Analysis Ia Report

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Problem 1. *Suppose:*

$$d_2(f,g) := \left(\int_a^b \{f(x) - g(x)\}^2 dx \right)^{\frac{1}{2}}, f : [a,b] \to \mathbb{R} \tag{1}$$

we will show $(C([a,b]), d_2)$ *is a* ***metric space****.*

(a) ***Positivity***
Take: $h = \int_a^b \{f(x) - g(x)\}^2 dx$*, then equation* (1) *becomes:*

$$d_2(f,g) = \sqrt{h}$$

Since the value of $(f(x) - g(x))^2 \geq 0$ *then* $\sqrt{h} \geq 0,\ \forall h \in C[a,b]$*, then:*

$$d_2(f,g) \geq 0$$

(b) ***Definiteness***
$(\Leftarrow)$ *put* $f(x) = g(x)$*, then equation* (1) *becomes:*

$$\begin{aligned} d_2(f,g) &= \sqrt{\int_a^b \{f(x) - g(x)\}^2 dx} \\ &= \sqrt{\int_a^b \{g(x) - g(x)\}^2 dx} \\ d_2(f,g) &= 0 \end{aligned}$$

$(\Rightarrow)$ *put* $d_2(f,g) = 0$*, then equation* (1) *becomes:*

$$\begin{aligned} 0 &= \sqrt{\int_a^b \{f(x) - g(x)\}^2 dx} \\ 0^2 &= \left(\sqrt{\int_a^b \{f(x) - g(x)\}^2 dx}\right)^2 \\ 0 &= \int_a^b \{f(x) - g(x)\}^2 dx \end{aligned}$$

to satisfies $(\Rightarrow),\ \forall a,b \in \mathbb{R}$

$$\begin{aligned} f(x) - g(x) &= 0 \\ \therefore f(x) &= g(x) \end{aligned}$$

(c) ***Symmetry***

$$d_2(f,g) = \sqrt{\int_a^b \{f(x)-g(x)\}^2 dx}$$

$$d_2(f,g) = \sqrt{\int_a^b \Big(f(x)^2 - 2f(x)g(x) + g(x)^2\Big) dx}$$

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$$d_2(f,g) = d_2(g,f),\ \forall f,g \in C$$

(d) ***Triangle Inequality***

Put $g(x) \le h(x) \le f(x)$

$$d_2(f,g) = \sqrt{\int_a^b \{f(x)-h(x)+h(x)-g(x)\}^2 dx}$$

then we take $r(x) = f(x) - h(x)$,*and* $s(x) = h(x) - g(x)$

$$d_2(f,g) = \sqrt{\int_a^b r(x)^2 + 2r(x)s(x) + s(x)^2 dx}$$

$$= \sqrt{\int_a^b r(x)^2 dx + 2\int_a^b r(x)s(x)dx + \int_a^b s(x)^2 dx}$$

according to Cauchy-Schwartz inequality:

$$\int_a^b r(x)s(x)dx \le \sqrt{\int_a^b r(x)^2 dx}\sqrt{\int_a^b s(x)^2 dx}$$

thus:

$$d_2(f,g) \le \sqrt{\int_a^b r(x)^2 dx + 2\sqrt{\int_a^b r(x)^2 dx}\sqrt{\int_a^b s(x)^2 dx} + \int_a^b s(x)^2 dx}$$

$$\le \sqrt{\left(\sqrt{\int_a^b r(x)^2 dx} + \sqrt{\int_a^b s(x)^2 dx}\right)^2}$$

$$\le \sqrt{\int_a^b r(x)^2 dx} + \sqrt{\int_a^b s(x)^2 dx}$$

$$d_2(f,g) \le \sqrt{\int_a^b \{f(x)-h(x)\}^2 dx} + \sqrt{\int_a^b \{h(x)-g(x)\}^2 dx}$$

$$\therefore d_2(f,g) \le d_2(f,h) + d_2(h,g)$$

Problem 2. *Let* (X,d) *be a metric space, suppose:*

$$\tilde{d}(x,y) := \frac{d(x,y)}{1+d(x,y)}\ (x,y \in X) \tag{2}$$

we will show that $\tilde{d}(x,y)$ *is also* ***metric*** *on* X

(a) ***Positivity***
Since $d(x,y)$ *is a metric, it satisfies* $d(x,y) \geq 0$*, thus:*

$$\tilde{d}(x,y) = \frac{d(x,y)}{1+d(x,y)} \geq 0$$

(b) ***Definiteness***
$(\Leftarrow)$ *put* $x = y$
Since $d(x,y)$ *is a metric, it satisfies* $x = y \Rightarrow d(x,y) = 0$*, thus:*

$$\tilde{d}(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{0}{1+0} = 0$$

$$\therefore \tilde{d}(x,y) = 0 \Leftarrow x = y$$

$(\Rightarrow)$ *put* $\tilde{d}(x,y) = 0$

$$\begin{aligned} \tilde{d}(x,y) &= \frac{d(x,y)}{1+d(x,y)} \\ 0 &= \frac{d(x,y)}{1+d(x,y)} \end{aligned}$$

to satisfies the equation, $d(x,y) = 0$
and since $d(x,y)$ *is a metric, it satisfies* $d(x,y) = 0 \Rightarrow x = y$ *then* x *should be equal to* y

$$\therefore \tilde{d}(x,y) = 0 \Rightarrow x = y$$

(c) ***Symmetry***
Since $d(x,y)$ *is a metric, it satisfies* $d(x,y) = d(y,x)$*,*
then we can rewrite equation (2) as:

$$\begin{aligned} \tilde{d}(x,y) &= \frac{d(x,y)}{1+d(x,y)} \\ &= \frac{d(y,x)}{1+d(y,x)} = \tilde{d}(y,x) \\ \therefore \tilde{d}(x,y) &= \tilde{d}(y,x) \end{aligned}$$

(d) ***Triangle Inequality***
Since $d(x,y)$ *is a metric, it satisfies* $d(x,y) \leq d(x,z) + d(z,y), \ \forall x,y,z \in X$*,*
then we can rewrite equation (2) as:

$$\begin{aligned} \tilde{d}(x,y) = \frac{d(x,y)}{1+d(x,y)} &\leq \frac{d(x,z)+d(z,y)}{1+d(x,z)+d(z,y)} \\ &\leq \frac{d(x,z)}{1+d(x,z)+d(z,y)} + \frac{d(z,y)}{1+d(x,z)+d(z,y)} \\ &\leq \tilde{d}(x,z) + \tilde{d}(z,y) \\ \therefore \tilde{d}(x,y) &\leq \tilde{d}(x,z) + \tilde{d}(z,y) \end{aligned}$$