Basics of Applied Analysis A
Report

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Problem 1. *Let $N \in \mathbb{N}, N \geq 2$, and $\alpha > 0$ be given.*
We define $A = (aij)$ as finite difference discretization of :

$$\begin{cases} \alpha u - u'' = f \ \ in \ (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{1}$$

with entries

$$a_{ij} = \begin{cases} \alpha + \frac{2}{h^2}, & i = j \\ -\frac{1}{h^2}, & |i - j| = 1 \\ 0 & otherwise \end{cases} \tag{2}$$

where $h = \frac{1}{N}$

  (a) *To find eigenvalues λ s.t. $Av = \lambda v$, We define $v_i := \sin\left(\frac{m\pi i}{N}\right)$ for some $m \in \mathbb{N}$, hence the discrete form of eigen problem becomes:*

$$\sum_{i=1}^{N} a_{ij} v_j = \lambda v_i$$

$$\left(\alpha + \frac{2}{h^2}\right) v_i - \frac{1}{h^2} v_{i+1} - \frac{1}{h^2} v_{i-1} = \lambda v_i$$

$$\left(\alpha + \frac{2}{h^2} - \lambda\right) v_i - \frac{1}{h^2}(v_{i+1} + v_{i-1}) = 0$$

substitute v_i to the discrete eigen problem, it becomes

$$\left(\alpha + \frac{2}{h^2} - \lambda\right) \sin\left(\frac{m\pi i}{N}\right) - \frac{1}{h^2}\left(\sin\left(\frac{m\pi(i+1)}{N}\right) + \sin\left(\frac{m\pi(i-1)}{N}\right)\right) = 0$$

$$\left(\alpha + \frac{2}{h^2} - \lambda\right) \sin\left(\frac{m\pi i}{N}\right) - \frac{1}{h^2}\left(2\sin\left(\frac{m\pi i}{N}\right)\cos\left(\frac{m\pi}{N}\right)\right) = 0$$

$$\sin\frac{m\pi i}{N}\left(\alpha + \frac{2}{h^2} - \lambda - \frac{1}{h^2}2\cos\left(\frac{m\pi}{N}\right)\right) = 0$$

to find λ, we set $\sin\frac{m\pi i}{N}$ equal to some constant C, so that:

$$\left(\alpha + \frac{2}{h^2} - \lambda - \frac{2}{h^2}\cos\left(\frac{m\pi}{N}\right)\right) = 0$$

$$\therefore \lambda = \alpha + \frac{2}{h^2}\left(1 - \cos\left(\frac{m\pi}{N}\right)\right) \tag{3}$$

With Eigenvectors:

$$\therefore v = \sin\left(\frac{m\pi i}{N}\right)$$

(b) We define spectral radius $\sigma(A) := \max\{|\lambda|\}$, using λ computed in (3), we get:

$$\sigma(A) = \max\{|\lambda|\}$$
$$= \alpha + \frac{2}{h^2}\left(1 - \cos\left(\frac{m\pi}{N}\right)\right)$$

then, thanks to the symmetricity of λ, we can use taylor expansion to approximate the value of

$$\cos\left(\frac{m\pi}{N}\right) \approx 1 - \frac{1}{2}\left(\frac{\pi}{N}\right)^2$$

, hence the spectral radius $\sigma(A)$ can be calculated by:

$$\sigma(A) \approx \alpha + \frac{2}{h^2}\left(1 - \left(1 - \frac{m^2\pi^2}{2N^2}\right)\right)$$
$$\approx \alpha + \frac{2}{h^2}\frac{m^2\pi^2}{2N^2}$$
$$\therefore \sigma(A) \approx \alpha + (m\pi)^2$$

(c) Let:

$$R := -D^{-1}(L + U)$$

be the Jacobi iteration matrix, we want to find the eigenvalues and eigenvectors s.t. $Rv = \lambda v$.

$$Rv = \lambda v$$
$$-D^{-1}(L + U)v = \lambda v$$
$$-(L + U)v = \lambda D v$$

the discrete form of the eigen problem is:

$$\frac{1}{h^2}(v_{i-1} + v_{i+1}) = \lambda\left(\alpha + \frac{2}{h^2}\right)v_i$$

and then we set, $v_i := \sin\left(\frac{m\pi i}{N}\right)$ and substitute to the discrete form of the eigen problem, thus:

$$\frac{1}{h^2}\left(\sin\left(\frac{m\pi(i+1)}{N}\right) + \sin\left(\frac{m\pi(i-1)}{N}\right)\right) = \lambda\left(\alpha + \frac{2}{h^2}\right)\sin\left(\frac{m\pi i}{N}\right)$$
$$\frac{1}{h^2}\left(2\sin\left(\frac{m\pi i}{N}\right)\cos\left(\frac{m\pi}{N}\right)\right) = \lambda\left(\alpha + \frac{2}{h^2}\right)\sin\left(\frac{m\pi i}{N}\right)$$
$$\sin\frac{m\pi i}{N}\left(\frac{2}{h^2}\cos\left(\frac{m\pi}{N}\right) - \lambda\left(\alpha + \frac{2}{h^2}\right)\right) = 0$$

to find λ, we set $\sin\frac{m\pi i}{N}$ equal to some constant D, so that:

$$\left(\frac{2}{h^2}\cos\left(\frac{m\pi}{N}\right) - \lambda\left(\alpha + \frac{2}{h^2}\right)\right) = 0$$
$$\lambda\left(\alpha + \frac{2}{h^2}\right) = \frac{2}{h^2}\cos\left(\frac{m\pi}{N}\right)$$

with $h = \frac{1}{N}$

$$\lambda = \frac{\frac{2}{h^2}\cos\left(\frac{m\pi}{N}\right)}{\alpha + \frac{2}{h^2}}$$

$$(4)$$

$$2$$

with $h = \frac{1}{N}$,

$$\therefore \lambda = \frac{2N^2 \cos\left(\frac{m\pi}{N}\right)}{\alpha + 2N^2}$$

(d) We define spectral radius $\sigma(R) := \max\{|\lambda|\}$, using λ computed in (4), we get:

$$\sigma(R) = \max\{|\lambda|\}$$
$$= \frac{2N^2 \cos\left(\frac{m\pi}{N}\right)}{\alpha + 2N^2}$$

again, thanks to the symmetricity of λ, we can use taylor expansion to approximate the value of

$$\cos\left(\frac{m\pi}{N}\right) \approx 1 - \frac{1}{2}\left(\frac{m\pi}{N}\right)^2$$

, hence the spectral radius $\sigma(R)$ can be calculated by:

$$\therefore \sigma(R) \approx \frac{2N^2\left(1 - \frac{m^2\pi^2}{2N^2}\right)}{\alpha + 2N^2}$$

Problem 2. *Let $N, h = \frac{1}{N}, \alpha$ and A be as in Problem 1. We define:*

$$\begin{cases} \alpha u - u'' = \sin(\pi x) \ in \ (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{5}$$

(a) *To find the exact solution of u, we use general solution for Ordinary Differential Equation (ODE):*

$$u(x) = A\sin(\pi x) + B\cos(\pi x) \tag{6}$$

and take the second derivative of u

$$u''(x) = -A\pi^2 \sin(\pi x) + B\pi^2 \cos(\pi x) \tag{7}$$

then substitute (6) and (7) to (5), we get:

$$\begin{aligned} \alpha(A\sin(\pi x) + B\cos(\pi x)) + A\pi^2 \sin(\pi x) + B\pi^2 \cos(\pi x) &= \sin(\pi x) \\ A(\alpha + \pi^2)\sin(\pi x) + B(\alpha + \pi^2)\cos(\pi x) &= \sin(\pi x) \end{aligned} \tag{8}$$

from (8) we know that

$$A(\alpha + \pi^2)\sin(\pi x) = \sin(\pi x)$$

and

$$B(\alpha + \pi^2)\cos(\pi x) = 0$$

$$\therefore A = \frac{1}{(\alpha + \pi^2)} \quad \therefore B = 0 \tag{9}$$

then we substitute (9) to (6), we get the exact solution as:

$$u(x) = \frac{\sin(\pi x)}{(\alpha + \pi^2)} \tag{10}$$

(b) *To find exact solution s.t. $Av = b$, with $b_i = \sin(\pi h i)$, we set*

$$v_i = C\sin(\pi h i)$$

thus,

$$Av = b$$

$$\sum_{i=1}^{N-1} a_{ij} v_i = b_i$$

$$\left(\left(\alpha + \frac{2}{h^2} \right) v_i - \frac{1}{h^2}\left(v_{i+1} + v_{i-1} \right) \right) = \sin(\pi h i)$$

$$\left(\left(\alpha + \frac{2}{h^2} \right) C\sin(\pi h i) - \frac{1}{h^2}\left(C\sin(\pi h(i+1)) + C\sin(\pi h(i-1)) \right) \right) = \sin(\pi h i)$$

$$\left(\left(\alpha + \frac{2}{h^2} \right) C\sin(\pi h i) - C\frac{1}{h^2}\left(2\sin(\pi h i)\cos(\pi h) \right) \right) = \sin(\pi h i)$$

divide by $\sin(\pi h i)$ on the both side we get:

$$C\left(\left(\alpha + \frac{2}{h^2} \right) - \frac{2}{h^2}\cos(\pi h) \right) = 1$$

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$$\therefore C = \frac{1}{\left(\left(\alpha + \frac{2}{h^2}\right) - \frac{2}{h^2}\cos(\pi h)\right)}$$

so the exact solution in terms of v_i is:

$$v_i = \frac{\sin(\pi h i)}{\left(\left(\alpha + \frac{2}{h^2}\right) - \frac{2}{h^2}\cos(\pi h)\right)}$$

(c) We define

$$\epsilon(h) := \max_{1 \leq i \leq N-1} |u(hi) - v_i|$$

then we use $u(x)$ and v_i that we derived before to find the explicit formula of $\epsilon(h)$:

$$\epsilon(h) := \max_{1 \leq i \leq N-1} \left| \frac{\sin(\pi h i)}{(\alpha + \pi^2)} - \frac{\sin(\pi h i)}{\left(\left(\alpha + \frac{2}{h^2}\right) - \frac{2}{h^2}\cos(\pi h)\right)} \right|$$

$$\epsilon(h) := \max_{1 \leq i \leq N-1} \left| \sin(\pi h i) \left(\frac{1}{(\alpha + \pi^2)} - \frac{1}{\left(\left(\alpha + \frac{2}{h^2}\right) - \frac{2}{h^2}\cos(\pi h)\right)} \right) \right|$$

for simplicity, we rewrite:

$$D := \sin(\pi h i)$$

$$E := \left(\frac{1}{(\alpha + \pi^2)} - \frac{1}{\left(\left(\alpha + \frac{2}{h^2}\right) - \frac{2}{h^2}\cos(\pi h)\right)} \right)$$

$$\max_{1 \leq i \leq N-1} |DE| = \max_{1 \leq i \leq N-1} |D| \max_{1 \leq i \leq N-1} |E|$$

$$= 1 \left(\frac{1}{(\alpha + \pi^2)} - \frac{1}{\left(\left(\alpha + \frac{2}{h^2}\right) - \frac{2}{h^2}\cos(\pi h)\right)} \right)$$

then, the explicit formula for $\epsilon(h)$ is:

$$\epsilon(h) = \left(\frac{1}{(\alpha + \pi^2)} - \frac{1}{\left(\left(\alpha + \frac{2}{h^2}\right) - \frac{2}{h^2}\cos(\pi h)\right)} \right) \tag{11}$$

around $h = 0$, leading order of taylor expansion of $\epsilon(h)$ can be obtained by expand $\cos(\pi h)$, so that

$$\cos(\pi h) \approx \cos(a) + \frac{\cos'(a)(\pi h - a)}{1!} + \frac{\cos''(a)(\pi h - a)^2}{2!} + \cdots$$

then we take $a = 0$, we get

$$\cos(\pi h) \approx 1 - \frac{1}{2}(\pi h)^2$$

substitute it to (11) we get:

$$\epsilon(h) = \left(\frac{1}{(\alpha + \pi^2)} - \frac{1}{\left(\left(\alpha + \frac{2}{h^2}\right) - \frac{2}{h^2}(1 - \frac{1}{2}(\pi h)^2)\right)} \right)$$

$$\epsilon(h) = \left(\frac{1}{(\alpha + \pi^2)} - \frac{1}{(\alpha + \pi^2)} \right)$$

which is just a constant, so the leading order is depend on what order of taylor expansion we choose to approximate the value of $\cos(\pi h)$ around $h = 0$. In this case if we choose taylor expansion orde 2, we get the leading term of ϵ is $\mathbb{O}(F)$ with F is a constant.

Problem 3. *We consider system of linear equations:*

$$\begin{cases} -v_{i-1,j} - v_{i+1,j} - v_{i,j-1} - v_{i,j+1} + 4v_{i,j} = b_{i,j} & i,j = 1, \cdots, N-1 \\ v_{0,j} = v_{0,N} = v_{i,0} = v_{i,N} = 0 & i,j = 1, \cdots, N-1 \end{cases} \tag{12}$$

for unknowns $v_{i,j}, i,j = 1, \cdots, N-1$.

(a) *We want to find the eigenvalues and eigenvectors of the matrix A for the system (12). Using the idea of discrete separation variables, we set $w_{i,j} = v_i \tilde{v}_j$ s.t.*

$$Aw_{i,j} = \lambda w_{i,j}$$

$$4w_{i,j} - w_{i-1,j} - w_{i+1,j} - w_{i,j-1} - w_{i,j+1} = \lambda w_{i,j}$$
$$4(v_i\tilde{v}_j) - (v_{i-1}\tilde{v}_j) - (v_{i+1}\tilde{v}_j) - (v_i\tilde{v}_{j-1}) - (v_i\tilde{v}_{j+1}) = \lambda(v_i\tilde{v}_j)$$
$$4(v_i\tilde{v}_j) - \tilde{v}_j(v_{i-1} + v_{i+1}) - v_i(\tilde{v}_{j-1} + \tilde{v}_{j+1}) = \lambda(v_i\tilde{v}_j)$$
$$(4-\lambda)(v_i\tilde{v}_j) - \tilde{v}_j(v_{i-1} + v_{i+1}) - v_i(\tilde{v}_{j-1} + \tilde{v}_{j+1}) = 0$$

then, divide by $v_i\tilde{v}_j$ on both side, we get:

$$(4-\lambda) - \frac{(v_{i-1} + v_{i+1})}{v_i} - \frac{(\tilde{v}_{j-1} + \tilde{v}_{j+1})}{\tilde{v}_j} = 0$$

$$(4-\lambda) - \frac{(v_{i-1} + v_{i+1})}{v_i} = \frac{(\tilde{v}_{j-1} + \tilde{v}_{j+1})}{\tilde{v}_j}$$

now, to satisfies equality, both side should be equal to some constant, let say G, hence:

$$(4-\lambda) - \frac{(v_{i-1} + v_{i+1})}{v_i} = G = \frac{(\tilde{v}_{j-1} + \tilde{v}_{j+1})}{\tilde{v}_j}$$

First, we solve for:

$$\frac{(\tilde{v}_{j-1} + \tilde{v}_{j+1})}{\tilde{v}_j} = G$$

$$(\tilde{v}_{j-1} + \tilde{v}_{j+1}) = \tilde{v}_j G$$

set: $\tilde{v}_j = \varphi^j$

$$\varphi^{j-1} + \varphi^{j+1} = \varphi^j G$$
$$\varphi^{j-1} + \varphi^{j+1} - \varphi^j G = 0$$

divide both side by φ^{j-1}

$$1 + \varphi^2 - \varphi G = 0$$
$$\therefore \varphi_{\pm} = \frac{G \pm \sqrt{(G^2 - 4)}}{2} \tag{13}$$

here we recall that eigenvector of $\tilde{v}_j$ should be a linear combination of φ s.t.

$$\tilde{v}_j = c_1\varphi_+^j + c_2\varphi_-^j$$

and we know the boundary condition on (12), when $j = N$ then $\tilde{v}_N = 0$, to satisfies this condition, then $c_1 = -c_2$, then

$$\tilde{v}_j = c_1(\varphi_+^j - \varphi_-^j)$$

$\therefore \varphi_+^j$ and φ_-^j should be distinguished.
Hence, we choose $G < 2$ for (13) we then rewrite (13) as:

$$\varphi_{\pm} = \frac{G \pm \sqrt{-(4 - G^2)}}{2}$$

$$\varphi_{\pm} = \frac{G \pm i\sqrt{(4 - G^2)}}{2}$$

then, the possible solution is

$$\varphi_\pm^j = \cos(\theta j) \pm i \sin(\theta j)$$

$$\therefore \tilde{v}_j = c_1(\cos(\theta j) + i \sin(\theta j) - (\cos(\theta j) - i \sin(\theta j)))$$
$$\tilde{v}_j = c_1(2i \sin(\theta j))$$

we choose $c_1 = -\frac{i}{2}$ so that

$$\tilde{v}_j = \sin(\theta j)$$

inserting boundary condition, when $j = N$ then $\tilde{v}_N = 0$, we get:

$$0 = \sin(\theta N)$$
$$\theta = m\pi, \ \forall m : 2, 4, 6 \cdots N, \ even$$
$$\theta = \frac{m\pi}{N}$$

then, we know that from imaginer triangle

$$\cos\theta = \frac{G}{2}$$

$\therefore G = 2\cos(\theta) = 2\cos(\frac{m\pi}{N})$

Second, we solve for:

$$(4 - \lambda) - \frac{(v_{i-1} + v_{i+1})}{v_i} = G$$

multiplied both side by v_i, we get:

$$v_i(4 - \lambda) - (v_{i-1} + v_{i+1}) = v_i G$$
$$v_i(4 - \lambda) - (v_{i-1} + v_{i+1}) - v_i G = 0$$
$$v_i(4 - \lambda - G) - (v_{i-1} + v_{i+1}) = 0$$

set $v_i = \xi^i$

$$\xi^i(4 - \lambda - G) - (\xi^{i-1} + \xi^{i+1}) = 0$$

divide both side by ξ^{i-1}

$$\xi(4 - \lambda - G) - (1 + \xi^2) = 0$$
$$\xi^2 - \xi(4 - \lambda - G) + 1 = 0$$
$$\therefore \xi_\pm = \frac{(4 - \lambda - G) \pm \sqrt{(4 - \lambda - G)^2 - 4}}{2}$$

take $2H = (4 - \lambda - G)$

$$\xi_\pm = \frac{2H \pm \sqrt{(2H)^2 - 4}}{2} \tag{14}$$
$$\xi_\pm = H \pm \sqrt{H^2 - 1}$$

here we recall that eigenvector of v_i should be a linear combination of ξ s.t.

$$v_i = d_1 \xi_+^i + d_2 \xi_-^i$$

and we know the boundary condition on (12), when $i = N$ then $v_N = 0$, to satisfies this condition, then $d_1 = -d_2$, then

$$v_i = d_1(\xi +^j - \xi_-^j)$$

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∴ ξ_+^j and ξ_-^j should be distinguished.

Hence, we choose $H < 1$ for (14) we then rewrite (14) as:

$$\xi_\pm = H \pm \sqrt{-(1 - H^2)}$$
$$\xi_\pm = H \pm i\sqrt{(1 - H^2)}$$

then, the possible solution is

$$\xi_\pm^i = \cos(\theta i) \pm I\sin(\theta i)$$

here i use "I" as imaginer number, so it is not confusing between "i" index and "I" imaginer number.

$$\therefore v_i = d_1(\cos(\theta i) + I\sin(\theta i) - (\cos(\theta i) - I\sin(\theta i)))$$
$$v_i = d_1(2I\sin(\theta i))$$

we choose $d_1 = -\frac{I}{2}$ so that

$$v_i = \sin(\theta i)$$

inserting boundary condition, when $i = N$ then $v_N = 0$, we get:

$$0 = \sin(\theta N)$$
$$\theta = m\pi, \quad \forall m : 2, 4, 6 \cdots N, \quad even$$
$$\theta = \frac{m\pi}{N}$$

then, we know that from imaginer triangle

$$\cos\theta = H$$

$\therefore H = \cos(\theta) = \cos(\frac{m\pi}{N})$

substitute H and G to $2H = (4 - \lambda - G)$, we get

$$2\cos\left(\frac{m\pi}{N}\right) = 4 - \lambda - 2\cos\left(\frac{m\pi}{N}\right)$$
$$\lambda = 4 - 2\cos\left(\frac{m\pi}{N}\right) - 2\cos\left(\frac{m\pi}{N}\right)$$
$$\therefore \lambda = 4 - 2\left(\cos\left(\frac{m\pi}{N}\right) + \cos\left(\frac{m\pi}{N}\right)\right)$$

for the eigenvectors, we already know that

$$w_{i,j} = v_i\tilde{v}_j$$
$$v_i = \sin(\theta i)$$
$$\tilde{v}_j = \sin(\theta j)$$
$$\therefore w_{i,j} = \sin\left(\frac{m\pi i}{N}\right)\sin\left(\frac{m\pi j}{N}\right)$$

(b) I attached the Python Code that solves (12) in 1

(c) Minimal Number of Iterations K such the:

$$\max_{1 \le i,j \le N-1}\left|v_{i,j}^{(K+1)} - v_{i,j}^{(K)}\right| \le 10^{-4}$$

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(a) $N = 10$

    i. Jacobi, $K = 53$

    ii. Gauss-Seidel, $K = 33$

    iii. SOR (omega=1.5), $K = 12$

(b) $N = 20$

    i. Jacobi, $K = 137$

    ii. Gauss-Seidel, $K = 94$

    iii. SOR (omega=1.5), $K = 45$

(c) $N = 50$

    i. Jacobi, $K = 1$

    ii. Gauss-Seidel, $K = 173$

    iii. SOR (omega=1.5), $K = 148$

(d) Optimal ω_b for SOR:

    (a) $N = 10$, $\omega_b = 1.16527511683289$

    (b) $N = 20$, $\omega_b = 1.2933223938735654$

    (c) $N = 50$, $\omega_b = 1.3746381523099196$

1 Attachment

```python
import numpy as np

#Function Definition
#Function to create Right Hand Side and Left Hand Side Matrix
def const_mat(N):
    h = 1/N
    A = np.zeros((N,N))
    b = np.zeros((N,N))
    for i in range(N):
        for j in range(N):
            A[i][i] = 4
            b[i][j] = h*h*(np.sin(np.pi*i/N)*np.sin(np.pi*j/N))
            #b[i][j] = 1
            if (i==j-1 and i%4==3) or (i==j+1 and i%4==0):
                A[i][j] = 0
            elif i==j-1:
                A[i][j] = -1
            elif i==j+1:
                A[i][j] = -1
            elif i==j-4:
                A[i][j] = -1
            elif i==j+4:
                A[i][j] = -1
    return A,b

#Jacobi
def jacobi(b):
    v = np.zeros_like(b)
    N = len(b)
    for k in range(1000):
        v_old = np.copy(v)
        for i in range(N-2):
            for j in range(N-2):
                v[i+1][j+1] = 0.25 * (v_old[i][j+1] + v_old[i+2][j+1] + v_old[i+1][j] + v_old[i+1][j+2] + b[i+1][j+1])
        #print(np.max(v))
        if np.max(np.abs(v - v_old)) < 1e-4:
            break
    return (k)

#Gauss-Seidel
def gauss_seidel(b):
    v = np.zeros_like(b)
    N = len(b)
    for k in range(1000):
        v_old = np.copy(v)
        for i in range(N-2):
            for j in range(N-2):
                v[i+1][j+1] = 0.25 * (v[i][j+1] + v[i+2][j+1] + v[i+1][j] + v[i+1][j+2] + b[i+1][j+1])
        #print(np.max(v))
        if np.max(np.abs(v - v_old)) < 1e-4:
            break
    return (k)

#SOR
def sor(omega,b):
    v = np.zeros_like(b)
    N = len(b)
    for k in range(1000):
        v_old = np.copy(v)
        for i in range(N-2):
            for j in range(N-2):
                v[i+1][j+1] = (1-omega)*v_old[i+1][j+1] + omega*(0.25 * (v[i][j+1] + v[i+2][j+1] + v[i+1][j] + v[i+1][j+2] + b[i+1][j+1]))
        #print(np.max(v))
        if np.max(np.abs(v - v_old)) < 1e-4:
            break
    return (k)

#Calculate Optimum Omega for SOR
def calc_op_omega(A):
    D = np.diag(np.diag(A))
    L = np.tril(A) - D
    U = np.triu(A) - D
    T = 1/np.diag(A)*(L+U)
    rho = np.max(np.linalg.eigvals(T))
    opt_omega = 2/(1+np.sqrt(1-rho*rho))

    return opt_omega

#main
Ns = [10,20,30,40,50]
om = 1.5
for n in Ns:
    matA , matB = const_mat(n)
    iterj = jacobi(matB)
    iterg = gauss_seidel(matB)
    iters = sor(om,matB)
    op_om = calc_op_omega(matA)
    print("N =",n)
    print("Minimum Iteration Jacobi =",iterj)
    print("Minimum Iteration Gauss-Seidel =",iterg)
    print("Minimum Iteration SOR using omega(1.5) =",iters)
    print("Optimal omega for SOR =",op_om)
```