Helix Derivatives

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September 4, 2012

1 Track based alignment

Each hit measurement, y_i, is assumed to be described by a (non-)linear track model $f(x_i, \mathbf{q})$ which depends on a small number of parameter $\mathbf{q}$. Typically these are the five track parameters (see above) for tracks in a uniform magnetic field,

$$y_i = f\left(x_i, \mathbf{q}\right) + \epsilon_i. \tag{1}$$

The coordinate x_i is the coordiante of the hit y_i and ϵ is the uncertainty on y_i. The local-fit function f is linearized, if needed, by expressing it as a linear function of the local parameter corrections $\mathbf{\Delta q}$ at some reference value $\mathbf{q}_k$,

$$f\left(x_i, \mathbf{q}_k + \mathbf{\Delta q}\right) = f\left(x_i, \mathbf{q}_k\right) + \frac{\partial f}{\partial q_1}\Delta \mathrm{q}_1 + \frac{\partial f}{\partial q_2}\Delta \mathrm{q}_2 + ... \tag{2}$$

The local fit relies on minimizing the measured residual z_i for each hit,

$$z_i = y_i - f\left(x_i, \mathbf{q}_k\right). \tag{3}$$

By solving for $\mathbf{\Delta q}$ for each iteration k and updating with $q_{k+1} = q_k + \mathbf{\Delta q}$ convergence and optimal $\mathbf{\Delta q}$ can be obtained. These local parameter corrections $\mathbf{\Delta q}$ is used in the global fit. The derivatives of f w.r.t. the local parameters $\mathbf{q}$ is calculated in Sec. 3.

Describe the addition of global parameters here.

2 Helical Track Equations

Equations for trajectories in the XY plane

Point on helix (x, y) satisfies,

$$\mathrm{R}^2 = \left(x - x_c\right)^2 + \left(y - y_c\right)^2 \tag{4}$$

and the coordinate of the centre of the circle can be written,

$$x_c = x + \mathrm{R}\sin\phi \tag{5}$$

$$y_c = y - \mathrm{R}\cos\phi. \tag{6}$$

Equation for trajectory in XZ plane

$$z = z_0 + \mathrm{s} \times \mathrm{slope} \tag{7}$$

3 Local Parameters

Start by expressing ϕ as a function of the track parameters d_0, ϕ_0, R and the interaction point along the beam line x. From 4, $y = y_c + R \times \cos\phi$ and $x_c = (R - d_0)\sin\phi_0$

$$\begin{aligned}
(\cos\phi)^2 &= \frac{1}{R^2}\left(R^2 - (x - x_c)^2\right) \\
(\cos\phi)^2 &= \frac{1}{R^2}\left(R^2 - (x - (R - d_0)\sin\phi_0)^2\right) \\
\phi &= \cos^{-1}\left(\sqrt{1 - \left(\frac{x - (R - d_0)\sin\phi_0}{R}\right)^2}\right)
\end{aligned} \tag{8}$$

Using,

$$\begin{aligned}
x &= x_c - R\sin\phi = (R - d_0)\sin\phi_0 - R\sin\phi \\
y &= y_c + R\cos\phi = -(R - d_0)\cos\phi_0 + R\cos\phi \\
z &= z_0 + s \times \mathrm{slope} = z_0 - R \times \mathrm{slope}\,(\phi - \phi_0)
\end{aligned} \tag{9}$$

we can calculate the local derivatives $\frac{\partial f_{x_i}(\mathbf{q})}{\partial \mathbf{q}}$ where $i = x, y, z$.

$\frac{\partial f_x(\mathbf{q})}{\partial \mathbf{q}}$:

$$\begin{aligned}
\frac{\partial f_x(\mathbf{q})}{\partial d_0} &= -\sin\phi_0 - R\cos\phi\frac{\partial\phi}{\partial d_0} \\
\frac{\partial f_x(\mathbf{q})}{\partial z_0} &= -R\cos\phi\frac{\partial\phi}{\partial z_0} \\
\frac{\partial f_x(\mathbf{q})}{\partial phi_0} &= (R - d_0)\cos\phi_0 - R\cos\phi\frac{\partial\phi}{\partial phi_0} \\
\frac{\partial f_x(\mathbf{q})}{\partial R} &= \sin\phi_0 - R\cos\phi\frac{\partial\phi}{\partial R} \\
\frac{\partial f_x(\mathbf{q})}{\partial \mathrm{slope}} &= R\cos\phi\frac{\partial\phi}{\partial \mathrm{slope}},
\end{aligned} \tag{10}$$

$\frac{\partial f_y(\mathbf{q})}{\partial \mathbf{q}}$:

$$\begin{aligned}
\frac{\partial f_y(\mathbf{q})}{\partial d_0} &= \cos\phi_0 - R\sin\phi\frac{\partial\phi}{\partial d_0} \\
\frac{\partial f_y(\mathbf{q})}{\partial z_0} &= -R\sin\phi\frac{\partial\phi}{\partial z_0} \\
\frac{\partial f_y(\mathbf{q})}{\partial phi_0} &= (R-d_0)\sin\phi_0 - R\sin\phi\frac{\partial\phi}{\partial phi_0} \\
\frac{\partial f_y(\mathbf{q})}{\partial R} &= -\cos\phi_0 - R\sin\phi\frac{\partial\phi}{\partial R} \\
\frac{\partial f_y(\mathbf{q})}{\partial \mathrm{slope}} &= -R\sin\phi\frac{\partial\phi}{\partial \mathrm{slope}} \qquad (11)
\end{aligned}$$

$\frac{\partial f_y(\mathbf{q})}{\partial \mathbf{q}}$:

$$\begin{aligned}
\frac{\partial f_z(\mathbf{q})}{\partial d_0} &= -R\times\mathrm{slope}\frac{\partial\phi}{\partial d_0} \\
\frac{\partial f_z(\mathbf{q})}{\partial z_0} &= 1 \\
\frac{\partial f_z(\mathbf{q})}{\partial phi_0} &= -R\times\mathrm{slope}\left(\frac{\partial\phi}{\partial phi_0} - 1\right) \\
\frac{\partial f_z(\mathbf{q})}{\partial R} &= -\mathrm{slope}\times\left(\phi - \phi_0 + R\frac{\partial\phi}{\partial R}\right) \\
\frac{\partial f_z(\mathbf{q})}{\partial \mathrm{slope}} &= s = -R(\phi-\phi_0) \qquad (12)
\end{aligned}$$

The $\frac{\partial\phi}{\partial\mathbf{q}}$ are given by the rather involved:

$$\begin{aligned}
\frac{\partial\phi}{\partial d_0} &= \frac{\sin\phi_0\,(x+(d_0-R)\sin\phi_0)}{R^2\sqrt{\frac{(x+(d_0-R)\sin\phi_0)^2\left(1-\frac{(x+(d_0-R)\sin\phi_0)^2}{R^2}\right)}{R^2}}} \\
\frac{\partial\phi}{\partial \phi_0} &= \frac{(d_0-R)\cos\phi_0\,(x+(d_0-R)\sin\phi_0)}{R^2\sqrt{\frac{(x+(d_0-R)\sin\phi_0)^2\left(1-\frac{(x+(d_0-R)\sin\phi_0)^2}{R^2}\right)}{R^2}}} \\
\frac{\partial\phi}{\partial R} &= -\left(\frac{(x+d_0\sin\phi_0)\,(x+(d_0-R)\sin\phi_0)}{R^3\sqrt{\frac{(x+(d_0-R)\sin\phi_0)^2\left(1-\frac{(x+(d_0-R)\sin\phi_0)^2}{R^2}\right)}{R^2}}}\right) \\
\frac{\partial\phi}{\partial slope} &= 0 \\
\frac{\partial\phi}{\partial z0} &= 0 \qquad (13)
\end{aligned}$$

4 Global Transformations

4.1 Translation

Using Eq. 4, 5, 6,we can write,

$$\begin{aligned} x &= x_c + \mathrm{sign}\left(\mathrm{R}\right) \times \mathrm{R} \times \sin\phi && (14) \\ y &= y_c + \mathrm{sign}\left(\mathrm{R}\right) \times \mathrm{R} \times \cos\phi && (15) \\ z &= z_0 - \mathrm{R} \times \mathrm{slope} \times \Delta\phi. && (16) \end{aligned}$$

Note that,

$$\begin{aligned} \phi &= \mathrm{atan2}\left(\sin\phi_0 - \frac{x - x_0}{\mathrm{R}}, \cos\phi_0 + \frac{y - y_0}{\mathrm{R}}\right) \\ \frac{\partial\phi}{\partial x} &= -\frac{1}{R} \times \frac{\cos\phi_0 + \frac{y-y_0}{\mathrm{R}}}{\left(\cos\phi_0 + \frac{y-y_0}{\mathrm{R}}\right)^2 + \left(\sin\phi_0 - \frac{x-x_0}{\mathrm{R}}\right)^2} \\ \frac{\partial\phi}{\partial y} &= -\frac{1}{R} \times \frac{\sin\phi_0 - \frac{x-x_0}{\mathrm{R}}}{\left(\cos\phi_0 + \frac{y-y_0}{\mathrm{R}}\right)^2 + \left(\sin\phi_0 - \frac{x-x_0}{\mathrm{R}}\right)^2} \\ \frac{\partial\phi}{\partial z} &= -\frac{1}{\mathrm{R} \times \mathrm{slope}}. && (17) \end{aligned}$$

4.1.1 Translation in x

Using Eq. 4, 5, 6,

$$\begin{aligned} \frac{\partial x}{\partial x} &= 1 \\ \frac{\partial y}{\partial x} &= -\mathrm{sign}\left(\mathrm{R}\right) \times \mathrm{R}\sin\phi\frac{\partial\phi}{\partial x} \\ \frac{\partial z}{\partial x} &= -\mathrm{R} \times \mathrm{slope}\frac{\partial\phi}{\partial x}, \end{aligned} \tag{18}$$

with $\frac{\partial\phi}{\partial x}$ given byEq. 17.

4.1.2 Translation in y

Using Eq. 4, 5, 6,

$$\begin{aligned} \frac{\partial x}{\partial y} &= \mathrm{sign}\left(\mathrm{R}\right) \times \mathrm{R}\cos\phi\frac{\partial\phi}{\partial y} \\ \frac{\partial y}{\partial y} &= 1 \\ \frac{\partial z}{\partial y} &= -\mathrm{R} \times \mathrm{slope}\frac{\partial\phi}{\partial y}, \end{aligned} \tag{19}$$

with $\frac{\partial\phi}{\partial y}$ given byEq. 17.

4.1.3 Translation in z

Using Eq. 4, 5, 6,

$$
\begin{aligned}
\frac{\partial x}{\partial z} &= \mathrm{sign}\left(\mathrm{R}\right) \times \mathrm{R} \cos\phi \frac{\partial \phi}{\partial z} \\
\frac{\partial y}{\partial z} &= -\mathrm{sign}\left(\mathrm{R}\right) \times \mathrm{R} \times \sin\phi \frac{\partial \phi}{\partial z} \\
\frac{\partial z}{\partial z} &= 1,
\end{aligned}
\tag{20}
$$

with $\frac{\partial \phi}{\partial z}$ given byEq. 17.

4.2 Rotation

4.2.1 Rotation around x - α

4.2.2 Rotation around y - β

4.2.3 Rotation around z - γ

Appendix

Practical Formulas

$$y = y_c + \operatorname{sign}(()\,\mathrm{R}) \times \sqrt{\left(\mathrm{R}^2 - (x - x_c)^2\right)} \tag{21}$$

$$x = x_c + \operatorname{sign}(()\,\mathrm{R}) \times \sqrt{\left(\mathrm{R}^2 - (y - y_c)^2\right)} \tag{22}$$

$$\Delta\phi = \operatorname{atan2}(x - x_c, y - y_c) - \operatorname{atan2}(x_0 - x_c, y_0 - y_c) \tag{23}$$

Arc length, s, can be written as:

$$\mathrm{s} = -\Delta\phi \times \mathrm{R}. \tag{24}$$

$$\frac{\partial y_c}{\partial d_0} = \frac{\operatorname{sign}(\mathrm{R})^2 \sin(\phi_0)(x - x_c)}{\sqrt{\mathrm{R}^2 - \operatorname{sign}(\mathrm{R})^2\left(\mathrm{R}^2 - (x - x_c)^2\right)}} \tag{25}$$