

## PHYS644 Problem Set 6

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### Problem 1: Photons in Today's Universe

We are told that the CMB  $T_0 = 2.725$  K (Kelvin isn't a degree!). We are asked to calculate the number density of photons in the universe today AND the energy density of photons -  $\Omega_{\gamma,0}H^2$ .

From lecture notes we know for bosons (photons are bosons)

$$n_\gamma = \frac{g\zeta(3)T^3}{\pi^2} \quad (1)$$

In natural units of  $k_b = c = \hbar = 1$ ,  $g$  is the number of internal degrees of freedom per particle in case  $g = 2$ .

To add back in units, we know that  $n$  should have units of  $1/\text{volume}^3$ ,

$$\text{cm}^{-3} = T^3 \quad (2)$$

but the right hand side has units of temperature.  $k_bT$  gets us units of energy, and then we need to get energy to volume, recall that  $\hbar c$  has units of energy \* length.

$$n_\gamma = \frac{g\zeta(3)}{\pi^2} \left( \frac{k_bT}{\hbar c} \right)^3 \quad (3)$$

Now we can plug in our value of  $T$ .

$$n_\gamma = 411 \text{ cm}^{-3}.$$

### Problem 1b: $\Omega_{\gamma,0}h^2$

From lecture notes we know:

the energy density of photons is

$$\rho_{\gamma,0} = 4\sigma T^4 \quad (4)$$

where  $\sigma$  is the stefan-boltzman constant, but in natural units.

and  $\rho_{\text{crit},0}$  is given by:

$$\rho_{\text{crit},0} = \frac{3H_0^2}{8\pi G} \quad (5)$$

This as units of  $kgR^{-3}$  but should have units of energy density -  $\rho = KgR^{-1}t^{-2}$ . We are off by  $c^2$ .

and  $\Omega_{\gamma,0}$  is given by:

$$\Omega_{\gamma,0} = \frac{\rho_{\gamma,0}}{\rho_{\text{crit},0}} \quad (6)$$

So  $\Omega_{\gamma,0}h^2$  is given by:

$$\Omega_{\gamma,0}h^2 = \frac{8\pi^3G}{45(100 \text{ km s}^{-1} \text{ Mpc}^{-1})^2} \frac{(k_B T_0)^4}{\hbar^3 c^3} \quad (7)$$

To add the units back I remember from stat mech that there is a factor of  $c$ , and then its just  $\rho_{\text{crit}}$  which we found and then using the unit form of  $\sigma$  (from wiki).

$$\Omega_{\gamma,0}h^2 = 2 \times 10^{12}$$

## Problem 2: Acceleration Redshift

$z_{acc}$  happens when the universe expansion just starts (postively) accelerating IE when  $\ddot{a} = 0$ .

The second Friedmann equation becomes

$$\frac{\ddot{a}}{a} = 0 = -\frac{4\pi G}{3}(\rho + 3p) \quad (8)$$

Therefor our condition is when  $\rho + 3p = 0$ . We are absorbing  $\Lambda$  into its own effect  $\rho$  and  $p$ .

Baryonic matter gives  $p = 0$ , and dark energy has  $p = w\rho_{DE}$ .

Now we have

$$0 = \rho_m + \rho_{DE}(1 + 3w) \Rightarrow \rho_m = -\rho_{DE}(1 + 3w) \quad (9)$$

recall that

$$\rho_{crit,0} = \frac{3H_0^2}{8\pi G} \quad (10)$$

$$\Omega_{m,0} = \frac{\rho_{m,0}}{\rho_{crit,0}} \quad (11)$$

$$\Omega_{\Lambda} = \Omega_{DE,0} = \frac{\rho_{DE,0}}{\rho_{crit,0}} \quad (12)$$

We now insert these into our condition, we convert to values at  $z$  recall that  $a = 1/(1+z)$

$$\Omega_{m,0} \rho_{crit,0} (1+z)^3 = -(1+3w) \Omega_{\Lambda} \rho_{crit,0} (1+z)^{3(1+w)}. \quad (13)$$

Now we solve for  $z$  — this will be  $z_{acc}$ .

$$(1+z)^{-3w} = -(1+3w) \frac{\Omega_{\Lambda}}{\Omega_{m,0}} \quad (14)$$

$$z = \left[ -(1+3w) \frac{\Omega_{\Lambda}}{\Omega_{m,0}} \right]^{1/3w} - 1 \quad (15)$$

We learned from class (and from  $a(t) = 1/(1+z)$  that  $z = 0$  is now, and  $z = -1$  is  $t = \infty$ . Aka  $z > 0$  if we are looking backwards in time. We see that the first term on the left is always positive when  $w < -\frac{1}{3}$ .

## Problem 4: Distance Measures

### Problem 4A:

We are told that  $\Omega_{m,0} = 0.3$ ,  $\Omega_{r,0} = 8.5 \times 10^{-5}$ ,  $\Omega_{\Lambda} = 1 - \Omega_{r,0} - \Omega_{m,0}$

We know that the physical distance  $R$  is given by

$$R(t) = a(t)\chi \quad (16)$$

and that  $a(t)$  is the scale factor and  $\chi$  is the comoving distance which is fixed if objects are just drifting due to  $a(t)$ .  $\chi$

Co-moving distance  $\chi$  is also written as  $D_c$ , We know that the co-moving distance is

$$D_c(z) = c \int_0^z H(z)^{-1} dz \quad (17)$$

and we can write  $H(z)$  as

$$H(z) = H_0 \sqrt{\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_k(1+z)^2 + \Omega_{\Lambda}} \quad (18)$$

This is the corrected for  $z$  version! In our case  $\Omega_k = 0$  because like the Earth<sup>1</sup> - the universe is flat. We now have:

$$D_c(z) = c \int_0^z \frac{1}{H_0 \sqrt{\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_{\Lambda}}} dz \quad (19)$$

While I could solve this analytically, we are asked to plot so why bother?

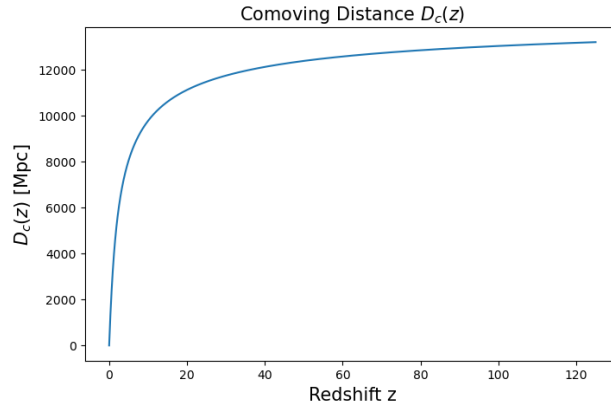


Figure 1: I'm not writing a caption

Here is our plot figure 1! Most of the evolution happens at “low” redshift and then it looks like we start to approach an asymptote as we go on.

### Problem 4B:

We do the same thing now until the edge of the observable universe! This is  $z = \infty$

Here is our plot figure 2. We can see we run into a numerical issue at high  $z$ , but that  $D_c$  asymptotically approaches roughly a constant value.

<sup>1</sup>This is a meme.

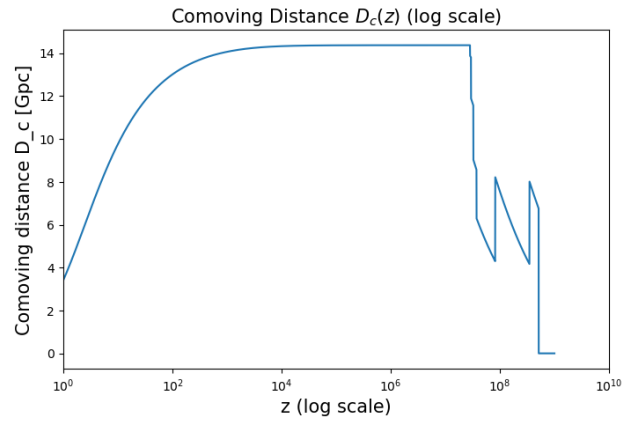


Figure 2: no caption!

**Problem 4C**

From figure 2 we can read the asymptotically value as  $\sim 14$  Gpc. So half of this value is  $\sim 7$  Gpc — by my eye that is around  $z = 10$  ish. (Its actually easier to see on 4A!)

**Problem 5:**