \cdot \varphi_j^{(K)} & \varphi_j^{(K)} \cdot \varphi_k^{(K)} \\ \varphi_k^{(K)} \cdot \varphi_i^{(K)} & \varphi_k^{(K)} \cdot \varphi_j^{(K)} & \varphi_k^{(K)} \cdot \varphi_k^{(K)} \end{pmatrix} \; dx \begin{pmatrix} v_i \\ v_j \\ v_k \end{pmatrix}} \\
&= \sqrt{\sum_K \begin{pmatrix} v_i & v_j & v_k \end{pmatrix} \frac{|K|}{12} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} v_i \\ v_j \\ v_k \end{pmatrix}} \\
&= \sqrt{\sum_K \frac{|K|}{12} \; v^T \cdot M \cdot v}
\end{aligned}
$$

with

$$v = \begin{pmatrix} v_i \\ v_j \\ v_k \end{pmatrix} \qquad\qquad M = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

In our problem, we need to find $\|v_h\|_{L^2(\Omega)} = \|u_h - \Pi_h u_e\|_{L^2(\Omega)}$. Then we subtitute

$$v_h(P) = u_h(P) - u_e(P)$$

for every point P.

3.5.3 $H_0^1(\Omega)$ norm

$$\begin{aligned}
\|v_h\|_{H_1^0(\Omega)} &= \sqrt{\int_\Omega \nabla v_h(x) \cdot \nabla v_h(x)\ dx} \\
&= \sqrt{\int_\Omega |\nabla v_h(x)|^2\ dx} \\
&= \sqrt{\sum_K |\nabla v_h|_K(x)|^2 |K|}
\end{aligned}$$

with $\nabla v_h|_K(x)$ does not depend on x

$$\nabla v_h|_K(x) = v_i \begin{pmatrix} c_1^{(i)} \\ c_2^{(i)} \end{pmatrix} + v_j \begin{pmatrix} c_1^{(j)} \\ c_2^{(j)} \end{pmatrix} + v_k \begin{pmatrix} c_1^{(i)} \\ c_2^{(i)} \end{pmatrix}.$$

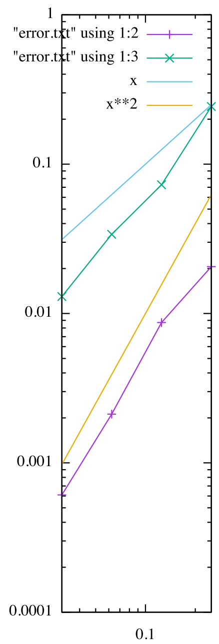


Figure 16

4 13 February 2018

4.1 Heat Equation

4.1.1 Problem

Find $u : \Omega \times (0, T) \to \mathbb{R}$ such that

$$\begin{cases} u_t - \nu \triangle u = f & \text{in } \Omega \times (0,T) \\ u = g & \text{on } \partial\Omega \times (0,T) \\ u = u^0 & \text{in } \Omega \text{ at } t = 0 \end{cases}$$

where f, g, u^0 is given.

4.1.2 Weak Form

We want to find $\{u(t) \in V(g(t)); t \in (0,T)\}$ with $V := H_0^1(\Omega)$, $g : \partial\Omega \to \mathbb{R}$ smooth function, and $V(g) := \{v + g \in H^1(\Omega); v \in V\}$ such that

$$\begin{cases} (u_t(t), v) + a(u(t), v) = (f(t), v) & , \forall v \in V \\ u(0) = u^0 & \in V(g(0)) \end{cases}$$

where $a(u,v) := \nu \int_\Omega \nabla u \cdot \nabla v \; dx$

4.1.3 Finite Element Method

For $n = 1, \ldots, N_T$, $\triangle t > 0$, $N_T := \lfloor \frac{T}{\triangle t} \rfloor \in \mathbb{N}$, we looking for $\{u_h^n \in V_h(g^n)\}$ such that

$$\begin{cases} \Big(\dfrac{u_h^n - u_h^{n-1}}{\triangle t}, v_h \Big) + a(u_h^n, v_h) = (f^n, v_h) & , \forall v_h \in V_h \\ u_h^0 = \Pi_h u^0 & \in V_h(g^0) \end{cases}$$

where

$$\begin{array}{rcl} V_h \subset V & : & P1 - FEspace \\ V_h(g) & := & \{v_h + g \in V(g); v_h \in V_h\} \\ \Pi_h & : & C(\bar{\Omega}) \to X_h \\ X_h \subset H^1(\Omega) & : & P1 - FEspace \\ (V_h \subset X_h) & \Rightarrow & V_h \subset H_0^1(\Omega), \; X_h \subset H^1(\Omega) \\ (\Pi_h u^0)(P) & = & u^0(P), \; P : \text{nodal point} \end{array}$$

Then the problem become

$$\begin{array}{rcl} \dfrac{1}{\triangle t}(u_h^n, v_h) + a(u_h^n, v_h) & = & \dfrac{1}{\triangle t}(u_h^{n-1}, v_h) + (f^n, v_h), \; \forall v_h \in V_h \\ \Big[\dfrac{1}{\triangle t} M_{Np \times Np} + A_{Np \times Np} \Big] \big[U_{Np \times 1} \big] & = & b_{Np \times 1}. \end{array}$$

4.1.4 Flowchart

The algorithm used to solve heat equation problem is shown as flowchart in Figure 17 and 18

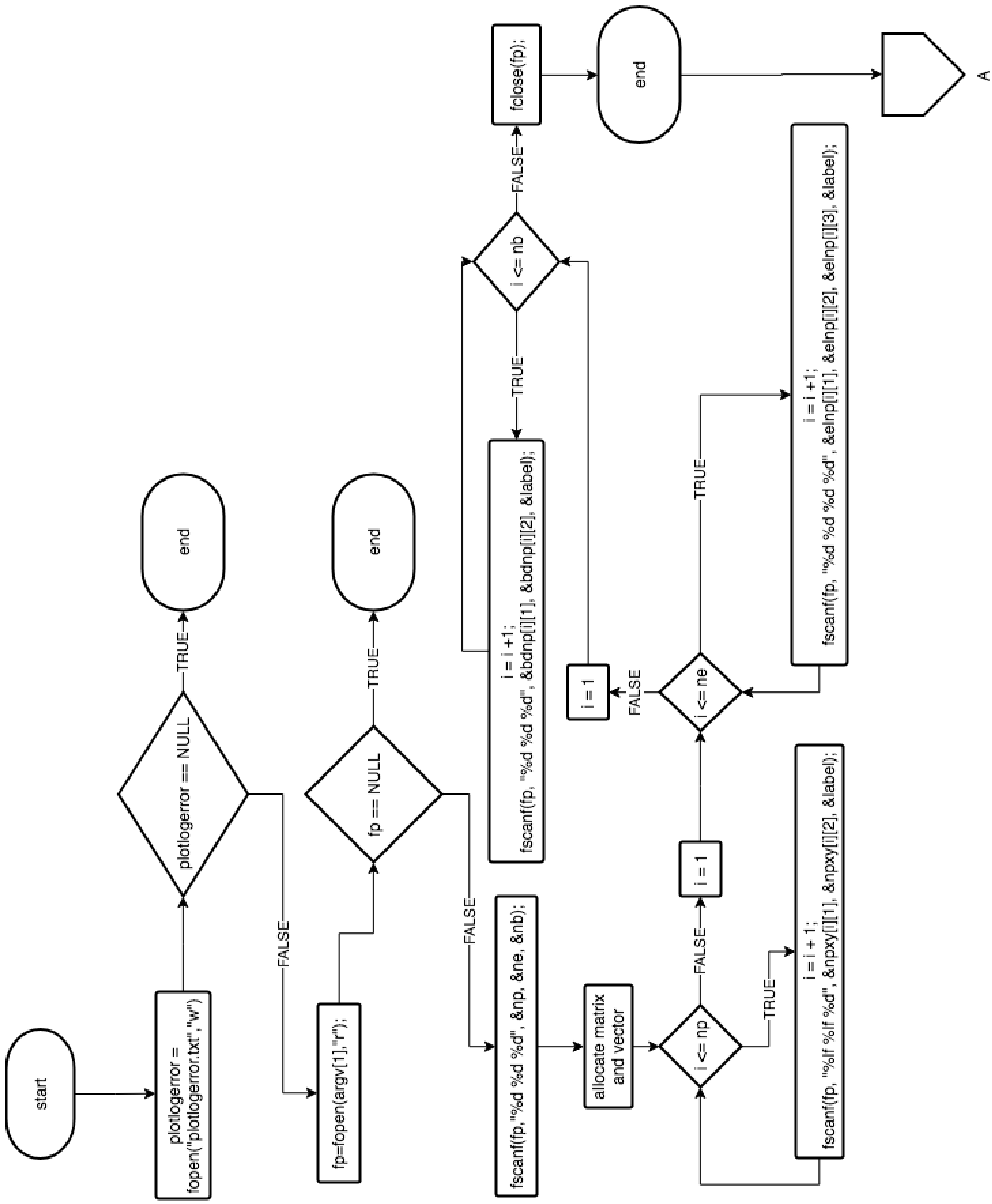


Figure 17

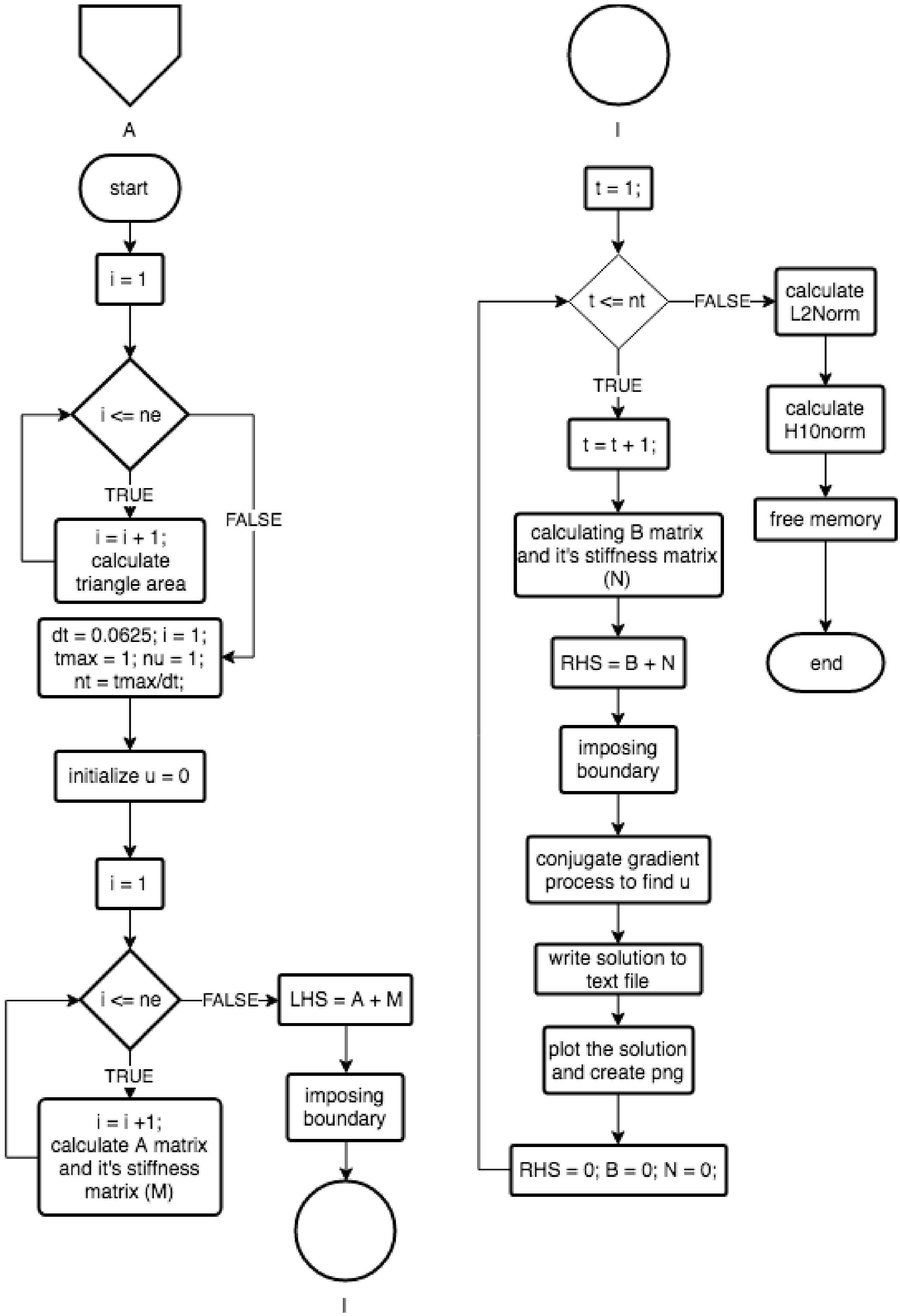

A
start
i = 1
i <= ne
TRUE
FALSE
i = i + 1; calculate triangle area
dt = 0.0625; i = 1; tmax = 1; nu = 1; nt = tmax/dt;
initialize u = 0
i = 1
i <= ne
FALSE
LHS = A + M
TRUE
i = i +1; calculate A matrix and it's stiffness matrix (M)
imposing boundary
I
I
t = 1;
t <= nt
FALSE
calculate L2Norm
TRUE
t = t + 1;
calculate H10norm
free memory
calculating B matrix and it's stiffness matrix (N)
end
RHS = B + N
imposing boundary
conjugate gradient process to find u
write solution to text file
plot the solution and create png
RHS = 0; B = 0; N = 0;

Figure 18

4.1.5 Numerical experiment

We consider problem

$$\begin{cases} u_t - \nu \triangle u = (1 + 2\nu\pi^2 t) sin(\pi x_1) sin(\pi x_2) & , \Omega \times (0,T) \\ u = 0 & , \partial\Omega \times (0,T) \\ u = 0 & , \Omega \text{ at } t = 0 \end{cases}$$

with exact solution

$$u(x_1, x_2, t) = t\ sin(\pi x_1) sin(\pi x_2).$$

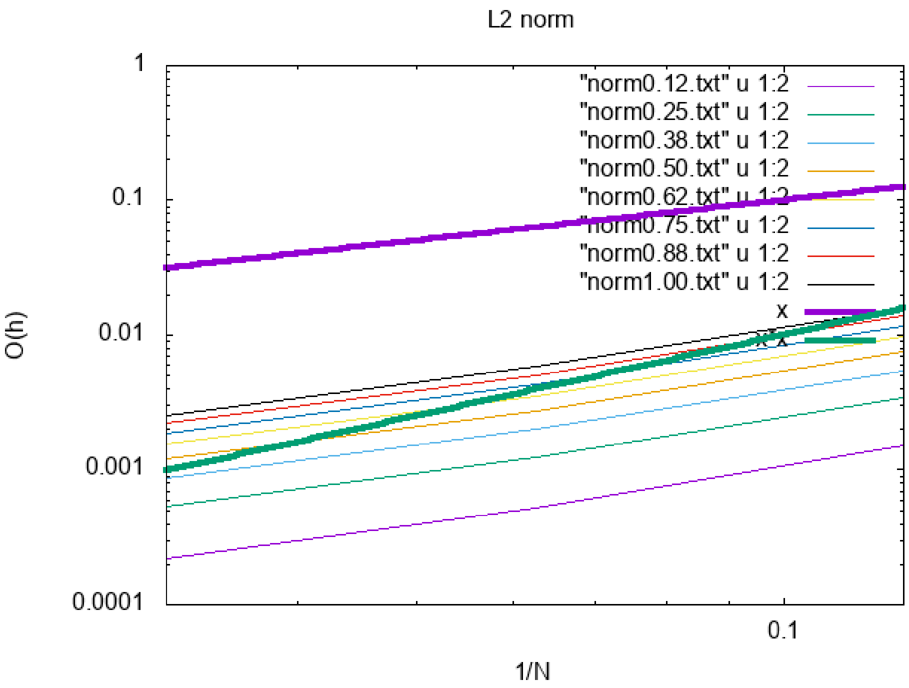


Figure 19

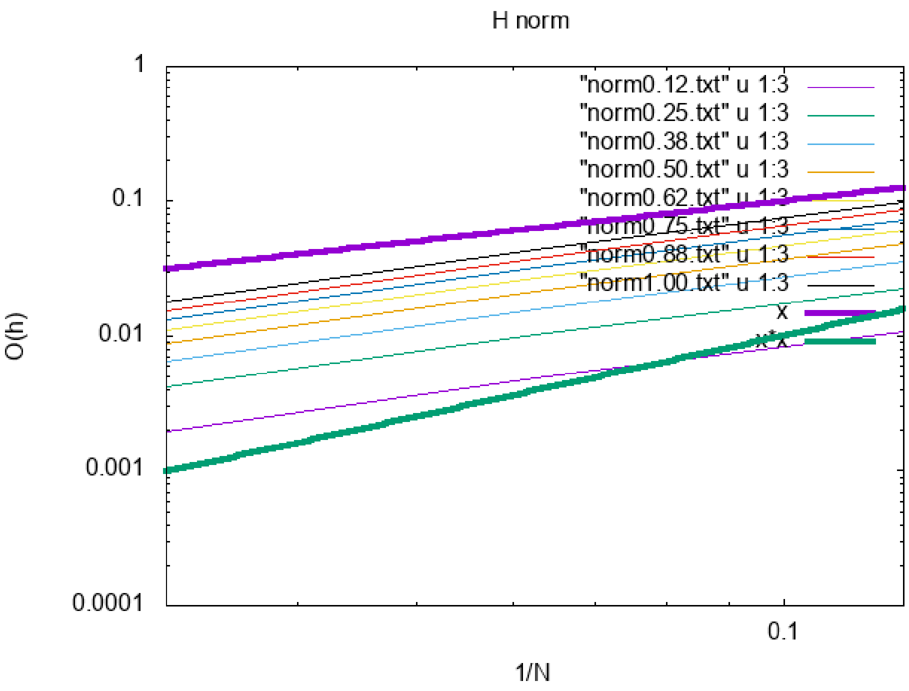
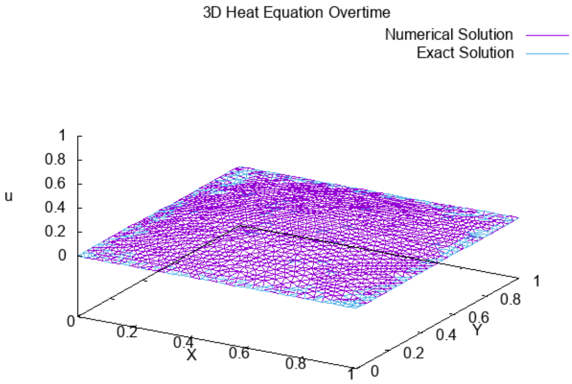
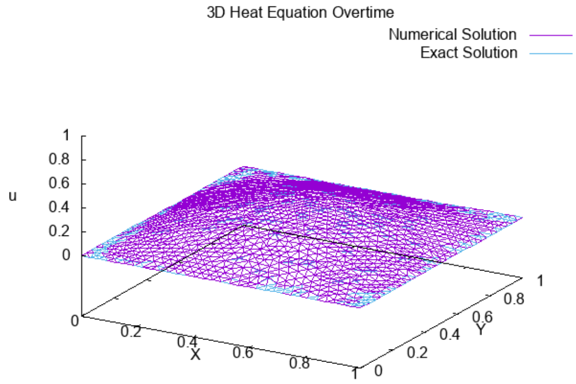


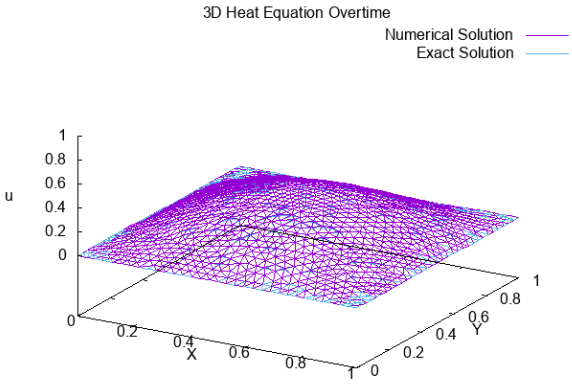
Figure 20



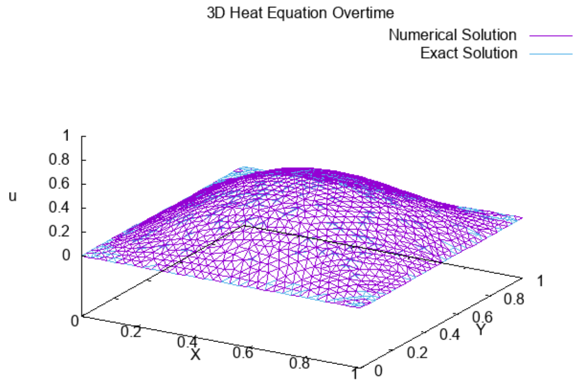
(a) 4 dt



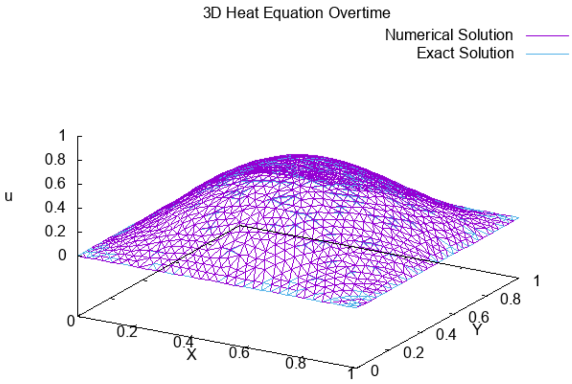
(b) 8 dt



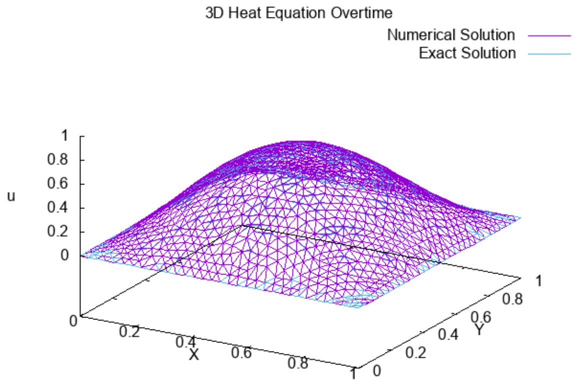
(c) 12 dt



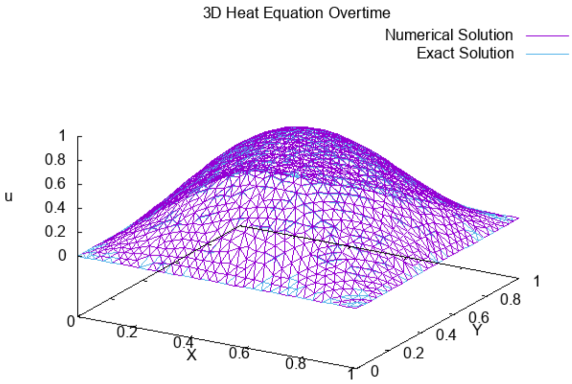
(d) 16 dt



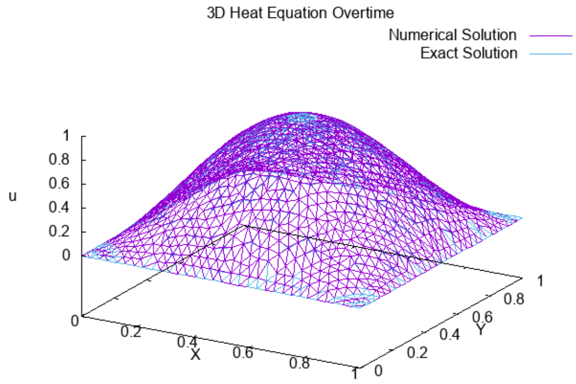
(e) 20 dt



(f) 24 dt



(g) 28 dt



(h) 32 dt