Topics in Computational Science Report
Derivation of Position Function on Random Walk Process

Alifian Mahardhika Maulana

June 8, 2018

Problem 1. *For $\alpha \in \mathbb{R}^+$ and $x(t) : \mathbb{R}^+ \to \mathbb{R}$, Let:*

$$\frac{dx}{dt} = \alpha(1-x)x \tag{1}$$

Proof the following:

$$x(t) = \frac{1}{1+Ae^{-\alpha t}}, \text{ with } A = \frac{1-x(0)}{x(0)}$$

Proof.

$$\begin{aligned} \frac{dx}{dt} &= \alpha(1-x)x \\ \frac{dx}{(1-x)x} &= \alpha dt \\ \left(\frac{1}{x} + \frac{1}{(1-x)}\right)dx &= \alpha dt \end{aligned}$$

taking integral on the both side, we obtain:

$$\begin{aligned} \int \frac{1}{x}dx + \int \frac{1}{(1-x)}dx &= \int \alpha dt \\ \ln(x) - \ln(1-x) &= \alpha t + C \\ \ln\left(\frac{x}{(1-x)}\right) &= \alpha t + C \end{aligned}$$

multiplied by $e(\exp)$ on the both side, we obtain:

$$\begin{aligned} \exp\left(\ln\left(\frac{x}{(1-x)}\right)\right) &= \exp(\alpha t + C) \\ \frac{x}{(1-x)} &= e^{\alpha t + C} \end{aligned}$$

let $e^{\alpha t + C} = e^{\alpha t}e^{C} = Ce^{\alpha t}$, thus:

$$\frac{x}{(1-x)} = Ce^{\alpha t} \tag{2}$$

for $t = 0$, we get:

$$\frac{x(0)}{(1-x(0))} = C$$

insert C to equation (2), hence the equation becomes:

$$\begin{aligned} \frac{x}{(1-x)} &= \frac{x(0)}{(1-x(0))}e^{\alpha t} \\ \frac{(1-x)}{x} &= \frac{(1-x(0))}{x(0)}e^{-\alpha t} \\ \frac{1}{x} - 1 &= \frac{(1-x(0))}{x(0)}e^{-\alpha t} \end{aligned}$$

let $A = \frac{(1-x(0))}{x(0)}$ thus the equation becomes:

$$\begin{aligned} \frac{1}{x} - 1 &= Ae^{-\alpha t} \\ \frac{1}{x} &= 1 + Ae^{-\alpha t} \\ \therefore x &= \frac{1}{1+Ae^{-\alpha t}} \end{aligned}$$

□