Seminar Notes Alifian

Alifian Mahardhika Maulana

May 16, 2018

1 3D Linear Elasticity

$$\begin{aligned}\Omega &\subset \mathbb{R}^d (d = 2, 3)\\ u = \Omega &\to \mathbb{R}^2 (\text{small displacement})\\ x &\mapsto u(x)\end{aligned}$$

1.1 Strain Tensor

$$\begin{aligned}e[u] &= (e_{ij}[u]) \in \mathbb{R}^{dxd}_{sym}\\ e[u] &:= \frac{1}{2}(\bigtriangledown^T u + (\bigtriangledown^T u)^T)\end{aligned} \tag{1}$$

1.2 Stress Tensor

$$\sigma[u] = (\sigma ij[u]) \in \mathbb{R}^{dxd}_{sym} \tag{2}$$

Based on Hook's Law, stress tensor must have equality with strain so that

$$\begin{aligned}\sigma &= \mathbf{C}e\\ \text{with } \mathbf{C} &= \mathbf{C}_{ijkl} (\text{is a 4th order elasticity tensor})\\ \sigma ij &= \mathbf{C}ijkle_{kl}\\ \mathbf{C}_{ijkl} &= \mathbf{C}_{ijlk} = \mathbf{C}_{klij} (\text{symmetry})\\ \mathbf{C}_{ijkl}\xi_{ij}\xi_{kl} &\geq \mathbf{C}_*|\xi|^2\end{aligned}$$

1.3 Boundary Value Problem

$$\begin{cases} -\partial_i \sigma_{ij}[u] & = f_j(x), x \in \Omega \\ u & = g(x), x \in \Gamma_D \\ \sigma[u]_\nu & = q(x), x \in \Gamma_N \end{cases} \tag{3}$$

1.4 Equilibrium Equations of Force in Ω and on Γ_N

1.4.1 Strain Energy Density

$$\omega[u](x) := \frac{1}{2}\sigma[u] : e[u] \tag{4}$$

Solving using Sobolev Space in Isotropic Case, equation 4 becomes

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

with λ, μ called Lame Constant

$$\delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$

$$\begin{aligned} \sigma[u] &= (\sigma_{ij}[u]) \\ \sigma_{ij}[u] &= c_{ijkl}e_{kl}[u] \\ &= \lambda(\delta_k u_k)\delta_{ij} + \mu(\delta_i u_j + \delta_j u_i) \\ &= \lambda(\text{div } u)I + 2\mu e[u] \\ \omega[u] &= \frac{1}{2}(\lambda(\text{div } u)I + 2\mu e[u]) : e[u] \\ \omega[u] &= \frac{1}{2}(\lambda(\text{div } u)^2 + \mu|e[u]|^2 \end{aligned}$$

Remark 1. *Positivity of* $\boldsymbol{C}$

$$\begin{aligned} (\boldsymbol{C}\xi) : \xi &\geq \boldsymbol{C}_*|\xi|^2(\forall\xi \in \mathbb{R}^{dxd}_{sym}) \\ (\boldsymbol{C}\xi) : \xi &= \lambda|tr\ \xi|^2 + 2\mu|\xi|^2 \end{aligned}$$

If $\lambda \geq 0, \mu > 0$*, then* $C_* = 2\mu$

$$\xi = (\xi_{ij}), |\xi|^2 = \xi_{ij}\xi_{ij} = \sum_{i=1...d}^{d}\sum_{j=1...d}^{d}|\xi_{ij}|^2$$

1.5 Elasticity Problem

$$\begin{cases} -\text{div } \sigma[u] & = f(x) \text{ in } \Omega \subset \mathbb{R}^d \\ u & = g(x) \text{ on } \Gamma_D \\ \sigma[u]v & = q(x) \text{ on } \Gamma_N \end{cases} \tag{5}$$

1.6 Crack Problem

$$\begin{cases} -\text{div } \sigma[u] & = f(x) \text{ in } \Omega \setminus \Sigma \subset \mathbb{R}^d \\ u & = g(x) \text{ on } \Gamma_D \\ \sigma[u]v & = q(x) \text{ on } \Gamma_N \\ \sigma[u]v & = 0 \text{ on } \Sigma^+ \cup \Sigma^- \end{cases} \tag{6}$$

1.7 Lebesque Measurable Theory

$$L^p(\Omega) := \left\{ v : \Omega \to \mathbb{R} \middle| \begin{cases} v = \text{Lebesque measurable} \\ \int_\Omega |v(x)|^p dx < \infty \end{cases} \right\} \tag{7}$$

Remark 2. *for $u, v \in \mathbb{L}^p(\Omega)$, if $\exists N \subset \Omega$ such that $\begin{cases} u(x) = v(x)(x \in \Omega \setminus N) \\ \mathcal{L}^d(N) = 0, \end{cases}$ then we identify u and v, $\mathcal{L}^d(N) = 0 \Leftrightarrow$ volume of $N = 0$ for simplicity, we also can say that $u(x) = v(x)$ for a.e. $x \in \Omega$*

for example

$$\begin{aligned} & v : \mathbb{R} \to \mathbb{R} \\ & v(x) = \begin{cases} 1, x \in \mathbb{Q} \\ 0, x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \\ & \int_{\mathbb{R}} v dx = 0, \mathcal{L}^1(\mathbb{Q}) = 0 \\ & v(x) = 0 \text{ on } \mathbb{R} \setminus \mathbb{Q}, \text{ or we can say } v = 0 \text{ a.e. in } \mathbb{R} \end{aligned} \tag{8}$$

1.8 Sobolev Space

$$\mathbb{W}^{1,p}(\Omega) := \left\{ v \in \mathbb{L}^p(\Omega) \frac{\partial v}{\partial x_j}_{(j=1...d)} \in \mathbb{L}^p(\Omega) \right\} \tag{9}$$

such $\frac{\partial v}{\partial x_j}$ we called it distribution sence.
example of Sobolev Space is as follow:

$$v \in \mathbb{L}^p(\Omega) \text{ if } \exists \omega_j \in \mathbb{L}(\Omega)$$

such that

$$\int_\Omega v \frac{\partial \varphi}{\partial x_j} dx = -\int_\Omega \omega_j \varphi dx (\forall \varphi \in \mathbb{C}_0^\infty(\Omega))$$
$$\Rightarrow \frac{\partial \varphi}{\partial x_j} = \omega_j \text{ in distribution sence}$$

for

$$v \in \mathbb{C}^1(\Omega), \frac{\partial v}{\partial x_j}(x) = \omega_j(x)$$
$$\Updownarrow$$
$$\int_\Omega \omega_j \varphi dx = -\int_\Omega v \frac{\partial \varphi}{\partial x_j} dx (\forall \varphi \in \mathbb{C}_0^\infty(\Omega))$$

In particular,

$$\mathbb{H}^1(\Omega) := \mathbb{W}^{1,2}(\Omega), \triangledown u = \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_d} \end{pmatrix}$$

inner product

$$(u, v)_{\mathbb{H}^1(\Omega)} := \int_\Omega uv \; dx + \int_\Omega \triangledown u \cdot \triangledown v \; dx$$

norm

$$||u||_{\mathbb{H}^1(\Omega)} := \sqrt{(u, v)_{\mathbb{H}^1(\Omega)}} = \sqrt{\int_\Omega |u|^2 dx + \int_\Omega | \triangledown u|^2 dx}$$

$\mathbb{H}^1(\Omega)$ is complete ($\mathbb{H}^1(\Omega)$ is a Hilbert Space)

$$(u,v)_{\mathbb{L}^2(\Omega)} = \int_\Omega uvdx$$

1.9 Incomplete Hilbert Space

$\mathbb{V}:$ a vector space in $\mathbb{R}$

$$\begin{cases} u,v \ \in \mathbb{V} \Rightarrow \alpha u + \beta v \in \mathbb{V} \\ \alpha,\beta \ \in \mathbb{R} \end{cases}$$

If $(\cdot,\cdot) : \mathbb{V}\times\mathbb{V} \to \mathbb{R}$ satisfies

$$\begin{cases} (u\cdot v) \geq 0 \text{ and } (u,u) = 0 \Leftrightarrow u = 0_v \in \mathbb{V} \\ (u,v) = (v,u) \\ (\alpha u + \beta v, \omega) = \alpha(u,\omega) + \beta(v,\omega) \end{cases}$$

then we call $[\mathbb{V}\times\mathbb{V}]$ pre Hilbert space or incomplete Hilbert Space.

1.10 Property of $\mathbb{L}^2(\Omega)$

For $v \in \mathbb{C}^1(\Omega)$,

$$\frac{\partial v}{\partial x_j}(x) = w_j(x)$$

$$\Updownarrow$$

$$\int_\Omega w_j \varphi dx = -\int \Omega v \frac{\partial \varphi}{\partial x_j} dx \ \left(\forall \varphi \in \mathbb{C}_0^\infty(\Omega)\right)$$

$(u,v)_{\mathbb{L}^2(\Omega)} = \int_\Omega uvdx$

$$\Rightarrow \left|\int_\Omega uvdx\right| \leq \int_\Omega |u||v|dx \leq ||u||_{\mathbb{L}^2(\Omega)}||v||_{\mathbb{L}^2(\Omega)}$$

$u, v \in \mathbb{H}^1(\Omega)$

$$\Rightarrow \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j} \in \mathbb{L}^2(\Omega)$$

$$\left| \int_\Omega \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j} dx \right| \leq \int_\Omega \left| \frac{\partial u}{\partial x_j} \right| \left| \frac{\partial v}{\partial x_j} \right| dx \leq \left\| \frac{\partial u}{\partial x_j} \right\|_{\mathbb{L}^2(\Omega)} \left\| \frac{\partial v}{\partial x_j} \right\|_{\mathbb{L}^2(\Omega)}$$

$$\nabla u \cdot \nabla v = \sum_{j=1}^{d} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j}$$

$$\begin{aligned}
\left| \int_\Omega \nabla u \cdot \nabla v dx \right| &= \left| \int \sum_{j=1}^{d} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j} dx \right| \leq \sum_{j=1}^{d} \int \left| \frac{\partial u}{\partial x_j} \right| \left| \frac{\partial v}{\partial x_j} \right| dx \\
&\leq \sum_{j=1}^{d} \left\| \frac{\partial u}{\partial x_j} \right\|_{\mathbb{L}^2(\Omega)} \left\| \frac{\partial v}{\partial x_j} \right\|_{\mathbb{L}^2(\Omega)} \\
&\leq \sqrt{\sum_{j=1}^{d} \left\| \frac{\partial u}{\partial x_j} \right\|_{\mathbb{L}^2(\Omega)}} \sqrt{\sum_{j=1}^{d} \left\| \frac{\partial v}{\partial x_j} \right\|_{\mathbb{L}^2(\Omega)}} \\
&= \sqrt{\sum_{j=1}^{d} \int_\Omega \left| \frac{\partial u}{\partial x_j} \right|^2 dx} \sqrt{\sum_{j=1}^{d} \int_\Omega \left| \frac{\partial v}{\partial x_j} \right|^2 dx} \\
&= \sqrt{\int_\Omega \left(\sum_{j=1}^{d} \left| \frac{\partial u}{\partial x_j} \right|^2 \right) dx} \sqrt{\int_\Omega \left(\sum_{j=1}^{d} \left| \frac{\partial v}{\partial x_j} \right|^2 \right) dx} \\
&= \sqrt{\int_\Omega \left| \nabla u \right|^2 dx} \sqrt{\int_\Omega \left| \nabla v \right|^2 dx}
\end{aligned}$$

$$\therefore \left| \int_\Omega \nabla u \cdot \nabla v dx \right| \leq \sqrt{\int_\Omega \left| \nabla u \right|^2 dx} \sqrt{\int_\Omega \left| \nabla v \right|^2 dx} \tag{10}$$

1.11 Energy (Revisited)

$$E(u) := \frac{1}{2} \int_\Omega \sigma[u] : e[u] dx - \int_\Omega f \cdot u dx - \int_{\Gamma_N} q \cdot u ds \tag{11}$$

with u is a vector of the elasticity problem define by:

$$\begin{aligned}
u \in \mathbb{H}^1(\Omega : \mathbb{R}^d) &:= \{ u : \Omega \to \mathbb{R}^d | u = (u_i, \cdots, u_d), u_i \in \mathbb{H}^1(\Omega) \} \\
&\Rightarrow E(u) < \infty
\end{aligned}$$

u : become solution $\Leftrightarrow u = argmin_{v \in \mathbb{H}^1(\Omega : \mathbb{R}^d)} E(v)$ such a technique we call it variational principle.

1.12 Variational Principle

Let's consider a Poisson Equation Problem:

$$\Omega \subset \mathbb{R}^d \begin{cases} -\triangle u & = f(x) \in \Omega \\ u & = g(x) \text{ on } \Gamma_D \; f \in L^2(\Omega), g \in H^1(\Omega), q \in L^2(\Gamma_N) \\ \frac{\partial u}{\partial v} & = q(x) \text{ on } \Gamma_N \end{cases} \tag{12}$$

Remark 3.
$$v \in H^1(\Omega) \Rightarrow \exists v|_\Gamma \in L^2(\Gamma)$$
we choose v *on* L^2 *because it will has value on the boundary*

1.12.1 Definition of Weak Solution

$$u \in H^1(\Omega) \text{ s.t. } \begin{cases} \int_\Omega \triangle u \cdot \triangle v dx = \int_\Omega f v dx + \int_{\Gamma_N} q v ds \\ \left(\forall v \in V := v \in H^1(\Omega) \middle| v|_{\Gamma_D} \right) \\ v|_{\Gamma_D} = g|_{\Gamma_D} (v - g \in V) \end{cases}$$

$(v - g \in V)$ mean $(v \in V + g := v + g | v \in V)$ with V is an affine space.

1.12.2 Definition of Strong Solution

$u \in H^2(\Omega)$ and u satisfies (12)

Remark 4.
$$H^2(\Omega) := \{u \in L^2(\Omega) \frac{\partial u}{\partial x_j}, \frac{\partial^2 u}{\partial x_i x_j} \in L^2(\Omega)\}$$
$$u \in H^2(\Omega) \Rightarrow \frac{\partial u}{\partial x_j} \in H^1(\Omega)$$
$$\frac{\partial u}{\partial v} = \sum v_i \frac{\partial v}{\partial x_i}$$

Properties 1.
$$u: \ \textit{strong solution} \ \Leftrightarrow \begin{cases} u: \ \textit{weak solution} \\ u \in H^2(\Omega) \end{cases}$$

Proof. Let's consider one dimension for simplicity, then:
$$(\Rightarrow) - \triangle u = f(x)$$
$$-\frac{\partial^2 u}{\partial x^2} = f(x)$$

Suppose we take $v, \forall v \in V := v \in H^1(\Omega) | v|_{\Gamma_D} = 0$, if v is smooth enough, then we take integral for both side, thus the equation becomes:
$$\int_\Omega -\frac{\partial^2 u}{\partial x^2} v dx = \int_\Omega f(x) v dx$$
$$\int_\Omega -\frac{\partial}{\partial x}\frac{\partial u}{\partial x} v dx = \int_\Omega f(x) v dx$$
using integration by parts, the left handside equation becomes:
$$\int_\Omega -\frac{\partial}{\partial x}\frac{\partial u}{\partial x} v dx = \left[-v \cdot \frac{du}{dx} \right]_{\partial\Omega} + \int_\Omega \frac{du}{dx}\frac{dv}{dx} dx$$
consider dirichlet boundary condition, thus the equation becomes:
$$-\int_{\Gamma_N} q v ds + \int_\Omega \frac{du}{dx}\frac{dv}{dx} dx = \int_\Omega f v dx$$
in general
$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx + \int_{\Gamma_N} q v ds$$
$\square$

$$\text{Energy } E(v) := \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega fv dx - \int_{\Gamma_N} qv ds$$

Theorem 1.

$$u: \textit{ weak solution } \Leftrightarrow u = \textit{ argmin}_{v\in V+g} E(v)$$

Theorem 2.

$$\exists! u = \textit{ argmin}_{v\in V+g} E(v)$$

Proof. of Theorem 1

$$(\Leftarrow) \text{ If } u = \text{ argmin }_{w\in V+g} E(w)$$

$$\text{since } u + fv \in V + g \ (\forall v \in V, \forall t \in \mathbb{R})$$

$$\frac{d}{dt} E(u+tv)|_{t=0} = 0 \text{ (First Variation)}$$

$$
\begin{aligned}
0 &= \frac{d}{dt} E(u+tv)|_{t=0} \\
&= \frac{d}{dt}\left[\frac{1}{2}\int_\Omega |\nabla u|^2 + 2t\nabla u\cdot\nabla v + t^2|\nabla v|^2 dx - \int_\Omega fu dx - \int_{\Gamma_N} qu ds - t(\int_\Omega fv dx + \int_{\Gamma_N} qv ds)\right]_{t=0} \\
&= \int_\Omega \nabla u \cdot \nabla v dx - \int_\Omega fv dx - \int_{\Gamma_N} qv ds
\end{aligned}
$$

$\therefore u =$ weak solution

□

Proof. of Theorem 1 $\Leftrightarrow$ Theorem 2
($\Rightarrow$) If u is a weak solution
for any $w \in V + g, (v := w - u \in V)$

$$
\begin{aligned}
E(w) - E(u) &= E(u+v) - E(u) \\
&= \int_\Omega (\nabla u \cdot \nabla v + \frac{1}{2}|\nabla v|^2) dx - \int_\Omega fv dx - \int_{\Gamma_N} qv ds \\
&= \frac{1}{2}\int_\Omega |\nabla v|^2 dx \geq 0 \\
&\therefore E(w) \geq E(u) (\forall w \in V + g) \\
&\therefore u = \text{ argmin}_{w\in V+g} E(w)
\end{aligned}
$$

□

2 Abstract Theory

X: a real Hilbert Space (ex: $H^1(\Omega)$)
V: a closed subspace of X (ex: $V \subset H^1(\Omega)$) in case of Poisson Equation (Linear)

2.1 Definition

1. a : X × X → $\mathbb{R}$ is a bilinear form, if
   $$\begin{cases} u \mapsto a(u,v) \text{ is linear for all } v \in X \\ v \mapsto a(u,v) \text{ is linear for all } u \in X \end{cases}$$
2. a bilinear form $a(u,v)$ is <u>bounded</u>, if $\exists a_0 > 0$ s.t. $|a(u,v)| \leq a_0 ||u||_x ||v||_x (\forall u, v \in X)$

3. a bilinear form $a(\cdot,\cdot)$ is symmetric, if $a(u,v) = a(v,u)(\forall u, v \in X)$

4. a bilinear form $a(\cdot,\cdot)$ is coercive, if $\exists \alpha > 0$ s.t. $a(u,u) \geq \alpha ||u||_x^2 (\forall u \in X)$

Remark 5. *A bilinear form $a(\cdot,\cdot)$ is bounded iff $a : X \times X \to \mathbb{R}$ is continuous*

2.2 Definition

1. $l : x \to \mathbb{R}$ is a linear form, if $l : x \to \mathbb{R}$ $(\in u \mapsto l(u))$ is linear

2. A linear form l is bounded, if $|l(u)| \leq \exists c||u||_x$ $(\forall u \in x)$

Remark 6. *A linear form l is bounded iff $l : x \to \mathbb{R}$ is continuous.*

2.3 Theorem (Lax - Milgram)

We suppose $a(\cdot,\cdot)$ is a bounded bilinear form on $X \times X$, and l is a bounded linear form on X.
If $a(\cdot,\cdot)$ is coercive on $V \times V$, then for any $g \in X$,
$\exists! u \in V + g$ s.t. $a(u,v) = l(v)(\forall v \in V)$

2.3.1 Example

$$X = H^1(\Omega), V = v \in H^1(\Omega)|v|_{\Gamma_D} = 0$$
$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v dx$$
$$l(v) := \int_\Omega f v dx + \int_{\Gamma_N} q v ds$$

Coercivity

$$\exists \alpha_0 > 0 : \int_\Omega |\nabla v|^2 dx \geq \alpha_0 \Big(\int_\Omega |\nabla v|^2 dx + \int_\Omega |v|^2 dx \Big)$$
$$\Updownarrow \text{ If we choose } \alpha_0 \text{ so small}$$
$$\exists \alpha_1 > 0 : \int_\Omega |\nabla v|^2 dx \geq \alpha_1 \int_\Omega |v|^2 dx (\forall v \in V)$$

Boundedness

$$|a(u,v)| \leq \sqrt{\int_\Omega |\nabla u|^2 dx} \sqrt{\int_\Omega |\nabla v|^2 dx} \leq ||u||_x ||v||_x$$

$$|l(v)| \leq ||f||_{L^2(\Omega)} ||v||_{L^2(\Omega)} + ||q||_{L^2(\Gamma_N)} ||v||_{L^2(\Gamma_N)} \leq \Big(||f||_{L^2(\Omega)} + C||q||_{L^2(\Gamma_N)} \Big) ||v||_x$$

Another Example:
Energy $E(u) := \frac{1}{2} a(u,u) - l(u)$

Theorem 3. *(Variational Principle) for $g \in X$,*

$$\begin{cases} u \in V + g \\ a(u,v) = l(v)(\forall v \in V) \Leftrightarrow \ argmin_{w \in V+g} E(w) \end{cases}$$

Proof.

$$
\begin{aligned}
&(\Leftarrow)\forall t \in \mathbb{R}, \forall v \in V, u + tv \in V + g \\
&\frac{d}{dt}E(u+tv)|_{t=0} = 0 \\
&E(u+tv) = \frac{1}{2}a(u+tv, u+tv) - l(u+tv) \\
&\qquad = \frac{1}{2}\Big(a(u,u) + 2ta(u,v) + t^2 a(v,v) - l(u) - tl(u)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&a(u,v) = l(v)(\forall v \in V) \\
\Rightarrow\ &\text{For } w \in V + g\ (v := w - u \in V) \\
&E(w) - E(v) = \frac{1}{2}a(v,v) \geq \frac{1}{2}\alpha||v||_x^2 \geq 0
\end{aligned}
$$

□

Remark 7. *Uniqueness in Lax-Milgram If u and w are both the minimizer of E among $V + g$, then*

$$0 = E(w) - E(u) = \frac{1}{2}a(v,v) \geq \frac{\alpha}{2}||v||_x^2 \geqq 0$$

$\therefore v = 0,\ \therefore w = u$

3 Linear Elasticity

We define:

$$
\begin{aligned}
&\Omega \subset \mathbb{R}^d\ (d = 2, 3) \\
&u : \Omega \to \mathbb{R}^d \text{ (displacement)} \\
&e[u] := \frac{1}{2}(\nabla^T u + \nabla u^T) \text{ (strain)} \\
&\sigma[u] := \mathcal{C}e[u] \\
&\mathcal{C} = (C_{ijkl}) \begin{cases} C_{ijkl} = C_{klij} = C_{jikl} \\ (C_\xi) : \xi \geq C_*|\xi|^2 (\forall \xi \in \mathbb{R}^{d\times d}_{sym}) \end{cases}
\end{aligned}
$$

Let's consider linear elasticity problem:

$$(**) \begin{cases} -div\ \sigma[u] = f(x),\ \text{in } \Omega \\ u = g(x) \text{ on } \Gamma_D \\ \sigma[u]\nu = q(x) \text{ on } \Gamma_N \end{cases} \tag{13}$$

$$f \in L^2(\Omega : \mathbb{R}^d,\ g \in H^1(\Omega : \mathbb{R}^d),\ q \in L^2(\Gamma_N : \mathbb{R}^d))$$

3.1 Strong Solution

$u \in H^2(\Omega : \mathbb{R}^d)$ satisfies $(**)$ then we call u : a strong solution

3.2 Weak Solution

$$\begin{cases} \int_\Omega \sigma[u] : e[v]dx = \int_\Omega f \cdot v dx + \int_{\Gamma_N} q \cdot v ds \big(\forall v \in V := \{v \in H^1\ (\Omega : \mathbb{R}^d)\ |v|_{\Gamma_D} = 0\}\big) \\ u \in V + g \end{cases}$$

3.3 Properties

$$u: \text{ strong solution } \Leftrightarrow \begin{cases} u: \text{ weak solution} \\ u \in H^2\ (\Omega : \mathbb{R}^d) \end{cases}$$

$$X := H^1(\Omega : \mathbb{R}^d)$$

$$a(u,v) := \int_\Omega \sigma[u] : e[v]dx$$

$$l(v) := \int_\Omega f \cdot v dx + \int_{\Gamma_N} q \cdot v ds$$

$$\begin{aligned} a(u,v) &= \int_\Omega (\mathcal{C}e[u]) : e[v]dx \\ &= \int_\Omega e[v] : (\mathcal{C}e[u])dx \\ &= a(v,u) \end{aligned}$$

For $v \in V$

$$\begin{aligned} a(v,v) &= \int_\Omega (\mathcal{C}e[v]) : e[v]dx \\ &\geqq C_* \int_\Omega |e[v]|^2 dx \\ &\geqq C_* ||v||_x^2 \end{aligned}$$

Properties 2.
- *$a(\cdot,\cdot)$ is bounded symmetric, bilinear form on $X \times X$.*
- *$a(\cdot,\cdot)$ is coercive on $V \times V$.*
- *l is bounded linear form on X.*

Theorem 4. *For any $g \in H^1(\Omega : \mathbb{R}^d)$,*

$$\exists! u : \textit{a weak solution of } (**), \textit{ and } \begin{cases} u = \operatorname{argmin}_{w \in V+g} E(w) \end{cases}$$