Assignment 3
Topics of Mathematical Science

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1. Prove by Cauchy's Product formula that:

$$e^{z+w} = e^z e^w, \quad \forall z, w \in \mathbb{C} \tag{1}$$

Answer:

Theorem 1. *Cauchy Product Rule*
Let's consider these power series: $\sum\limits_{i=0}^{\infty} a_i x_i$ *and* $\sum\limits_{j=0}^{\infty} b_j x_j$ *With* a_i *and* b_j *be a complex coefficient. The Cauchy product of these power series are as follows:*

$$\left(\sum_{i=0}^{\infty} a_i x_i\right) \cdot \left(\sum_{j=0}^{\infty} b_j x_j\right) = \sum_{k=0}^{\infty} \sum_{l=0}^{k} (a_l b_{k-l}) x^k \tag{2}$$

then, we define e^z and e^w in form of Formal Power Series:

$$e^z = \sum_{i=0}^{\infty} \frac{z^i}{i!} \quad e^w = \sum_{j=0}^{\infty} \frac{w^j}{j!} \tag{3}$$

hence, the cauchy product of (3) are:

$$\begin{aligned} e^z e^w = \left(\sum_{i=0}^{\infty} \frac{z^i}{i!}\right) \cdot \left(\sum_{j=0}^{\infty} \frac{w^j}{j!}\right) &= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{z^i}{i!} \frac{y^{n-i}}{(n-i)!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i=0}^{n} \binom{n}{i} z^i w^{n-1} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = e^{z+w} \end{aligned} \tag{4}$$

2. Prove:
   Suppose FPS:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{5}$$

   converges absolutely on any compact set on $|z - z_0| < R$. Then (5) is complex differentiable on $|z - z_0| < R$ and $f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$

   Answer: (5) is complex differentiable at z if the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \tag{6}$$

   exist. This limit is denoted by $f'(z)$ or $\frac{df}{dz}$ Then, we calculate $f'(z)$ of (5) as follows:

$$\begin{aligned}
f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h \to 0} \frac{\sum_{n=0}^{\infty} a_n (z + h - z_0)^n - \sum_{n=0}^{\infty} a_n (z - z_0)^n}{h} \\
&= \lim_{h \to 0} \sum_{n=0}^{\infty} a_n \frac{(z + h - z_0)^n - (z - z_0)^n}{h} \\
&= \lim_{h \to 0} \sum_{n=0}^{\infty} a_n \frac{(z - z_0)^n + \sum_{k=1}^{n} \binom{n}{k} (z - z_0)^{n-1} h^k}{h} \\
&= \sum_{n=0}^{\infty} a_n \lim_{h \to 0} \frac{hn(z - z_0)^{n-1} + h^2 n((z - z_0)^{n-1} + \cdots)}{h} \\
&= \sum_{n=0}^{\infty} a_n n (z - z_0)^{n-1} + \lim_{h \to 0} hn((z - z_0)^{n-1}) + \cdots \\
&= \sum_{n=0}^{\infty} a_n n (z - z_0)^{n-1}
\end{aligned} \tag{7}$$