

Seminar Notes Alifian

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1 3D Linear Elasticity

$$\begin{aligned}\Omega &\subset \mathbb{R}^d (d = 2, 3) \\ u &= \Omega \rightarrow \mathbb{R}^2 (\text{small displacement}) \\ x &\mapsto u(x)\end{aligned}$$

1.1 Strain Tensor

$$\begin{aligned}e[u] &= (e_{ij}[u]) \in \mathbb{R}_{sym}^{d \times d} \\ e[u] &:= \frac{1}{2}(\nabla^T u + (\nabla^T u)^T)\end{aligned}\tag{1}$$

1.2 Stress Tensor

$$\sigma[u] = (\sigma_{ij}[u]) \in \mathbb{R}_{sym}^{d \times d}\tag{2}$$

Based on Hook's Law, stress tensor must have equality with strain so that

$$\begin{aligned}\sigma &= \mathbf{C}e \\ \text{with } \mathbf{C} &= \mathbf{C}_{ijkl} (\text{is a 4th order elasticity tensor}) \\ \sigma_{ij} &= \mathbf{C}_{ijkl} e_{kl} \\ \mathbf{C}_{ijkl} &= \mathbf{C}_{ijlk} = \mathbf{C}_{klij} (\text{symmetry}) \\ \mathbf{C}_{ijkl} \xi_{ij} \xi_{kl} &\geq C_* |\xi|^2\end{aligned}$$

1.3 Boundary Value Problem

$$\begin{cases} -\partial_i \sigma_{ij}[u] &= f_j(x), x \in \Omega \\ u &= g(x), x \in \Gamma_D \\ \sigma[u]_\nu &= q(x), x \in \Gamma_N \end{cases}\tag{3}$$

1.4 Equilibrium Equations of Force in Ω and on Γ_N

1.4.1 Strain Energy Density

$$\omega[u](x) := \frac{1}{2} \sigma[u] : e[u] \quad (4)$$

Solving using Sobolev Space in Isotropic Case, equation 4 becomes

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

with λ, μ called Lamé Constant

$$\delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$

$$\begin{aligned} \sigma[u] &= (\sigma_{ij}[u]) \\ \sigma_{ij}[u] &= c_{ijkl} e_{kl}[u] \\ &= \lambda (\delta_k u_k) \delta_{ij} + \mu (\delta_i u_j + \delta_j u_i) \\ &= \lambda (\operatorname{div} u) I + 2\mu e[u] \end{aligned}$$

$$\begin{aligned} \omega[u] &= \frac{1}{2} (\lambda (\operatorname{div} u) I + 2\mu e[u]) : e[u] \\ \omega[u] &= \frac{1}{2} (\lambda (\operatorname{div} u)^2 + \mu |e[u]|^2) \end{aligned}$$

Remark 1. *Positivity of C*

$$\begin{aligned} (C\xi) : \xi &\geq C_* |\xi|^2 (\forall \xi \in \mathbb{R}_{sym}^{d \times d}) \\ (C\xi) : \xi &= \lambda |\operatorname{tr} \xi|^2 + 2\mu |\xi|^2 \end{aligned}$$

If $\lambda \geq 0, \mu > 0$, then $C_* = 2\mu$

$$\xi = (\xi_{ij}), |\xi|^2 = \xi_{ij} \xi_{ij} = \sum_{i=1 \dots d} \sum_{j=1 \dots d} |\xi_{ij}|^2$$

1.5 Elasticity Problem

$$\begin{cases} -\operatorname{div} \sigma[u] &= f(x) \text{ in } \Omega \subset \mathbb{R}^d \\ u &= g(x) \text{ on } \Gamma_D \\ \sigma[u]v &= q(x) \text{ on } \Gamma_N \end{cases} \quad (5)$$

1.6 Crack Problem

$$\begin{cases} -\operatorname{div} \sigma[u] &= f(x) \text{ in } \Omega \setminus \Sigma \subset \mathbb{R}^d \\ u &= g(x) \text{ on } \Gamma_D \\ \sigma[u]v &= q(x) \text{ on } \Gamma_N \\ \sigma[u]v &= 0 \text{ on } \Sigma^+ \cup \Sigma^- \end{cases} \quad (6)$$

1.7 Lebesgue Measurable Theory

$$L^p(\Omega) := \left\{ v : \Omega \rightarrow \mathbb{R} \mid \begin{cases} v = \text{Lebesgue measurable} \\ \int_{\Omega} |v(x)|^p dx < \infty \end{cases} \right\} \quad (7)$$

Remark 2. for $u, v \in \mathbb{L}^p(\Omega)$, if $\exists N \subset \Omega$ such that $\begin{cases} u(x) = v(x) (x \in \Omega \setminus N) \\ \mathcal{L}^d(N) = 0, \end{cases}$ then we identify u and v , $\mathcal{L}^d(N) = 0 \Leftrightarrow$ volume of $N = 0$ for simplicity, we also can say that $u(x) = v(x)$ for a.e. $x \in \Omega$

for example

$$\begin{aligned} v : \mathbb{R} &\rightarrow \mathbb{R} \\ v(x) &= \begin{cases} 1, x \in \mathbb{Q} \\ 0, x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \\ \int_{\mathbb{R}} v dx &= 0, \mathcal{L}^1(\mathbb{Q}) = 0 \\ v(x) &= 0 \text{ on } \mathbb{R} \setminus \mathbb{Q}, \text{ or we can say } v = 0 \text{ a.e. in } \mathbb{R} \end{aligned} \quad (8)$$

1.8 Sobolev Space

$$\mathbb{W}^{1,p}(\Omega) := \left\{ v \in \mathbb{L}^p(\Omega) \mid \frac{\partial v}{\partial x_j} \Big|_{(j=1\dots d)} \in \mathbb{L}^p(\Omega) \right\} \quad (9)$$

such $\frac{\partial v}{\partial x_j}$ we called it distribution sence.

example of Sobolev Space is as follow:

$$v \in \mathbb{L}^p(\Omega) \text{ if } \exists \omega_j \in \mathbb{L}(\Omega)$$

such that

$$\begin{aligned} \int_{\Omega} v \frac{\partial \varphi}{\partial x_j} dx &= - \int_{\Omega} \omega_j \varphi dx (\forall \varphi \in \mathbb{C}_0^\infty(\Omega)) \\ \Rightarrow \frac{\partial \varphi}{\partial x_j} &= \omega_j \text{ in distribution sence} \end{aligned}$$

for

$$\begin{aligned} v \in \mathbb{C}^1(\Omega), \frac{\partial v}{\partial x_j}(x) &= \omega_j(x) \\ \Updownarrow \\ \int_{\Omega} \omega_j \varphi dx &= - \int_{\Omega} v \frac{\partial \varphi}{\partial x_j} dx (\forall \varphi \in \mathbb{C}_0^\infty(\Omega)) \end{aligned}$$

In particular,

$$\mathbb{H}^1(\Omega) := \mathbb{W}^{1,2}(\Omega), \nabla u = \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_d} \end{pmatrix}$$

inner product

$$(u, v)_{\mathbb{H}^1(\Omega)} := \int_{\Omega} uv \, dx + \int_{\Omega} \nabla u \cdot \nabla v \, dx$$

norm

$$\|u\|_{\mathbb{H}^1(\Omega)} := \sqrt{(u, u)_{\mathbb{H}^1(\Omega)}} = \sqrt{\int_{\Omega} |u|^2 dx + \int_{\Omega} |\nabla u|^2 dx}$$

$\mathbb{H}^1(\Omega)$ is complete ($\mathbb{H}^1(\Omega)$ is a Hilbert Space)

$$(u, v)_{\mathbb{L}^2(\Omega)} = \int_{\Omega} uv dx$$

1.9 Incomplete Hilbert Space

\mathbb{V} : a vector space in \mathbb{R}

$$\begin{cases} u, v \in \mathbb{V} \Rightarrow \alpha u + \beta v \in \mathbb{V} \\ \alpha, \beta \in \mathbb{R} \end{cases}$$

If $(\cdot, \cdot) : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{R}$ satisfies

$$\begin{cases} (u \cdot v) \geq 0 \text{ and } (u, u) = 0 \Leftrightarrow u = 0_v \in \mathbb{V} \\ (u, v) = (v, u) \\ (\alpha u + \beta v, \omega) = \alpha(u, \omega) + \beta(v, \omega) \end{cases}$$

then we call $[\mathbb{V} \times \mathbb{V}]$ pre Hilbert space or incomplete Hilbert Space.

1.10 Property of $\mathbb{L}^2(\Omega)$

For $v \in \mathbb{C}^1(\Omega)$,

$$\frac{\partial v}{\partial x_j}(x) = w_j(x)$$

$$\Updownarrow$$

$$\int_{\Omega} w_j \varphi dx = - \int_{\Omega} \Omega v \frac{\partial \varphi}{\partial x_j} dx \quad (\forall \varphi \in \mathbb{C}_0^\infty(\Omega))$$

$$(u, v)_{\mathbb{L}^2(\Omega)} = \int_{\Omega} uv dx$$

$$\Rightarrow \left| \int_{\Omega} uv dx \right| \leq \int_{\Omega} |u| |v| dx \leq \|u\|_{\mathbb{L}^2(\Omega)} \|v\|_{\mathbb{L}^2(\Omega)}$$

$$u, v \in \mathbb{H}^1(\Omega)$$

$$\begin{aligned}
& \Rightarrow \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j} \in \mathbb{L}^2(\Omega) \\
& \left| \int_{\Omega} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j} dx \right| \leq \int_{\Omega} \left| \frac{\partial u}{\partial x_j} \right| \left| \frac{\partial v}{\partial x_j} \right| dx \leq \left\| \frac{\partial u}{\partial x_j} \right\|_{\mathbb{L}^2(\Omega)} \left\| \frac{\partial v}{\partial x_j} \right\|_{\mathbb{L}^2(\Omega)} \\
& \nabla u \cdot \nabla v = \sum_{j=1}^d \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j} \\
& \left| \int_{\Omega} \nabla u \cdot \nabla v dx \right| = \left| \int_{\Omega} \sum_{j=1}^d \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j} dx \right| \leq \sum_{j=1}^d \int_{\Omega} \left| \frac{\partial u}{\partial x_j} \right| \left| \frac{\partial v}{\partial x_j} \right| dx \\
& \leq \sum_{j=1}^d \left\| \frac{\partial u}{\partial x_j} \right\|_{\mathbb{L}^2(\Omega)} \left\| \frac{\partial v}{\partial x_j} \right\|_{\mathbb{L}^2(\Omega)} \\
& \leq \sqrt{\sum_{j=1}^d \left\| \frac{\partial u}{\partial x_j} \right\|_{\mathbb{L}^2(\Omega)}^2} \sqrt{\sum_{j=1}^d \left\| \frac{\partial v}{\partial x_j} \right\|_{\mathbb{L}^2(\Omega)}^2} \\
& = \sqrt{\sum_{j=1}^d \int_{\Omega} \left| \frac{\partial u}{\partial x_j} \right|^2 dx} \sqrt{\sum_{j=1}^d \int_{\Omega} \left| \frac{\partial v}{\partial x_j} \right|^2 dx} \\
& = \sqrt{\int_{\Omega} \left(\sum_{j=1}^d \left| \frac{\partial u}{\partial x_j} \right|^2 \right) dx} \sqrt{\int_{\Omega} \left(\sum_{j=1}^d \left| \frac{\partial v}{\partial x_j} \right|^2 \right) dx} \\
& = \sqrt{\int_{\Omega} |\nabla u|^2 dx} \sqrt{\int_{\Omega} |\nabla v|^2 dx} \\
& \therefore \left| \int_{\Omega} \nabla u \cdot \nabla v dx \right| \leq \sqrt{\int_{\Omega} |\nabla u|^2 dx} \sqrt{\int_{\Omega} |\nabla v|^2 dx} \tag{10}
\end{aligned}$$

1.11 Energy (Revisited)

$$E(u) := \frac{1}{2} \int_{\Omega} \sigma[u] : e[u] dx - \int_{\Omega} f \cdot u dx - \int_{\Gamma_N} q \cdot u ds \tag{11}$$

with u is a vector of the elasticity problem define by:

$$\begin{aligned}
u \in \mathbb{H}^1(\Omega; \mathbb{R}^d) &:= \{u : \Omega \rightarrow \mathbb{R}^d \mid u = (u_1, \dots, u_d), u_i \in \mathbb{H}^1(\Omega)\} \\
&\Rightarrow E(u) < \infty
\end{aligned}$$

u : become solution $\Leftrightarrow u = \operatorname{argmin}_{v \in \mathbb{H}^1(\Omega; \mathbb{R}^d)} E(v)$ such a technique we call it variational principle.

1.12 Variational Principle

Let's consider a Poisson Equation Problem:

$$\Omega \subset \mathbb{R}^d \begin{cases} -\Delta u &= f(x) \in \Omega \\ u &= g(x) \text{ on } \Gamma_D \quad f \in L^2(\Omega), g \in H^1(\Omega), q \in L^2(\Gamma_N) \\ \frac{\partial u}{\partial \nu} &= q(x) \text{ on } \Gamma_N \end{cases} \tag{12}$$

Remark 3.

$$v \in H^1(\Omega) \Rightarrow \exists v|_{\Gamma} \in L^2(\Gamma)$$

we choose v on L^2 because it will has value on the boundary

1.12.1 Definition of Weak Solution

$$u \in H^1(\Omega) \text{ s.t. } \begin{cases} \int_{\Omega} \Delta u \cdot \Delta v dx = \int_{\Omega} f v dx + \int_{\Gamma_N} q v ds \\ \left(\forall v \in V := v \in H^1(\Omega) | v|_{\Gamma_D} \right) \\ v|_{\Gamma_D} = g|_{\Gamma_D} (v - g \in V) \end{cases}$$

$(v - g \in V)$ mean $(v \in V + g := v + g | v \in V)$ with V is an affine space.

1.12.2 Definition of Strong Solution

$u \in H^2(\Omega)$ and u satisfies (12)

Remark 4.

$$\begin{aligned} H^2(\Omega) &:= \{u \in L^2(\Omega) \mid \frac{\partial u}{\partial x_j}, \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^2(\Omega)\} \\ u \in H^2(\Omega) &\Rightarrow \frac{\partial u}{\partial x_j} \in H^1(\Omega) \\ \frac{\partial u}{\partial v} &= \sum v_i \frac{\partial v}{\partial x_i} \end{aligned}$$

Properties 1.

$$u : \text{ strong solution } \Leftrightarrow \begin{cases} u : \text{ weak solution} \\ u \in H^2(\Omega) \end{cases}$$

$$\text{Energy } E(v) := \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx - \int_{\Omega} f v dx - \int_{\Gamma_N} q v ds$$

Theorem 1.

$$u : \text{ weak solution } \Leftrightarrow u = \operatorname{argmin}_{v \in V+g} E(v)$$

Theorem 2.

$$\exists! u = \operatorname{argmin}_{v \in V+g} E(v)$$

Proof. of Theorem 1

$$(\Leftarrow) \text{ If } u = \operatorname{argmin}_{w \in V+g} E(w)$$

since $u + f v \in V + g$ ($\forall v \in V, \forall t \in \mathbb{R}$)

$$\frac{d}{dt} E(u + t v)|_{t=0} = 0 \text{ (First Variation)}$$

$$\begin{aligned} 0 &= \frac{d}{dt} E(u + t v)|_{t=0} \\ &= \frac{d}{dt} \left[\frac{1}{2} \int_{\Omega} |\nabla u|^2 + 2t \nabla u \cdot \nabla v + t^2 |\nabla v|^2 dx - \int_{\Omega} f u dx - \int_{\Gamma_N} q u ds - t \left(\int_{\Omega} f v dx + \int_{\Gamma_N} q v ds \right) \right]_{t=0} \\ &= \int_{\Omega} \nabla u \cdot \nabla v dx - \int_{\Omega} f v dx - \int_{\Gamma_N} q v ds \\ &\therefore u = \text{ weak solution} \end{aligned}$$

□

Proof. of Theorem 1 \Leftrightarrow Theorem 2

(\Rightarrow) If u is a weak solution

for any $w \in V + g, (v := w - u \in V)$

$$\begin{aligned}
 E(w) - E(u) &= E(u + v) - E(u) \\
 &= \int_{\Omega} (\nabla u \cdot \nabla v + \frac{1}{2} |\nabla v|^2) dx - \int_{\Omega} f v dx - \int_{\Gamma_N} q v ds \\
 &= \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx \geq 0 \\
 \therefore E(w) &\geq E(u) (\forall w \in V + g) \\
 \therefore u &= \operatorname{argmin}_{w \in V + g} E(w)
 \end{aligned}$$

□

2 Abstract Theory

X : a real Hilbert Space (ex: $H^1(\Omega)$)

V : a closed subspace of X (ex: $V \subset H^1(\Omega)$) in case of Poisson Equation (Linear)

2.1 Definition

1. $a : X \times X \rightarrow \mathbb{R}$ is a bilinear form, if

$$\begin{cases} u \mapsto a(u, v) \text{ is linear for all } v \in X \\ v \mapsto a(u, v) \text{ is linear for all } u \in X \end{cases}$$
2. a bilinear form $a(u, v)$ is bounded, if $\exists a_0 > 0$ s.t. $|a(u, v)| \leq a_0 \|u\|_x \|v\|_x (\forall u, v \in X)$
3. a bilinear form $a(\cdot, \cdot)$ is symmetric, if $a(u, v) = a(v, u) (\forall u, v \in X)$
4. a bilinear form $a(\cdot, \cdot)$ is coercive, if $\exists \alpha > 0$ s.t. $a(u, u) \geq \alpha \|u\|_x^2 (\forall u \in X)$

Remark 5. A bilinear form $a(\cdot, \cdot)$ is bounded iff $a : X \times X \rightarrow \mathbb{R}$ is continuous

2.2 Definition

1. $l : x \rightarrow \mathbb{R}$ is a linear form, if $l : x \rightarrow \mathbb{R} (\in u \mapsto l(u))$ is linear
2. A linear form l is bounded, if $|l(u)| \leq \exists c \|u\|_x (\forall u \in x)$

Remark 6. A linear form l is bounded iff $l : x \rightarrow \mathbb{R}$ is continuous.

2.3 Theorem (Lax - Milgram)

We suppose $a(\cdot, \cdot)$ is a bounded bilinear form on $X \times X$, and l is a bounded linear form on X .

If $a(\cdot, \cdot)$ is coercive on $V \times V$, then for any $g \in X$,

$\exists! u \in V + g$ s.t. $a(u, v) = l(v) (\forall v \in V)$

2.3.1 Example

$$X = H^1(\Omega), V = v \in H^1(\Omega) | v|_{\Gamma_D} = 0$$

$$a(u, v) := \int_{\Omega} \nabla u \cdot \nabla v dx$$

$$l(v) := \int_{\Omega} f v dx + \int_{\Gamma_N} q v ds$$

Coercivity

$$\exists \alpha_0 > 0 : \int_{\Omega} |\nabla v|^2 dx \geq \alpha_0 \left(\int_{\Omega} |\nabla v|^2 dx + \int_{\Omega} |v|^2 dx \right)$$

\Updownarrow If we choose α_0 so small

$$\exists \alpha_1 > 0 : \int_{\Omega} |\nabla v|^2 dx \geq \alpha_1 \int_{\Omega} |v|^2 dx (\forall v \in V)$$

Boundedness

$$|a(u, v)| \leq \sqrt{\int_{\Omega} |\nabla u|^2 dx} \sqrt{\int_{\Omega} |\nabla v|^2 dx} \leq \|u\|_x \|v\|_x$$

$$|l(v)| \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|q\|_{L^2(\Gamma_N)} \|v\|_{L^2(\Gamma_N)} \leq \left(\|f\|_{L^2(\Omega)} + C \|q\|_{L^2(\Gamma_N)} \right) \|v\|_x$$

Another Example:

$$\text{Energy } E(u) := \frac{1}{2} a(u, u) - l(u)$$

Theorem 3. (*Variational Principle*) for $g \in X$,

$$\begin{cases} u \in V + g \\ a(u, v) = l(v) (\forall v \in V) \end{cases} \Leftrightarrow \operatorname{argmin}_{w \in V+g} E(w)$$

Proof.

$$(\Leftarrow) \forall t \in \mathbb{R}, \forall v \in V, u + tv \in V + g$$

$$\frac{d}{dt} E(u + tv) |_{t=0} = 0$$

$$\begin{aligned} E(u + tv) &= \frac{1}{2} a(u + tv, u + tv) - l(u + tv) \\ &= \frac{1}{2} \left(a(u, u) + 2ta(u, v) + t^2 a(v, v) - l(u) - tl(v) \right) \end{aligned}$$

$$a(u, v) = l(v) (\forall v \in V)$$

$$\Rightarrow \text{For } w \in V + g \text{ (} v := w - u \in V \text{)}$$

$$E(w) - E(u) = \frac{1}{2} a(v, v) \geq \frac{1}{2} \alpha \|v\|_x^2 \geq 0$$

□

Remark 7. *Uniqueness in Lax-Milgram* If u and w are both the minimizer of E among $V + g$, then

$$0 = E(w) - E(u) = \frac{1}{2} a(v, v) \geq \frac{\alpha}{2} \|v\|_x^2 \geq 0$$

$$\therefore v = 0, \therefore w = u$$

3 Linear Elasticity

We define:

$$\begin{aligned}\Omega &\subset \mathbb{R}^d \ (d = 2, 3) \\ u &: \Omega \rightarrow \mathbb{R}^d \ (\text{displacement}) \\ e[u] &:= \frac{1}{2}(\nabla^T u + \nabla u^T) \ (\text{strain}) \\ \sigma[u] &:= \mathcal{C}e[u] \\ \mathcal{C} &= (C_{ijkl}) \begin{cases} C_{ijkl} = C_{klij} = C_{jikl} \\ (C_\xi) : \xi \geq C_* |\xi|^2 (\forall \xi \in \mathbb{R}_{sym}^{d \times d}) \end{cases}\end{aligned}$$

Let's consider linear elasticity problem:

$$(**) \begin{cases} -\operatorname{div} \sigma[u] = f(x), & \text{in } \Omega \\ u = g(x) & \text{on } \Gamma_D \\ \sigma[u]\nu = q(x) & \text{on } \Gamma_N \end{cases} \quad (13)$$

$$f \in L^2(\Omega : \mathbb{R}^d), \ g \in H^1(\Omega : \mathbb{R}^d), \ q \in L^2(\Gamma_N : \mathbb{R}^d)$$

3.1 Strong Solution

$u \in H^2(\Omega : \mathbb{R}^d)$ satisfies $(**)$ then we call u : a strong solution

3.2 Weak Solution

$$\begin{cases} \int_{\Omega} \sigma[u] : e[v] dx = \int_{\Omega} f \cdot v dx + \int_{\Gamma_N} q \cdot v ds \ (\forall v \in V := \{v \in H^1(\Omega : \mathbb{R}^d) \mid v|_{\Gamma_D} = 0\}) \\ u \in V + g \end{cases}$$

3.3 Properties

$$\begin{aligned} u : \text{strong solution} &\Leftrightarrow \begin{cases} u : \text{weak solution} \\ u \in H^2(\Omega : \mathbb{R}^d) \end{cases} \\ X &:= H^1(\Omega : \mathbb{R}^d) \\ a(u, v) &:= \int_{\Omega} \sigma[u] : e[v] dx \\ l(v) &:= \int_{\Omega} f \cdot v dx + \int_{\Gamma_N} q \cdot v ds \\ a(u, v) &= \int_{\Omega} (\mathcal{C}e[u]) : e[v] dx \\ &= \int_{\Omega} e[v] : (\mathcal{C}e[u]) dx \\ &= a(v, u) \end{aligned}$$