Assignment 3
Analysis Ia Report

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July 3, 2018

1. Let (X, d) be a complete metric space and let $f : X \to X$ be a map. Suppose the iterated map
$$f^k = f \circ \cdots \circ f \quad \text{(k times)} \tag{1}$$
is a contraction for some $k \geq 2$. Prove that f has a unique fixed point $x \in X$.

Answer:
By the contraction mapping theorem, f^k has a unique fixed point, let's call it x, so that
$$f^k(x) = x \tag{2}$$
by (2) and (1) we note that
$$f^k(f(x)) = f(f^k(x)) = f(x)$$
Therefore, $f(x)$ and x are both fixed points of f^k. Since f^k has a unique fixed point, $f(x) = x$. Now, we show that for any $x_0 \in X$ the points $f^k(x_0)$ converges to x as $k \to \infty$. Let's consider $f^k(x_0)$ as k runs through some iteration until N. i.e. pick $0 \leq i \leq N - 1$ look at the points $f^{kN+i}(x_0)$ as $k \to \infty$. Since
$$f^{kN+i}(x_0) = f^{kN}(f^i(x_0)) = (f^k)^N(f^i(x_0))$$
and f^k is a contraction, it must be tend to x by the contraction mapping theorem. So all k sequences $\{f^{kN+i}(x_0)\}_{k\geq 1}$ tend to x.
$\therefore f$ has a unique fixed point $x \in X$.

2. Complete the proof of Picard-Lindelof Theorem by showing that $A : M \to M$ is a contraction if h is small

Answer:
We define:

$$\text{for } x \in M, \quad A(x,t) = x_0 + \int_{t_0}^{t} f(s, x(t))ds \tag{3}$$

and also consider Lipschitz-continuous

$$d(f(x,t) - f(y,t)) \leq Ld(x-y)\Big|\forall(x,t),(y,t) \in S \quad \text{for some } L > 0 \tag{4}$$

Let's consider two function $x, y \in M$, we want to show

$$d(A(x), A(y)) \leq Kd(x,y) \quad \text{for some } K \in (0,1). \tag{5}$$

So let t be such that

$$d(A(x), A(y)) = d(A(x,t), A(y,t))$$

then using definition of A

$$\begin{aligned} d(A(x,t), A(y,t)) &= d\bigg(\int_{t_0}^{t}(f(s,x(t)) - f(s,y(t)))ds\bigg) \\ &= \int_{t_0}^{t} d(f(s,x(t)) - f(s,y(t)))ds \\ &\leq L\int_{t_0}^{t} d(x-y)ds, \quad f \text{ is Lipschitz-continuous} \\ &\leq Lhd(x-y) \end{aligned} \tag{6}$$

based on (4) we know that $L > 0$, therefore (6) is a contraction if $h < \frac{1}{L}$(h is small enough).