Helix Derivatives

P. Hansson Adrian[1], M. Graham[1], S. Uemura[1]

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1 Introduction

This note describes the implementation of the alignment procedure for the HPS SVT. Most of this follows what is described in more depth in Ref. [1] and in the MillepedeII manual (ref).

Updates needed:

- Need to update the mix of nomenclature of q for track parameters and hit positions.
- Clean-up formulas that are not relevant
- improve organization in the descriptions

2 Track Based Alignment

Each hit measurement, y_i, is assumed to be described by a (non-)linear track model $f(x_i, \mathbf{q})$ which depends on a small number of parameter $\mathbf{q}$. In the case of a particle track in a homogenous magnetic field these are the five track parameters describing a helical track,

$$y_i = f\left(x_i, \mathbf{q}\right) + \epsilon_i. \tag{1}$$

The coordinate x_i is the coordinate of the hit y_i and ϵ is the uncertainty on y_i. The local-fit function f is linearized, if needed, by expressing it as a linear function of the local parameter corrections $\mathbf{\Delta q}$ at some reference value $\mathbf{q}_k$,

$$f\left(x_i, \mathbf{q}_k + \mathbf{\Delta q}\right) = f\left(x_i, \mathbf{q}_k\right) + \frac{\partial f}{\partial \mathrm{q}_1}\Delta \mathrm{q}_1 + \frac{\partial f}{\partial \mathrm{q}_2}\Delta \mathrm{q}_2 + ... \tag{2}$$

The local fit relies on minimizing the measured residual z_i for each hit,

$$z_i = y_i - f\left(x_i, \mathbf{q}_k\right). \tag{3}$$

By solving for $\mathbf{\Delta q}$ for each iteration k and updating with $q_{k+1} = q_k + \mathbf{\Delta q}$ convergence and optimal $\mathbf{\Delta q}$ can be obtained.

This so-called local fit is performed assuming the relative positions of the sensors providing the hit measurements. Each sensor has 6 degrees of freedom, 3 translation and 3 rotations, given by a vector $\mathbf{q}$ and the goal of track based alignment is to determine the corrections $\mathbf{\Delta q}$ to the nominal values. These global parameters can be incorporated in the residual by

$$z_i = y_i - f\left(x_i, \mathbf{q}_k\right) = \sum_j^{\nu} \frac{\partial f}{\partial \mathrm{q}_j} \Delta \mathrm{q}_1 + \sum_l \frac{\partial f}{\partial \mathrm{p}_l} \Delta \mathrm{p}_l. \tag{4}$$

where the best local (track) parameter correction $\Delta\mathbf{q}$ can be used in the global fit.

The alignment algorithm last step is to minimize these residuals w.r.t. the global parameters. This involves solving a system of linear equations. In our first strategy we make use of the MillepedeII software program. The input to MillepedeII are the residuals, local (track) derivatives and the global derivatives. Practically the program is split into the MILLE programs which gathers and prepares the input to the PEDE program which carries out the actual minimization.

Section describes the alignment parameterization and in particular how the global derivatives look like in the local sensor frame. The calculation of the global and local derivatives are carried out in Sec. 4 and Sec. 5, respectively.

3 Alignment Parameterization

`Note the change of nomenclature for q compared to previous section.`

A hit measurement vector $\mathbf{q}$ can be represented as,

$$\mathbf{r} = \mathbf{R}^{\mathbf{T}}\mathbf{q} + \mathbf{r_0}, \tag{5}$$

where $\mathbf{R}$ is a rotation matrix and $\mathbf{r_0}$ is the position of the sensor. The task of the alignment procedure is to provide correction to position and rotation of the sensor, $\mathbf{q_0}$ and $\mathbf{R}$, respectively,

$$\mathbf{r} = \mathbf{R}^{\mathbf{T}}\mathbf{\Delta R}\left(\mathbf{q} + \Delta\mathbf{q}\right) + \mathbf{r_0}. \tag{6}$$

The alignment parameters are the components of $\Delta\mathbf{q}$ and $\Delta\mathbf{R}$ and are often expressed in the local sensor coordinates as they are related to the individual sensor. The measured hit position components are $\mathbf{q} = (u, v, w)$, where the precisely measured coordinate on the sensor is separated from the less well-known coordinate. In a strip sensor u is typically the precisely measured coordinate and v is the un-measured coordinate. w is the direction normal to the sensor plane. For alignment typically the Δw is ignored as all hits happen at the sensor plane as will be evident later. The rotation correction matrix $\Delta\mathbf{R}$ are reduced to three angles around the u-, v- and w-axis and are denoted as α, β and γ (around the center of the sensors and thus do not induce a translation). Each sensor thus has 6 alignment parameters and following the notation in Ref. **??** it can be represented by a vector $\mathbf{a}$,

$$\mathbf{a} = \begin{pmatrix} \Delta u \\ \Delta v \\ \Delta w \\ \alpha \\ \beta \\ \gamma \end{pmatrix}$$

In order to solve the minimization problem we need to calculate the derivatives of the residuals w.r.t. the local and global parameters. The residual $\mathbf{z}$ is,

$$\mathbf{z} = \mathbf{q}_a - \mathbf{q}_p = \begin{pmatrix} u_m \\ v_m \\ w_m \end{pmatrix} - \begin{pmatrix} u_p \\ v_p \\ w_p \end{pmatrix}$$

where $\mathbf{q}_a$ is the alignment corrected hit,

$$\mathbf{q_a} = \mathbf{\Delta R}\mathbf{q}_h + \Delta\mathbf{q}, \tag{7}$$

where $\mathbf{q_h}$ is the measured hit position. For the minimization of the square of residuals the global derivatives,

$$\frac{\partial \mathbf{z}}{\partial \mathbf{a}} = \frac{\partial \mathbf{q}_a}{\partial \mathbf{a}} - \frac{\partial \mathbf{q}_p}{\partial \mathbf{a}}. \tag{8}$$

needs to be calculated. The partial derivatives w.r.t. to a translation Δu is,

$$\frac{\partial \mathbf{q}_a}{\partial \Delta u} = \frac{\partial}{\partial \mathbf{\Delta u}} (\mathbf{\Delta R}\mathbf{q}_h + \Delta\mathbf{q}) = \frac{\partial}{\partial \Delta u} \begin{pmatrix} \Delta u \\ \Delta v \\ \Delta w \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and similarly for the other translations in v and w,

$$\frac{\partial \mathbf{q}_a}{\partial \Delta v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \frac{\partial \mathbf{q}_a}{\partial \Delta w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The rotation matrix is given by

$$\mathbf{\Delta R} = \mathbf{R}_\gamma \times \mathbf{R}_\beta \times \mathbf{R}_\alpha$$

where,

$$\mathbf{R}_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \mathbf{R}_\beta = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}, \mathbf{R}_\gamma = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

All angles are assumed to be small and after linearization the derivatives become,

$$\frac{\partial \mathbf{\Delta R}}{\partial \alpha} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \frac{\partial \mathbf{\Delta R}}{\partial \beta} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \frac{\partial \mathbf{\Delta R}}{\partial \gamma} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

These can be written more compactly, and evaluated at the measured position $\mathbf{q}_h = \mathbf{q}_m = (u_m, v_m, w_m)$ as,

$$\frac{\partial \mathbf{q}_a}{\partial \mathbf{a}} = \begin{pmatrix} \mathbf{1} & \frac{\partial \Delta R}{\partial \alpha}\mathbf{q}_h & \frac{\partial \Delta R}{\partial \beta}\mathbf{q}_h & \frac{\partial \Delta R}{\partial \gamma}\mathbf{q}_h \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & -w_m & v_m \\ 0 & 1 & 0 & w_m & 0 & -u_m \\ 0 & 0 & 1 & -v_m & u_m & 0 \end{pmatrix}$$

As mentioned before $w_m = 0$ by construction (the hit $\mathbf{q}_h$ is on the sensor surface) and the un-measured direction v can be ignored by it kept here for consistency,

$$\frac{\partial \mathbf{q}_a}{\partial \mathbf{a}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & v_m \\ 0 & 1 & 0 & 0 & 0 & -u_m \\ 0 & 0 & 1 & -v_m & u_m & 0 \end{pmatrix}$$

The global derivative $\frac{\partial \mathbf{q}_p}{\partial \mathbf{a}}$ measures the effect of the predicted track position on the surface of the sensor. Note that a shift of the sensor in the u, v plane is equivalent to a shift of the measured hit position $\mathbf{q}_h$ and the only direction where the track propagation needs to be taken into account is the w direction. Using this we can write (this is unclear!!),

$$\frac{\partial \mathbf{q}_p}{\partial \mathbf{a}} = \frac{\partial \mathbf{q}_p}{\partial \mathbf{q}_a}\frac{\partial \mathbf{q}_a}{\partial \mathbf{a}} \tag{9}$$

where $\frac{\partial \mathbf{q}_p}{\partial \mathbf{q}_a}$ is

$$\frac{\partial \mathbf{q}_p}{\partial u_a} = 0, \frac{\partial \mathbf{q}_p}{\partial v_a} = 0, \frac{\partial \mathbf{q}_p}{\partial w_a} = \frac{\partial \mathbf{q}_p}{\partial w_h} = \begin{pmatrix} \frac{\partial u_p}{\partial w_h} \\ \frac{\partial v_p}{\partial w_h} \\ \frac{\partial w_p}{\partial w_h} \end{pmatrix}$$

since a shift in the u, v plane is equivalent to a shift in the hit position. Using this and $\frac{\partial \mathbf{q}_a}{\partial \mathbf{a}}$ calculated earlier,

$$\frac{\partial \mathbf{q}_p}{\partial \mathbf{q}_a}\frac{\partial \mathbf{q}_a}{\partial \mathbf{a}} = \begin{pmatrix} 0 & 0 & \frac{\partial u_p}{\partial w_h} \\ 0 & 0 & \frac{\partial v_p}{\partial w_h} \\ 0 & 0 & \frac{\partial w_p}{\partial w_h} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & v_m \\ 0 & 1 & 0 & 0 & 0 & -u_m \\ 0 & 0 & 1 & -v_m & u_m & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & \frac{\partial u_p}{\partial w_h} & -v_m \frac{\partial u_p}{\partial w_h} & u_m \frac{\partial u_p}{\partial w_h} & 0 \\ 0 & 0 & \frac{\partial v_p}{\partial w_h} & -v_m \frac{\partial v_p}{\partial w_h} & u_m \frac{\partial v_p}{\partial w_h} & 0 \\ 0 & 0 & \frac{\partial w_p}{\partial w_h} & -v_m \frac{\partial w_p}{\partial w_h} & u_m \frac{\partial w_p}{\partial w_h} & 0 \end{pmatrix}$$

Now we have all the ingredients to calculate Eq. 8:

$$\frac{\partial \mathbf{z}}{\partial \mathbf{a}} = \frac{\partial \mathbf{q}_a}{\partial \mathbf{a}} - \frac{\partial \mathbf{q}_p}{\partial \mathbf{a}} = \frac{\partial \mathbf{q}_a}{\partial \mathbf{a}} - \frac{\partial \mathbf{q}_p}{\partial \mathbf{q}_a}\frac{\partial \mathbf{q}_a}{\partial \mathbf{a}} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & v_m \\ 0 & 1 & 0 & 0 & 0 & -u_m \\ 0 & 0 & 1 & -v_m & u_m & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \frac{\partial u_p}{\partial w_h} & -v_m \frac{\partial u_p}{\partial w_h} & u_m \frac{\partial u_p}{\partial w_h} & 0 \\ 0 & 0 & \frac{\partial v_p}{\partial w_h} & -v_m \frac{\partial v_p}{\partial w_h} & u_m \frac{\partial v_p}{\partial w_h} & 0 \\ 0 & 0 & \frac{\partial w_p}{\partial w_h} & -v_m \frac{\partial w_p}{\partial w_h} & u_m \frac{\partial w_p}{\partial w_h} & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & -\frac{\partial u_p}{\partial w_h} & v_m \frac{\partial u_p}{\partial w_h} & -u_m \frac{\partial u_p}{\partial w_h} & v_m \\ 0 & 1 & -\frac{\partial v_p}{\partial w_h} & v_m \frac{\partial v_p}{\partial w_h} & -u_m \frac{\partial v_p}{\partial w_h} & -u_m \\ 0 & 0 & 1 - \frac{\partial w_p}{\partial w_h} & v_m \frac{\partial w_p}{\partial w_h} - v_m & u_m - u_m \frac{\partial w_p}{\partial w_h} & 0 \end{pmatrix}$$

Since $w_m = 0$ we can ignore the third component which means that we can write the global residual derivative as,

$$\frac{\partial \mathbf{z}}{\partial \mathbf{a}} = \begin{pmatrix} 1 & 0 & -\frac{\partial u_p}{\partial w_h} & v_m \frac{\partial u_p}{\partial w_h} & -u_m \frac{\partial u_p}{\partial w_h} & v_m \\ 0 & 1 & -\frac{\partial v_p}{\partial w_h} & v_m \frac{\partial v_p}{\partial w_h} & -u_m \frac{\partial v_p}{\partial w_h} & -u_m \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In the strip sensor case only the well-measured direction is important and thus only the first row is important.

`NOTE THAT IN THE REFERENCE I NEED TO GET -1* THE ABOVE??`

The calculation of the global derivatives $\frac{\partial q}{\partial \mathbf{a}}$ is detailed in Sec. 4.

4 Global Derivatives

The parameterization that is layer out in Sec. 3 provide information on the global derivatives that we need to calculate.

$$\frac{\partial \mathbf{z}}{\partial \mathbf{a}} = \begin{pmatrix} 1 & 0 & -\frac{\partial u_p}{\partial w_h} & v_m \frac{\partial u_p}{\partial w_h} & -u_m \frac{\partial u_p}{\partial w_h} & v_m \\ 0 & 1 & -\frac{\partial v_p}{\partial w_h} & v_m \frac{\partial v_p}{\partial w_h} & -u_m \frac{\partial v_p}{\partial w_h} & -u_m \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where only the first row is important for strip sensors. In the global frame, the derivatives involving e.g. u will be affected by translations and rotations around the global coordinate axis x, y, z. Our global derivatives are typically calculated in the global frame since that is where track fitting occurs. The next section shows how to determine the global derivatives in the global frame.

4.1 Global Derivatives in the Local Frame

Later in this section we show how to derive the derivatives of the predicted hit position $\mathbf{x}_p$ in the global frame. This is a natural calculation and is also where our alignment parameters come together to form a common detector geometry. However, previously we showed what global derivatives we need to calculate in the local sensor frame $\frac{\partial \mathbf{z}}{\partial \mathbf{a}}$.

In order to connect these two calculations we need to understand how the change of an alignment parameter in the global frame affect the alignment parameter in the local sensor frame that is used in the Millepede minimization.

Using integration by parts we have

$$\frac{\partial \mathbf{z}}{\partial \mathbf{b}} = \frac{\partial \mathbf{z}}{\partial \mathbf{a}} \frac{\partial \mathbf{a}}{\partial \mathbf{b}}$$

where

$$\mathbf{a} = \begin{pmatrix} \Delta u \\ \Delta v \\ \Delta w \\ \alpha \\ \beta \\ \gamma \end{pmatrix}, \mathbf{b} = \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \\ a \\ b \\ c \end{pmatrix}$$

are the alignment parameters in the local ($\mathbf{a}$) and global ($\mathbf{b}$) frame. The matrix $\frac{\partial \mathbf{a}}{\partial \mathbf{b}}$ is defined as

$$\frac{\partial \mathbf{a}}{\partial \mathbf{b}} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} & \frac{\partial u}{\partial a} & \frac{\partial u}{\partial b} & \frac{\partial u}{\partial c} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} & \frac{\partial v}{\partial a} & \frac{\partial v}{\partial b} & \frac{\partial v}{\partial c} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} & \frac{\partial w}{\partial a} & \frac{\partial w}{\partial b} & \frac{\partial w}{\partial c} \\ \frac{\partial \alpha}{\partial x} & \frac{\partial \alpha}{\partial y} & \frac{\partial \alpha}{\partial z} & \frac{\partial \alpha}{\partial a} & \frac{\partial \alpha}{\partial b} & \frac{\partial \alpha}{\partial c} \\ \frac{\partial \beta}{\partial x} & \frac{\partial \beta}{\partial y} & \frac{\partial \beta}{\partial z} & \frac{\partial \beta}{\partial a} & \frac{\partial \beta}{\partial b} & \frac{\partial \beta}{\partial c} \\ \frac{\partial \gamma}{\partial x} & \frac{\partial \gamma}{\partial y} & \frac{\partial \gamma}{\partial z} & \frac{\partial \gamma}{\partial a} & \frac{\partial \gamma}{\partial b} & \frac{\partial \gamma}{\partial c} \end{pmatrix}$$

where $\Delta x = x$ and $\Delta u = u$ (similarly in the other directions) are equivalent in the derivatives.

To calculate the matrix elements we start by expressing how a small translation $\mathbf{\Delta x}(= (\Delta x, \Delta y, \Delta z))$ and rotation $\Delta R'$ in the global frame affects the local translation $\mathbf{\Delta q} = (\Delta u, \Delta v, \Delta w)$ on a point in the global frame $\mathbf{x} = (x, y, z)$

$$\mathbf{\Delta q} = \mathbf{T}\mathbf{\Delta x} + \mathbf{T}\left(\mathbf{\Delta R}'\mathbf{x} - \mathbf{x}\right) = \mathbf{T}\mathbf{\Delta x} + \mathbf{T}\left(\mathbf{\Delta R}' - \mathbf{I}\right)\mathbf{x} \tag{10}$$

where $\mathbf{T}$ is the rotation matrix from the global to local frame. The rotation matrix $\mathbf{\Delta R}'$ is identical to $\mathbf{\Delta R}$ derived for the local frame earlier, only that the rotation axes are around x, y, z instead of u, v, w.

The other thing we need is to express the local sensor rotations $\mathbf{q} = (\alpha, \beta, \gamma)$ caused by a small rotation $\Delta R' = (a, b, c)$ on a point $\mathbf{x} = (x, y, z)$ around the global axes. Using $x = \mathbf{T}^{-1}\mathbf{q}$,

$$\begin{aligned} \mathbf{\Delta R}'\mathbf{x} &= \mathbf{T}^{-1}\mathbf{\Delta R}\mathbf{q} \\ \mathbf{\Delta R}'\mathbf{x} &= \mathbf{\Delta R}'\mathbf{T}^{-1}\mathbf{q} = \mathbf{T}^{-1}\mathbf{\Delta R}\mathbf{q} \\ \mathbf{T}\mathbf{\Delta R}'\mathbf{T}^{-1}\mathbf{q} &= \mathbf{\Delta R}\mathbf{q} \\ \mathbf{\Delta R} &= \mathbf{T}\mathbf{\Delta R}'\mathbf{T}^{-1} \end{aligned} \tag{11}$$

The calculation of the elements of $\frac{\partial \mathbf{a}}{\partial \mathbf{b}}$ is split up into the 4 3×3 quadrants and discussed below.

Upper left 3×3

Using Eq. 10 we can calculate the upper left 3×3 part of the $\frac{\partial \mathbf{a}}{\partial \mathbf{b}}$

$$\frac{\partial \mathbf{q}}{\partial x} = \frac{\partial}{\partial x}\mathbf{T}\mathbf{\Delta x} = \mathbf{T}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \frac{\partial \mathbf{q}}{\partial y} = \mathbf{T}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \frac{\partial \mathbf{q}}{\partial z} = \mathbf{T}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Lower right 3×3

This part of the $\frac{\partial \mathbf{a}}{\partial \mathbf{b}}$ is calculated from

$$\frac{\partial \mathbf{\Delta R}}{\partial (a,b,c)} = \frac{\partial}{\partial (a,b,c)} \left(\mathbf{T} \mathbf{\Delta R}' \mathbf{T}^{-1} \right)$$

Expressing $\mathbf{\Delta R}'$ as a product of Euler matrices and using $\frac{\partial \Delta R'}{\partial a,b,c}$ derived in the previous section the right hand side is given by,

$$\frac{\partial}{\partial a}(\Delta R') = \frac{\partial}{\partial a}\left(\mathbf{T}\left(R_g(c)R_b(b)R_a(a)\right)\mathbf{T}^{-1}\right) = \mathbf{T}\frac{\partial R_a(a)}{\partial a}\mathbf{T}^{-1} = \mathbf{T}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\mathbf{T}^{-1}$$

$$\frac{\partial}{\partial b}(\Delta R') = \frac{\partial}{\partial b}\left(\mathbf{T}\left(R_g(c)R_b(b)R_a(b)\right)\mathbf{T}^{-1}\right) = \mathbf{T}\frac{\partial R_b(b)}{\partial b}\mathbf{T}^{-1} = \mathbf{T}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\mathbf{T}^{-1}$$

$$\frac{\partial}{\partial c}(\Delta R') = \frac{\partial}{\partial c}\left(\mathbf{T}\left(R_g(c)R_b(b)R_a(b)\right)\mathbf{T}^{-1}\right) = \mathbf{T}\frac{\partial R_g(c)}{\partial c}\mathbf{T}^{-1} = \mathbf{T}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\mathbf{T}^{-1}$$

which can be written out more explicitly by also expressing $\mathbf{T}$ as a product of Euler angles $\mathbf{T} = R_g(k)R_b(j)R_a(i)$ (remember that $\mathbf{T}$ is the rotation matrix that takes a vector in the global frame to the sensor frame and thus i, j, k are the rotation angles),

$$\frac{\partial}{\partial a}(\Delta R') = \mathbf{T}\frac{\partial R_a(a)}{\partial a}\mathbf{T}^{-1} = \begin{pmatrix} 0 & s_j & c_j s_k \\ -s_j & 0 & c_j c_k \\ -c_j s_k & -c_j c_k & 0 \end{pmatrix}$$

$$\frac{\partial}{\partial b}(\Delta R') = \mathbf{T}\frac{\partial R_b(b)}{\partial b}\mathbf{T}^{-1} = \begin{pmatrix} 0 & -c_j s_i & -c_i c_k + s_i s_j s_k \\ c_j s_i & 0 & c_k s_i s_j + c_i s_k \\ c_i c_k - s_i s_j s_k & -c_k s_i s_j - c_i s_k & 0 \end{pmatrix}$$

$$\frac{\partial}{\partial b}(\Delta R') = \mathbf{T}\frac{\partial R_b(b)}{\partial b}\mathbf{T}^{-1} = \begin{pmatrix} 0 & c_i c_j & -c_k s_i - c_i s_j s_k \\ -c_i c_j & 0 & -c_i c_k s_j + s_i s_k \\ c_k s_i + c_i s_j s_k & c_i c_k s_j - s_i s_k & 0 \end{pmatrix}$$

These can be compared to $\mathbf{T}$ for fun,

$$\mathbf{T} = \begin{pmatrix} c_j c_k & c_k s_i s_j + c_i s_k & -c_i c_k s_j + s_i s_k \\ -c_j s_k & c_i c_k - s_i s_j s_k & c_k s_i + c_i s_j s_k \\ s_j & -c_j s_i & c_i c_j \end{pmatrix}$$

The left hand side can be written as

$$\frac{\partial}{\partial a}\mathbf{\Delta R} = \frac{\partial}{\partial a}\left(R_g(\gamma)R_b(\beta)R_a(\alpha)\right)\frac{\partial R_g(\gamma)}{\partial \gamma}\frac{\partial \gamma}{\partial a} + \frac{\partial R_b(\beta)}{\partial \beta}\frac{\partial \beta}{\partial a} + \frac{\partial R_a(\alpha)}{\partial \alpha}\frac{\partial \alpha}{\partial a}$$
$$\frac{\partial}{\partial b}\mathbf{\Delta R} = \frac{\partial}{\partial b}\left(R_g(\gamma)R_b(\beta)R_a(\alpha)\right)\frac{\partial R_g(\gamma)}{\partial \gamma}\frac{\partial \gamma}{\partial b} + \frac{\partial R_b(\beta)}{\partial \beta}\frac{\partial \beta}{\partial b} + \frac{\partial R_a(\alpha)}{\partial \alpha}\frac{\partial \alpha}{\partial b}$$
$$\frac{\partial}{\partial c}\mathbf{\Delta R} = \frac{\partial}{\partial c}\left(R_g(\gamma)R_b(\beta)R_a(\alpha)\right)\frac{\partial R_g(\gamma)}{\partial \gamma}\frac{\partial \gamma}{\partial c} + \frac{\partial R_b(\beta)}{\partial \beta}\frac{\partial \beta}{\partial c} + \frac{\partial R_a(\alpha)}{\partial \alpha}\frac{\partial \alpha}{\partial c} \quad (12)$$

and using the $\frac{\partial \Delta R'}{\partial a,b,c}$ derivatives,

$$\frac{\partial}{\partial a}\mathbf{\Delta R} = \frac{\partial\gamma}{\partial a}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{\partial\beta}{\partial a}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \frac{\partial\alpha}{\partial a}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\frac{\partial}{\partial b}\mathbf{\Delta R} = \frac{\partial\gamma}{\partial b}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{\partial\beta}{\partial b}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \frac{\partial\alpha}{\partial b}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\frac{\partial}{\partial c}\mathbf{\Delta R} = \frac{\partial\gamma}{\partial c}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{\partial\beta}{\partial c}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \frac{\partial\alpha}{\partial c}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

and simplifying,

$$\frac{\partial}{\partial a}\mathbf{\Delta R} = \begin{pmatrix} 0 & \frac{\partial\gamma}{\partial a} & 0 \\ -\frac{\partial\gamma}{\partial a} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\frac{\partial\beta}{\partial a} \\ 0 & 0 & 0 \\ \frac{\partial\beta}{\partial a} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\partial\alpha}{\partial a} \\ 0 & -\frac{\partial\alpha}{\partial a} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial\gamma}{\partial a} & -\frac{\partial\beta}{\partial a} \\ -\frac{\partial\gamma}{\partial a} & 0 & \frac{\partial\alpha}{\partial a} \\ \frac{\partial\beta}{\partial a} & -\frac{\partial\alpha}{\partial a} & 0 \end{pmatrix}$$

$$\frac{\partial}{\partial b}\mathbf{\Delta R} = \begin{pmatrix} 0 & \frac{\partial\gamma}{\partial b} & 0 \\ -\frac{\partial\gamma}{\partial b} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\frac{\partial\beta}{\partial b} \\ 0 & 0 & 0 \\ \frac{\partial\beta}{\partial b} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\partial\alpha}{\partial b} \\ 0 & -\frac{\partial\alpha}{\partial b} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial\gamma}{\partial b} & -\frac{\partial\beta}{\partial b} \\ -\frac{\partial\gamma}{\partial b} & 0 & \frac{\partial\alpha}{\partial b} \\ \frac{\partial\beta}{\partial b} & -\frac{\partial\alpha}{\partial b} & 0 \end{pmatrix}$$

$$\frac{\partial}{\partial c}\mathbf{\Delta R} = \begin{pmatrix} 0 & \frac{\partial\gamma}{\partial c} & 0 \\ -\frac{\partial\gamma}{\partial c} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\frac{\partial\beta}{\partial c} \\ 0 & 0 & 0 \\ \frac{\partial\beta}{\partial c} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\partial\alpha}{\partial c} \\ 0 & -\frac{\partial\alpha}{\partial c} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial\gamma}{\partial c} & -\frac{\partial\beta}{\partial c} \\ -\frac{\partial\gamma}{\partial c} & 0 & \frac{\partial\alpha}{\partial c} \\ \frac{\partial\beta}{\partial c} & -\frac{\partial\alpha}{\partial c} & 0 \end{pmatrix}$$

and finally, equating left- and right-hand side,

$$\frac{\partial}{\partial a}\mathbf{\Delta R} = \begin{pmatrix} 0 & \frac{\partial\gamma}{\partial a} & -\frac{\partial\beta}{\partial a} \\ -\frac{\partial\gamma}{\partial a} & 0 & \frac{\partial\alpha}{\partial a} \\ \frac{\partial\beta}{\partial a} & -\frac{\partial\alpha}{\partial a} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -c_j s_i & -c_i c_k + s_i s_j s_k \\ c_j s_i & 0 & c_k s_i s_j + c_i s_k \\ c_i c_k - s_i s_j s_k & -c_k s_i s_j - c_i s_k & 0 \end{pmatrix}$$

$$\frac{\partial}{\partial b}\mathbf{\Delta R} = \begin{pmatrix} 0 & \frac{\partial\gamma}{\partial b} & -\frac{\partial\beta}{\partial b} \\ -\frac{\partial\gamma}{\partial b} & 0 & \frac{\partial\alpha}{\partial b} \\ \frac{\partial\beta}{\partial b} & -\frac{\partial\alpha}{\partial b} & 0 \end{pmatrix} = \begin{pmatrix} 0 & c_i c_j & -c_k s_i - c_i s_j s_k \\ -c_i c_j & 0 & -c_i c_k s_j + s_i s_k \\ c_k s_i + c_i s_j s_k & c_i c_k s_j - s_i s_k & 0 \end{pmatrix}$$

$$\frac{\partial}{\partial c}\mathbf{\Delta R} = \begin{pmatrix} 0 & \frac{\partial\gamma}{\partial c} & -\frac{\partial\beta}{\partial c} \\ -\frac{\partial\gamma}{\partial c} & 0 & \frac{\partial\alpha}{\partial c} \\ \frac{\partial\beta}{\partial c} & -\frac{\partial\alpha}{\partial c} & 0 \end{pmatrix} = \begin{pmatrix} c_i 0 & c_i c_j & -c_k s_i - c_i s_j s_k \\ -c_i c_k & 0 & -c_i c_k s_j + s_i s_k \\ c_k s_i + c_i s_j s_k & c_i c_k s_k - s_i s_k & 0 \end{pmatrix}$$

which we can use to identify the derivatives $\frac{\partial(\alpha,\beta,\gamma)}{\partial(a,b,c)}$,

$$\frac{\partial\alpha}{\partial a} = c_k s_i s_j + c_i s_k, \frac{\partial\alpha}{\partial b} = -c_i c_k s_j + s_i s_k, \frac{\partial\alpha}{\partial c} = -c_i c_k s_j + s_i s_k \quad (13)$$

$$\frac{\partial\beta}{\partial a} = c_i c_k - s_i s_j s_k, \frac{\partial\beta}{\partial b} = c_k s_i + c_i s_j s_k, \frac{\partial\beta}{\partial c} = c_k s_i + c_i s_j s_k \quad (14)$$

$$\frac{\partial\gamma}{\partial a} = -c_j s_i, \frac{\partial\gamma}{\partial b} = c_i c_j \frac{\partial\gamma}{\partial c} = c_i c_j \quad (15)$$

Upper right 3×3

$$\frac{\partial \mathbf{\Delta q}}{\partial (a,b,c)} = \frac{\partial}{\partial (a,b,c)} \left(\mathbf{T\Delta x} + \mathbf{T} \left(\mathbf{\Delta R'} - \mathbf{I} \right) \mathbf{x} \right) = \frac{\partial}{\partial (a,b,c)} \left(\mathbf{T} \left(\mathbf{\Delta R'} - \mathbf{I} \right) \mathbf{x} \right) \tag{16}$$

where the derivatives $\frac{\partial \mathbf{\Delta R'}}{\partial a,b,c}$ are given in Sec. 3.

Lower left 3×3

Lower left quadrant is trivial as it comes down to the effect of a translation on the rotation which is zero,

$$\frac{\partial \mathbf{\Delta R}}{\partial (x,y,z)} = \frac{\partial}{\partial (x,y,z)} \left(\mathbf{T\Delta R'T}^{-1} \right) = \mathbf{0} \tag{17}$$

Putting the quadrants together

Finally we can present the form of the $\frac{\partial \mathbf{a}}{\partial \mathbf{b}}$:

4.2 Translation Derivative

Starting from Eq. 42 and 43, we can write,

$$\begin{aligned} x &= x_c \mathrm{R} \sin\phi & (18) \\ y &= y_c + \mathrm{R} \times \cos\phi & (19) \\ z &= z_0 - \mathrm{R} \times \mathrm{slope} \times \Delta\phi. & (20) \end{aligned}$$

Note that,

$$\phi = \mathrm{atan2}\left(\sin\phi_0 - \frac{x - x_0}{\mathrm{R}}, \cos\phi_0 + \frac{y - y_0}{\mathrm{R}} \right). \tag{21}$$

$$\tag{22}$$

4.2.1 Translation in x

To calculate the derivatives for a translation in x we express the track model equation as a function of x, i.e. $f(x, \mathbf{q})$. Using Eq. 21 and substituting,

$$\begin{aligned} y &= -(R - d_0)\cos\phi_0 + \mathrm{sign}\left(\mathrm{R}\right) \sqrt{R^2 - (x - (R - d_0)\sin\phi_0)^2} \\ y_0 &= d_0 \cos\phi_0 \\ x_0 &= -d_0 \sin\phi_0 & (23) \\ & & (24) \end{aligned}$$

we can calculate $\frac{\partial f_{x_i}(\mathbf{q})}{\partial x}$ where $x_i = x, y, z$.

$$
\begin{aligned}
\frac{\partial f_x}{\partial x} &= 1 \\
\frac{\partial f_y}{\partial x} &= -R \sin\phi \frac{\partial \phi}{\partial x} \\
\frac{\partial f_z}{\partial x} &= -\mathrm{R} \times \mathrm{slope} \frac{\partial \phi}{\partial x},
\end{aligned}
\tag{25}
$$

with $\frac{\partial \phi}{\partial x}$ given by,

$$
\frac{\partial \phi}{\partial x} = \frac{-R^2 \operatorname{sign}(\mathrm{R})}{\sqrt{R^2 - (x + (d_0 - R)s_0)^2}\left(-(x + (d_0 - R)s_0)^2 + \operatorname{sign}(\mathrm{R})^2\left(-R^2 + x^2 + 2(d_0 - R)x s_0 + (d_0 - R)^2 \sin^2\phi_0\right)\right)} \tag{26}
$$

where $s_0 = \sin\phi_0$.

4.2.2 Translation in y

Similarly as for x, using Eq. 21 and substituting,

$$
\begin{aligned}
x &= (R - d_0)\sin\phi_0 + \operatorname{sign}(\mathrm{R})\sqrt{R^2 - (y(R - d_0)\cos\phi_0)^2} \\
y_0 &= d_0 \cos\phi_0 \\
x_0 &= -d_0 \sin\phi_0 \qquad (27) \\
& \qquad (28)
\end{aligned}
$$

we can calculate $\frac{\partial f_{x_i}(\mathbf{q})}{\partial y}$ where $x_i = x, y, z$.

$$
\begin{aligned}
\frac{\partial x}{\partial y} &= \mathrm{R}\cos\phi \frac{\partial \phi}{\partial y} \\
\frac{\partial y}{\partial y} &= 1 \\
\frac{\partial z}{\partial y} &= -\mathrm{R} \times \mathrm{slope} \frac{\partial \phi}{\partial y},
\end{aligned}
\tag{29}
$$

with $\frac{\partial \phi}{\partial y}$ given by,

$$
\frac{\partial \phi}{\partial y} = \frac{R^2 \operatorname{sign}(\mathrm{R})}{\sqrt{R^2 - (y + (-d_0 + R)c_0)^2}\left(-(y + (-d_0 + R)c_0)^2 + (-R^2 + y^2 - 2(d_0 - R)y c_0 + (d_0 - R)^2 c_0^2)\operatorname{sign}(\mathrm{R})^2\right)} \tag{30}
$$

where $c_0 = \cos\phi_0$.

4.2.3 Translation in z

The derivatives are given by,

$$\begin{aligned}
\frac{\partial x}{\partial z} &= \mathrm{sign}\,(\mathrm{R}) \times \mathrm{R} \cos\phi \frac{\partial \phi}{\partial z} \\
\frac{\partial y}{\partial z} &= -\mathrm{sign}\,(\mathrm{R}) \times \mathrm{R} \times \sin\phi \frac{\partial \phi}{\partial z} \\
\frac{\partial z}{\partial z} &= 1,
\end{aligned} \tag{31}$$

with $\frac{\partial \phi}{\partial z}$ given by

$$\frac{\partial \phi}{\partial z} = \frac{\partial}{\partial z}\left(-\frac{z - z_0}{R \times \mathrm{slope}} + \phi_0\right) = -\frac{1}{R \times \mathrm{slope}} \tag{32}$$

$$\tag{33}$$

4.3 Rotation Derivatives

We assume that all rotation angles are around the center of the sensor and small and use the small angle limit for rotations $\mathbf{k} = (\alpha, \beta, \gamma)$ corresponding to rotations around the three axis (x, y, z). The derivatives $\frac{\partial f_{x_i}}{\partial \mathbf{k}}$ are given by,

$$\frac{\partial f_{x_i}}{\partial \mathbf{k}} = \begin{pmatrix} \frac{\partial f_x}{\partial \alpha} & \frac{\partial f_y}{\partial \alpha} & \frac{\partial f_z}{\partial \alpha} \\ \frac{\partial f_x}{\partial \beta} & \frac{\partial f_y}{\partial \beta} & \frac{\partial f_z}{\partial \beta} \\ \frac{\partial f_x}{\partial \gamma} & \frac{\partial f_y}{\partial \gamma} & \frac{\partial f_z}{\partial \gamma} \end{pmatrix}$$

$$\frac{\partial f_{x_i}}{\partial \mathbf{k}} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

5 Local Derivatives

Start by expressing ϕ as a function of the track parameters d_0, ϕ_0, R and the interaction point along the beam line x. From 41, $y = y_c + R \times \cos\phi$ and $x_c = (R - d_0)\sin\phi_0$

$$\begin{aligned}
(\cos\phi)^2 &= \frac{1}{R^2}\left(R^2 - (x - x_c)^2\right) \\
(\cos\phi)^2 &= \frac{1}{R^2}\left(R^2 - (x - (R - d_0)\sin\phi_0)^2\right) \\
\phi &= \cos^{-1}\left(\sqrt{1 - \left(\frac{x - (R - d_0)\sin\phi_0}{R}\right)^2}\right)
\end{aligned} \tag{34}$$

or, equivalently, using Eq. 42,

$$
\begin{aligned}
\sin\phi &= \frac{1}{R}(x - x_c) = (x - (R - d_0)\sin\phi_0) \\
\phi &= \arcsin\left(\frac{x - (R - d_0)\sin\phi_0}{R}\right). \qquad (35)
\end{aligned}
$$

Using,

$$
\begin{aligned}
x &= x_c - R\sin\phi = (R - d_0)\sin\phi_0 - R\sin\phi \\
y &= y_c + R\cos\phi = -(R - d_0)\cos\phi_0 + R\cos\phi \\
z &= z_0 + s \times \text{slope} = z_0 - R \times \text{slope}\,(\phi - \phi_0)\,, \qquad (36)
\end{aligned}
$$

we can calculate the local derivatives $\frac{\partial f_{x_i}(\mathbf{q})}{\partial \mathbf{q}}$ where $i = x, y, z$.

$\frac{\partial f_x(\mathbf{q})}{\partial \mathbf{q}}$:

$$
\begin{aligned}
\frac{\partial f_x(\mathbf{q})}{\partial d_0} &= -\sin\phi_0 - R\cos\phi\frac{\partial\phi}{\partial d_0} \\
\frac{\partial f_x(\mathbf{q})}{\partial z_0} &= -R\cos\phi\frac{\partial\phi}{\partial z_0} \\
\frac{\partial f_x(\mathbf{q})}{\partial phi_0} &= (R - d_0)\cos\phi_0 - R\cos\phi\frac{\partial\phi}{\partial phi_0} \\
\frac{\partial f_x(\mathbf{q})}{\partial R} &= \sin\phi_0 - R\cos\phi\frac{\partial\phi}{\partial R} \\
\frac{\partial f_x(\mathbf{q})}{\partial \text{slope}} &= R\cos\phi\frac{\partial\phi}{\partial \text{slope}}, \qquad (37)
\end{aligned}
$$

$\frac{\partial f_y(\mathbf{q})}{\partial \mathbf{q}}$:

$$
\begin{aligned}
\frac{\partial f_y(\mathbf{q})}{\partial d_0} &= \cos\phi_0 - R\sin\phi\frac{\partial\phi}{\partial d_0} \\
\frac{\partial f_y(\mathbf{q})}{\partial z_0} &= -R\sin\phi\frac{\partial\phi}{\partial z_0} \\
\frac{\partial f_y(\mathbf{q})}{\partial phi_0} &= (R - d_0)\sin\phi_0 - R\sin\phi\frac{\partial\phi}{\partial phi_0} \\
\frac{\partial f_y(\mathbf{q})}{\partial R} &= -\cos\phi_0 - R\sin\phi\frac{\partial\phi}{\partial R} \\
\frac{\partial f_y(\mathbf{q})}{\partial \text{slope}} &= -R\sin\phi\frac{\partial\phi}{\partial \text{slope}} \qquad (38)
\end{aligned}
$$

$\frac{\partial f_y(\mathbf{q})}{\partial \mathbf{q}}$:

$$
\begin{aligned}
\frac{\partial f_z(\mathbf{q})}{\partial d_0} &= -R \times \text{slope}\frac{\partial \phi}{\partial d_0} \\
\frac{\partial f_z(\mathbf{q})}{\partial z_0} &= 1 \\
\frac{\partial f_z(\mathbf{q})}{\partial phi_0} &= -R \times \text{slope}\left(\frac{\partial \phi}{\partial phi_0} - 1\right) \\
\frac{\partial f_z(\mathbf{q})}{\partial R} &= -\text{slope} \times \left(\phi - \phi_0 + R\frac{\partial \phi}{\partial R}\right) \\
\frac{\partial f_z(\mathbf{q})}{\partial \text{slope}} &= s = -R(\phi - \phi_0)
\end{aligned} \tag{39}
$$

$\frac{\partial \phi}{\partial \mathbf{q}}$ **using Eq. 35**

$$
\begin{aligned}
\frac{\partial \phi}{\partial d_0} &= \frac{\sin \phi_0}{R\sqrt{\frac{(x+(d_0-R)\sin\phi_0)^2}{R^2}}} \\
\frac{\partial \phi}{\partial phi_0} &= \frac{(d_0 - R)\cos\phi_0}{R\sqrt{1 - \frac{(x+(d_0-R)\sin\phi_0)^2}{R^2}}} \\
\frac{\partial \phi}{\partial R} &= \frac{-x - d_0 \sin\phi_0}{R^2\sqrt{1 - \frac{(x+(d_0-R)\sin\phi_0)^2}{R^2}}} \\
\frac{\partial \phi}{\partial slope} &= 0 \\
\frac{\partial \phi}{\partial z0} &= 0
\end{aligned} \tag{40}
$$

The calculation of $\frac{\partial \phi}{\partial \mathbf{q}}$ using Eq. 34 is shown in appendix.

6 Helical Track Equations

Equations for trajectories in the XY plane

Point on helix (x, y) satisfies,

$$\mathrm{R}^2 = (x - x_c)^2 + (y - y_c)^2 \tag{41}$$

and the coordinate of the centre of the circle can be written,

$$
\begin{aligned}
x_c &= x + \mathrm{R}\sin\phi & (42)\\
y_c &= y - \mathrm{R}\cos\phi. & (43)
\end{aligned}
$$

Equation for trajectory in XZ plane

$$z = z_0 + \mathrm{s} \times \mathrm{slope} \tag{44}$$

Appendix

Practical Formulas

$$y = y_c + \mathrm{sign}\left(()\,\mathrm{R}\right) \times \sqrt{\left(\mathrm{R}^2 - (x - x_c)^2\right)} \tag{45}$$

$$x = x_c + \mathrm{sign}\left(()\,\mathrm{R}\right) \times \sqrt{\left(\mathrm{R}^2 - (y - y_c)^2\right)} \tag{46}$$

$$\Delta\phi = \mathrm{atan2}(x - x_c, y - y_c) - \mathrm{atan2}(x_0 - x_c, y_0 - y_c) \tag{47}$$

Arc length, s, can be written as:

$$\mathrm{s} = -\Delta\phi \times \mathrm{R}. \tag{48}$$

$$\frac{\partial y_c}{\partial d_0} = \frac{\mathrm{sign}\left(\mathrm{R}\right)^2 \sin\left(\phi_0\right)\left(x - x_c\right)}{\sqrt{\mathrm{R}^2 - \mathrm{sign}\left(\mathrm{R}\right)^2 \left(\mathrm{R}^2 - (x - x_c)^2\right)}} \tag{49}$$

$\frac{\partial\phi}{\partial\mathbf{q}}$ using Eq. 34

$$\begin{aligned}
\frac{\partial\phi}{\partial d_0} &= \frac{\sin\phi_0\left(x + (d_0 - R)\sin\phi_0\right)}{R^2\sqrt{\frac{(x+(d_0-R)\sin\phi_0)^2\left(1-\frac{(x+(d_0-R)\sin\phi_0)^2}{R^2}\right)}{R^2}}} \\
\frac{\partial\phi}{\partial\phi_0} &= \frac{(d_0 - R)\cos\phi_0\left(x + (d_0 - R)\sin\phi_0\right)}{R^2\sqrt{\frac{(x+(d_0-R)\sin\phi_0)^2\left(1-\frac{(x+(d_0-R)\sin\phi_0)^2}{R^2}\right)}{R^2}}} \\
\frac{\partial\phi}{\partial R} &= -\left(\frac{\left(x + d_0\sin\phi_0\right)\left(x + (d_0 - R)\sin\phi_0\right)}{R^3\sqrt{\frac{(x+(d_0-R)\sin\phi_0)^2\left(1-\frac{(x+(d_0-R)\sin\phi_0)^2}{R^2}\right)}{R^2}}}\right) \\
\frac{\partial\phi}{\partial slope} &= 0 \\
\frac{\partial\phi}{\partial z0} &= 0
\end{aligned} \tag{50}$$

References

[1] Markus Stoye, *Calibration and Alignment of the CMS Silicon Tracking Detector*, Dissertation, 2007.