Linear Elasticity Modelling in 2D and 3D Using Finite Element Method

Alifian Mahardhika Maulana

May 26, 2018

1 Basic Theory

We define:

$$
\begin{aligned}
&\Omega \subset \mathbb{R}^d \ (d = 2, 3) \\
&u : \Omega \to \mathbb{R}^d \text{ (displacement)} \\
&e[u] := \frac{1}{2}(\nabla^T u + \nabla u^T) \text{ (strain)} \\
&\sigma[u] := \mathcal{C}e[u] \\
&\mathcal{C} = (C_{ijkl}) \begin{cases} C_{ijkl} = C_{klij} = C_{jikl} \\ (C_\xi) : \xi \geq C_* |\xi|^2 (\forall \xi \in \mathbb{R}^{d\times d}_{sym}) \end{cases}
\end{aligned}
$$

Let's consider linear elasticity problem:

$$
(**) \begin{cases} -div\ \sigma[u] = f(x), \text{ in } \Omega \\ u = g(x) \text{ on } \Gamma_D \\ \sigma[u]\nu = q(x) \text{ on } \Gamma_N \end{cases} \tag{1}
$$

$$
f \in L^2(\Omega : \mathbb{R}^d,\ g \in H^1(\Omega : \mathbb{R}^d),\ q \in L^2(\Gamma_N : \mathbb{R}^d))
$$

1.1 Strong Solution

$$
u \in H^2(\Omega : \mathbb{R}^d) \text{ satisfies } (**) \text{ then we call } u : \text{a strong solution}
$$

1.2 Weak Solution

$$
\begin{cases} \int_\Omega \sigma[u] : e[v]dx = \int_\Omega f \cdot vdx + \int_{\Gamma_N} q \cdot vds \big(\forall v \in V := \{v \in H^1\ (\Omega : \mathbb{R}^d)\ |v|_{\Gamma_D} = 0\}\big) \\ u \in V + g \end{cases}
$$

1.3 Properties

$$
u : \text{ strong solution } \Leftrightarrow \begin{cases} u : \text{ weak solution} \\ u \in H^2\ (\Omega : \mathbb{R}^d) \end{cases}
$$

$$
\begin{aligned}
&X := H^1(\Omega : \mathbb{R}^d) \\
&a(u, v) := \int_\Omega \sigma[u] : e[v]dx \\
&l(v) := \int_\Omega f \cdot vdx + \int_{\Gamma_N} q \cdot vds
\end{aligned}
\qquad
\begin{aligned}
a(u, v) &= \int_\Omega (\mathcal{C}e[u]) : e[v]dx \\
&= \int_\Omega e[v] : (\mathcal{C}e[u])dx \\
&= a(v, u)
\end{aligned}
$$

For $v \in V$

$$
\begin{aligned}
a(v, v) &= \int_\Omega (\mathcal{C}e[v]) : e[v]dx \\
&\geqq C_* \int_\Omega |e[v]|^2 dx \\
&\geqq C_* ||v||_x^2
\end{aligned}
$$

Properties 1.

- *$a(\cdot,\cdot)$ is bounded symmetric, bilinear form on $X \times X$.*
- *$a(\cdot,\cdot)$ is coercive on $V \times V$.*
- *l is bounded linear form on X.*

Theorem 1. *For any $g \in H^1(\Omega : \mathbb{R}^d)$,*

$$\exists! u : \textit{a weak solution of } (**), \textit{ and } \begin{cases} u = \operatorname{argmin}_{w \in V+g} E(w) \end{cases}$$