Linear Elasticity and Crack Modelling in 2D and 3D Using Finite Element Method

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1 Basic Theory

We define:

$$
\begin{aligned}
&\Omega \subset \mathbb{R}^d \ (d = 2, 3) \\
&u : \Omega \to \mathbb{R}^d \text{ (displacement)} \\
&e[v] := \frac{1}{2}(\nabla^T v + \nabla v^T) \text{ (strain)} \\
&\nabla^T v := \begin{pmatrix} \partial_1 v_1 & \partial_2 v_2 \\ \partial_1 v_2 & \partial_2 v_1 \end{pmatrix} \\
&\nabla v^T := (\nabla^T v)^T \\
&\sigma[u] := \mathcal{C} e[u] \\
&\mathcal{C} = (C_{ijkl}) \begin{cases} C_{ijkl} = C_{klij} = C_{jikl} \\ (C_\xi) : \xi \geq C_* |\xi|^2 (\forall \xi \in \mathbb{R}^{d \times d}_{sym}) \end{cases}
\end{aligned}
$$

Let's consider linear elasticity problem:

$$
(**) \begin{cases} -div\ \sigma[u] = f(x), \text{ in } \Omega \\ u = g(x) \text{ on } \Gamma_D \\ \sigma[u]\nu = q(x) \text{ on } \Gamma_N \end{cases} \tag{1}
$$

$$
f \in L^2(\Omega : \mathbb{R}^d,\ g \in H^1(\Omega : \mathbb{R}^d),\ q \in L^2(\Gamma_N : \mathbb{R}^d))
$$

1.1 Strong Solution

$$
u \in H^2(\Omega : \mathbb{R}^d) \text{ satisfies } (**) \text{ then we call } u : \text{a strong solution}
$$

1.2 Weak Solution

$$
\begin{cases} \int_\Omega \sigma[u] : e[v] dx = \int_\Omega f \cdot v dx + \int_{\Gamma_N} q \cdot v ds \big(\forall v \in V := \{v \in H^1\ (\Omega : \mathbb{R}^d)\ |v|_{\Gamma_D} = 0\}\big) \\ u \in V + g \end{cases}
$$

1.3 Proposition

$$
u : \text{ strong solution } \Leftrightarrow \begin{cases} u : \text{ weak solution} \\ u \in H^2\ (\Omega : \mathbb{R}^d) \end{cases}
$$

Proof. ($\Rightarrow$) Assume we choose $v \in V := \{v \in H^1\ (\Omega : \mathbb{R}^d)\ |v|_{\Gamma_D} = 0\}$, with v is a very smooth test function. Then we take integral over the domain for equation (1) on both side.

$$\int_\Omega -div\sigma[u] \cdot v dx = \int_\Omega f \cdot v dx$$
$$\int_\Omega \sigma[u] : \nabla v dx - \int_\Gamma \sigma[u]\nu \cdot v ds = \int_\Omega f \cdot v dx \text{ (by Divergence Formula)}$$
$$\int_\Omega \sigma[u] : \nabla v dx - \left(\int_{\Gamma_D} \sigma[u]\nu \cdot v ds + \int_{\Gamma_N} \sigma[u]\nu \cdot v ds \right) = \int_\Omega f \cdot v dx$$

From Boundary Condition we know that:

$$\begin{cases} v = 0 \text{ on } \Gamma_D \\ \sigma[u]\nu = q \text{ on } \Gamma_N \end{cases}$$

Hence, we have:

$$\begin{aligned} \int_\Omega \sigma[u] : e[v] dx - \int_{\Gamma_N} q \cdot v ds &= \int_\Omega f \cdot v dx \\ \int_\Omega \sigma[u] : e[v] dx &= \int_\Omega f \cdot v dx + \int_{\Gamma_N} q \cdot v ds \end{aligned} \tag{2}$$

with:

$$X := H^1(\Omega : \mathbb{R}^d)$$
$$a(u,v) := \int_\Omega \sigma[u] : e[v] dx$$
$$l(v) := \int_\Omega f \cdot v dx + \int_{\Gamma_N} q \cdot v ds$$

$$\begin{aligned} a(u,v) &= \int_\Omega (\mathcal{C}e[u]) : e[v] dx \\ &= \int_\Omega e[v] : (\mathcal{C}e[u]) dx \\ &= a(v,u) \end{aligned}$$

Then we rewrite equation (2) in a bilinear and linear form:

$$a(u,v) = l(v)$$

($\Leftarrow$) Since $u \in H^2\ (\Omega : \mathbb{R}^2), div(\sigma[u]) \in L^2(\Omega : \mathbb{R}^2)$ and $a(u,v) = 0$ for all $v \in V$, we have:

$$\begin{aligned} 0 &= \int_\Omega \sigma[u] : e[v] dx \\ 0 &= \int_\Omega \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} : \begin{pmatrix} \nabla v_1 & \nabla v_2 \end{pmatrix} dx \\ 0 &= \int_\Omega \begin{pmatrix} \sigma_{11} \\ \sigma_{21} \end{pmatrix} \cdot \nabla v_1 + \begin{pmatrix} \sigma_{12} \\ \sigma_{22} \end{pmatrix} \cdot \nabla v_2 dx \\ 0 &= \int_\Omega div \begin{pmatrix} \sigma_{11} \\ \sigma_{21} \end{pmatrix} v_1 + div \begin{pmatrix} \sigma_{12} \\ \sigma_{22} \end{pmatrix} v_2 - \int_{\partial\Omega} \begin{pmatrix} \sigma_{11} \\ \sigma_{21} \end{pmatrix} \cdot \nu v_1 + \begin{pmatrix} \sigma_{12} \\ \sigma_{22} \end{pmatrix} \cdot \nu v_2 ds \\ 0 &= \int_\Omega (f + div\sigma[u]) \cdot v dx - \int_{\partial\Omega} (\sigma[u]\nu) \cdot v ds \\ 0 &= \int_\Omega f \cdot v dx + \int_\Omega div\sigma[u] \cdot v dx - \int_{\partial\Omega} (\sigma[u]\nu) \cdot v ds \end{aligned}$$

then, assume we choose $v \in C_0^\infty(\Omega) \subset V, v = 0$ near $\partial\Omega$
$f \in L^1(\Omega)$, then we have:

$$\int_\Omega (f + div\sigma[u]) \cdot v dx = 0\ (\forall v \in C_0^\infty(\Omega, \mathbb{R}^2))$$
$$\therefore f + div\sigma[u] = 0 \text{ in } \Omega$$

then, $\forall v \in C_0^\infty(\bar{\Omega})$ s.t. $(supp(v) \cap \partial\Omega) \subset \Gamma_N$

$$\int_{\Gamma_N} (\sigma[u]\nu) \cdot v ds = 0$$
$$\sigma[u]\nu = 0 \text{ on } \Gamma_N$$

□

For $v \in V$

$$\begin{aligned} a(v,v) &= \int_\Omega (\mathcal{C}e[v]) : e[v] dx \\ &\geqq C_* \int_\Omega |e[v]|^2 dx \\ &\geqq C_* ||v||_x^2 \end{aligned}$$

Properties 1.

- *$a(\cdot,\cdot)$ is bounded symmetric, bilinear form on $X \times X$.*
- *$a(\cdot,\cdot)$ is coercive on $V \times V$.*
- *l is bounded linear form on X.*

Theorem 1. *For any $g \in H^1(\Omega : \mathbb{R}^d)$,*

$$\exists! u : \textit{a weak solution of } (**), \textit{ and } \Big\{ u = \operatorname{argmin}_{w \in V+g} E(w)$$

2 Modelling and Simulation

In this simulation, we use a cantilever beam (Mild Steel Material) as the domain, which is a thin rectangular cross section introduced by Timoshenkol Goodier (1970), then we must specify nondimensionalized value for simulation as shown in table 1.

Numerical Parameter	Typical Value [unit]	Nondimensionalized value
Young's modulus (E)	2.1×10^7 [Pa]	210
Poisson's ratio (ν)	0.3 [-]	0.3
Gravity constant (f)	9.80655 [N/kg]	9.80655
Weight (q)	0 [N]	0
Length (L)	3.0 [m]	3.0
Depth (h)	0.2 [m]	0.2
Width (b)	0.25 [m]	0.25

Table 1: Material properties and numerical parameters

With the help of FreeFem++ software, we created a 2D and 3D model as shown in figure 1,

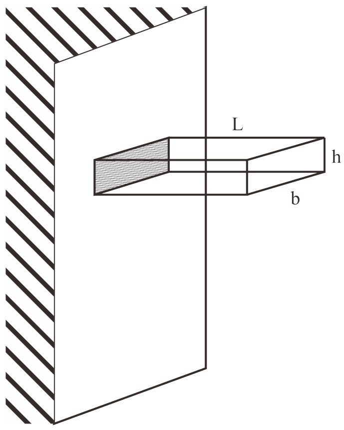


Figure 1: 3D Model of cantilever beam. We use gravity force as the body force **f** and fixed the left part of the beam, and then we give a 1 Newton weight force act on the right part of the beam as the neumann boundary condition **q**.

then we solve the displacement vector (u, v) after solving the displacement, we calculate σ which stand for stress force acting on surface of the cantilever beam using equation below:

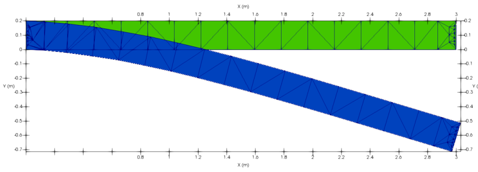
$$\sigma = (d\lambda^2 + 4\lambda\mu)div(u)^2 + (4\mu^2|e[u]|^2),\ d = 2, 3$$

$$\lambda \text{ (Lame's first parameter) } := \frac{E\nu}{(1+\nu)(1-2\nu)}$$

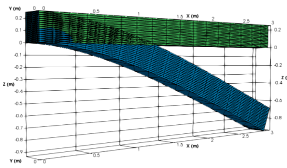
$$\mu \text{ Lame's second parameter } := \frac{E}{2(1+\nu)}$$

3 Result and Discussion

The simulation used mesh P1 finite element method on FreeFEM++, where u and v calculated for division number of mesh equal 16. In the figure 2 we can see the deformation of the cantilever beam in 2D and 3D graphics.



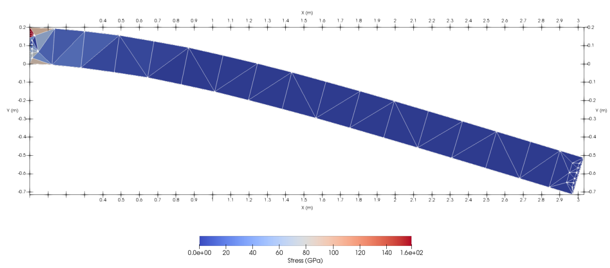
(a) Deformation in 2D. Green line show condition before gravity and weight force applied to the domain. Red line show condition after we solve linear elasticity with gravity and weight force applied to the domain. Maximal Displacement ($u = 0.03$ $[m]$)



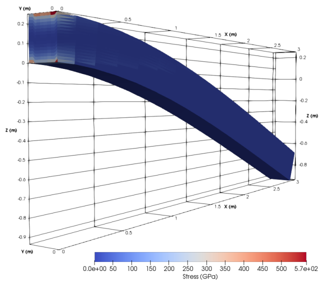
(b) Deformation in 3D. Green line show condition before gravity and weight force applied to the domain. Red line show condition after we solve linear elasticity with gravity and weight force applied to the domain. Maximal Displacement ($u = 0.05$ $[m]$)

Figure 2: Deformation in 2D and 3D

While in the figure 3 we can see result from calculating the stress tensor on 2D and 3D case.



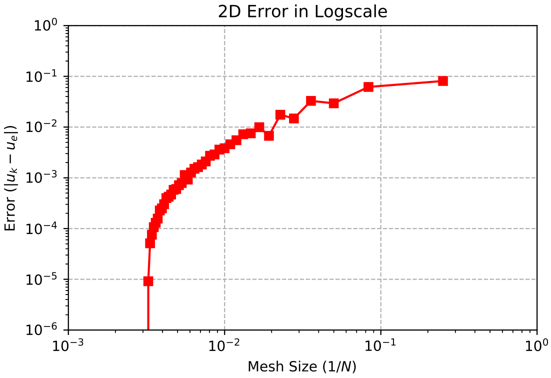
(a) Calculated σ on 2D case. The value of σ on the domain, maped by the color in the picture with respect to the color palette on the lower side of the graph. Maximal stress given on the surface ($\sigma = 158.612$ $[GPa]$)



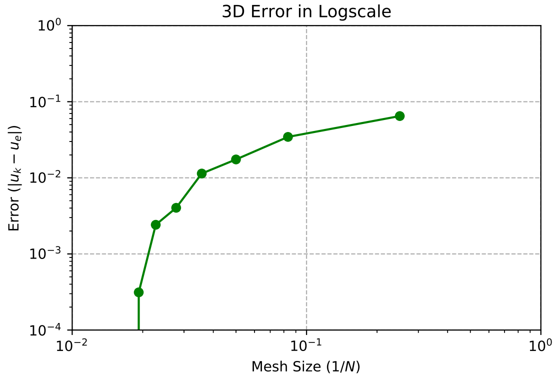
(b) Calculated σ on 3D case. The value of σ on the domain, maped by the color in the picture with respect to the color palette on the lower side of the graph. Maximal stress given on the surface ($\sigma = 567.034$ $[GPa]$)

Figure 3: Stress Tensor in 2D and 3D

In the figure 4 we can see convergence on 2D and 3D case.



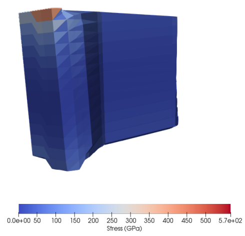
(a) Calculated error on 2D case using optimum displacement vector on mesh size $(1/316) = 0.00316456$



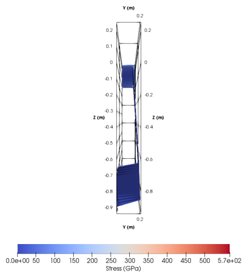
(b) Calculated error on 3D case using optimum displacement vector on mesh size $(1/60) = 0.016667$

Figure 4: Error Plot in 2D and 3D

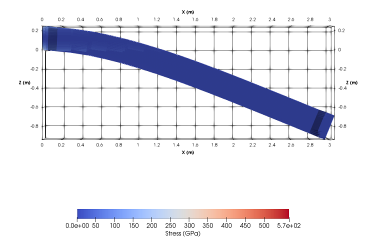
On the figure 5 below, we can see the result from sliced view on 3D case, in this case, we sliced through the Y-normal plane of the 3D model.



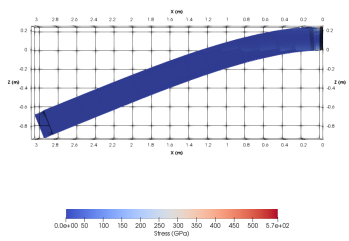
(a) Inside of the 3D model



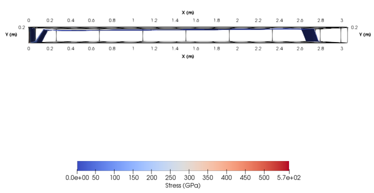
(b) Front View



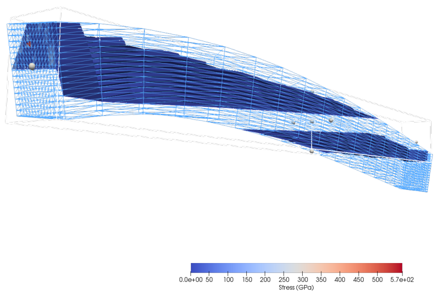
(c) Right View



(d) Left View



(e) Top View

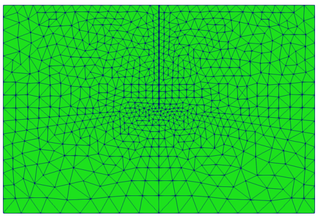


(f) Wireframed-Sliced View

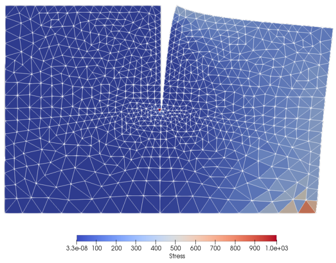
Figure 5: 3D Sliced View

4 Elasticity with Crack Problem

Below is the graph of Linear Elasticity Problem with Crack Model in 2D.



(a) Elasticity with Crack Problem Model in 2D



(b) Calculated σ on 2D crack case. The value of σ on the domain, maped by the color in the picture with respect to the color palette on the lower side of the graph. Maximal stress given on the surface ($\sigma = 100.921$ $[GPa]$)

Figure 6: Elasticity with Crack Problem in 2D

From figure 6, we calculated the elastic energy through various length of crack, below is the result of the energy profile from our model in 6.

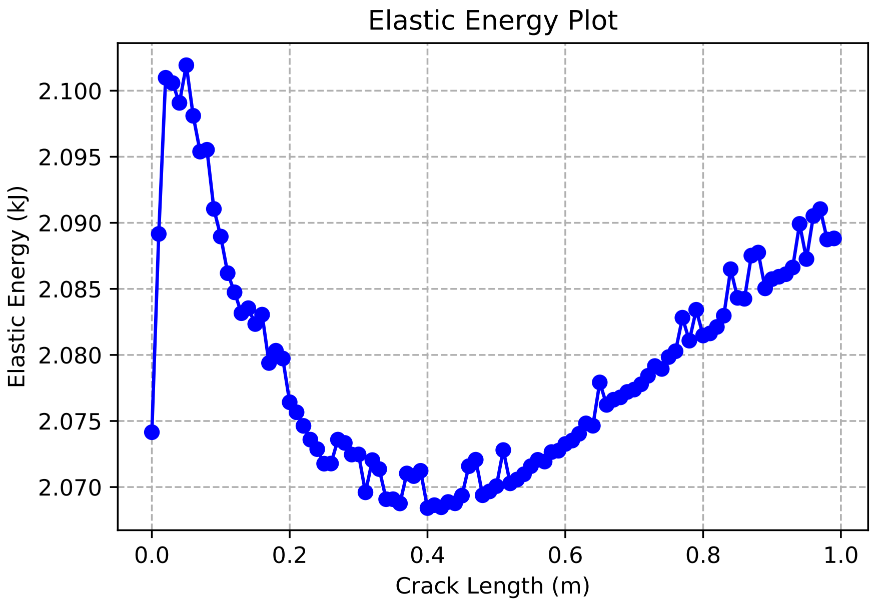


Figure 7: Elasticity Energy Profile Crack Problem Model in 2D