Seminar Notes Alifian

Alifian Mahardhika Maulana

April 18, 2018

1 3D Linear Elasticity

$$\begin{aligned}\Omega \subset \mathbb{R}^d (d = 2, 3)\\ u = \Omega \to \mathbb{R}^2 (\text{small displacement})\\ x \mapsto u(x)\end{aligned}$$

1.1 Strain Tensor

$$\begin{aligned}e[u] &= (e_{ij}[u]) \in \mathbb{R}^{dxd}_{sym}\\ e[u] &:= \frac{1}{2}(\bigtriangledown^T u + (\bigtriangledown^T u)^T)\end{aligned}$$

1.2 Stress Tensor

$$\sigma[u] = (\sigma ij[u]) \in \mathbb{R}^{dxd}_{sym}$$

Based on Hook's Law, stress tensor must have equality with strain so that

$$\begin{aligned}\sigma &= \mathbb{C}e\\ \text{with } \mathbb{C} &= \mathbb{C}_{ijkl} (\text{is a 4th order elasticity tensor})\\ \sigma ij &= \mathbb{C}_{ijkl} e_{kl}\\ \mathbb{C}_{ijkl} &= \mathbb{C}_{ijlk} = \mathbb{C}_{klij} (\text{symmetry})\\ \mathbb{C}_{ijkl}\xi_{ij}\xi_{kl} &\geq \mathbb{C}_* |\xi|^2\end{aligned}$$

1.3 Boundary Value Problem

$$\begin{cases} -\partial_i \sigma_{ij}[u] & = f_j(x), x \in \Omega \\ u & = g(x), x \in \Gamma_D \\ \sigma[u]_\nu & = q(x), x \in \Gamma_N \end{cases} \tag{1}$$

1.4 Equilibrium Equations of Force in Ω and on Γ_N

1.4.1 Strain Energy Density

$$\omega[u](x) := \frac{1}{2}\sigma[u] : e[u] \tag{2}$$

Solving using Sobolev Space in Isotropic Case, equation 2 becomes

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

with λ, μ called Lame Constant

$$\delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$

$$\begin{aligned} \sigma[u] &= (\sigma_{ij}[u]) \\ \sigma_{ij}[u] &= c_{ijkl}e_{kl}[u] \\ &= \lambda(\delta_k u_k)\delta_{ij} + \mu(\delta_i u_j + \delta_j u_i) \\ &= \lambda(\text{div } u)I + 2\mu e[u] \end{aligned}$$

$$\omega[u] = \frac{1}{2}(\lambda(\text{div } u)I + 2\mu e[u]) : e[u]$$

$$\omega[u] = \frac{1}{2}(\lambda(\text{div } u)^2 + \mu|e[u]|^2$$

Remark 1. *Positivity of* $\mathbb{C}$

$$\begin{aligned} (\mathbb{C}\xi) : \xi &\geq C_*|\xi|^2 (\forall \xi \in \mathbb{R}^{dxd}_{sym}) \\ (\mathbb{C}\xi) : \xi &= \lambda|tr\ \xi|^2 + 2\mu|\xi|^2 \end{aligned}$$

If $\lambda \geq 0, \mu > 0$*, then* $C_* = 2\mu$

$$\xi = (\xi_{ij}), |\xi|^2 = \xi_{ij}\xi_{ij} = \sum_{i=1...d}^{d}\sum_{j=1...d}^{d}|\xi_{ij}|^2$$

1.5 Elasticity Problem

$$\begin{cases} -\text{div } \sigma[u] & = f(x) \text{ in } \Omega \subset \mathbb{R}^d \\ u & = g(x) \text{ on } \Gamma_D \\ \sigma[u]v & = q(x) \text{ on } \Gamma_N \end{cases}$$

1.6 Crack Problem

$$\begin{cases} -\text{div } \sigma[u] & = f(x) \text{ in } \Omega \setminus \Sigma \subset \mathbb{R}^d \\ u & = g(x) \text{ on } \Gamma_D \\ \sigma[u]v & = q(x) \text{ on } \Gamma_N \\ \sigma[u]v & = 0 \text{ on } \Sigma^+ \cup \Sigma^- \end{cases}$$