Growth of Structure

What we have so far is that inflation seeds some density fluctuations for us, which eventually grow to form galaxies and the large scale structure we see today.

We defined the power spectrum to describe these fluctuations, quantifying for us how much of these spatial fluctuations are on small vs large scales.

Two versions:

① "The" power spectrum: $\langle \tilde{\delta}(\vec{k}) \tilde{\delta}(\vec{k}')^* \rangle = (2\pi)^3 \delta^D(\vec{k}-\vec{k}') P(k)$

$$\text{or} \quad P(k) = \frac{\langle |\tilde{\delta}(\vec{k})|^2 \rangle}{V}$$

← volume of survey.

Units: $(\text{Mpc})^3$, $\because$ two factors of volume from FT, divided by one factor of volume

$$\text{where} \quad \delta \equiv \frac{\rho - \bar{\rho}}{\bar{\rho}}.$$

② Dimensionless power spectrum:

$$\Delta^2(k) \equiv \frac{k^3 P(k)}{2\pi^2}$$

← Units: dimensionless!

Note that depending on the context, it's sometimes helpful to think about fluctuations in the gravitational potential Φ, and sometimes it's more helpful to consider fluctuations in the matter density ρ.

The two are related via the Poisson eqn:

$$\nabla^2 \Phi = 4\pi G \rho.$$

What I am interested in ~~*~~ are the perturbations in density and perturbations in potential and how they relate to one another. So we ~~perturbt~~ perturb this equation:

$$\nabla^2 \underbrace{\delta\Phi}_{\equiv \Phi - \bar{\Phi}} = 4\pi G \bar{\rho}\delta$$

Recall that $\bar{\rho}\delta = \rho - \bar{\rho}$

Now, suppose we were to try to solve this equation in Fourier space. In other words, suppose we let

$$\delta\Phi = A_k e^{i\vec{k}\cdot\vec{r}} \quad \text{and} \quad \delta \equiv B_k e^{i\vec{k}\cdot\vec{r}}.$$

Then $\nabla^2 \delta\Phi = A_k\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)e^{i\vec{k}\cdot\vec{r}} = k^2 A_k e^{i\vec{k}\cdot\vec{r}}$

So the action of ∇^2 in Fourier space is multiplication by k^2.

Note also that while an operator like ∇^2 mixes different pixels in position space, it just separately multiplies each Fourier mode by k. They don't "talk" to one another.

$$\Rightarrow \quad k^2 \widetilde{\delta\Phi} = 4\pi G \bar{\rho}\tilde{\delta} \quad \Rightarrow \quad k^2 \widetilde{\delta\Phi} \propto \tilde{\delta}$$

This means that we can translate between perturbations in Φ and ρ by just multiplying by k^2!

Need to square k^2 because $P \sim \tilde{\delta}^2$

$$\Rightarrow \quad P_\delta \propto k^4 P_\Phi.$$

We saw last time inflation predicts $\Delta^2_\Phi \propto k^{n_s - 1}$ (where $n_s \approx 1$) $\Rightarrow P_\Phi \propto k^{n_s - 4}$.

This means that $P_\delta(k) \propto k^{n_s}$

Sometimes I will omit this subscript. If I do, I mean the matter/density $P(k)$

We would like to compare this to observations to test our theories.

What do we see if we look at observations?

⇒ $\boxed{\text{Show slides with matter paper}}$

This doesn't exactly look like a single power law of the form k^{n_s}! What we are missing is a description of how all the fluctuations have evolved since being generated / seeded by inflation.

There are two things that have happened since the initial conditions were seeded:

i) Fluctuations have grown in amplitude due to gravitational clustering

ii) The power spectrum has acquired a more complicated dependence on k. This is not surprising — as we've discussed before, perturbations can't grow until the horizon has become on the same order as their wavelength (ie until they "reenter the horizon"), ~~so~~ and we expect this to happen at different times for different k's.

Often this is parametrized as

$$\tilde{\delta}(\vec{k}, \underset{\text{Scale factor}}{a}) \propto \underset{\text{Initial conditions}}{\tilde{\delta}_{\text{primordial}}(\vec{k})} \, \underset{\text{ii) "Transfer function"}}{T(k)} \, \underset{\text{i) "Growth factor"}}{D_1(a)}$$

Note that this suggests a linear relation between the primordial fluctuations and the final fluctuations. This is only true if the fluctuations are small (ie $|\delta| \ll 1$) so that we can make a linear approximation of our evolution equations.

We'll do this today, doing linear perturbation theory. We'll concentrate mostly on the growth factor, which means we can ignore the effect of the horizon. Working deep inside the horizon, we can use Newtonian mechanics. (Yay!)

Newtonian linear perturbation theory

First, let's list the relevant equations that we would like to solve. They're the equations of fluid mechanics:

Poisson equation : $\nabla^2 \Phi = 4\pi G \rho$ ("Gravity")

Continuity equation : $\frac{\partial}{\partial t}\rho = -\vec{\nabla}\cdot(\rho\vec{v})$ ("If density goes down here, it's because stuff flowed out")

Euler equation : $\rho\frac{\partial\vec{v}}{\partial t} + \rho\vec{v}\cdot\vec{\nabla}(\vec{v}) = -\rho\vec{\nabla}\Phi - \vec{\nabla}p.$

("F=ma for a fluid").

We now linearize these equations by writing

$$\rho(\vec{r},t) = \bar{\rho}(t)\left[1 + \delta(\vec{r},t)\right]$$

Follows "usual" evolution Eg $\bar{\rho} \propto \frac{1}{a^3}$ for matter

$$\vec{v}(\vec{r},t) = \vec{v}_0 + \delta\vec{v}(\vec{r},t)$$ Peculiar velocity

Hubble flow! $\vec{v}_0 = H\vec{r}$

$$\Phi(\vec{r},t) = \Phi_0(\vec{r},t) + \delta\Phi(\vec{r},t).$$

What we need to do now is to substitute these expressions into our equations and then discard higher order terms like $(\delta v)^2$.

We already did the Poisson equation: $\nabla^2 \delta\Phi = 4\pi G \bar{\rho} \delta$.

If you crank through the algebra, you get

Sound speed from $v_s^2 = \left(\frac{\partial p}{\partial \rho}\right)$

where $(\dot{\ })$ means $\frac{\partial}{\partial t}$

$$\begin{cases} \dot{\delta} + \vec{\nabla} \cdot \delta\vec{v} + \vec{v}_0 \cdot \vec{\nabla}\delta = 0 \\ \delta\dot{\vec{v}} + (\vec{v}_0 \cdot \vec{\nabla})\delta\vec{v} + \left(\frac{\dot{a}}{a}\right)\delta\vec{v} = -\vec{\nabla}\delta\Phi - v_s^2 \vec{\nabla}\delta \end{cases}$$

These are three coupled PDEs. Yuck! Once again, we can make our lives easier by going into Fourier space.

If we substitute into this expressions like

$$\delta(\vec{r},t) = \int \frac{d^3 \vec{k}}{(2\pi)^3} \tilde{\delta}(\vec{k},t) \exp\left[-i \frac{\vec{k}\cdot\vec{r}}{a(t)}\right]$$

This factor ensures that $\vec{k}$ is a wavevector in comoving coords, since $\frac{\vec{r}}{a}$ is the comoving distance (we want to focus on clustering, not expansion)

Then we can get expressions like:

$$\boxed{\ddot{\tilde{\delta}} + 2\left(\frac{\dot{a}}{a}\right)\dot{\tilde{\delta}} = \tilde{\delta}\left(4\pi G\bar{\rho} - \frac{v_s^2 k^2}{a^2}\right).} \quad \ldots (*)$$

This is what we want! A good ol' ODE telling us how $\tilde{\delta}(k)$ grows with time!

Three competing effects

(*) is a wonderful equation because each of the terms really has a nice physical interpretation, ~~and~~ affecting $\tilde{\delta}$. Let's examine them one by one, turning the others off if necessary.

① **Gravity**. This is the $4\pi G\bar{\rho}$ term. If the other terms weren't there, we would have

$$\ddot{\tilde{\delta}} = 4\pi G \bar{\rho}\, \tilde{\delta}$$ $\Rightarrow$ Positive feedback! Exponential growth because clumping amplifies gravitational effects!

(2) Hubble expansion. This is the $2\left(\frac{\dot{a}}{a}\right)\delta$ term, and we see that it enters the differential equation like a drag term, inhibiting the growth. The quicker the expansion, the greater this drag term, because the more the expansion is trying to ~~press~~ pull apart clustering.

(3) Pressure. If I compress a gas, it pushes back. This fights clustering. Turning off the other terms, I have

$$\ddot{\tilde{\delta}} = -\frac{v_s^2 k^2}{a^2}\tilde{\delta}$$

This is a harmonic oscillator! If pressure wins, this essentially means that I set up sound waves in the medium rather than having gravitational collapse.

When does pressure win? The crucial thing is the sign of the R.H.S. The tipping point is when it's zero, when

$$4\pi G \bar{\rho} = v_s^2 \left(\frac{k}{a}\right)^2$$

k is the comoving wavenumber; $\frac{k}{a}$ is the physical wavenumber.

Let $\frac{k}{a} = \frac{2\pi}{\lambda_J}$ "Jeans length"

$$\Rightarrow \boxed{\lambda_J = \sqrt{\frac{\pi v_s^2}{G\bar{\rho}}}}$$

→ Perturbations on scales $> \lambda_J$ collapse

→ Perturbations on scales $< \lambda_J$ form stable oscillations

Now, $H^2 = \frac{8\pi G\bar{\rho}}{3}$, so we can write this as

$$\lambda_J = 2\pi\left(\frac{2}{3}\right)^{1/2}\frac{v_s}{H}.$$

How big is this? What wins? Gravity or pressure?

It depends on the substance, because different things have different sound speeds.

Photons: Recall that $p = \frac{1}{3}\rho c^2$, so $v_s^2 = \frac{\partial p}{\partial \rho} = \frac{c^2}{3} \Rightarrow v_s = \frac{c}{\sqrt{3}}$

This means $\lambda_J = \frac{2\pi\sqrt{2}}{3}\frac{c}{H}$. Larger than the Hubble length c/H!

$\Rightarrow$ Photons don't really collapse gravitationally.

Baryons: This one is interesting!

Prior to recombination, the baryons are ionized, and this charged particle soup scatters photons easily via Thomson scattering, so the photons and baryons act as a coupled photon-baryon fluid. The photons "lend" their pressure to the baryons and they can't collapse.

After recombination, the sound speed is given by the usual formula you are used to from thermo:

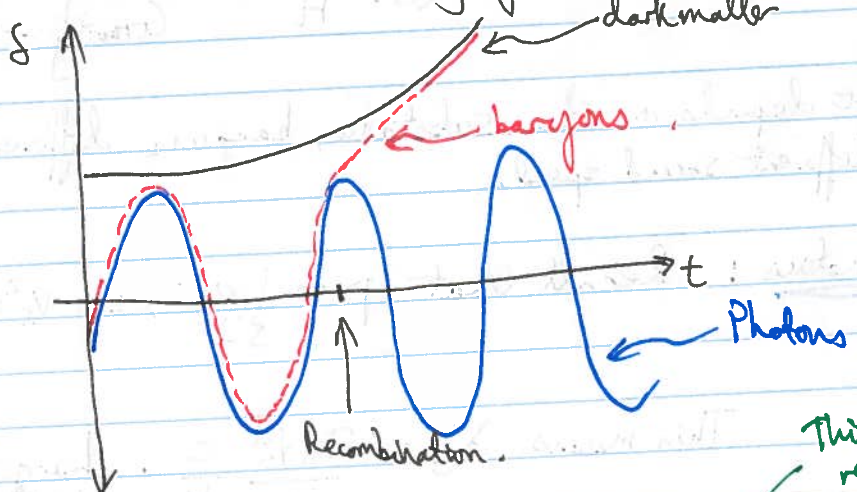
$$c_s = \left(\frac{k_B T}{m}\right)^{1/2} \sim 1.5\times10^{-5}\,c$$

for T at recombo

The sound speed plummets, and all of a sudden the baryons can cluster!

Dark matter: this is (to first approximation) pressureless, so gravity always wins!

Here's a ~~cute~~ cartoon summary of the different components:



This is of course relative

The dark matter gets a "head start" on seeding structure. Once baryons decouple they can fall into the deep potential wells seeded by the dark matter. Without this seeding effect we wouldn't get structure growth to happen fast enough to explain the galaxies we see today!

Note two important things from the intuition we've established:

i) The key to understanding structure growth is to understand how dark matter clusters, since the baryons (roughly speaking) just follow along.

ii) The photons don't really cluster, so the gravity term in (*) is really sourced by matter.

To really emphasize ii), we can write our equation from before as:

$$\ddot{\tilde{\delta}} + 2H\dot{\tilde{\delta}} - \frac{3}{2}\Omega_m H^2 \tilde{\delta} = 0$$

No pressure term because I'm now thinking about dark matter

In different phases of evolution, we get different behaviour:

Radiation-dominated: $\Omega_m \ll 1$, $a \sim t^{1/2}$, $H = \frac{1}{2t}$

$$\Rightarrow \ddot{\tilde{\delta}} + \frac{1}{t}\dot{\tilde{\delta}} \simeq 0$$

There's growth, but it's slow!

$$\Rightarrow \tilde{\delta}(k,t) \simeq B_1 + B_2 \ln t$$

Dark energy dominated: $\Omega_m \ll 1$, $H \simeq H_0$ (const).

$$\ddot{\tilde{\delta}} + 2H_0\dot{\tilde{\delta}} \simeq 0$$

$$\Rightarrow \tilde{\delta}(k,t) \simeq C_1 + C_2 e^{-2H_0 t}$$

No growth

Decaying mode. Unimportant

But wait. We see galaxies accreting today. What do you mean there is no growth during dark energy domination? Our analysis assumes that we are in the linear regime. Small, nonlinear scales like galaxies aren't applicable to this analysis.

Matter dominated: $\Omega_m \approx 1$, $a \sim t^{2/3}$, $H = \frac{2}{3t}$.

$$\Rightarrow \ddot{\tilde{\delta}} + \frac{4}{3t}\dot{\tilde{\delta}} - \frac{2}{3t^2}\tilde{\delta} = 0.$$

Try power law solution: $\tilde{\delta} = At^n$

$$n(n-1)At^{n-2} + \frac{4}{3t}nAt^{n-1} - \frac{2}{3t^2}At^n = 0$$

$$\Rightarrow n(n-1) + \frac{4}{3}n - \frac{2}{3} = 0.$$

Solutions are $n = -1$ and $n = 2/3$.

$$\Rightarrow \tilde{\delta}(k,t) \approx A_1 t^{2/3} + A_2 t^{-1}$$

decaying mode; unimportant.

So during matter domination, $\tilde{\delta} \propto a(t)$

Not the runaway exponential growth of a static universe :(of expansion, but still, this is the only era with substantial growth.