Assignment 3
Analysis Ia Report

Alifian Mahardhika Maulana

July 2, 2018

1. Let (X, d) be a complete metric space and let $f : X \to X$ be a map. Suppose the iterated map
$$f^k = f \circ \cdots \circ f \quad \text{(k times)} \tag{1}$$
is a contraction for some $k \geq 2$. Prove that f has a unique fixed point $x \in X$.

Answer:
By the contraction mapping theorem, f^k has a unique fixed point, let's call it x, so that
$$f^k(x) = x \tag{2}$$
by (2) and (1) we note that
$$f^k(f(x)) = f(f^k(x)) = f(x)$$
Therefore, $f(x)$ and x are both fixed points of f^k. Since f^k has a unique fixed point, $f(x) = x$. Now, we show that for any $x_0 \in X$ the points $f^k(x_0)$ converges to x as $k \to \infty$. Let's consider $f^k(x_0)$ as k runs through some iteration until N. i.e. pick $0 \leq i \leq N - 1$ look at the points $f^{kN+i}(x_0)$ as $k \to \infty$. Since
$$f^{kN+i}(x_0) = f^{kN}(f^i(x_0)) = (f^k)^N(f^i(x_0))$$
and f^k is a contraction, it must be tend to x by the contraction mapping theorem. So all k sequences $\{f^{kN+i}(x_0)\}_{k \geq 1}$ tend to x.
$\therefore f$ has a unique fixed point $x \in X$.