Assignment 7
Topics of Mathematical Science

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Formula for the residue of f at z_0 pole of order m is following,

$$res(f;z_0) = \lim_{z\to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right] \tag{1}$$

1. Find the poles and residues of the following:

   (a) $\frac{1}{z^2+5z+6}$, poles $z_0 = \begin{cases} -3, m=1 \\ -2, m=1 \end{cases}$, using formula (1), we can compute the residues as follow:

   $$res(f;-2) = \lim_{z\to -2} \frac{1}{(0)!}\left[(z+2)\frac{1}{(z+2)(z+3)}\right] = 1$$
   $$res(f;-3) = \lim_{z\to -3} \frac{1}{(0)!}\left[(z+3)\frac{1}{(z+2)(z+3)}\right] = -1$$

   (b) $\frac{z}{(z^2-1)^2}$, it has poles $z_0 = \begin{cases} 1, m=2 \\ -1, m=2 \end{cases}$, using formula (1), we can compute the residues as follow:

   $$res(f;1) = \lim_{z\to 1} \frac{1}{(1)!}\frac{d}{dz}\left[(z-1)^2\frac{z}{(z+1)^2(z-1)^2}\right] = 0$$
   $$res(f;-1) = \lim_{z\to -1} \frac{1}{(1)!}\frac{d}{dz}\left[(z+1)^2\frac{z}{(z+1)^2(z-1)^2}\right] = 0$$

   (c) $\frac{1}{(z+1)^2(z+2)}$, it has poles $z_0 = \begin{cases} 1, m=2 \\ -2, m=1 \end{cases}$, using formula (1), we can compute the residues as follow:

   $$res(f;-1) = \lim_{z\to -1} \frac{1}{(1)!}\frac{d}{dz}\left[(z+1)^2\frac{1}{(z+1)^2(z+2)}\right] = -1$$
   $$res(f;-2) = \lim_{z\to -2} \frac{1}{(0)!}\left[(z+2)\frac{1}{(z+1)^2(z+2)}\right] = 1$$

   (d) $\frac{z^4+2z+1}{(z-1)^2}$, it has poles $z_0 = 1, m = 2$, using formula (1), we can compute the residues as follow:

   $$res(f;1) = \lim_{z\to 1} \frac{1}{(1)!}\frac{d}{dz}\left[(z-1)^2\frac{z^4+2z+1}{(z-1)^2}\right] = 6$$

   (e) $\frac{z^3+z+1}{(z+1)^2(z+2)}$, it has poles $z_0 = \begin{cases} -1, m=2 \\ -2, m=1 \end{cases}$, using formula (1), we can compute the residues as follow:

   $$res(f;-1) = \lim_{z\to -1} \frac{1}{(1)!}\frac{d}{dz}\left[(z+1)^2\frac{z^3+z+1}{(z+1)^2(z+2)}\right] = 5$$
   $$res(f;-2) = \lim_{z\to -2} \frac{1}{(0)!}\left[(z+2)\frac{z^3+z+1}{(z+1)^2(z+2)}\right] = -10$$

(f) $\frac{1}{z^m(1-z)^n}$, it has poles $z_0 = \begin{cases} 0, m = m \\ 1, m = n \end{cases}$, using formula (1), we can compute the residues as follow:

$$res(f;0) = \lim_{z\to 0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z)^m \frac{1}{(z)^m(1-z)^n} \right] = \frac{(n+m-2)!}{(m-1)!(n-1)!}$$
$$res(f;1) = \lim_{z\to 1} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[-(1-z)^n \frac{1}{(z)^m(1-z)^n} \right] = -\frac{(m+n-2)!}{(n-1)!(m-1)!}$$

Cauchy Residue Theorem: the integral of $f(z)$ along $B(z_0, r) = \{z \in \mathbb{C} | |z - z_0| < r\}$ is equal to $2\pi i$ times the sum of the residues of the singularities in the interior of the contour,

$$\int_{\partial B(z_0,r)} f(z)dz = 2\pi i \sum_{z_k \in \partial B} res(f; zk) \tag{2}$$

2. Calculate the following line integrals with the Residue theorem (2)

(a)

$$\int_{\partial B(0,1)} \frac{1}{z^2 + 5z + 6} dz$$

it has poles, $z = \{-3, -2\}$, with $B(0,1) = \{z \in \mathbb{C} | |z| < 1\}$, because the poles outside of the ball,

$$\int_{\partial B(0,1)} \frac{1}{z^2 + 5z + 6} dz = 0$$

(b)

$$\int_{\partial B(0,2)} \frac{z}{(z^2 - 1)^2} dz$$

it has poles $z = \begin{cases} 1, m = 2 \\ -1, m = 2 \end{cases}$, with $B(0,2) = \{z \in \mathbb{C} | |z| < 2\}$, because the poles inside of the ball, hence,

$$\int_{\partial B(0,2)} \frac{z}{(z^2 - 1)^2} dz = 2\pi i (res(f;1) + res(f;-1))$$

with each residue, we computed as follow:

$$res(f;1) = \lim_{z\to 1} \frac{1}{(1)!} \frac{d}{dz} \left[(z-1)^2 \frac{z}{(z-1)^2(z+1)^2} \right] = 0$$
$$res(f;-1) = \lim_{z\to -1} \frac{1}{(1)!} \frac{d}{dz} \left[(z+1)^2 \frac{z}{(z-1)^2(z+1)^2} \right] = 0$$

therefore,

$$\int_{\partial B(0,2)} \frac{z}{(z^2 - 1)^2} dz = 2\pi i (res(f;1) + res(f;-1)) = 2\pi i (0 + 0) = 0$$

(c)

$$\int_{\partial B(0,2)} \frac{1}{(z+1)^2(z+3)} dz$$

it has poles $z = \begin{cases} -1, m=2 \\ -3, m=1 \end{cases}$, with $B(0,2) = \{z \in \mathbb{C} | |z| < 2\}$, because the poles inside of the ball is just $z = -1$, hence,

$$\int_{\partial B(0,2)} \frac{1}{(z+1)^2(z+3)} dz = 2\pi i(res(f;-1))$$

with residue, we computed as follow:

$$res(f;-1) = \lim_{z \to -1} \frac{1}{(1)!} \frac{d}{dz} \left[(z+1)^2 \frac{1}{(z+1)^2(z+3)} \right] = -\frac{1}{4}$$

thus,

$$\int_{\partial B(0,2)} \frac{1}{(z+1)^2(z+3)} dz = 2\pi i(res(f;-1)) = \frac{i\pi}{2}$$

Method of residues, **Application to compute improper integral**: Let $R(x) = P(x)/Q(x)$ be a rational function of a real variable satisfying the following two criteria: $Q(x) \neq 0$ and $deg(Q)deg(P) \geq 2$

$$\int_{-\infty}^{\infty} R(x)dx = 2\pi i \sum_{Im(\alpha_k)>0} res(R(x); \alpha_k) \tag{3}$$

3. Evaluate the following integrals by the method of residues.

(a)

$$\int_0^{2\pi} \frac{d\theta}{a + \sin\theta}, \quad (a > 1)$$

we change the form into,

$$\int_{|z|=1} \frac{2}{z^2 + i2az - 1} dz$$

, it has poles $\alpha = \begin{cases} i\left(-\frac{a}{2} - \frac{1}{2}\sqrt{a^2-1}\right) \\ i\left(-\frac{a}{2} + \frac{1}{2}\sqrt{a^2-1}\right) \end{cases}$,thus the residue is,

$$res(f;\alpha) = \lim_{z \to \alpha} (z-\alpha)f(z) = \lim_{z \to \alpha} \frac{2}{\alpha - \beta} = \frac{1}{i\sqrt{a^2-1}}$$

therefore

$$\int_0^{2\pi} \frac{d\theta}{a + \sin\theta} = 2\pi i res(f;\alpha) = \frac{2\pi}{\sqrt{a^2-1}}$$

(b)

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4 + 5x^2 + 6} dx$$

, it has poles $\alpha = \begin{cases} i\sqrt{2}, m=1 \\ i\sqrt{3}, m=1 \\ -i\sqrt{2}, m=1 \\ -i\sqrt{3}, m=1 \end{cases}$,thus the residue is,

$$res(f; i\sqrt{2}) = \lim_{z \to i\sqrt{2}} \left(\frac{(z - i\sqrt{2})z^2}{(z^2+2)(z^2+3)} \right) = -\frac{\sqrt{2}}{i2}$$

$$res(f; i\sqrt{3}) = \lim_{z \to i\sqrt{3}} \left(\frac{(z - i\sqrt{3})z^2}{(z^2+2)(z^2+3)} \right) = \frac{\sqrt{3}}{i2}$$

therefore,

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+5x^2+6} dx = 2\pi i(res(f; i\sqrt{2}) + res(f; i\sqrt{3})) = \pi\left(\sqrt{3}-\sqrt{2}\right)$$

(c)

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2} dx, \quad (a \neq 0)$$

we can rewrite as,

$$Im\left(\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+a^2} dx\right)$$

, it has poles $\alpha = \{ia, -ia\}$,thus the residue is,

$$res(f; ia) = \lim_{z \to ia}\left((z-ia)\frac{ze^{iz}}{(z-ia)(z+ia)}\right) = \frac{e^{-\alpha}}{2}$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2} dx = Im\left(\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+a^2} dx\right) = Im(i\pi e^{-a}) = \pi e^{-a}$$

4. Let $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$, compute

$$\int_{\partial\mathbb{D}} \frac{|dz|}{|z-a|^2}, \quad (|a| < 1)$$

we can rewrite as,

$$\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$$

on $a > 1, |\alpha| < 1$, then α is the only pole of $R(z)$ in $\{|z| < 1\}$, therefore, we get the integral

$$\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2} = R(z) = \int_{|z|=1} \frac{1}{i}\frac{-1}{az^2-(1+a^2)z+a} dz$$

, it has poles $\alpha = \{\frac{1}{a}, a\}$, for $|a| < 1$, the only pole is on $\{|z| < 1\}$, thus the residue is,

$$res(R(z); a) = \lim_{z \to a}(z-a)\frac{-1}{a(z-\frac{1}{a})(z-a)} = \frac{1}{i}\frac{-1}{a(a-\frac{1}{a})} = \frac{-1}{i(a^2-1)}$$

Therefore,

$$\begin{aligned}\int_{\partial\mathbb{D}} \frac{|dz|}{|z-a|^2} &= \int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2} \\ &= 2\pi i res(R(z); \alpha) = 2\pi i\left(\frac{-1}{i(a^2-1)}\right) \\ &= \frac{-2\pi}{a^2-1} = \frac{2\pi}{1-a^2}\end{aligned}$$