0.1 Problem

Consider Poisson Equation problem as shown below. We want to find u such that

$$\begin{cases} -\Delta u = f(x) & \text{in } \Omega \\ u = g(x) & \text{on } \Gamma = \partial\Omega. \end{cases} \tag{1}$$

0.2 Continuous (Partial Differential Equation)

We need to know some notation beforehand.

$$\begin{aligned}
X &:= H^1(\Omega) \\
V &:= H_0^1(\Omega) \subset X \\
H^1(\Omega) &\equiv \{v \in L^2(\Omega); \frac{\partial v}{\partial x} \in L^2(\Omega)\} \\
L^2(\Omega) &\equiv \{v : \Omega \to \mathbb{R}; \int_\Omega v^2(x)dx < \infty\} \\
V(g) &:= \{v \in X; v = g \text{ on } \Gamma \text{ or } v - g \in V\} \\
V &= V(0).
\end{aligned}$$

From the strong form in equation (1), we can obtain the weak form $\forall$ test function $v(x)$, where $v|_\Gamma = 0$, then,

$$\begin{aligned}
&\int_\Omega (-\Delta u)(x)v(x)dx \\
&= \int_\Omega \Big(-\frac{\partial^2 u}{\partial {x_1}^2}(x)v(x) - \frac{\partial^2 u}{\partial {x_2}^2}(x)v(x)\Big)\ dx \\
&= -\int_\Omega \frac{\partial^2 u}{\partial x_1^2}(x)v(x)\ dx - \int_\Omega \frac{\partial^2 u}{\partial x_2^2}(x)v(x)\ dx \\
&= -\Big(\int_{\partial\Omega} \frac{\partial u}{\partial x_1}(x)v(x)n_i\ ds - \int_\Omega \frac{\partial u}{\partial x_1}(x)\frac{\partial v}{\partial x_1}(x)\ dx\Big) - \Big(\int_{\partial\Omega} \frac{\partial u}{\partial x_2}(x)v(x)n_i\ ds - \int_\Omega \frac{\partial u}{\partial x_2}(x)\frac{\partial v}{\partial x_2}(x)\ dx\Big) \\
&= \Big(\int_\Gamma \frac{\partial u}{\partial x_1}(x)v(x)n_i + \frac{\partial u}{\partial x_2}(x)v(x)n_i\ ds\Big) + \int_\Omega \frac{\partial u}{\partial x_1}(x)\frac{\partial v}{\partial x_1}(x) + \frac{\partial u}{\partial x_2}(x)\frac{\partial v}{\partial x_2}(x)\ dx \\
&= \int_\Omega \nabla u(x) \cdot \nabla v(x)\ dx \\
&= \int_\Omega f(x)v(x)\ dx
\end{aligned}$$

such that

$$\int_\Omega \nabla u \cdot \nabla v\ dx = \int_\Omega fv\ dx.$$

For simplicity, we assume $\Omega = (0.1)^2$. Then we obtain weak form of equation (1),

$$\begin{cases} a(u,v) = l(v), \forall v \in V \\ u \in V(g). \end{cases} \tag{2}$$

where $a(u,v) := \int_\Omega \nabla u \cdot \nabla v \; dx$ is bilinear form and $l(v) := \int_\Omega f \;\; v \; dx$ is linear form.

To show that there is exist unique solution u, we can use the Remark below.

Remark 0.2.1 $\exists! u = \underset{v \in V(g)}{argmin} \; (\frac{1}{2}a(v,v) - l(v)) = \underset{w \in V(g)}{argmin} \; J(w)$

Using this Remark, the Proposition below is given with proof.

Proposition

For $\qquad J(v) := \frac{1}{2}a(v,v) - l(v), \qquad\qquad u = \underset{v \in V(g)}{argmin} \; J(v) \iff (2)$

Proof:

$(\Rightarrow)$ if $u = argmin \; J$ then $u + tv \in V(g), \;\; \forall t \in \mathbb{R}, \forall v \in V = H_0^1(\Omega)$. Since it is on boundary Γ, then $g = u = u + tv$.

$$\begin{array}{lll} J(u) & \leq J(w) & , \forall w \in V(g), \;\; w = u + tv \in V(g) \\ J(u) & \leq J(u+tv) & , \forall t \in \mathbb{R}, \;\; \forall v \in V \end{array}$$

Then

$$\begin{aligned} J(u+tv) &= \frac{1}{2}a(u+tv, u+tv) - l(u+tv) \\ &= \frac{1}{2}a(u,u) + ta(u,v) + \frac{t^2}{2}a(v,v) - l(u) - tl(v) \\ &= \frac{t^2}{2}a(v,v) + t(a(u,v) - l(v)) + J(u) \\ &=: \varphi(t) \end{aligned}$$

Because $\varphi(t)$ is in quadratic form, then its minimum obtained at $t = 0$. So that $\varphi = 0$ such that $a(u,v) - l(v) = 0$.

$(\Leftarrow)$ $\forall t \in \mathbb{R}, \forall v \in V$ we have

$$J(u,tv) = J(u) + \frac{t^2}{2}a(v,v) \geq J(u).$$

$\forall w \in V(g)$, we set $v := w - u \in V, \;\; t := 1, \;\; w = u + tv$

$$J(w) = J(u+tv) \geq J(u)$$

0.3 Discrete (Finite Element Method)

Notation

$$\begin{aligned} X_h &\subset X \text{ (usually } \dim X_h < \infty) \\ V_h &= X_h \cap V \\ g_h &\in X_h \text{(approximation of } g) \\ V_h(g_h) &= \{v_h \in X_h ; v_h - g_h \in V_h\}. \end{aligned}$$

Then the weak form is approximated with

$$\begin{cases} a(u_h, v_h) & = l(v_h),\ \forall v_h \in V_h \\ u_h & \in V_h(g_h) \end{cases} \iff u_h = \underset{w_h \in V_h(g_h)}{argmin}\ J(w_h)$$

Using Finite Element Method,

$$\begin{aligned} X_h &= \{v_h \in C^0(\overline{\Omega}); v_h|_K \text{ is linear}\} \\ V_h &= X_h \cap H_0^1(\Omega), \end{aligned}$$

or we could write

$$\begin{aligned} X_h &= \langle \varphi_1, \dots, \varphi_{N_p} \rangle \\ &= \{\sum_{i=1}^{N_p} c_i \varphi_i \ ;\ c_i \in \mathbb{R}\} \end{aligned}$$

where $\{\varphi_i\}_{i=1}^{N_p}$ become a basis of the vector space X_h. For nodal points $\{P_i\}_{i=1}^{N_p}$ and $\varphi_i \in X_n \ ;\ \varphi_i(P_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

For $x = (x_1, x_2) \in \mathbb{R}^2$ and $\forall v_h \in X_h$, we will have

$$\begin{aligned} v_h(\cdot) &= \sum_{i=1}^{N_p} v_h(P_i) \varphi_i(\cdot) \in X_h \\ w_h &:= \sum_{i=1}^{N_p} v_h(P_i) \varphi_i \in X_h \\ w_h(P_j) &= \sum_{i=1}^{N_p} v_h(P_i) \varphi_i(P_j) \\ &= \sum_{i=1}^{N_p} v_h(P_i) \delta_{ij} \\ &= v_h(P_j) \end{aligned}$$

A basis of V_h

$$\Omega \cap \Gamma = \Phi$$

$$\{\varphi_i; P_i \in \Omega\} \subset \{P_i\}_{i=1}^{N_p}$$

for simplicity, we assume

$$\{\varphi_i; P_i \in \Omega\} = \{P_i\}_{i=1}^{N}(N < N_p)$$

s.t.$\{P_i\}_{i=1}^{N} \subset \Omega$ and $\{P_i\}_{i=N+1}^{N_p} \subset \Gamma$

$$V_h = \langle \varphi_1, \cdots, \varphi_N \rangle$$

$$(**) a(u_h, v_h) = l(v_h)(\forall v_h \in V_h)$$

$$\Updownarrow$$

$$a(u_h, \varphi_i) = l(\varphi_i)(i = 1, \cdots, N)$$

$$\Downarrow \text{choose } v_h = \varphi_i \in V$$

$$\forall v_h \in V_h, c_i = v_h(P_i), v_h = \sum_{i=1}^{N} c_i \varphi_i$$

$$\begin{aligned} a(u_h, v_h) &= a(u_h, \sum_{i=1}^{N} c_i \varphi_i) \\ &= \sum_{i=1}^{N} c_i a(u_h, \varphi_i) = \sum_{i=1}^{N} c_i l(\varphi_i) \\ &= l(\sum_{i=1}^{N} c_i \varphi_i) = l(v_h) \end{aligned}$$

we set $u_j := u_h(P_j)$ $(j = 1, \cdots, N_p)$

$$\text{boundary} \to u_j = g_j = g_h(P_j)(j = N+1, \cdots, N_p)$$

$$u_h \in v_h(g_h)$$

$$P_j \in \Gamma$$

$$\text{unknown} : u_1, \cdots, u_N$$

$$(**) \Leftrightarrow \begin{cases} a(u_h, \varphi_i) = l(\varphi_i)(i = 1, \cdots, N) \\ u_h = \sum_{j=1}^{N} u_j \varphi_j + \sum_{j=N+1}^{N_p} g_j \varphi_j (u_h \in V_h(g_h)) \end{cases}$$

we set $a_{ij} := a(u_i, u_j) = a(u_j, u_i)$

s.t. $\sum_{j=1}^{N} a_{ij}u_j + \sum_{j=N+1}^{N_p} a_{ij}g_j = l(\varphi_i)(i = 1, \cdots, N)$

$$\textbf{we set } A := (a_{ij}) \in \mathbb{R}^{N\times N}_{\text{sym}}$$

$$\mathbf{u} := \begin{pmatrix} u_i \\ \vdots \\ u_N \end{pmatrix}$$

$$\mathbf{b} := (l(u_i) - \sum_{j=N+1}^{N_p} a_{ij}g_j)_{i=1,\cdots,N}(**) \Leftrightarrow \mathrm{A}\mathbf{u} = \mathbf{b}$$