Basics of Applied Analysis A Report

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Problem 1. *Discuss the stability of the difference schemes for the transport equation*

$$u_x + bu_x = 0 \tag{1}$$

using von Neumann stability analysis:

1. *"naive" explicit scheme*
$$\frac{v_k^{n+1} - v_k^n}{\tau} + b\frac{v_{k+1}^n - v_{k-1}^n}{2h} = 0 \tag{2}$$
2. *implicit scheme*
$$\frac{v_k^{n+1} - v_k^n}{\tau} + b\frac{v_{k+1}^{n+1} - v_{k-1}^{n+1}}{2h} = 0 \tag{3}$$
3. *Discuss the dissipation and dispersion properties of the implicit scheme in (b). Is it a satisfactory scheme for* (1)

Answer:

1. von Neumann stability analysis for "naive" explicit scheme:
we rewrite (2) become,
$$v_k^{n+1} = v_k^n - \frac{R}{2}(v_{k+1}^n - v_{k-1}^n), \quad R := \frac{b\tau}{h} \tag{4}$$
then substitute $v_{k+q} = e^{iq\xi}\hat{v}^n$ to (4), we get
$$\hat{v}^{n+1} = \hat{v}^n \left(1 - \frac{R}{2}(e^{i\xi} - e^{-i\xi})\right) \tag{5}$$
then we define, $g(\xi) = \left(1 - \frac{R}{2}(e^{i\xi} - e^{-i\xi})\right) = 1 - iR\sin(\xi)$, taking norm of $g(\xi)$ we get
$$|g(\xi)| = |1 - iR\sin(\xi)| = \sqrt{1 + R^2\sin^2(\xi)}, \quad \xi = (-\pi, \pi) \tag{6}$$
by (6) we get $|g(\xi)| > 1$, according to von Neumann stability, the explicit scheme (2) is unstable.
2. von Neumann stability analysis for implicit scheme:
we rewrite (3) become,
$$\frac{R}{2}\left(v_{k+1}^{n+1} - v_{k-1}^{n+1}\right) + v_k^{n+1} = v_k^n, \quad R := \frac{b\tau}{h} \tag{7}$$

then substitute $v_{k+q} = e^{iq\xi}\hat{v}^n$ to (7), we get

$$\begin{aligned} \left(1 + \frac{R}{2}\left(e^{i\xi} - e^{-i\xi}\right)\right)\hat{v}^{n+1} &= \hat{v}^n \\ \hat{v}^{n+1} &= \frac{1}{\left(1 + \frac{R}{2}\left(e^{i\xi} - e^{-i\xi}\right)\right)}\hat{v}^n \end{aligned} \tag{8}$$

then we define, $g(\xi) = \frac{1}{\left(1+\frac{R}{2}\left(e^{i\xi}-e^{-i\xi}\right)\right)} = \frac{1}{(1+iR\sin(\xi))}$, taking norm of $g(\xi)$ we get

$$|g(\xi)| = \left|\frac{1}{(1+iR\sin(\xi))}\right| = \frac{1}{\sqrt{1+R^2\sin^2(\xi)}}, \quad \xi = (-\pi, \pi) \tag{9}$$

by (9) we get $|g(\xi)| < 1$, according to von Neumann stability, the implicit scheme (3) is stable.

3. To analyze the dissipation and dispersion of (3), we substitute $v^{n+q}_{k+p} = e^{i(q\omega\tau + p\beta h)}$ to (7) we get,

$$\begin{aligned} \frac{R}{2}\left(e^{i(\omega\tau+\beta h)} - e^{i(\omega\tau-\beta h)}\right) + e^{i\omega\tau} &= 1, \quad R := \frac{b\tau}{h} \\ \left(\frac{R}{2}\left(e^{i\beta h} - e^{-i\beta h}\right) + 1\right)e^{i\omega\tau} &= 1 \\ (iR\sin(\beta h) + 1)\, e^{i\omega\tau} &= 1 \\ e^{i\omega\tau} &= \frac{1}{(iR\sin(\beta h) + 1)} \end{aligned} \tag{10}$$

taking norm of $e^{i\omega\tau}$, we get

$$|e^{i\omega\tau}| = e^{-\omega_2\tau} = \frac{1}{R^2\sin^2(\beta h) + 1} \tag{11}$$

by (11), $e^{-\omega_2\tau} < 1$, therefore according to von Neumann stability analysis, the implicit scheme on (3) is **dissipative**.
Then, to analyze the dispersion, we take $\arg(e^{i\omega\tau})$,

$$\begin{aligned} \arg(e^{i\omega\tau}) &= \arg(e^{i\omega_1\tau}) + \arg(e^{-\omega_2\tau}) \\ &= \omega_1\tau + 0 \end{aligned} \tag{12}$$

which, $\omega_1\tau = \arctan\left(\frac{Im(e^{i\omega\tau})}{Re(e^{i\omega\tau})}\right)$, we can get the real and imajiner part of $e^{i\omega\tau}$ by first multiplying it with it's rational factor,

$$e^{i\omega\tau} = \frac{1}{(iR\sin(\beta h)+1)}\frac{(iR\sin(\beta h)-1)}{(iR\sin(\beta h)-1)} = -\frac{iR\sin(\beta h)+1}{R^2\sin^2+1} \tag{13}$$

then, $\omega_1\tau = \arctan\left(\frac{Im(e^{i\omega\tau})}{Re(e^{i\omega\tau})}\right) = \frac{R\sin(\beta h)}{1} = R\sin(\beta h)$. Since $\omega_1\tau$ is not a constant, therefore, according to von Neumann stability analysis, the implicit scheme on (3) is **dispersive**.

Problem 2. *Show that the following implicit difference schemes for approximating the solution to*

$$u_t + bu_x = au_{xx} \tag{14}$$

are unconditionally stable using the von Neumann stability analysis. Here $R = b\frac{\tau}{h}, r = a\frac{\tau}{h^2}$.

1.

$$v_k^{n+1} + \frac{R}{2}(v_{k+1}^{n+1} - v_{k-1}^{n+1}) - r(v_{k+1}^{n+1} - 2v_k^{n+1} + v_{k-1}^{n+1}) = v_k^n \tag{15}$$

2.

$$v_k^{n+1} + \frac{R}{4}(v_{k+1}^{n+1} - v_{k-1}^{n+1}) - \frac{r}{2}(v_{k+1}^{n+1} - 2v_k^{n+1} + v_{k-1}^{n+1}) = v_k^n - \frac{R}{4}(v_{k+1}^n - v_{k-1}^n) + \frac{r}{2}(v_{k+1}^n - 2v_k^n + v_{k-1}^n) \tag{16}$$

Answer:

1. Substitute $v_{k+q} = e^{iq\xi}\hat{v}^n$ to (15) we get,

$$\begin{aligned}
\hat{v}^{n+1} + \frac{R}{2}(e^{i\xi} - e^{-i\xi})\hat{v}^{n+1} - r(e^{i\xi} - 2 + e^{-i\xi})\hat{v}^{n+1} &= \hat{v}^n \\
\hat{v}^{n+1}\left(1 + \frac{R}{2}(e^{i\xi} - e^{-i\xi}) - r(e^{i\xi} - 2 + e^{-i\xi})\right) &= \hat{v}^n \\
\hat{v}^{n+1} &= \frac{1}{\left(1 + \frac{R}{2}(e^{i\xi} - e^{-i\xi}) - r(e^{i\xi} - 2 + e^{-i\xi})\right)}\hat{v}^n
\end{aligned} \tag{17}$$

then, we define:

$$\begin{aligned}
g(\xi) &= \frac{1}{\left(1 + \frac{R}{2}(e^{i\xi} - e^{-i\xi}) - r(e^{i\xi} - 2 + e^{-i\xi})\right)} = \frac{1}{\left(1 + iR\sin(\xi) + r(2\cos(\xi) - 2)\right)} \\
&= \frac{1}{\left(1 + iR\sin(\xi) + r(-4\sin^2(\frac{\xi}{2}))\right)}
\end{aligned} \tag{18}$$

by (18), $g(\xi) < 1$, according to von Neumann stability analysis, if $g(\xi) < 1$ the scheme will be unconditionally stable.

2. Substitute $v_{k+q} = e^{iq\xi}\hat{v}^n$ to (16) we get,

$$\begin{aligned}
\hat{v}^{n+1} + \frac{R}{4}(e^{i\xi} - e^{-i\xi})\hat{v}^{n+1} - \frac{r}{2}(e^{i\xi} - 2 + e^{-i\xi})\hat{v}^{n+1} &= \hat{v}^n - \frac{R}{4}(e^{i\xi} - e^{-i\xi})\hat{v}^n + \frac{r}{2}(e^{i\xi} - 2 + e^{-i\xi})\hat{v}^n \\
\hat{v}^{n+1}\left(1 + \frac{R}{4}(e^{i\xi} - e^{-i\xi}) - \frac{r}{2}(e^{i\xi} - 2 + e^{-i\xi})\right) &= \hat{v}^n\left(1 - \frac{R}{4}(e^{i\xi} - e^{-i\xi}) + \frac{r}{2}(e^{i\xi} - 2 + e^{-i\xi})\right) \\
\hat{v}^{n+1} &= \frac{\left(1 - \frac{R}{4}(e^{i\xi} - e^{-i\xi}) + \frac{r}{2}(e^{i\xi} - 2 + e^{-i\xi})\right)}{\left(1 + \frac{R}{4}(e^{i\xi} - e^{-i\xi}) - \frac{r}{2}(e^{i\xi} - 2 + e^{-i\xi})\right)}\hat{v}^n
\end{aligned} \tag{19}$$

then, we define:

$$
\begin{aligned}
g(\xi) &= \frac{\left(1 - \frac{R}{4}(e^{i\xi} - e^{-i\xi}) + \frac{r}{2}(e^{i\xi} - 2 + e^{-i\xi})\right)}{\left(1 + \frac{R}{4}(e^{i\xi} - e^{-i\xi}) - \frac{r}{2}(e^{i\xi} - 2 + e^{-i\xi})\right)} = \frac{\left(1 - \frac{R}{4}(2i\sin(\xi)) + \frac{r}{2}(2\cos(\xi) - 2)\right)}{\left(1 + \frac{R}{4}(2i\sin(\xi)) - \frac{r}{2}(2\cos(\xi) - 2)\right)} \\
&= \frac{\left(1 - \frac{R}{4}(2i\sin(\xi)) + \frac{r}{2}(-4\sin^2(\frac{\xi}{2}))\right)}{\left(1 + \frac{R}{4}(2i\sin(\xi)) - \frac{r}{2}(-4\sin^2(\frac{\xi}{2}))\right)}
\end{aligned}
\tag{20}
$$

by (20), $g(\xi) < 1$, according to von Neumann stability analysis, if $g(\xi) < 1$ the scheme will be unconditionally stable.

Problem 3. *Discuss the dissipation and dispersion of the following implicit numerical schemes for the wave equation*

1.

$$
\frac{v_k^{n+1} - 2v_k^n + v_k^{n-1}}{\tau^2} = \frac{v_{k+1}^{n+1} - 2v_k^{n+1} + v_{k-1}^{n+1}}{h^2} \tag{21}
$$

2.

$$
\frac{v_k^{n+1} - 2v_k^n + v_k^{n-1}}{\tau^2} = \frac{v_{k+1}^{n+1} - 2v_k^{n+1} + v_{k-1}^{n+1}}{2h^2} + \frac{v_{k+1}^{n-1} - 2v_k^{n-1} + v_{k-1}^{n-1}}{2h^2} \tag{22}
$$

Answer:

1. Substitute $v_{k+p}^{n+q} = e^{i(q\omega\tau + p\beta h)}\hat{v}^n$ with $R = \frac{\tau}{h}$ to (21) we get,

$$
\begin{aligned}
\frac{(e^{i\omega\tau} - 2 + e^{-i\omega\tau})}{\tau^2}\hat{v}^n &= \frac{(e^{i(\omega\tau+\beta h)} - 2e^{i\omega\tau} + e^{i(\omega\tau-\beta h)})}{h^2}\hat{v}^n \\
e^{i\omega\tau} - 2 + e^{-i\omega\tau} &= R^2(2\cos(\beta h) - 2)e^{i\omega\tau}
\end{aligned}
\tag{23}
$$

take $g = e^{i\omega\tau}$, (23) become

$$
\begin{aligned}
-2 + g^{-1} + g\left(1 - R^2(2\cos(\beta h) - 2)\right) &= 0 \\
-2 + g^{-1} + g\left(1 - R^2(-4\sin^2(\frac{\beta h}{2}))\right) &= 0
\end{aligned}
\tag{24}
$$

multiple by g we get

$$
g^2(1 - R^2(-4\sin^2(\frac{\beta h}{2}))) - 2g + 1 = 0 \tag{25}
$$

solving (25) we get,

$$
g = e^{i\omega\tau} = \frac{1 \pm i2R\sin(\frac{\beta h}{2})}{(1 - R^2(-4\sin^2(\frac{\beta h}{2})))} \tag{26}
$$

taking norm of (26), we get

$$
|e^{i\omega\tau}| = e^{i\omega_2\tau} = \max_{-\pi \le \frac{\beta h}{2} \le \pi} \left| \frac{1 \pm i2R\sin(\frac{\beta h}{2})}{(1 - R^2(-4\sin^2(\frac{\beta h}{2})))} \right| = \frac{1 \pm 2R}{1 + 4R^2} \tag{27}
$$

from (27), $e^{i\omega_2\tau} < 1$, therefore according to von Neumann stability analysis, the implicit numerical scheme on (21) is **dissipative**.

Then, to analyze the dispersion, we take $\arg(e^{i\omega\tau})$,

$$\begin{aligned}\arg(e^{i\omega\tau}) &= \arg(e^{i\omega_1\tau}) + \arg(e^{-\omega_2\tau}) \\ &= \omega_1\tau + 0\end{aligned} \tag{28}$$

which, $\omega_1\tau = \arctan\left(\frac{Im(e^{i\omega\tau})}{Re(e^{i\omega\tau})}\right)$, then, $\omega_1\tau = \arctan\left(\frac{Im(e^{i\omega\tau})}{Re(e^{i\omega\tau})}\right) = \frac{2R\sin(\beta h)}{1} = 2R\sin(\beta h)$. Since $\omega_1\tau$ is not a constant, therefore, according to von Neumann stability analysis, the implicit scheme on (21) is **dispersive**.

2. Substitute $v_{k+p}^{n+q} = e^{i(q\omega\tau+p\beta h)}\hat{v}^n$ with $R = \frac{\tau^2}{2h^2}$ to (22) we get,

$$\begin{aligned}&\frac{(e^{i\omega\tau} - 2 + e^{-i\omega\tau})}{\tau^2}\hat{v}^n = \frac{(e^{i(\omega\tau+\beta h)} - 2e^{i\omega\tau} + e^{i(\omega\tau-\beta h)})}{2h^2}\hat{v}^n + \frac{(e^{i(-\omega\tau+\beta h)} - 2e^{-i\omega\tau} + e^{-i(\omega\tau+\beta h)})}{2h^2}\hat{v}^n \\ &(e^{i\omega\tau} - 2 + e^{-i\omega\tau}) = R\left((2\cos(\beta h) - 2)e^{i\omega\tau} + (2\cos(\beta h) - 2)e^{-i\omega\tau}\right) \\ &(e^{i\omega\tau} - 2 + e^{-i\omega\tau}) - R\left((2\cos(\beta h) - 2)e^{i\omega\tau} + (2\cos(\beta h) - 2)e^{-i\omega\tau}\right) = 0 \\ &e^{i\omega\tau}\left(1 - R(2\cos(\beta h) - 2)\right) + e^{-i\omega\tau}\left(1 - R(2\cos(\beta h) - 2)\right) - 2 = 0\end{aligned} \tag{29}$$

take $g = e^{i\omega\tau}$, (29) become

$$\begin{aligned}&g\left(1 - R(2\cos(\beta h) - 2)\right) + g^{-1}\left(1 - R(2\cos(\beta h) - 2)\right) - 2 = 0 \\ &g\left(1 - R(-4\sin^2(\frac{\beta h}{2})\right) + g^{-1}\left(1 - R(-4\sin^2(\frac{\beta h}{2})\right) - 2 = 0\end{aligned} \tag{30}$$

multiple by g, we get

$$g^2\left(1 - R(-4\sin^2(\frac{\beta h}{2})\right) + \left(1 - R(-4\sin^2(\frac{\beta h}{2})\right) - 2g = 0 \tag{31}$$

solving (31) we get,

$$g = e^{i\omega\tau} = \frac{2 \pm \sqrt{4 - 4(1 + 4R\sin^2(\frac{\beta h}{2}))^2}}{2(1 + 4R\sin^2(\frac{\beta h}{2}))} \tag{32}$$

taking norm of (32), we get

$$|e^{i\omega\tau}| = e^{i\omega_2\tau} = \max_{-\pi\leq\frac{\beta h}{2}\leq\pi}\left|\frac{2 \pm \sqrt{4 - 4(1 + 4R\sin^2(\frac{\beta h}{2}))^2}}{2(1 + 4R\sin^2(\frac{\beta h}{2}))}\right| = \frac{2 \pm (1 + 4R)}{2(1 + 4R)} \tag{33}$$

from (33), $e^{i\omega_2\tau} = 1$, therefore according to von Neumann stability analysis, the implicit numerical scheme on (22) is **non-dissipative**.

Then, to analyze the dispersion, we take $\arg(e^{i\omega\tau})$,

$$\begin{aligned}\arg(e^{i\omega\tau}) &= \arg(e^{i\omega_1\tau}) + \arg(e^{-\omega_2\tau}) \\ &= \omega_1\tau + 0\end{aligned} \tag{34}$$

which, $\omega_1\tau = \arctan\left(\frac{Im(e^{i\omega\tau})}{Re(e^{i\omega\tau})}\right)$, then, $\omega_1\tau = \arctan\left(\frac{Im(e^{i\omega\tau})}{Re(e^{i\omega\tau})}\right) = 0$. Since $\omega_1\tau$ is 0 because there is no imaginer part, therefore, according to von Neumann stability analysis, the implicit scheme on (22) is **non-dispersive**.