

Applied Analysis Notes

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1 Finite Element Method

1.1 Strong Form

Find $u : \Omega \rightarrow \mathbb{R}$ s.t.

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \quad (1)$$

where $f : \Omega \rightarrow \mathbb{R}, g : \partial\Omega \rightarrow \mathbb{R}$ are given functions

1.2 Weak Form

Find $u \in V(g)$ s.t.

$$a(u, v) = (f, v), \forall v \in V,$$

where

$$\begin{aligned} (u, v) &\equiv \int_{\Omega} u(x)v(x)dx, \\ a(u, v) &\equiv (\nabla u, \nabla v) = \int_{\Omega} \nabla u \cdot \nabla v dx, \\ V(g) &\equiv v \in H^1(\Omega); v = g \text{ on } \partial\Omega, \\ V &\equiv V(0) \end{aligned}$$

1.3 FEM

$$\text{Find } u_h \in V_h(g_h) \text{ s.t.}$$

$$a(u_h, v_h) = (f, v_h), \forall v_h \in V_h$$

where

$$V_h \subset V, \dim V_h < +\infty$$

$$g_h, \text{ approximation of } g$$

$$u_h, \text{ approximation solution of } u$$

$$h : \text{mesh size}$$

1.4 1-Dimension Case

Let us define basis functions $\varphi_i, \varphi_i : (0, 1) \rightarrow \mathbb{R}$

$\varphi_i : \text{piecewiselinear}$

$$\varphi_i(x_j) = \delta_{ij} = \begin{cases} 1(i = j) \\ 0(i \neq j) \end{cases}$$

$$X_h = \langle \varphi_0, \dots, \varphi_4 \rangle = \sum_{i=0}^4 c_i \varphi_i(x); c_i \in \mathbb{R}, i = 0, \dots, 4$$

$$V_h(g) = v_h \in X_h; v_h(x_0) = g_0, v_h(x_1) = g_1$$

$$= g_0 \varphi_0(x) + \sum_{i=1}^3 c_i \varphi_i(x) + g_1 \varphi_4(x); c_i \in \mathbb{R}, i = 1, 2, 3$$

$$\dim X_h = 5, \dim V_h(g) = 3$$

$$V_h = V_h(0) = \sum_{i=1}^3 c_i \varphi_i(x) c_i \in \mathbb{R}, i = 1, 2, 3$$

Rewrite FEM

$$u_h(x) = g_0 \varphi_0(x) + \sum_{i=1}^3 c_i \varphi_i(x) + g_1 \varphi_4(x) \begin{cases} \text{Find } c_{i=1}^3 \subset \mathbb{R} \text{ s.t.} \\ a(u_h, v_h), \forall v_h \in V_h \end{cases} \quad \begin{cases} \text{Find } c_{i=1}^3 \\ a(u_h, \varphi_i) = (f, \varphi_i), i = 1, 2, 3, \end{cases} \quad \text{problem}$$