Basics of Applied Analysis A Report

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Problem 1. *Discuss the stability of the difference schemes for the transport equation*

$$u_x + bu_x = 0 \tag{1}$$

using von Neumann stability analysis:

1. *"naive" explicit scheme*
$$\frac{v_k^{n+1} - v_k^n}{\tau} + b\frac{v_{k+1}^n - v_{k-1}^n}{2h} = 0 \tag{2}$$

2. *implicit scheme*
$$\frac{v_k^{n+1} - v_k^n}{\tau} + b\frac{v_{k+1}^{n+1} - v_{k-1}^{n+1}}{2h} = 0 \tag{3}$$

3. *Discuss the dissipation and dispersion properties of the implicit scheme in (b). Is it a satisfactory scheme for* (1)

Answer:

1. von Neumann stability analysis for "naive" explicit scheme:
we rewrite (2) become,
$$v_k^{n+1} = v_k^n - \frac{R}{2}(v_{k+1}^n - v_{k-1}^n), \quad R := \frac{b\tau}{h} \tag{4}$$
then substitute $v_{k+q} = e^{iq\xi}\hat{v}^n$ to (4), we get
$$\hat{v}^{n+1} = \hat{v}^n\left(1 - \frac{R}{2}(e^{i\xi} - e^{-i\xi})\right) \tag{5}$$
then we define, $g(\xi) = \left(1 - \frac{R}{2}(e^{i\xi} - e^{-i\xi})\right) = 1 - iR\sin(\xi)$, taking norm of $g(\xi)$ we get
$$|g(\xi)| = |1 - iR\sin(\xi)| = \sqrt{1 + R^2\sin^2(\xi)}, \quad \xi = (-\pi, \pi) \tag{6}$$
by (6) we get $|g(\xi)| > 1$, according to von Neumann stability, the explicit scheme (2) is unstable.

2. von Neumann stability analysis for implicit scheme:
we rewrite (3) become,
$$\frac{R}{2}\left(v_{k+1}^{n+1} - v_{k-1}^{n+1}\right) + v_k^{n+1} = v_k^n, \quad R := \frac{b\tau}{h} \tag{7}$$

then substitute $v_{k+q} = e^{iq\xi}\hat{v}^n$ to (7), we get

$$\begin{aligned} \left(1 + \frac{R}{2}\left(e^{i\xi} - e^{-i\xi}\right)\right)\hat{v}^{n+1} &= \hat{v}^n \\ \hat{v}^{n+1} &= \frac{1}{\left(1 + \frac{R}{2}\left(e^{i\xi} - e^{-i\xi}\right)\right)}\hat{v}^n \end{aligned} \tag{8}$$

then we define, $g(\xi) = \frac{1}{\left(1+\frac{R}{2}\left(e^{i\xi}-e^{-i\xi}\right)\right)} = \frac{1}{(1+iR\sin(\xi))}$, taking norm of $g(\xi)$ we get

$$|g(\xi)| = \left|\frac{1}{(1 + iR\sin(\xi))}\right| = \frac{1}{\sqrt{1 + R^2\sin^2(\xi)}}, \quad \xi = (-\pi, \pi) \tag{9}$$

by (9) we get $|g(\xi)| < 1$, according to von Neumann stability, the implicit scheme (3) is stable.

3. To analyze the dissipation and dispersion of (3), we substitute $v_{k+p}^{n+q} = e^{i(q\omega\tau + p\beta h)}$ to (7) we get,

$$\begin{aligned} \frac{R}{2}\left(e^{i(\omega\tau+\beta h)} - e^{i(\omega\tau-\beta h)}\right) + e^{i\omega\tau} &= 1, \quad R := \frac{b\tau}{h} \\ \left(\frac{R}{2}\left(e^{i\beta h} - e^{-i\beta h}\right) + 1\right)e^{i\omega\tau} &= 1 \\ \left(iR\sin(\beta h) + 1\right)e^{i\omega\tau} &= 1 \\ e^{i\omega\tau} &= \frac{1}{(iR\sin(\beta h) + 1)} \end{aligned} \tag{10}$$

taking norm of $e^{i\omega\tau}$, we get

$$|e^{i\omega\tau}| = e^{-\omega_2\tau} = \frac{1}{R^2\sin^2(\beta h) + 1} \tag{11}$$

by (11), $e^{-\omega_2\tau} < 1$, therefore according to von Neumann stability analysis, the implicit scheme on (3) is **dissipative**.
Then, to analyze the dispersion, we take $\arg(e^{i\omega\tau})$,

$$\begin{aligned} \arg(e^{i\omega\tau}) &= \arg(e^{i\omega_1\tau}) + \arg(e^{-\omega_2\tau}) \\ &= \omega_1\tau + 0 \end{aligned} \tag{12}$$

which, $\omega_1\tau = \arctan\left(\frac{Im(e^{i\omega\tau})}{Re(e^{i\omega\tau})}\right)$, we can get the real and imajiner part of $e^{i\omega\tau}$ by first multiplying it with it's rational factor,

$$e^{i\omega\tau} = \frac{1}{(iR\sin(\beta h) + 1)}\frac{(iR\sin(\beta h) - 1)}{(iR\sin(\beta h) - 1)} = -\frac{iR\sin(\beta h) + 1}{R^2\sin^2 + 1} \tag{13}$$

then, $\omega_1\tau = \arctan\left(\frac{Im(e^{i\omega\tau})}{Re(e^{i\omega\tau})}\right) = \frac{R\sin(\beta h)}{1} = R\sin(\beta h)$. Since $\omega_1\tau$ is not a constant, therefore, according to von Neumann stability analysis, the implicit scheme on (3) is **dispersive**.

Problem 2. *Show that the following implicit difference schemes for approximating the solution to*

$$u_t + bu_x = au_{xx} \tag{14}$$

are unconditionally stable using the von Neumann stability analysis. Here $R = b\frac{\tau}{h}, r = a\frac{\tau}{h^2}$.

1.

$$v_k^{n+1} + \frac{R}{2}(v_{k+1}^{n+1} - v_{k-1}^{n+1}) - r(v_{k+1}^{n+1} - 2v_k^{n+1} + v_{k-1}^{n+1}) = v_k^n \tag{15}$$

2.

$$v_k^{n+1}+\frac{R}{4}(v_{k+1}^{n+1}-v_{k-1}^{n+1})-\frac{r}{2}(v_{k+1}^{n+1}-2v_k^{n+1}+v_{k-1}^{n+1}) = v_k^n-\frac{R}{4}(v_{k+1}^n-v_{k-1}^n)+\frac{r}{2}(v_{k+1}^n-2v_k^n+v_{k-1}^n) \tag{16}$$

Answer:

1. Substitute $v_{k+q} = e^{iq\xi}\hat{v}^n$ to (15) we get,

$$\begin{aligned}
\hat{v}^{n+1} + \frac{R}{2}(e^{i\xi} - e^{-i\xi})\hat{v}^{n+1} - r(e^{i\xi} - 2 + e^{-i\xi})\hat{v}^{n+1} &= \hat{v}^n \\
\hat{v}^{n+1}\left(1 + \frac{R}{2}(e^{i\xi} - e^{-i\xi}) - r(e^{i\xi} - 2 + e^{-i\xi})\right) &= \hat{v}^n \\
\hat{v}^{n+1} &= \frac{1}{\left(1 + \frac{R}{2}(e^{i\xi} - e^{-i\xi}) - r(e^{i\xi} - 2 + e^{-i\xi})\right)}\hat{v}^n
\end{aligned} \tag{17}$$

then, we define:

$$\begin{aligned}
g(\xi) &= \frac{1}{\left(1 + \frac{R}{2}(e^{i\xi} - e^{-i\xi}) - r(e^{i\xi} - 2 + e^{-i\xi})\right)} = \frac{1}{\left(1 + iR\sin(\xi) + r(2\cos(\xi) - 2)\right)} \\
&= \frac{1}{\left(1 + iR\sin(\xi) + r(-4\sin^2(\frac{\xi}{2}))\right)}
\end{aligned} \tag{18}$$

by (18), $g(\xi) < 1$, according to von Neumann stability analysis, if $g(\xi) < 1$ the scheme will be unconditionally stable.

2. Substitute $v_{k+q} = e^{iq\xi}\hat{v}^n$ to (16) we get,

$$\begin{aligned}
\hat{v}^{n+1} + \frac{R}{4}(e^{i\xi} - e^{-i\xi})\hat{v}^{n+1} - \frac{r}{2}(e^{i\xi} - 2 + e^{-i\xi})\hat{v}^{n+1} &= \hat{v}^n - \frac{R}{4}(e^{i\xi} - e^{-i\xi})\hat{v}^n + \frac{r}{2}(e^{i\xi} - 2 + e^{-i\xi})\hat{v}^n \\
\hat{v}^{n+1}\left(1 + \frac{R}{4}(e^{i\xi} - e^{-i\xi}) - \frac{r}{2}(e^{i\xi} - 2 + e^{-i\xi})\right) &= \hat{v}^n\left(1 - \frac{R}{4}(e^{i\xi} - e^{-i\xi}) + \frac{r}{2}(e^{i\xi} - 2 + e^{-i\xi})\right) \\
\hat{v}^{n+1} &= \frac{\left(1 - \frac{R}{4}(e^{i\xi} - e^{-i\xi}) + \frac{r}{2}(e^{i\xi} - 2 + e^{-i\xi})\right)}{\left(1 + \frac{R}{4}(e^{i\xi} - e^{-i\xi}) - \frac{r}{2}(e^{i\xi} - 2 + e^{-i\xi})\right)}\hat{v}^n
\end{aligned} \tag{19}$$

then, we define:

$$g(\xi) = \frac{\left(1 - \frac{R}{4}(e^{i\xi} - e^{-i\xi}) + \frac{r}{2}(e^{i\xi} - 2 + e^{-i\xi})\right)}{\left(1 + \frac{R}{4}(e^{i\xi} - e^{-i\xi}) - \frac{r}{2}(e^{i\xi} - 2 + e^{-i\xi})\right)} = \frac{\left(1 - \frac{R}{4}(2i\sin(\xi)) + \frac{r}{2}(2\cos(\xi) - 2)\right)}{\left(1 + \frac{R}{4}(2i\sin(\xi)) - \frac{r}{2}(2\cos(\xi) - 2)\right)}$$
$$= \frac{\left(1 - \frac{R}{4}(2i\sin(\xi)) + \frac{r}{2}(-4\sin^2(\frac{\xi}{2}))\right)}{\left(1 + \frac{R}{4}(2i\sin(\xi)) - \frac{r}{2}(-4\sin^2(\frac{\xi}{2}))\right)} \tag{20}$$

by (20), $g(\xi) < 1$, according to von Neumann stability analysis, if $g(\xi) < 1$ the scheme will be unconditionally stable.

Problem 3. *Discuss the dissipation and dispersion of the following implicit numerical schemes for the wave equation*

1.

$$\frac{v_k^{n+1} - 2v_k^n + v_k^{n-1}}{\tau^2} = \frac{v_{k+1}^{n+1} - 2v_k^{n+1} + v_{k-1}^{n+1}}{h^2} \tag{21}$$

2.

$$\frac{v_k^{n+1} - 2v_k^n + v_k^{n-1}}{\tau^2} = \frac{v_{k+1}^{n+1} - 2v_k^{n+1} + v_{k-1}^{n+1}}{2h^2} + \frac{v_{k+1}^{n-1} - 2v_k^{n-1} + v_{k-1}^{n-1}}{2h^2} \tag{22}$$

Answer:

1. Substitute $v_{k+p}^{n+q} = e^{i(q\omega\tau + p\beta h)}\hat{v}^n$ with $R = \frac{\tau}{h}$ to (21) we get,

$$\frac{(e^{i\omega\tau} - 2 + e^{-i\omega\tau})}{\tau^2}\hat{v}^n = \frac{(e^{i(\omega\tau+\beta h)} - 2e^{i\omega\tau} + e^{i(\omega\tau-\beta h)})}{h^2}\hat{v}^n$$
$$e^{i\omega\tau} - 2 + e^{-i\omega\tau} = R^2(2\cos(\beta h) - 2)e^{i\omega\tau} \tag{23}$$

take $g = e^{i\omega\tau}$, (23) become

$$-2 + g^{-1} + g\left(1 - R^2(2\cos(\beta h) - 2)\right) = 0$$
$$-2 + g^{-1} + g\left(1 - R^2(-4\sin^2(\frac{\beta h}{2}))\right) = 0 \tag{24}$$

multiple by g we get

$$g^2(1 - R^2(-4\sin^2(\frac{\beta h}{2}))) - 2g + 1 = 0 \tag{25}$$

solving (25) we get,

$$g = e^{i\omega\tau} = \frac{1 \pm i2R\sin(\frac{\beta h}{2})}{(1 - R^2(-4\sin^2(\frac{\beta h}{2})))} \tag{26}$$

taking norm of (26), we get

$$|e^{i\omega\tau}| = e^{i\omega_2\tau} = \max_{-\pi \le \frac{\beta h}{2} \le \pi} \left| \frac{1 \pm i2R\sin(\frac{\beta h}{2})}{(1 - R^2(-4\sin^2(\frac{\beta h}{2})))} \right| = \frac{1 \pm 2R}{1 + 4R^2} \tag{27}$$

from (27), $e^{i\omega_2\tau} < 1$, therefore according to von Neumann stability analysis, the implicit numerical scheme on (21) is **dissipative**.

Then, to analyze the dispersion, we take $\arg(e^{i\omega\tau})$,

$$
\begin{aligned}
\arg(e^{i\omega\tau}) &= \arg(e^{i\omega_1\tau}) + \arg(e^{-\omega_2\tau}) \\
&= \omega_1\tau + 0
\end{aligned} \tag{28}
$$

which, $\omega_1\tau = \arctan\left(\frac{Im(e^{i\omega\tau})}{Re(e^{i\omega\tau})}\right)$, then, $\omega_1\tau = \arctan\left(\frac{Im(e^{i\omega\tau})}{Re(e^{i\omega\tau})}\right) = \frac{2R\sin(\beta h)}{1} = 2R\sin(\beta h)$. Since $\omega_1\tau$ is not a constant, therefore, according to von Neumann stability analysis, the implicit scheme on (21) is **dispersive**.

2. Substitute $v_{k+p}^{n+q} = e^{i(q\omega\tau+p\beta h)}\hat{v}^n$ with $R = \frac{\tau^2}{2h^2}$ to (22) we get,

$$
\begin{aligned}
&\frac{(e^{i\omega\tau} - 2 + e^{-i\omega\tau})}{\tau^2}\hat{v}^n = \frac{(e^{i(\omega\tau+\beta h)} - 2e^{i\omega\tau} + e^{i(\omega\tau-\beta h)})}{2h^2}\hat{v}^n + \frac{(e^{i(-\omega\tau+\beta h)} - 2e^{-i\omega\tau} + e^{-i(\omega\tau+\beta h)})}{2h^2}\hat{v}^n \\
&(e^{i\omega\tau} - 2 + e^{-i\omega\tau}) = R\left((2\cos(\beta h) - 2)e^{i\omega\tau} + (2\cos(\beta h) - 2)e^{-i\omega\tau}\right) \\
&(e^{i\omega\tau} - 2 + e^{-i\omega\tau}) - R\left((2\cos(\beta h) - 2)e^{i\omega\tau} + (2\cos(\beta h) - 2)e^{-i\omega\tau}\right) = 0 \\
&e^{i\omega\tau}\left(1 - R(2\cos(\beta h) - 2)\right) + e^{-i\omega\tau}\left(1 - R(2\cos(\beta h) - 2)\right) - 2 = 0
\end{aligned} \tag{29}
$$

take $g = e^{i\omega\tau}$, (29) become

$$
\begin{aligned}
&g\left(1 - R(2\cos(\beta h) - 2)\right) + g^{-1}\left(1 - R(2\cos(\beta h) - 2)\right) - 2 = 0 \\
&g\left(1 - R(-4\sin^2(\frac{\beta h}{2})\right) + g^{-1}\left(1 - R(-4\sin^2(\frac{\beta h}{2})\right) - 2 = 0
\end{aligned} \tag{30}
$$

multiple by g, we get

$$
g^2\left(1 - R(-4\sin^2(\frac{\beta h}{2})\right) + \left(1 - R(-4\sin^2(\frac{\beta h}{2})\right) - 2g = 0 \tag{31}
$$

solving (31) we get,

$$
g = e^{i\omega\tau} = \frac{2 \pm \sqrt{4 - 4(1 + 4R\sin^2(\frac{\beta h}{2}))^2}}{2(1 + 4R\sin^2(\frac{\beta h}{2}))} \tag{32}
$$

taking norm of (32), we get

$$
|e^{i\omega\tau}| = e^{i\omega_2\tau} = \max_{-\pi \le \frac{\beta h}{2} \le \pi}\left|\frac{2 \pm \sqrt{4 - 4(1 + 4R\sin^2(\frac{\beta h}{2}))^2}}{2(1 + 4R\sin^2(\frac{\beta h}{2}))}\right| = \frac{2 \pm (1 + 4R)}{2(1 + 4R)} \tag{33}
$$

from (33), $e^{i\omega_2\tau} = 1$, therefore according to von Neumann stability analysis, the implicit numerical scheme on (22) is **non-dissipative**.

Then, to analyze the dispersion, we take $\arg(e^{i\omega\tau})$,

$$
\begin{aligned}
\arg(e^{i\omega\tau}) &= \arg(e^{i\omega_1\tau}) + \arg(e^{-\omega_2\tau}) \\
&= \omega_1\tau + 0
\end{aligned} \tag{34}
$$

which, $\omega_1\tau = \arctan\left(\frac{Im(e^{i\omega\tau})}{Re(e^{i\omega\tau})}\right)$, then, $\omega_1\tau = \arctan\left(\frac{Im(e^{i\omega\tau})}{Re(e^{i\omega\tau})}\right) = 0$. Since $\omega_1\tau$ is 0 because there is no imaginer part, therefore, according to von Neumann stability analysis, the implicit scheme on (22) is **non-dispersive**.

Problem 4. *Implement the "naive" finite difference scheme and the implicit scheme to solve*

$$\begin{cases} u_t + u_x = 0 \quad in\ (0,1)x(0,\infty) \\ u(x,0) = f(x) \quad for\ x \in (0,1) \\ u(0,t) = 0 \quad for\ t \geq 0 \end{cases} \tag{35}$$

Answer:

1. Exact Solution: We want to find the exact solution of (35) using general solution

$$u(x,t) = f(x)(x-t)$$

   (a) For

$$f(x) = \begin{cases} 0.25 - |x - 0.25|, \quad x < 0.5 \\ 0, \quad x \geq 0.5 \end{cases}$$

   we have the exact solution as follow:

$$u(x,t) = (0.25 - |x - 0.25|)(x-t) \tag{36}$$

   (b) For

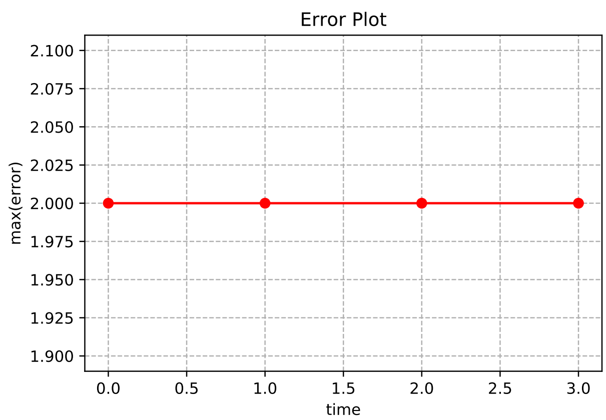
$$f(x) = xe^{-100(x-0.25)^2}$$

   the exact solution is:

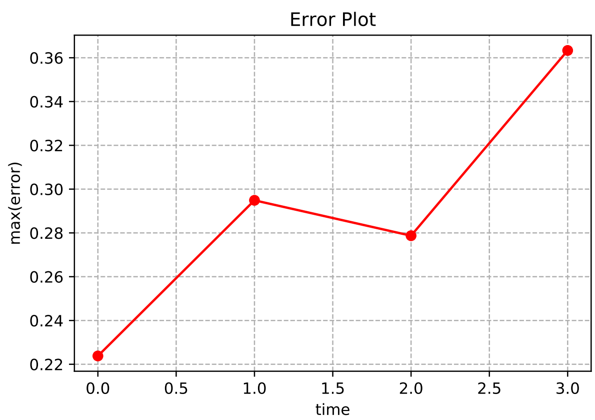
$$u(x,t) = (x^2 - xt)e^{-100(x-0.25)^2} \tag{37}$$

2. Error plot for naive scheme with error defined by:

$$\max_{0<k<M} |v_k^n - u(kh, n\tau)|$$

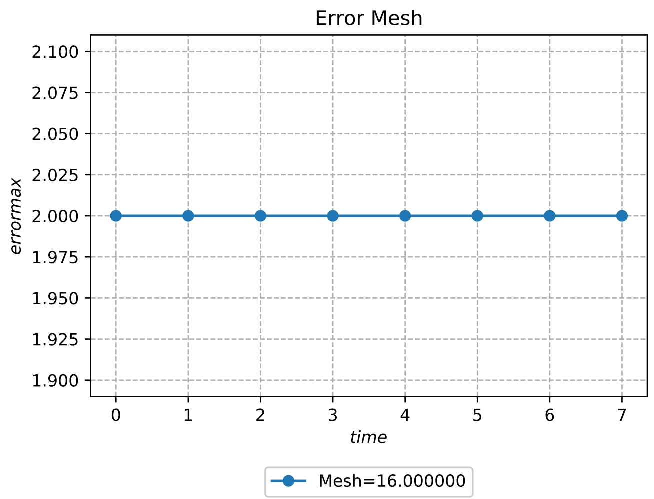


(a) Error Plot using (i) initial data

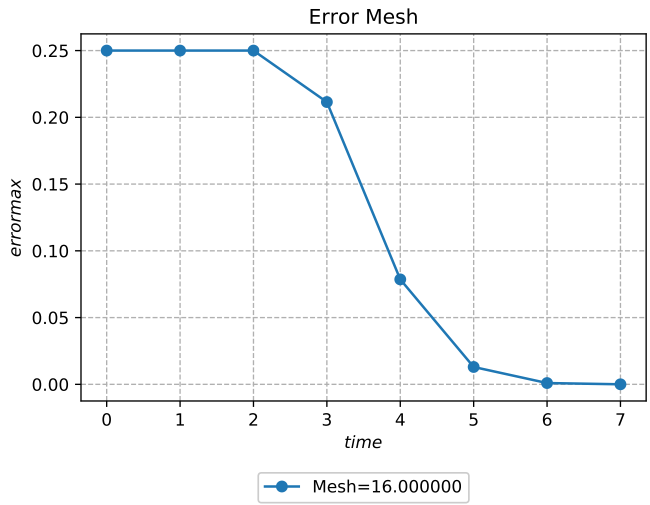


(b) Error Plot using (ii) initial data

Error Plot for Implicit Scheme:



(a) Error Plot using (i) initial data for implicit scheme

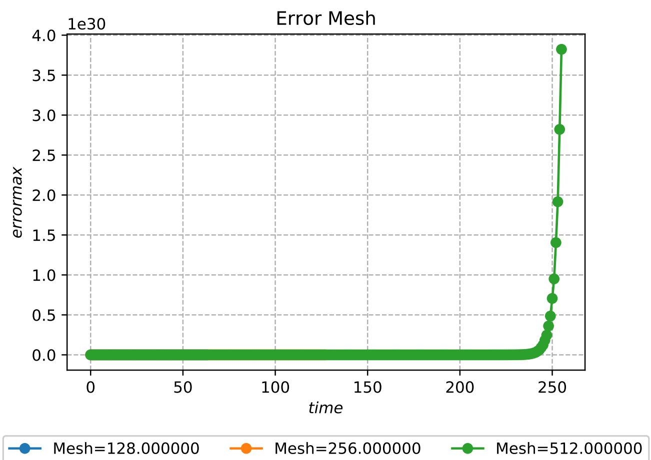


(b) Error Plot using (ii) initial data for implicit scheme

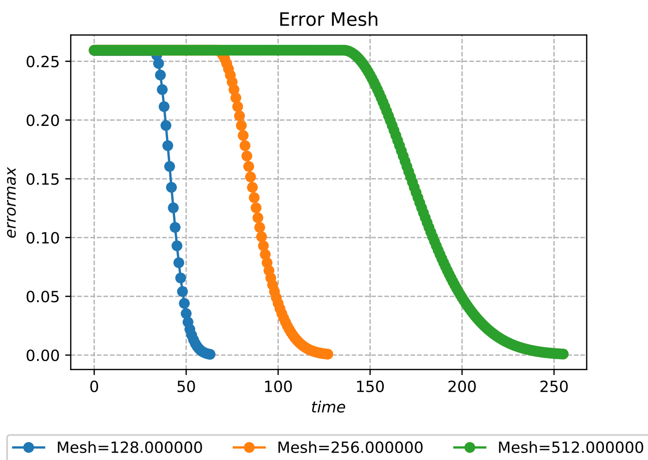
3. Check the maximum error in the naive scheme

4. Is the error increasing with the same rate if use the initial condition (ii) above? **Answer:** The error is not increasing if using the initial condition (ii), vice versa, the solution is better if using initial condition (ii)

5. How does the maximum of the error behave if the mesh size $M = 128, 256, 512$? As shown in the figure below, using "naive" scheme, the solution always blowup, at the other han, using implicit scheme, resulting in stable and convergent result.



(a) Error plot overtime in Naive Scheme



(b) Error plot overtime in Implicit Scheme