Helix Derivatives

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1 Introduction

This note describes the implementation of the alignment procedure for the HPS SVT. Most of this follows what is described in more depth in Ref. [1] and in the MillepedeII manual (ref).

Updates needed:

- Need to update the mix of nomenclature of q for track parameters and hit positions.
- Clean-up formulas that are not relevant
- improve organization in the descriptions

2 Track Based Alignment

Each hit measurement, y_i, is assumed to be described by a (non-)linear track model $f(x_i, \tau)$ which depends on a small number of parameter τ. In the case of a particle track in a homogenous magnetic field these are the five track parameters describing a helical track,

$$y_i = f\left(x_i, \tau\right) + \epsilon_i. \tag{1}$$

The coordinate x_i is the coordinate of the hit y_i and ϵ is the uncertainty on y_i. The local-fit function f is linearized, if needed, by expressing it as a linear function of the local parameter corrections $\Delta\tau$ at some reference value τ_k,

$$f\left(x_i, \tau_k + \Delta\tau\right) = f\left(x_i, \tau_k\right) + \frac{\partial f}{\partial \mathrm{q}_1}\Delta \mathrm{q}_1 + \frac{\partial f}{\partial \mathrm{q}_2}\Delta \mathrm{q}_2 + ... \tag{2}$$

The local fit relies on minimizing the measured residual z_i for each hit,

$$z_i = y_i - f\left(x_i, \tau_k\right). \tag{3}$$

By solving for $\Delta\tau$ for each iteration k and updating with $\tau_{k+1} = \tau_k + \Delta\tau$ convergence and optimal $\Delta\tau$ can be obtained.

This so-called local fit is performed assuming the relative positions of the sensors providing the hit measurements. Each sensor has 6 degrees of freedom, 3 translation and 3 rotations, given by a vector $\mathbf{p}$ and the goal of track based alignment is to determine the corrections $\mathbf{\Delta p}$ to the nominal values. These global parameters can be incorporated in the residual by

$$z_i = y_i - f\left(x_i, \tau_k\right) = \sum_j^{\nu} \frac{\partial f}{\partial \mathrm{q}_j} \Delta \mathrm{q}_j + \sum_l \frac{\partial f}{\partial \mathrm{p}_l} \Delta \mathrm{p}_l. \tag{4}$$

where l is the number of global parameters and the best local (track) parameter correction $\Delta\tau$ can be used in the global fit.

The alignment algorithm last step is to minimize these residuals w.r.t. the global parameters which involves solving a system of linear equations using the MillepedeII software program.

In order to run the minimization we have to provide three inputs to MillepedeII:

- The track residuals $\mathbf{z}$,
- the local track derivatives $\frac{\partial f}{\partial \tau}$,
- and the global derivatives $\frac{\partial f}{\partial \mathbf{p}}$.

This note describes the calculation of these inputs.

Section 3 introduces some useful nomenclature and definitions used in the later sections. The calculation of all the local track derivatives are described in Sec. 6. Section describes the way we setup the alignment parameterization w.r.t. the global parameters and the actual derivatives are calculated in Sec. 5.

3 Coordinates, Frames and Helical Tracks

In the following there are two coordinate systems that are used. The first one is what we call the *local (sensor) frame* which is a coordinate system based on the sensor plane itself where the three axis u, v, w are typically defined with the u direction being the well measured coordinate on the sensor and v the less measured coordinate and finally w orthogonal and in this case normal to the sensor plane.

The other coordinate system is what we call the *global (tracking) frame* which is the coordinate system in which our tracks are fitted. This is defined with the three axis x, y, z where the magnetic filed is parallel to the z axis. Charged particles thus follow helical trajectories with y as the bend-plane and x, in our case, coincides with the the beam line direction.

Equations for trajectories in the XY plane

Point on helix (x, y) satisfies,

$$\mathrm{R}^2 = \left(x - x_c\right)^2 + \left(y - y_c\right)^2 \tag{5}$$

and the coordinate of the centre of the circle can be written,

$$x_c = x + \mathrm{R}\sin\phi \tag{6}$$
$$y_c = y - \mathrm{R}\cos\phi. \tag{7}$$

Equation for trajectory in XZ plane

$$z = z_0 + \mathrm{s} \times \mathrm{slope} \tag{8}$$

4 Alignment Parameterization

`Note the change of nomenclature for q compared to previous section.`

A hit measurement vector $\mathbf{q}$ can be represented as,

$$\mathbf{r} = \mathbf{R^T q} + \mathbf{r_0}, \tag{9}$$

where $\mathbf{R}$ is a rotation matrix and $\mathbf{r_0}$ is the position of the sensor. The task of the alignment procedure is to provide correction to position and rotation of the sensor, $\mathbf{q_0}$ and $\mathbf{R}$, respectively,

$$\mathbf{r} = \mathbf{R^T \Delta R}\,(\mathbf{q} + \Delta\mathbf{q}) + \mathbf{r_0}. \tag{10}$$

The alignment parameters are the components of $\Delta\mathbf{q}$ and $\Delta\mathbf{R}$ and are often expressed in the local sensor coordinates as they are related to the individual sensor. As noted in the previous section, the measured hit position components are $\mathbf{q} = (u, v, w)$, where the precisely measured coordinate on the sensor is separated from the less well-known coordinate. In a strip sensor u is typically the precisely measured coordinate and v is the un-measured coordinate. w is the direction normal to the sensor plane. For alignment typically the Δw is ignored as all hits happen at the sensor plane as will be evident later. The rotation correction matrix $\Delta\mathbf{R}$ are reduced to three angles around the u-, v- and w-axis and are denoted as α, β and γ (around the center of the sensors and thus do not induce a translation). Each sensor thus has 6 alignment parameters and following the notation in Ref. **??** it can be represented by a vector $\mathbf{a}$,

$$\mathbf{a} = \begin{pmatrix} \Delta u \\ \Delta v \\ \Delta w \\ \alpha \\ \beta \\ \gamma \end{pmatrix}$$

In order to solve the minimization problem we need to calculate the derivatives of the residuals w.r.t. the local and global parameters. The residual $\mathbf{z}$ is,

$$\mathbf{z} = \mathbf{q}_a - \mathbf{q}_p = \begin{pmatrix} u_m \\ v_m \\ w_m \end{pmatrix} - \begin{pmatrix} u_p \\ v_p \\ w_p \end{pmatrix}$$

where $\mathbf{q}_a$ is the alignment corrected hit,

$$\mathbf{q_a} = \mathbf{\Delta R q}_h + \Delta \mathbf{q}, \tag{11}$$

where $\mathbf{q_h}$ is the measured hit position. For the minimization of the square of residuals the global derivatives,

$$\frac{\partial \mathbf{z}}{\partial \mathbf{a}} = \frac{\partial \mathbf{q}_a}{\partial \mathbf{a}} - \frac{\partial \mathbf{q}_p}{\partial \mathbf{a}}. \tag{12}$$

needs to be calculated.

Starting with $\frac{\partial \mathbf{q}_a}{\partial \mathbf{a}}$ the partial derivatives w.r.t. to a translation Δu is,

$$\frac{\partial \mathbf{q}_a}{\partial \Delta u} = \frac{\partial}{\partial \Delta u} \left(\mathbf{\Delta R q}_h + \Delta \mathbf{q}\right) = \frac{\partial}{\partial \Delta u} \begin{pmatrix} \Delta u \\ \Delta v \\ \Delta w \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and similarly for the other translations in v and w,

$$\frac{\partial \mathbf{q}_a}{\partial \Delta v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \frac{\partial \mathbf{q}_a}{\partial \Delta w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Next we calculate the derivatives w.r.t. the rotations i.e. $\frac{\partial \mathbf{q}_a}{\partial \alpha}, \frac{\partial \mathbf{q}_a}{\partial \beta}, \frac{\partial \mathbf{q}_a}{\partial \gamma}$. By observing that

$$\frac{\partial \mathbf{q}_a}{\partial \alpha} = \frac{\partial}{\partial \alpha} \left(\mathbf{\Delta R q}_h + \Delta \mathbf{q}\right) = \frac{\partial}{\partial \alpha} \left(\mathbf{\Delta R q}_h\right) \tag{13}$$

$$\frac{\partial \mathbf{q}_a}{\partial \beta} = \frac{\partial}{\partial \beta} \left(\mathbf{\Delta R q}_h + \Delta \mathbf{q}\right) = \frac{\partial}{\partial \beta} \left(\mathbf{\Delta R q}_h\right) \tag{14}$$

$$\frac{\partial \mathbf{q}_a}{\partial \gamma} = \frac{\partial}{\partial \gamma} \left(\mathbf{\Delta R q}_h + \Delta \mathbf{q}\right) = \frac{\partial}{\partial \gamma} \left(\mathbf{\Delta R q}_h\right) \tag{15}$$

where the rotation matrix is given by

$$\mathbf{\Delta R} = \mathbf{R}_\gamma \times \mathbf{R}_\beta \times \mathbf{R}_\alpha$$

and each rotation is described by the normal $3x3$ rotation matrices:

$$\mathbf{R}_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \mathbf{R}_\beta = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}, \mathbf{R}_\gamma = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

All angles are assumed to be small and after linearization the derivatives become,

$$\frac{\partial \mathbf{\Delta R}}{\partial \alpha} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \frac{\partial \mathbf{\Delta R}}{\partial \beta} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \frac{\partial \mathbf{\Delta R}}{\partial \gamma} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

These can be written more compactly, and evaluated at the measured position $\mathbf{q}_h = \mathbf{q}_m = (u_m, v_m, w_m)$ as,

$$\frac{\partial \mathbf{q}_a}{\partial \mathbf{a}} = \begin{pmatrix} \mathbf{1} & \frac{\partial \Delta R}{\partial \alpha}\mathbf{q}_h & \frac{\partial \Delta R}{\partial \beta}\mathbf{q}_h & \frac{\partial \Delta R}{\partial \gamma}\mathbf{q}_h \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & -w_m & v_m \\ 0 & 1 & 0 & w_m & 0 & -u_m \\ 0 & 0 & 1 & -v_m & u_m & 0 \end{pmatrix}$$

As mentioned before $w_m = 0$ by construction (the hit $\mathbf{q}_h$ is on the sensor surface) and the un-measured direction v can be ignored by it kept here for consistency,

$$\frac{\partial \mathbf{q}_a}{\partial \mathbf{a}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & v_m \\ 0 & 1 & 0 & 0 & 0 & -u_m \\ 0 & 0 & 1 & -v_m & u_m & 0 \end{pmatrix}$$

The global derivative $\frac{\partial \mathbf{q}_p}{\partial \mathbf{a}}$ measures the effect of the predicted track position on the surface of the sensor. Note that a shift of the sensor in the u, v plane is equivalent to a shift of the measured hit position $\mathbf{q}_h$ and the only direction where the track propagation needs to be taken into account is the w direction. Using this we can write (this is unclear!!),

$$\frac{\partial \mathbf{q}_p}{\partial \mathbf{a}} = \frac{\partial \mathbf{q}_p}{\partial \mathbf{q}_a}\frac{\partial \mathbf{q}_a}{\partial \mathbf{a}} \tag{16}$$

where $\frac{\partial \mathbf{q}_p}{\partial \mathbf{q}_a}$ is

$$\frac{\partial \mathbf{q}_p}{\partial u_a} = 0, \frac{\partial \mathbf{q}_p}{\partial v_a} = 0, \frac{\partial \mathbf{q}_p}{\partial w_a} = \frac{\partial \mathbf{q}_p}{\partial w_h} = \begin{pmatrix} \frac{\partial u_p}{\partial w_h} \\ \frac{\partial v_p}{\partial w_h} \\ \frac{\partial w_p}{\partial w_h} \end{pmatrix}$$

since a shift in the u, v plane is equivalent to a shift in the hit position. Using this and $\frac{\partial \mathbf{q}_a}{\partial \mathbf{a}}$ calculated earlier,

$$\frac{\partial \mathbf{q}_p}{\partial \mathbf{q}_a}\frac{\partial \mathbf{q}_a}{\partial \mathbf{a}} = \begin{pmatrix} 0 & 0 & \frac{\partial u_p}{\partial w_h} \\ 0 & 0 & \frac{\partial v_p}{\partial w_h} \\ 0 & 0 & \frac{\partial w_p}{\partial w_h} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & v_m \\ 0 & 1 & 0 & 0 & 0 & -u_m \\ 0 & 0 & 1 & -v_m & u_m & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & \frac{\partial u_p}{\partial w_h} & -v_m\frac{\partial u_p}{\partial w_h} & u_m\frac{\partial u_p}{\partial w_h} & 0 \\ 0 & 0 & \frac{\partial v_p}{\partial w_h} & -v_m\frac{\partial v_p}{\partial w_h} & u_m\frac{\partial v_p}{\partial w_h} & 0 \\ 0 & 0 & \frac{\partial w_p}{\partial w_h} & -v_m\frac{\partial w_p}{\partial w_h} & u_m\frac{\partial w_p}{\partial w_h} & 0 \end{pmatrix}$$

Now we have all the ingredients to calculate Eq. 12:

$$\frac{\partial \mathbf{z}}{\partial \mathbf{a}} = \frac{\partial \mathbf{q}_a}{\partial \mathbf{a}} - \frac{\partial \mathbf{q}_p}{\partial \mathbf{a}} = \frac{\partial \mathbf{q}_a}{\partial \mathbf{a}} - \frac{\partial \mathbf{q}_p}{\partial \mathbf{q}_a}\frac{\partial \mathbf{q}_a}{\partial \mathbf{a}} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & v_m \\ 0 & 1 & 0 & 0 & 0 & -u_m \\ 0 & 0 & 1 & -v_m & u_m & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \frac{\partial u_p}{\partial w_h} & -v_m\frac{\partial u_p}{\partial w_h} & u_m\frac{\partial u_p}{\partial w_h} & 0 \\ 0 & 0 & \frac{\partial v_p}{\partial w_h} & -v_m\frac{\partial v_p}{\partial w_h} & u_m\frac{\partial v_p}{\partial w_h} & 0 \\ 0 & 0 & \frac{\partial w_p}{\partial w_h} & -v_m\frac{\partial w_p}{\partial w_h} & u_m\frac{\partial w_p}{\partial w_h} & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & -\frac{\partial u_p}{\partial w_h} & v_m \frac{\partial u_p}{\partial w_h} & -u_m \frac{\partial u_p}{\partial w_h} & v_m \\ 0 & 1 & -\frac{\partial v_p}{\partial w_h} & v_m \frac{\partial v_p}{\partial w_h} & -u_m \frac{\partial v_p}{\partial w_h} & -u_m \\ 0 & 0 & 1-\frac{\partial w_p}{\partial w_h} & v_m \frac{\partial w_p}{\partial w_h} - v_m & u_m - u_m \frac{\partial w_p}{\partial w_h} & 0 \end{pmatrix}$$

Since $w_m = 0$ we can ignore the third component which means that we can write the global residual derivative as,

$$\frac{\partial \mathbf{z}}{\partial \mathbf{a}} = \begin{pmatrix} 1 & 0 & -\frac{\partial u_p}{\partial w_h} & v_m \frac{\partial u_p}{\partial w_h} & -u_m \frac{\partial u_p}{\partial w_h} & v_m \\ 0 & 1 & -\frac{\partial v_p}{\partial w_h} & v_m \frac{\partial v_p}{\partial w_h} & -u_m \frac{\partial v_p}{\partial w_h} & -u_m \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In the case of a strip sensor only the well-measured direction is important and thus only the first row is important.

The calculation of the global derivatives $\frac{\partial q}{\partial \mathbf{a}}$ is detailed in Sec. 5.

5 Global Derivatives

The parameterization that is layed out in Sec. 4 provide information on what global derivatives are needed,

$$\frac{\partial \mathbf{z}}{\partial \mathbf{a}} = \begin{pmatrix} 1 & 0 & -\frac{\partial u_p}{\partial w_h} & v_m \frac{\partial u_p}{\partial w_h} & -u_m \frac{\partial u_p}{\partial w_h} & v_m \\ 0 & 1 & -\frac{\partial v_p}{\partial w_h} & v_m \frac{\partial v_p}{\partial w_h} & -u_m \frac{\partial v_p}{\partial w_h} & -u_m \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where only the first row is important for strip sensors. In the global frame, the derivatives involving e.g. u will be affected by translations and rotations around the global coordinate axis x, y, z. Our global derivatives are typically calculated in the global frame since that is where track fitting occurs. The next section shows how to determine the global derivatives in the global frame.

5.1 Global Derivatives in the Local Frame

Later in this section we show how to derive the derivatives of the predicted hit position $\mathbf{x}_p$ in the global frame. This is a natural calculation and is also where our alignment parameters come together to form a common detector geometry. However, previously we showed what global derivatives we need to calculate in the local sensor frame $\frac{\partial \mathbf{z}}{\partial \mathbf{a}}$.

In order to connect these two calculations we need to understand how the change of an alignment parameter in the global frame affect the alignment parameter in the local sensor frame that is used in the Millepede minimization.

Using integration by parts we have

$$\frac{\partial \mathbf{z}}{\partial \mathbf{b}} = \frac{\partial \mathbf{z}}{\partial \mathbf{a}} \frac{\partial \mathbf{a}}{\partial \mathbf{b}}$$

where

$$\mathbf{a} = \begin{pmatrix} \Delta u \\ \Delta v \\ \Delta w \\ \alpha \\ \beta \\ \gamma \end{pmatrix}, \mathbf{b} = \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \\ a \\ b \\ c \end{pmatrix}$$

are the alignment parameters in the local ($\mathbf{a}$) and global ($\mathbf{b}$) frame. The matrix $\frac{\partial \mathbf{a}}{\partial \mathbf{b}}$ is defined as

$$\frac{\partial \mathbf{a}}{\partial \mathbf{b}} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} & \frac{\partial u}{\partial a} & \frac{\partial u}{\partial b} & \frac{\partial u}{\partial c} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} & \frac{\partial v}{\partial a} & \frac{\partial v}{\partial b} & \frac{\partial v}{\partial c} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} & \frac{\partial w}{\partial a} & \frac{\partial w}{\partial b} & \frac{\partial w}{\partial c} \\ \frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial y} & \frac{\partial \alpha}{\partial z} & \frac{\partial \alpha}{\partial a} & \frac{\partial \alpha}{\partial b} & \frac{\partial \alpha}{\partial c} \\ \frac{\partial \beta}{\partial x} & \frac{\partial \beta}{\partial y} & \frac{\partial \beta}{\partial z} & \frac{\partial \beta}{\partial a} & \frac{\partial \beta}{\partial b} & \frac{\partial \beta}{\partial c} \\ \frac{\partial \gamma}{\partial x} & \frac{\partial \gamma}{\partial y} & \frac{\partial \gamma}{\partial z} & \frac{\partial \gamma}{\partial a} & \frac{\partial \gamma}{\partial b} & \frac{\partial \gamma}{\partial c} \end{pmatrix}$$

where $\Delta x = x$ and $\Delta u = u$ (similarly in the other directions) are equivalent in the derivatives.

To calculate the matrix elements we start by expressing how a small translation $\mathbf{\Delta x}(= (\Delta x, \Delta y, \Delta z))$ and rotation $\mathbf{\Delta R'}$ in the global frame affects the local translation $\mathbf{\Delta q} = (\Delta u, \Delta v, \Delta w)$ on a point in the global frame $\mathbf{x} = (x, y, z)$

$$\mathbf{\Delta q} = \mathbf{T\Delta x} + \mathbf{T}\left(\mathbf{\Delta R'x} - \mathbf{x}\right) = \mathbf{T\Delta x} + \mathbf{T}\left(\mathbf{\Delta R'} - \mathbf{I}\right)\mathbf{x} \tag{17}$$

where $\mathbf{T}$ is the rotation matrix from the global to local frame. The rotation matrix $\mathbf{\Delta R'}$ is identical to $\mathbf{\Delta R}$ derived for the local frame earlier, only that the rotation axes are around x, y, z instead of u, v, w.

The other thing we need is to express the local sensor rotations $\mathbf{q} = (\alpha, \beta, \gamma)$ caused by a small rotation $\mathbf{\Delta R'} = (a, b, c)$ on a point $\mathbf{x} = (x, y, z)$ around the global axes. Using $x = \mathbf{T}^{-1}\mathbf{q}$,

$$\begin{aligned} \mathbf{\Delta R'x} &= \mathbf{T}^{-1}\mathbf{\Delta Rq} \\ \mathbf{\Delta R'x} &= \mathbf{\Delta R'T}^{-1}\mathbf{q} = \mathbf{T}^{-1}\mathbf{\Delta Rq} \\ \mathbf{T\Delta R'T}^{-1}\mathbf{q} &= \mathbf{\Delta Rq} \\ \mathbf{\Delta R} &= \mathbf{T\Delta R'T}^{-1} \end{aligned} \tag{18}$$

The calculation of the elements of $\frac{\partial \mathbf{a}}{\partial \mathbf{b}}$ is split up into the 4 3×3 quadrants and discussed below.

Upper left 3×3

Using Eq. 17 we can calculate the upper left 3×3 part of the $\frac{\partial \mathbf{a}}{\partial \mathbf{b}}$

$$\frac{\partial \mathbf{q}}{\partial x} = \frac{\partial}{\partial x}\mathbf{T\Delta x} = \mathbf{T}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \frac{\partial \mathbf{q}}{\partial y} = \mathbf{T}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \frac{\partial \mathbf{q}}{\partial z} = \mathbf{T}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Lower right 3×3

This part of the $\frac{\partial \mathbf{a}}{\partial \mathbf{b}}$ is calculated from

$$\frac{\partial \mathbf{\Delta R}}{\partial (a, b, c)} = \frac{\partial}{\partial (a, b, c)} \left(\mathbf{T \Delta R' T}^{-1} \right)$$

Expressing $\mathbf{\Delta R'}$ as a product of Euler matrices and using $\frac{\partial \mathbf{\Delta R'}}{\partial a,b,c}$ derived in the previous section the right hand side is given by,

$$\frac{\partial}{\partial a} \left(\mathbf{T \Delta R' T}^{-1} \right) = \frac{\partial}{\partial a} \left(\mathbf{T} \left(R_g(c) R_b(b) R_a(a) \right) \mathbf{T}^{-1} \right) = \mathbf{T} \frac{\partial R_a(a)}{\partial a} \mathbf{T}^{-1} = \mathbf{T} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \mathbf{T}^{-1}$$

$$\frac{\partial}{\partial b} \left(\mathbf{T \Delta R' T}^{-1} \right) = \frac{\partial}{\partial b} \left(\mathbf{T} \left(R_g(c) R_b(b) R_a(b) \right) \mathbf{T}^{-1} \right) = \mathbf{T} \frac{\partial R_b(b)}{\partial b} \mathbf{T}^{-1} = \mathbf{T} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \mathbf{T}^{-1}$$

$$\frac{\partial}{\partial c} \left(\mathbf{T \Delta R' T}^{-1} \right) = \frac{\partial}{\partial c} \left(\mathbf{T} \left(R_g(c) R_b(b) R_a(b) \right) \mathbf{T}^{-1} \right) = \mathbf{T} \frac{\partial R_g(c)}{\partial c} \mathbf{T}^{-1} = \mathbf{T} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{T}^{-1}$$

which can be written out more explicitly by also expressing $\mathbf{T}$ as a product of Euler angles $\mathbf{T} = R_g(k) R_b(j) R_a(i)$ (remember that $\mathbf{T}$ is the rotation matrix that takes a vector in the global frame to the sensor frame and thus i, j, k are the rotation angles),

$$\frac{\partial}{\partial a} \left(\mathbf{T \Delta R' T}^{-1} \right) = \mathbf{T} \frac{\partial R_a(a)}{\partial a} \mathbf{T}^{-1} = \begin{pmatrix} 0 & s_j & c_j s_k \\ -s_j & 0 & c_j c_k \\ -c_j s_k & -c_j c_k & 0 \end{pmatrix}$$

$$\frac{\partial}{\partial b} \left(\mathbf{T \Delta R' T}^{-1} \right) = \mathbf{T} \frac{\partial R_b(b)}{\partial b} \mathbf{T}^{-1} = \begin{pmatrix} 0 & -c_j s_i & -c_i c_k + s_i s_j s_k \\ c_j s_i & 0 & c_k s_i s_j + c_i s_k \\ c_i c_k - s_i s_j s_k & -c_k s_i s_j - c_i s_k & 0 \end{pmatrix}$$

$$\frac{\partial}{\partial c} \left(\mathbf{T \Delta R' T}^{-1} \right) = \mathbf{T} \frac{\partial R_g(c)}{\partial c} \mathbf{T}^{-1} = \begin{pmatrix} 0 & c_i c_j & -c_k s_i - c_i s_j s_k \\ -c_i c_j & 0 & -c_i c_k s_j + s_i s_k \\ c_k s_i + c_i s_j s_k & c_i c_k s_j - s_i s_k & 0 \end{pmatrix}$$

The left hand side can be written as

$$\begin{aligned}
\frac{\partial}{\partial a} \mathbf{\Delta R} &= \frac{\partial}{\partial a} \left(R_g(\gamma) R_b(\beta) R_a(\alpha) \right) \frac{\partial R_g(\gamma)}{\partial \gamma} \frac{\partial \gamma}{\partial a} + \frac{\partial R_b(\beta)}{\partial \beta} \frac{\partial \beta}{\partial a} + \frac{\partial R_a(\alpha)}{\partial \alpha} \frac{\partial \alpha}{\partial a} \\
\frac{\partial}{\partial b} \mathbf{\Delta R} &= \frac{\partial}{\partial b} \left(R_g(\gamma) R_b(\beta) R_a(\alpha) \right) \frac{\partial R_g(\gamma)}{\partial \gamma} \frac{\partial \gamma}{\partial b} + \frac{\partial R_b(\beta)}{\partial \beta} \frac{\partial \beta}{\partial b} + \frac{\partial R_a(\alpha)}{\partial \alpha} \frac{\partial \alpha}{\partial b} \\
\frac{\partial}{\partial c} \mathbf{\Delta R} &= \frac{\partial}{\partial c} \left(R_g(\gamma) R_b(\beta) R_a(\alpha) \right) \frac{\partial R_g(\gamma)}{\partial \gamma} \frac{\partial \gamma}{\partial c} + \frac{\partial R_b(\beta)}{\partial \beta} \frac{\partial \beta}{\partial c} + \frac{\partial R_a(\alpha)}{\partial \alpha} \frac{\partial \alpha}{\partial c} \qquad (19)
\end{aligned}$$

and using the $\frac{\partial \mathbf{\Delta R}'}{\partial a,b,c}$ derivatives,

$$\frac{\partial}{\partial a}\mathbf{\Delta R} = \frac{\partial \gamma}{\partial a}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{\partial \beta}{\partial a}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \frac{\partial \alpha}{\partial a}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\frac{\partial}{\partial b}\mathbf{\Delta R} = \frac{\partial \gamma}{\partial b}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{\partial \beta}{\partial b}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \frac{\partial \alpha}{\partial b}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\frac{\partial}{\partial c}\mathbf{\Delta R} = \frac{\partial \gamma}{\partial c}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{\partial \beta}{\partial c}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \frac{\partial \alpha}{\partial c}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

and simplifying,

$$\frac{\partial}{\partial a}\mathbf{\Delta R} = \begin{pmatrix} 0 & \frac{\partial \gamma}{\partial a} & 0 \\ -\frac{\partial \gamma}{\partial a} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\frac{\partial \beta}{\partial a} \\ 0 & 0 & 0 \\ \frac{\partial \beta}{\partial a} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\partial \alpha}{\partial a} \\ 0 & -\frac{\partial \alpha}{\partial a} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial \gamma}{\partial a} & -\frac{\partial \beta}{\partial a} \\ -\frac{\partial \gamma}{\partial a} & 0 & \frac{\partial \alpha}{\partial a} \\ \frac{\partial \beta}{\partial a} & -\frac{\partial \alpha}{\partial a} & 0 \end{pmatrix}$$

$$\frac{\partial}{\partial b}\mathbf{\Delta R} = \begin{pmatrix} 0 & \frac{\partial \gamma}{\partial b} & 0 \\ -\frac{\partial \gamma}{\partial b} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\frac{\partial \beta}{\partial b} \\ 0 & 0 & 0 \\ \frac{\partial \beta}{\partial b} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\partial \alpha}{\partial b} \\ 0 & -\frac{\partial \alpha}{\partial b} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial \gamma}{\partial b} & -\frac{\partial \beta}{\partial b} \\ -\frac{\partial \gamma}{\partial b} & 0 & \frac{\partial \alpha}{\partial b} \\ \frac{\partial \beta}{\partial b} & -\frac{\partial \alpha}{\partial b} & 0 \end{pmatrix}$$

$$\frac{\partial}{\partial c}\mathbf{\Delta R} = \begin{pmatrix} 0 & \frac{\partial \gamma}{\partial c} & 0 \\ -\frac{\partial \gamma}{\partial c} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\frac{\partial \beta}{\partial c} \\ 0 & 0 & 0 \\ \frac{\partial \beta}{\partial c} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\partial \alpha}{\partial c} \\ 0 & -\frac{\partial \alpha}{\partial c} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial \gamma}{\partial c} & -\frac{\partial \beta}{\partial c} \\ -\frac{\partial \gamma}{\partial c} & 0 & \frac{\partial \alpha}{\partial c} \\ \frac{\partial \beta}{\partial c} & -\frac{\partial \alpha}{\partial c} & 0 \end{pmatrix}$$

and finally, equating left- and right-hand side,

$$\frac{\partial}{\partial a}\mathbf{\Delta R} = \begin{pmatrix} 0 & \frac{\partial \gamma}{\partial a} & -\frac{\partial \beta}{\partial a} \\ -\frac{\partial \gamma}{\partial a} & 0 & \frac{\partial \alpha}{\partial a} \\ \frac{\partial \beta}{\partial a} & -\frac{\partial \alpha}{\partial a} & 0 \end{pmatrix} = \begin{pmatrix} 0 & s_j & c_j s_k \\ -s_j & 0 & c_j c_k \\ -c_j s_k & -c_j c_k & 0 \end{pmatrix}$$

$$\frac{\partial}{\partial b}\mathbf{\Delta R} = \begin{pmatrix} 0 & \frac{\partial \gamma}{\partial b} & -\frac{\partial \beta}{\partial b} \\ -\frac{\partial \gamma}{\partial b} & 0 & \frac{\partial \alpha}{\partial b} \\ \frac{\partial \beta}{\partial b} & -\frac{\partial \alpha}{\partial b} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -c_j s_i & -c_i c_k + s_i s_j s_k \\ c_j s_i & 0 & c_k s_i s_j + c_i s_k \\ c_i c_k - s_i s_j s_k & -c_k s_i s_j - c_i s_k & 0 \end{pmatrix}$$

$$\frac{\partial}{\partial c}\mathbf{\Delta R} = \begin{pmatrix} 0 & \frac{\partial \gamma}{\partial c} & -\frac{\partial \beta}{\partial c} \\ -\frac{\partial \gamma}{\partial c} & 0 & \frac{\partial \alpha}{\partial c} \\ \frac{\partial \beta}{\partial c} & -\frac{\partial \alpha}{\partial c} & 0 \end{pmatrix} = \begin{pmatrix} 0 & c_i c_j & -c_k s_i - c_i s_j s_k \\ -c_i c_j & 0 & -c_i c_k s_j + s_i s_k \\ c_k s_i + c_i s_j s_k & c_i c_k s_j - s_i s_k & 0 \end{pmatrix}$$

which we can use to identify the derivatives $\frac{\partial(\alpha,\beta,\gamma)}{\partial(a,b,c)}$,

$$\frac{\partial \alpha}{\partial a} = c_j c_k, \frac{\partial \alpha}{\partial b} = c_k s_i s_j + c_i s_k, \frac{\partial \alpha}{\partial c} = -c_i c_k s_j + s_i s_k \tag{20}$$

$$\frac{\partial \beta}{\partial a} = -c_j s_k, \frac{\partial \beta}{\partial b} = c_i c_k - s_i s_j s_k, \frac{\partial \beta}{\partial c} = c_k s_i + c_i s_j s_k \tag{21}$$

$$\frac{\partial \gamma}{\partial a} = s_j, \frac{\partial \gamma}{\partial b} = -c_j s_i, \frac{\partial \gamma}{\partial c} = c_i c_j \tag{22}$$

This can be identified simply as $\mathbf{T}$,

$$\frac{\partial \mathbf{\Delta R}}{\partial(a,b,c)} = \frac{\partial(\alpha,\beta,\gamma)}{\partial(a,b,c)} = \begin{pmatrix} c_j c_k & c_k s_i s_j + c_i s_k & -c_i c_k s_j + s_i s_k \\ -c_j s_k & c_i c_k - s_i s_j s_k & c_k s_i + c_i s_j s_k \\ s_j & -c_j s_i & c_i c_j \end{pmatrix} = \mathbf{T}$$

This can be understood by the following: a small rotation correction (a,b,c) to a vector in the global frame induces a small correction rotation in the local frame. The rate of that induced change, i.e. the derivative, must be proportional to the rotation matrix $\mathbf{T}$. If not, a rotation in the global frame would deform an object in the local frame as not all points on the object would transform according to the rotation matrix $\mathbf{T}$.

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Need a more intuitive explanation
```

Upper right 3×3

$$\frac{\partial \mathbf{\Delta q}}{\partial\,(a,b,c)} = \frac{\partial}{\partial\,(a,b,c)}\left(\mathbf{T\Delta x} + \mathbf{T}\left(\mathbf{\Delta R}' - \mathbf{I}\right)\mathbf{x}\right) = \frac{\partial}{\partial\,(a,b,c)}\left(\mathbf{T}\left(\mathbf{\Delta R}' - \mathbf{I}\right)\mathbf{x}\right) \tag{23}$$

where the assumption is that $\frac{\partial \mathbf{\Delta x}}{\partial a,b,c} \approx 0$. The derivatives $\frac{\partial \mathbf{\Delta R}'}{\partial a,b,c}$ are given in Sec. 4.

Lower left 3×3

Lower left quadrant is trivial as it comes down to the effect of a translation on the rotation which is zero,

$$\frac{\partial \mathbf{\Delta R}}{\partial\,(x,y,z)} = \frac{\partial}{\partial\,(x,y,z)}\left(\mathbf{T\Delta R}'\mathbf{T}^{-1}\right) = \mathbf{0} \tag{24}$$

Putting the quadrants together

The above calculation of $\frac{\partial \mathbf{a}}{\partial \mathbf{b}}$ is then used to finally calculate

$$\frac{\partial \mathbf{z}}{\partial \mathbf{b}} = \frac{\partial \mathbf{z}}{\partial \mathbf{a}}\frac{\partial \mathbf{a}}{\partial \mathbf{b}} =$$

$$\begin{pmatrix} 1 & 0 & -\frac{\partial u_p}{\partial w_h} & v_m \frac{\partial u_p}{\partial w_h} & -u_m \frac{\partial u_p}{\partial w_h} & v_m \\ 0 & 1 & -\frac{\partial v_p}{\partial w_h} & v_m \frac{\partial v_p}{\partial w_h} & -u_m \frac{\partial v_p}{\partial w_h} & -u_m \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} & \frac{\partial u}{\partial a} & \frac{\partial u}{\partial b} & \frac{\partial u}{\partial c} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} & \frac{\partial v}{\partial a} & \frac{\partial v}{\partial b} & \frac{\partial v}{\partial c} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} & \frac{\partial w}{\partial a} & \frac{\partial w}{\partial b} & \frac{\partial w}{\partial c} \\ \frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial y} & \frac{\partial \alpha}{\partial z} & \frac{\partial \alpha}{\partial a} & \frac{\partial \alpha}{\partial b} & \frac{\partial \alpha}{\partial c} \\ \frac{\partial \beta}{\partial x} & \frac{\partial \beta}{\partial y} & \frac{\partial \beta}{\partial z} & \frac{\partial \beta}{\partial a} & \frac{\partial \beta}{\partial b} & \frac{\partial \beta}{\partial c} \\ \frac{\partial \gamma}{\partial x} & \frac{\partial \gamma}{\partial y} & \frac{\partial \gamma}{\partial z} & \frac{\partial \gamma}{\partial a} & \frac{\partial \gamma}{\partial b} & \frac{\partial \gamma}{\partial c} \end{pmatrix}$$

where for a strip sensor we only care about the top row of $\frac{\partial z}{\partial a}$ and thus

$$\frac{\partial \mathbf{z}}{\partial \mathbf{b}} = \begin{pmatrix} \frac{\partial \mathbf{z}}{\partial x} & \frac{\partial \mathbf{z}}{\partial y} & \frac{\partial \mathbf{z}}{\partial \mathbf{z}} & \frac{\partial \mathbf{z}}{\partial a} & \frac{\partial \mathbf{z}}{\partial b} & \frac{\partial \mathbf{z}}{\partial c} \end{pmatrix} =$$

$$
\begin{pmatrix} 1 & 0 & -\frac{\partial u_p}{\partial w_h} & v_m \frac{\partial u_p}{\partial w_h} & -u_m \frac{\partial u_p}{\partial w_h} & v_m \end{pmatrix}
\begin{pmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} & \frac{\partial u}{\partial a} & \frac{\partial u}{\partial b} & \frac{\partial u}{\partial c} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} & \frac{\partial v}{\partial a} & \frac{\partial v}{\partial b} & \frac{\partial v}{\partial c} \\
\frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} & \frac{\partial w}{\partial a} & \frac{\partial w}{\partial b} & \frac{\partial w}{\partial c} \\
\frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial y} & \frac{\partial \alpha}{\partial z} & \frac{\partial \alpha}{\partial a} & \frac{\partial \alpha}{\partial b} & \frac{\partial \alpha}{\partial c} \\
\frac{\partial \beta}{\partial x} & \frac{\partial \beta}{\partial y} & \frac{\partial \beta}{\partial z} & \frac{\partial \beta}{\partial a} & \frac{\partial \beta}{\partial b} & \frac{\partial \beta}{\partial c} \\
\frac{\partial \gamma}{\partial x} & \frac{\partial \gamma}{\partial y} & \frac{\partial \gamma}{\partial z} & \frac{\partial \gamma}{\partial a} & \frac{\partial \gamma}{\partial b} & \frac{\partial \gamma}{\partial c}
\end{pmatrix}
$$

Normally one calculate the derivatives in the global frame and transforms them into the local frame. In this case the previous chain rule would read

$$
\frac{\partial \mathbf{z}}{\partial \mathbf{a}} = \frac{\partial \mathbf{z}}{\partial \mathbf{b}} \frac{\partial \mathbf{b}}{\partial \mathbf{a}} = \frac{\partial \mathbf{z}}{\partial \mathbf{b}} \left(\frac{\partial \mathbf{a}}{\partial \mathbf{b}} \right)^{-1}
$$

Since we are interested in only the measurement direction u in $\frac{\partial \mathbf{z}}{\partial \mathbf{a}}$ we can write down the terms that will enter into the computation. The first row in $\frac{\partial \mathbf{z}}{\partial \mathbf{a}}$ is, with $\mathbf{z}_x$ is the residual in the x direction,

$$
\begin{pmatrix} 1 & 0 & -\frac{\partial u_p}{\partial w_h} & v_m \frac{\partial u_p}{\partial w_h} & -u_m \frac{\partial u_p}{\partial w_h} & v_m \end{pmatrix} =
$$

$$
\begin{pmatrix} \frac{\partial \mathbf{z}_x}{\partial x} & \frac{\partial \mathbf{z}_x}{\partial y} & \frac{\partial \mathbf{z}_x}{\partial z} & \frac{\partial \mathbf{z}_x}{\partial a} & \frac{\partial \mathbf{z}_x}{\partial b} & \frac{\partial \mathbf{z}_x}{\partial c} \end{pmatrix}
\begin{pmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} & \frac{\partial x}{\partial \alpha} & \frac{\partial x}{\partial \beta} & \frac{\partial x}{\partial \gamma} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} & \frac{\partial y}{\partial \alpha} & \frac{\partial y}{\partial \beta} & \frac{\partial y}{\partial \gamma} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} & \frac{\partial z}{\partial \alpha} & \frac{\partial z}{\partial \beta} & \frac{\partial z}{\partial \gamma} \\
\frac{\partial a}{\partial u} & \frac{\partial a}{\partial v} & \frac{\partial a}{\partial w} & \frac{\partial a}{\partial \alpha} & \frac{\partial a}{\partial \beta} & \frac{\partial a}{\partial \gamma} \\
\frac{\partial b}{\partial u} & \frac{\partial b}{\partial v} & \frac{\partial b}{\partial w} & \frac{\partial b}{\partial \alpha} & \frac{\partial b}{\partial \beta} & \frac{\partial b}{\partial \gamma} \\
\frac{\partial c}{\partial u} & \frac{\partial c}{\partial v} & \frac{\partial c}{\partial w} & \frac{\partial c}{\partial \alpha} & \frac{\partial c}{\partial \beta} & \frac{\partial c}{\partial \gamma}
\end{pmatrix}
$$

To summarize, we have shown in this section how to calculate the derivatives of the residual $\mathbf{z}$ w.r.t. to the six alignment parameters $\mathbf{a}$. It is calculated from the derivatives of the residual in the global frame $\frac{\partial \mathbf{z}}{\partial \mathbf{b}}$ and transformed using the Jacobian $\frac{\partial \mathbf{b}}{\partial \mathbf{a}}$ into the local frame. The Jacobian was derived in this section and the derivatives in the global frame are calculated in Sec. 5.2.

5.2 Global Derivatives in the Global Frame

In this section we calculate the global derivatives in the global frame $\frac{\partial \mathbf{z}}{\partial \mathbf{b}}$ that is used to compute the global derivatives in the local frame, $\frac{\partial \mathbf{z}}{\partial \mathbf{a}}$ which is the input to the minimization. Following the same logic as in Sec. 4 we will express the residual $\mathbf{z}^{gl}$, with superscript gl denoting that this is in the global frame now, by

$$
\mathbf{z}^{gl} = \mathbf{q}_a^{gl} - \mathbf{q}_p^{gl} = \begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix} - \begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix}
$$

where $\mathbf{q}_a^{gl}$ is the alignment corrected hit,

$$
\mathbf{q}_a^{gl} = \mathbf{\Delta R}' \mathbf{q}_h^{gl} + \mathbf{\Delta q^{gl}}, \tag{25}
$$

$\mathbf{q}_p^{gl}$ is the predicted point of interaction with the sensor. We can write $\frac{\partial \mathbf{z}^{gl}}{\partial \mathbf{b}}$ as,

$$\frac{\partial \mathbf{z}^{gl}}{\partial \mathbf{b}} = \frac{\partial \mathbf{q}_a^{gl}}{\partial \mathbf{b}} - \frac{\partial \mathbf{q}_p^{gl}}{\partial \mathbf{b}} \tag{26}$$

where the first term was already derived for the local frame in Sec. 4 ($\frac{\partial \mathbf{q}_a}{\partial \mathbf{a}}$). This derivative is the same in the global frame, evaluated at the measured position (x_m, y_m, z_m),

$$\frac{\partial \mathbf{q}_a^{gl}}{\partial \mathbf{b}} = \begin{pmatrix} \mathbf{1} & \frac{\partial \mathbf{\Delta R}'}{\partial a}\mathbf{q}_h^{gl} & \frac{\partial \mathbf{\Delta R}'}{\partial b}\mathbf{q}_h^{gl} & \frac{\partial \mathbf{\Delta R}'}{\partial c}\mathbf{q}_h^{gl} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & -z_m & y_m \\ 0 & 1 & 0 & z_m & 0 & -x_m \\ 0 & 0 & 1 & -y_m & x_m & 0 \end{pmatrix}$$

The second term in Eq. 26 is the derivative of the predicted position $\mathbf{q}_p^{gl} = (x_p, y_p, z_p)$ of the track model w.r.t. to the alignment parameters $\mathbf{b}$. The track model prediction at x_i for hit i is given by $\mathbf{q}_p^{gl} = f(x_i, \tau)$ where τ are the track parameters $\tau = (d_0, \phi_0, R, z_0, \mathit{slope})$. Starting from Eq. 6 and 7, a generic position (x, y, z) on a helix can be written,

$$\begin{aligned} x &= x_c - \mathrm{R}\sin\phi \\ y &= y_c + \mathrm{R} \times \cos\phi \\ z &= z_0 - \mathrm{R} \times \mathrm{slope} \times \Delta\phi. \end{aligned} \tag{27}$$

where,

$$\phi = \operatorname{atan2}\left(\sin\phi_0 - \frac{x - x_0}{\mathrm{R}}, \cos\phi_0 + \frac{y - y_0}{\mathrm{R}}\right). \tag{28}$$

A translation in x is calculated by expressing the track model equation as a function of x. Using Eq. 28 and substituting,

$$\begin{aligned} y(x) &= -(R - d_0)\cos\phi_0 + \operatorname{sign}(\mathrm{R})\sqrt{R^2 - (x - (R - d_0)\sin\phi_0)^2} \\ y_0 &= d_0\cos\phi_0 \\ x_0 &= -d_0\sin\phi_0 \end{aligned} \tag{29}$$

we can calculate $\frac{\partial \mathbf{q}_p^{gl}}{\partial x}$.

$$\begin{aligned} \frac{\partial x_p}{\partial x} = \frac{\partial f(x,\tau)}{\partial x} &= 1 \\ \frac{\partial y_p}{\partial x} = \frac{\partial y(x)}{\partial x} &= -R\sin\phi\frac{\partial \phi}{\partial x} \\ \frac{\partial z_p}{\partial x} = \frac{\partial z(x)}{\partial x} &= -\mathrm{R} \times \mathrm{slope}\frac{\partial \phi}{\partial x}, \end{aligned}$$

with $\frac{\partial \phi}{\partial x}$ given by,

$$\frac{\partial \phi}{\partial x} = \frac{-R^2 \operatorname{sign}(\mathrm{R})}{\sqrt{R^2 - (x + (d_0 - R)s_0)^2}\left(-(x + (d_0 - R)s_0)^2 + \operatorname{sign}(\mathrm{R})^2\left(-R^2 + x^2 + 2(d_0 - R)xs_0 + (d_0 - R)^2\sin^2\phi_0\right)\right)} \tag{30}$$

and $s_0 = \sin\phi_0$.

Similarly, a **translation in** y can be calculated using Eq. 28 and substituting,

$$\begin{aligned} x(y) &= (R-d_0)\sin\phi_0 + \mathrm{sign}\,(\mathrm{R})\sqrt{R^2 - (y(R-d_0)\cos\phi_0)^2} \\ y_0 &= d_0\cos\phi_0 \\ x_0 &= -d_0\sin\phi_0 \end{aligned} \tag{31}$$

$$\tag{32}$$

we can calculate $\frac{\partial \mathbf{q}_p^{gl}}{\partial y}$.

$$\begin{aligned} \frac{\partial x_p}{\partial y} = \frac{\partial x(y)}{\partial y} &= \mathrm{R}\cos\phi\frac{\partial\phi}{\partial y} \\ \frac{\partial y_p}{\partial y} = \frac{\partial f\,(y,\tau)}{\partial y} &= 1 \\ \frac{\partial z_p}{\partial y} = \frac{\partial z(y)}{\partial y} &= -\mathrm{R}\times \mathrm{slope}\frac{\partial\phi}{\partial y}, \end{aligned} \tag{33}$$

with $\frac{\partial\phi}{\partial y}$ given by,

$$\frac{\partial\phi}{\partial y} = \frac{R^2\mathrm{sign}\,(\mathrm{R})}{\sqrt{R^2-(y+(-d_0+R)c_0)^2}\left(-(y+(-d_0+R)c_0)^2+(-R^2+y^2-2(d_0-R)yc_0+(d_0-R)^2c_0^2)\mathrm{sign}\,(\mathrm{R})^2\right)} \tag{34}$$

and $c_0 = \cos\phi_0$.

A **translation in** z is also given in a similar way,

$$\begin{aligned} \frac{\partial x_p}{\partial z} = \frac{\partial x(z)}{\partial z} &= \mathrm{sign}\,(\mathrm{R})\times\mathrm{R}\cos\phi\frac{\partial\phi}{\partial z} \\ \frac{\partial y_p}{\partial z} = \frac{\partial y(z)}{\partial z} &= -\mathrm{sign}\,(\mathrm{R})\times\mathrm{R}\times\sin\phi\frac{\partial\phi}{\partial z} \\ \frac{\partial z_p}{\partial z} = \frac{\partial f\,(z,\tau)}{\partial z} &= 1, \end{aligned} \tag{35}$$

with $\frac{\partial\phi}{\partial z}$ given by

$$\frac{\partial\phi}{\partial z} = \frac{\partial}{\partial z}\left(-\frac{z-z_0}{R\times \mathrm{slope}}+\phi_0\right) = -\frac{1}{R\times\mathrm{slope}} \tag{36}$$

$$\tag{37}$$

Rotations

We assume that all rotation angles are around the center of the sensor and small and use the small angle limit for rotations $\mathbf{k} = (\alpha, \beta, \gamma)$ corresponding to rotations around the three axis (x, y, z). The derivatives $\frac{\partial f_{x_i}}{\partial \mathbf{k}}$ are given by,

$$\frac{\partial f_{x_i}}{\partial \mathbf{k}} = \begin{pmatrix} \frac{\partial f_x}{\partial \alpha} & \frac{\partial f_y}{\partial \alpha} & \frac{\partial f_z}{\partial \alpha} \\ \frac{\partial f_x}{\partial \beta} & \frac{\partial f_y}{\partial \beta} & \frac{\partial f_z}{\partial \beta} \\ \frac{\partial f_x}{\partial \gamma} & \frac{\partial f_y}{\partial \gamma} & \frac{\partial f_z}{\partial \gamma} \end{pmatrix}$$

$$\frac{\partial f_{x_i}}{\partial \mathbf{k}} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

6 Local Derivatives

This section describes the calculation of how a small change in the track parameters impacts the predicted hit position $\mathbf{q}_p$ i.e. the derivatives of the track model $f(\mathbf{q})$ w.r.t. the track model parameters τ. Note that we are interested in the derivatives in the local sensor plane as this is where the residual is computed. Thus we want to calculate the partial derivatives $\frac{\partial \mathbf{z}}{\partial vec\tau}$ for each hit i where $\mathbf{z}$ is there residual in the (u, v, w) coordinates. Since the transformation from the global to local frame do not depend on the track parameters we can write

$$\frac{\partial \mathbf{z}}{\partial \tau} = \frac{\partial}{\partial \tau}(\mathbf{y} - f(x, \tau)) = -\frac{\partial f(x, \tau)}{\partial \tau} = -\mathbf{T}\frac{\partial f(x, \tau)}{\partial \tau} \tag{38}$$

where the measured hit position in the x, y, z coordinates is $\mathbf{y}$ and the function is evaluated at a point x along the x-axis. $\mathbf{T}$ is the rotation matrix from the global to local frame.

Express ϕ as a function of the track parameters τ and the interaction point along the beam line x, using Eq. 6,

$$\begin{aligned} \sin\phi &= \frac{1}{R}(x_c - x) = \frac{1}{R}((R - d_0)\sin\phi_0 - x) \\ \phi &= \arcsin\left(\frac{1}{R}((R - d_0)\sin\phi_0 - x)\right). \end{aligned} \tag{39}$$

Then using,

$$\begin{aligned} x &= x_c - R\sin\phi = (R - d_0)\sin\phi_0 - R\sin\phi \\ y &= y_c + R\cos\phi = -(R - d_0)\cos\phi_0 + R\cos\phi \\ z &= z_0 + s \times \text{slope} = z_0 - R \times \text{slope}(\phi - \phi_0), \end{aligned} \tag{40}$$

we can calculate the local derivatives $\frac{\partial f(\mathbf{x}, \tau)}{\partial \tau}$ where $\mathbf{x} = (x, y, z)$.

$\frac{\partial f(x,\tau)}{\partial \mathbf{q}}$:

$$\begin{aligned}
\frac{\partial f\left(x,\tau\right)}{\partial d_0} &= -\sin\phi_0 - R\cos\phi\frac{\partial\phi}{\partial d_0} \\
\frac{\partial f\left(x,\tau\right)}{\partial z_0} &= -R\cos\phi\frac{\partial\phi}{\partial z_0} \\
\frac{\partial f\left(x,\tau\right)}{\partial \phi_0} &= \left(R-d_0\right)\cos\phi_0 - R\cos\phi\frac{\partial\phi}{\partial phi_0} \\
\frac{\partial f\left(x,\tau\right)}{\partial R} &= \sin\phi_0 - R\cos\phi\frac{\partial\phi}{\partial R} - \sin\phi \\
\frac{\partial f\left(x,\tau\right)}{\partial R} &= \sin\phi_0 - R\cos\phi\frac{\partial\phi}{\partial R} \\
\frac{\partial f\left(x,\tau\right)}{\partial \mathrm{slope}} &= R\cos\phi\frac{\partial\phi}{\partial \mathrm{slope}},
\end{aligned} \tag{41}$$

$\frac{\partial f(y,\tau)}{\partial \mathbf{q}}$:

$$\begin{aligned}
\frac{\partial f\left(y,\tau\right)}{\partial d_0} &= \cos\phi_0 - R\sin\phi\frac{\partial\phi}{\partial d_0} \\
\frac{\partial f\left(y,\tau\right)}{\partial z_0} &= -R\sin\phi\frac{\partial\phi}{\partial z_0} \\
\frac{\partial f\left(y,\tau\right)}{\partial \phi_0} &= \left(R-d_0\right)\sin\phi_0 - R\sin\phi\frac{\partial\phi}{\partial \phi_0} \\
\frac{\partial f\left(y,\tau\right)}{\partial R} &= -\cos\phi_0 - R\sin\phi\frac{\partial\phi}{\partial R} + \cos\phi \\
\frac{\partial f\left(y,\tau\right)}{\partial \mathrm{slope}} &= -R\sin\phi\frac{\partial\phi}{\partial \mathrm{slope}}
\end{aligned} \tag{42}$$

$\frac{\partial f(z,\tau)}{\partial \mathbf{q}}$:

$$\begin{aligned}
\frac{\partial f\left(z,\tau\right)}{\partial d_0} &= -R\times\mathrm{slope}\frac{\partial\phi}{\partial d_0} \\
\frac{\partial f\left(z,\tau\right)}{\partial z_0} &= 1 \\
\frac{\partial f\left(z,\tau\right)}{\partial \phi_0} &= -R\times\mathrm{slope}\left(\frac{\partial\phi}{\partial phi_0} - 1\right) \\
\frac{\partial f\left(z,\tau\right)}{\partial R} &= -\mathrm{slope}\times\left(\phi - \phi_0 + R\frac{\partial\phi}{\partial R}\right) \\
\frac{\partial f\left(z,\tau\right)}{\partial \mathrm{slope}} &= s = -R(\phi-\phi_0)
\end{aligned} \tag{43}$$

where $\frac{\partial\phi}{\partial \mathbf{q}}$ above is calculated using Eq. 39,

$$
\begin{aligned}
\frac{\partial \phi}{\partial d_0} = &= \frac{-\sin\phi_0}{R\sqrt{1-\frac{(x+(d_0-R)\sin\phi_0)^2}{R^2}}} \\
\frac{\partial \phi}{\partial \phi_0} &= \frac{-(d_0-R)\cos\phi_0}{R\sqrt{1-\frac{(x+(d_0-R)\sin\phi_0)^2}{R^2}}} \\
\frac{\partial \phi}{\partial R} &= \frac{x+d_0\sin\phi_0}{R^2\sqrt{1-\frac{(x+(d_0-R)\sin\phi_0)^2}{R^2}}} \\
\frac{\partial \phi}{\partial slope} &= 0 \\
\frac{\partial \phi}{\partial z_0} &= 0
\end{aligned}
\tag{44}
$$

Appendix

Practical Formulas

$$y = y_c + \operatorname{sign}(()\,\mathrm{R}) \times \sqrt{\left(\mathrm{R}^2 - (x - x_c)^2\right)} \tag{45}$$

$$x = x_c + \operatorname{sign}(()\,\mathrm{R}) \times \sqrt{\left(\mathrm{R}^2 - (y - y_c)^2\right)} \tag{46}$$

$$\Delta\phi = \mathrm{atan2}(x - x_c, y - y_c) - \mathrm{atan2}(x_0 - x_c, y_0 - y_c) \tag{47}$$

Arc length, s, can be written as:

$$\mathrm{s} = -\Delta\phi \times \mathrm{R}. \tag{48}$$

$$\frac{\partial y_c}{\partial d_0} = \frac{\operatorname{sign}(\mathrm{R})^2 \sin(\phi_0)(x - x_c)}{\sqrt{\mathrm{R}^2 - \operatorname{sign}(\mathrm{R})^2 \left(\mathrm{R}^2 - (x - x_c)^2\right)}} \tag{49}$$

Alternative calcualtion of $\frac{\partial\phi}{\partial\mathbf{q}}$

We can express ϕ as a function of the track parameter τ using Eq. 5 together with $y = y_c + R \times \cos\phi$ and $x_c = (R - d_0)\sin\phi_0$

$$\begin{aligned}
(\cos\phi)^2 &= \frac{1}{R^2}\left(R^2 - (x - x_c)^2\right) \\
(\cos\phi)^2 &= \frac{1}{R^2}\left(R^2 - (x - (R - d_0)\sin\phi_0)^2\right) \\
\phi &= \cos^{-1}\left(\sqrt{1 - \left(\frac{x - (R - d_0)\sin\phi_0}{R}\right)^2}\right) \qquad (50)
\end{aligned}$$

and the local track derivatives are becomes,

$$\begin{aligned}
\frac{\partial \phi}{\partial d_0} &= \frac{\sin\phi_0 \left(x + (d_0 - R)\sin\phi_0\right)}{R^2 \sqrt{\frac{(x+(d_0-R)\sin\phi_0)^2 \left(1 - \frac{(x+(d_0-R)\sin\phi_0)^2}{R^2}\right)}{R^2}}} \\
\frac{\partial \phi}{\partial \phi_0} &= \frac{(d_0 - R)\cos\phi_0 \left(x + (d_0 - R)\sin\phi_0\right)}{R^2 \sqrt{\frac{(x+(d_0-R)\sin\phi_0)^2 \left(1 - \frac{(x+(d_0-R)\sin\phi_0)^2}{R^2}\right)}{R^2}}} \\
\frac{\partial \phi}{\partial R} &= -\left(\frac{\left(x + d_0\sin\phi_0\right)\left(x + (d_0 - R)\sin\phi_0\right)}{R^3 \sqrt{\frac{(x+(d_0-R)\sin\phi_0)^2 \left(1 - \frac{(x+(d_0-R)\sin\phi_0)^2}{R^2}\right)}{R^2}}}\right) \\
\frac{\partial \phi}{\partial slope} &= 0 \\
\frac{\partial \phi}{\partial z0} &= 0
\end{aligned} \tag{51}$$

References

[1] Markus Stoye, *Calibration and Alignment of the CMS Silicon Tracking Detector*, Dissertation, 2007.