Poisson Problem in 2D with Modified Cassini Egg Domain

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1 Poisson Problem

1.1 Strong Form

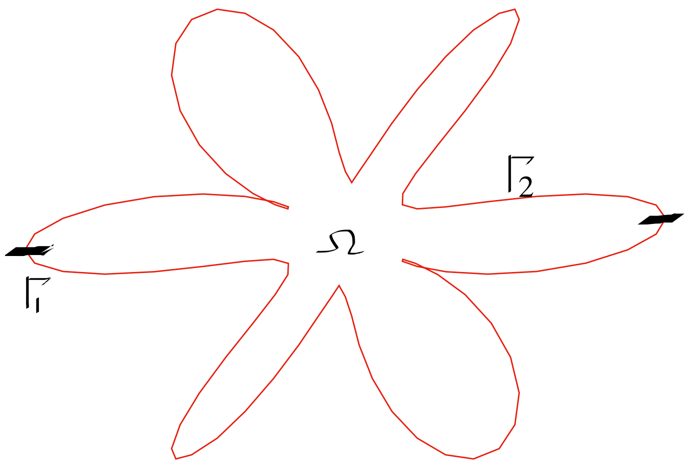


Figure 1: Modified Cassini Egg Domain

Let's consider $f : \Omega \to \mathbb{R}$ and $g : \Gamma \to \mathbb{R}$ be given.
Then we define Poisson Problem, Find $u : \Omega \to \mathbb{R}$ s.t.

$$\begin{cases} -\triangle u = f \text{ in } \Omega \\ u = g \text{ on } \Gamma_1 \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_2 \end{cases} \tag{1}$$

1.2 Weak Form

We define Sobolev spaces,

$$L^2(\Omega) := \{u : \Omega \to \mathbb{R}; \int_\Omega |u(x)|^2 dx < \infty\}$$
$$H^1(\Omega) := \{u \in L^2(\Omega); \nabla u \in L^2(\Omega)^2\}.$$

For a given function $g : \Gamma_1 \to \mathbb{R}$ we define function spaces,

$$V(g) := \{v \in H^1(\Omega)|\ v|_{\Gamma_1} = g\}, \quad V := V(0) = \{v \in H^1(\Omega)|\ v|_{\Gamma_1} = 0\}$$

$\forall v \in V$, we multiplied it to both side of (1) and integrating over Ω, we get

$$-\int_\Omega \triangle u v dx = \int_\Omega f v dx \tag{2}$$

by divergence theorem, the left side of (2) becomes:

$$
\begin{aligned}
\int_\Omega \triangle uv dx &= \int_\Omega \nabla \cdot (\nabla uv) dx - \int_\Omega \nabla u \cdot \nabla v dx \\
&= \int_{\partial\Omega} \nu(v \nabla u) dx - \int_\Omega \nabla u \cdot \nabla v dx \\
&= \int_\Gamma v \frac{\partial u}{\partial \nu} ds - \int_\Omega \nabla u \cdot \nabla v dx \\
&= \int_{\Gamma_1} v \frac{\partial u}{\partial \nu} ds + \int_{\Gamma_2} v \frac{\partial u}{\partial \nu} ds - \int_\Omega \nabla u \cdot \nabla v dx
\end{aligned}
\tag{3}
$$

from boundary condition, we know that:

$$
v|_{\Gamma_1} = 0, \forall v \in V \quad \tfrac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_2
$$

hence, (3) becomes:

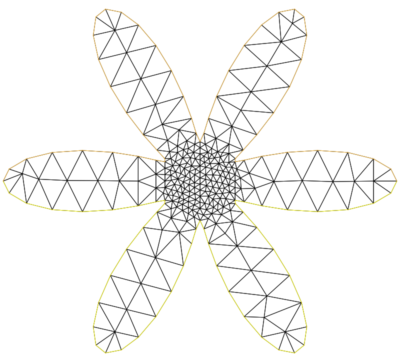
$$
\int_\Omega \triangle uv dx = -\int_\Omega \nabla u \cdot \nabla v dx
$$

Then, we obtain the weak formulation of (1); find $u \in V(g)$ s.t.

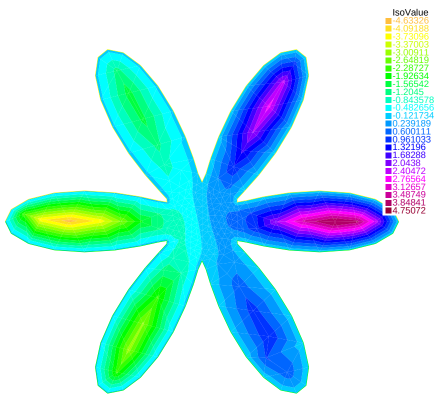
$$
\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx \tag{4}
$$

2 FreeFEM++ Modelling and Result

After we obtain weak form of (1) as shown in (4), we then create the domain and then solve the Poisson problem using FreeFEM++ Software. The result is:



(a) Mesh created by FreeFEM++ with division number=50



(b) Graphics of solution u from 2D Poisson Problem, color palette on the right side show the value of u on each point.

Figure 2: The Mesh and Result of 2D Poisson Problem