Assignment 5
Analysis I Report

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Problem 1. *Prove that every sequentially compact subset of a metric space is bounded and closed.*

Answer: Let X be a metric space and $K \subset X$ be compact. If x_n is a convergent sequence in K with limit $x \in X$, then every subsequence of x_n converges to x. Since K is compact, some subsequence of x_n converges to a limit in K, so $x \in K$ and K is closed.

We use contradiction to prove bounded. Assume K is closed and bounded. Take sequence $X_n \subset \mathbb{R}^n$ where $X_n = (x_1^n, \cdots, x_N^n)$. Since X_n is bounded, each of the sequences $(x_j^n), i \leq j \leq N$, is bounded. Since every bounded sequence in $\mathbb{R}$ has a converging subsequence in K, then seting $X(x_1, \cdots, x_N)$, we have that $x_{n_k} \to x \in \mathbb{R}^N$. Since K is closed, $X \in K$ and K is compact.

Problem 2. *Let X be a Banach space and let f, g be linear operators on X, $f : X \to X$ is compact and $g : X \to X$ is continuous. Prove that the composition maps $g \circ f$ and $f \circ g$ are compact.*

Answer:

Proposition 1. *Let $(X, d_x), (Y, d_y)$: metric space $f : X \to Y$ continuous, then $M \subset X$: compact $\Rightarrow f(M)$ is compact in Y*

Definition 1. X, Y : Banach space, $f : M \subset X \to Y$. f is called a compact mapping if f is continuous and $\forall B \subset M$ bounded, $f(B)$ is relatively compact

1. *$f \circ g$ is compact.* Since f is compact, by definition (1), $f(x)$ is continuous and by proposition (1), a continuous function map compact sets into compact sets, therefore $f \circ g$ is compact. □

2. *$g \circ f$ is compact.* Since g is continuous, by proposition (1), a continuous function map compact sets into compact sets, moreover f is compact, hence subset of f is also compact, therefore $g \circ f$ is compact. □

Problem 3. *Let* $X = C[0,1]$ *and* $||u|| := \max_{0 \le x \le 1} |u(x)|$. *For given* $\alpha \in \mathbb{R}$ *and* $f \in X$, *consider the nonlinear integral equation:*

$$u(x) = \alpha \int_0^1 \sin u(x) dx + f(x) \quad (*)$$

1. *Show that if* $|\alpha| < 1$, $(*)$ *has a unique solution* $u \in X$. *(Hint: Contraction mapping principle)*
2. *(extra) Consider the case* $|\alpha| \ge 1$. *Does* $(*)$ *has a solution* $u \in X$?

Answer:

1. Let $g(u(x)) = \alpha \int_0^1 \sin u(x) dx + f(x)$, and $g(v(x)) = \alpha \int_0^1 \sin v(x) dx + f(x)$, taking distance of $g(u(x))$ and $g(v(x))$, we get

$$\begin{aligned} d(g(u(x)) - g(v(x))) &= \left| \alpha \int_0^1 \sin u(x) dx + f(x) - \alpha \int_0^1 \sin v(x) dx - f(x) \right| \\ &= \left| \alpha \int_0^1 (\sin u(x) - \sin v(x)) dx \right| \end{aligned} \tag{1}$$

   by triangle inequality,

$$d(g(u(x)) - g(v(x))) \le |\alpha| \left| \int_0^1 (\sin u(x) - \sin v(x)) dx \right| \tag{2}$$

   Since $X = C[0,1]$ and $||u|| := \max_{0 \le x \le 1} |u(x)|$, we can rewrite (2) as follows:

$$\begin{aligned} d(g(u(x)) - g(v(x))) &\le |\alpha| \max_{0 \le x \le 1} \left| \int_0^1 (\sin u(x) - \sin v(x)) dx \right| \\ &\le |\alpha| \, ||u(x) - v(x)|| \end{aligned} \tag{3}$$

   Therefore, (*) satisfied contraction mapping principle, so that $u(x)$ has unique solution $u \in X$ if $|\alpha| < 1$.

2. Assume $u(x)$ and $v(x)$ are fixed point, then it should satisfies

$$0 \le d(u(x) - v(x)) = d(g(u(x)) - g(v(x))) < \alpha d(g(u(x)) - g(v(x))), \quad \alpha \in [0,1] \tag{4}$$

   if we choose $|\alpha| \ge 1$, then $u(x)$ doesn't satisfied (4), therefore $u(x)$ doesn't have an unique solution $u \in X$.