Analysis Ia Report

Alifian Mahardhika Maulana

May 22, 2018

Problem 1. *Suppose:*

$$d_2(f,g) := \left(\int_a^b \{f(x) - g(x)\}^2 dx \right)^{\frac{1}{2}}, f : [a,b] \to \mathbb{R} \tag{1}$$

we will show $(C([a,b]), d_2)$ *is a* ***metric space****.*

(a) ***Positivity***
Take: $h = \int_a^b \{f(x) - g(x)\}^2 dx$*, then equation* (1) *becomes:*

$$d_2(f,g) = \sqrt{h}$$

Since the value of $(f(x) - g(x))^2 \geq 0$ *then* $\sqrt{h} \geq 0,\ \forall h \in C[a,b]$*, then:*

$$d_2(f,g) \geq 0$$

(b) ***Definiteness***
$(\Leftarrow)$ *put* $f(x) = g(x)$*, then equation* (1) *becomes:*

$$\begin{aligned} d_2(f,g) &= \sqrt{\int_a^b \{f(x) - g(x)\}^2 dx} \\ &= \sqrt{\int_a^b \{g(x) - g(x)\}^2 dx} \\ d_2(f,g) &= 0 \end{aligned}$$

$(\Rightarrow)$ *put* $d_2(f,g) = 0$*, then equation* (1) *becomes:*

$$\begin{aligned} 0 &= \sqrt{\int_a^b \{f(x) - g(x)\}^2 dx} \\ 0^2 &= \left(\sqrt{\int_a^b \{f(x) - g(x)\}^2 dx} \right)^2 \\ 0 &= \int_a^b \{f(x) - g(x)\}^2 dx \end{aligned}$$

to satisfies $(\Rightarrow),\ \forall a,b \in \mathbb{R}$

$$\begin{aligned} f(x) - g(x) &= 0 \\ \therefore f(x) &= g(x) \end{aligned}$$

(c) ***Symmetry***

$$
\begin{aligned}
d_2(f,g) &= \sqrt{\int_a^b \{f(x)-g(x)\}^2 dx} \\
d_2(f,g) &= \sqrt{\int_a^b \Big(f(x)^2 - 2f(x)g(x) + g(x)^2\Big) dx} \\
d_2(f,g) &= \sqrt{\int_a^b \Big(g(x)^2 - 2g(x)f(x) + f(x)^2\Big) dx} \\
d_2(f,g) &= \sqrt{\int_a^b \{g(x)-f(x)\}^2 dx} \\
d_2(f,g) &= d_2(g,f),\ \forall f,g \in C
\end{aligned}
$$

(d) ***Triangle Inequality***
Put $g(x) \le h(x) \le f(x)$

$$
d_2(f,g) = \sqrt{\int_a^b \{f(x)-h(x)+h(x)-g(x)\}^2 dx}
$$

then we take $r(x) = f(x) - h(x)$,*and* $s(x) = h(x) - g(x)$

$$
\begin{aligned}
d_2(f,g) &= \sqrt{\int_a^b r(x)^2 + 2r(x)s(x) + s(x)^2 dx} \\
&= \sqrt{\int_a^b r(x)^2 dx + 2\int_a^b r(x)s(x)dx + \int_a^b s(x)^2 dx}
\end{aligned}
$$

according to Cauchy-Schwartz inequality:

$$
\int_a^b r(x)s(x)dx \le \sqrt{\int_a^b r(x)^2 dx}\sqrt{\int_a^b s(x)^2 dx}
$$

thus:

$$
\begin{aligned}
d_2(f,g) &\le \sqrt{\int_a^b r(x)^2 dx + 2\sqrt{\int_a^b r(x)^2 dx}\sqrt{\int_a^b s(x)^2 dx} + \int_a^b s(x)^2 dx} \\
&\le \sqrt{\left(\sqrt{\int_a^b r(x)^2 dx} + \sqrt{\int_a^b s(x)^2 dx}\right)^2} \\
&\le \sqrt{\int_a^b r(x)^2 dx} + \sqrt{\int_a^b s(x)^2 dx} \\
d_2(f,g) &\le \sqrt{\int_a^b \{f(x)-h(x)\}^2 dx} + \sqrt{\int_a^b \{h(x)-g(x)\}^2 dx} \\
\therefore d_2(f,g) &\le d_2(f,h) + d_2(h,g)
\end{aligned}
$$

Problem 2. *Let (X, d) be a metric space, suppose:*

$$\tilde{d}(x, y) := \frac{d(x, y)}{1 + d(x, y)} \ (x, y \in X) \tag{2}$$

*we will show that $\tilde{d}(x, y)$ is also **metric** on X*

(a) ***Positivity***
Since $d(x, y)$ is a metric, it satisfies $d(x, y) \geq 0$, thus:

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)} \geq 0$$

(b) ***Definiteness***
($\Leftarrow$) *put $x = y$*
Since $d(x, y)$ is a metric, it satisfies $x = y \Rightarrow d(x, y) = 0$, thus:

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)} = \frac{0}{1 + 0} = 0$$

$$\therefore \tilde{d}(x, y) = 0 \Leftarrow x = y$$

($\Rightarrow$) *put $\tilde{d}(x, y) = 0$*

$$\begin{aligned} \tilde{d}(x, y) &= \frac{d(x, y)}{1 + d(x, y)} \\ 0 &= \frac{d(x, y)}{1 + d(x, y)} \end{aligned}$$

to satisfies the equation, $d(x, y) = 0$
and since $d(x, y)$ is a metric, it satisfies $d(x, y) = 0 \Rightarrow x = y$ then x should be equal to y

$$\therefore \tilde{d}(x, y) = 0 \Rightarrow x = y$$

(c) ***Symmetry***
Since $d(x, y)$ is a metric, it satisfies $d(x, y) = d(y, x)$,
then we can rewrite equation (2) as:

$$\begin{aligned} \tilde{d}(x, y) &= \frac{d(x, y)}{1 + d(x, y)} \\ &= \frac{d(y, x)}{1 + d(y, x)} = \tilde{d}(y, x) \\ \therefore \tilde{d}(x, y) &= \tilde{d}(y, x) \end{aligned}$$

(d) ***Triangle Inequality***
Since $d(x, y)$ is a metric, it satisfies $d(x, y) \leq d(x, z) + d(z, y)$, $\forall x, y, z \in X$,
then we can rewrite equation (2) as:

$$\begin{aligned} \tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)} &\leq \frac{d(x, z) + d(z, y)}{1 + d(x, z) + d(z, y)} \\ &\leq \frac{d(x, z)}{1 + d(x, z) + d(z, y)} + \frac{d(z, y)}{1 + d(x, z) + d(z, y)} \\ &\leq \tilde{d}(x, z) + \tilde{d}(z, y) \\ \therefore \tilde{d}(x, y) &\leq \tilde{d}(x, z) + \tilde{d}(z, y) \end{aligned}$$