Helix Derivatives

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1 Introduction

This note describes the implementation of the alignment procedure for the HPS SVT. Most of this follows what is described in more depth in Ref. [1].

Updates needed:

- Need to update the mix of nomenclature of q for track parameters and hit positions.
- Clean-up formulas that are not relevant
- improve organization in the descriptions

2 Track based alignment

Each hit measurement, y_i, is assumed to be described by a (non-)linear track model $f(x_i, \mathbf{q})$ which depends on a small number of parameter $\mathbf{q}$. In the case of a particle track in a homogenous magnetic filed these are five track parameters (see Sec. 4),

$$y_i = f\left(x_i, \mathbf{q}\right) + \epsilon_i. \tag{1}$$

The coordinate x_i is the coordinate of the hit y_i and ϵ is the uncertainty on y_i. The local-fit function f is linearized, if needed, by expressing it as a linear function of the local parameter corrections $\Delta\mathbf{q}$ at some reference value $\mathbf{q}_k$,

$$f\left(x_i, \mathbf{q}_k + \Delta\mathbf{q}\right) = f\left(x_i, \mathbf{q}_k\right) + \frac{\partial f}{\partial \mathrm{q}_1}\Delta \mathrm{q}_1 + \frac{\partial f}{\partial \mathrm{q}_2}\Delta \mathrm{q}_2 + ... \tag{2}$$

The local fit relies on minimizing the measured residual z_i for each hit,

$$z_i = y_i - f\left(x_i, \mathbf{q}_k\right). \tag{3}$$

By solving for $\Delta\mathbf{q}$ for each iteration k and updating with $q_{k+1} = q_k + \Delta\mathbf{q}$ convergence and optimal $\Delta\mathbf{q}$ can be obtained. These local parameter corrections $\Delta\mathbf{q}$ is used in the global fit. The derivatives of f w.r.t. the local parameters $\mathbf{q}$ is calculated in Sec. 6.

Including the effect of the global alignment parameters $\mathbf{p}$ we can write the residual as,

$$z_i = y_i - f\left(x_i, \mathbf{q}_k\right) = \sum_j^{\nu} \frac{\partial f}{\partial \mathrm{q}_j} \Delta \mathrm{q}_1 + \sum_l \frac{\partial f}{\partial \mathrm{p}_l} \Delta \mathrm{p}_l. \tag{4}$$

The alignment algorithm last step is to minimize these residuals w.r.t. the global parameters. This involves solving a system of linear equations. In our first strategy we make use of the MillepedeII software program. The input to MillepedeII are the residuals, local (track) derivatives and the global derivatives. Practically the program is split into the MILLE programs which gathers and prepares the input to the PEDE program which carries out the actual minimization.

In the following section the parameterization of the alignment is described to be able to calculate the relevant global derivatives. The details of the local and global derivatives are layer out in Sec. 6 and 5.

3 Alignment Parameterization

The hit measurement is done in a local sensor frame and are transformed into a global frame to connect hits together to form tracks. A hit measurement vector $\mathbf{q}$ can be represented in the global frame as,

$$\mathbf{r} = \mathbf{R}^{\mathbf{T}} \mathbf{q} + \mathbf{r_0}, \tag{5}$$

where $\mathbf{R}$ is a rotation matrix and $\mathbf{r_0}$ is the position of the sensor. The task of the alignment procedure is to provide correction to position and rotation of the sensor, $\mathbf{q_0}$ and $\mathbf{R}$, respectively,

$$\mathbf{r} = \mathbf{R}^{\mathbf{T}} \mathbf{\Delta R} \left(\mathbf{q} + \Delta \mathbf{q}\right) + \mathbf{r_0}. \tag{6}$$

The alignment parameters are the components of $\Delta\mathbf{q}$ and $\Delta\mathbf{R}$ and are often expressed in the local sensor coordinates as they are related to the individual sensor. The measured hit position components are $\mathbf{q} = (u, v, w)$, where the precise measured coordinate on the sensor is separated from the less well-known positions. In a strip sensor u is the precisely measured coordinate and v is the un-measured coordinate. w is the direction normal to the sensor plane. For alignment typically the Δw is ignored as all hits happen at the sensor plane. For strip sensors in general Δv is essentially not measured and can also be ignored. The rotation correction matrix $\Delta\mathbf{R}$ are reduced to three angles around the u-, v- and w-axis and are denoted as α, β and γ (around the center of the sensors and thus do not induce a translation). Each sensor thus has 6 alignment parameters and following the notation in Ref.X it can be represented by a vector **a**,

$$\mathbf{a} = \begin{pmatrix} \Delta u \\ \Delta v \\ \Delta w \\ \alpha \\ \beta \\ \gamma \end{pmatrix}$$

In order to solve the minimization problem we need to calculate the derivatives of the residuals w.r.t. the local and global parameters. The residual **z** is,

$$\mathbf{z} = \mathbf{q}_a - \mathbf{q}_p = \begin{pmatrix} u_m \\ v_m \\ w_m \end{pmatrix} - \begin{pmatrix} u_p \\ v_p \\ w_p \end{pmatrix} = \begin{pmatrix} u_m \end{pmatrix} - \begin{pmatrix} u_p \end{pmatrix}$$

where $\mathbf{q}_a$ is the alignment corrected hit,

$$\mathbf{q_a} = \mathbf{\Delta R} \mathbf{q}_h + \Delta \mathbf{q}, \tag{7}$$

where $\mathbf{q_h}$ is the measured hit position. For the minimization of the square of residuals the global derivatives,

$$\frac{\partial \mathbf{z}}{\partial \mathbf{a}} = \frac{\partial \mathbf{q}_a}{\partial \mathbf{a}} - \frac{\partial \mathbf{q}_p}{\partial \mathbf{a}}. \tag{8}$$

needs to be calculated. The partial derivatives w.r.t. to a translation Δu is,

$$\frac{\partial \mathbf{q}_a}{\partial \Delta u} = \frac{\partial}{\partial \mathbf{\Delta u}} \left(\mathbf{\Delta R} \mathbf{q}_h + \Delta \mathbf{q} \right) = \frac{\partial}{\partial \Delta u} \begin{pmatrix} \Delta u \\ \Delta v \\ \Delta w \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and similarly for the other translations in v and w,

$$\frac{\partial \mathbf{q}_a}{\partial \Delta v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \frac{\partial \mathbf{q}_a}{\partial \Delta w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The rotation matrix is given by

$$\mathbf{\Delta R} = \mathbf{R}_\gamma \times \mathbf{R}_\beta \times \mathbf{R}_\alpha nonumber \tag{9}$$

where,

$$\mathbf{R}_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \mathbf{R}_\beta = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}, \mathbf{R}_\beta = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

All angles are assumed to be small and after linearization the derivatives become,

$$\frac{\partial \mathbf{\Delta R}}{\partial \alpha} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \frac{\partial \mathbf{\Delta R}}{\partial \beta} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \frac{\partial \mathbf{\Delta R}}{\partial \gamma} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

These can be written more compactly, and evaluated at the measured position $\mathbf{q}_h = \mathbf{q}_m = (u_m, v_m, w_m)$ as,

$$\frac{\partial \mathbf{q}_a}{\partial \mathbf{a}} = \begin{pmatrix} \mathbf{1} & \frac{\partial \Delta R}{\partial \alpha}\mathbf{q}_h & \frac{\partial \Delta R}{\partial \beta}\mathbf{q}_h & \frac{\partial \Delta R}{\partial \gamma}\mathbf{q}_h \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & -w_m & v_m \\ 0 & 1 & 0 & w_m & 0 & -u_m \\ 0 & 0 & 1 & -v_m & u_m & 0 \end{pmatrix}$$

As mentioned before $w_m = 0$ by construction (the hit $\mathbf{q}_h$ is on the sensor surface) and the un-measured direction v can be ignored by it kept here for consistency,

$$\frac{\partial \mathbf{q}_a}{\partial \mathbf{a}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & v_m \\ 0 & 1 & 0 & 0 & 0 & -u_m \\ 0 & 0 & 1 & -v_m & u_m & 0 \end{pmatrix}$$

The global derivative $\frac{\partial \mathbf{q}_p}{\partial \mathbf{a}}$ measures the effect of the predicted track position on the surface of the sensor. Note that a shift of the sensor in the u, v plane is equivalent to a shift of the measured hit position $\mathbf{q}_h$ and the only direction where the track propagation needs to be taken into account is the w direction. Using this we can write (this is unclear!!),

$$\frac{\partial \mathbf{q}_p}{\partial \mathbf{a}} = \frac{\partial \mathbf{q}_p}{\partial \mathbf{q}_a}\frac{\partial \mathbf{q}_a}{\partial \mathbf{a}} \tag{10}$$

where $\frac{\partial \mathbf{q}_p}{\partial \mathbf{q}_a}$ is

$$\frac{\partial \mathbf{q}_p}{\partial u_a} = 0, \frac{\partial \mathbf{q}_p}{\partial v_a} = 0, \frac{\partial \mathbf{q}_p}{\partial w_a} = \frac{\partial \mathbf{q}_p}{\partial w_h} = \begin{pmatrix} \frac{\partial u_p}{\partial w_h} \\ \frac{\partial v_p}{\partial w_h} \\ \frac{\partial w_p}{\partial w_h} \end{pmatrix}$$

since a shift in the u, v plane is equivalent to a shift in the hit position. Using this and $\frac{\partial \mathbf{q}_a}{\partial \mathbf{a}}$ calculated earlier,

$$\frac{\partial \mathbf{q}_p}{\partial \mathbf{q}_a}\frac{\partial \mathbf{q}_a}{\partial \mathbf{a}} = \begin{pmatrix} 0 & 0 & \frac{\partial u_p}{\partial w_h} \\ 0 & 0 & \frac{\partial v_p}{\partial w_h} \\ 0 & 0 & \frac{\partial w_p}{\partial w_h} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & v_m \\ 0 & 1 & 0 & 0 & 0 & -u_m \\ 0 & 0 & 1 & -v_m & u_m & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & \frac{\partial u_p}{\partial w_h} & -v_m \frac{\partial u_p}{\partial w_h} & u_m \frac{\partial u_p}{\partial w_h} & 0 \\ 0 & 0 & \frac{\partial v_p}{\partial w_h} & -v_m \frac{\partial v_p}{\partial w_h} & u_m \frac{\partial v_p}{\partial w_h} & 0 \\ 0 & 0 & \frac{\partial w_p}{\partial w_h} & -v_m \frac{\partial w_p}{\partial w_h} & u_m \frac{\partial w_p}{\partial w_h} & 0 \end{pmatrix}$$

Now we have all the ingredients to calculate Eq. 8:

$$\frac{\partial \mathbf{z}}{\partial \mathbf{a}} = \frac{\partial \mathbf{q}_a}{\partial \mathbf{a}} - \frac{\partial \mathbf{q}_p}{\partial \mathbf{a}} = \frac{\partial \mathbf{q}_a}{\partial \mathbf{a}} - \frac{\partial \mathbf{q}_p}{\partial \mathbf{q}_a}\frac{\partial \mathbf{q}_a}{\partial \mathbf{a}} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & v_m \\ 0 & 1 & 0 & 0 & 0 & -u_m \\ 0 & 0 & 1 & -v_m & u_m & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \frac{\partial u_p}{\partial w_h} & -v_m \frac{\partial u_p}{\partial w_h} & u_m \frac{\partial u_p}{\partial w_h} & 0 \\ 0 & 0 & \frac{\partial v_p}{\partial w_h} & -v_m \frac{\partial v_p}{\partial w_h} & u_m \frac{\partial v_p}{\partial w_h} & 0 \\ 0 & 0 & \frac{\partial w_p}{\partial w_h} & -v_m \frac{\partial w_p}{\partial w_h} & u_m \frac{\partial w_p}{\partial w_h} & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & -\frac{\partial u_p}{\partial w_h} & v_m\frac{\partial u_p}{\partial w_h} & -u_m\frac{\partial u_p}{\partial w_h} & v_m \\ 0 & 1 & -\frac{\partial v_p}{\partial w_h} & v_m\frac{\partial v_p}{\partial w_h} & -u_m\frac{\partial v_p}{\partial w_h} & -u_m \\ 0 & 0 & 1-\frac{\partial w_p}{\partial w_h} & v_m\frac{\partial w_p}{\partial w_h}-v_m & u_m-u_m\frac{\partial w_p}{\partial w_h} & 0 \end{pmatrix}$$

Since $w_m = 0$ we can ignore the third component which means that we can write the global residual derivative as,

$$\frac{\partial \mathbf{z}}{\partial \mathbf{a}} = \begin{pmatrix} 1 & 0 & -\frac{\partial u_p}{\partial w_h} & v_m\frac{\partial u_p}{\partial w_h} & -u_m\frac{\partial u_p}{\partial w_h} & v_m \\ 0 & 1 & -\frac{\partial v_p}{\partial w_h} & v_m\frac{\partial v_p}{\partial w_h} & -u_m\frac{\partial v_p}{\partial w_h} & -u_m \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In the strip sensor case only the well-measured direction is important and thus only the first row is important.

`NOTE THAT IN THE REFERENCE I NEED TO GET -1* THE ABOVE??`

The calculation of the global derivatives $\frac{\partial q}{\partial \mathbf{a}}$ is detailed in Sec. 5.

4 Helical Track Equations

Equations for trajectories in the XY plane

Point on helix (x, y) satisfies,

$$\mathrm{R}^2 = (x - x_c)^2 + (y - y_c)^2 \tag{11}$$

and the coordinate of the centre of the circle can be written,

$$x_c = x + \mathrm{R}\sin\phi \tag{12}$$
$$y_c = y - \mathrm{R}\cos\phi. \tag{13}$$

Equation for trajectory in XZ plane

$$z = z_0 + \mathrm{s} \times \mathrm{slope} \tag{14}$$

5 Global Derivatives

The parameterization that is layer out in Sec. 3 provide information on the global derivatives that we need to calculate. These derivatives, $\frac{\partial \mathbf{z}}{\partial \mathbf{a}}$, are all expressed in the local sensor frame. In the global frame, the derivatives involving e.g. u will be affected by translations and rotations around the global coordinate axis x, y, z. The global derivatives $\frac{\partial \mathbf{z}}{\partial \mathbf{a}}$ described in Sec. 3 are naturally calculated in the global frame and then transformed into the corresponding local sensor frame.

As we saw earlier the relevant global derivatives are

$$\frac{\partial \mathbf{z}}{\partial \mathbf{a}} = \begin{pmatrix} 1 & 0 & -\frac{\partial u_p}{\partial w_h} & v_m \frac{\partial u_p}{\partial w_h} & -u_m \frac{\partial u_p}{\partial w_h} & v_m \\ 0 & 1 & -\frac{\partial v_p}{\partial w_h} & v_m \frac{\partial v_p}{\partial w_h} & -u_m \frac{\partial v_p}{\partial w_h} & -u_m \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where only the first row is important for strip sensors. To calculate these derivatives the derivatives w.r.t. the global translation and rotations needs to be calculated and then properly transformed to the the relevant sensor frame quantitiess.

`Need to change nomenclature below so that it corresponds to the predicted position in the earlier section q_p.`

5.1 Translation

Starting from Eq. 12 and 13, we can write,

$$\begin{aligned} x &= x_c \mathrm{R} \sin\phi & (15) \\ y &= y_c + \mathrm{R} \times \cos\phi & (16) \\ z &= z_0 - \mathrm{R} \times \mathrm{slope} \times \Delta\phi. & (17) \end{aligned}$$

Note that,

$$\phi = \mathrm{atan2}\left(\sin\phi_0 - \frac{x - x_0}{\mathrm{R}}, \cos\phi_0 + \frac{y - y_0}{\mathrm{R}}\right). \quad (18)$$

$$(19)$$

5.1.1 Translation in x

To calculate the derivatives for a translation in x we express the track model equation as a function of x, i.e. $f(x, \mathbf{q})$. Using Eq. 18 and substituting,

$$\begin{aligned} y &= -(R - d_0)\cos\phi_0 + \mathrm{sign}\,(\mathrm{R})\sqrt{R^2 - (x - (R - d_0)\sin\phi_0)^2} \\ y_0 &= d_0 \cos\phi_0 \\ x_0 &= -d_0 \sin\phi_0 & (20) \\ & & (21) \end{aligned}$$

we can calculate $\frac{\partial f_{x_i}(\mathbf{q})}{\partial x}$ where $x_i = x, y, z$.

$$\begin{aligned} \frac{\partial f_x}{\partial x} &= 1 \\ \frac{\partial f_y}{\partial x} &= -R \sin\phi \frac{\partial \phi}{\partial x} \\ \frac{\partial f_z}{\partial x} &= -\mathrm{R} \times \mathrm{slope} \frac{\partial \phi}{\partial x}, \end{aligned} \quad (22)$$

with $\frac{\partial \phi}{\partial x}$ given by,

$$\frac{\partial \phi}{\partial x} = \frac{-R^2 \operatorname{sign}(\mathrm{R})}{\sqrt{R^2 - (x + (d_0 - R)s_0)^2} \left(-(x + (d_0 - R)s_0)^2 + \operatorname{sign}(\mathrm{R})^2 \left(-R^2 + x^2 + 2(d_0 - R)xs_0 + (d_0 - R)^2 \sin^2 \phi_0\right)\right)} \tag{23}$$

where $s_0 = \sin \phi_0$.

5.1.2 Translation in y

Similarly as for x, using Eq. 18 and substituting,

$$\begin{aligned} x &= (R - d_0) \sin \phi_0 + \operatorname{sign}(\mathrm{R}) \sqrt{R^2 - (y(R - d_0) \cos \phi_0)^2} \\ y_0 &= d_0 \cos \phi_0 \\ x_0 &= -d_0 \sin \phi_0 \end{aligned} \tag{24}$$

$$\tag{25}$$

we can calculate $\frac{\partial f_{x_i}(\mathbf{q})}{\partial y}$ where $x_i = x, y, z$.

$$\begin{aligned} \frac{\partial x}{\partial y} &= \mathrm{R} \cos \phi \frac{\partial \phi}{\partial y} \\ \frac{\partial y}{\partial y} &= 1 \\ \frac{\partial z}{\partial y} &= -\mathrm{R} \times \text{slope} \frac{\partial \phi}{\partial y}, \end{aligned} \tag{26}$$

with $\frac{\partial \phi}{\partial y}$ given by,

$$\frac{\partial \phi}{\partial y} = \frac{R^2 \operatorname{sign}(\mathrm{R})}{\sqrt{R^2 - (y + (-d_0 + R)c_0)^2} \left(-(y + (-d_0 + R)c_0)^2 + (-R^2 + y^2 - 2(d_0 - R)yc_0 + (d_0 - R)^2 c_0^2) \operatorname{sign}(\mathrm{R})^2\right)} \tag{27}$$

where $c_0 = \cos \phi_0$.

5.1.3 Translation in z

The derivatives are given by,

$$\begin{aligned} \frac{\partial x}{\partial z} &= \operatorname{sign}(\mathrm{R}) \times \mathrm{R} \cos \phi \frac{\partial \phi}{\partial z} \\ \frac{\partial y}{\partial z} &= -\operatorname{sign}(\mathrm{R}) \times \mathrm{R} \times \sin \phi \frac{\partial \phi}{\partial z} \\ \frac{\partial z}{\partial z} &= 1, \end{aligned} \tag{28}$$

with $\frac{\partial \phi}{\partial z}$ given by

$$\frac{\partial \phi}{\partial z} = \frac{\partial}{\partial z} \left(-\frac{z - z_0}{R \times \text{slope}} + \phi_0 \right) = -\frac{1}{R \times \text{slope}} \tag{29}$$

$$\tag{30}$$

5.2 Rotation

We assume that all rotation angles are around the center of the sensor and small and use the small angle limit for rotations $\mathbf{k} = (\alpha, \beta, \gamma)$ corresponding to rotations around the three axis (x, y, z). The derivatives $\frac{\partial f_{x_i}}{\partial \mathbf{k}}$ are given by,

$$\frac{\partial f_{x_i}}{\partial \mathbf{k}} = \begin{pmatrix} \frac{\partial f_x}{\partial \alpha} & \frac{\partial f_y}{\partial \alpha} & \frac{\partial f_z}{\partial \alpha} \\ \frac{\partial f_x}{\partial \beta} & \frac{\partial f_y}{\partial \beta} & \frac{\partial f_z}{\partial \beta} \\ \frac{\partial f_x}{\partial \gamma} & \frac{\partial f_y}{\partial \gamma} & \frac{\partial f_z}{\partial \gamma} \end{pmatrix}$$

$$\frac{\partial f_{x_i}}{\partial \mathbf{k}} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

6 Local Derivatives

Start by expressing ϕ as a function of the track parameters d_0, ϕ_0, R and the interaction point along the beam line x. From 11, $y = y_c + R \times \cos\phi$ and $x_c = (R - d_0)\sin\phi_0$

$$\begin{aligned} (\cos\phi)^2 &= \frac{1}{R^2}\left(R^2 - (x - x_c)^2\right) \\ (\cos\phi)^2 &= \frac{1}{R^2}\left(R^2 - (x - (R - d_0)\sin\phi_0)^2\right) \\ \phi &= \cos^{-1}\left(\sqrt{1 - \left(\frac{x - (R - d_0)\sin\phi_0}{R}\right)^2}\right) \end{aligned} \tag{31}$$

or, equivalently, using Eq. 12,

$$\begin{aligned} \sin\phi &= \frac{1}{R}(x - x_c) = (x - (R - d_0)\sin\phi_0) \\ \phi &= \arcsin\left(\frac{x - (R - d_0)\sin\phi_0}{R}\right). \end{aligned} \tag{32}$$

Using,

$$\begin{aligned} x &= x_c - R\sin\phi = (R - d_0)\sin\phi_0 - R\sin\phi \\ y &= y_c + R\cos\phi = -(R - d_0)\cos\phi_0 + R\cos\phi \\ z &= z_0 + s \times \text{slope} = z_0 - R \times \text{slope}\,(\phi - \phi_0)\,, \end{aligned} \tag{33}$$

we can calculate the local derivatives $\frac{\partial f_{x_i}(\mathbf{q})}{\partial \mathbf{q}}$ where $i = x, y, z$.

$\frac{\partial f_x(\mathbf{q})}{\partial \mathbf{q}}$:

$$\begin{aligned}
\frac{\partial f_x(\mathbf{q})}{\partial d_0} &= -\sin\phi_0 - R\cos\phi\frac{\partial\phi}{\partial d_0} \\
\frac{\partial f_x(\mathbf{q})}{\partial z_0} &= -R\cos\phi\frac{\partial\phi}{\partial z_0} \\
\frac{\partial f_x(\mathbf{q})}{\partial phi_0} &= (R-d_0)\cos\phi_0 - R\cos\phi\frac{\partial\phi}{\partial phi_0} \\
\frac{\partial f_x(\mathbf{q})}{\partial R} &= \sin\phi_0 - R\cos\phi\frac{\partial\phi}{\partial R} \\
\frac{\partial f_x(\mathbf{q})}{\partial \mathrm{slope}} &= R\cos\phi\frac{\partial\phi}{\partial \mathrm{slope}},
\end{aligned} \tag{34}$$

$\frac{\partial f_y(\mathbf{q})}{\partial \mathbf{q}}$:

$$\begin{aligned}
\frac{\partial f_y(\mathbf{q})}{\partial d_0} &= \cos\phi_0 - R\sin\phi\frac{\partial\phi}{\partial d_0} \\
\frac{\partial f_y(\mathbf{q})}{\partial z_0} &= -R\sin\phi\frac{\partial\phi}{\partial z_0} \\
\frac{\partial f_y(\mathbf{q})}{\partial phi_0} &= (R-d_0)\sin\phi_0 - R\sin\phi\frac{\partial\phi}{\partial phi_0} \\
\frac{\partial f_y(\mathbf{q})}{\partial R} &= -\cos\phi_0 - R\sin\phi\frac{\partial\phi}{\partial R} \\
\frac{\partial f_y(\mathbf{q})}{\partial \mathrm{slope}} &= -R\sin\phi\frac{\partial\phi}{\partial \mathrm{slope}}
\end{aligned} \tag{35}$$

$\frac{\partial f_y(\mathbf{q})}{\partial \mathbf{q}}$:

$$\begin{aligned}
\frac{\partial f_z(\mathbf{q})}{\partial d_0} &= -R\times\mathrm{slope}\frac{\partial\phi}{\partial d_0} \\
\frac{\partial f_z(\mathbf{q})}{\partial z_0} &= 1 \\
\frac{\partial f_z(\mathbf{q})}{\partial phi_0} &= -R\times\mathrm{slope}\left(\frac{\partial\phi}{\partial phi_0} - 1\right) \\
\frac{\partial f_z(\mathbf{q})}{\partial R} &= -\mathrm{slope}\times\left(\phi - \phi_0 + R\frac{\partial\phi}{\partial R}\right) \\
\frac{\partial f_z(\mathbf{q})}{\partial \mathrm{slope}} &= s = -R(\phi-\phi_0)
\end{aligned} \tag{36}$$

$\frac{\partial\phi}{\partial\mathbf{q}}$ **using Eq. 32**

$$
\begin{aligned}
\frac{\partial\phi}{\partial d_0} &= \frac{\sin\phi_0}{R\sqrt{\frac{(x+(d_0-R)\sin\phi_0)^2}{R^2}}} \\
\frac{\partial\phi}{\partial phi_0} &= \frac{(d_0-R)\cos\phi_0}{R\sqrt{1-\frac{(x+(d_0-R)\sin\phi_0)^2}{R^2}}} \\
\frac{\partial\phi}{\partial R} &= \frac{-x-d_0\sin\phi_0}{R^2\sqrt{1-\frac{(x+(d_0-R)\sin\phi_0)^2}{R^2}}} \\
\frac{\partial\phi}{\partial slope} &= 0 \\
\frac{\partial\phi}{\partial z0} &= 0
\end{aligned}
\tag{37}
$$

The calculation of $\frac{\partial\phi}{\partial\mathbf{q}}$ using Eq. 31 is shown in appendix.

7 Transformation to the Sensor Frame

In the above sections all equations have been in what we call the “tracking frame” which is described in Sec. **??**. Normally residuals, and thus the derivatives, are measured on the so-called “sensor frame” which is a coordinate system local to each sensor or plane. Typically this is defined with u being the well-measured coordinate, v being the less well-measured coordinate and w normal to the sensor plane. In our case it means that v is parallel to the strips and u is the measurement direction.

When aligning a strip sensor there is only one measured direction of the hit for each plane/sensor, the u direction. Thus, the only residual that will go into the alignment is in that direction. In the case of the HPS experiment the u direction is close to parallel with the tracking frame z direction for the axial sensors.

Thus to properly prepare the input to the alignment all the derivatives (and residuals) have to be transformed into the corresponding values in the u direction. Practically this means that for axial sensors, that have u close to parallel with z, the z contribution will dominate. For the stereo sensors the stereo angle will cause contributions from x, y direction to contribute more.

The translation and rotation matrices, T are built using the detector geometry package which takes as input a xml file describing the position and rotations of the sensors. To take a vector of predicted hit positions on a sensor, $\mathbf{q_p^t}$, from the tracking to the sensor frame:

$$\mathbf{q}_p^s = \mathbf{T}_{T->S}\mathbf{q}_p \tag{38}$$

where $\mathbf{T}_{T->S}$ is the transformation matrix going from the tracking to sensor frame for a given sensor. In the current software this is practically done by going to the global coordinate system, called the "detector frame",

$$\begin{aligned}
\mathbf{T}_{T->S} &= \mathbf{T}_{D->S}\mathbf{T}^{-1}_{T->D} \\
\mathbf{q}^s_p &= \mathbf{T}_{D->S}\mathbf{T}^{-1}_{T->D}\mathbf{q}_p
\end{aligned} \tag{39}$$

where $\mathbf{T}^{-1}_{T->D}$ is the inverse of the transformation matrix going from tracking frame to detector frame and $\mathbf{T}_{D->S}$ is the transformation from the detector to the sensor frame.

These transformation are applied to all the vector of derivatives, both the local $\frac{\partial \mathbf{f}(\mathbf{q})}{\partial \mathbf{q}}$ and global $\frac{\partial \mathbf{f}(\mathbf{q})}{\partial \mathbf{x}}$ in order to extract their contribution to the u direction residual which is used in the alignment.

Appendix

Practical Formulas

$$y = y_c + \mathrm{sign}\left(()\,\mathrm{R}\right) \times \sqrt{\left(\mathrm{R}^2 - (x - x_c)^2\right)} \tag{40}$$

$$x = x_c + \mathrm{sign}\left(()\,\mathrm{R}\right) \times \sqrt{\left(\mathrm{R}^2 - (y - y_c)^2\right)} \tag{41}$$

$$\Delta\phi = \mathrm{atan2}(x - x_c, y - y_c) - \mathrm{atan2}(x_0 - x_c, y_0 - y_c) \tag{42}$$

Arc length, s, can be written as:

$$\mathrm{s} = -\Delta\phi \times \mathrm{R}. \tag{43}$$

$$\frac{\partial y_c}{\partial d_0} = \frac{\mathrm{sign}\left(\mathrm{R}\right)^2 \sin\left(\phi_0\right)(x - x_c)}{\sqrt{\mathrm{R}^2 - \mathrm{sign}\left(\mathrm{R}\right)^2 \left(\mathrm{R}^2 - (x - x_c)^2\right)}} \tag{44}$$

$\frac{\partial\phi}{\partial\mathbf{q}}$ using Eq. 31

$$\begin{aligned}
\frac{\partial\phi}{\partial d_0} &= \frac{\sin\phi_0\,(x + (d_0 - R)\sin\phi_0)}{R^2\sqrt{\frac{(x+(d_0-R)\sin\phi_0)^2\left(1-\frac{(x+(d_0-R)\sin\phi_0)^2}{R^2}\right)}{R^2}}} \\
\frac{\partial\phi}{\partial\phi_0} &= \frac{(d_0 - R)\cos\phi_0\,(x + (d_0 - R)\sin\phi_0)}{R^2\sqrt{\frac{(x+(d_0-R)\sin\phi_0)^2\left(1-\frac{(x+(d_0-R)\sin\phi_0)^2}{R^2}\right)}{R^2}}} \\
\frac{\partial\phi}{\partial R} &= -\left(\frac{(x + d_0\sin\phi_0)\,(x + (d_0 - R)\sin\phi_0)}{R^3\sqrt{\frac{(x+(d_0-R)\sin\phi_0)^2\left(1-\frac{(x+(d_0-R)\sin\phi_0)^2}{R^2}\right)}{R^2}}}\right) \\
\frac{\partial\phi}{\partial slope} &= 0 \\
\frac{\partial\phi}{\partial z0} &= 0
\end{aligned} \tag{45}$$

References

[1] Markus Stoye, *Calibration and Alignment of the CMS Silicon Tracking Detector*, Dissertation, 2007.