Variational Approach to Crack Propagation in a Cantilever Beam

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1 Linear Elasticity Basic Theory

We define:

$$
\begin{aligned}
&\Omega \subset \mathbb{R}^d \ (d = 2, 3)\\
&u : \Omega \to \mathbb{R}^d \text{ (displacement)}\\
&e[v] := \frac{1}{2}(\nabla^T v + \nabla v^T) \text{ (strain)}\\
&\nabla^T v := \begin{pmatrix} \partial_1 v_1 & \partial_2 v_2 \\ \partial_1 v_2 & \partial_2 v_1 \end{pmatrix}\\
&\nabla v^T := (\nabla^T v)^T\\
&\sigma[u] := \mathcal{C} e[u]\\
&\mathcal{C} = (C_{ijkl}) \begin{cases} C_{ijkl} = C_{klij} = C_{jikl} \\ (C_\xi) : \xi \geq C_* |\xi|^2 (\forall \xi \in \mathbb{R}^{d\times d}_{sym}) \end{cases}
\end{aligned}
$$

Let's consider linear elasticity problem:

$$
(**) \begin{cases} -div\ \sigma[u] = f(x), \text{ in } \Omega \\ u = g(x) \text{ on } \Gamma_D \\ \sigma[u]\nu = q(x) \text{ on } \Gamma_N \end{cases} \tag{1}
$$

$$
f \in L^2(\Omega : \mathbb{R}^d,\ g \in H^1(\Omega : \mathbb{R}^d),\ q \in L^2(\Gamma_N : \mathbb{R}^d))
$$

1.1 Strong Solution

$$
u \in H^2(\Omega : \mathbb{R}^d) \text{ satisfies } (**) \text{ then we call } u : \text{a strong solution} \tag{2}
$$

1.2 Weak Solution

$$
\begin{cases} \int_\Omega \sigma[u] : e[v] dx = \int_\Omega f \cdot v dx + \int_{\Gamma_N} q \cdot v ds \big(\forall v \in V := \{v \in H^1\ (\Omega : \mathbb{R}^d)\ |v|_{\Gamma_D} = 0\}\big) \\ u \in V + g \end{cases} \tag{3}
$$

1.3 Proposition

$$
u : \text{ strong solution } \Leftrightarrow \begin{cases} u : \text{ weak solution} \\ u \in H^2\ (\Omega : \mathbb{R}^d) \end{cases}
$$

Proof. ($\Rightarrow$) Assume we choose $v \in V := \{v \in H^1\ (\Omega : \mathbb{R}^d)\ |v|_{\Gamma_D} = 0\}$, with v is a very smooth test function. Then we take integral over the domain for equation (1) on both side.

$$\int_\Omega -div\sigma[u] \cdot vdx = \int_\Omega f \cdot vdx$$
$$\int_\Omega \sigma[u] : \nabla vdx - \int_\Gamma \sigma[u]\nu \cdot vds = \int_\Omega f \cdot vdx \text{ (by Divergence Formula)}$$
$$\int_\Omega \sigma[u] : \nabla vdx - \left(\int_{\Gamma_D} \sigma[u]\nu \cdot vds + \int_{\Gamma_N} \sigma[u]\nu \cdot vds \right) = \int_\Omega f \cdot vdx$$

From Boundary Condition we know that:

$$\begin{cases} v = 0 \text{ on } \Gamma_D \\ \sigma[u]\nu = q \text{ on } \Gamma_N \end{cases}$$

Hence, we have:

$$\begin{aligned} \int_\Omega \sigma[u] : e[v]dx - \int_{\Gamma_N} q \cdot vds &= \int_\Omega f \cdot vdx \\ \int_\Omega \sigma[u] : e[v]dx &= \int_\Omega f \cdot vdx + \int_{\Gamma_N} q \cdot vds \end{aligned} \tag{4}$$

with:

$$X := H^1(\Omega : \mathbb{R}^d)$$
$$a(u,v) := \int_\Omega \sigma[u] : e[v]dx$$
$$l(v) := \int_\Omega f \cdot vdx + \int_{\Gamma_N} q \cdot vds$$

$$\begin{aligned} a(u,v) &= \int_\Omega (\mathcal{C}e[u]) : e[v]dx \\ &= \int_\Omega e[v] : (\mathcal{C}e[u])dx \\ &= a(v,u) \end{aligned}$$

Then we rewrite equation (4) in a bilinear and linear form:

$$a(u,v) = l(v)$$

($\Leftarrow$) Since $u \in H^2\ (\Omega : \mathbb{R}^2), div(\sigma[u]) \in L^2(\Omega : \mathbb{R}^2)$ and $a(u,v) = 0$ for all $v \in V$, we have:

$$\begin{aligned} 0 &= \int_\Omega \sigma[u] : e[v]dx \\ 0 &= \int_\Omega \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} : \begin{pmatrix} \nabla v_1 & \nabla v_2 \end{pmatrix} dx \\ 0 &= \int_\Omega \begin{pmatrix} \sigma_{11} \\ \sigma_{21} \end{pmatrix} \cdot \nabla v_1 + \begin{pmatrix} \sigma_{12} \\ \sigma_{22} \end{pmatrix} \cdot \nabla v_2 dx \\ 0 &= \int_\Omega div \begin{pmatrix} \sigma_{11} \\ \sigma_{21} \end{pmatrix} v_1 + div \begin{pmatrix} \sigma_{12} \\ \sigma_{22} \end{pmatrix} v_2 - \int_{\partial\Omega} \begin{pmatrix} \sigma_{11} \\ \sigma_{21} \end{pmatrix} \cdot \nu v_1 + \begin{pmatrix} \sigma_{12} \\ \sigma_{22} \end{pmatrix} \cdot \nu v_2 ds \\ 0 &= \int_\Omega (f + div\sigma[u]) \cdot vdx - \int_{\partial\Omega} (\sigma[u]\nu) \cdot vds \\ 0 &= \int_\Omega f \cdot vdx + \int_\Omega div\sigma[u] \cdot vdx - \int_{\partial\Omega} (\sigma[u]\nu) \cdot vds \end{aligned}$$

then, assume we choose $v \in C_0^\infty(\Omega) \subset V, v = 0$ near $\partial\Omega$
$f \in L^1(\Omega)$, then we have:

$$\int_\Omega (f + div\sigma[u]) \cdot vdx = 0\ (\forall v \in C_0^\infty(\Omega, \mathbb{R}^2))$$
$$\therefore f + div\sigma[u] = 0 \text{ in } \Omega$$

then, $\forall v \in C_0^\infty(\bar{\Omega})$ s.t. $(supp(v) \cap \partial\Omega) \subset \Gamma_N$

$$\int_{\Gamma_N} (\sigma[u]\nu) \cdot v ds = 0$$
$$\sigma[u]\nu = 0 \text{ on } \Gamma_N$$

$\square$

For $v \in V$

$$\begin{aligned} a(v,v) &= \int_\Omega (\mathcal{C}e[v]) : e[v] dx \\ &\geqq C_* \int_\Omega |e[v]|^2 dx \\ &\geqq C_* ||v||_x^2 \end{aligned}$$

Properties 1.
- *$a(\cdot,\cdot)$ is bounded symmetric, bilinear form on $X \times X$.*
- *$a(\cdot,\cdot)$ is coercive on $V \times V$.*
- *l is bounded linear form on X.*

Theorem 1. *For any $g \in H^1(\Omega : \mathbb{R}^d)$,*

$$\exists! u : \textit{a weak solution of } (**), \textit{ and } \begin{cases} u = \operatorname{argmin}_{w \in V+g} E(w) \end{cases}$$

2 Linear Elasticity with Crack Propagation Case

2.1 Strong Form

Let's consider Linear Elasticity with Crack Propagation Case:

$$\begin{cases} -div\sigma[u] = f(x), & (x \in \Omega \setminus \Sigma(L)) \\ u = g & (x \in \Gamma_D) \\ \sigma[u]\nu = q & (x \in \Gamma_l) \\ \sigma[u]\nu = 0 & (x \in \Gamma_0 \cup \Sigma^+(L) \cup \Sigma^-(L)) \end{cases} \tag{5}$$

2.2 Weak Form

We define

$$V_L := v \in H^1(\Omega_L : \mathbb{R}^2) | v = 0 \text{ on } \Gamma_D$$

then the problem (5) becomes: Find $u \in V_L$ such that,

$$\int_{\Omega_L} \sigma[u] : e[v] dx = \int_{\Omega_L} f v dx + \int_{\Gamma_l} q v ds \ (\forall v \in V_L) \tag{6}$$

then, we define elastic energy:

$$E(v, L) := \frac{1}{2} a_L(v,v) - l(v), \ (v \in V_L) \tag{7}$$

then, Francfort and Marigo introduced the total energy:

$$\tilde{\varepsilon}(L, g) := \tilde{E}(L, g) + \gamma L \text{ (L:=}|\Sigma|) \tag{8}$$

where $\tilde{E}(L,g)$ is elastic energy and γL is surface energy. The condition $(G \geq \gamma)$ is also equivalent to $\frac{\partial \tilde{\varepsilon}}{\partial L}(L, g(t)) \leq 0$. Then they consider

$$\begin{cases} L(t) := \operatorname{argmin}_{L^-(t) \leq L \leq L_\infty} \\ L^-(t) := \sup_{L<t} L(s) \end{cases} \tag{9}$$

Suppose the boundary condition $g(t)$ on Γ_D is $g(t) = tg_0(x)$ $(x \in \Gamma_D)$, then the elastic energy becomes

$$\tilde{E}(L, tg_0) = t^2 \tilde{E}(L, g_0) \tag{10}$$

and the total energy becomes

$$\tilde{\varepsilon}(L, tg_0) = t^2 \tilde{E}(L, g_0) + \gamma L \tag{11}$$

3 Modelling and Simulation

In this simulation, we use a cantilever beam (Mild Steel Material) as the domain, which is a thin rectangular cross section introduced by Timoshenkol Goodier (1970), then we specify nondimensionalized parameter for simulation as shown in table **??**.

Numerical Parameter	Typical Value [unit]	Nondimensionalized value
Poisson's ratio (ν)	0.3 [-]	0.3
Young's modulus (E)	4.0×10^{10} [Pa]	4.0×10^{7}
Gravity constant (f)	9.80655 [N/kg]	9.80655
Weight (q)	0 [N]	0
Length (L)	3.0 [m]	3.0
Depth (h)	0.3 [m]	0.3
Width (b)	0.25 [m]	0.25
Effective Fracture Toughness (γ_{eff})	0.5[-]	0.5

Table 1: Material properties and numerical parameters

With the help of FreeFem++ software, we created a 2D and 3D model as shown in figure 1,

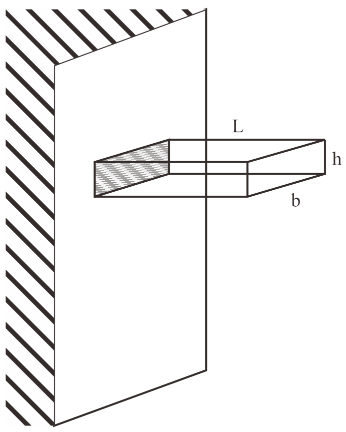


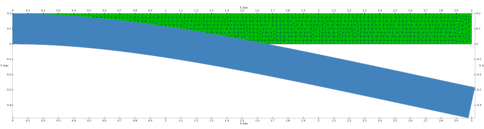
Figure 1: 3D Model of cantilever beam. We use gravity force as the body force **f** and fixed the left part of the beam, and then we give a 1 Newton weight force act on the right part of the beam as the neumann boundary condition **q**.

then we solve the displacement vector (u, v). After solving the displacement, we calculate σ which stand for stress force acting on surface of the cantilever beam using equation below:

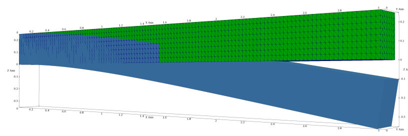
$$\sigma = (d\lambda^2 + 4\lambda\mu)div(u)^2 + (4\mu^2|e[u]|^2),\ d = 2, 3$$
$$\lambda \text{ (Lame's first parameter) } := \frac{E\nu}{(1+\nu)(1-2\nu)}$$
$$\mu \text{ Lame's second parameter } := \frac{E}{2(1+\nu)}$$

4 Result and Discussion

We solved the problem in (3) by using P1 finite element method on FreeFEM++, where u and v calculated for division number of mesh = 32. In the figure 2 we can see the deformation of the cantilever beam in 2D and 3D graphics.



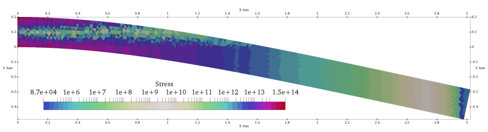
(a) Deformation in 2D. Green line show condition before gravity and weight force applied to the domain. Red line show condition after we solve linear elasticity with gravity and weight force applied to the domain. Maximal Displacement ($u = 0.03$ $[m]$)



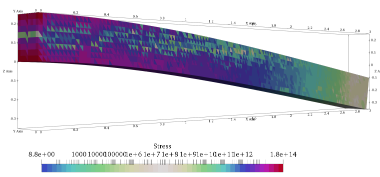
(b) Deformation in 3D. Green line show condition before gravity and weight force applied to the domain. Red line show condition after we solve linear elasticity with gravity and weight force applied to the domain. Maximal Displacement ($u = 0.05$ $[m]$)

Figure 2: Deformation in 2D and 3D

While in the figure 3 we can see result from calculating the stress tensor on 2D and 3D case.



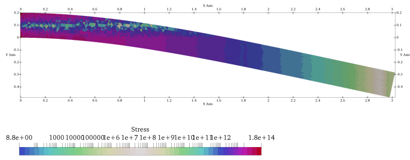
(a) Calculated σ on 2D case. The value of σ on the domain, maped by the color in the picture with respect to the color palette on the lower side of the graph. Maximal stress given on the surface ($\sigma = 158.612$ $[GPa]$)



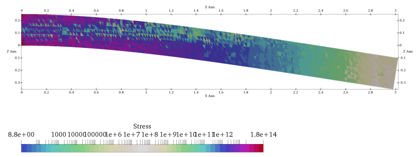
(b) Calculated σ on 3D case. The value of σ on the domain, maped by the color in the picture with respect to the color palette on the lower side of the graph. Maximal stress given on the surface ($\sigma = 567.034$ $[GPa]$)

Figure 3: Stress Tensor in 2D and 3D

On the figure 4, we can see comparison of calculated surface stress on 2D and 3D case (sliced on the side).



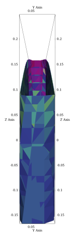
(a) Calculated surface stress on 2D



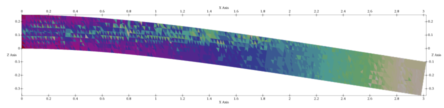
(b) Calculated surface stress on 3D

Figure 4: Comparison of Stress Tensor in 2D and 3D

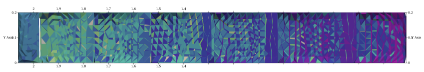
On the figure 5 below, we can see the result from sliced view on 3D case, in this case, we sliced through the Y-normal plane of the 3D model.



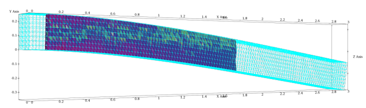
(a) Front View



(b) Right View



(c) Top View



(d) Wireframed-Sliced View

Figure 5: 3D Sliced View

4.1 Convergence Analysis

Since we don't have an exact solution for problem (1), we will define

$$u_h := u^k \iff \max |u^k - u^{k-1}| \leq \epsilon$$

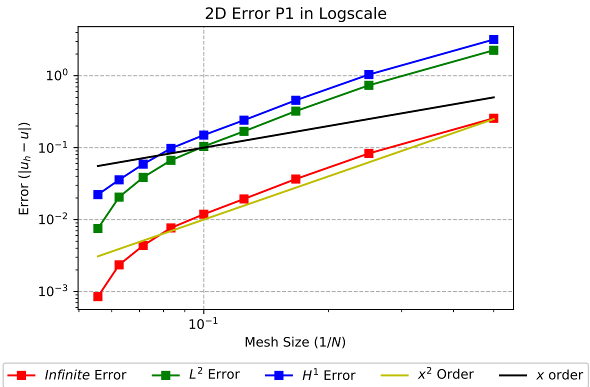
k is index of the current solution and ϵ is a small number, $\epsilon > 0$.
There are three types of error that we will compute, Infinity Error, $H^1(\Omega)$ and $L^2(\Omega)$, each of them defined by:

$$||u_h - u||_\infty = \max |u_h - u| \tag{12}$$

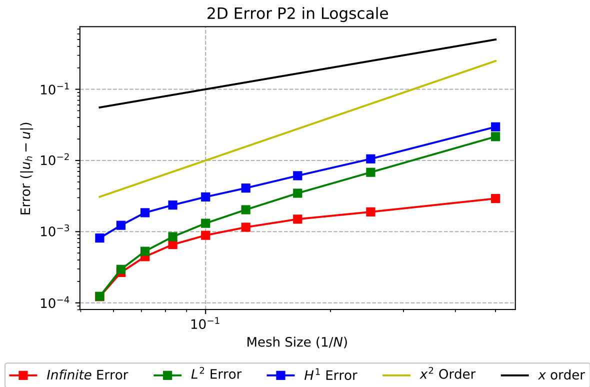
$$||u||^2_{H^1(\Omega)} = \int_\Omega |u|^2 dx + \int_\Omega |\nabla u|^2 dx \tag{13}$$

$$||u||^2_{L^2(\Omega)} = \int_\Omega |u|^2 dx \tag{14}$$

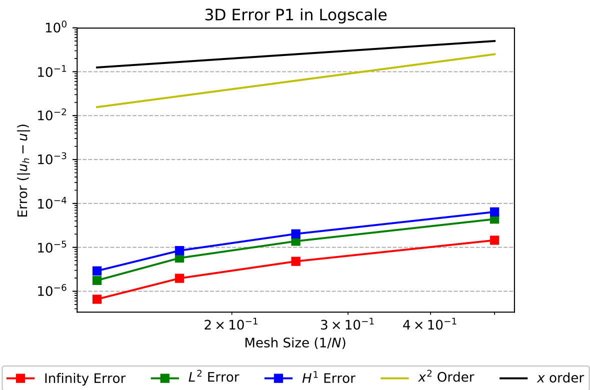
We can see the result from 2D and 3D case in the figure 6 below:



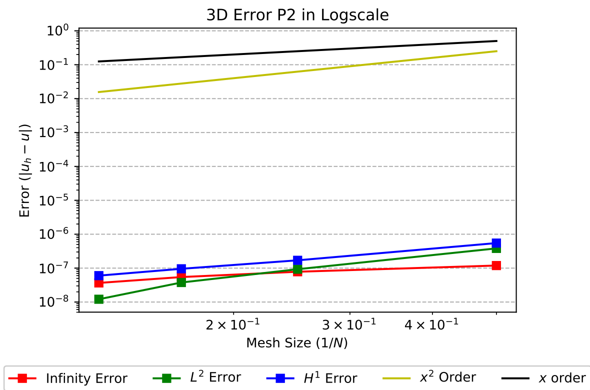
(a) 2D Error P1 Plot in Logscale



(b) 2D Error P2 Plot in Logscale



(c) 3D Error P1 Plot in Logscale



(d) 3D Error P2 Plot in Logscale

Figure 6: Error Plot in Logscale

5 Manufactured Solution

We define error estimation

$$error := ||u_n - u_{exact}|| = O(h^{\alpha}) \quad \alpha = \text{EOC(Experimental Order of Convergence)} \tag{15}$$

We try to solve linear elasticity based on exact solution defined by:

$$u_{exact} = \begin{pmatrix} u1 \\ u2 \end{pmatrix} = \begin{pmatrix} \sin(x_1) \\ -\sin(x_2) \end{pmatrix} \tag{16}$$

Using (16) we build linear elasticity problem as follows:

$$\begin{cases} -div\ \sigma[u] = f(x), \quad f(x) = \begin{pmatrix} \lambda \sin x_1 + 2\mu \sin x_1 \\ -\lambda \sin x_2 - 2\mu \sin x_2 \end{pmatrix} & \text{in } \Omega \\ u = g(x), \quad g(x) = \begin{pmatrix} \sin(x_1) \\ -\sin(x_2) \end{pmatrix} = \begin{pmatrix} \pm 1 \\ -\sin(x_2) \end{pmatrix} & \text{on } \Gamma_D \\ \sigma[u]\nu = q(x), \quad q(x) = \begin{pmatrix} 0 \\ \pm\lambda\cos(x_1) \end{pmatrix} & \text{on } \Gamma_N \end{cases} \tag{17}$$

Solving (17) we get the result as shown in Figure 7 and 8

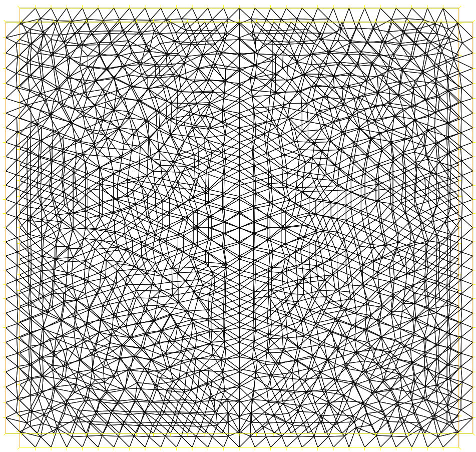


Figure 7: Deformation of Manufactured Solution

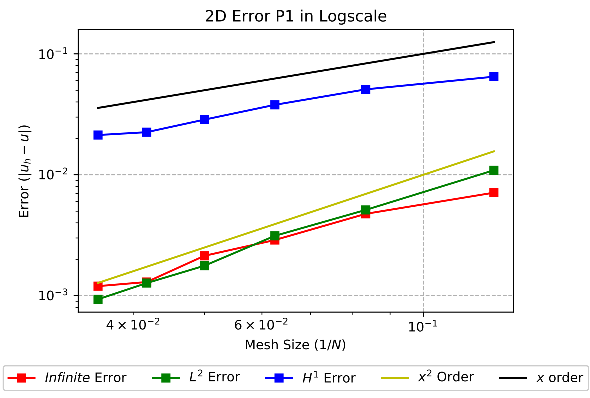


Figure 8: Error plot in Logscale

6 The Convergence of variational Approach to Crack Propagation

6.1 Simulation

We solved problem (6) using model as shown in figure 9 and numerical parameter like in table 1, the difference from linear elasticity problem is the domain. In the crack propagation case, we disturb the domain with adding some so called "crack path" into the domain. For this case, we use n(division number) = 32 and d(width of the crack) = 0.01 [m]. The result of our simulation shown the figure 10.

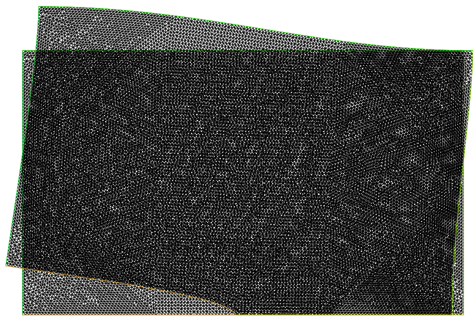


Figure 9: Deformation of Cracked Domain

6.1.1 Convergence Analysis

Since we don't have an exact solution for problem (6), we will define

$$u_h := u^k \iff \max |u^k - u^{k-1}| \leq \epsilon$$

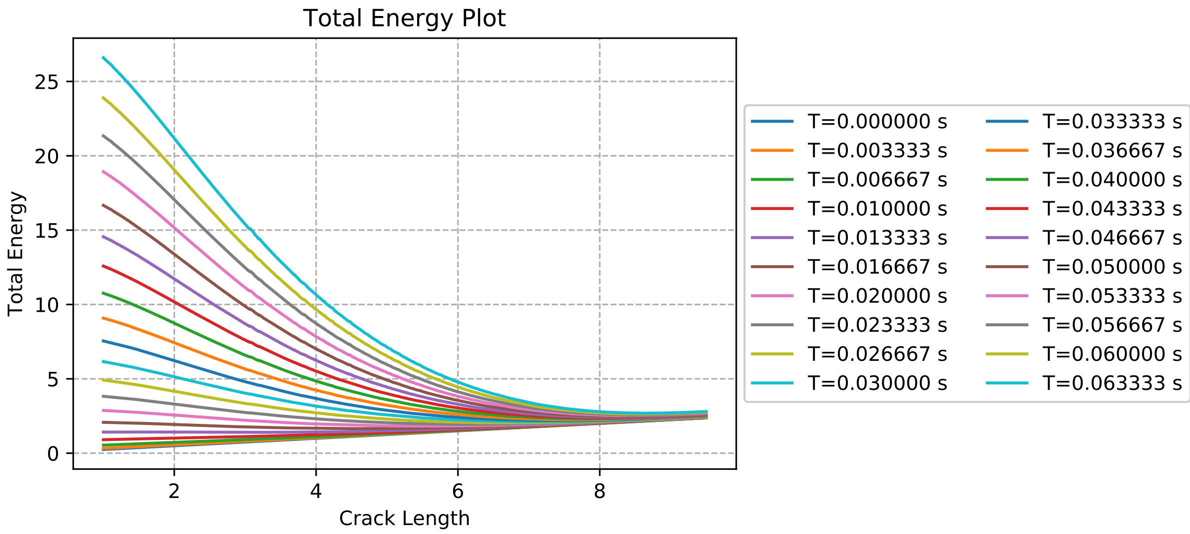
k is index of the current solution and ϵ is a small number, $\epsilon > 0$.
There are three types of error that we will compute, Infinity Error, $H^1(\Omega)$ and $L^2(\Omega)$, each of them defined by:

$$||u_h - u||_\infty = \max |u_h - u| \tag{18}$$

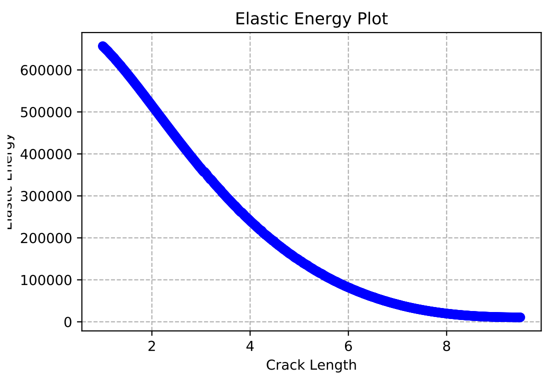
$$||u||^2_{H^1(\Omega)} = \int_\Omega |u|^2 dx + \int_\Omega |\nabla u|^2 dx \tag{19}$$

$$||u||^2_{L^2(\Omega)} = \int_\Omega |u|^2 dx \tag{20}$$

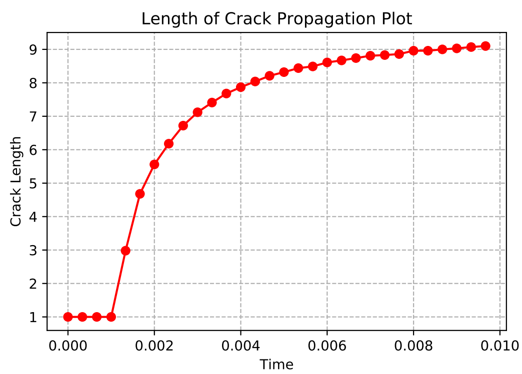
We can see the result from 2D case in the figure 11 below:



(a) Total Energy



(b) Elastic Energy



(c) Crack Length

Figure 10: Elastic Energy, Crack Length, and Total Energy

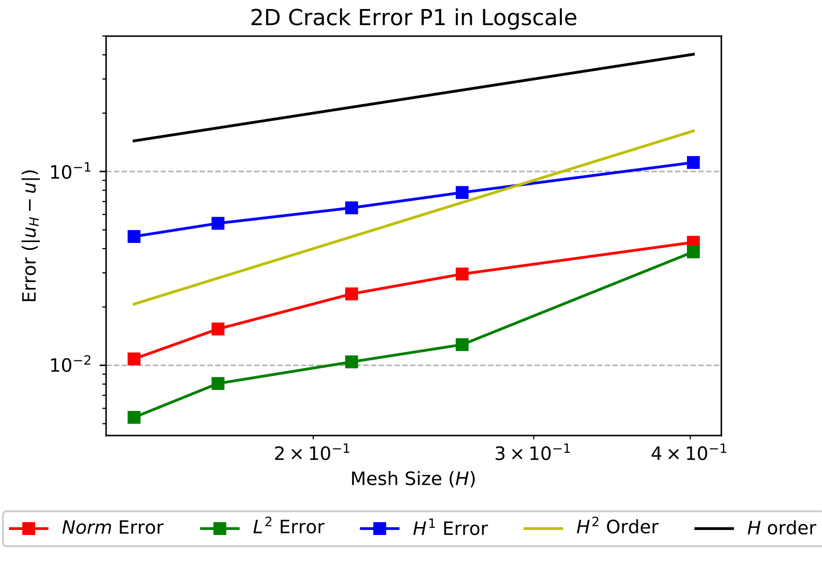


Figure 11: Convergence on 2D Cracked Domain

6.2 Case Study

We solved problem (6) using model as shown in figure 12 and numerical parameter like in table 1, the difference from simulation case is the domain. In this case study, we change the width of the cantilever beam to 0.3[m]. The result is shown the figure 13.

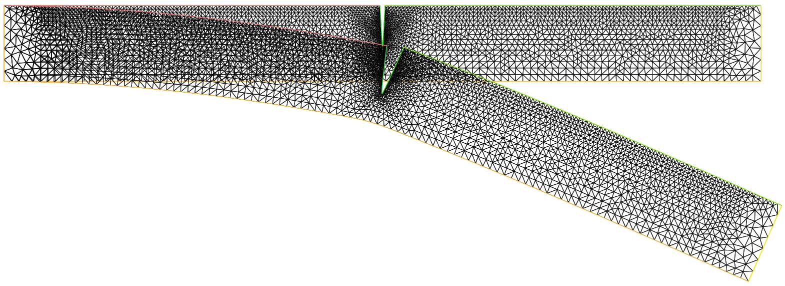


Figure 12: Deformation of Cracked Domain

6.3 Convergence Analysis

Since we don't have an exact solution for problem (6), we will define

$$u_h := u^k \iff \max |u^k - u^{k-1}| \le \epsilon$$

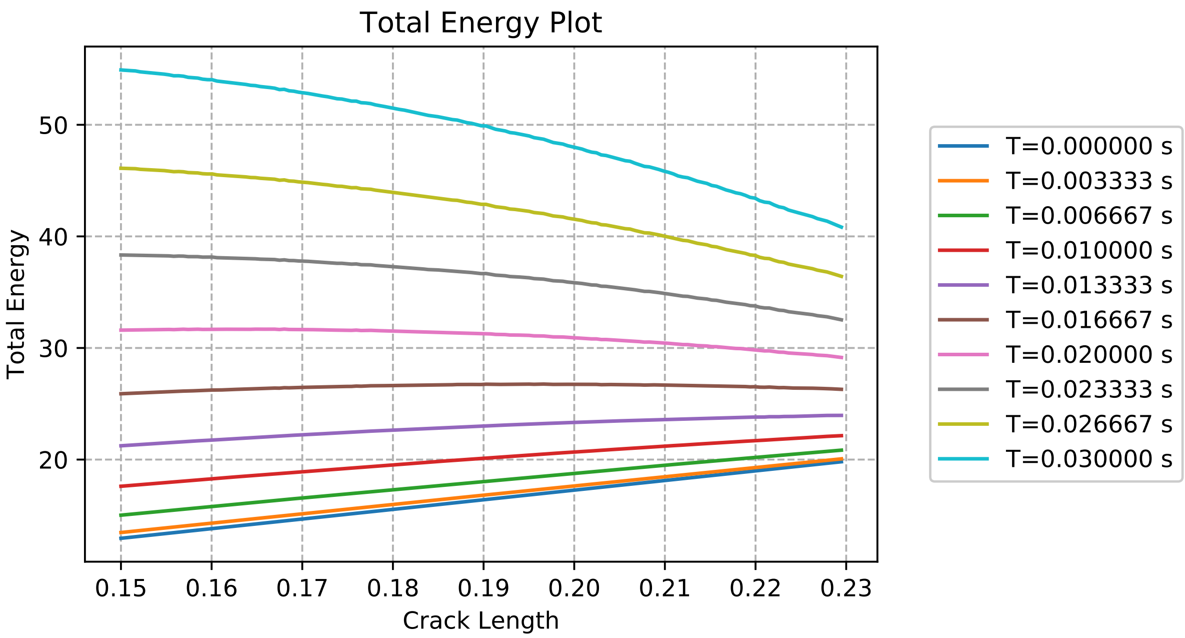
k is index of the current solution and ϵ is a small number, $\epsilon > 0$.
There are three types of error that we will compute, Infinity Error, $H^1(\Omega)$ and $L^2(\Omega)$, each of them defined by:

$$||u_h - u||_\infty = \max |u_h - u| \tag{21}$$

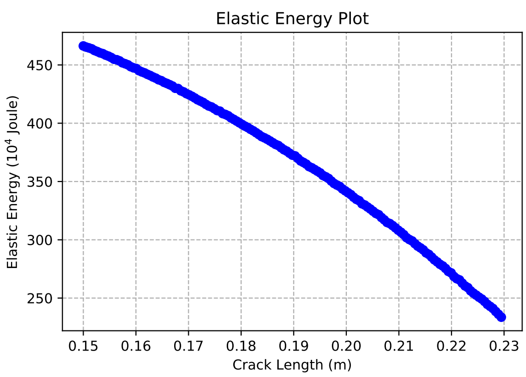
$$||u||^2_{H^1(\Omega)} = \int_\Omega |u|^2 dx + \int_\Omega |\nabla u|^2 dx \tag{22}$$

$$||u||^2_{L^2(\Omega)} = \int_\Omega |u|^2 dx \tag{23}$$

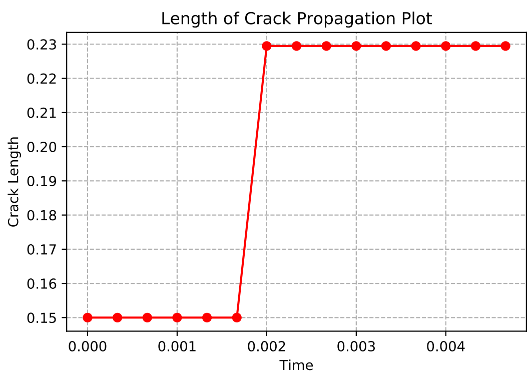
We can see the result from 2D case in the figure 14a below:



(a) Total Energy

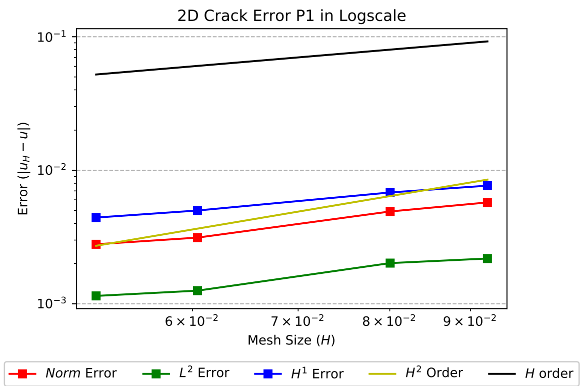


(b) Elastic Energy

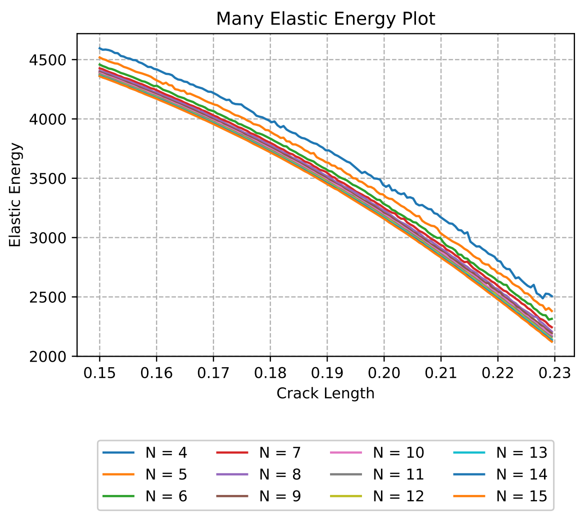


(c) Crack Length

Figure 13: Elastic Energy, Crack Length, and Total Energy



(a) Convergence on 2D Cracked Domain



(b) Elastic Energy of Various Division Number (N)

Figure 14: Convergence Analysis

6.4 3D Crack Propagation Study Case

We solved problem (6) in 3D using model as shown in figure 15 and numerical parameter like in table 1. The result is shown the figure 16.

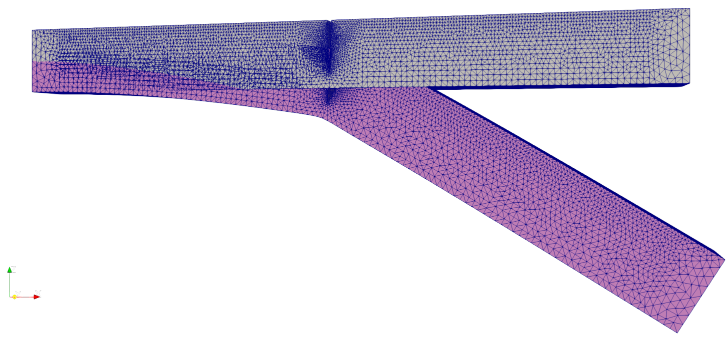
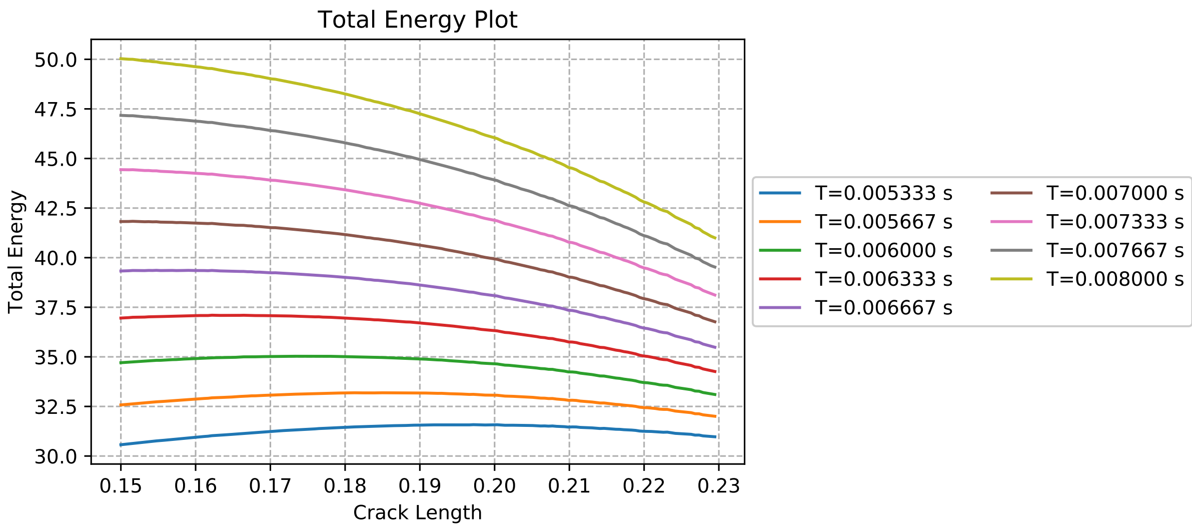
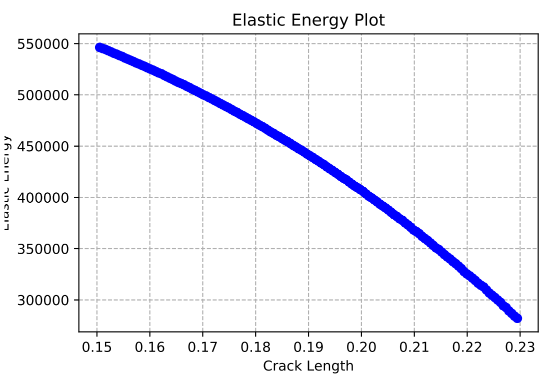


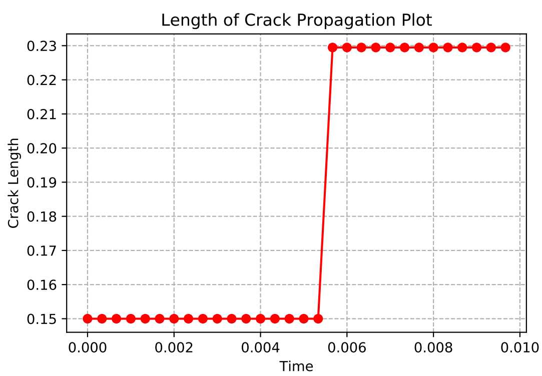
Figure 15: Deformation of Cracked Domain in 3D



(a) Total Energy in 3D



(b) Elastic Energy in 3D



(c) Crack Length

Figure 16: Elastic Energy, Crack Length, and Total Energy

6.5 Crack Location and Geometry Analysis

Figure 18 show the result from various type of crack location based on table 2 and crack dimension in figure 17.

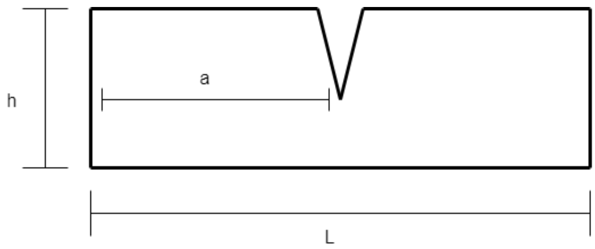
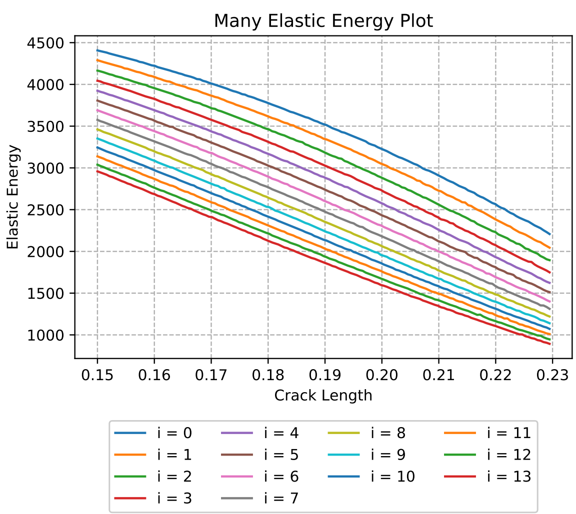


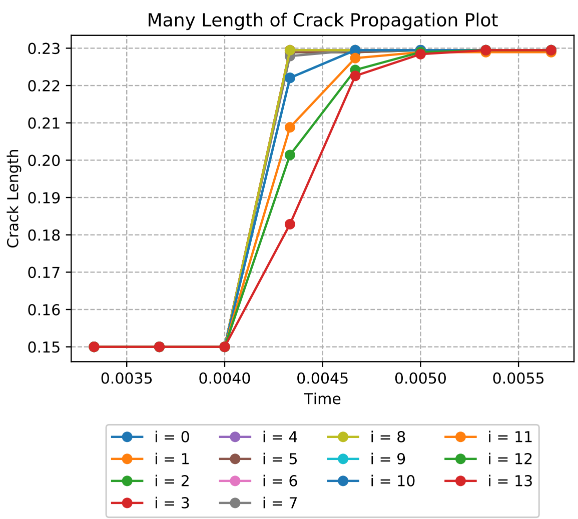
Figure 17: Crack Dimension

Number (i)	a[m]
0	1.5
1	1.4
2	1.3
3	1.2
4	1.1
5	1
6	0.9
7	0.8
8	0.7
9	0.6
10	0.5
11	0.4
12	0.3
13	0.2

Table 2: Variation of Crack Location (a)



(a) Various Elastic Energy



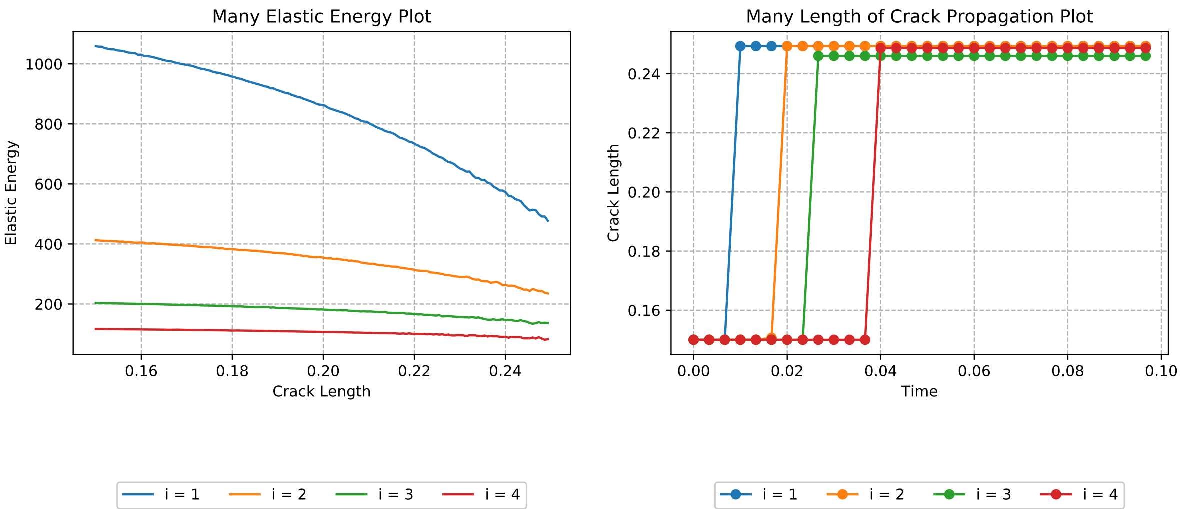
(b) Various Crack Length

Figure 18: Elastic Energy, Crack Length of Various Crack Location

While figure 19 show the result of various geometry ratio in table 3.

Number (i)	L (m)	h (m)	Ratio
0	3	0.3	10
1	5	0.3	16.67
2	7	0.3	23.33
3	9	0.3	30
4	11	0.3	36.67

Table 3: Variation of Geometry Ratio



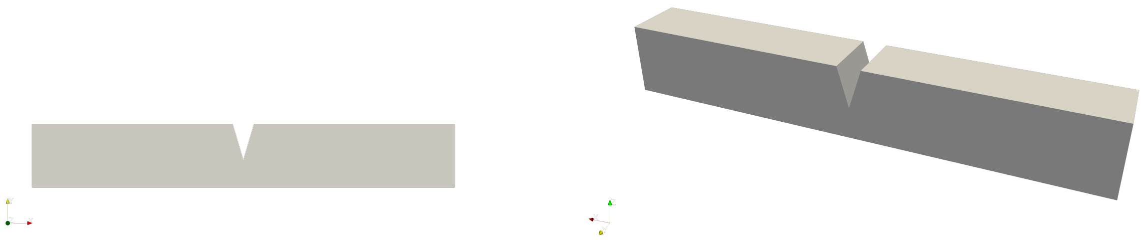
(a) Various Elastic Energy

(b) Various Crack Length

Figure 19: Elastic Energy, Crack Length of Various Geometry Ratio

6.6 Comparison 3D and 2D Model

In this section we will discuss about relevance between 2 dimension crack model and 3 dimension crack model, how they related one and each other, we will focus on the elastic energy to compare. As seen in figure 20 the difference in 3D and 2D model is just in the width of the beam (w).



(a) 2D Crack Model (b) 3D Crack Model

Figure 20: 2D and 3D Crack Model

Therefore, we can write the solution of deformation in 3D as:

$$u = \begin{pmatrix} u_1(x_1, x_2, x_3) & \cdots & \cdots \\ u_2(x_1, x_2, x_3) & \ddots & \vdots \\ u_3(x_1, x_2, x_3) & \cdots & \cdots \end{pmatrix}$$

however in 2D we write the solution as:

$$\tilde{u} = \begin{pmatrix} \tilde{u_1}(x_1, x_2) & \cdots \\ \tilde{u_2}(x_1, x_2) & \cdots \end{pmatrix}$$

Moreover, the elastic energy in 3D can be written as,

$$E(u) = \frac{1}{2} \int_{\Omega \backslash \Sigma} \sigma : e \; dx = E(A) + \gamma A$$

while in 2D we write the elastic energy as,

$$\tilde{E}(\tilde{u}) = \frac{1}{2} \int_{\Omega' \backslash \Sigma'} \tilde{\sigma} : \tilde{e} \; d\tilde{x} = \tilde{E}(L) + \gamma L$$

Using above notation, we assume 3D model and 2D model related as,

$$\begin{aligned} E(u) = E(A) + \gamma A &= E(wL) + \gamma(wL) \\ &\approx w\tilde{E}(L) + w\gamma L \\ &\approx w(\tilde{E}(L) + \gamma L) \end{aligned}$$

then, we try to prove this consideration by multiplying elastic energy profile in 2D by w as shown in 21,

6.7 Dependency on Stress Intensity Factor (K_{IC})

In previous section, we defined elastic energy (E) which can be calculated by an independent variable γ, which later we call γ as the *Fracture Toughness* of a material, in some references, it also depend on the Stress Intensity Factor (K_{IC}) addressed as,

$$\gamma = \frac{K_{IC}^2}{E}$$

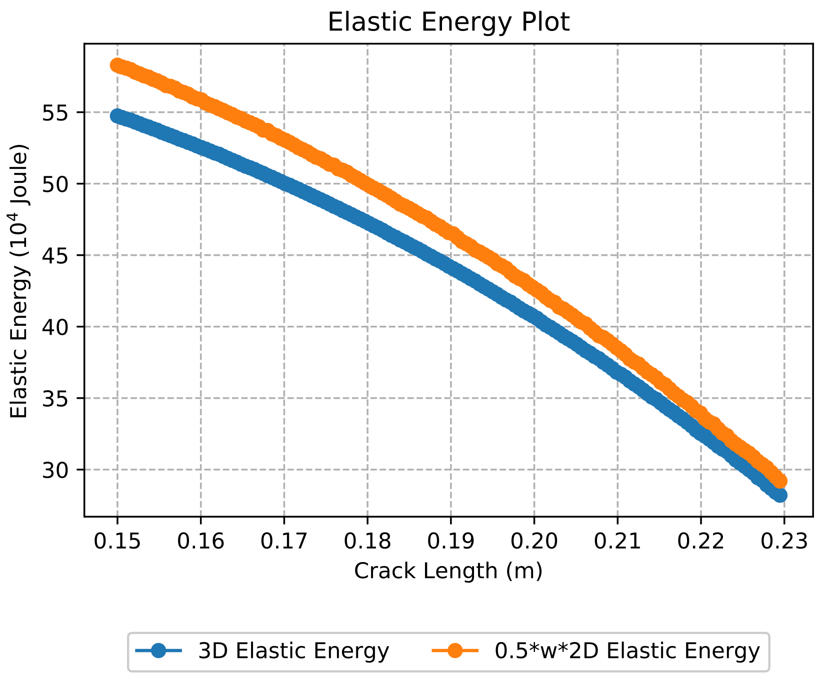
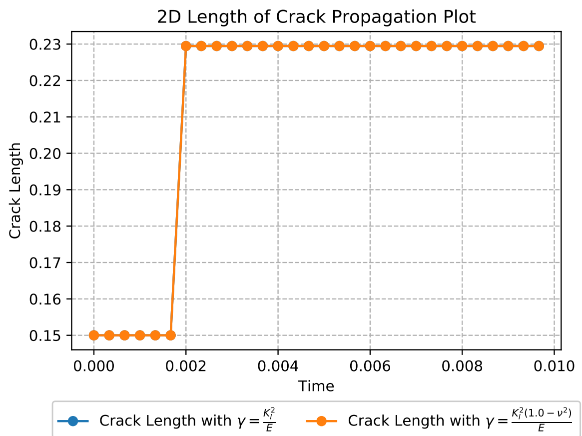


Figure 21: Elastic Energy Comparison 2D and 3D

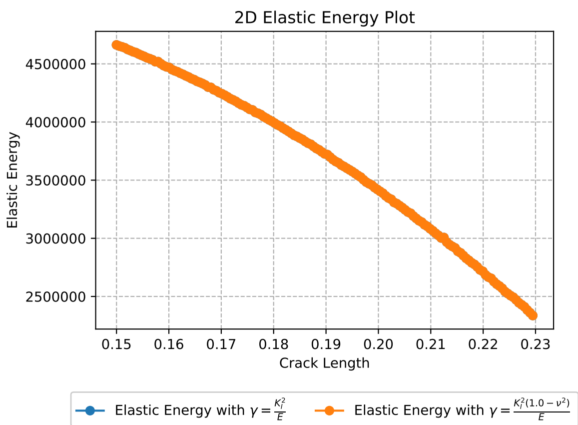
in 3D, while in 2D, we calculated γ as follows,

$$\tilde{\gamma} = \frac{K_{IC}^2}{E'} = \frac{K_{IC}^2(1-\nu^2)}{E}$$

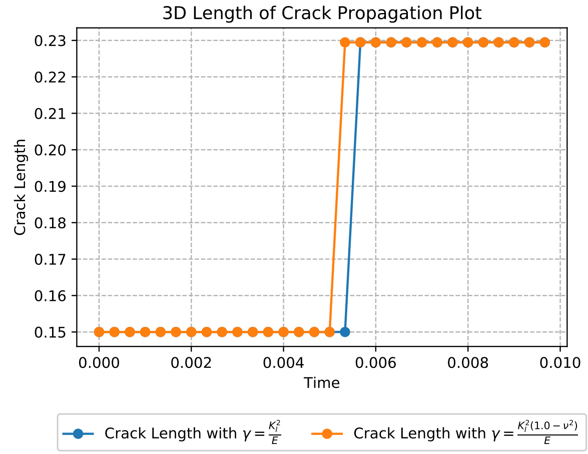
below is the result, where we compare elastic energy in 2D and 3D with and without involved K_{IC} in solving the crack propagation problem.



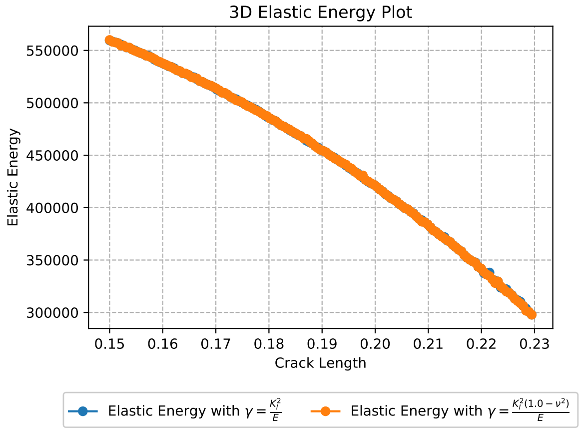
(a) Comparison on 2D Crack Length with Stress Intensity Factor (K_{IC})



(b) Comparison on 2D Elastic Energy with Stress Intensity Factor (K_{IC})



(a) Comparison on 3D Crack Length with Stress Intensity Factor (K_{IC})



(b) Comparison on 3D Elastic Energy with Stress Intensity Factor (K_{IC})

6.8 Multiple Crack Location

In this section, we try to give crack path in two location, like shown in figure 24, then solve the same problem as written in equation 6. The deformation is shown in figure 25 Below is the

Figure 24: Model of Multiple Crack Location

energy profile, total energy, and crack length calculated from the left crack,

6.8.1 Contour Plot Elastic Energy

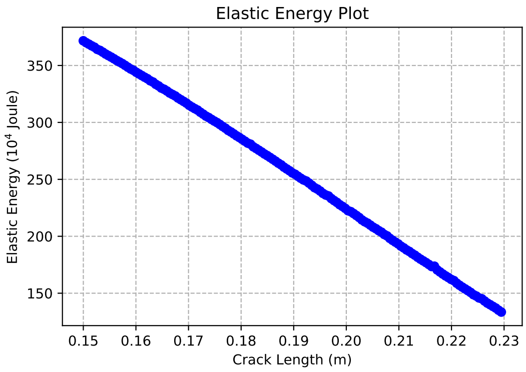
For better understanding about the elastic energy profile of multiple crack location, we try to use contour plot, like shown in figure from figure 28 we consider a "gradient flow" (a curve of maximal slope) of E w.r.t. $|.|_p$

$$x = (x_1, x_2), \begin{cases} |x|_p := (|x_1|^p + |x_2|^p)^{\frac{1}{2}} \ 1 \leq p < \infty \\ |x|_\infty := \max(|x_1|, |x_2|) \ p = \infty \end{cases} \tag{24}$$

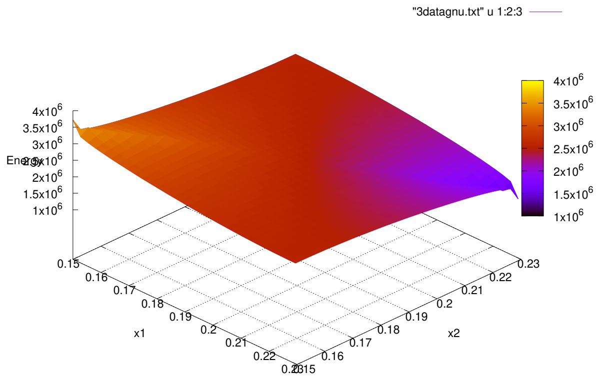
When $x(t) = (x_1(t), x_2(t))$ is "gradient flow"?
in $\underline{\text{p=2 case}}$,
$\frac{d}{dt}E(x(t))$: How the energy changes along the curve $x(t)$.
$\frac{d}{dt}E(x(t)) = \nabla E(x(t)).x'(t)$

To find the velocity (= speed and direction): $x'(t) = |x'(t)|\frac{x'(t)}{|x'(t)|} = vn \begin{cases} v \geq 0 \\ n \in \mathbb{R}^2 \\ |n| = 1 \end{cases}$

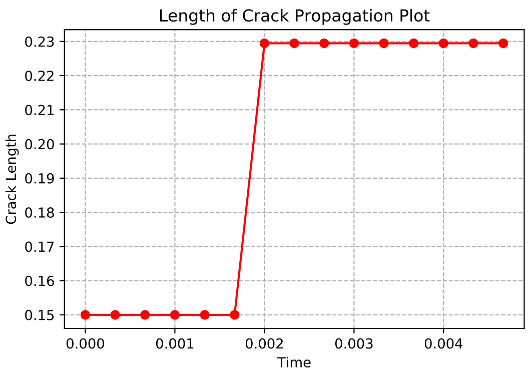
Figure 25: Deformation of Multiple Crack Location



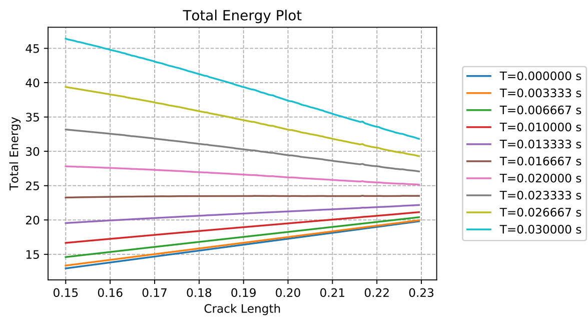
(a) Multiple Crack Elastic Energy Profile



(b) Multiple Crack Location Surface Energy Profile



(a) Multiple Crack Location Crack Length



(b) Multiple Crack Total Energy

which direction is the most effective to reduce the energy?

$$\min_{|n|=1}\left[\frac{d}{dt}E(x(t))\right] = \min_{|n|=1}\left[\nabla E(x(t)).(vn)\right] = v\min_{|n|=1}\left[\nabla E(x(t)).n\right] \tag{25}$$

before we proceed to our problem, let's take an example with $a := (a_1, a_2)$:

$$\min_{|n=1|}(a.n) = \min_{n_1^2+n_2^2=1}(a_1n_1 + a_2n_2) \tag{26}$$

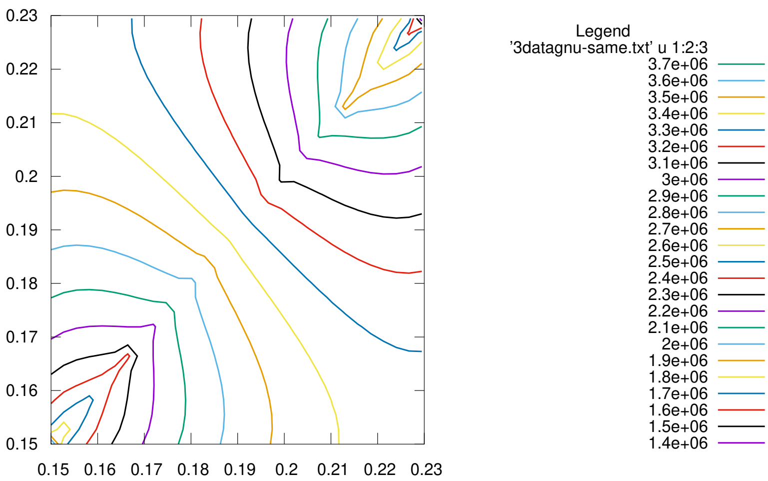


Figure 28: Contour Plot of Multiple Crack Elastic energy with tip 1 and tip 2 moving at the same time

if $a//n$ in the same direction,

$$\begin{aligned} \Rightarrow a &= |a|n \\ n &= \frac{a}{|a|} \\ a.n &= a.\left(\frac{a}{|a|}\right) = \frac{|a|^2}{|a|} = |a| \end{aligned} \tag{27}$$

suppose that $f(n) = (a.n)$, to find the minimizer, we take "gradient" of the function $(f(n))$

$$\begin{aligned} -\nabla f(n) &= -\left(\frac{\partial f}{\partial n_1}, \frac{\partial f}{\partial n_2}\right) \\ &= -\left(\frac{\partial (a.n)}{\partial n_1}, \frac{\partial (a.n)}{\partial n_2}\right) \\ &= -\left(\frac{\partial (a_1 n_1 + a_2 n_2)}{\partial n_1}, \frac{\partial (a_1 n_1 + a_2 n_2)}{\partial n_2}\right) \\ &= -(a_1, a_2) \end{aligned} \tag{28}$$

because we know that $n = \frac{a}{|a|}$, the direction of minimizer of $f(n)$ is

$$n = \frac{a}{|a|} = \frac{-(a_1, a_2)}{\sqrt{a_1^2 + a_2^2}} \tag{29}$$

we applied the same technique to find minimizer of E,
suppose that $f(n) = (\nabla E(x(t)).n)$, to find the minimizer, we take "gradient" of the function $(f(n))$

$$\begin{aligned} -\nabla f(n) &= -\left(\frac{\partial f}{\partial n_1}, \frac{\partial f}{\partial n_2}\right) \\ &= -\left(\frac{\partial (\nabla E(x(t)).n)}{\partial n_1}, \frac{\partial (\nabla E(x(t)).n)}{\partial n_2}\right) \end{aligned} \tag{30}$$

Because we have to guess which direction give the most effective path to reduce the energy, we try to use some k value to look at the graph of the elastic energy like shown in the figure 29 below.

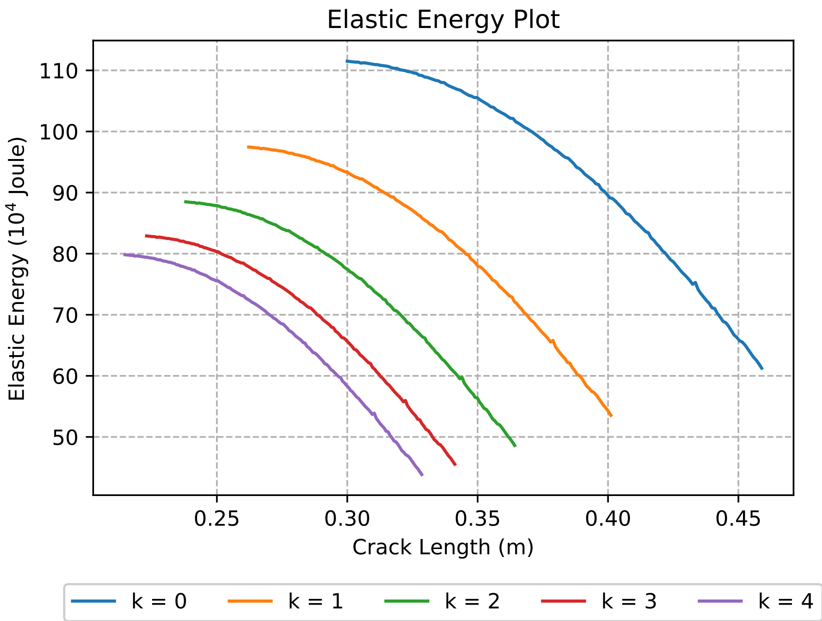


Figure 29: Elastic Energy Profile on some k value

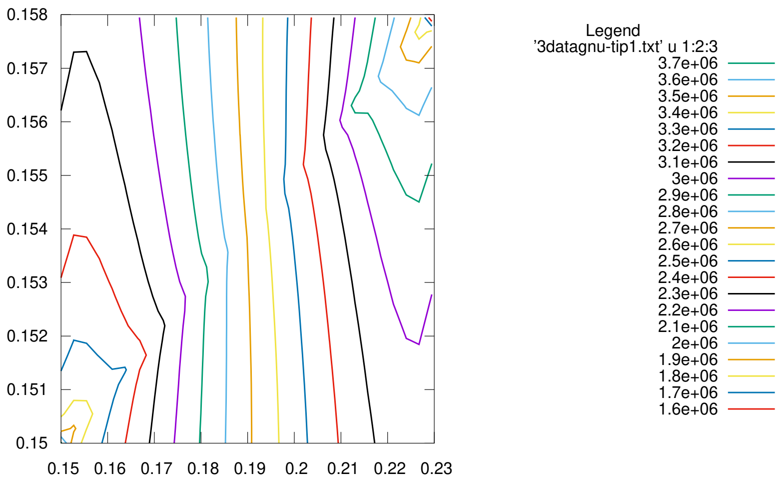


Figure 30: Contour Plot of Multiple Crack Elastic energy with tip 1 move 10 percent faster than tip 2

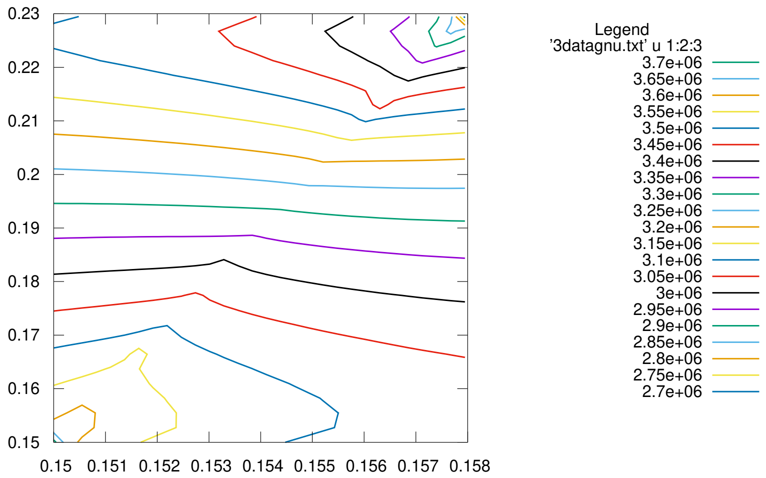


Figure 31: Contour Plot of Multiple Crack Elastic energy with tip 1 move 10 percent slower than tip 2

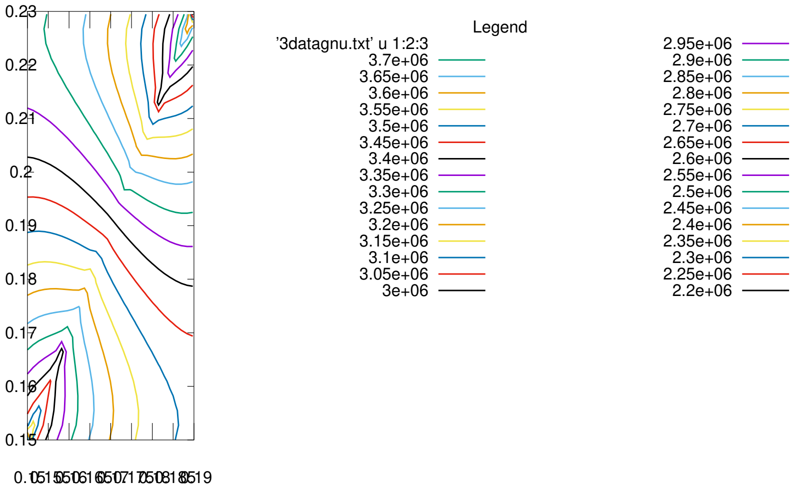


Figure 32: Contour Plot of Multiple Crack Elastic energy with tip 1 move 50 percent slower than tip 2