PHY644 Notes

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Contents

1 Galaxies As Collisionless Fluids

We cannot think of stars as individual elements, but rather the collection of stars a s **collision-less fluid**.

1.1 Phase Space Density

We can define a Phase Space Density,

$$f(\vec{x}, \vec{v}, t)\, d^3\vec{x}\, d^3\vec{v} \tag{1}$$

This is the number of stars in volume $d^3\vec{x}$, centred on $\vec{x}$., and velocities in small range $d^3\vec{v}$ centred on $\vec{v}$.aka density in 6D phase space given by $\vec{w} = (\vec{x}, \vec{v})$

We can think of a swarm (or blob) of stars (particles) moving in this phase space with velocity:

$$\dot{\vec{w}} = (\dot{\vec{x}}, \dot{\vec{v}}) = (\vec{v}, -\nabla\Phi) \tag{2}$$

Here Φ is the gravitational potential not gravitational potential energy (IE it is per unit mass).

1.2 Fluid Mechanics for Stars

Let's apply Newtonion Mechanics (Fluid mechanics). We assume that stars are collision-less, and that no stars form or die.

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \frac{\partial(f\dot{w}_\alpha)}{\partial w_\alpha} = 0 \tag{3}$$

$f\dot{w}_\alpha$ is the phase space current. Using the product rule, we can express the term with the sum as:

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} f\frac{\partial \dot{w}_\alpha}{\partial w_\alpha} + \dot{w}_\alpha \frac{\partial f}{\partial w_\alpha} = 0 \tag{4}$$

The flow described by $\dot{w}$ is special because:

$$\sum_{\alpha}^{6} \frac{\partial \dot{w}_\alpha}{\partial w_\alpha} = \sum_{i}^{3}\left(\frac{\partial v_i}{\partial x_i} + \frac{\partial \dot{v}_i}{\partial v_i}\right) = \sum_{i}^{3}\left(0 + \frac{-\partial}{\partial v_i}\frac{d\Phi}{\partial x_i}\right) = 0 \tag{5}$$

Because v_i is independent of x_i, and we can write $\dot{v}_i$ as the gradient of the potential, and then the partial derivative of that is equal to 0 because it is not dependent on v_i.

$$\boxed{\sum_{i}^{3}\left(0 + \frac{-\partial}{\partial v_i}\frac{d\Phi}{\partial x_i}\right) = 0} \tag{6}$$

Throwing this back into equation 4, we have the colisionless Boltzman equation:

$$\boxed{\frac{\partial f}{\partial t} + \sum_{\alpha}^{6} \dot{w}_\alpha \frac{\partial f}{\partial w_\alpha} = 0} \tag{7}$$

There are a few equivalent variations of this using the above equations.

1.3 Co-moving fluid mechanic Equation

Recasting into Lagrangian or curvature forms.

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{\alpha}^{6} \dot{w}_\alpha \frac{\partial}{\partial w_\alpha} \tag{8}$$

This is the form of the equation for someone (or a reference frame) flowing along a trajectory in phase space. It says that I can get a change either from some explicit time dependence (1st term) or because it moved to a different part of phase space (2nd term).

The Boltzman equation is then just $\boxed{\frac{df}{dt} = 0}$. In other words, for an observer moving along with a star's path $\dot{w}$ would not see the local phase space density change.

1.4 Moments of Phase space

We do not observe phase space directly, but we can observe the moments of phase space.

1.4.1 0th Moment

The zeroth moment - gives the spacial number density of stars:

$$n(\vec{x}) = \int_{-\infty}^{+\infty} f(\vec{x}, \vec{v}, t) d^3\vec{v} \tag{9}$$

The first moment - gives the average velocity:

$$\mathbb{E}[\vec{v}(\vec{x})] = \frac{1}{n} \int_{-\infty}^{+\infty} \vec{v} f(\vec{x}, \vec{v}, t) d^3\vec{v} \tag{10}$$

The second moment - is related to the velocity dispersion tensor:

$$\langle v_i(\vec{x}) v_j(\vec{x}) \rangle = \frac{1}{n} \int_{-\infty}^{+\infty} (v_i v_j) f(\vec{x}, \vec{v}, t) d^3\vec{v} = \langle v_i(\vec{x}) \rangle \langle v_j(\vec{x}) \rangle + \sigma_{ij}^2 \tag{11}$$

The σ_{ij}^2 is the velocity dispersion.

1.5 Getting Useful forms

To actually get the equations of this, we can take the moments of the Boltsman equation.

For the 0th moment:

$$0 = \int_{-\infty}^{+\infty} [\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f - (\vec{\nabla}_x \Phi) \cdot (\vec{\nabla}_v f)] d^3\vec{v} \tag{12}$$

Use Integration by parts...

The 0th moment is the continuity equation:

$$\boxed{\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n(\langle \vec{v} \rangle)) = 0} \tag{13}$$

This says that stars are conserved.

1.5.1 1st moment: The Jeans Equation

$$0 = \int_{-\infty}^{+\infty} [\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f - (\vec{\nabla}_x \Phi) \cdot (\vec{\nabla}_v f)] \vec{v} d^3\vec{v} \tag{14}$$

break into terms, and use integration by parts!

$$\boxed{\partial_t \langle \vec{v_j} \rangle + \sum_i \langle \vec{v_i} \rangle \vec{\nabla}_{x,i} \langle \vec{v_j} \rangle = -\vec{\nabla}_{x,j} \Phi - \sum_i \frac{\vec{\nabla}_{x,i}(n\sigma_{ij}^2)}{n}} \tag{15}$$

From left to right, we label the terms as - Bulk accretion, velocity sheer, grav force, and pressure. This is the Jeans equation and is the equivalent of the Euler equation in classical fluid mechanics - the $F = MA$ of fluid dynamics.

The right hand term - the pressure, means that a collisionless fluid will still have a pressure as long as the velocity dispersion is non-zero.

Squashing this fluid increases the velocity dispersion increasing the pressure for it to spread back out.

1.6 Stability Analysis

The Jeans equation tells us some information about the stability of a gas trying to collapse (if it collapses or if it bounces back).

Assume the gas consists of particles of mass m, we can turn $n \Rightarrow \rho$ by $\rho = nm$. We assume the velocity dispersion is diagonal and that $\rho\sigma^2 = P$ gives the pressure. This means that our analysis applies to normal gases as well.

All gradients are spacial so we can drop the x $\vec{\nabla}_x \Rightarrow \vec{\nabla}$.

For a small perturbation we assume a static background (the subscripts with 0) and use:

- $\rho = \rho_0 + \epsilon\rho_1$
- $\vec{v} = \vec{v}_0 + \epsilon\vec{v}_1$
- $P = P_0 + \epsilon P_1$
- $\Phi = \Phi_0 + \epsilon\Phi_1$

Now we insert into the continuity equation, and the Jeans equation and group powers of ϵ. The ϵ^0 terms cancel. The ϵ^1 terms give.

$$\partial_t \rho_1 + \rho_0 \vec{\nabla} \cdot \vec{v}_1 = 0 \tag{16}$$

$$\partial_t \vec{v}_1 = -\vec{\nabla}\Phi - \frac{\vec{\nabla} P_1}{\rho} \tag{17}$$

Recall that sound speed is given by:

$$\boxed{\vec{\nabla} P = \frac{\partial P}{\partial \rho}\frac{\partial \rho}{\partial \vec{x}} = c_s^2 \vec{\nabla}\rho} \tag{18}$$

Jeans swindle: We substitute in the c_s version, and use Poission's equation for gravity for $\nabla^2\Phi_0 = 4\pi G\rho_0$ which cannot be true for Φ_0 and ρ_0 both constant when $\rho_0 = 0$.

The brings us to a wave equation of the form:

$$\partial_t^2 \rho_1 - 4\pi G \rho_0 \rho_1 - c_s^2 \nabla^2 \rho_1 = 0 \tag{19}$$

The general solution is:

$$\rho_1 = \rho_1(0) e^{i(\omega t \pm K x)} \tag{20}$$

where ω is the angular velocity, and $k = 2\pi/\lambda$.

2 Geometry of Our Universe

Now we begin cosmology! In Galaxy evolution we cared about individual objects, but here we care about population statistics. It is like being a public health official vs a doctor.

Quick note: this section uses natural units $\hbar = c = k_b = 1$. This is different from geometrical units where $G = c = 1$.

$$M_{\text{planck}} = \sqrt{\frac{\hbar c}{G}} \tag{21}$$

In Natural units, this becomes $G^{-1/2}$.

2.1 The Cosmological Principle

Describing the spacetime geometry of our universe is a difficult task, but we can make three simplifications we make and test empirically.

The Cosmological Principle: the Universe is **statistically homogeneous** and **isotropic**. Homogeneity implies that the Universe has the same average properties at every point in space, whereas isotropy implies that the Universe looks the same in all directions from any given vantage point. Together, these symmetries constrain the possible forms of the spacetime metric, leading naturally to the **Friedmann–Lemaître–Robertson–Walker (FLRW)** metric as the general solution for a universe governed by general relativity.

The cosmological principle does not hold for time, because the universe is expanding — time marches forward.

2.2 Hubble Law

Hubble Law: Hubble discovered that the redshift recession velocity of distance galaxies follows the relationship:

$$v = H_0 d \tag{22}$$

$H_0 \sim 70\,\text{kms}^{-1}\text{Mpc}$ The subscript 0 means today value in the present universe.

With the current Hubble tension it is easier to write $H_0 = 100\,\text{kms}^{-1}\text{Mpc}$, and then use $h = 0.7$ to remain noncommittal.

Another way to think about the expansion of the universe is as a **scaling factor**, and that if we measure the universe with some meter stick that the scaling on the meter stick is changing by some function $a(t)$.

By convention we say $a_0 = a(t_0) = 1$ — that today's time is the age of the universe.

Scaling Factors: It's annoying to describe distance if it is constantly changing so we go instead with comoving distances:

$$d_{\text{physical}} = a(t)\chi_{cosmo} \tag{23}$$

Where d_{physical} is what we measure on our meter stick, and χ_{cosmo} "takes out the expansion". If two objects have a changing cosmological distance then there motion is not just due to the expansion of the universe. Remember that $a_0 = a(t_0) = 1$ — that today's time is the age of the universe.

Let's differentiate both sides:

$$\vec{v}(t) = \frac{d\vec{d}_{\text{physical}}}{dt} = \frac{da}{dt}\vec{\chi}_{cosmo} = \frac{da}{dt}\frac{1}{a}\vec{d}_{\text{physical}}(t) \tag{24}$$

This is the Hubble Law! $\vec{v} \propto d$ but at a different time, thus Hubble's law holds for all time but with different values of H_0.

$$\boxed{H(t) = H = \frac{\dot{a}}{a}} \tag{25}$$

This is called the **Hubble parameter** H_0 is today's value.

In general $H(t)$ is some complicated function of time. However, an interesting special case is a universe where the expansion just makes galaxies coast along at a constant speed.

In this case $\dot{a} = const \Rightarrow H(t) = \frac{const}{a} = \frac{H_0}{a}$. The last substitution is because $H(t_0) = H_0$ as $a_0 = 1$. Because $H = \frac{\dot{a}}{a}$ always we can say $\dot{a} = H_0 \Rightarrow \int_0^1 da = H_0 \int_0^{t_0} dt$.

This is how we take the inverse of H_0 as the age of the universe! **This is the Hubble Time $T_h = \frac{1}{H_0} \approx 14.5\,\text{Gyr}$.**

2.3 Allowed Expansion Rates

While objects cannot travel faster then c, space can expand faster then c, this is known as **super luminal expansion** and defines the observable universe.

Using a comoving coordinates makes this problem easy as we are factoring out the expansion.

In time dt light moves a physical distance cdt, so the comoving distance is $\frac{cdt}{a}$ and for a finite time integral we have $\chi = \int \frac{1}{a} dt$ (recall $c = 1$).

Basically its $c \to \frac{c}{a}$. In GR, space itself can expand or contract.

Particle (Causal) Horizon: This is the furthest distance unimpeded light can have travelled from $t = 0$ to $t = t$

$$\chi = \int_0^t \frac{1}{a} dt' \tag{26}$$

To get the physical distance we multiply by $a(t)$.

$$d_p = a(t)\chi = a(t)\int_0^t \frac{1}{a} dt' \tag{27}$$

In the case of subluminal expansion: we have $a(t) \sim t^p$ where $0 < p < 1$ meaning that $\ddot{a} < 0$ (slowing down).

Given enough time, light will see the entire universe.

in the case of superluminal expansion we have $a(t) \sim t^p$ where $p > 1$ meaning that $\ddot{a} > 0$ (getting faster).

You only see finite χ regardless of how long you wait. Light cannot catch up with the expansion of the universe. We live in this type of universe presently.

Intuition: $p = 1$ where $a \propto t$ is the dividing line because the amount of distance light can travel is ct so if $a \propto t$, the expansion balances this effect.

2.4 Friedmann–Lemaître–Robertson–Walker (FLRW) metric

Friedmann–Lemaître–Robertson–Walker (FLRW) metric: This is the metric of space-time for our universe as a whole, result of the cosmological principle.

$$ds^2 = -dt^2 + a(t)^2[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2] \tag{28}$$

Where $d\Omega^2 = d\theta^2 + sm^2\theta d\phi^2$, and K is the spatial curative and it can be positive, zero, or negative. (Curvature of space, not of space-time).

2.5 Observables

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With the FLRW metric we can make predictions, and observe. one of the primary ones is the redshift.

$$z = \frac{\lambda_{obs} - \lambda_0}{\lambda_0} \tag{29}$$

The elongation of wavelengths can be shown to be proportional to $a(t)$ therefore:

$$\frac{\lambda_{obs}}{\lambda_0} = \frac{a(t_0)}{a(t)} \tag{30}$$

$a(t_0) = 1$

$$1 + z = \frac{1}{a} \tag{31}$$

redshift is both a measure of distance, and time.

3 Dynamics of Our Universe

In the last section we talked about $a(t)$ in kinematic terms, and this time we will talk about in dynamic terms — what causes it move the way it does. These are given by the two Friedman equations.

3.1 The Friedmann Equations

First Friedmann equation:

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\,\rho}{3} - \frac{k}{a^2},$$

where ρ is the energy (mass–energy) density (units: energy per volume, e.g. $Jm(^{-3})$; in $c = 1$ units ρ carries same units as mass density).

Second (acceleration) Friedmann equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right),$$

where p is the pressure. (Again written in $c = 1$ units; with a cosmological constant Λ add $+\Lambda/3$ on the right.) Another usefull equation is $\dot{\rho} = -3(\rho + p)(\frac{\dot{a}}{a})$

The first equation is analogous to an energy equation in Newtonian mechanics.

If you ignore the pressure term in the second equation, it looks similar to "$F = Ma$". The negative sign in the second equation says that matter causes the expansion to decrease.

The second equation says that pressure is source of gravity as well, not just density —- this is a property from general relativity.

3.1.1 Deriving the first Friedman equation from Newton's Cosmology

We can derive the first Friedman equation from Newtonian thinking. Imagine a homogeneous universe filled with matter.

Draw a little circle around a test mass - the test mass is on the shell.

we have:

$$m\frac{d^2r}{dt^2} = -\frac{GM_{\text{enc}}m}{R(t)^2} \Rightarrow \ddot{R} = -\frac{GM_{\text{enc}}}{R^2} \tag{32}$$

Multiply both sides by $\dot{R}$ and integrate $\int \dot{R}\ddot{R}\,dt = \frac{\dot{R}^2}{2} + c$.

$$\frac{1}{2}\dot{R}^2 - \frac{GM_{\text{enc}}}{R} = \text{const} \equiv \frac{-R_0^2k}{2}. \tag{33}$$

The first term on the left hand side is the kinetic energy per mass, and the second term on the left hand side is the Gravitational Potential Energy per unit mass. The constant we chose is well chosen in advance.

$$M_{\text{enc}} = \frac{4}{3}\pi\rho(t)R(t)^3 \tag{34}$$

and we can let $R(t) = a(t)R_0$ filling in we have:

$$\frac{1}{2}(\dot{a}(t)^2 R_0^2) - \frac{4G\pi\rho(t)a(t)^2 R_0^2}{3} = \text{const} \equiv \frac{-R_0^2 k}{2}. \tag{35}$$

finally we have

$$\boxed{(\frac{\dot{a}}{a})^2 = \frac{8}{3}\pi G\rho - \frac{\kappa}{a^2}} \tag{36}$$

This is the first Friedman equation!

3.1.2 Getting the second Friedman Equation

To get the second Friedman equation we appeal to thermodynamics. The first law of thermodynamics says:

$$dE = dQ - pdV \tag{37}$$

where dE is the change in energy of the system, dQ is heat added to the system, and $-pdV$ is the work done by the system on its surroundings. Due to our assumption of the cosmology principle $dQ = 0$ as if any small parcel of gas was non-zero it would be special.

We can write $dE = pdV$, but we can also write $dE = d(\rho a^3)$ as this is what is changing where ρ is the energy density. This is the co-moving volume so $v = a^3$.

Rewriting we have:

$$a^3 d\rho + 3a^2\rho da = -3a^2 p da \tag{38}$$

rearranging we have:

$$\boxed{\dot{\rho} = -3(\rho + p)(\frac{\dot{a}}{a})} \tag{39}$$

Differentiating the first Frieddman equation we have $2a\dot{a} = \frac{8}{3}\pi G(2a\dot{a}\rho + a^2\dot{\rho})$ we use this and wearrange:

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G(\rho + 3p)} \tag{40}$$

3.2 The Parameters of the Friedman equations

We have derived the equations that govern the dynamics of $a(t)$, and they depend on the stuff in the universe ρ, p etc.

Here are the common parameters.

$$p = w\rho \tag{41}$$

w is known as the equation of state parameter and is dimensionless, this works because ρ and p have the same units, its similar to an ideal gas.

plugging this into the equation for $\dot{\rho}$ we have:

$$\dot{\rho} = -3(1 + w)\rho(\frac{\dot{a}}{a}) \Rightarrow \boxed{\rho \sim a^{-3(1+w)}} \tag{42}$$

In a flat universe $k = 0$, we have

$$\boxed{(\frac{\ddot{a}}{a})^2 \propto a^{-3(1+w)} \Rightarrow a(t) \propto t^{\frac{2}{3(1+w)}}} \tag{43}$$

3.2.1 Normal Matter

For normal matter , and "dust" $w = 0$. If $k_bT << mc^2$ then the rest mass energy dominates, and we can safely neglect pressure.

For $p = 0$, $w = 0$, $\rho_m \sim a^{-3}$.

3.2.2 Vacuum Energy

Vacuum Energy $w = -1$: The energy of a vacuum just comes from the vacumm... the density is a const

$$\rho_\Lambda = \text{const} \Rightarrow w = -1 \tag{44}$$

from $\rho \sim a^{-3(1+w)}$ so $a \sim 1$ when $w = -1$.

in this case $a(t) \propto e^{H_0 t}$. This is an accelerated expansion.

w needs to be $w < -\frac{1}{3}$ for an accelerated expansion.

3.2.3 An Empty Universe

Here $\rho = 0$, and $(\frac{\dot{a}}{a})^2 = -\frac{1}{2}k \Rightarrow \dot{a} = \text{const}$

aka

$$a(t) \propto t \tag{45}$$

3.2.4 Light Domination or Radiation

In this case $w = \frac{1}{3}$ this comes from thermodynamics.

$P = \frac{4\sigma}{3c}T^4$ and $\rho = 4\sigma T^4$. so $w = \frac{1}{3}$ From the Stefan-Boltzman equations. This assumes thermal equilibrium

in this case

$$\boxed{a \propto t^{1/2}} \tag{46}$$

$$\boxed{\rho_r \propto a^4} \tag{47}$$

Another way to think of $\rho_r \propto a^4$ this is you get the the a^{-3} from the normal volume dilution, and an extra a^{-1} from $E = \frac{hc}{\lambda}$ and λ is changing as well due to redshift.

3.2.5 de sitter space

de sitter space: An universe with constant energy density per space

3.3 Subtle Parts to Highlight

3.3.1 CMB in thermal equilibrium

When talking about radiation, we assumed it was in thermal equilibrium so we could invoke the Stefan-Boltzman equation.

The CMB looks like it is in thermal equilibrium but it is not, but it still follows the Black-Body radiation curve so its alright, but with a modified temperature $\boxed{T_f = \frac{a_i}{a_f} T_i}$.

$$T \propto \frac{1}{a} \tag{48}$$

3.3.2 Normal Matter is pressure-less

We saw that Galaxies have an effective pressure — from the peculiar motions of the stars, even if it is not from the thermal motion. Does normal matter not have a pressure?

The interesting thing about peculiar motions is that they decay. As the universe expands their peculiar velocities get damped to 0.

4 Cosmological Parameters & Distances

From Sec. 3 we know how the universe evolves for single-component cases, but in reality everything is mixed!

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{\text{radiation},0}\left(\frac{1}{a^4}\right) + \frac{8\pi G}{3}\rho_{\text{mass},0}\left(\frac{1}{a^3}\right) - \frac{\kappa}{a^2} + \frac{8\pi G}{3}\rho_{\text{vacuum}} \tag{49}$$

Subscript zero means "value today."

In reality it is a bit messy, but we can make it easier by neglecting the terms that do not dominate. Our universe evolves from **radiation**-dominated to **matter**-dominated to **curvature**-dominated (although our universe appears to have $\kappa = 0$), and finally to **dark energy**-dominated.

> **de Sitter Space**: A universe with constant energy density per space; it is a good approximation to the universe of today, but in reality $\rho_m \neq 0$.

4.1 Ratios: Big Ω

We can write the mixed-component universe today as:

$$H_0^2 = \frac{8\pi G}{3}\left(\rho_{r,0} + \rho_{m,0} + \rho_\Lambda\right) - \frac{\kappa}{a_0^2}. \tag{50}$$

Recall that the subscript 0 means "evaluated today." Next we divide by H_0^2:

$$1 = \frac{8\pi G}{3H_0^2}\rho_{r,0} + \frac{8\pi G}{3H_0^2}\rho_{m,0} + \frac{8\pi G}{3H_0^2}\rho_\Lambda - \frac{\kappa}{H_0^2 a_0^2} \tag{51}$$

$$1 = \Omega_{r,0} + \Omega_{m,0} + \Omega_{\Lambda,0} + \Omega_{k,0} \tag{52}$$

The value with physical meaning for mass is $\rho_{m,0} \propto \Omega_{m,0}H_0^2$. This is a sum rule that must add up to one; in other words, each term is expressed as a ratio to the critical density.

> **Critical Density**: The critical density is
>
> $$\rho_{\text{crit}} = \frac{3H_0^2}{8\pi G}, \tag{53}$$
>
> and is approximately one atom per cubic meter. In general, for the ith component of matter or energy, we have:
>
> $$\Omega_{i,0} = \frac{\rho_{i,0}}{\rho_{\text{crit}}}. \tag{54}$$
>
> The critical density is interesting because if we add up the different sources of energy density in the universe and we get ρ_{crit}, then $\Omega_k = 0$, implying a spatially flat universe ($k = 0$). Otherwise, the universe is open or closed depending on the sign of Ω_k.

We can evaluate the Ωs for non-present times by looking at how the energy density dilutes with scale factor a. For example, note how $\rho_{\text{radiation}} = \rho_{\text{radiation},0}/a^4$, so $\Omega_\gamma = \Omega_{\gamma,0}/a^4$. You might see Ω_{tot}, which means everything except the $\Omega_{k,0}$ term.

4.2 Parameters of Our Universe

Here is what we currently believe the values are, mostly from the *Planck* satellite:

Parameter	**Symbol**	**Current Beliefs**
Spatial Curvature	$\Omega_{k,0}$	0.001 ± 0.002
Matter Density	$\Omega_{m,0}\, h^2$	0.14240 ± 0.00087
Baryon Density	$\Omega_{b,0}\, h^2$	0.02242 ± 0.00014
Hubble Constant	H_0	(67.9 ± 0.7) km s^{-1} Mpc^{-1}
Amplitude of Matter Fluctuations	A_S	$(2.10 \pm 0.03) \times 10^{-9}$
Scalar Spectral Index	n_S	0.966 ± 0.005
CMB Optical Depth	τ	0.0561 ± 0.0071
CMB Temperature	$T_{\text{CMB},0}$	2.7255 ± 0.0006 K
Dark Energy Equation of State	w	-1.04 ± 0.06

Notes from this table:

- Our universe is flat
- The value with physical meaning for mass is $\rho_{m,0} \propto \Omega_{m,0} H_0^2$.
- Ω_m includes all types of matter including cold dark matter
- Ω_b (Baryonic matter) is $\sim \frac{1}{6}$ or $\sim \frac{1}{7}$ Ω_m.
- these parameters describe the lumpiness of the matter distribution
- small, but non-zero prob that a CMB photon is scattered on the way to our telescope

Things We Haven't Detected Yet

Parameter	**Symbol**	**Current Beliefs**
Sum of Neutrino Masses	$\sum m_\nu$	< 0.12 eV (95% credibility)
Effective Number of Neutrino Species	N_{eff}	$2.99^{+0.34}_{-0.33}$ (95% credibility)
Tensor-to-Scalar Ratio	r	< 0.106 (95% credibility)

Neutrino masses affect cosmology and can be constrained through observations of the cosmic microwave background and large-scale structure. Gravitational waves can also leave imprints on the CMB polarization.

4.3 Notes from the parameters

- Our universe is flat
- The value with physical meaning for mass is $\rho_{m,0} \propto \Omega_{m,0} H_0^2$.
- Ω_m includes all types of matter including cold dark matter
- Ω_b (Baryonic matter) is $\sim \frac{1}{6}$ or $\sim \frac{1}{7}$ Ω_m.
- these parameters describe the lumpiness of the matter distribution
- small, but non-zero prob that a CMB photon is scattered on the way to our telescope

4.3.1 A digression on ΛCDM

Sometimes people look at matter density, baryon density, the hubble constant, A_S, n_S, τ and say "Our universe is well-fit by a 6-parameter ΛCDM model". Λ is the vacuum energy (what we have been investigating with $w = -1.$)

We have freedom to choose the 6 parameters. For example, where is Λ? Where are those 6 parameters? Well, we can choose them.

$1 = \Omega_{m,0} + \Omega_\Lambda + \Omega_k$

$\Omega_\Lambda = 1 - \Omega_{m,0}$ $(Omega_k = 0)$

$\Omega_\Lambda = 1 - \frac{\Omega_{m,0} h_0^2}{h_0^2}$

Additionally, why 6 parameters? It's only 6 parameters because we chose them. For example, we assumed $\Omega_k = 0$, or might assume a specific value of T_{CMB} because we have measured it well. You may go beyond vanilla ΛCDM to add add more parameters. For example, you may assume a nonzero neutrino mass now that we know it from experiments.

For example, changing neutrino mass might affect how much they get caught by black holes, and therefore change black hole growth predictions slightly.

4.3.2 Parameters we can derive

CMB Photon Density:
$\rho_\gamma = 4\sigma T^4_{CMB,0}$
$\Omega_{\gamma,0} h^2 = 2.47 \cdot 10^{-5}$

Neutrino Density: $\Omega_{\nu,0} h^2 = 1.68 \cdot 10^{-5}$

Radiation Density: CMB photon density + neutrino density
$\Omega_r = \Omega_\gamma + \Omega_\nu$
$\Omega_{r,0} = 8.5 * 10^{-5}$

This is small today, but when a is small in the early universe, $1/a^4$ wins over $1/a^3$.

So at one point early on, the radiation density was higher than matter density. We can work out the redshift when those densities were equal, z_{eq} recall that $1 + z = 1/a$)

4.3.3 The age of our Universe

We have to take into account the evolution of each component.

$$H^2(z) = \frac{8\pi G}{3}(\rho_m(1+z)^3 + \rho_r(1+z)^4 + \rho_\Lambda - \kappa(1+z)^2) \tag{55}$$

$$H^2(z) = \frac{8\pi G}{3}(H_0^2\Omega_m(1+z)^3 + H_0^2\Omega_r(1+z)^4 + H_0^2\Omega_\Lambda - \kappa(1+z)^2) \tag{56}$$

Do to the evolution of the universe the approx $t \sim 1/H_0$ works better then we might naively expect.

4.4 Distances

4.4.1 Comoving Distance

We see a thing at redshift z. How far is it now?

Use the FLRW metric

$$ds^2 = -dt^2 + a^2 dr^2/(1 - \kappa r^2 \tag{57}$$

From relativity photons travel on paths $ds^2 = 0$

$$a^2 dr^2/(1 - \kappa r^2) = dt^2 \tag{58}$$

Assume $\kappa = 0$

$$dr = dt/a(t) \tag{59}$$

and integrate

$$r = \int \frac{dt}{a(t)} = \int \frac{da}{a^2(t)H(z)} = -\int_0^z \frac{dz'}{H(z')} \tag{60}$$

this is the same as

Note that times in cosmology are proportional to c/H_0 and times seem to be proportional to cH_0

4.4.2 Luminosity Distance

One way we might define a distance is the luminosity Distance:

> **luminosity Distance**: We know the flux follows an inverse square law: $f = \frac{L}{4\pi d_L^2}$ If we know the observed flux F and the intrinsic luminosity L, the luminosity distance d_L is the distance that makes this true.

What is the value of dL as a function of z?

This is tricky for 3 reasons:

1. As photons spread out from a source, they spread out over an area $A = 4\pi S_\kappa(\chi)^2$, where that S_κ is from the FLRW metric's second form (if flat universe, $= R_0\chi$) 2. Photons redshift. they loose energy by a factor of $(1 + z)$ when received. 3. Distances stretch by a factor of (1+z) photons take longer to arrive.

(skipped)

5 Principles of Early Universe Thermodynamics

To an excellent approximation, we have so far only only dealt with with **gravity** — we have asked how different forms of energy density **gravitate** and effect our universe's expansion.

But particles have interesting interactions of their own — especially in the hot cauldron of the early universe!

In the early universe, particles and antiparticles were in thermal equilibrium, constantly being created and annihilated through various fundamental interactions. Some representative processes include:

$$e^+ + e^- \rightleftharpoons 2\gamma \qquad \text{(Annihilation / Pair Production)}$$

$$n \to p + e^- + \bar{\nu}_e \qquad \text{(Beta Decay)}$$

$$e^+ + e^- \rightleftharpoons \nu_e + \bar{\nu}_e \qquad \text{(Weak Nuclear Interaction)}$$

$$\mathrm{DM} + \mathrm{DM} \rightleftharpoons X + X \qquad \text{(Dark Matter Annihilation)}$$

Here, X represents standard model particles (excluding photons), since dark matter does not couple directly to light.

The early universe is **radiation dominate**d so recall

$$\rho_r \propto T^4 \tag{61}$$

$$H = \left(\frac{8\pi\rho_r}{3m_{pl}^2}\right)^{1/2} \propto T^2 \tag{62}$$

$$a(t) \propto t^{1/2} \tag{63}$$

$$t \propto T^{-2} \tag{64}$$

Recall that in Natural units, $G = \frac{1}{m_{pl}^2}$. Temperature T is another way to keep time. This is handy because particles have reaction rates that depend on temperature. The relationship for Temp, and a is consistent with a Black-body radiation curve which we defined earlier.

5.1 Basic Principles

1. Phase Space Density Is No Longer Collisionless

We are no longer dealing with a "gas" of stars, but a real gas of particles in a high-energy environment. We will be using momentum instead of velocity for the phase space — which is more natural for relativistic particles.

From quantum statistical mechanics we have that each state / particle takes up a volume of h^3. In natural units $\hbar = 1$ so $h = 2\pi$.

Putting this together we have:

$$\frac{d^6 N}{d^3p d^3x} = \frac{g}{(2\pi)^3} f(\vec{p}, t) \tag{65}$$

g here is called the **degeneracy factor** to account for the internal degrees of freedom of a particles, like spin which allow for multiple particles to occupy the same position-momentum state and still be in a different quantum state.

Recall the phase-space current equation 3 defined as

$$\frac{1}{dt} = \frac{\partial}{\partial t} + \vec{v}\frac{d}{d\vec{x}} + \frac{d\vec{p}}{dt}\frac{d}{d\vec{p}} \tag{66}$$

and we said that $\dot{f} = 0$ — that says that as we follow some particles along a phase-space trajectory that the density f is preserved. This means we had a **collisionless** system with only smooth evolution. **No particles were created or destroyed**.

Recall also the 0th moment – the continuity equation.

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\langle\vec{v}\rangle) = 0 \Rightarrow \frac{\partial n}{\partial t} = 0 \tag{67}$$

In our case, $\langle\vec{v}\rangle$ is 0 due to the cosmological principle.

Problem This gives $n =$ constant — appropriate for a static universe, but irrational for an expanding universe.

The issue is our time derivative equation! There is a correction for GR which is hard... the easier solution is to say

$$na^3 = \text{constant} \tag{68}$$

If no particles are created or destroyed then the particle density is proportional to scale volume. This implies:

$$\frac{d}{dt}(na^3) = 0 = a^3\frac{dn}{dt} + n3a^2\frac{da}{dt} \Rightarrow \dot{n} + 3Hn = 0 \tag{69}$$

Compared to a static universe / before — there is an extra Hubble dilution term.

2. There is another issue — Particles are created, and destroyed.

$$\dot{n} + 3Hn = \text{Reaction Terms} \tag{70}$$

We apply quantum statistical mechanics, if in thermal equilibrium.

$$f(p) = \frac{1}{\exp[\frac{E-\mu}{T}] \pm 1} \tag{71}$$

Positive for the Fermi-Dirac distribution used for Fermions, Negative for Bose-Einstein distribution for Bosons. The Classical Boltzmann distribution is when the 1 goes to 0. It gives the occupation number the average number of particles per quantum state of momentum in thermal equilibrium for systems of indistinguishable particles obeying quantum statistics.

E is the total relativistic energy $E = \sqrt{p^2 + m^2}$, and μ is the chemical potential.

Recall that:

$$du = Tds + \mu dN - PdV \tag{72}$$

Holding s, and v constant, $\mu = (\frac{du}{dN})_{s,v}$ — the chemical potential is the change in energy of a system to add a particle.

Photons automatically have $\mu_\gamma = 0$ because photons can be created for free without changing the energy of the system.

$$e^- + p^+ \rightleftharpoons e^- + p + +\gamma \tag{73}$$

.

A lot of the time we can set $\mu \approx 0$ at early times when $\mu << T$.
Once we have f the prescription is the same as before.
Number density is given by:

$$n = \frac{g}{(2\pi)^3} \int f(\vec{p}) d^3p \tag{74}$$

and the energy density is given by

$$\rho = \frac{g}{(2\pi)^3} \int f(\vec{p}) E d^3p \tag{75}$$

Recall that $E = \sqrt{p^2 + m^2}$.

5.1.1 Relativistic and non-Relativistic particles

Particles can be relativistic sometimes and non-relativistic at other times.

If a particle is in thermal equilibrium in the hot particle soup of the early universe then at early times when $T >> M$ it will be relativistic. At later times when the universe has cooled more $T << M$, and it is now non-relativistic. Remember these equations have a hidden factor of k_b and mc^2.

There is another important effect that happens as T drops consider something like:

$$e^+e^- \rightleftharpoons 2\gamma \tag{76}$$

When $T >> M$ it is easy to do the backward reaction (pair production), and when $T << M$ the forward reaction (annihilation) dominates and we "lose" the species.

$\boxed{\text{Particle species not only become less relativistic as time goes on, but we lose species.}}$

In the relativistic case we have:
Number density is given by:

$$n = \frac{g}{(2\pi)^3} \int_0^\infty f(\vec{p}) d^3p = \frac{gT^3}{(2\pi)^3} \int_0^\infty \frac{y^2}{e^y \pm 1} dy \tag{77}$$

and the energy density is given by

$$\rho = \frac{g}{(2\pi)^3} \int_0^\infty f(\vec{p}) E d^3p = \frac{gT^4}{(2\pi)^3} \int_0^\infty \frac{y^3}{e^y \pm 1} dy \tag{78}$$

with $T >> m$, and $y = p/T$. These integrals can be solved analytically.
The generalized Stefan-Boltzman law becomes:

$$\rho = \frac{\pi^2 g T^4}{30} \begin{cases} 1, & \text{for bosons} \\ \frac{7}{8}, & \text{for fermions} \end{cases} \tag{79}$$

This is for one particle species, for multiple types of particles we have:

$$\rho_r = \sum_i \rho_i = \frac{\pi^2 g_* T^4}{30} \tag{80}$$

where g_* is given by:

$$= \sum_{\text{Bosons}} g_i (\frac{T_i}{T})^4 + \sum_{\text{Fermions}} (\frac{7}{8}) g_i (\frac{T_i}{T})^4 \tag{81}$$

g_* lets us account for all relativistic species without doing each one by hand — its an effective number. We average Fermions, and Bosons and each species can be at a different temperature. At some point some of them may fall out of equilibrium.

We can similarly do the integral for n:

$$n = \frac{\zeta(3)\, g T^3}{\pi^2} \begin{cases} 1, & \text{for bosons} \\ \frac{3}{4}, & \text{for fermions} \end{cases} \tag{82}$$

Other useful quantities are the entropy S, and the entropy density $s = \frac{S}{V}$.

From thermodynamics:

$$s = \frac{S}{V} = \frac{\rho + P}{T} \tag{83}$$

This P is pressure! Not momentum. For relativistic particles $P = \frac{1}{3}\rho$ recall that $w_r = \frac{1}{3}$.

thus:

$$s = \sum_i \frac{\rho_i + P_i}{T_i} = \frac{4}{3} \sum_i \frac{\rho_i}{T_i} = \frac{2\pi^2}{45} g_{*s} T^3 \tag{84}$$

where g_{*s} is:

$$g_{*s} = \sum_{\text{Bosons}} g_i (\frac{T_i}{T})^3 + \sum_{\text{Fermions}} (\frac{7}{8}) g_i (\frac{T_i}{T})^3 \tag{85}$$

Because $S = sV$ is conserved, $g_{*s}(T) T^3 a^3 = \text{constant}$. the $a^3 \propto V$, and $g_{*s}(T)T^3 \propto S$. We will use this to figure out how the temperature changes as species fall out of equilibrium.

Another useful quantity is the number density to entropy density ratio for a species.

$$Y_i = \frac{n_i}{s_i} \tag{86}$$

Since $s \propto a^{-3}$, $Y_i \propto n_i a^3$ — it is the measure of the number of particles of a species i in a **comoving volume**. If no particles for this species are created or destroyed Y_i is a constant.

6 Relics

We are still in the early universe, where radiation dominates.

Particles can have interactions that are "on" or "off" at different times.

Interactions are how species maintain thermal-equilibrium, and are important for equations like $\dot{n} + 3Hn =$ (interactions).

Suppose we have a sea of particles and we ask how likely it is to interact with a particle — this leads into the general arguments for cross sectional area, and mean free path. Think of a column of particles of cross sectional area A, and density n. The target particle has a cross-sectional area ρ. To get a flux of particles through the cylinder we can multiply nv where v is the velocity of the particles. nv has units of $\frac{\#}{\text{area}\times\text{time}}$. To convert this flux into a rate of hits we multiply by the cross sectional area ρ of the target.

$$\Gamma = n\langle\sigma v\rangle \tag{87}$$

The angle brackets represent a thermal average.

If a reaction rate is high then it is important! We can compare it to H the cosmological parameter that has units of inverse time. You can also think of H as the time it takes for particles to escape each other due to the expansion of the universe. If we take the inverse it is a comparison of the age of the universe at that time, vs the time per interaction.

Keep in mind that H is itself time-dependent, so the reaction rates become important or less important with time as well.

Relic Abundance: Relic abundance depends on when interactions stop, and weather a particle was relativistic or non-relativistic at the time. By relic abundance, we mean how many particles are left over when annihilation stops.

When this happens is, and if a particle was relativistic or not is important. This is also known as **freeze-out.**.

6.1 Hot Relics

Hot Relics: Particles that froze-out while relativistic — meaning that Γ dropped below H, while $T >> m$.

Relativistic particles tend to be quite abundant just before freeze-out, the interactions are still "on", so the particles are in thermal-equilibrium with the rest of the universe. This means that we can use the thermal equilibrium equation from the last section.

$$n \propto gT^3 \tag{88}$$

Annihilation is happening, but there are enough reverse process reactions (pair production) that the particles are replenished.

At some point Γ drops below H and the interactions stop — so the number of particles na^3 becomes a constant.

Explicitly:

$$n_{\text{freezeout}} = \frac{\zeta(3)\, g\, T_{\text{freezeout}}^3}{\pi^2} \times \begin{cases} 1, & \text{for Bosons} \\ \frac{3}{4}, & \text{for Fermions} \end{cases} \tag{89}$$

Then n is:

$$n = n_{\text{freezeout}}(\frac{a_{\text{freezeout}}}{a})^3 \tag{90}$$

Recalling that $T \sim \frac{1}{a}$

$$n = n_{\text{freezeout}}(\frac{T}{T_{\text{freezeout}}})^3 \tag{91}$$

After freeze-out, the particle species no longer in thermal-equilibrium with the rest of universe, so the T is not the same. This is known as being **decoupled** .

6.2 Cold Relics

Cold Relics: A Cold Relics is when freeze-out - $\Gamma < H$ happens when the species is **non-relativistic** — $T_{\text{freezeout}} << m$. For cold relics plots of $Y = \frac{n}{s}$ — the number density to entropy density vs $\frac{m}{T}$.

If a species of particle is still in thermal-equilibrium when it becomes non-relativistic we can use the non-relativistic limit of the quantum statistical mechanics equation. We still generally assume $\mu = 0$.

$$n = g(\frac{mT}{2\pi})^{3/2} \exp[-\frac{m}{T}] \tag{92}$$

and

$$\rho \approx mn \tag{93}$$

The energy density is dominated by the rest mass. In this cold-regime annihilation dominates, and the **abundance is exponentially suppressed**. At some point $\Gamma << H$ and the interactions cease so the particles do not get destroyed anymore. At what Y that interactions stop depends on $\langle \sigma v \rangle$.

The rate Γ is for n particles hitting a single target, but there are really n possible targets — the collision rate is $\propto n^2$.

Writing out the equation:

$$\dot{n} + 3Hn = -n^2\langle \sigma v \rangle + q\langle \sigma v \rangle \tag{94}$$

The first bracket term on the RHS is the depletion of species x, and the second is the replenishing of the species with an unknown factor of q. The second term does not depend on n.

We do not know what q is, but we know in thermal-equilibrium that the LHS $= 0$, then we can say:

$$0 - n_{eq}^2\langle \sigma v \rangle + q\langle \sigma v \rangle \Rightarrow q = n_{eq}^2 \tag{95}$$

and therefore:

$$\dot{n} + 3Hn = (n_{eq}^2 - n^2)\langle \sigma v \rangle \tag{96}$$

This is known as the Ricatti differential equation. Which is solved by doing the following substitions.

$$Y = \frac{n}{s}, \tag{97}$$

$$x = \frac{m}{T}, \tag{98}$$

$$\lambda = \frac{2\pi^2}{45} g_{*s} \frac{m^3 \langle \sigma v \rangle}{H(T = m)}. \tag{99}$$

Where the Hubble parameter is evaluated at the transition to non-relativistic.

$$\frac{dY}{dx} = -\frac{\lambda}{x}(Y^2 - Y_{eq}^2) \tag{100}$$

This cannot be solved analytically. But after freezeout we know that $Y >> Y_{eq}$ — as there is an exponetial drop off.

This gives us:

$$Y_\infty \approx \frac{x_{\text{freezeout}}}{\lambda} \tag{101}$$

Recall that $\lambda \propto \langle \sigma v \rangle$, we see that the higher the cross-section area for annihilation, the lower the relic abundance.

6.3 WIMP Dark Matter (Cold Relic Application)

Weakly Interacting Massive Particles (WIMP) is a theory of dark matter. Suppose it follows a relation like:

$$X + X \rightleftharpoons SM + SM \tag{102}$$

Where SM is a standard model particle. We know that the relic abundance is set by $\langle \sigma v \rangle$. We might ask the following if we set the relic abundance to be the observationally known $\Omega_{DM} \approx 5\Omega_b$ do we get something reasonable for $\langle \sigma v \rangle$, or something ridiculous? This is an Order-of-magnitude problem.

At freeze out $\Gamma = \langle \sigma v \rangle \sim H$ so we can say:

$$n_f \langle \sigma v \rangle \sim \frac{T_f^2}{m_{pl}} \tag{103}$$

The m_{pl} comes from the expression for H during radiation domination. The subscript f stands for at freeze-out.

We know that at matter-radiation equality (MRE) the energy density of matter and photons were equal.

$$\rho_m = \rho_r \Rightarrow m_{DM} n_{MRE} \approx \frac{\pi^2 g_*}{30} T^4 \tag{104}$$

The left hand side after the arrow is the DM number density at MRE, and the pre-factor on the RHS is ≈ 1, $g_* \approx 3$.

$$n_{MRE} \approx \frac{T^4}{m_{DM}} \tag{105}$$

Since freezout, n dilutes as a^{-3}.

(skipped)

7 Neutrinos and Dark Matter

Last time there was an example for Cold Relics — the hypothetical WIMP dark matter, this time lets start with a Hot relic example

7.1 Neutrinos (A Hot Relic Application)

At $t \lesssim 1\,\mathrm{s}$, $T \gtrsim 1\,\mathrm{MeV}$ ($10^{10}k$) neutrino interactions are active, here we use neutrino to refer to neutrinos and anti-neutrinos.

$$e^- + \nu_e \rightleftharpoons e^- + \nu_e \tag{106}$$
$$e^+ + e^- \rightleftharpoons \nu_e + \bar{\nu}_e \tag{107}$$
$$...\mathrm{etc} \tag{108}$$

The bar represents a photon from an annihilation process. These types of processes are also felt by muons $\mu^\pm$ talking to muon neutrinos ν_μ, and $\bar{\nu}_\mu$. But $m_\mu \sim 105\,\mathrm{MeV}$, these reactions long since froze out / annilhated.

To figure out when the neutrino interactions turn off — and decouple from being in thermal equilibrium from everything else, we compare Γ and H. This means we need to know σ for these interactions.

We can use:

$$n = n_{\mathrm{freeze}}(\frac{a_{\mathrm{freezeout}}}{a})^3 \tag{109}$$

and recall that $a \sim \frac{1}{T}$.

$$\Gamma = \ n\langle\sigma v\rangle \approx T^3 G_f^2 T^2 = G_f^2 T^5 \tag{110}$$

$$\boxed{\Gamma \sim G^2 T^N} \tag{111}$$

This is because $v \approx c = 1$, and $n_\nu \propto T^3$ because it is relativistic. I do not know what G_f means but it is a G-fermi constant that comes from particle physics, and dimensional analysis gives us the power.

For a radiation dominated universe like we are in, $H \sim \frac{T^2}{\mathrm{Mpc}}$. Then we can write:

$$\frac{\Gamma}{H} = \frac{G_f^2 T^5}{\frac{T^2}{\mathrm{Mpc}}} \approx (\frac{T}{\mathrm{MeV}})^3 \tag{112}$$

This means that neutrinos decouple around $T = 1\,\mathrm{MeV}$. Once they decouple their number density goes down as the $n \sim a^{-3}$. Their distribution function is a snapshot of what it looked like at decoupling, but the T also goes down as $T \sim 1/a$. So

$$T_\nu(a) = T_{\nu,\mathrm{decouple}}(\frac{a_{\mathrm{decouple}}}{a}) \tag{113}$$

However if we evaluate $T\nu$ at $a = 1$ (Today) we do not get the CMB temp, because soon after neutrinos decuople, we have...

Electron-Positron Annihilation

Even though by $T \sim 1\,\mathrm{MeV}$ there are no more electron - neutrino interactions, there are still electron interactions because electrons are relativistic, and feel electro-magnetism (They react to make photons).

$m_e \approx 0.5\,\mathrm{MeV}$. so no long after neutrino decoupling they become non-relativistic and are exponentially suppressed as annihilation dominates. This causes their entropy to dumped into the photons — heating them up. Let's work out the new photon temperature.

recall the entropy density s

$$s = \frac{2\pi^2}{45} g_{*,s} T^3 \tag{114}$$

and that $sa^3 \propto g_{*,s}a^3T^3$ is a conserved quantity — because total entropy is conserved. If $g_{*,s}$ chances due to positron - electron annihilation, then T must change.

Right before they annihilate we can write $g_{*,s}$, and compare it to after, they are at the same temperature for this. Neutrinos already decoupled so its only the electrons, positrons, and photons that we count.

Before $g_{*,s} = \frac{11}{2}$, and after we have $g_{*,s} = 2$

Therefore we can write:

$$T_{\gamma,\text{after}} = (\frac{11}{4})^{1/3} T_{\gamma,\text{before}} \tag{115}$$

Then we can relate this back to the neutrino temperature, The neutrino T_ν was scaling at the same rate as T_γ before the annihilation of positrons, and electrons, and $T_\nu = T_{\gamma,\text{before}}$. And $T_\nu \sim 1.96k$, as $T_{\gamma,\text{after}} = 2.75k$

Entropy conservation is actually:

$$T_\gamma \propto \frac{1}{g_{*,s}^{1/3} a} \tag{116}$$

This is important when $g_{*,s}$ changes! which happens in the early universe.

For neutrinos — we have a hot relic that still exists today, and have figured out its background temperature today. We know that there are three types of neutrino, and their anti-particle represented by a bar.

(skipped) After electron - positron annilhation the effective number of degrees of freedom aka $g_* \approx 3$

We typically fit for the number of degrees of freedom in cosmological models to determine if there are unknown particles.

We also know that neutrinos have mass, but not exactly what it is, we have an upper limit.

(skipped)

Roughly speaking, Neutrinos are light and travel close to the speed of light — and do not interact much. They stream out of over-densities in the universe inhibiting large structure formations.

7.2 Dark Matter

So far we have discussed two examples of relics, neutrinos (hot relics), and WIMPs (cold relics) — could either be Dark Matter?

neutrino relics cannot be dark matter, they would wash out too much dark matter and have other issues (see hw 7)

We know this about Dark Matter

- Ω_m, Ω_b imply a limit on Ω_{DM}
- Locally $\rho_{DM} \sim 0.4 \pm 0.1\,\mathrm{GeV/cm^3}$
- Should be stable $\Gamma >> H_0^{-1}$

- upper bounds on interaction rates
- upper bound on DM velocity streaming (Like neutrinos)

These combine to give us some interesting constraints on m_{DM} the mass of a dark matter particle, and upper and lower bound on mass. This gives us about 90 orders of magnitude on the dark matter particle mass.

8 Big Bang Nuclear Synthesis (BBN)

So far we have talked about some of the key events that have happened in our universe at $t \sim 1\,\mathrm{s}$, $T \sim$ 1MeV, and $T \sim 10^{10}k$

We now press on with the story and talk about **Big Bang Nuclear Synthesis (BBN)**. This is where the hot cauldron of the universe makes the first (stable) atoms, and produce a lot of ^{4}He, and small amounts of elements up to ^{7}Li.

Context. BBN begins around $T \sim 0.1\,\mathrm{MeV}$, so not long after e^+/e^- annihilation. At this point $g_* \approx 3.38$ (photons and neutrinos), which means that $t \approx 132\mathrm{s}(\frac{0.1\,\mathrm{MeV}}{T})^2$. This means that BBN starts a few seconds to a few minutes after the big bang.

8.1 Neutrinos

Neutrons are the main character. The goal is to work out the elemental abundances, conceptually the most important thing is to keep in mind in BBN is to “follow the neutrinos”.

Neutrons are slightly more massive than protons $Q = m_n - m_p = 1.29\,\mathrm{MeV}$ this disfavors them. Neutrons and Protons can convert into each other via weak interactions

$$n + \nu_e \rightleftharpoons p + e^- \tag{117}$$
$$n + e^+ \rightleftharpoons p + \bar{\nu}_e \tag{118}$$

To make heavy nuclei, we need both neutrons and protons. A nucleus made of just protons or neutrons is unstable and decays. Here we consider an nucleus stable if it is long-lived compared to the age of the universe at this time — a few seconds to a few minutes.

- **Hydrogen Isotopes:**
  - Proton(s) $^1_1$H: stable
  - Deuterium $^2_1$H: stable
  - Tritium $^3_1$H: $t_{1/2} = 12\,\mathrm{yr}$
- **Helium Isotopes:**
  - Helium-3 $^3_2$He: stable
  - Helium-4 $^4_2$He: stable
- **Lithium Isotopes:**
  - Lithium-7 $^7_3$Li: stable
- **Beryllium Isotopes:**
  - Beryllium-7 $^7_4$Be: $t_{1/2} = 52\,\mathrm{days}$

Starting prior to BBN — say $T \gtrsim 10\,\mathrm{MeV}$, the neutrons and protons are in equilibrium via their interactions with the weak force. Recall that $m_p, m_n \sim 1\,\mathrm{GeV}$, and **they are non-relativistic** at this point. We use the non-relativistic limit of the equilibrium equations.

$$n_p = g_p \left(\frac{m_p T}{2\pi}\right)^{3/2} \exp\left[-\frac{m_p - \mu_p}{T}\right], \tag{119}$$

$$n_n = g_n \left(\frac{m_n T}{2\pi}\right)^{3/2} \exp\left[-\frac{m_n - \mu_n}{T}\right]. \tag{120}$$

We can then write the ratio of $\frac{n_n}{n_p}$

$$\frac{n_n}{n_p} = \frac{g_n \left(\frac{m_n T}{2\pi}\right)^{3/2} \exp\left[-\frac{m_n - \mu_n}{T}\right]}{g_p \left(\frac{m_p T}{2\pi}\right)^{3/2} \exp\left[-\frac{m_p - \mu_p}{T}\right]} = (\frac{m_n}{m_p})^{3/2} \exp -\frac{Q}{T} \tag{121}$$

Where $Q = m_n - m_p$. Informally the neutron and proton are like two states of a nucleon, with a Boltzmann factor governing the relative number between the two, IE this changes as a function of T.

We see that as the universe cools to $T \sim 1\,\text{MeV}$, and then $T \sim 0.1\,\text{MeV}$ that there are more protons than neutrons. Like all the interactions we have studied so far, this competes with H.

Neutron freeze-out happens at around $T \sim 0.8\,\text{MeV}$. and results in $\frac{n_n}{n_p} = 0.2$ at freeze out. So when the interactions stop, only $\sim 1/6$ nucleons are neutrinos, but worse free neutrons are not stable and they decay via **Beta Decay**

$$n \to p + e^- + \bar{\nu}_e. \tag{122}$$

This has a decay lifetime of $\tau_n \sim 15\,\text{min}$, and **is** relevant to BBN. This number is not easy to study because neutrons are neutral, and is a key uncertainty in BBN codes.

Neutrons then determine the outcome of BBN, to a good approximation all neutrons end up in ^{4}He, and this is how we predict He abundances.

BBN is a race against time, all the nuclear reactions need to happen before the neutrons decay. Now we need to look at the nuclear reactions.

8.2 Steps in BNN

There are a handful of important reactions in BBN. Deuterium production is what kicks off BBN,

$$p + n \rightleftharpoons D + \gamma. \tag{123}$$

This gives a lightly bound nucleus with binding energy of $B_D = (m_n + m_p - m_p) = 2.22\,\text{MeV}$

Using similar non-relativistic equilibrium distribution math we have:

$$\frac{n_D}{n_p n_n} = \frac{g_D}{g_p g_n}(\frac{m_D}{m_p m_n})^{3/2}(\frac{T}{2\pi})^{-3/2} \exp(\frac{B_D}{T}) \tag{124}$$

We can get rid of the gs because $g_D = 3$ it is a spin 1 particle, and $g_p = g_n = 2$, and $m_n \approx m_p \approx \frac{m_d}{2}$.

This means we can write it as:

$$\frac{n_D}{n_p n_n} \approx 6(\frac{m_n T}{\pi})^{-3/2} \exp(\frac{B_D}{T}) \tag{125}$$

It is custom to write express this in terms of the **Baryon-to-photon ratio** η

$$\eta = \frac{n_B}{n_\gamma} = 6.12 \times 10^{-10} \tag{126}$$

It is tiny because matter, antimatter asymmetry is tiny.

After neutron freeze out, 5/6 of Baryons are protons therefore

$$n_p \approx 0.83 n_B \approx 0.83 \eta n_\gamma \approx 0.2 n_\gamma \eta T^3 \tag{127}$$

And we can write the $\frac{n_D}{n_n}$ as :

$$\frac{n_D}{n_n} \approx 6.7\eta(\frac{T}{m_N})^{3/2} \exp(\frac{B_D}{T}) \tag{128}$$

This ratio starts out $<< 1$ at the beginning of BBN where there is barely any D, it becomes $>> 1$ as BBN proceeds and the number of free neutrons n_n plummets as neutrons get incorporated into nuclei.

When does a significant amount of D get produced? This is basically the same as asking when ^{4}He is produced because the later reactions are very quick.

A reasonable metric is when $\frac{n_D}{n_n} \sim 1$, with our expressions this happens at $T_{\text{BBN}} \sim 0.07\,\text{MeV}$ and $t_{\text{BBN}} \sim 300\,\text{s}$ This gives us when T_{BBN} that we stated at the opening of this section.

This happens at later times / cooler temp then one would naively expect from looking at nuclear reactions with $B_D \sim 2\,\text{MeV}$. The reason is the **very small** η. There are so many photons around that the reverse reaction — breaking up of D is favoured.

A more detailed calculation gives $t_{\text{BBN}} \sim 200\,\text{s}$. With the time of BBN, we can figure out how many neutrons are around (after decays) to be bound up in nuclei. The Neutron fraction is given by:

$$X_n(t) = \frac{n_n}{n_p + n_n} = X_n(t_{\text{nfreeze}}) \exp(-\frac{t_{\text{BNN}}}{\tau_N}) \tag{129}$$

$$X_n(t) = (\frac{1}{6}) \exp(\frac{-200}{890}) \approx 0.13 \tag{130}$$

Again, to an excellent approximation all of this ends up in ^{4}He, and it is conventional to express this amount as a mass fraction.

The primordial fraction Y_P:

$$Y_p = \frac{\rho_{\text{He}}}{\rho_B} = \frac{m_{\text{He}} n_{\text{He}}}{m_H n_H + m_{\text{He}} n_{\text{He}}} \approx \frac{4 n_{\text{He}}}{n_H + 4 n_{\text{He}}} = \frac{4 n_{\text{He}}}{n_p + n_n} \tag{131}$$

We used $m_{\text{He}} \approx 4 m_{\text{H}}$. n_H = protons that remain free, and n_0 = total number of protons.

$n_{He} = \frac{n_n}{2}$ if $\approx$ all neutrons end up in ^{4}He.

$$n_{\text{He}} = \frac{2 n_n}{n_p + n_n} = 2 X_n \approx 0.26 \tag{132}$$

The heavier elements exist only in trace amounts, our universe is 75% Hydrogen, and 25% Helium. Hard to make heavier elements because it is not as dense as the core of stars.

8.3 Testing BBN

The only free parameter is η, a good test is to see if η is consistent based on the different elements abundances. This works extremely well.

You can also then use BBN as a probe of standard cosmology turning the problem around.

9 Recombination & Reionization

We have talked about various events in the early universe.

After BBN, we are left with a big soup of electrons, protons, Helium nuclei, and photons as far as the coupled thermodynamic soup goes.

What happens to this soup? That is the story for today, and the answer is not much for a while, and the some pivotal events.

- **Matter–radiation equality:**
  $z \sim 3250$, $t \approx 50{,}000\,\mathrm{yr}$, $T \approx 9000\,\mathrm{K}$.
- **Recombination** (protons and e^- form neutral hydrogen):
  $z \sim 1275$, $t \approx 290{,}000\,\mathrm{yr}$, $T \approx 3500\,\mathrm{K}$.
- **Decoupling and last scattering** (photons free to roam the universe):
  $z \sim 1090$, $t \approx 380{,}000\,\mathrm{yr}$, $T \approx 3000\,\mathrm{K}$.
- **Reionization** (first galaxies separate p and e^- again):
  $z \sim 7$ (uncertain by a factor of ~ 2), $t \approx 800\,\mathrm{Myr}$, $T \approx 25\,\mathrm{K}$.

The middle events are particular important because they give rise to the **CMB.**

9.1 Cartoon Of What Happens

Tight coupling, and decoupling. At $z >> 1000$, we have photons and baryons in a tightly coupled **photon-baryon** fluid.

This capacity is driven by **Thomson scattering** of photons off electrons:

$$\gamma + e^- \rightleftharpoons \gamma + e^- \tag{133}$$

The Thomson-cross section σ_t is :

$$\sigma_t = \frac{8\pi}{3}(\frac{\alpha\hbar}{m_e c})^2 = 6.65 \times 10^{-29}\,\mathrm{m}^2\,(\mathrm{SI\,units}) \tag{134}$$

the e^- then drag nuclei around via **Coulomb attraction** so it is all tightly coupled.

In this era, the average distance a photon travels between collisions — the mean free path λ is short:

$$\lambda = \frac{1}{n_e \sigma_t} \tag{135}$$

and the interaction rate is high:

$$\Gamma = c n_e \sigma_t \tag{136}$$

Here the photons travel at c.

We can define the ionization fraction as:

$$X_e = \frac{n_e}{n_b} = \frac{n_p}{n_b} \tag{137}$$

This is assuming net charge neutrality, and n_p is fro free protons, not the total. The ionization fraction goes from $X_e = 1$ early on to $X_e << 1$ later. With this definition we have:

$$n_e = n_B x_e = n_{B,0}(1+z)^3 x_e(z) \tag{138}$$

The z factor comes from the normal dilution of matter as a^{-3}, using this as our n_e we can put in for Γ

$$\Gamma = n_{B,0}(1+z)^3 x_e(z) c\sigma_t \approx \tag{139}$$

Now we do the usual thing and compare to H, we use the $H(z)$ equation, recall that we are no longer in the radiation dominated universe.

$$H = H_0\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4} \approx 10^{-18}(1+z)^{3/2}\,\mathrm{s}^{-1} \tag{140}$$

The Γ increases at $(1+z)^3$ which is steeper then H, so $\Gamma >> H$ at high redshift — where the interactions are "on", and they are "off" at low redshifts. It is crucial that we account or X_e as it will change by orders of magnitude.

The redshift of decoupling can be found by the usual $\Gamma(z) = H(z)$ at $z = z_{decouple}$.This gives

$$1 + z_{decouple} = \frac{38.3}{X_e(z_{decouple})^{2/3}} \tag{141}$$

This is X_e as a function of z! If Hydrogen just magically stayed ionized, decoupling would happen at latter times.

9.2 Recombination

This should really be called "combination" a its when neutral atoms where formed for the first time. It is called recombination for historical reasons.

This is the when the reaction:

$$H + \gamma \rightleftharpoons p + e^- \tag{142}$$

gets pushed to the left — more H atoms. To a good approximation, we can just ignore He, even thought it is 25% of the baryonic universe.H recombination happens at $z \lesssim 1$.

He recombines at an earlier z, conceptually you can think of the binding energy of a He nuclei is higher — it has 2 protons and can stick to electrons more easily.

To calculate the z of recombination for H, we use the equilibrium equation for non-relativistic particles:

$$n_i = g_i(\frac{m_i T}{2\pi})^{3/2} \exp -\frac{(m_i - \mu_i)}{T} \tag{143}$$

in the case of chemical equilbrium $\mu_H + \mu_\gamma = \mu_p + \mu_e$, $\mu_\gamma = 0$ always.

This gives:

$$\frac{n_H}{n_p n_e} = \frac{g_H}{g_p g_e}(\frac{m_H}{m_p m_e})^{3/2}(\frac{T}{2\pi})^{-3/2} \exp(\frac{m_p + m_e - m_H}{T}) \tag{144}$$

The binding energy $Q = m_p + m_e - m_H = 13.6\,\mathrm{eV}$, and using recalling that $m_p \approx m_h$, and $g_e = g_p = \frac{1}{2}g_H = 2$, we get the Saha equation:

$$\frac{n_H}{n_p n_e} = (m_e\frac{T}{2\pi})^{-3/2} \exp(\frac{Q}{T}) \tag{145}$$

The goal was to find the free electron fraction x_e, ignoring He we can write:

$$x_e = \frac{n_e}{n_B} = \frac{n_p}{n_B} = \frac{n_p}{n_p + n_H} \tag{146}$$

Which we can rewrite as:

$$\frac{1 - x_e}{x_e} = \frac{n_H}{n_p} = n_e(\frac{m_e T}{2\pi})^{-3/2} \exp(\frac{Q}{T}) \tag{147}$$

For n_e, we have $\eta = \frac{n_B}{n_\gamma} = \frac{n_e}{x_e n_\gamma}$. Plugging this in, we get:

$$\frac{1 - x_e}{x_e} = n_\gamma \eta (\frac{m_e T}{2\pi})^{-3/2} \exp(\frac{Q}{T}) \tag{148}$$

Finally, using the fact that $n_\gamma \propto T^3$ and plugging in numbers we have:

$$\frac{1 - x_e}{x_e} = 3.84\eta(\frac{T}{m_e})^{3/2} \exp(\frac{Q}{T}). \tag{149}$$

Since we know the CMB temperature today T_0, and we know $T = T_0(1 + z)$, we can solve this equation numerically for $x_e(z)$.

(skipped)

What the Saha equation gets right is that recombination is actually a **fairly drawn out process.** It doesn't have the final details right right, though, which needs to involve modelling all of the energy levels in H, and the different paths down to the ground state.

Now that we know $X_e(z)$, we can find out when $z_{\text{decoupling}}$ happens and the photons decouple. Plugging this into our earlier relations we get:

$$z_{\text{decouple}} \approx 1090 \tag{150}$$

At this point photons scatter one last time before free streaming to our telescopes. The range of z is $\Delta z \sim \pm 100$.

9.3 Reionization

About 5% of photons have been scattered since the CMB (before arriving in our telescopes with the CMB). This is because of **reionization — a pivotal event in our universe** when the first galaxies systematically ionized the neutral hydrogen, returning hydrogen in the IGM to an ionized state.

The probability of scattering is given by the optical depth τ which is dimensionless.

$$P \sim 1 - e^{-\tau} \approx \tau \tag{151}$$

We are in the limit where $\tau << 1$, to first order we can assume instantaneous z for reionization.

$$\tau = \int_{\tau_{reion}}^{\tau_0} \Gamma(t)\, dt = 0.054 \pm 0.008 \tag{152}$$

From Plank, and this is hard to measure. Another way to think of it is that $\Gamma = c n_e \sigma_t$, and τ is distance in units of the free path multiplying by dt gives us distance in units of mean free path as $n_e \sigma_t$ is the mean fee path, and cdt is the distance traveled.

Writing it in the integral way allows us to account for z being spread out as we can write $n_e(z)$ - you have to model the ionization history for a more accurate number.

This does not happen in equilibruim, and there is not a simple analytical form to do this as we have to take into account galaxies!

(skipped)

10 Inflation Part 1

At even higher redshifts now! This will lead into discussing universe with perturbations, up until now the interesting quantities have been a function of time / scale factor / redshift as opposed to position **r** this will change soon.

10.1 The Hot Big Bang Model

The Hot Big Bang model aka a review of cosmology until this section, We have a framework for describing the expansion of our universe given its contents (the first and second Friedmann equations).

With the right mix of components ρ, we can fit the data to our expansion history. Since each component scales differently with the scale factor a, different parts of the energy budget take turns dominating ρ: first radiation, then matter, and finally dark energy. Curvature domination could have occurred, but observations indicate that our universe has $k = 0$.

The main components are:

- **Radiation**: $\rho \propto \frac{1}{a^4}$, $a \propto t^{1/2}$, decelerating
- **Matter**: $\rho \propto \frac{1}{a^3}$, $a \propto t^{2/3}$, decelerating
- **Dark Energy**: $\rho = \text{const.}$, $a \propto e^{H_0 t}$, accelerating

With this in mind, we can take the known laws of nuclear physics and thermodynamics to explore the past, leading to two key predictions:

1. **Cosmic Microwave Background (CMB)**: There exists a CMB with a temperature of $T \sim 3\,\text{K}$ ($T \sim 10^{-13}\,\text{GeV}$) today.
2. **Big Bang Nucleosynthesis (BBN)**:This correctly predicts the primordial abundances of elements.

This framework is known as the **Hot Big Bang Model**. It is incredibly successful in the ways described above and works remarkably well up until $t \sim 1, \text{sec}$—the time when energies were at nuclear scales. Earlier than this, the universe was dominated by nuclear physics.

10.2 Issues with the Hot big Bang Model

However, despite its success, the Hot Big Bang model has some unresolved problems...

10.2.1 The Flatness Problem

The Friedman equations say:

$$H^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2}, \tag{153}$$

$$1 = \frac{8\pi G\rho}{3H^2} - \frac{k}{a^2H^2}, \tag{154}$$

$$1 = \Omega_{\text{Tot}} - \Omega_k. \tag{155}$$

Today, Ω_k is measured to be consistent with 0 (Planck 2018, Galaxy surveys). This is strange to see!

$|\Omega_k|$ evolves as:

$$|\Omega_k|_0 = |\Omega_k|_{\text{Earlier}} \frac{a_e^2 H_e^2}{a_0^2 H_0^2} \tag{156}$$

We can write $H \sim 1/t$, and we get

$$|\Omega_k|_0 = |\Omega_k|_{\text{Earlier}} \frac{a_e^2/t_e^2}{a_0^2/t_0^2} \tag{157}$$

To good approximation, we can write this as $\dot{a}$.

$$\boxed{|\Omega_k|_0 = |\Omega_k|_{\text{Earlier}} \frac{\dot{a}_e^2}{\dot{a}_0^2}} \tag{158}$$

For most of our universe history the expansion has been decelerating. So the rate of change of the scale factor (the expansion rate) today $\dot{a}_0$ is lower then $\dot{a}_e$ earlier. This means that $\frac{\dot{a}_e}{\dot{a}_0} > 1$, so is our universe is seen as flat today, it must have been **really** flat in earlier times. But how early are we talking?

In the early radiation dominate universe we have: $a \sim t^{1/2}$, $\dot{a} \sim t^{-1/2} = a^{-1}$. But we have $T \sim \frac{1}{a}$, so $\dot{a} \sim T$, lets plug in some numbers using the CMB temperature today, and the Grand Unified Theory scale.

$$\frac{\dot{a}_e^2}{\dot{a}_0^2} = \frac{T_e}{T_0} = 10^{56} \tag{159}$$

Today we can measure the universe is flat to within 10^{-3}, but this means it must have been flat to within 10^{-59} in the early universe!

This is incredibly **fine-tuned** which is not a sign of a good theory. We either want a theory that is robust to the exact values of parameters or a theory that explains why a parameter has such a fine-tuned value.

10.2.2 The Horizon Problem

Why is our universe so homogeneous? The CMB is incredibly uniform, the fluctuations of amplitude are 1 part in 10^5. Our theory of recombination (when the CMB was released) happened at around $t_{rec} \sim 400,000$ years. Today the universe is 14 Gyr. So CMB photons have basically been travelling towards us since the universe began. However, this raises some issues about thermal equilibrium photons from point A, and B on opposite sides of the universe were too far away to have influenced each-other. So how did they "know" to be at the same temperature?

Lets work with some numbers. First lets work out the radius of our observable universe, this is the total distance that light can travel in the whole age of our universe.

FRW Metric: $ds^2 = -c^2dt^2 + a^2(t)d\chi^2$, light travels on null gradients IE $ds^2 = 0$, therefore $\chi = \int \frac{c}{a(t)}dt$. This is the comoving distance! To get the proper distance we need to multiply by $a(t)$. The proper name for this is the **particle horizon**.

Particle Horizon:
The particle horizon is the maximum comoving distance from which light has had time to reach us since the Big Bang.

$$= a(t) \int_0^t \frac{c}{a(t)} dt \sim \frac{c}{H_0} \tag{160}$$

This approximation is accurate to factors of order unity today. At any time t, we can use this to figure out how big of a region could be in causal contact

The past light-cone (what we actually observe) is the observable universe, we can compare the particle horizon for points a and b, and the observable size of the universe. IE the ratio of the radius of the universe today scaled back to a_e, over the particle horizon at time t_e.

$$= \frac{(\frac{a_e}{a_0})\frac{c}{H_0}}{a_e \int_0^{t_e} \frac{c}{a(t)} dt} \tag{161}$$

Now in the various universe dynamics we have looked at, we have had $a(t) \sim t^{\gamma}$, with $0 < \gamma < 1$, this means we can write the denominator as $\sim ct_e$. Pluggin this in, we get

$$= \frac{(\frac{a_e}{a_0})\frac{c}{H_0}}{a_e \int_0^{t_e} \frac{c}{a(t)} dt} = \frac{(\frac{a_e}{a_0})\frac{1}{H_0}}{t_e} \sim \frac{(\frac{a_e}{a_0})t_0}{t_e} \sim \frac{\dot{a}_e}{\dot{a}_0} \tag{162}$$

$\frac{\dot{a}_e}{\dot{a}_0} = 10^{28}$, this is a big number. This means that point a, and b were not within eachothers particle horison, and thus not in thermal contact. As we go back further in time, we get more and more causal islands.

10.2.3 The Monopole Problem

Monopoles exist in many grand-unified theories (GUTs), at even earlier / hotter times. This is an issue because we do not see them! They should effect BBN. They are also massive particles, and could cause the universe to be matter dominated early.

Magnetic monopoles are not traditional particles, they are examples of **topological defects**. As the universe descends from higher energies to lower ones, it undergoes various phase transitions (fancy names are electro weak symmetry breaking).

We are going to compare this to crystallization, if different parts of something start to crystallize from seed crystals they do not "coordinate", and when they grow enough to meet they have defects along the boundaries. In our universe, it happens that we would some of these defects as magnetic monopoles. (There is only one **B** field, and it connects from the isolated casual islands). The boundaries depend on a^3!.

The number of magnetic monopoles is $\sim \dot{a}_e^3/\dot{a}_0$, and is a huge number! Just like before so not seeing them is bad.

10.3 The Solution - Inflation

We have a few issues with the Hot Big Bang model, and they all seem to arise from the decelerating universe IE that $\dot{a}_e/\dot{a}_0 >> 1$.

The inflation solution is to postulate that there was an **early** period of accelerated expansion.

It solves the **flatness problem** — our universe is flat today because everything we see today was close together on a very small patch IE we are zoomed very close into a manifold.

It solves the **Horizon / Homogeneous Problem** — inflation takes a small patch that was homogeneous and stretches it out.

It solves the **Monopole problem** — the monopole density has been diluted to a negligible density, we come from a patch that had $n_{\text{monopole}} << 1$.

This picture dramatically increases the variables of what we might think of our universe. In the pre-inflation universe, it can be inhomogeneous and lumpy, and look like anything. It could also have been around for a long time, when we say the universe is 14gyr old we really mean since our patch inflated. Inflation is the bang of the big bang.

10.4 Realizing Inflation

We cannot simply declare that the universe behaves like this, we need an mechanism to cause inflation. That's a topic for next time, but this mechanism needs to be able to:

- must produce accelerated expansion
- must go on for long enough $\sim e^{60}$ to solve the issues
- must end and connect back to radiation dominate universe around $t \sim 1\,\text{sec}$, that is were the traditional Big Bang model works well, a cosmological constant wont work as it does not end
- must repopulate the universe with particles, otherwise particles will go the way of magnetic monopoles

11 Inflation Part 2

Now lets come up with some inflationary mechanisms!

The Friedman equation says:

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} \tag{163}$$

In our case, We know what we want on the left hand side now we need to invent a good ρ for the right hand side to make it all work. We can probably neglect the k term as well because inflation will make it irrelevant.

11.1 What could it be?

11.1.1 Dark Energy

No. This is a constant ρ, which does fit the exponential growth required; however, there is nothing to turn it off. And it does not grow fast enough anyway.

11.2 Scaler Field

We can get something that works if we have inflation driven by a scaler field.

Scaler field: Some field $\phi(\mathbf{r}, t)$ that has a scaler value in every point in space $\mathbf{r}$, and time t. The Higgs field is an example of a scaler field. (Inflation is not the Higgs field tho)

.

We don't know what scaler field causes inflation, we simply postulate that one exists and call it the inflation field, but that is just a name. There is still a lot of freedom for the field to change the observables, it only has to satisfy our requirements for inflation.

Inflation is a paradigm, not a theory. It does not provide the details of the inflation field, you provide that and inflation tells you what the observable consequences are.

11.3 Scaler field dynamics

We are now going to see if a scaler field gives us the right expansion dynamics for inflation.

The key is to identify w, the equation of state parameter for the scaler field:

$$p = w\rho \tag{164}$$

because different w values give different solutions and hence $a(t)$ in the Friedman equation above.

For a scaler field ϕ, the stress-energy tensor $T_{\mu\nu}$ is:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)) \tag{165}$$

The V at the end is a potential energy term. The energy density, and pressure are different components that can be read off the stress-energy tensor.

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{166}$$

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{167}$$

In the second equation, it looks like a "Kinetic" Energy term and a "Potential" energy term. We can now find w by using $w = \frac{p}{\rho}$.

$$w = \frac{p}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \tag{168}$$

The form of the potential energy is where you can pretty much draw what you want and see what the observational consequences are there after inflation.

The value of ϕ can change with time. If we use the Friedman equation, we have:

We can always say that $H = \frac{\dot{a}}{a}$, and that $\dot{H} = \frac{\ddot{a}}{a} - H^2$.

Using the second Friedman equation we can write:

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3}(\rho + 3p) = \frac{-8\pi G}{3}(\dot{\phi}^2 - V) \tag{169}$$

Which means that $\dot{H}$ is:

$$\dot{H} = \frac{-8\pi G}{3}(\frac{3}{2}\dot{\phi}^2) \tag{170}$$

And then we can plug into the $2H\dot{H}$ equation to get:

$$\boxed{\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0} \tag{171}$$

This is the equation of of motion for ϕ. It tells us how ϕ evolves with time. This is good — well need this because we need time variability to have inflation end.

The dynamics of the inflation field are very much like that of a ball rolling on a hill. The force is $-\frac{\partial V}{\partial \phi}$, and the Hubble expansion causes a friction term.

What conditions must $V(\phi)$ satisfy for our inflation to work?

1) From equation 169, in order to get a rapidly accelerating universe the potential energy must dominate. so we need $\frac{1}{2}\dot{\phi}^2 << V(\phi)$.

2) For inflation to persist long enough to provide a factor of e^{60} of expansion, we cannot have $\dot{\phi}$ change very much or we might violate the above. So $|\ddot{\phi}|$ must be small compared to the other terms in the equation of motion $|3H\dot{\phi}|$, and $|\frac{\partial V}{\partial \phi}|$.

If we can find or invent a $V(\phi)$ that satisfies these conditions then you will have come up with an inflationary model!

It is conventional to encapsulate these in a non-dimensional form, we need to satisfy the **slow-roll condition**.

$$\epsilon = \frac{1}{16\pi G}(\frac{\frac{\partial\partial V}{\partial \phi}}{V})^2 \tag{172}$$

$$|\eta| = \frac{1}{8\pi G}\frac{|\frac{\partial^2 V}{\partial \phi^2}|}{V} \tag{173}$$

If ϵ, $|\eta| << 1$, we satisfy the slow roll conditions and inflation happens.

What sorts of potential satisfy slow roll? Really flat potentials satisfy slow rolls, and higher side of a little valley as well. (there is a diagram in the actual notes).

what happens is that the ball rolls slowly down the potential, and eventually we exit the region that satisfies the slow roll conditions. The ball settles down and oscillates at the bottom of the potential well. At this point, the inflation field couples to ordinary particle fields and populates these fields. This creation of standard particles is known as **preheating** and is not well understood aka sketchy.

Aside from the last step, we have a inflation model that works!.

11.4 Eternal Inflation

Here is an interesting thing. Generally speaking, inflation is **eternal**. Other patches of the pre-inflation universe may have a ϕ that also satisfies **slow-roll** and hence inflate. A crazier thing that can happen. Because ϕ is a quantum field, there is a non-zero chance that that the ball will climb uphill (quantum tunnel), back to where **slow-roll** works, and we start inflating again. This is of course exponentially suppressed by quantum mechanics, but inflation makes e^{60} more volume so these rare events still matter.

It's hard to gauge how likely this is because the universe is infinite.

12 Seed Fluctuations And Two Point Stats

This lecture note, and the one before should have a hand drawn plot that I have not added yet

Inflation is the "default" model of the very early universe because it provides a mechanism for generating the **spatial fluctuations** in density that we see in our universe.

The cartoon picture is that the initially quantum level fluctuations in the scaler field get blown up into macroscopic sized. These fluctuations seed the initial conditions (the inhomogeneous) that give rise to large scale structure and galaxies.

Initially parts of the universe are in causal contact, get yonked out of causal contact and then as time goes on more and more distant parts of the universe come back into causal contact again. As light has more and more time to transverse our universe.

Suppose we have some quantum fluctuations that get stretched out to cosmological scales, what happens to them afterwords? Normally, If we gave something like that the amplitude of the fluctuation will grow — they may be over dense and therefore attract more matter!. We must be careful, we are stretching the perturbations so that the "wavelength" of these spatial fluctuations is larger then the horizon scale. That probably changes things.

We can be super rigorous and do relativistic perturbation theory, but the final result is quite intuitive.

Fourier modes with spatial wavelengths larger then the horizon are **Frozen**. Causalilty means the amplitude of these perturbations cannot grow. This is a gauge-dependent statement, and in GR its possible to pick coordinates where this is not the case.

So these quantum fluctuations are stretched out to superhorizon scales, meaning that they exit the horizon, where they are frozen and gravitational collapse cannot happen. Eventually, the size of the horizon gets large enough to match the scale of the fluctuation and they reenter the horizon, and only then does gravitational clustering begin for that mode.

Let's take this a step further:

Quantum fluctuation in the inflation field that occurred early on in the inflationary expansion exited the horizon early and got stretched out to the largest scales (because inflation is still going strong $\Rightarrow$ plenty of time to stretch). These are the modes that reenter the horizon the latest.

Quantum fluctuations that occurred later in the inflationary process existed the horizon later, and got stretched less. They are therefore fluctuations on a smaller scales. These ar the modes that reenter the horizon the earliest.

In other words, first out is last in. Or that small scale modes are sensitive to the end of inflation, and that the large scale ones are sensitive to the beginning of inflation.

12.1 Describing the Fluctuations - 2 Point Statistics

Before was just a conceptual overview, now we do some of the math. Suppose we have some field $\rho(\mathbf{r})$. We can define the overdensity of this field as:

$$\delta(\mathbf{r}) = \frac{\rho(\mathbf{r}) - \bar{\rho}}{\bar{\rho}} \tag{174}$$

Where $\bar{\rho}$ is the average value of the field. This could be anything, but we have picked the notation ρ because we are often concerned with the matter density. It could be other scaler fields though.

Now, no theory of our universe predicts exactly what $\delta(\mathbf{r})$ is, because our universe is a random place — we cannot predict how the quantum fluctuations in inflation would go, for example, because otherwise they wouldn't be random. All we can do is predict the statistical properties of the field.

We can look at the moments of the field.

- $\langle \delta(\mathbf{r}) \rangle$ — **Mean (0th moment)**: Identically zero by construction, since δ is the overdensity field.
- $\langle \delta^2(\mathbf{r}) \rangle$ — **Variance:** Measures the amplitude of the fluctuations.
- $\langle \delta(\mathbf{r}_1)\, \delta(\mathbf{r}_2) \rangle$ — **Covariance (two-point correlation function):** Quantifies how density fluctuations at two different points are correlated, providing insight into the underlying physics.

The Correlation function $\xi(\mathbf{r}_1, \mathbf{r}_2)$:

$$\xi(\mathbf{r}_1, \mathbf{r}_2) = \langle \delta(\mathbf{r}_1)\, \delta(\mathbf{r}_2) \rangle \tag{175}$$

We can simplify the left hand side, by writing it as:

$$\xi(\mathbf{r}_1, \mathbf{r}_2) = \xi(\mathbf{r}_1 - \mathbf{r}_2) = \xi(|\mathbf{r}_1 - \mathbf{r}_2|) = \xi(\mathbf{r}) \tag{176}$$

We can write the first simplification because of homogeneity. The universe does not care what we call our origin. There is a **statistical translation invariance** it should only care about the difference in coordinates.

The second simplification (the absolute value bars) comes from isotropy, there is no preferred direction.

The correlation function is what we call the **2-point function in configuration space.**

We can also compute a 2 point function in Fourier space, the Fourier transform convention we use is:

$\delta(\mathbf{r}) = \int \frac{1}{(2\pi)^3} d^3\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}} \tilde{\delta}(\mathbf{k})$

So we can write the two point statistic as:

$$\langle \tilde{\delta}(\mathbf{k}_1)\, \tilde{\delta}(\mathbf{k}_2)^* \rangle = \langle \int d^3\mathbf{r}_1 e^{-i\mathbf{k}_1\cdot\mathbf{r}_1} \delta(\mathbf{r}_1) \int d^3\mathbf{r}_2 e^{i\mathbf{k}_2\cdot\mathbf{r}_2} \delta(\mathbf{r}_2) \rangle = \int d^3\mathbf{r}_1 d^3\mathbf{r}_2 e^{-i\mathbf{k}_1\cdot\mathbf{r}_1} e^{i\mathbf{k}_2\cdot\mathbf{r}_2} \xi(\mathbf{r}_1 - \mathbf{r}_2) \tag{177}$$

Now we can do a change of coordinates, let $\mathbf{r}_2 = \mathbf{r}_1 + \mathbf{x}$. Then we can write:

$$\langle \tilde{\delta}(\mathbf{k}_1)\, \tilde{\delta}(\mathbf{k}_2)^* \rangle = \int d^3\mathbf{r}_1 d^3\mathbf{x} e^{-i\mathbf{k}_1\cdot\mathbf{r}_1} e^{i\mathbf{k}_2\cdot(\mathbf{r}_1+\mathbf{x})} \xi(\mathbf{x}) \tag{178}$$

We can insert a Dirac Delta function δ_d.

$$= \int d^3\mathbf{x} (2\pi)^3 \delta_d(\mathbf{k}_1 - \mathbf{k}_2) \xi(\mathbf{x}) e^{i\mathbf{k}_2\cdot\mathbf{x}} = (2\pi)^3 \delta_d(\mathbf{k}_1 - \mathbf{k}_2) \int d^3\mathbf{x} e^{i\mathbf{k}\cdot\mathbf{x}} \xi(\mathbf{x}) \tag{179}$$

We define the term inside of the integral in the last line as $P(\mathbf{k})$ the **Power-spectrum**. This means that the power-spectrum is the Fourier transform of the correlation function:

$$\boxed{\langle \tilde{\delta}(\mathbf{k}_1)\, \tilde{\delta}(\mathbf{k}_2)^* \rangle = (2\pi)^3 \delta_d(\mathbf{k}_1 - \mathbf{k}_2) P(\mathbf{k}_1)} \tag{180}$$

Here we also assume isotropy so $P(\mathbf{k}) \Rightarrow P(k)$.

The power spectrum, and the correlation function are two equivalent ways to look at fluctuations.

The correlation function does not depend on the "origin" of $\mathbf{r}_1$, and $\mathbf{r}_2$, but different postions are correlated.

The power-spectrum does depend on the size of k, but different Fourier modes are **not** correlated.

The power-spectrum can be thought of as the variance of the fluctuations on different k-scales. (skipped)

Large spatial scale r corresponds to small k.

We often write $\Delta^2(k) = \frac{k^3 P(k)}{2\pi^2}$ as the "dimensionless power-spectrum".

13 Initial Fluctuations

Today we close the book on inflation by computing some of its predictions regarding the seed fluctuations. First, we need to build one more piece of mathematical machinery. Suppose we wanted to compute the contributions to the variance in configuration space from Fourier modes of various length scales.

(skipped)

$\Delta^2(k) = \frac{k^3 P(k)}{2\pi^2}$ as the "dimensionless power-spectrum". But it has units of the square of whatever units the original field might be. We can see that $\Delta^2(k)$ is the contribution to the variance of fluctuations $<\delta^2>$ per ln interval in k. We can write this mathematically as $\Delta^2(k) = \frac{d<\delta^2>}{d\ln k}$.

13.1 Density Fluctuations From Inflation

What we want to do now is to go back and predict the amplitude of density fluctuations. Quantum fluctuations in inflation $\Rightarrow$ (the arrow is reheating) density fluctuations.

If we were doing this "properly" doing relativistic perturbation theory, the relevant quantity would be the **gravitational potential** Φ, so we are trying to compute $\Delta^2_\Phi(k) = \frac{d<\Phi^2>}{d\ln k}$. This turns out to be a constant - $\Delta^2_\Phi(k)$ is independent of k.

We have approximately exponential expansion:

$$a(t) \propto \exp(Ht) \tag{181}$$

Now, **exponential expansion has no preferred origin in time.** If we were to blindfold you and then take off your blindfold at some random time during inflation, you wouldn't be able to tell whether inflation had just started or had been going on for a while. Translating the origin of time gives:

$$\exp(H(t - t_*) = \exp(-Ht_*)\exp(Ht) \propto \exp(Ht) \tag{182}$$

and we can just absorb the translation offset into our proportionality constant.

But we just said that the behaviour is the same at all times, so the fluctuation amplitude must not depend on the k scale!

This means that all we have left to do is determine the amplitude of fluctuations ie the constant. We would ideally do this from some quantum field theory, but we do not really want to assume QFT for this class. What do physicists do when there is a piece of physics they do not know how to do / do not want to do? — **Dimensional Analysis**!

One way to think about why there are density fluctuations sourced by inflation is to say that because of quantum fluctuations, different parts of our universe finish inflating at slightly different times. Different bits exit the slow-roll part of the inflation potential at slightly different times. They start reheating at slightly different times. Bits that finish inflation a little earlier have had a little extra time to expand and cool off $\Rightarrow$ these become the parts of our universe that are a tiny bit below average in energy density — the **Cold Spots**.

We can write for the inflation field:

$$\delta t = \frac{\delta\phi}{\dot{\phi}} \tag{183}$$

Suppose we want to find the RMS fluctuation amplitude of the gravitational potential $\sqrt{\Delta^2_\Phi}$. This is a dimensionless number, so we need to combine δt with something that makes it dimensionless. We do have such a thing, the only scale in the problem is H, and this has dimensions of inverse time, so we can say.

$$\sqrt{\Delta_\Phi^2} \sim H\delta t \sim \frac{H\delta\phi}{\dot{\phi}} \tag{184}$$

To proceed, we need to know the typical fluctuation amplitude of the inflation field $d\phi$. This we can do with dimensional analysis as well. We have to be careful though — we are using natural units IE we have set $\hbar = c = 1$ so we need to remember that.

In natural units, we can express all quantities in terms of energy.

$$[t] = [\frac{1}{E}] = [L] \tag{185}$$

You can see this from the Plank formula for the energy of a photon. This means that H has units of energy. The Length one is similarly from the Plank equation.

Now lets move onto the units of $\delta\phi$. Recall that $\rho \sim \frac{1}{2}\dot{\phi}^2$ is the kinetic energy. The LHS has units of energy density, and so the right hand side also needs to have units of energy density. Anyway the conclusion is that ϕ has units of energy or $\sim H$ as H is the only variable we have. The exact QFT scale is $d\phi = \frac{H}{2\pi}$.

This means that:

$$\boxed{\sqrt{\Delta_\Phi^2} = \frac{H^2}{2\pi\dot{\phi}}} \tag{186}$$

The full QFT result says:

$$\boxed{\Delta_\Phi^2(k) = \frac{H^4}{(2\pi\dot{\phi})^2}} \tag{187}$$

With the dependence of k restored. This is more exact then what our dimensional analysis might have suspected.

We can simplify this into a nice form by remembering that under the slow-roll approximation, we have:

$$H^2 \approx \frac{8\pi G}{3} V(\phi) \tag{188}$$

$$3H\dot{\phi} = -V'(\phi) \tag{189}$$

Then we can write:

$$\Delta_\Phi^2(k) = \frac{128\pi G^3}{3}(\frac{V^3}{V'^2}) \tag{190}$$

Now suppose we parameterize $\Delta_\Phi^2(k)$ as a power law, such that:

$$\Delta_\Phi^2(k) \propto k^{n_s - 1} \tag{191}$$

The **spectral index** $n - s$, controls how close to scale invariant we are. We expect $n_s \approx 1$, but lets see!

$$n_s - 1 = \frac{\partial \ln \Delta_\Phi^2}{\partial \ln k} \tag{192}$$

The question is how do we compute this derivative. Recall that we really want to understand what the amplitudes of the fluctuations are when modes cross the horizon, because that is when they are frozen in. Meaning that we want to evaluate this when the physical wavelength λ is comparable to the size of the horizon $\frac{c}{H}$ where $c = 1$. The middle part is the comoving wavenumber, and the scale factor.

$$\lambda \sim \frac{a}{k} \sim H^{-1} \tag{193}$$

(skipped)

$$\boxed{n_s = 1 - 6\epsilon + 2\eta} \tag{194}$$

Where these are the slow-roll parameters. A key prediction of inflation is that there is a nearly scale-invariant spectrum of fluctuations.

14 Growth of Structures

What we have so far is that inflation seeds primordial density fluctuations, which eventually grow to form galaxies and the large-scale structure we observe today.

We define the power spectrum to describe these fluctuations, quantifying how much power exists on small versus large spatial scales.

14.1 Power Spectrum Review

There are two commonly used versions of the power spectrum.

Version One:

$$\langle \tilde{\delta}(\mathbf{k})\tilde{\delta}^*(\mathbf{k}')\rangle = (2\pi)^3 \delta_D(\mathbf{k}-\mathbf{k}')P(k) \tag{195}$$

Here,

$$\delta = \frac{\rho - \bar{\rho}}{\bar{\rho}}, \tag{196}$$

where ρ is the matter density and $\bar{\rho}$ is its mean value.

Equivalently, one may write:

$$P(k) = \frac{\langle |\tilde{\delta}(\mathbf{k})|^2\rangle}{V}, \tag{197}$$

where V is the survey volume. The power spectrum $P(k)$ has units of $(\mathrm{Mpc})^3$, arising from two factors of volume from the Fourier transform and one factor in the denominator.

Version Two: The Dimensionless Power Spectrum

$$\Delta^2(k) = \frac{k^3 P(k)}{2\pi^2}. \tag{198}$$

In this case, $\Delta^2(k)$ is dimensionless.

Depending on the context, it is sometimes useful to consider fluctuations in the gravitational potential Φ, and other times it is more useful to consider fluctuations in the matter density ρ.

These two are related by the Poisson equation:

$$\nabla^2\Phi = 4\pi G\rho. \tag{199}$$

Since we are interested in perturbations in both the density and the gravitational potential—and how they relate to one another—we now perturb this equation.

14.2 Power Spectrum Perturbations

The perturbed Poisson equation is

$$\Delta^2 \delta\Phi = 4\pi G\bar{\rho}\delta \tag{200}$$

Here, $\delta\Phi = \Phi - \bar{\Phi}$, and $\bar{\rho}\delta = \rho - \bar{\rho}$.

No suppose we were to try and solve this equation in Fourier space, in other words let:

$$\delta\Phi = A_k e^{i\mathbf{k}\cdot\mathbf{r}} \tag{201}$$

$$\delta = B_k e^{i\mathbf{k}\cdot\mathbf{r}} \tag{202}$$

Then we can write:

$$\Delta^2\Phi = A_k(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})e^{i\mathbf{k}\cdot\mathbf{r}} = k^2 A_k e^{i\mathbf{k}\cdot\mathbf{r}} \tag{203}$$

So the action of Δ^2 in Fourier space is multiplication by k^2.

$$k^2\tilde{\delta\Phi} = 4\pi G\bar{\delta}\tilde{\rho} \Rightarrow k^2\tilde{\delta\Phi} \propto \tilde{\delta} \tag{204}$$

This means that we can translate between perturbations in Φ, and ρ by just multiplying by k^2.

$$P_\delta \propto k^4 P_\Phi \tag{205}$$

This time we square k^2 because $P \sim \tilde{\delta}^2$.

We saw last time that inflation predicts $\Delta^2_\Phi \propto k^{n_s-1} \Rightarrow P_\phi \propto k^{n_s-4}$. This means that $P_\delta(k) \propto k^{ns}$. Going forward we will sometimes omit the subscripts on P, when we do I mean the matter / density $P(k)$.

We would like to compare this to observations to test our theories.

skipped (a plot should be here)

14.3 Comparison with data

This doesn't exactly look like a single power law of the form k^{n_s}. What we are missing is a description of how all the fluctuations have **evolved** since being generated / seeded by inflation.

There are two things that have happened since the initial conditions were seeded:

- Fluctuations have **grown** in amplitude due to gravitational clustering
- The power spectrum has acquired a more complicated dependence on k. This is not surprising — as we have discussed before, perturbations cannot grow until the horizon has become on the same order as their wavelength IE until they reenter the horizon, and we expect this to happen at different times for different ks.

Often this is paramterized as:

$$\tilde{\delta}(\mathbf{k}, a) \propto \tilde{\delta}_{prim}(\mathbf{k})T(\mathbf{k})D_1(a) \tag{206}$$

Where a is the scale factor, and the primordial is the initial conditions, $\mathbf{T}$ is the transfer function and D_1 is the growth factor.

Note that this suggests a linear relation between the primordial fluctuations and the final fluctuations. This is only true if the fluctuations are small $|\delta| << 1$. So that we can make a linear approximation of our evolution equations.

We will do this today, doing linear perturbation theory. Well concentrate mostly on the growth factor, which means we can ignore the effect of the horizon. Working deep inside the horizon, we can use Newtonian mechanics.

14.4 Newtonian Linear Perturbation Theory

Lets list the relevant equations that we would like to solve. They are the equations of fluid mechanics.

- **Poisson Equation (Gravity):**
$$\nabla^2\Phi = 4\pi G\rho \tag{207}$$

- **Continuity Equation:** If density decreases locally, matter has flowed out.
$$\frac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot(\rho\mathbf{v}) \tag{208}$$

- **Euler Equation ($F = ma$ for a fluid):**
$$\rho\frac{\partial\mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\vec{\nabla})\mathbf{v} = -\rho\vec{\nabla}\Phi - \vec{\nabla}\rho \tag{209}$$

Now we linearize these equations by writing:

$$\rho(\mathbf{r},t) = \bar{\rho}(t)[1+\delta(\mathbf{r},t)] \tag{210}$$

The $\bar{\rho}(t)$ follows the usual $\propto a^{-3}$ for matter.

$$\mathbf{v}(\mathbf{r},t) = \mathbf{v}_0 + \delta\mathbf{v}(\mathbf{r},t) \tag{211}$$

$\mathbf{v}_0$ is the Hubble flow $= H\mathbf{r}$, and the other term is the Peculiar velocity.

Lastly, we have:
$$\Phi(\mathbf{r},t) = \Phi_0(\mathbf{r},t) + \delta\Phi(\mathbf{r},t) \tag{212}$$

What we need to do now is to substitute these expressions into our equations and then discard higher order terms like $(\delta\mathbf{v})^2$

We already did the Poisson equation, if you crank through the algebra you get:

$$\dot{\delta} + \vec{\nabla}\cdot\delta\mathbf{v} + \mathbf{v}_0\cdot\vec{\nabla}\delta = 0 \tag{213}$$

$$\dot{\delta\mathbf{v}} + (\mathbf{v}\cdot\vec{\nabla})\delta\mathbf{v} + (\frac{\dot{a}}{a})\delta\mathbf{v} = -\vec{\nabla}\delta\Phi - v_s^2\vec{\nabla}\delta \tag{214}$$

Where $v_s^2 = \frac{\partial p}{\partial\rho}$ is the sound speed.

These are the coupled Partial Differential equations (PDEs), Which are non-trivial to solve. once again we can make our lives easier by going to Fourier space.

So we write:
$$\delta(\mathbf{r},t) = \int \frac{1}{(2\pi)^3}\tilde{\delta}(\mathbf{k},t)\exp\left(-i\mathbf{k}/a(t)\cdot\mathbf{r}\right)d^3\mathbf{k} \tag{215}$$

The inclusion of the a factor in the exponential ensures that $\mathbf{k}$ is a wavevector in comoving coordinates, as $\mathbf{r}/a$ is the commoving distance. We want to focus on clustering, not expansion.

Fourier space allows us to write expressions like (higher-order terms are dropped):

$$\boxed{\ddot{\tilde{\delta}} + 2\left(\frac{\dot{a}}{a}\right)\dot{\tilde{\delta}} = \tilde{\delta}\left(4\pi G\bar{\rho} - \frac{v_s^2k^2}{a^2}\right)} \tag{216}$$

This is a wonderful equation as each of the terms really has nice physical interpretation, effecting $\ddot{\tilde{\delta}}$. Let's examine them one by one, turning the others off if necessary.

14.5 Three Competing effects

Fourier space allows us to write expressions like (higher-order terms are dropped):

$$\boxed{\ddot{\tilde{\delta}} + 2\left(\frac{\dot{a}}{a}\right)\dot{\tilde{\delta}} = \tilde{\delta}\left(4\pi G\bar{\rho} - \frac{v_s^2 k^2}{a^2}\right)} \tag{217}$$

This is a wonderful equation as each of the terms really has nice physical interpretation, effecting $\ddot{\tilde{\delta}}$. Let's examine them one by one, turning the others off if necessary.

There are three competing effects.

The first is **Gravity**. This is the $4\pi G\bar{\rho}$ term. If the other terms were not there, we would have $\ddot{\tilde{\delta}} = 4\pi G\bar{\rho}$. This is a positive feedback, which leads to exponential growth because clumping amplifies gravitational effects.

The next is **The Hubble expansion** in the form of the $2\left(\frac{\dot{a}}{a}\right)\dot{\tilde{\delta}}$ term. We see that it enters the differential equation like a drag term, inhibiting the growth. The quicker the expansion, the greater this drag term becomes the more the expansion is trying to pull apart clustering.

Pressure. If we compress a gas, it pushes back. This oscillates like a simple harmonic oscillator. we have $\ddot{\tilde{\delta}} = -\frac{v_s^2 k^2}{a^2}\tilde{\delta}$. If pressure wins, this eventually means that I set up sound waves in the medium rather then having gravitational collapse.

When does pressure win? The crucial thing is the sign of on the RHS. The tipping point is when it is 0, and therefor when:

$$\frac{v_s^2 k^2}{a^2} = 4\pi G\bar{\rho} \tag{218}$$

Recall that k is the comoving wavenumber, and k/a is the physical wavenumber.

Lets set k/a to the Jeans length:

$$\frac{k}{a} = \frac{2\pi}{\lambda_j} \tag{219}$$

Where the Jeans length is given by $\lambda_j = \sqrt{\frac{\pi v_s^2}{G\bar{\rho}}}$. Perturbations on scales much larger then λ_j collapse, while perturbations on scales $< \lambda_j$ form stable oscillations.

Now, $H^2 = \frac{8\pi G\bar{\rho}}{3}$, so we can write this as:

$$\lambda_j = 2\pi(\frac{2}{3})^{1/2}\frac{v_s}{H} \tag{220}$$

We can evaluate this. Well almost, we need the v_s which depends on the substance.

For **photons**: recall that $p = \frac{1}{3}\rho c^2$, so $v_s^2 = c^2/3$. Meaning that $\lambda_j = \frac{2\sqrt{2}\pi c}{3H}$. This is larger then the Hubble length c/H. This means that photons don't collapse gravitationally.

For **Baryons**: This time it is interesting. **Prior to recombination** baryons are ionized, and this charged environment scatters photons easily via Thomson scattering, so the photons and baryons act as coupled photon-baryon fluid. The photons "lend" their pressure to the charged baryons and they do not collapse.

After recombination, the sound speed is given by the usual formula from thermo $c_s = (\frac{k_b T}{m})^{1/2} \sim 1.5 \times 10^{-5}c$ for T at recombination. The sound speed plummets, and all of a sudden the baryons can cluster.

Dark Matter: to a first approximation, dark matter is pressurelesss so gravity always wins. In other words Dark Matter gets a "head start" on seeding structure. Once baryons decouple they can fall into a relative deep potential wells seeded by Dark matter. Without this seeding effect we wouldn't get structure growth to happen fast enough to explain galaxies we see today.

Note two important things from the intuition we have established:

- The key to understanding structure growth is to understand how dark matter clusters, since the baryons (roughly speaking) just follow along
- The photons don't really cluster, so the gravity term is really sourced by matter.

To really emphasize the second point, we can write our equation from before as:

$$\ddot{\tilde{\delta}} + 2H\dot{\tilde{\delta}} - \frac{3}{2}\Omega_m H^2 \tilde{\delta} = 0 \tag{221}$$

Here we neglected the pressure term because we are thinking about Dark matter. In different phases of evolution, we get different behaviour:

Radiation-Dominated: $\Omega_m << 1$, $a \sim t^{1/2}$, $H = \frac{1}{2t}$.

This leads to:

$$\ddot{\tilde{\delta}} + \frac{1}{t}\dot{\tilde{\delta}} = 0 \tag{222}$$

solving this differential equation, we have:

$$\tilde{\delta} = B_1 + B_2 \ln t \tag{223}$$

There is growth, but it is small.

Dark-Energy Dominated: $\Omega_m << 1$, $H \approx H_0$.

$$\ddot{\tilde{\delta}} + 2H_0\dot{\tilde{\delta}} = 0 \tag{224}$$

solving this equation leads to:

$$\tilde{\delta}(k,t) = C_1 + C_2 e^{-2H_0 t} \tag{225}$$

The first term has no growth, and the second is a decaying mode which is unimportant.

This is odd, because we see galaxies today (in dark matter domination). Our analysis assumes that we are in the **linear** regime. Small, non-linear scales like galaxies aren't applicable to this analysis.

Matter Dominated: $\Omega_m = 1$, $a \sim t^{2/3}$, $H = \frac{2}{3t}$.

This leads to the equation:

$$\ddot{\tilde{\delta}} + \frac{4}{3t}\dot{\tilde{\delta}} - \frac{2}{3t^2}\tilde{\delta} = 0 \tag{226}$$

This time we apply a power-law solution.

$$\tilde{\delta}(k,t) = A_1 t^{2/3} + A_2 t^{-1} \tag{227}$$

The second term again is a decaying mode, which is unimportant, and so during matter domination $\tilde{\delta} \sim a(t)$. Not the runaway exponential growth of a static universe; however, this is the only era with substantial growth of structure.

15 Matter Power-Spectrum

We have been building the pieces for understanding the form of the matter power spectrum.

Recall we have:

$$\tilde{\delta}(\mathbf{k}, a) \propto \tilde{\delta}_{\text{prim}}(\mathbf{k})T(k)D_1(a) \tag{228}$$

Where $\tilde{\delta}_{\text{prim}}$ is the initial conditions, T is the transfer function, and D is the growth factor.

To get to $T(k)$, we need to refine our discussion of what happens to modes as they enter/exit the horizon.

Previously, we had carelessly said that modes outside the horizon are **frozen**. But we need to be more precise. What is frozen is the fluctuations in gravitational potential Φ, **not density**.

The Poisson equation gives:

$$\nabla^2 d\Phi = 4\pi G\bar{\rho}\delta \Rightarrow \frac{k^2 \tilde{\delta\Phi}}{a^2} \propto \bar{\rho}\tilde{\delta} \tag{229}$$

Where we are using the comoving k. If $\tilde{\delta\Phi}$ is frozen for super horizon modes, then $\tilde{\delta}$ is not because $\bar{\rho}$ evolves with time ($\bar{\rho} \propto \frac{1}{a^3}$ during matter domination, and $\frac{1}{a^4}$ in radiation domination).

For super-horizon perturbations:

- **Matter Domination:**$\tilde{\delta} \propto a$
- **Radiation Domination:** $\tilde{\delta} \propto a^2$

Once the modes reenter the horizon, they evolve according to the Newtonian expressions we found before. What we need to know is the amplitude as they reenter the horizon.

$$< \delta^2_{\mathrm{rms,R_H}} >\sim \int_0^{\frac{1}{R_H}} \frac{k^2(4\pi)}{(2\pi)^3} P(k) \tag{230}$$

Here, R_H is the cutoff $k_H \sim \frac{1}{R_H}$, we do not care about the fluctuations on smaller scales. We solve this with $u = kR_H$, $du = R_H dk$, and assuming $P(k) \propto k^{n_s}$ which we know from inflation.

the result is:

$$< \delta^2_{\mathrm{rms,R_H}} >\sim R_H^{-(n_s+3)} \Rightarrow \delta \sim R_H^{-(n_s+3)/2} \tag{231}$$

When a mode enters the horizon, the amplitude is $\sim$ super-horizon growth $\times$ primordial rms.

Therefore:

Epoch	Co-moving horizon scale	Super-horizon growth	Amplitude at horizon ent
Radiation Domination	$R_H \sim \frac{ct}{a} \propto a \quad (a \propto t^{1/2})$	$\tilde{\delta} \propto a^2 \propto R_H^2$	$\tilde{\delta}_{\text{entry}} \sim R_H^2\, R_H^{-(n_s+3)/2}$
Matter Domination	$R_H \sim \frac{ct}{a} \propto a^{1/2} \quad (a \propto t^{2/3})$	$\tilde{\delta} \propto a \propto R_H^2$	$\tilde{\delta}_{\text{entry}} \sim R_H^2\, R_H^{-(n_s+3)/2}$

Table 3: Scaling of the co-moving horizon, super-horizon growth, and perturbation amplitude at horizon entry during radiation- and matter-dominated epochs.

Regardless of when we enter the horizon, the amplitude of fluctuation goes like:

$$\tilde{\delta}_{\text{entry}} \sim R_H^{(1-n_s)/2} \tag{232}$$

Notice that if n_s were perfectly $= 1$, all density nodes would reenter the horizon at the same amplitude, this is another sense in which people refer to $n_s = 1$ as the the "scale invariant spectrum".

What happens after horizon entry? Recall from our Newtonian results, that during radiation domination growth is negligible (logarithmic). Thus a mode that enters during radiation domination will miss out on the super horizon growth that larger modes are still experiences.

Since super-horizon growth goes like $\tilde{\delta} \propto R_H^2$, if the large k modes miss out on this because they entered the horizon, their amplitude is supressed by:

$$\frac{\tilde{\delta}(k)}{\tilde{\delta}(k_{\text{enter}})} = (\frac{k}{k_{\text{enter}}})^{-2} \tag{233}$$

This means that:

- For $k << k_{eq}, T(k) = 1$
- For k^{-2}, for $k >> k_{eq}$

Since, $\tilde{\delta}(\mathbf{k,}a) \propto \tilde{\delta}_{\text{prim}}(\mathbf{k})T(k)$ we have:

$$P(k) \propto P_{\text{prim}}T^2(k) \tag{234}$$

We finally have:

- for $k << k_{eq}$, $P(k) \propto k^{n_s}$
- for $k >> k_{eq}$, $P(k) \propto k^{n_s-4}$

15.1 What Physics Do we Learn?

What Physics do we learn from measuring the matter power spectrum?

- primordial power spectrum at low $k \Rightarrow n_s$
- The turnover k wavenumber depends on when matter-radiation equalilty takes place. A higher Ω_m means the turnover is shifted rightward on a plot of $P(k)$ (y axis) vs k (x axis).
- If we zoom into the high k region, we notice some wiggles, this are the baryonic acoustic oscillations (BAOs).

15.2 Baryonic Acoustic Oscillations (BAO):

The CDM pulls on the Baryons sure, but our early story was too simple. The baryons pull on the CDM as well. Whatever the baryons were doing with the photons before recombination gets imprinted a little bit onto the power spectrum. The pressure from the photon-baryon fluid before recombination pushes out against clumping.

Details of DM physics matter on small scales. Cold dark matter clumps easily and forms lots of structure on small scales. Warm dark matter is more able to stream out of shallow potential wells, inhibiting small structure.

Looking at possible suppression on small scales can be very helpful. But you need to be **careful** — eventually you get to high enough k values that nonlinear effects become important.

15.3 How do we measure the matter power spectrum in practice

The most conventional way is to do galaxy surveys. We measure the locations of galaxies, and point-by-point we build up a picture of the matter distribution.

Central Assumption: Galaxies are Biased Tracers of δ. Where galaxies are concentrated, so is matter. Galaxies only form at peaks in density.

We can model this as:

$$\delta_g(\mathbf{r}) = b(\delta(\mathbf{r})) \approx b_0 + b_1(\delta(\mathbf{r})) + ... \tag{235}$$

Where b is some function, and we assume $b_0 \approx 0$. AKA a linear model $\delta_g = b\delta$. The b is known as the bias parameter.

An incomplete list of things to worry about for observations:

- The Bias parameter — extra nuisance param
- Shot Noise — discrete counting does not always capture the underlying distribution, this can be mitigated be a large sample size
- Cosmic Variance — our universe is random, what if we get lucky and sample an unrepresentative part? Mitigated by doing large volume surveys

The true power spectrum involves an ensemble average:

$$P(k) = \frac{< |\tilde{\delta}(\mathbf{k})| >}{V} \tag{236}$$

Where the LHS is the true power-spectrum. We cannot do a true ensemble average in real life, so we replace it with an average over different directions, assuming isotropy.

$$\hat{P}(k) = \frac{< |\tilde{\delta}(\mathbf{k})| >}{V}, \mathbf{k} \in k \tag{237}$$

Here $\hat{P}(k)$ is the estimator on P, and has a random error. If you imagine bins of constant k, the small k are disproportionally effected by the volume (they only occur when volume is small). The second and third effects can be traded for each-other.

If you use **dim galaxies** $\Rightarrow$ they are more common and hence lower shot noise; but they are hard to see at further distances so the survey volume is limited (higher cosmic variances).

If you use **bright objects** like quasars $\Rightarrow$ they are more rare meaning higher shot noise, but you can have a larger volume.

A goldilocks galaxy are luminous red galaxies.

16 Cosmic Microwave Background (CMB) Part 1

Today we are going to start talking about one of the most successful probes of cosmology, the cosmic microwave background.

When we look at the CMB, we get to see early in our universe. Why do the anisotropies exist? We say they are fluctuations, but what causes them? They come from **inhomogeneities** in the matter power-spectrum and are imprinted on the CMB as **anisotropies**.

This tells us that to predict the CMB, we have to worry about several things:

- **Primordial** CMB fluctuations — connected to the density fluctuations, and should be consistent with galaxy surveys
- **Projection Effects** — from Fourier modes to sphere
- **Secondary Effects** — that affect photons as they travel to our telescopes

We will give a semi-qualitative treatment of each contribution.

Useful table: Y_{lm} are spherical harmonics.

Table 4: Correspondence between spatial inhomogeneities and angular anisotropies. Y_{lm} are spherical harmonics.

Label	**Inhomogeneities**	**Anisotr**
Field	$\delta(\mathbf{r})$ (overdensity)	$T_f(\hat{r}) = \frac{\delta T}{T}$ (fractional t
Harmonic space	$\tilde{\delta}(\mathbf{k}) = \int d^3\mathbf{r}\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, \delta(\mathbf{r})$,inverse has a factor $\frac{1}{(2\pi)^3}$	$a_{lm} = \int d\Omega\, Y^*_{lm}(\hat{r})\, T_f(\hat{r})$,
Length / angular scales	$\lambda \sim \frac{2\pi}{k}$	$\theta \sim$
Power spectrum	$\langle \tilde{\delta}(\mathbf{k}) \tilde{\delta}^*(\mathbf{k}') \rangle = (2\pi)^3 \delta_D(\mathbf{k}-\mathbf{k}')\, P(k)$	$\langle a_{lm} a^*_{l'm'} \rangle =$
"Practical estimator"	$\hat{P}(k) = \frac{\sum_{\tilde{k}\in k} \lvert\tilde{\delta}(\mathbf{k})\rvert^2}{V}$	$\hat{C}_l = \frac{1}{2l+1} \sum$
"Dimensionless" power spectrum	$\Delta^2(k) = \frac{k^3 P(k)}{2\pi^2}$	$D_l = \frac{l(l}{2}$

Just as with the matter power spectrum, there is good physics governing the shape of C_l that we can use to learn about our universe.

The CMB from 2018 diagram would be here, it peaks at $l \sim 200$, which corresponds to $\theta \sim 1$.

If we ignore the projection, and secondary effects we can write the CMB as:

$$T_f(\hat{r}) = \frac{\delta T(\hat{r})}{T} = \delta\Phi(\mathbf{r}) - \hat{r} \cdot \mathbf{v}(\mathbf{r}) + \frac{1}{3}\delta(\mathbf{r}) \tag{238}$$

$\mathbf{v}$ here is the peculiar velocity, Φ is the gravitational potential.

The $\delta\Phi$ term arises because if a photon finds itself in a deep gravitational well, it looses energy as it climbs out. **A potential well has $\delta\Phi < 0$, so we see a lower temperature.** The $\frac{1}{3}\delta(\mathbf{r})$ term is because overdensitiy regions are hotter, this counter acts the first term. The $\hat{r} \cdot \mathbf{v}(\mathbf{r})$ term is due to peculiar velocities causing photons to be blueshifted or redshifted.

THe factor of 1/3 is due to how photons and baryons were tightly coupled before recombination. (skipped).

The terms that win depend on the angular scale that you look at. In a plot of $\frac{l(l+1)}{2\pi}C_l$ (y axis) vs l (x axis) the x-axis can be broken from left-to-right into: Saehces - Waffle plateau, Acoustic Peaks, Damping Tail.

Just as with the matter power spectrum, these scales correspond to the largest scales that are only barely beginning to cluster $\Rightarrow$ meaning there is not much movement and we can neglect the Doppler term.

Recall that we can think of the different hot and cold spots as being due to our universe having different amounts of time to cool via expansion.

In particular:

- inside a potential well due to the GR time dilation
- "universe younger" inside well
- spot hasn't had much time to expand $\Rightarrow$ denser $\Rightarrow$ *hotter*

We can write:

$$\frac{dt}{t} = d\Phi \tag{239}$$
$$a \sim t^{2/3} \tag{240}$$
$$T \sim \frac{1}{a} \tag{241}$$
$$\tag{242}$$

This leads to:

$$\frac{dT}{T} = -\frac{2}{3}\frac{dt}{t} = -\frac{2}{3}\delta\Phi \tag{243}$$

we can add this to the gravitational redshift to get:

$$T_f\hat{r} = \frac{dT}{T} = \delta\Phi - \frac{2}{3}\delta\Phi = \frac{1}{3}\delta\Phi \tag{244}$$

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