Topics in Computational Science Report
Norbert Pozar Class

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1. Suppose that u is a twice continuously differentiable positive solution of the porous medium equation:

$$u_t - \triangle(u^m) = 0, \quad x \in \mathbb{R}^n, t > 0 \tag{1}$$

In the pressure form:

$$p_t - (m-1)p\triangle p - |\nabla p|^2 = 0 \tag{2}$$

Then, we define:

$$p(x,t) := \frac{m}{m-1} u^{m-1}(x,t) \tag{3}$$

Show that (3) is a solution of (1) in (2) form.

Answer:
First we compute:

$$p_t = \frac{d}{dt}p(x,t), \quad \triangle p = \tfrac{d^2}{dx^2}p(x,t) = \tfrac{d}{dx}\nabla p \tag{4}$$

$$\nabla p = \frac{d}{dx}p(x,t), \qquad \triangle(u^m) = \tfrac{d^2}{dx^2}u^m \tag{5}$$

Applied Chain Rule to (4),(5), we get:

$$\begin{aligned} p_t &= \frac{d}{dt}p(x,t) \\ &= \frac{dp}{du}\frac{du}{dt} \\ &= \frac{m}{(m-1)}(m-1)u^{(m-2)}\frac{d}{dt}u \\ &= mu^{(m-2)}u_t \end{aligned} \tag{6}$$

$$\begin{aligned} \triangle p &= \frac{d}{dx}\nabla p \\ &= \frac{d}{dx}\big(mu^{(m-2)}\nabla u\big) \\ &= \big(m(m-2)u^{(m-3)}\nabla u\big)\nabla u + mu^{(m-2)}\triangle u \\ &= m(m-2)u^{(m-3)}|\nabla u|^2 + mu^{(m-2)}\triangle u \end{aligned} \tag{8}$$

$$\begin{aligned} \nabla p &= \frac{d}{dx}p(x,t) \\ &= \frac{dp}{du}\frac{du}{dx} \\ &= \frac{m}{(m-1)}(m-1)u^{(m-2)}\frac{d}{dx}u \\ &= mu^{(m-2)}\nabla u \end{aligned} \tag{7}$$

$$\begin{aligned} \triangle(u^m) &= \frac{d^2}{dx^2}u^m \\ &= \frac{d}{dx}\left(\frac{d}{dx}u^m\right) \\ &= \frac{d}{dx}\left(mu^{(m-1)}\nabla u\right) \\ &= \big(m(m-1)u^{(m-2)}\nabla u\big)\nabla u + (mu^{(m-1)})\triangle u \\ &= m(m-1)u^{(m-2)}|\nabla u|^2 + mu^{(m-1)} \end{aligned} \tag{9}$$

Substitute (6), (7) $\to$ (2), we get:

$$mu^{(m-2)}u_t - mu^{(m-1)}\triangle p - m^2u^{2(m-2)}|\nabla u|^2 = 0 \tag{10}$$

Divide (10) with $mu^{(m-2)}$, we get:

$$u_t - u\triangle p - mu^{(m-2)}|\nabla u|^2 = 0 \tag{11}$$

Substitute (8) → (11), we get:

$$\begin{aligned}
u_t - u\left(m(m-2)u^{(m-3)}|\nabla u|^2 + mu^{(m-2)}\triangle u\right) - mu^{(m-2)}|\nabla u|^2 = 0 \\
u_t - \left(m(m-2)u^{(m-2)}|\nabla u|^2 + mu^{(m-1)}\triangle u\right) - mu^{(m-2)}|\nabla u|^2 = 0 \\
u_t - mu^{(m-2)}|\nabla u|^2\left((m-2)+1\right) - mu^{(m-1)}\triangle u = 0 \\
u_t - m(m-1)u^{(m-2)}|\nabla u|^2 - mu^{(m-1)}\triangle u = 0
\end{aligned} \tag{12}$$

Substitute (9) → (12),

$$\therefore u_t - \triangle(u^m) = 0 \tag{13}$$