Applied Analysis Report
Gronwall's Inequality

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1. Prove the discrete Gronwall Inequality

   Theorem 1. *Discrete Gronwall Inequality*
   Let $\{x^n\}_{n\geq 0}, \{y^n\}_{n\geq 1}, \{z^n\}_{n\geq 1}, a > 0, \triangle t$ *: small* $\left(\triangle t \leq \frac{1}{2a}\right)$ *be non-negative sequences,* $T > 0, N_T \leqq \left[\frac{T}{\triangle t}\right]$

   $$\frac{x^n - x^{n-1}}{\triangle t} + y^n \leq ax^n + z^n; \quad n \geq 1 \tag{1}$$

   $\Rightarrow \exists c > 0$ *independent of* $\triangle t$ *s.t.*

   $$\max_{n=0,\cdots,N_T} x^n + \triangle t \sum_{n=1}^{N_T} y^n \leq c\left(x^0 + \triangle t \sum_{n=1}^{N_T} z^n\right) \tag{2}$$

   Answer:

   Proof. We multiplied (1) by $\triangle t$ we get:

   $$x^n - x^{n-1} + \triangle t y^n \leq a\triangle t x^n + \triangle t z^n \tag{3}$$

   take summation over N_T for (3) as follows:

   $$\sum_{n=1}^{N_T} x^n - \sum_{n=1}^{N_T} x^{n-1} + \triangle t \sum_{n=1}^{N_T} y^n \leq a\triangle t \sum_{n=1}^{N_T} x^n + \triangle t \sum_{n=1}^{N_T} z^n \tag{4}$$

   from (4) we know that x^n and x^{n-1} in the left hand side cancelled out one and each other except for the first and last part, so (4) becomes:

   $$x^{N_T} - x^0 + \triangle t \sum_{n=1}^{N_T} y^n \leq a\triangle t \sum_{n=1}^{N_T} x^n + \triangle t \sum_{n=1}^{N_T} z^n$$

   $$x^{N_T} + \triangle t \sum_{n=1}^{N_T} y^n \leq x^0 + a\triangle t \sum_{n=1}^{N_T} x^n + \triangle t \sum_{n=1}^{N_T} z^n \tag{5}$$

   $$x^{N_T} - a\triangle t \sum_{n=1}^{N_T} x^n + \triangle t \sum_{n=1}^{N_T} y^n \leq x^0 + \triangle t \sum_{n=1}^{N_T} z^n$$

then, for $a > 0, \triangle t$: small $\left(\triangle t \leq \frac{1}{2a}\right), T > 0, N_T \leqq \left[\frac{T}{\triangle t}\right]$ and x^n, y^n, z^n are a non-negative sequences, we define,

$$A := \frac{x^{N_T} - a\triangle t \sum_{n=1}^{N_T} x^n + \triangle t \sum_{n=1}^{N_T} y^n}{x^0 + \triangle t \sum_{n=1}^{N_T} z^n}$$

$$B := \frac{\max_{n=0,\cdots,N_T} x^n + \triangle t \sum_{n=1}^{N_T} y^n}{x^0 + \triangle t \sum_{n=1}^{N_T} z^n}$$

First, take a condition where $A \geq B$, then,

$$\begin{aligned} \frac{\max_{n=0,\cdots,N_T} x^n + \triangle t \sum_{n=1}^{N_T} y^n}{x^0 + \triangle t \sum_{n=1}^{N_T} z^n} &\leq \max_{n=0,\cdots,N_T} x^n + N_T \triangle t \max_{n=0,\cdots,N_T} y^n \\ &\leq \max_{n=0,\cdots,N_T} x^n + T y^n = c \end{aligned}$$

Second, take a condition where $A < B$, it is clear that

$$\frac{\max_{n=0,\cdots,N_T} x^n + \triangle t \sum_{n=1}^{N_T} y^n}{x^0 + \triangle t \sum_{n=1}^{N_T} z^n} \leq 1 = c$$

then, we can find $c = \max\{1, \max_{n=0,\cdots,N_T}(x^n + T y^n)\}$ such that:

$$\max_{n=0,\cdots,N_T} x^n + \triangle t \sum_{n=1}^{N_T} y^n \leq c\left(x^0 + \triangle t \sum_{n=1}^{N_T} z^n\right) \tag{6}$$

$\square$