Assignment 2
Topics of Mathematical Science

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Exercise: Q1

1. Show that:

$$
\begin{aligned}
&\frac{d^2}{dt^2}F(x(t), y(t)) \\
=&F_{xx}(x(t), y(t))\left(\frac{dx}{dt}(t)\right)^2 + 2F_{xy}(x(t), y(t))\frac{dx}{dt}(t)\frac{dy}{dt}(t) \\
+&F_{yy}(x(t), y(t))\left(\frac{dy}{dt}(t)\right)^2 + F_x(x(t), y(t))\frac{d^2x}{dt^2}(t) + F_y(x(t), y(t))\frac{d^2y}{dt^2}(t)
\end{aligned}
$$

Use fact that $F_{xy}(a, b) = F_{yx}(a, b)$ since F is of class C^2 around a point $P(a, b)$.

Answer:
Take second derivative of $F(x(t), y(t))$ over time, and use chain rule, so that:

$$
\begin{aligned}
\frac{d^2}{dt^2}F(x(t), y(t)) =&\frac{d}{dt}\left(\frac{d}{dt}\big(F(x(t), y(t))\big)\right) \\
=&\frac{d}{dt}\left(F_x(x(t), y(t))\frac{dx}{dt}(t) + F_y(x(t), y(t))\frac{dy}{dt}(t)\right) \\
=&F_{xx}(x(t), y(t))\left(\frac{dx}{dt}(t)\right)^2 + F_x(x(t), y(t))\frac{d^2x}{dt^2}(t) + F_{xy}(x(t), y(t))\frac{dx}{dt}(t) \\
&+ F_{yx}(x(t), y(t))\frac{dy}{dt}(t) + F_{yy}(x(t), y(t))\left(\frac{dy}{dt}(t)\right)^2 + F_y(x(t), y(t))\frac{d^2y}{dt^2}(t)
\end{aligned}
$$

Use fact that $F_{xy} = F_{yx}$

$$
\begin{aligned}
\therefore \frac{d^2}{dt^2}F(x(t), y(t)) =&F_{xx}(x(t), y(t))\left(\frac{dx}{dt}(t)\right)^2 + F_x(x(t), y(t))\frac{d^2x}{dt^2}(t) + 2F_{xy}(x(t), y(t))\frac{dx}{dt}(t)\frac{dy}{dt}(t) \\
&+ F_{yy}(x(t), y(t))\left(\frac{dy}{dt}(t)\right)^2 + F_y(x(t), y(t))\frac{d^2y}{dt^2}(t)
\end{aligned}
$$

2. Show that:

$$\varphi''(a) = -\frac{F_{xx}(a,b)F_y(a,b)^2 - 2F_{xy}(a,b)F_x(a,b)F_y(a,b) + F_{yy}(a,b)F_x(a,b)^2}{F_y(a,b)^3}$$

Answer:
Take second derivative of $F(x, \varphi(x))$ over x, and use chain rule, so that:

$$\begin{aligned}
\frac{d^2}{dx^2}F(x,\varphi(x)) =& \frac{d}{dx}\left(\frac{d}{dx}F(x,\varphi(x))\right)\\
=& \frac{d}{dx}\left(F_x(x,\varphi(x))F_y(x,\varphi(x)) + F_y(x,\varphi(x))\varphi'(x)\right)\\
=& F_{xx}(x,\varphi(x))F_y(x,\varphi(x))^2 + F_{xy}(x,\varphi(x))F_y(x,\varphi(x))\\
&+ F_{yx}(x,\varphi(x))F_y(x,\varphi(x)) + F_{yy}(x,\varphi(x))F_x(x,\varphi(x))^2 + \varphi''(x)F_y(x,\varphi(x))
\end{aligned}$$

Use fact that $F_{xy} = F_{yx}$

And bring all derivation of $F(x, \varphi(x))$ to the left side, hence:

$$\varphi''(x) = -\frac{F_{xx}(x,\varphi(x))F_y(x,\varphi(x))^2 + 2F_{xy}(x,\varphi(x))F_y(x,\varphi(x))F_x(x,\varphi(x)) + F_{yy}(x,\varphi(x))F_x(x,\varphi(x))^2}{F_y(x,\varphi(x))^3}$$

take $x = a$,

$$\therefore \varphi''(a) = -\frac{F_{xx}(a,b)F_y(a,b)^2 + 2F_{xy}(a,b)F_y(a,b)F_x(a,b) + F_{yy}(a,b)F_x(a,b)^2}{F_y(a,b)^3}$$

3. Let

$$D_1 = (x, y) \in \mathbb{R}^2 | 0 \leq x \leq 1,\ 0 \leq y \leq 2$$
$$D_2 = (x, y) \in \mathbb{R}^2 | 1 \leq x \leq 2,\ 0 \leq y \leq 1 + x^2$$

Calculate the following integral:

(a)

$$\iint_{D_1} xy^2 dxdy = \int_0^2 \int_0^1 xy^2 dxdy = \int_0^2 \frac{1}{2}x^2y^2 \Big|_0^1 dy = \int_0^2 \frac{1}{2}y^2 dy = \frac{1}{6}y^3 \Big|_0^2 = \frac{4}{3} \approx 1.333$$

(b)

$$\begin{aligned}
\iint_{D_1} (x+y)^2 dxdy &= \int_0^2 \int_0^1 (x+y)^2 dxdy \\
&= \int_0^2 \int_0^1 x^2 + 2xy + y^2 dxdy \\
&= \int_0^2 \frac{1}{3}x^3 + x^2y + y^2 \Big|_0^1 dy \\
&= \int_0^2 \frac{1}{3} + y + y^2 dy \\
&= \frac{1}{3}y + \frac{1}{2}y^2 + \frac{1}{3}y^3 \Big|_0^2 = \frac{16}{3} \approx 5.333
\end{aligned}$$

(c)

$$\begin{aligned}
\iint_{D_2} (x^2+y)^2 dxdy &= \int_1^2 \int_0^{1+x^2} (x^2+y)^2 dydx \\
&= \int_1^2 \int_0^{1+x^2} x^4 + 2x^2y + y^2 dydx \\
&= \int_1^2 x^4y + x^2y^2 + \frac{1}{3}y^3 \Big|_0^{1+x^2} dx \\
&= \int_1^2 x^4(1+x^2) + x^2(1+x^2)^2 + \frac{1}{3}(1+x^2)^3 dx \\
&= \int_1^2 4x^4 + \frac{7}{3}x^6 + 2x^2 + \frac{1}{3} dx \\
&= \frac{4}{5}x^5 + \frac{1}{3}x^7 + \frac{2}{3}x^3 + \frac{1}{3}x \Big|_1^2 \approx 72.133
\end{aligned}$$

4. Suppose D is a bounded domain with smooth boundary. Using Green Theorem, show that the line integral:

$$\int_{\partial D} -ydx + xdy \tag{1}$$

equal to the area of D.

Answer:

Theorem 1. *Green Theorem*
Let C be a positively oriented, piecewise smooth, simple closed curve in a plane, and let D be the region bounded by C. If L and M are functions of (x, y) defined on an open region containing D and have continuous partial derivatives there, then:

$$\oint_C (Ldx + Mdy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dxdy$$

We define area of D as:

$$\iint_D dxdy$$

using Green Theorem, we can rewrite (1) becomes:

$$\begin{aligned}\int_{\partial D} -ydx + xdy &= \iint_D \left(\frac{\partial(x)}{\partial x} - \frac{\partial(-y)}{\partial y}\right) dxdy \\ &= \iint_D 1 + 1dxdy \\ &=2 \iint_D dxdy = 2 \times \text{Area of } D\end{aligned}$$