Basics of Applied Analysis A
Report

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Problem 1. *Let $N \in \mathbb{N}, N \geq 2$, and $\alpha > 0$ be given.*
We define $A = (aij)$ as finite difference discretization of :

$$\begin{cases} \alpha u - u'' = f \ \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{1}$$

with entries

$$a_{ij} = \begin{cases} \alpha + \frac{2}{h^2}, & i = j \\ -\frac{1}{h^2}, & |i-j| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where $h = \frac{1}{N}$

(a) *To find eigenvalues λ s.t. $Av = \lambda v$, We define $v_i := \sin\left(\frac{m\pi i}{N}\right)$ for some $m \in \mathbb{N}$, hence the discrete form of eigen problem becomes:*

$$\sum_{i=1}^{N} a_{ij} v_j = \lambda v_i$$

$$\left(\alpha + \frac{2}{h^2}\right) v_i - \frac{1}{h^2} v_{i+1} - \frac{1}{h^2} v_{i-1} = \lambda v_i$$

$$\left(\alpha + \frac{2}{h^2} - \lambda\right) v_i - \frac{1}{h^2}(v_{i+1} + v_{i-1}) = 0$$

substitute v_i to the discrete eigen problem, it becomes

$$\left(\alpha + \frac{2}{h^2} - \lambda\right) \sin\left(\frac{m\pi i}{N}\right) - \frac{1}{h^2}\left(\sin\left(\frac{m\pi(i+1)}{N}\right) + \sin\left(\frac{m\pi(i-1)}{N}\right)\right) = 0$$

$$\left(\alpha + \frac{2}{h^2} - \lambda\right) \sin\left(\frac{m\pi i}{N}\right) - \frac{1}{h^2}\left(2\sin\left(\frac{m\pi i}{N}\right)\cos\left(\frac{m\pi}{N}\right)\right) = 0$$

$$\sin\frac{m\pi i}{N}\left(\alpha + \frac{2}{h^2} - \lambda - \frac{1}{h^2} 2\cos\left(\frac{m\pi}{N}\right)\right) = 0$$

to find λ, we set $\sin\frac{m\pi i}{N}$ equal to some constant C, so that:

$$\left(\alpha + \frac{2}{h^2} - \lambda - \frac{2}{h^2}\cos\left(\frac{m\pi}{N}\right)\right) = 0$$

$$\therefore \lambda = \alpha + \frac{2}{h^2}\left(1 - \cos\left(\frac{m\pi}{N}\right)\right) \tag{3}$$

With Eigenvectors:

$$\therefore v = \sin\left(\frac{m\pi i}{N}\right)$$

(b) *We define spectral radius* $\sigma(A) := \max\{|\lambda|\}$, *using* λ *computed in* (3), *we get:*

$$\begin{aligned}\sigma(A) &= \max\{|\lambda|\} \\ &= \alpha + \frac{2}{h^2}\left(1 - \cos\left(\frac{m\pi}{N}\right)\right)\end{aligned}$$

then, thanks to the symmetricity of λ, *we can use taylor expansion to approximate the value of*

$$\cos\left(\frac{m\pi}{N}\right) \approx 1 - \frac{1}{2}\left(\frac{\pi}{N}\right)^2$$

, *hence the spectral radius* $\sigma(A)$ *can be calculated by:*

$$\begin{aligned}\sigma(A) &\approx \alpha + \frac{2}{h^2}\left(1 - \left(1 - \frac{m^2\pi^2}{2N^2}\right)\right) \\ &\approx \alpha + \frac{2}{h^2}\frac{m^2\pi^2}{2N^2} \\ &\therefore \sigma(A) \approx \alpha + (m\pi)^2\end{aligned}$$

(c) *Let:*

$$R := -D^{-1}(L+U)$$

be the Jacobi iteration matrix, we want to find the eigenvalues and eigenvectors s.t. $Rv = \lambda v$.

$$\begin{aligned}Rv =& \lambda v \\ -D^{-1}(L+U)v =& \lambda v \\ -(L+U)v =& \lambda D v\end{aligned}$$

the discrete form of the eigen problem is:

$$\frac{1}{h^2}(v_{i-1} + v_{i+1}) = \lambda\left(\alpha + \frac{2}{h^2}\right)v_i$$

and then we set, $v_i := \sin\left(\frac{m\pi i}{N}\right)$ *and substitute to the discrete form of the eigen problem, thus:*

$$\begin{aligned}\frac{1}{h^2}\left(\sin\left(\frac{m\pi(i+1)}{N}\right) + \sin\left(\frac{m\pi(i-1)}{N}\right)\right) &= \lambda\left(\alpha + \frac{2}{h^2}\right)\sin\left(\frac{m\pi i}{N}\right) \\ \frac{1}{h^2}\left(2\sin\left(\frac{m\pi i}{N}\right)\cos\left(\frac{m\pi}{N}\right)\right) &= \lambda\left(\alpha + \frac{2}{h^2}\right)\sin\left(\frac{m\pi i}{N}\right) \\ \sin\frac{m\pi i}{N}\left(\frac{2}{h^2}\cos\left(\frac{m\pi}{N}\right) - \lambda\left(\alpha + \frac{2}{h^2}\right)\right) &= 0\end{aligned}$$

to find λ, *we set* $\sin\frac{m\pi i}{N}$ *equal to some constant* D, *so that:*

$$\begin{aligned}\left(\frac{2}{h^2}\cos\left(\frac{m\pi}{N}\right) - \lambda\left(\alpha + \frac{2}{h^2}\right)\right) &= 0 \\ \lambda\left(\alpha + \frac{2}{h^2}\right) &= \frac{2}{h^2}\cos\left(\frac{m\pi}{N}\right) \\ \text{with } h &= \frac{1}{N} \\ \lambda &= \frac{\frac{2}{h^2}\cos\left(\frac{m\pi}{N}\right)}{\alpha + \frac{2}{h^2}}\end{aligned} \tag{4}$$

with $h = \frac{1}{N}$,

$$\therefore \lambda = \frac{2N^2 \cos\left(\frac{m\pi}{N}\right)}{\alpha + 2N^2}$$

(d) *We define spectral radius* $\sigma(R) := \max\{|\lambda|\}$*, using* λ *computed in* (4)*, we get:*

$$\begin{aligned}\sigma(R) &= \max\{|\lambda|\} \\ &= \frac{2N^2 \cos\left(\frac{m\pi}{N}\right)}{\alpha + 2N^2}\end{aligned}$$

again, thanks to the symmetricity of λ*, we can use taylor expansion to approximate the value of*

$$\cos\left(\frac{m\pi}{N}\right) \approx 1 - \frac{1}{2}\left(\frac{m\pi}{N}\right)^2$$

, hence the spectral radius $\sigma(R)$ *can be calculated by:*

$$\therefore \sigma(R) \approx \frac{2N^2\left(1 - \frac{m^2\pi^2}{2N^2}\right)}{\alpha + 2N^2}$$

Problem 2. *Let $N, h = \frac{1}{N}, \alpha$ and A be as in Problem 1. We define:*

$$\begin{cases} \alpha u - u'' = \sin(\pi x) \ \textit{in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{5}$$

(a) To find the exact solution of u, we use general solution for Ordinary Differential Equation (ODE):

$$u(x) = A\sin(\pi x) + B\cos(\pi x) \tag{6}$$

and take the second derivative of u

$$u''(x) = -A\pi^2 \sin(\pi x) + B\pi^2 \cos(\pi x) \tag{7}$$

then substitute (6) *and* (7) *to* (5)*, we get:*

$$\begin{aligned} \alpha(A\sin(\pi x) + B\cos(\pi x)) + A\pi^2 \sin(\pi x) + B\pi^2 \cos(\pi x) &= \sin(\pi x) \\ A(\alpha + \pi^2)\sin(\pi x) + B(\alpha + \pi^2)\cos(\pi x) &= \sin(\pi x) \end{aligned} \tag{8}$$

from (8) *we know that*

$$A(\alpha + \pi^2)\sin(\pi x) = \sin(\pi x)$$

and

$$B(\alpha + \pi^2)\cos(\pi x) = 0$$

$$\therefore A = \frac{1}{(\alpha + \pi^2)} \quad \therefore B = 0 \tag{9}$$

then we substitute (9) *to* (6)*, we get the exact solution as:*

$$u(x) = \frac{\sin(\pi x)}{(\alpha + \pi^2)} \tag{10}$$

(b) To find exact solution s.t. $Av = b$, with $b_i = \sin(\pi h i)$, we set

$$v_i = C\sin(\pi h i)$$

thus,

$$\begin{aligned} Av &= b \\ \sum_{i=1}^{N-1} a_{ij} v_i &= b_i \\ \left(\left(\alpha + \frac{2}{h^2}\right) v_i - \frac{1}{h^2}(v_{i+1} + v_{i-1})\right) &= \sin(\pi h i) \\ \left(\left(\alpha + \frac{2}{h^2}\right) C\sin(\pi h i) - \frac{1}{h^2}(C\sin(\pi h(i+1)) + C\sin(\pi h(i-1)))\right) &= \sin(\pi h i) \\ \left(\left(\alpha + \frac{2}{h^2}\right) C\sin(\pi h i) - C\frac{1}{h^2}(2\sin(\pi h i)\cos(\pi h))\right) &= \sin(\pi h i) \end{aligned}$$

divide by $\sin(\pi h i)$ on the both side we get:

$$C\left(\left(\alpha + \frac{2}{h^2}\right) - \frac{2}{h^2}\cos(\pi h)\right) = 1$$

$$\therefore C = \frac{1}{\left(\left(\alpha + \frac{2}{h^2}\right) - \frac{2}{h^2}\cos(\pi h)\right)}$$

so the exact solution in terms of v_i is:

$$v_i = \frac{\sin(\pi h i)}{\left(\left(\alpha + \frac{2}{h^2}\right) - \frac{2}{h^2}\cos(\pi h)\right)}$$

(c) *We define*

$$\epsilon(h) := \max_{1 \leq i \leq N-1} |u(hi) - v_i|$$

then we use $u(x)$ and v_i that we derived before to find the explicit formula of $\epsilon(h)$:

$$\epsilon(h) := \max_{1 \leq i \leq N-1} \left| \frac{\sin(\pi h i)}{(\alpha + \pi^2)} - \frac{\sin(\pi h i)}{\left(\left(\alpha + \frac{2}{h^2}\right) - \frac{2}{h^2}\cos(\pi h)\right)} \right|$$

$$\epsilon(h) := \max_{1 \leq i \leq N-1} \left| \sin(\pi h i) \left(\frac{1}{(\alpha + \pi^2)} - \frac{1}{\left(\left(\alpha + \frac{2}{h^2}\right) - \frac{2}{h^2}\cos(\pi h)\right)} \right) \right|$$

for simplicity, we rewrite:

$$D := \sin(\pi h i)$$

$$E := \left(\frac{1}{(\alpha + \pi^2)} - \frac{1}{\left(\left(\alpha + \frac{2}{h^2}\right) - \frac{2}{h^2}\cos(\pi h)\right)} \right)$$

using triangle inequality, we get:

$$\begin{aligned}\max_{1 \leq i \leq N-1} \left|DE\right| &\leq \max_{1 \leq i \leq N-1} \left|D\right| \max_{1 \leq i \leq N-1} \left|E\right| \\ &\leq 1 \left(\frac{1}{(\alpha + \pi^2)} - \frac{1}{\left(\left(\alpha + \frac{2}{h^2}\right) - \frac{2}{h^2}\cos(\pi h)\right)} \right)\end{aligned}$$

then, the explicit formula for $\epsilon(h)$ is:

$$\epsilon(h) \leq \left(\frac{1}{(\alpha + \pi^2)} - \frac{1}{\left(\left(\alpha + \frac{2}{h^2}\right) - \frac{2}{h^2}\cos(\pi h)\right)} \right) \tag{11}$$

around $h = 0$, leading order of taylor expansion of $\epsilon(h)$ can be obtained by expand $\cos(\pi h)$, so that

$$\cos(\pi h) \approx \cos(a) + \frac{\cos'(a)(\pi h - a)}{1!} + \frac{\cos''(a)(\pi h - a)^2}{2!} + \cdots$$

then we take $a = 0$, we get

$$\cos(\pi h) \approx 1 - \frac{1}{2}(\pi h)^2$$

substitute it to (11) *we get:*

$$\epsilon(h) \leq \left(\frac{1}{(\alpha + \pi^2)} - \frac{1}{\left(\left(\alpha + \frac{2}{h^2} \right) - \frac{2}{h^2}(1 - \frac{1}{2}(\pi h)^2) \right)} \right)$$

$$\epsilon(h) \leq \left(\frac{1}{(\alpha + \pi^2)} - \frac{1}{(\alpha + \pi^2)} \right)$$

which is just a constant, so the leading order is depend on what order of taylor expansion we choose to approximate the value of $\cos(\pi h)$ *around* $h = 0$. *In this case if we choose taylor expansion orde 2, we get the leading term of* ϵ *is* $\mathbb{O}(F)$ *with* F *is a constant.*

Problem 3. *We consider system of linear equations:*

$$\begin{cases} -v_{i-1,j} - v_{i+1,j} - v_{i,j-1} - v_{i,j+1} + 4v_{i,j} = b_{i,j} & i,j = 1,\cdots,N-1 \\ v_{0,j} = v_{0,N} = v_{i,0} = v_{i,N} = 0 & i,j = 1,\cdots,N-1 \end{cases} \tag{12}$$

for unknowns $v_{i,j}, i,j = 1,\cdots,N-1$.

(a) *We want to find the eigenvalues and eigenvectors of the matrix A for the system (12). Using the idea of discrete separation variables, we set $w_{i,j} = v_i\tilde{v}_j$ s.t.*

$$Aw_{i,j} = \lambda w_{i,j}$$

$$\begin{aligned} 4w_{i,j} - w_{i-1,j} - w_{i+1,j} - w_{i,j-1} - w_{i,j+1} &= \lambda w_{i,j} \\ 4(v_i\tilde{v}_j) - (v_{i-1}\tilde{v}_j) - (v_{i+1}\tilde{v}_j) - (v_i\tilde{v}_{j-1}) - (v_i\tilde{v}_{j+1}) &= \lambda(v_i\tilde{v}_j) \\ 4(v_i\tilde{v}_j) - \tilde{v}_j(v_{i-1} + v_{i+1}) - v_i(\tilde{v}_{j-1} + \tilde{v}_{j+1}) &= \lambda(v_i\tilde{v}_j) \\ (4-\lambda)(v_i\tilde{v}_j) - \tilde{v}_j(v_{i-1} + v_{i+1}) - v_i(\tilde{v}_{j-1} + \tilde{v}_{j+1}) &= 0 \end{aligned}$$

then, divide by $v_i\tilde{v}_j$ on both side, we get:

$$\begin{aligned} (4-\lambda) - \frac{(v_{i-1} + v_{i+1})}{v_i} - \frac{(\tilde{v}_{j-1} + \tilde{v}_{j+1})}{\tilde{v}_j} &= 0 \\ (4-\lambda) - \frac{(v_{i-1} + v_{i+1})}{v_i} &= \frac{(\tilde{v}_{j-1} + \tilde{v}_{j+1})}{\tilde{v}_j} \end{aligned}$$

now, to satisfies equality, both side should be equal to some constant, let say G, hence:

$$(4-\lambda) - \frac{(v_{i-1} + v_{i+1})}{v_i} = G = \frac{(\tilde{v}_{j-1} + \tilde{v}_{j+1})}{\tilde{v}_j}$$

First, we solve for:

$$\begin{aligned} \frac{(\tilde{v}_{j-1} + \tilde{v}_{j+1})}{\tilde{v}_j} &= G \\ (\tilde{v}_{j-1} + \tilde{v}_{j+1}) &= \tilde{v}_j G \end{aligned}$$

set: $\tilde{v}_j = \varphi^j$

$$\begin{aligned} \varphi^{j-1} + \varphi^{j+1} &= \varphi^j G \\ \varphi^{j-1} + \varphi^{j+1} - \varphi^j G &= 0 \end{aligned}$$

divide both side by φ^{j-1}

$$\begin{aligned} 1 + \varphi^2 - \varphi G &= 0 \\ \therefore \varphi_\pm &= \frac{G \pm \sqrt{(G^2 - 4)}}{2} \end{aligned} \tag{13}$$

here we recall that eigenvector of $\tilde{v}_j$ should be a linear combination of φ s.t.

$$\tilde{v}_j = c_1\varphi_+^j + c_2\varphi_-^j$$

and we know the boundary condition on (12), when $j = N$ then $\tilde{v}_N = 0$, to satisfies this condition, then $c_1 = -c_2$, then

$$\tilde{v}_j = c_1(\varphi_+^j - \varphi_-^j)$$

$\therefore$ φ_+^j and φ_-^j should be distinguished.
Hence, we choose $G < 2$ for (13) we then rewrite (13) as:

$$\begin{aligned} \varphi_\pm &= \frac{G \pm \sqrt{-(4 - G^2)}}{2} \\ \varphi_\pm &= \frac{G \pm i\sqrt{(4 - G^2)}}{2} \end{aligned}$$

then, the possible solution is

$$\varphi_{\pm}^{j} = \cos(\theta j) \pm i\sin(\theta j)$$

$$\begin{aligned}\therefore \tilde{v}_j &= c_1(\cos(\theta j) + i\sin(\theta j) - (\cos(\theta j) - i\sin(\theta j)))\\ \tilde{v}_j &= c_1(2i\sin(\theta j))\end{aligned}$$

we choose $c_1 = -\frac{i}{2}$ *so that*

$$\tilde{v}_j = \sin(\theta j)$$

inserting boundary condition, when $j = N$ *then* $\tilde{v}_N = 0$*, we get:*

$$\begin{aligned}0 &= \sin(\theta N)\\ \theta &= m\pi,\ \forall m : 2, 4, 6 \cdots N,\ \textit{even}\\ \theta &= \frac{m\pi}{N}\end{aligned}$$

then, we know that from imaginer triangle

$$\cos\theta = \frac{G}{2}$$

$\therefore G = 2\cos(\theta) = 2\cos(\frac{m\pi}{N})$
Second, we solve for:

$$(4 - \lambda) - \frac{(v_{i-1} + v_{i+1})}{v_i} = G$$

multiplied both side by v_i*, we get:*

$$\begin{aligned}v_i(4 - \lambda) - (v_{i-1} + v_{i+1}) &= v_i G\\ v_i(4 - \lambda) - (v_{i-1} + v_{i+1}) - v_i G &= 0\\ v_i(4 - \lambda - G) - (v_{i-1} + v_{i+1}) &= 0\end{aligned}$$

set $v_i = \xi^i$

$$\xi^i(4 - \lambda - G) - (\xi^{i-1} + \xi^{i+1}) = 0$$

divide both side by ξ^{i-1}

$$\begin{aligned}\xi(4 - \lambda - G) - (1 + \xi^2) &= 0\\ \xi^2 - \xi(4 - \lambda - G) + 1 &= 0\\ \therefore \xi_{\pm} &= \frac{(4 - \lambda - G) \pm \sqrt{(4 - \lambda - G)^2 - 4}}{2}\end{aligned}$$

take $2H = (4 - \lambda - G)$

$$\begin{aligned}\xi_{\pm} &= \frac{2H \pm \sqrt{(2H)^2 - 4}}{2}\\ \xi_{\pm} &= H \pm \sqrt{H^2 - 1}\end{aligned} \tag{14}$$

here we recall that eigenvector of v_i *should be a linear combination of* ξ *s.t.*

$$v_i = d_1\xi_+^i + d_2\xi_-^i$$

and we know the boundary condition on (12)*, when* $i = N$ *then* $v_N = 0$*, to satisfies this condition, then* $d_1 = -d_2$*, then*

$$v_i = d_1(\xi +^j - \xi_-^j)$$

$\therefore\ \xi_+^j$ *and* ξ_-^j *should be distinguished.*
Hence, we choose $H < 1$ *for* (14) *we then rewrite* (14) *as:*

$$\xi_\pm = H \pm \sqrt{-(1 - H^2)}$$
$$\xi_\pm = H \pm i\sqrt{(1 - H^2)}$$

then, the possible solution is

$$\xi_\pm^i = \cos(\theta i) \pm I \sin(\theta i)$$

here i use "I" as imaginer number, so it is not confusing between "i" index and "I" imaginer number.

$$\therefore v_i = d_1(\cos(\theta i) + I\sin(\theta i) - (\cos(\theta i) - I\sin(\theta i)))$$
$$v_i = d_1(2I\sin(\theta i))$$

we choose $d_1 = -\frac{I}{2}$ *so that*

$$v_i = \sin(\theta i)$$

inserting boundary condition, when $i = N$ *then* $v_N = 0$*, we get:*

$$0 = \sin(\theta N)$$
$$\theta = m\pi,\ \ \forall m : 2, 4, 6 \cdots N,\ \ even$$
$$\theta = \frac{m\pi}{N}$$

then, we know that from imaginer triangle

$$\cos\theta = H$$

$\therefore H = \cos(\theta) = \cos(\frac{m\pi}{N})$
substitute H *and* G *to* $2H = (4 - \lambda - G)$*, we get*

$$2\cos\left(\frac{m\pi i}{N}\right) = 4 - \lambda - 2\cos\left(\frac{m\pi j}{N}\right)$$
$$\lambda = 4 - 2\cos\left(\frac{m\pi i}{N}\right) - 2\cos\left(\frac{m\pi j}{N}\right)$$
$$\therefore \lambda = 4 - 2\Bigg(\cos\left(\frac{m\pi i}{N}\right) + \cos\left(\frac{m\pi j}{N}\right)\Bigg)$$

for the eigenvectors, we already know that

$$w_{i,j} = v_i\tilde{v}_j$$
$$v_i = \sin(\theta i)$$
$$\tilde{v}_j = \sin(\theta j)$$
$$\therefore w_{i,j} = \sin\left(\frac{m\pi i}{N}\right)\sin\left(\frac{m\pi j}{N}\right)$$

(b) *I attached the Python Code that solves* (12) *in 1*

(c) *Minimal Number of Iterations K such the:*

$$\max_{1\le i,j\le N-1}\left|v_{i,j}^{(K+1)} - v_{i,j}^{(K)}\right| \le 10^{-4}$$

(a) $N = 10$
   i. Jacobi, $K = 53$
   ii. Gauss-Seidel, $K = 33$
   iii. SOR (omega=1.5), $K = 12$
(b) $N = 20$
   i. Jacobi, $K = 137$
   ii. Gauss-Seidel, $K = 94$
   iii. SOR (omega=1.5), $K = 45$
(c) $N = 50$
   i. Jacobi, $K = 1$
   ii. Gauss-Seidel, $K = 173$
   iii. SOR (omega=1.5), $K = 148$

(d) Optimal ω_b for SOR:
   (a) $N = 10$, $\omega_b = 1.0446$
   (b) $N = 20$, $\omega_b = 1.0446$
   (c) $N = 50$, $\omega_b = 1.0446$

1 Attachment

```
import numpy as np

#Function Definition
#Function to create Right Hand Side and Left Hand Side Matrix
def const_mat(N):
    h = 1/N
    A = np.zeros((N,N))
    b = np.zeros((N,N))
    for i in range(N):
        for j in range(N):
            A[i][i] = 4
            b[i][j] = h*h*(np.sin(np.pi*i/N)*np.sin(np.pi*j/N))
            #b[i][j] = 1
            if (i==j-1 and i%4==3) or (i==j+1 and i%4==0):
                A[i][j] = 0
            elif i==j-1:
                A[i][j] = -1
            elif i==j+1:
                A[i][j] = -1
            elif i==j-4:
                A[i][j] = -1
            elif i==j+4:
                A[i][j] = -1
    return A,b

#Jacobi
def jacobi(b):
    v = np.zeros_like(b)
    N = len(b)
    for k in range(1000):
        v_old = np.copy(v)
        for i in range(N-2):
            for j in range(N-2):
                v[i+1][j+1] = 0.25 * (v_old[i][j+1] + v_old[i+2][j+1] + v_old[i+1][j] + v_old[i+1][j+2] + b[i+1][j+1])
        #print(np.max(v))
        if np.max(np.abs(v - v_old)) < 1e-4:
            break
    return(k)

#Gauss-Seidel
def gauss_seidel(b):
    v = np.zeros_like(b)
    N = len(b)
    for k in range(1000):
        v_old = np.copy(v)
        for i in range(N-2):
            for j in range(N-2):
                v[i+1][j+1] = 0.25 * (v[i][j+1] + v[i+2][j+1] + v[i+1][j] + v[i+1][j+2] + b[i+1][j+1])
        #print(np.max(v))
        if np.max(np.abs(v - v_old)) < 1e-4:
            break
    return(k)

#SOR
def sor(omega,b):
    v = np.zeros_like(b)
    N = len(b)
    for k in range(1000):
        v_old = np.copy(v)
        for i in range(N-2):
            for j in range(N-2):
                v[i+1][j+1] = (1-omega)*v_old[i+1][j+1] + omega*(0.25 * (v[i][j+1] + v[i+2][j+1] + v[i+1][j] + v[i+1][j+2] + b[i+1][j+1]))
        #print(np.max(v))
        if np.max(np.abs(v - v_old)) < 1e-4:
            break
    return(k)

#Calculate Optimum Omega for SOR
def calc_op_omega(A):
    L = np.diag(np.diag(A,-1),-1)
    U = np.diag(np.diag(A,1),1)
    T = 1/np.diag(A)*(L+U)
    rho = np.max(np.linalg.eigvals(T))
    opt_omega = 2/(1+np.sqrt(1-rho*rho))

    return opt_omega

#main
Ns = [10,20,30,40,50]
om = 1.5
for n in Ns:
    matA, matB = const_mat(n)
    iterj = jacobi(matB)
    iterg = gauss_seidel(matB)
    iters = sor(om,matB)
    op_om = calc_op_omega(matA)
    print("N =",n)
    print("Minimum Iteration Jacobi =",iterj)
    print("Minimum Iteration Gauss-Seidel =",iterg)
    print("Minimum Iteration SOR using omega(1.5) =",iters)
    print("Optimal omega for SOR =",op_om)
```