

Seminar Notes Alifian

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1 3D Linear Elasticity

$$\begin{aligned}\Omega &\subset \mathbb{R}^d (d = 2, 3) \\ u &= \Omega \rightarrow \mathbb{R}^2 (\text{small displacement}) \\ x &\mapsto u(x)\end{aligned}$$

1.1 Strain Tensor

$$\begin{aligned}e[u] &= (e_{ij}[u]) \in \mathbb{R}_{sym}^{d \times d} \\ e[u] &:= \frac{1}{2}(\nabla^T u + (\nabla^T u)^T)\end{aligned}\tag{1}$$

1.2 Stress Tensor

$$\sigma[u] = (\sigma_{ij}[u]) \in \mathbb{R}_{sym}^{d \times d}\tag{2}$$

Based on Hook's Law, stress tensor must have equality with strain so that

$$\begin{aligned}\sigma &= \mathbf{C}e \\ \text{with } \mathbf{C} &= \mathbf{C}_{ijkl} (\text{is a 4th order elasticity tensor}) \\ \sigma_{ij} &= \mathbf{C}_{ijkl} e_{kl} \\ \mathbf{C}_{ijkl} &= \mathbf{C}_{ijlk} = \mathbf{C}_{klij} (\text{symmetry}) \\ \mathbf{C}_{ijkl} \xi_{ij} \xi_{kl} &\geq C_* |\xi|^2\end{aligned}$$

1.3 Boundary Value Problem

$$\begin{cases} -\partial_i \sigma_{ij}[u] &= f_j(x), x \in \Omega \\ u &= g(x), x \in \Gamma_D \\ \sigma[u]_\nu &= q(x), x \in \Gamma_N \end{cases}\tag{3}$$

1.4 Equilibrium Equations of Force in Ω and on Γ_N

1.4.1 Strain Energy Density

$$\omega[u](x) := \frac{1}{2} \sigma[u] : e[u] \quad (4)$$

Solving using Sobolev Space in Isotropic Case, equation 4 becomes

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

with λ, μ called Lamé Constant

$$\delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$

$$\begin{aligned} \sigma[u] &= (\sigma_{ij}[u]) \\ \sigma_{ij}[u] &= c_{ijkl} e_{kl}[u] \\ &= \lambda (\delta_k u_k) \delta_{ij} + \mu (\delta_i u_j + \delta_j u_i) \\ &= \lambda (\operatorname{div} u) I + 2\mu e[u] \end{aligned}$$

$$\begin{aligned} \omega[u] &= \frac{1}{2} (\lambda (\operatorname{div} u) I + 2\mu e[u]) : e[u] \\ \omega[u] &= \frac{1}{2} (\lambda (\operatorname{div} u)^2 + \mu |e[u]|^2) \end{aligned}$$

Remark 1. *Positivity of C*

$$\begin{aligned} (C\xi) : \xi &\geq C_* |\xi|^2 (\forall \xi \in \mathbb{R}_{sym}^{d \times d}) \\ (C\xi) : \xi &= \lambda |\operatorname{tr} \xi|^2 + 2\mu |\xi|^2 \end{aligned}$$

If $\lambda \geq 0, \mu > 0$, then $C_* = 2\mu$

$$\xi = (\xi_{ij}), |\xi|^2 = \xi_{ij} \xi_{ij} = \sum_{i=1 \dots d} \sum_{j=1 \dots d} |\xi_{ij}|^2$$

1.5 Elasticity Problem

$$\begin{cases} -\operatorname{div} \sigma[u] &= f(x) \text{ in } \Omega \subset \mathbb{R}^d \\ u &= g(x) \text{ on } \Gamma_D \\ \sigma[u]v &= q(x) \text{ on } \Gamma_N \end{cases} \quad (5)$$

1.6 Crack Problem

$$\begin{cases} -\operatorname{div} \sigma[u] &= f(x) \text{ in } \Omega \setminus \Sigma \subset \mathbb{R}^d \\ u &= g(x) \text{ on } \Gamma_D \\ \sigma[u]v &= q(x) \text{ on } \Gamma_N \\ \sigma[u]v &= 0 \text{ on } \Sigma^+ \cup \Sigma^- \end{cases} \quad (6)$$

1.7 Lebesgue Measurable Theory

$$L^p(\Omega) := \left\{ v : \Omega \rightarrow \mathbb{R} \mid \begin{cases} v = \text{Lebesgue measurable} \\ \int_{\Omega} |v(x)|^p dx < \infty \end{cases} \right\} \quad (7)$$

Remark 2. for $u, v \in \mathbb{L}^p(\Omega)$, if $\exists N \subset \Omega$ such that $\begin{cases} u(x) = v(x) (x \in \Omega \setminus N) \\ \mathcal{L}^d(N) = 0, \end{cases}$ then we identify u and v , $\mathcal{L}^d(N) = 0 \Leftrightarrow \text{volume of } N = 0$ for simplicity, we also can say that $u(x) = v(x)$ for a.e. $x \in \Omega$

for example

$$\begin{aligned} v : \mathbb{R} &\rightarrow \mathbb{R} \\ v(x) &= \begin{cases} 1, x \in \mathbb{Q} \\ 0, x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \\ \int_{\mathbb{R}} v dx &= 0, \mathcal{L}^1(\mathbb{Q}) = 0 \\ v(x) &= 0 \text{ on } \mathbb{R} \setminus \mathbb{Q}, \text{ or we can say } v = 0 \text{ a.e. in } \mathbb{R} \end{aligned} \quad (8)$$

1.8 Sobolev Space

$$\mathbb{W}^{1,p}(\Omega) := \left\{ v \in \mathbb{L}^p(\Omega) \mid \frac{\partial v}{\partial x_j} \Big|_{(j=1\dots d)} \in \mathbb{L}^p(\Omega) \right\} \quad (9)$$

such $\frac{\partial v}{\partial x_j}$ we called it distribution sence.

example of Sobolev Space is as follow:

$$v \in \mathbb{L}^p(\Omega) \text{ if } \exists \omega_j \in \mathbb{L}(\Omega)$$

such that

$$\begin{aligned} \int_{\Omega} v \frac{\partial \varphi}{\partial x_j} dx &= - \int_{\Omega} \omega_j \varphi dx (\forall \varphi \in \mathbb{C}_0^\infty(\Omega)) \\ \Rightarrow \frac{\partial \varphi}{\partial x_j} &= \omega_j \text{ in distribution sence} \end{aligned}$$

for

$$\begin{aligned} v \in \mathbb{C}^1(\Omega), \frac{\partial v}{\partial x_j}(x) &= \omega_j(x) \\ \Updownarrow \\ \int_{\Omega} \omega_j \varphi dx &= - \int_{\Omega} v \frac{\partial \varphi}{\partial x_j} dx (\forall \varphi \in \mathbb{C}_0^\infty(\Omega)) \end{aligned}$$

In particular,

$$\mathbb{H}^1(\Omega) := \mathbb{W}^{1,2}(\Omega), \nabla u = \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_d} \end{pmatrix}$$

inner product

$$(u, v)_{\mathbb{H}^1(\Omega)} := \int_{\Omega} uv \, dx + \int_{\Omega} \nabla u \cdot \nabla v \, dx$$

norm

$$\|u\|_{\mathbb{H}^1(\Omega)} := \sqrt{(u, u)_{\mathbb{H}^1(\Omega)}} = \sqrt{\int_{\Omega} |u|^2 dx + \int_{\Omega} |\nabla u|^2 dx}$$

$\mathbb{H}^1(\Omega)$ is complete ($\mathbb{H}^1(\Omega)$ is a Hilbert Space)

$$(u, v)_{\mathbb{L}^2(\Omega)} = \int_{\Omega} u v dx$$

1.9 Incomplete Hilbert Space

\mathbb{V} : a vector space in \mathbb{R}

$$\begin{cases} u, v \in \mathbb{V} \Rightarrow \alpha u + \beta v \in \mathbb{V} \\ \alpha, \beta \in \mathbb{R} \end{cases}$$

If $(\cdot, \cdot) : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{R}$ satisfies

$$\begin{cases} (u \cdot v) \geq 0 \text{ and } (u, u) = 0 \Leftrightarrow u = 0_v \in \mathbb{V} \\ (u, v) = (v, u) \\ (\alpha u + \beta v, \omega) = \alpha(u, \omega) + \beta(v, \omega) \end{cases}$$

then we call $[\mathbb{V} \times \mathbb{V}]$ pre Hilbert space or incomplete Hilbert Space.

1.10 Property of $\mathbb{L}^2(\Omega)$

$u, v \in \mathbb{L}^2(\Omega)$

$$\Rightarrow \left| \int_{\Omega} u v dx \right| \leq \int_{\Omega} |u| |v| dx \leq \|u\|_{\mathbb{L}^2(\Omega)} \|v\|_{\mathbb{L}^2(\Omega)}$$

$u, v \in \mathbb{H}^1(\Omega)$

$$\Rightarrow \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j} \in \mathbb{L}^2(\Omega)$$

$$\left| \int_{\Omega} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j} dx \right| \leq \int_{\Omega} \left| \frac{\partial u}{\partial x_j} \right| \left| \frac{\partial v}{\partial x_j} \right| dx \leq \left\| \frac{\partial u}{\partial x_j} \right\|_{\mathbb{L}^2(\Omega)} \left\| \frac{\partial v}{\partial x_j} \right\|_{\mathbb{L}^2(\Omega)}$$

$$\nabla u \cdot \nabla v = \sum_{j=1}^d \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j}$$

$$\begin{aligned} \left| \int_{\Omega} \nabla u \cdot \nabla v dx \right| &= \left| \int \sum_{j=1}^d \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j} dx \right| \leq \sum_{j=1}^d \int \left| \frac{\partial u}{\partial x_j} \right| \left| \frac{\partial v}{\partial x_j} \right| dx \\ &\leq \sum_{j=1}^d \left\| \frac{\partial u}{\partial x_j} \right\|_{\mathbb{L}^2(\Omega)} \left\| \frac{\partial v}{\partial x_j} \right\|_{\mathbb{L}^2(\Omega)} \\ &\leq \sqrt{\sum_{j=1}^d \left\| \frac{\partial u}{\partial x_j} \right\|_{\mathbb{L}^2(\Omega)}^2} \sqrt{\sum_{j=1}^d \left\| \frac{\partial v}{\partial x_j} \right\|_{\mathbb{L}^2(\Omega)}^2} \\ &= \sqrt{\sum_{j=1}^d \int_{\Omega} \left| \frac{\partial u}{\partial x_j} \right|^2 dx} \sqrt{\sum_{j=1}^d \int_{\Omega} \left| \frac{\partial v}{\partial x_j} \right|^2 dx} \\ &= \sqrt{\int_{\Omega} \left(\sum_{j=1}^d \left| \frac{\partial u}{\partial x_j} \right|^2 \right) dx} \sqrt{\int_{\Omega} \left(\sum_{j=1}^d \left| \frac{\partial v}{\partial x_j} \right|^2 \right) dx} \\ &= \sqrt{\int_{\Omega} |\nabla u|^2 dx} \sqrt{\int_{\Omega} |\nabla v|^2 dx} \end{aligned}$$

$$\therefore \left| \int_{\Omega} \nabla u \cdot \nabla v dx \right| \leq \sqrt{\int_{\Omega} |\nabla u|^2 dx} \sqrt{\int_{\Omega} |\nabla v|^2 dx} \quad (10)$$