Final Report
Basic of Discrete Mathematics

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Problem 1. *Let I be a $\mathbb{Q}[x,y]$-ideal generated by $f = (1-x-y)^2 - 4xy$ and $g = y + x^2 - 1$. Find a Grobener basis of I.*

Answer: To find the Grobener basis of an Ideal, we use Buchberger's Algorithm described in Algorithm 1. Before we proceed to the algorithm, let me introduce the notion of the least common multiple of a pair of monomials. Consider monomials x^α and x^β, where as usual this is short for $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ and similar for the other monomial. Then, the least common multiple (LCM) of these can be found by taking the maximum of each index,

$$LCM(x^\alpha, x^\beta) = x_1^{\max(\alpha_1,\beta_1)} x_2^{\max(\alpha_2,\beta_2)} \cdots x_n^{\max(\alpha_n,\beta_n)}$$

Now we define the S-polynomial of a pair of polynomials,

$$S(f_1, f_2) = \frac{x^\gamma}{LT(f_1)} f_1 - \frac{x^\gamma}{LT(f_2)} f_2$$

where $x^\gamma = LCM(LM(f_1), LM(f_2))$ is the least common multiple of the leading monomials of the polynomials.

Algorithm 1 Buchberger's Algorithm

Input: $F = (f_1, f_2, \cdots, f_s)$, a list of generators for a non-zero ideal.
Output: $G = (g_1, g_2, \cdots, g_t)$, a Grobener basis for the ideal with $F \subset G$
  Initialise $G := F$,
  repeat
    $G' := G$
    for every pair $\{g_i, g_j\}$ $i \neq j$ **do**
      $x^\gamma \leftarrow LCM(LM(g_i), LM(g_j))$
      $S(g_i, g_j) \leftarrow \frac{x^\gamma}{LT(g_i)} g_i - \frac{x^\gamma}{LT(g_j)} g_j$
      compute the remainder S on dividing $S(g_i, g_i)$ by G'.
      if $S \neq 0$ **then**
        $G := G \cup \{S\}$,
      end if
    end for
  until $G = G'$ **return** G

Using lex order $x > y$ and by Algorithm 1, Grobener basis of the ideal $I = \langle f, g \rangle$ is following,

$$G = \{x^2 - 2xy - 2x + y^2 - 2y + 1, x^2 + y - 1\}$$

Problem 2. *Let $I = \langle x^2 + y^2 + z^2 - 3xyz, (3y + z - 2)(2x - 1)\rangle$ be a $\mathbb{Q}[x, y, z]$-ideal. Find a Grobener basis of I.*

Answer: By using lex order $x > y > z$ and Algorithm 1, we can find Grobener basis of the ideal $I = \langle x^2 + y^2 + z^2 - 3xyz, (3y + z - 2)(2x - 1)\rangle$ is following,

$$G = \{x^2 - 3xyz + y^2 + z^2, 6xy + 2xz - 4x - 3y - z + 2\}$$

Problem 3. *Solve the following system of algebraic equations in the complex domain:*

$$x^4 - 4x^2 + 5y^2 - 11 = 0, \quad x^2y - 2y + 5 = 0. \tag{1}$$

Answer: By using application of Grobener basis, we want to solve (1) as follows,
Let I be an ideal defined by:

$$I = \langle x^4 - 4x^2 + 5y^2 - 11, x^2y - 2y + 5\rangle \tag{2}$$

By using lex order $x > y$ and Algorithm 1, we can find Grobener basis of (2) as follows,

$$G = \langle x^4 - 4x^2 + 5y^2 - 11, x^2y - 2y + 5, y^3 - x^2 - 3y + 2\rangle \tag{3}$$

Thus, we can rewrite (3) as follows:

$$\begin{bmatrix} x^4 - 4x^2 + 5y^2 - 11 \\ x^2y - 2y + 5 \\ y^3 - x^2 - 3y + 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

Then, by using Gauss-Elimination method, we solve (4) in terms of y, we get,

$$y^4 - 3y^2 + 12y + 5 = 0$$
$$(y^2 - 3y + 5)(y^2 + 3y + 1) = 0 \tag{5}$$

Then we solve (5), we get,

$$\begin{aligned} y_1 &= 1.5 + 1.658i, & y_2 &= 1.5 - 1.658i \\ y_3 &= -1.5 + 1.118i, & y_4 &= -1.5 - 1.658i \end{aligned} \tag{6}$$

Substitute each $y := \{y_1, y_2, y_3, y_4\}$ on (6) to (4). To get,
(*)For $y = y_1 = 1.5 + 1.658i$, (4) becomes:

$$\begin{bmatrix} x^4 - 4x^2 - 13.5 + 24.875i \\ (1.5 + 1.658i)x^2 + 2 - 3.317i \\ -x^2 - 11.5 + 1.658i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{7}$$

Then, by using Gauss-Elimination method, we solve (7) in terms of x, we get,

$$x^4 - 4x^2 - 31.5 + 5i = 0 \tag{8}$$

By solving (8), we get one of the solution of (1) as follows,

$$x_1 = \pm(2.825 - 0.074i), \quad x_2 = \pm(0.105 + 1.996i) \tag{9}$$

Using the same step as (*) for y_2, y_3, y_4 to find x for the solution of (1), therefore the solution of (1) is as follows,

$$\begin{array}{lll} y = 1.5 + 1.658i & x = \pm(2.825 - 0.074i), & x = \pm(0.105 + 1.996i) \\ y = 1.5 - 1.658i & x = \pm(2.957 + 0.567i), & x = \pm(0.751 - 2.233i) \\ y = -1.5 + 1.118i & x = \pm(2.57 + 0.289i), & x = \pm(0.449 - 1.65i) \\ y = -1.5 - 1.658i & x = \pm(2.825 - 0.074i), & x = \pm(0.105 + 1.996i) \end{array} \tag{10}$$

Problem 4. *On a polynomial ring $\mathbb{Q}[x, y]$, let $I = \langle x, y^2 \rangle$ and $J = \langle x^3, y \rangle$. Find the intersection $I \cap J$.*

Answer: If I and J are two ideals generated respectively by $\{f_1, \cdots, f_m\}$ and $\{g_1, \cdots, g_k\}$, then a grobener basis of $I \cap J$ consists in the polynomials that do not contain t, in the Grobener basis of the ideal,

$$K = \langle tf_1, \cdots, tf_m, (1-t)g_1, \cdots, (1-t)g_k \rangle.$$

In other words, $I \cap J$ is obtained by eliminating t in K. Therefore,

$$I \cap J := G \cap \mathbb{Q}[x, y] \tag{11}$$

Hence, for this problem, K is define by,

$$K = \langle tx, ty^2, (1-t)x^3, (1-t)y \rangle$$

By using lex order $x > y > t$ and Algorithm 1, we can find the grobener basis of K-ideal as following,

$$G = \{tx^3, txy, tx^3y^2, ty^2\}$$

Thus, by Equation (11), we get:

$$I \cap J = \{x^3, xy, y^2\}$$

Problem 5. *Consider the monomials $M_2 = \{x_1^{a_1} x_2^{a_2} | a_1, a_2 \in \mathbb{N}_0\}$ of two variables. Prove that, on M_2, the graded lexicographic order (grlex) coincides with the graded reverse lexicographic order (grevlex).*

Answer:

Definition 1. (Graded Lex Order). Let $\alpha, \beta \in \mathbb{Z}_{\geq 0}^n$. We say $\alpha >_{grlex} \beta$ if

$$|\alpha| = \sum_{i=1}^n \alpha_i > |\beta| = \sum_{i=1}^n \beta_i, \quad \text{or} \quad |\alpha| = |\beta| \text{ and } \alpha >_{lex} \beta$$

Definition 2. (Graded Reverse Lex Order). Let $\alpha, \beta \in \mathbb{Z}_{\geq 0}^n$. We say $\alpha >_{grevlex} \beta$ if

$$|\alpha| = \sum_{i=1}^n \alpha_i > |\beta| = \sum_{i=1}^n \beta_i, \quad \text{or} \quad |\alpha| = |\beta|$$

By definition 1 and 2, both $>_{grlex}$ and $>_{grevlex}$ use total degree in the same way. To break a tie, $>_{grlex}$ uses lex order, so that it looks at the leftmost (or largest) variable and favors the larger power. In contrast, when $>_{grevlex}$ finds the same total degree, it looks at the rightmost (or smallest) variable and favors the smaller power.

Therefore, in M_2 it is clear that $>_{grlex}$ coincides with $>_{grevlex}$ because, if we have a monomial of two variable and arrange by $>_{grlex}$ which looks at the leftmost (or largest) variable and favors larger power $\Rightarrow$ the rightmost (or smaller) variable is favors the smaller power.

Proof. Let P be a polynomial defined by:

$$P = x^3y + x^2y^2$$

By graded lexicographic order,

$$x^3y >_{grlex} x^2y^2$$

since both monomials have total degree 4 and $x^3y >_{lex} x^2y^2$. In this case, we also have

$$x^3y >_{grevlex} x^2y^2$$

but for a different reason: x^3y is larger because the smaller variable y appears to a smaller power. $\blacksquare$

Problem 6. *Let $\leq$ be a monomial order and let G be a finite set of polynomials. Prove that G is a Grobener basis of $I = \langle G \rangle$ with respect to $\leq$ if and only if*

$$S(f, g) \xrightarrow[G]{*} 0 \tag{12}$$

for any $f, g \in G$

Answer:

Proposition 1. *Let $G = \{g_1, \cdots, g_t\}$ be a Groebner basis for an ideal $I \subset k[x_1, \cdots, x_n]$ and let $f \in k[x_1, \cdots, x_n]$. Then there is a unique $r \in k[x_1, \cdots, x_n]$ with the following two properties:*

(i) *No term of r is divisible by any of $LT(g_1), \cdots, LT(g_t)$.*

(ii) *There is $g \in I$ such that $f = g + r$.*

In particular, r is the remainder on division of f by G no matter how the elements of G are listed when using the division algorithm.

By using Proposition 1 we want to proof (12),

Proof. If the remainder is zero, then we have already observed that $f \in I$. Conversely, given $f \in I$ then $f = f + 0$ satisfies the two conditions of Proposition 1. It follows that 0 is the remainder of f on division by G. ∎