Seminar Notes Alifian

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1 3D Linear Elasticity

$$\Omega \subset \mathbb{R}^d (d = 2, 3)$$
$$u = \Omega \to \mathbb{R}^2 (\text{small displacement})$$
$$x \mapsto u(x)$$

1.1 Strain Tensor

$$e[u] = (e_{ij}[u]) \in \mathbb{R}^{dxd}_{sym}$$
$$e[u] := \frac{1}{2}(\nabla^T u + (\nabla^T u)^T) \tag{1}$$

1.2 Stress Tensor

$$\sigma[u] = (\sigma ij[u]) \in \mathbb{R}^{dxd}_{sym} \tag{2}$$

Based on Hook's Law, stress tensor must have equality with strain so that

$$\sigma = \mathbf{C}e$$
$$\text{with } \mathbf{C} = \mathbf{C}_{ijkl}(\text{is a 4th order elasticity tensor})$$
$$\sigma ij = \mathbf{C}ijkle_{kl}$$
$$\mathbf{C}_{ijkl} = \mathbf{C}_{ijlk} = \mathbf{C}_{klij}(\text{symmetry})$$
$$\mathbf{C}_{ijkl}\xi_{ij}\xi_{kl} \geq \mathbf{C}_*|\xi|^2$$

1.3 Boundary Value Problem

$$\begin{cases} -\partial_i \sigma_{ij}[u] & = f_j(x), x \in \Omega \\ u & = g(x), x \in \Gamma_D \\ \sigma[u]_\nu & = q(x), x \in \Gamma_N \end{cases} \tag{3}$$

1.4 Equilibrium Equations of Force in Ω and on Γ_N

1.4.1 Strain Energy Density

$$\omega[u](x) := \frac{1}{2}\sigma[u] : e[u] \tag{4}$$

Solving using Sobolev Space in Isotropic Case, equation 4 becomes

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

with λ, μ called Lame Constant

$$\delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$

$$\sigma[u] = (\sigma_{ij}[u])$$
$$\sigma_{ij}[u] = c_{ijkl}e_{kl}[u]$$
$$= \lambda(\delta_k u_k)\delta_{ij} + \mu(\delta_i u_j + \delta_j u_i)$$
$$= \lambda(\text{div } u)I + 2\mu e[u]$$
$$\omega[u] = \frac{1}{2}(\lambda(\text{div } u)I + 2\mu e[u]) : e[u]$$
$$\omega[u] = \frac{1}{2}(\lambda(\text{div } u)^2 + \mu|e[u]|^2$$

Remark 1. *Positivity of C*

$$(C\xi) : \xi \geq C_*|\xi|^2 (\forall \xi \in \mathbb{R}_{sym}^{dxd})$$
$$(C\xi) : \xi = \lambda|tr\ \xi|^2 + 2\mu|\xi|^2$$

If $\lambda \geq 0, \mu > 0$, then $C_* = 2\mu$

$$\xi = (\xi_{ij}), |\xi|^2 = \xi_{ij}\xi_{ij} = \sum_{i=1...d}^{d}\sum_{j=1...d}^{d}|\xi_{ij}|^2$$

1.5 Elasticity Problem

$$\begin{cases} -\text{div } \sigma[u] & = f(x) \text{ in } \Omega \subset \mathbb{R}^d \\ u & = g(x) \text{ on } \Gamma_D \\ \sigma[u]v & = q(x) \text{ on } \Gamma_N \end{cases} \tag{5}$$

1.6 Crack Problem

$$\begin{cases} -\text{div } \sigma[u] & = f(x) \text{ in } \Omega \setminus \Sigma \subset \mathbb{R}^d \\ u & = g(x) \text{ on } \Gamma_D \\ \sigma[u]v & = q(x) \text{ on } \Gamma_N \\ \sigma[u]v & = 0 \text{ on } \Sigma^+ \cup \Sigma^- \end{cases} \tag{6}$$

1.7 Lebesque Measurable Theory

$$L^p(\Omega) := \left\{ v : \Omega \to \mathbb{R} \middle| \begin{cases} v = \text{Lebesque measurable} \\ \int_\Omega |v(x)|^p dx < \infty \end{cases} \right\} \tag{7}$$

Remark 2. *for* $u, v \in \mathbb{L}^p(\Omega)$, *if* $\exists N \subset \Omega$ *such that* $\begin{cases} u(x) = v(x)(x \in \Omega \setminus N) \\ \mathcal{L}^d(N) = 0, \end{cases}$ *then we*
identify u *and* v, $\mathcal{L}^d(N) = 0 \Leftrightarrow$ *volume of* $N = 0$ *for simplicity, we also can say that*
$u(x) = v(x)$ *for a.e.* $x \in \Omega$

for example

$$v : \mathbb{R} \to \mathbb{R}$$

$$v(x) = \begin{cases} 1, x \in \mathbb{Q} \\ 0, x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \tag{8}$$

$$\int_\mathbb{R} v dx = 0, \mathcal{L}^1(\mathbb{Q}) = 0$$

$v(x) = 0$ on $\mathbb{R} \setminus \mathbb{Q}$, or we can say $v = 0$ a.e. in $\mathbb{R}$

1.8 Sobolev Space

$$\mathbb{W}^{1,p}(\Omega) := \left\{ v \in \mathbb{L}^p(\Omega) \frac{\partial v}{\partial x_j}_{(j=1...d)} \in \mathbb{L}^p(\Omega) \right\} \tag{9}$$

such $\frac{\partial v}{\partial x_j}$ we called it distribution sence.
example of Sobolev Space is as follow:

$$v \in \mathbb{L}^p(\Omega) \text{ if } \exists \omega_j \in \mathbb{L}(\Omega)$$

such that

$$\int_\Omega v \frac{\partial \varphi}{\partial x_j} dx = - \int_\Omega \omega_j \varphi dx (\forall \varphi \in \mathbb{C}_0^\infty(\Omega))$$

$$\Rightarrow \frac{\partial \varphi}{\partial x_j} = \omega_j \text{ in distribution sence}$$

for

$$v \in \mathbb{C}^1(\Omega), \frac{\partial v}{\partial x_j}(x) = \omega_j(x)$$

$$\Updownarrow$$

$$\int_\Omega \omega_j \varphi dx = - \int_\Omega v \frac{\partial \varphi}{\partial x_j} dx (\forall \varphi \in \mathbb{C}_0^\infty(\Omega))$$

In particular,

$$\mathbb{H}^1(\Omega) := \mathbb{W}^{1,2}(\Omega), \nabla u = \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_d} \end{pmatrix}$$

inner product

$$(u, v)_{\mathbb{H}^1(\Omega)} := \int_\Omega uv \ dx + \int_\Omega \nabla u \cdot \nabla v \ dx$$

norm

$$||u||_{\mathbb{H}^1(\Omega)} := \sqrt{(u, v)_{\mathbb{H}^1(\Omega)}} = \sqrt{\int_\Omega |u|^2 dx + \int_\Omega |\nabla u|^2 dx}$$

$\mathbb{H}^1(\Omega)$ is complete ($\mathbb{H}^1(\Omega)$ is a Hilbert Space)

$$(u, v)_{\mathbb{L}^2(\Omega)} = \int_\Omega uv dx$$

1.9 Incomplete Hilbert Space

$\mathbb{V}$: a vector space in $\mathbb{R}$

$$\begin{cases} u, v \in \mathbb{V} \Rightarrow \alpha u + \beta v \in \mathbb{V} \\ \alpha, \beta \in \mathbb{R} \end{cases}$$

If $(\cdot, \cdot) : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ satisfies

$$\begin{cases} (u \cdot v) \geq 0 \text{ and } (u, u) = 0 \Leftrightarrow u = 0_v \in \mathbb{V} \\ (u, v) = (v, u) \\ (\alpha u + \beta v, \omega) = \alpha(u, \omega) + \beta(v, \omega) \end{cases}$$

then we call $[\mathbb{V} \times \mathbb{V}]$ pre Hilbert space or incomplete Hilbert Space.

1.10 Property of $\mathbb{L}^2(\Omega)$

For $v \in \mathbb{C}^1(\Omega)$,

$$\frac{\partial v}{\partial x_j}(x) = w_j(x)$$

$$\Updownarrow$$

$$\int_\Omega w_j \varphi dx = - \int \Omega v \frac{\partial \varphi}{\partial x_j} dx \ \left(\forall \varphi \in \mathbb{C}_0^\infty(\Omega) \right)$$

$(u, v)_{\mathbb{L}^2(\Omega)} = \int_\Omega uv dx$

$$\Rightarrow \left| \int_\Omega uv dx \right| \leq \int_\Omega |u||v| dx \leq ||u||_{\mathbb{L}^2(\Omega)} ||v||_{\mathbb{L}^2(\Omega)}$$

$u, v \in \mathbb{H}^1(\Omega)$

$$\Rightarrow \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j} \in \mathbb{L}^2(\Omega)$$

$$\left| \int_\Omega \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j} dx \right| \leq \int_\Omega \left| \frac{\partial u}{\partial x_j} \right| \left| \frac{\partial v}{\partial x_j} \right| dx \leq \left\| \frac{\partial u}{\partial x_j} \right\|_{\mathbb{L}^2(\Omega)} \left\| \frac{\partial v}{\partial x_j} \right\|_{\mathbb{L}^2(\Omega)}$$

$$\nabla u \cdot \nabla v = \sum_{j=1}^d \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j}$$

$$\left| \int_\Omega \nabla u \cdot \nabla v dx \right| = \left| \int \sum_{j=1}^d \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j} dx \right| \leq \sum_{j=1}^d \int \left| \frac{\partial u}{\partial x_j} \right| \left| \frac{\partial v}{\partial x_j} \right| dx$$

$$\leq \sum_{j=1}^d \left\| \frac{\partial u}{\partial x_j} \right\|_{\mathbb{L}^2(\Omega)} \left\| \frac{\partial v}{\partial x_j} \right\|_{\mathbb{L}^2(\Omega)}$$

$$\leq \sqrt{\sum_{j=1}^d \left\| \frac{\partial u}{\partial x_j} \right\|_{\mathbb{L}^2(\Omega)}} \sqrt{\sum_{j=1}^d \left\| \frac{\partial v}{\partial x_j} \right\|_{\mathbb{L}^2(\Omega)}}$$

$$= \sqrt{\sum_{j=1}^d \int_\Omega \left| \frac{\partial u}{\partial x_j} \right|^2 dx} \sqrt{\sum_{j=1}^d \int_\Omega \left| \frac{\partial v}{\partial x_j} \right|^2 dx}$$

$$= \sqrt{\int_\Omega \left(\sum_{j=1}^d \left| \frac{\partial u}{\partial x_j} \right|^2 \right) dx} \sqrt{\int_\Omega \left(\sum_{j=1}^d \left| \frac{\partial v}{\partial x_j} \right|^2 \right) dx}$$

$$= \sqrt{\int_\Omega |\nabla u|^2 dx} \sqrt{\int_\Omega |\nabla v|^2 dx}$$

$$\therefore \left| \int_\Omega \nabla u \cdot \nabla v dx \right| \leq \sqrt{\int_\Omega |\nabla u|^2 dx} \sqrt{\int_\Omega |\nabla v|^2 dx} \tag{10}$$

1.11   Energy (Revisited)

$$E(u) := \frac{1}{2} \int_\Omega \sigma[u] : e[u] dx - \int_\Omega f \cdot u dx - \int_{\Gamma_N} q \cdot u ds \tag{11}$$

with u is a vector of the elasticity problem define by:

$$u \in \mathbb{H}^1(\Omega : \mathbb{R}^d) := \{u : \Omega \to \mathbb{R}^d | u = (u_i, \cdots, u_d), u_i \in \mathbb{H}^1(\Omega)\}$$
$$\Rightarrow E(u) < \infty$$

u : become solution $\Leftrightarrow u = argmin_{v \in \mathbb{H}^1(\Omega : \mathbb{R}^d)} E(v)$ such a technique we call it variational principle.

1.12   Variational Principle

Let's consider a Poisson Equation Problem:

$$\Omega \subset \mathbb{R}^d \begin{cases} -\triangle u & = f(x) \in \Omega \\ u & = g(x) \text{ on } \Gamma_D \quad f \in L^2(\Omega), g \in H^1(\Omega), q \in L^2(\Gamma_N) \\ \frac{\partial u}{\partial v} & = q(x) \text{ on } \Gamma_N \end{cases} \tag{12}$$

Remark 3.
$$v \in H^1(\Omega) \Rightarrow \exists v|_\Gamma \in L^2(\Gamma)$$
we choose v on L^2 because it will has value on the boundary

1.12.1 Definition of Weak Solution

$$u \in H^1(\Omega) \text{ s.t. } \begin{cases} \int_\Omega \triangle u \cdot \triangle v dx = \int_\Omega fv dx + \int_{\Gamma_N} qv ds \\ \left(\forall v \in V := v \in H^1(\Omega) \big| v|_{\Gamma_D} \right) \\ v|_{\Gamma_D} = g|_{\Gamma_D} (v - g \in V) \end{cases}$$

$(v - g \in V)$ mean $(v \in V + g := v + g | v \in V)$ with V is an affine space.

1.12.2 Definition of Strong Solution

$u \in H^2(\Omega)$ and u satisfies (12)

Remark 4.
$$H^2(\Omega) := \{ u \in L^2(\Omega) \frac{\partial u}{\partial x_j}, \frac{\partial^2 u}{\partial x_i x_j} \in L^2(\Omega) \}$$

$$u \in H^2(\Omega) \Rightarrow \frac{\partial u}{\partial x_j} \in H^1(\Omega)$$

$$\frac{\partial u}{\partial v} = \sum v_i \frac{\partial v}{\partial x_i}$$

Properties 1.
$$u : \text{ strong solution } \Leftrightarrow \begin{cases} u : \text{ weak solution} \\ u \in H^2(\Omega) \end{cases}$$

Proof. Let's consider one dimension for simplicity, then:
$$(\Rightarrow) - \triangle u = f(x)$$
$$-\frac{\partial^2 u}{\partial x^2} = f(x)$$

Suppose we take $v, \forall v \in V := v \in H^1(\Omega) | v|_{\Gamma_D} = 0$, if v is smooth enough, then we take integral for both side, thus the equation becomes:

$$\int_\Omega -\frac{\partial^2 u}{\partial x^2} v dx = \int_\Omega f(x) v dx$$
$$\int_\Omega -\frac{\partial}{\partial x} \frac{\partial u}{\partial x} v dx = \int_\Omega f(x) v dx$$

using integration by parts, the left handside equation becomes:

$$\int_\Omega -\frac{\partial}{\partial x} \frac{\partial u}{\partial x} v dx = \left[-v \cdot \frac{du}{dx} \right]_{\partial \Omega} + \int_\Omega \frac{du}{dx} \frac{dv}{dx} dx$$

consider dirichlet boundary condition, thus the equation becomes:

$$-\int_{\Gamma_N} qv ds + \int_\Omega \frac{du}{dx} \frac{dv}{dx} dx = \int_\Omega fv dx$$

in general

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega fv dx + \int_{\Gamma_N} qv ds$$

$\square$

$$\text{Energy } E(v) := \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega fv dx - \int_{\Gamma_N} qv ds$$

Theorem 1.
$$u: \text{ weak solution } \Leftrightarrow u = \text{argmin}_{v \in V+g} E(v)$$

Theorem 2.
$$\exists! u = \text{argmin}_{v \in V+g} E(v)$$

Proof. of Theorem 1
$$(\Leftarrow) \text{ If } u = \text{argmin }_{w \in V+g} E(w)$$
$$\text{since } u + fv \in V + g \; (\forall v \in V, \forall t \in \mathbb{R})$$
$$\frac{d}{dt} E(u+tv)|_{t=0} = 0 \text{ (First Variation)}$$

$$0 = \frac{d}{dt} E(u+tv)|_{t=0}$$
$$= \frac{d}{dt}\left[\frac{1}{2}\int_\Omega |\nabla u|^2 + 2t\nabla u \cdot \nabla v + t^2 |\nabla v|^2 dx - \int_\Omega fu dx - \int_{\Gamma_N} qu ds - t(\int_\Omega fv dx + \int_{\Gamma_N} qv ds)\right]_{t=0}$$
$$= \int_\Omega \nabla u \cdot \nabla v dx - \int_\Omega fv dx - \int_{\Gamma_N} qv ds$$
$$\therefore u = \text{ weak solution}$$

$\square$

Proof. of Theorem 1 $\Leftrightarrow$ Theorem 2
($\Rightarrow$) If u is a weak solution
for any $w \in V + g, (v := w - u \in V)$

$$E(w) - E(u) = E(u+v) - E(u)$$
$$= \int_\Omega (\nabla u \cdot \nabla v + \frac{1}{2}|\nabla v|^2)dx - \int_\Omega fv dx - \int_{\Gamma_N} qv ds$$
$$= \frac{1}{2}\int_\Omega |\nabla v|^2 dx \geq 0$$
$$\therefore E(w) \geq E(u)(\forall w \in V + g)$$
$$\therefore u = \text{argmin}_{w \in V+g} E(w)$$

$\square$

2 Abstract Theory

X: a real Hilbert Space (ex: $H^1(\Omega)$)
V: a closed subspace of X (ex: $V \subset H^1(\Omega)$) in case of Poisson Equation (Linear)

2.1 Definition

1. a : X $\times$ X $\to \mathbb{R}$ is a bilinear form, if
$$\begin{cases} u \mapsto a(u,v) \text{ is linear for all } v \in X \\ v \mapsto a(u,v) \text{ is linear for all } u \in X \end{cases}$$

2. a bilinear form $a(u,v)$ is <u>bounded</u>, if $\exists a_0 > 0$ s.t. $|a(u,v)| \leq a_0 ||u||_x ||v||_x (\forall u, v \in X)$

3. a bilinear form $a(\cdot, \cdot)$ is <u>symmetric</u>, if $a(u, v) = a(v, u)(\forall u, v \in X)$

4. a bilinear form $a(\cdot, \cdot)$ is <u>coercive</u>, if $\exists \alpha > 0$ s.t. $a(u, u) \geq \alpha \|u\|_x^2 (\forall u \in X)$

Remark 5. *A bilinear form $a(\cdot, \cdot)$ is bounded iff $a : X \times X \to \mathbb{R}$ is continuous*

2.2 Definition

1. $l : x \to \mathbb{R}$ is a linear form, if $l : x \to \mathbb{R}$ ($\in u \mapsto l(u)$) is linear

2. A linear form l is bounded, if $|l(u)| \leq \exists c \|u\|_x$ $(\forall u \in x)$

Remark 6. *A linear form l is bounded iff $l : x \to \mathbb{R}$ is continuous.*

2.3 Theorem (Lax - Milgram)

We suppose $a(\cdot, \cdot)$ is a bounded bilinear form on $X \times X$, and l is a bounded linear form on X.
If $a(\cdot, \cdot)$ is coercive on $V \times V$, then for any $g \in X$,
$\exists! u \in V + g$ s.t. $a(u, v) = l(v)(\forall v \in V)$

2.3.1 Example

$$X = H^1(\Omega), V = v \in H^1(\Omega)|v|_{\Gamma_D} = 0$$

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v dx$$

$$l(v) := \int_\Omega f v dx + \int_{\Gamma_N} q v ds$$

Coercivity

$$\exists \alpha_0 > 0 : \int_\Omega |\nabla v|^2 dx \geq \alpha_0 \left(\int_\Omega |\nabla v|^2 dx + \int_\Omega |v|^2 dx \right)$$

$$\Updownarrow \text{ If we choose } \alpha_0 \text{ so small}$$

$$\exists \alpha_1 > 0 : \int_\Omega |\nabla v|^2 dx \geq \alpha_1 \int_\Omega |v|^2 dx (\forall v \in V)$$

Boundedness

$$|a(u, v)| \leq \sqrt{\int_\Omega |\nabla u|^2 dx} \sqrt{\int_\Omega |\nabla v|^2 dx} \leq \|u\|_x \|v\|_x$$

$$|l(v)| \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|q\|_{L^2(\Gamma_N)} \|v\|_{L^2(\Gamma_N)} \leq \left(\|f\|_{L^2(\Omega)} + C\|q\|_{L^2(\Gamma_N)} \right) \|v\|_x$$

Another Example:
Energy $E(u) := \frac{1}{2} a(u, u) - l(u)$

Theorem 3. *(Variational Principle) for $g \in X$,*

$$\begin{cases} u \in V + g \\ a(u, v) = \underline{l(v)}(\forall v \in V) \Leftrightarrow argmin_{w \in V+g} E(w) \end{cases}$$

Proof.

$$(\Leftarrow) \forall t \in \mathbb{R}, \forall v \in V, u + tv \in V + g$$

$$\frac{d}{dt} E(u + tv)|_{t=0} = 0$$

$$E(u + tv) = \frac{1}{2} a(u + tv, u + tv) - l(u + tv)$$

$$= \frac{1}{2} \left(a(u, u) + 2t a(u, v) + t^2 a(v, v) - l(u) - t l(u) \right)$$

$$a(u, v) = l(v) (\forall v \in V)$$

$$\Rightarrow \text{ For } w \in V + g \ (v := w - u \in V)$$

$$E(w) - E(v) = \frac{1}{2} a(v, v) \geq \frac{1}{2} \alpha \|v\|_x^2 \geq 0$$

$$\square$$

Remark 7. *Uniqueness in Lax-Milgram If u and w are both the minimizer of E among $V + g$, then*

$$0 = E(w) - E(u) = \frac{1}{2} a(v, v) \geq \frac{\alpha}{2} \|v\|_x^2 \geqq 0$$

$\therefore v = 0, \ \therefore w = u$

3 Linear Elasticity

We define:

$$\Omega \subset \mathbb{R}^d \ (d = 2, 3)$$

$$u : \Omega \to \mathbb{R}^d \ (\text{displacement})$$

$$e[u] := \frac{1}{2} (\nabla^T u + \nabla u^T) \ (\text{strain})$$

$$\sigma[u] := \mathcal{C} e[u]$$

$$\mathcal{C} = (C_{ijkl}) \begin{cases} C_{ijkl} = C_{klij} = C_{jikl} \\ (C_\xi) : \xi \geq C_* |\xi|^2 (\forall \xi \in \mathbb{R}_{sym}^{d \times d}) \end{cases}$$

Let's consider linear elasticity problem:

$$(**) \begin{cases} -div \ \sigma[u] = f(x), \text{ in } \Omega \\ u = g(x) \text{ on } \Gamma_D \\ \sigma[u]\nu = q(x) \text{ on } \Gamma_N \end{cases} \tag{13}$$

$$f \in L^2(\Omega : \mathbb{R}^d), \ g \in H^1(\Omega : \mathbb{R}^d), \ q \in L^2(\Gamma_N : \mathbb{R}^d))$$

3.1 Strong Solution

$u \in H^2(\Omega : \mathbb{R}^d)$ satisfies $(**)$ then we call u : a strong solution

3.2 Weak Solution

$$\begin{cases} \int_\Omega \sigma[u] : e[v] dx = \int_\Omega f \cdot v dx + \int_{\Gamma_N} q \cdot v ds (\forall v \in V := \{v \in H^1 \ (\Omega : \mathbb{R}^d) \ |v|_{\Gamma_D} = 0\}) \\ u \in V + g \end{cases}$$

3.3 Properties

$$u: \text{ strong solution } \Leftrightarrow \begin{cases} u: \text{ weak solution} \\ u \in H^2 \ (\Omega : \mathbb{R}^d) \end{cases}$$

$$X := H^1(\Omega : \mathbb{R}^d)$$

$$a(u,v) := \int_\Omega \sigma[u] : e[v] dx$$

$$l(v) := \int_\Omega f \cdot v dx + \int_{\Gamma_N} q \cdot v ds$$

$$a(u,v) = \int_\Omega (\mathcal{C}e[u]) : e[v] dx$$
$$= \int_\Omega e[v] : (\mathcal{C}e[u]) dx$$
$$= a(v,u)$$

For $v \in V$

$$a(v,v) = \int_\Omega (\mathcal{C}e[v]) : e[v] dx$$
$$\geqq C_* \int_\Omega |e[v]|^2 dx$$
$$\geqq C_* ||v||_x^2$$

Properties 2. • $a(\cdot,\cdot)$ *is bounded symmetric, bilinear form on* $X \times X$.

• $a(\cdot,\cdot)$ *is coercive on* $V \times V$.

• l *is bounded linear form on* X.

Theorem 4. *For any* $g \in H^1(\Omega : \mathbb{R}^d)$,

$$\exists! u : \text{ a weak solution of } (**), \text{ and } \begin{cases} u = \operatorname{argmin}_{w \in V+g} E(w) \end{cases}$$

3.4 Abstract Theory (cont.)

X : a Hilbert Space, $V \subset X$: a closed subspace, $a : X \times X \to \mathbb{R}$ a bilinear form.

$$(*) \begin{cases} \text{Continuous} \\ \text{Symmetric} \\ \text{Coercive } a(u,u) \geq \text{ on } V \times V, \alpha ||u||_x^2 (\forall u \in V) \end{cases}$$

3.5 Lax-Milgram

If a satisfies $(*)$, $\forall l : x \to \mathbb{R}$ continuous, linear form,

$$\forall g \in X, \ \exists! u \in V + g$$

s.t. $a(u,v) = l(v)(\forall v \in V)$
Moreover,

$$u = \operatorname*{argmin}_{w \in V+g}$$

where $E(w) := \frac{1}{2}a(w,w) - l(w)$

3.5.1   Example 1

$$\begin{cases} |\Gamma_D| > 0 \\ X = H^1(\Omega) \\ V = v \in H^1(\Omega)| \ v|_{\Gamma_D} = 0 \\ a(u,v) := \int_\Omega \nabla u \cdot \nabla v dx \\ l(v) := \int_\Omega fv dx + \int_{\Gamma_N} qv ds \end{cases}$$

Using poincare inequality,

$$\exists c > 0 \text{ s.t. } \int_\Omega |\nabla v|^2 dx \geqq c \int_\Omega |v|^2 dx (\forall v \in V)$$

$$\begin{aligned} \Rightarrow v \in V \Rightarrow a(v,v) &= \frac{1}{2} \int_\Omega |\nabla v|^2 dx \\ &\geqq \frac{1}{2} \int_\Omega |\nabla v|^2 dx + \frac{1}{2} c \int_\Omega |v|^2 dx \\ &\geqq \frac{1}{2} \min(1,c) \left(\int_\Omega |\nabla v|^2 dx + \int_\Omega |v|^2 dx \right) \\ &= \alpha ||v||^2_{H^1(\Omega)} \end{aligned}$$

$\therefore a$ is coercive on $V \times V$

L-M $\Rightarrow \exists! u :$ a weak solution for:

$$\begin{cases} -\triangle u = f \text{ in } \Omega \\ u = g \text{ on } \Gamma_D \\ \frac{\partial u}{\partial \nu} = q \text{ on } \Gamma_N \end{cases}$$

$$u = \underset{w \in V+g}{\operatorname{argmin}} \left(\frac{1}{2} \int_\Omega |\nabla w|^2 dx - \int_\Omega fw dx - \int_{\Gamma_N} qw ds \right)$$

3.5.2   Example 2

$$\begin{cases} X = H^1(\Omega : \mathbb{R}^d), d = 2,3 \\ V = v \in H^1(\Omega : \mathbb{R}^d) : v|_{\Gamma_D} = 0 \\ a(u,v) := \int_\Omega \sigma[u] : e[v] dx \\ l(v) := \int_\Omega f \cdot v dx + \int_{\Gamma_N} q \cdot v ds \begin{cases} f \in L^2(\Omega : \mathbb{R}^d) \\ q \in L^2(\Gamma_N : \mathbb{R}^d), q \in X \end{cases} \end{cases}$$