Analysis Ia Report

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Problem 1. *Let $M = (S, d)$ be a metric space.*
Let $G \subseteq S$. Then, G is open in $M \iff$ it is a union of open balls.

Proof. ($\Rightarrow$) Let G be open set in M. Let $x \in G$.
by definition of open set:

$$\exists \delta_x \in \mathbb{R}^n : B_{\delta_x}(x, d) \subseteq G$$

where $B_{\delta_x}(x, d)$ is the open δ_x-ball of x in M.

$$\therefore G = \bigcap_{x \in G} B_{\delta_x}(x, d)$$

($\Leftarrow$) Let G be an union of open balls in M.
Let the outer of the open balls be the elements of an indexing set I.
Then G can be written:

$$G = \bigcap_{x \in I} B_{\delta_x}(x, d)$$

where $\delta_x \in \mathbb{R}^n$ is the radius of open ball-of x.
Let $y \in G$. By definition of union:

$$\exists x \in I : y \in B_{\delta_x}(x, d)$$

because an open ball is neighborhood of all points inside, we can say that $B_{\delta_x}(x, d)$ is neighborhood of y, by set: $B_{\delta_x}(x, d) \subseteq G$,from theory of superset of neighborhood in Metric Space, it follows that G is a neighborhood of y.
Since y is arbitrary, it follows that G is a neighborhood of its point. Hence, by definition:

$$\therefore G \text{ is open in } M$$

$\square$

Problem 2. *Let $C([0,1])$ be the set of all continuous functions $f : [0,1] \to \mathbb{R}$, for $f, g \in C([0,1])$. Show that $(C([0,1]), d_1)$ is not complete.*

Proof. Suppose that:

$$d_1(f,g) := \int_0^1 |f(x) - g(x)|dx, \ \ f, g \in C[0,1]$$

Let's consider a sequence $\{f_n\}_{n \geq 3}$:

$$f_n(x) = \begin{cases} 0, & 0 \leq x < \dfrac{1}{2} - \dfrac{1}{n}, \\ n\left(x + \dfrac{1}{n} - \dfrac{1}{2}\right), & \dfrac{1}{2} - \dfrac{1}{n} \leq x < \dfrac{1}{2}, \\ 1, & \dfrac{1}{2} \leq x \leq 1 \end{cases}$$

It shows that the sequence (f_n) converges to discontinuous function $f(x) := 0$ for $0 \leq x < \frac{1}{2}$ and $f(x) := 1$ for $\frac{1}{2} \leq x \leq 1$. Hence, $f \notin C[0,1]$;

$$\therefore \text{ there is no } g \in C[0,1] \text{ s.t. } d_1(f_n, g) \to 0$$

□