Helix Derivatives

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1 Track based alignment

Each hit measurement, y_i, is assumed to be described by a (non-)linear track model $f(x_i, \mathbf{q})$ which depends on a small number of parameter $\mathbf{q}$. Typically these are the five track parameters (see above) for tracks in a uniform magnetic field,

$$y_i = f(x_i, \mathbf{q}) + \epsilon_i. \tag{1}$$

The coordinate x_i is the coordiante of the hit y_i and ϵ is the uncertainty on y_i. The local-fit function f is linearized, if needed, by expressing it as a linear function of the local parameter corrections $\Delta\mathbf{q}$ at some reference value $\mathbf{q}_k$,

$$f(x_i, \mathbf{q}_k + \Delta\mathbf{q}) = f(x_i, \mathbf{q}_k) + \frac{\partial f}{\partial q_1}\Delta \mathrm{q}_1 + \frac{\partial f}{\partial q_2}\Delta \mathrm{q}_2 + ... \tag{2}$$

The local fit relies on minimizing the measured residual z_i for each hit,

$$z_i = y_i - f(x_i, \mathbf{q}_k). \tag{3}$$

By solving for $\Delta\mathbf{q}$ for each iteration k and updating with $q_{k+1} = q_k + \Delta\mathbf{q}$ convergence and optimal $\Delta\mathbf{q}$ can be obtained. These local parameter corrections $\Delta\mathbf{q}$ is used in the global fit. The derivatives of f w.r.t. the local parameters $\mathbf{q}$ is calculated in Sec. 4.

Describe the addition of global parameters here.

2 Coordinate Systems

Describe the detector, tracking and sensor frames.

3 Helical Track Equations

Equations for trajectories in the XY plane

Point on helix (x, y) satisfies,

$$\mathrm{R}^2 = (x - x_c)^2 + (y - y_c)^2 \tag{4}$$

and the coordinate of the centre of the circle can be written,

$$
\begin{aligned}
x_c &= x + \mathrm{R}\sin\phi & (5)\\
y_c &= y - \mathrm{R}\cos\phi. & (6)
\end{aligned}
$$

Equation for trajectory in XZ plane

$$z = z_0 + \mathrm{s} \times \mathrm{slope} \tag{7}$$

4 Local Parameters

Start by expressing ϕ as a function of the track parameters d_0, ϕ_0, R and the interaction point along the beam line x. From 4, $y = y_c + R \times \cos\phi$ and $x_c = (R - d_0)\sin\phi_0$

$$
\begin{aligned}
(\cos\phi)^2 &= \frac{1}{R^2}\left(R^2 - (x - x_c)^2\right)\\
(\cos\phi)^2 &= \frac{1}{R^2}\left(R^2 - (x - (R - d_0)\sin\phi_0)^2\right)\\
\phi &= \cos^{-1}\left(\sqrt{1 - \left(\frac{x - (R - d_0)\sin\phi_0}{R}\right)^2}\right) & (8)
\end{aligned}
$$

or, equivalently, using Eq. 5,

$$
\begin{aligned}
\sin\phi &= \frac{1}{R}(x - x_c) = (x - (R - d_0)\sin\phi_0)\\
\phi &= \arcsin\left(\frac{x - (R - d_0)\sin\phi_0}{R}\right). & (9)
\end{aligned}
$$

Using,

$$
\begin{aligned}
x &= x_c - R\sin\phi = (R - d_0)\sin\phi_0 - R\sin\phi\\
y &= y_c + R\cos\phi = -(R - d_0)\cos\phi_0 + R\cos\phi\\
z &= z_0 + s \times \mathrm{slope} = z_0 - R \times \mathrm{slope}\,(\phi - \phi_0)\,, & (10)
\end{aligned}
$$

we can calculate the local derivatives $\frac{\partial f_{x_i}(\mathbf{q})}{\partial \mathbf{q}}$ where $i = x, y, z$.

$\frac{\partial f_x(\mathbf{q})}{\partial \mathbf{q}}$:

$$\begin{aligned}
\frac{\partial f_x(\mathbf{q})}{\partial d_0} &= -\sin\phi_0 - R\cos\phi\frac{\partial\phi}{\partial d_0} \\
\frac{\partial f_x(\mathbf{q})}{\partial z_0} &= -R\cos\phi\frac{\partial\phi}{\partial z_0} \\
\frac{\partial f_x(\mathbf{q})}{\partial phi_0} &= (R-d_0)\cos\phi_0 - R\cos\phi\frac{\partial\phi}{\partial phi_0} \\
\frac{\partial f_x(\mathbf{q})}{\partial R} &= \sin\phi_0 - R\cos\phi\frac{\partial\phi}{\partial R} \\
\frac{\partial f_x(\mathbf{q})}{\partial \mathrm{slope}} &= R\cos\phi\frac{\partial\phi}{\partial \mathrm{slope}},
\end{aligned} \tag{11}$$

$\frac{\partial f_y(\mathbf{q})}{\partial \mathbf{q}}$:

$$\begin{aligned}
\frac{\partial f_y(\mathbf{q})}{\partial d_0} &= \cos\phi_0 - R\sin\phi\frac{\partial\phi}{\partial d_0} \\
\frac{\partial f_y(\mathbf{q})}{\partial z_0} &= -R\sin\phi\frac{\partial\phi}{\partial z_0} \\
\frac{\partial f_y(\mathbf{q})}{\partial phi_0} &= (R-d_0)\sin\phi_0 - R\sin\phi\frac{\partial\phi}{\partial phi_0} \\
\frac{\partial f_y(\mathbf{q})}{\partial R} &= -\cos\phi_0 - R\sin\phi\frac{\partial\phi}{\partial R} \\
\frac{\partial f_y(\mathbf{q})}{\partial \mathrm{slope}} &= -R\sin\phi\frac{\partial\phi}{\partial \mathrm{slope}}
\end{aligned} \tag{12}$$

$\frac{\partial f_y(\mathbf{q})}{\partial \mathbf{q}}$:

$$\begin{aligned}
\frac{\partial f_z(\mathbf{q})}{\partial d_0} &= -R\times\mathrm{slope}\frac{\partial\phi}{\partial d_0} \\
\frac{\partial f_z(\mathbf{q})}{\partial z_0} &= 1 \\
\frac{\partial f_z(\mathbf{q})}{\partial phi_0} &= -R\times\mathrm{slope}\left(\frac{\partial\phi}{\partial phi_0} - 1\right) \\
\frac{\partial f_z(\mathbf{q})}{\partial R} &= -\mathrm{slope}\times\left(\phi - \phi_0 + R\frac{\partial\phi}{\partial R}\right) \\
\frac{\partial f_z(\mathbf{q})}{\partial \mathrm{slope}} &= s = -R(\phi - \phi_0)
\end{aligned} \tag{13}$$

$\frac{\partial\phi}{\partial\mathbf{q}}$ using Eq. 9

$$\begin{aligned}
\frac{\partial\phi}{\partial d_0} &= \frac{\sin\phi_0}{R\sqrt{\frac{(x+(d_0-R)\sin\phi_0)^2}{R^2}}} \\
\frac{\partial\phi}{\partial phi_0} &= \frac{(d_0-R)\cos\phi_0}{R\sqrt{1-\frac{(x+(d_0-R)\sin\phi_0)^2}{R^2}}} \\
\frac{\partial\phi}{\partial R} &= \frac{-x-d_0\sin\phi_0}{R^2\sqrt{1-\frac{(x+(d_0-R)\sin\phi_0)^2}{R^2}}} \\
\frac{\partial\phi}{\partial slope} &= 0 \\
\frac{\partial\phi}{\partial z0} &= 0
\end{aligned} \tag{14}$$

The calculation of $\frac{\partial\phi}{\partial\mathbf{q}}$ using Eq. 8 is shown in appendix.

5 Global Parameters

5.1 Translation

Starting from Eq. 5 and 6, we can write,

$$x = x_c \mathrm{R}\sin\phi \tag{15}$$

$$y = y_c + \mathrm{R}\times\cos\phi \tag{16}$$

$$z = z_0 - \mathrm{R}\times\mathrm{slope}\times\Delta\phi. \tag{17}$$

Note that,

$$\phi = \mathrm{atan2}\left(\sin\phi_0 - \frac{x-x_0}{\mathrm{R}}, \cos\phi_0 + \frac{y-y_0}{\mathrm{R}}\right). \tag{18}$$

$$\tag{19}$$

5.1.1 Translation in x

To calculate the derivatives for a translation in x we express the track model equation as a function of x, i.e. $f(x,\mathbf{q})$. Using Eq. 18 and substituting,

$$\begin{aligned}
y &= -(R-d_0)\cos\phi_0 + \mathrm{sign}\,(\mathrm{R})\sqrt{R^2-(x-(R-d_0)\sin\phi_0)^2} \\
y_0 &= d_0\cos\phi_0 \\
x_0 &= -d_0\sin\phi_0
\end{aligned} \tag{20}$$

$$\tag{21}$$

we can calculate $\frac{\partial f_{x_i}(\mathbf{q})}{\partial x}$ where $x_i = x, y, z$.

$$
\begin{aligned}
\frac{\partial f_x}{\partial x} &= 1 \\
\frac{\partial f_y}{\partial x} &= -R \sin \phi \frac{\partial \phi}{\partial x} \\
\frac{\partial f_z}{\partial x} &= -\mathrm{R} \times \mathrm{slope} \frac{\partial \phi}{\partial x},
\end{aligned} \tag{22}
$$

with $\frac{\partial \phi}{\partial x}$ given by,

$$
\frac{\partial \phi}{\partial x} = \frac{-R^2 \operatorname{sign}(\mathrm{R})}{\sqrt{R^2 - (x + (d_0 - R)s_0)^2}\left(-(x + (d_0 - R)s_0)^2 + \operatorname{sign}(\mathrm{R})^2\left(-R^2 + x^2 + 2(d_0 - R)xs_0 + (d_0 - R)^2 \sin^2 \phi_0\right)\right)} \tag{23}
$$

where $s_0 = \sin \phi_0$.

5.1.2 Translation in y

Similarly as for x, using Eq. 18 and substituting,

$$
\begin{aligned}
x &= (R - d_0) \sin \phi_0 + \operatorname{sign}(\mathrm{R}) \sqrt{R^2 - (y(R - d_0) \cos \phi_0)^2} \\
y_0 &= d_0 \cos \phi_0 \\
x_0 &= -d_0 \sin \phi_0 && (24) \\
& && (25)
\end{aligned}
$$

we can calculate $\frac{\partial f_{x_i}(\mathbf{q})}{\partial y}$ where $x_i = x, y, z$.

$$
\begin{aligned}
\frac{\partial x}{\partial y} &= \mathrm{R} \cos \phi \frac{\partial \phi}{\partial y} \\
\frac{\partial y}{\partial y} &= 1 \\
\frac{\partial z}{\partial y} &= -\mathrm{R} \times \mathrm{slope} \frac{\partial \phi}{\partial y},
\end{aligned} \tag{26}
$$

with $\frac{\partial \phi}{\partial y}$ given by,

$$
\frac{\partial \phi}{\partial y} = \frac{R^2 \operatorname{sign}(\mathrm{R})}{\sqrt{R^2 - (y + (-d_0 + R)c_0)^2}\left(-(y + (-d_0 + R)c_0)^2 + (-R^2 + y^2 - 2(d_0 - R)yc_0 + (d_0 - R)^2 c_0^2) \operatorname{sign}(\mathrm{R})^2\right)} \tag{27}
$$

where $c_0 = \cos \phi_0$.

5.1.3 Translation in z

The derivatives are given by,

$$\begin{aligned}
\frac{\partial x}{\partial z} &= \text{sign}\,(\text{R}) \times \text{R} \cos\phi \frac{\partial \phi}{\partial z} \\
\frac{\partial y}{\partial z} &= -\text{sign}\,(\text{R}) \times \text{R} \times \sin\phi \frac{\partial \phi}{\partial z} \\
\frac{\partial z}{\partial z} &= 1,
\end{aligned} \tag{28}$$

with $\frac{\partial \phi}{\partial z}$ given by

$$\frac{\partial \phi}{\partial z} = \frac{\partial}{\partial z}\left(-\frac{z - z_0}{R \times \text{slope}} + \phi_0\right) = -\frac{1}{R \times \text{slope}} \tag{29}$$

$$\tag{30}$$

5.2 Rotation

We assume that all rotation angles are around the center of the sensor and small and use the small angle limit for rotations $\mathbf{k} = (\alpha, \beta, \gamma)$ corresponding to rotations around the three axis (x, y, z). The derivatives $\frac{\partial f_{x_i}}{\partial \mathbf{k}}$ are given by,

$$\frac{\partial f_{x_i}}{\partial \mathbf{k}} = \begin{pmatrix} \frac{\partial f_x}{\partial \alpha} & \frac{\partial f_y}{\partial \alpha} & \frac{\partial f_z}{\partial \alpha} \\ \frac{\partial f_x}{\partial \beta} & \frac{\partial f_y}{\partial \beta} & \frac{\partial f_z}{\partial \beta} \\ \frac{\partial f_x}{\partial \gamma} & \frac{\partial f_y}{\partial \gamma} & \frac{\partial f_z}{\partial \gamma} \end{pmatrix}$$

$$\frac{\partial f_{x_i}}{\partial \mathbf{k}} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

6 Transformation to the Sensor Frame

In the above sections all equations have been in what we call the "tracking frame" which is described in Sec. 2. Normally residuals, and thus the derivatives, are measured on the so-called "sensor frame" which is a coordinate system local to each sensor or plane. Typically this is defined with u being the well-measured coordinate, v being the less well-measured coordinate and w normal to the sensor plane. In our case it means that v is parallel to the strips and u is the measurement direction.

When aligning a strip sensor there is only one measured direction of the hit for each plane/sensor, the u direction. Thus, the only residual that will go into the alignment is in that direction. In the case of the HPS experiment the u direction is close to parallel with the tracking frame z direction for the axial sensors.

Thus to properly prepare the input to the alignment all the derivatives (and residuals) have to be transformed into the corresponding values in the u direction. Practically this means that for axial sensors, that have u close to parallel with z, the z contribution will dominate. For the stereo sensors the stereo angle will cause contributions from x, y direction to contribute more.

The translation and rotation matrices, T are built using the detector geometry package which takes as input a xml file describing the position and rotations of the sensors. To take a vector of predicted hit positions on a sensor, $\mathbf{q_p^t}$, from the tracking to the sensor frame:

$$\mathbf{q}_p^s = \mathbf{T}_{T->S}\mathbf{q}_p \tag{31}$$

where $\mathbf{T}_{T->S}$ is the transformation matrix going from the tracking to sensor frame for a given sensor. In the current software this is practically done by going to the global coordinate system, called the "detector frame",

$$\begin{aligned} \mathbf{T}_{T->S} &= \mathbf{T}_{D->S}\mathbf{T}_{T->D}^{-1} \\ \mathbf{q}_p^s &= \mathbf{T}_{D->S}\mathbf{T}_{T->D}^{-1}\mathbf{q}_p \end{aligned} \tag{32}$$

where $\mathbf{T}_{T->D}^{-1}$ is the inverse of the transformation matrix going from tracking frame to detector frame and $\mathbf{T}_{D->S}$ is the transformation from the detector to the sensor frame.

These transformation are applied to all the vector of derivatives, both the local $\frac{\partial \mathbf{f}(\mathbf{q})}{\partial \mathbf{q}}$ and global $\frac{\partial \mathbf{f}(\mathbf{q})}{\partial \mathbf{x}}$ in order to extract their contribution to the u direction residual which is used in the alignment.

Appendix

Practical Formulas

$$y = y_c + \operatorname{sign}(()\,\mathrm{R}) \times \sqrt{\left(\mathrm{R}^2 - (x - x_c)^2\right)} \tag{33}$$

$$x = x_c + \operatorname{sign}(()\,\mathrm{R}) \times \sqrt{\left(\mathrm{R}^2 - (y - y_c)^2\right)} \tag{34}$$

$$\Delta\phi = \operatorname{atan2}(x - x_c, y - y_c) - \operatorname{atan2}(x_0 - x_c, y_0 - y_c) \tag{35}$$

Arc length, s, can be written as:

$$\mathrm{s} = -\Delta\phi \times \mathrm{R}. \tag{36}$$

$$\frac{\partial y_c}{\partial d_0} = \frac{\operatorname{sign}(\mathrm{R})^2 \sin(\phi_0)(x - x_c)}{\sqrt{\mathrm{R}^2 - \operatorname{sign}(\mathrm{R})^2 \left(\mathrm{R}^2 - (x - x_c)^2\right)}} \tag{37}$$

$\frac{\partial \phi}{\partial \mathbf{q}}$ using Eq. 8

$$\begin{aligned}
\frac{\partial \phi}{\partial d_0} &= \frac{\sin\phi_0 \left(x + (d_0 - R)\sin\phi_0\right)}{R^2 \sqrt{\frac{(x + (d_0 - R)\sin\phi_0)^2 \left(1 - \frac{(x + (d_0 - R)\sin\phi_0)^2}{R^2}\right)}{R^2}}} \\
\frac{\partial \phi}{\partial \phi_0} &= \frac{(d_0 - R)\cos\phi_0 \left(x + (d_0 - R)\sin\phi_0\right)}{R^2 \sqrt{\frac{(x + (d_0 - R)\sin\phi_0)^2 \left(1 - \frac{(x + (d_0 - R)\sin\phi_0)^2}{R^2}\right)}{R^2}}} \\
\frac{\partial \phi}{\partial R} &= -\left(\frac{(x + d_0\sin\phi_0)\left(x + (d_0 - R)\sin\phi_0\right)}{R^3 \sqrt{\frac{(x + (d_0 - R)\sin\phi_0)^2 \left(1 - \frac{(x + (d_0 - R)\sin\phi_0)^2}{R^2}\right)}{R^2}}}\right) \\
\frac{\partial \phi}{\partial slope} &= 0 \\
\frac{\partial \phi}{\partial z0} &= 0
\end{aligned} \tag{38}$$