PHYS 644 Lecture #11: Relics

Last time we started to think about interactions between particle species in the early, radiation-dominated universe

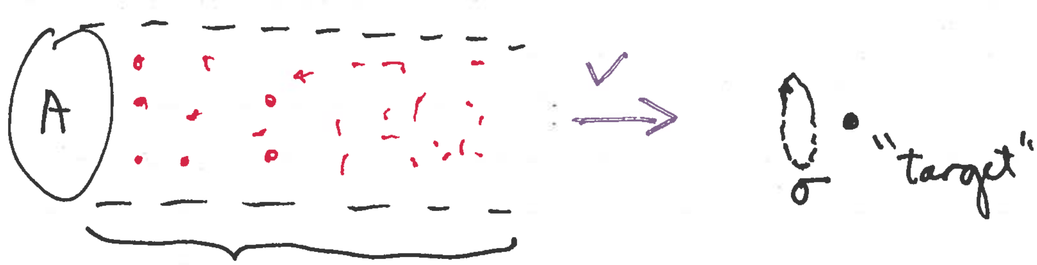
$\Rightarrow$ $\boxed{\text{Review Slides}}$

We now introduce two more principles that will guide our thinking in this era.

(3) Particles have interactions that can be "on" or "off" at different times

Let's study the interactions more carefully. There are two reasons for this. First, we need to include it in equations like $\dot{n} + 3Hn = (\text{interactions})$. Second, for a lot of our expressions (like assuming Fermi-Dirac or Bose-Einstein), we relied on the assumption of thermal equilibrium. Interactions are how particles "talk to each other" and are how species can maintain thermal equilibrium.

Suppose we have ~~to~~ a sea of particles and we ask how likely it is to interact with a particle (a "target"):



Now, nv is a flux: $\dfrac{\#}{\text{area} \times \text{time}}$. To get a rate of "hits", we multiply by an area σ, the "cross-section" which is an effective target area.

The probability for particles to hit the target is $\sim \sigma / A$, which means

$$\boxed{\Gamma \equiv \text{interaction rate} \equiv n \langle \sigma v \rangle}$$

What is the $\langle \cdots \rangle$ average? There's of course no universal "v" that all the particles travel at. Also, in general, σ itself is a function of v. The $\langle \cdots \rangle$ denotes a thermal average.

If the interaction rate is high, then a particular process is important and we need to consider it as something that happens. But what is high? The units of Γ are $(\text{time})^{-1}$. It therefore makes sense to compare this to the other quantity that we know that has dimensions of $(\text{time})^{-1}$: the Hubble parameter.

$\Rightarrow$ $\boxed{\text{Slide Q on } H \text{ vs } \Gamma}$

It's easier to think about H^{-1} vs Γ^{-1} instead. We know Γ^{-1} as the Hubble time at that time, so it's roughly the age of our Universe. If Γ measures # of interactions per second, then Γ^{-1} is the time per interaction. Obviously if the time per interaction is longer than the age of our Universe, it essentially never happens and we can ignore it!

Another way to phrase this is that Γ^{-1} is "time for particles to find each other" and H^{-1} is "time for particles to escape each other" due to expansion

Importantly, this is not a yes/no once-and-for-all question because H itself is a time-dependent (and therefore equivalently ~~as~~ a temperature T or energy scale-dependent quantity). Whether or not an interaction is important or not therefore ~~depends~~ is a function of time.

$\Rightarrow$ $\boxed{\text{Slide Q on temperature dependence}}$

In summary, $\boxed{\text{interactions are only "on" when } \Gamma \gg H^{-1}\text{, and this depends on what epoch of our Universe since both sides depend on time / temperature}}$

(4) The relic abundance depends on when interactions stop and whether a particle was relativistic or non-relativistic at that time.

Note that it doesn't have to be annihilation to photons. Eg DM doesn't talk to electromagnetism, so cannot annihilate to photons.

By "relic abundance", we mean how much is left over once annihilations like

$$X + \bar{X} \to 2\gamma$$

turn off.

Precisely when this happens — and whether the particle is relativistic or not when this freezeout occurs — is important.

Hot relics

These are particles that ~~fro~~ froze out while they were still relativistic. In other words, Γ dropped below H at some temperature T where $T \gg m$.

Relativistic particles tend to be quite abundant! Right before freezeout the interactions are still "on", ~~so~~ ~~so the number~~ ~~los~~: so the particles can "talk to the rest of the universe" and be in thermal equilibrium. This means the $n \propto g T^3$ result ~~holds. Annihilation~~ for equilibrium relativistic particles holds.

Annihilations are happening, but because $k_B T \gg m c^2$, the reverse process of pair production $2\gamma \to X + \bar{X}$ can replenish the particles. They're abundant.

Now, at some point Γ drops below H. With no more interactions, the # of particles doesn't change any more! After that, their number density just dilutes as $\frac{1}{a^3}$.

Explicitly, if $n_{freezeout} = \frac{\zeta(3)}{\pi^2} g T_{freezeout}^3 (\cdots)$ 1 or $3/4$ for bosons vs fermions

Then $n = n_{freezeout} \left(\frac{a_{freezeout}}{a}\right)^3$

$= n_{freezeout} \left(\frac{T}{T_{freezeout}}\right)^3$ Using $T \sim \frac{1}{a}$

Note that in this subsequent evolution, the temperature T of this species has no obligation to be the same as the rest of the universe — after all, interactions are "off" at this point and the species is decoupled. If something happens to the rest of our Universe to change its temperature after this species has decoupled, it's not going to "know".

We'll see an example of this with neutrinos soon!

Cold relics

If interactions Γ drop below H only after a species has become non-relativistic ($T_{freezeout} \ll m$) then we have a cold relic.

To discuss cold relics, we tend to look at a plot of $Y \equiv \frac{n}{s}$ (# density to entropy density ratio) vs. $\frac{m}{T}$.

$\Rightarrow$ Slide Qs for plot intuition

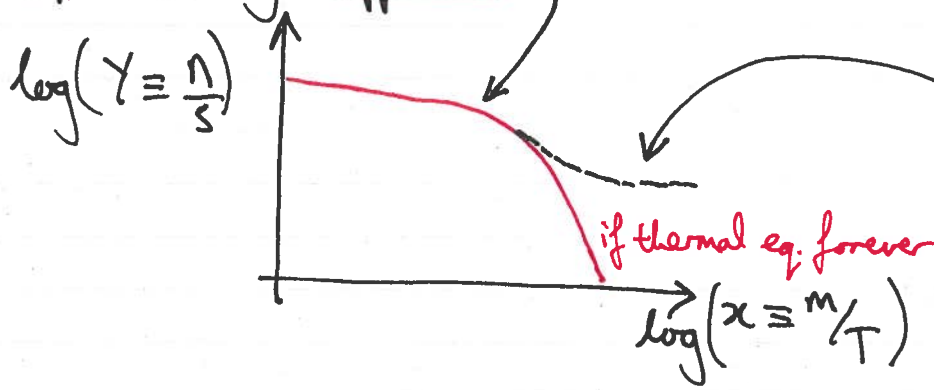
If a particle is still in thermal equilibrium when it becomes non-relativistic, then we take the non-relativistic limit of Fermi-Dirac / Bose-Einstein. This gives

$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} \exp\left[-\frac{m}{T}\right]$$

(still approximating $\mu \simeq 0$, although this isn't true forever)

$$\rho \simeq mn$$ (energy density dominated by rest mass)

Here, $m \gg T$, so annihilation dominates over the (now very difficult) process of pair production, and the abundance is exponentially suppressed.



Of course, at some point Γ falls below H, and the interactions cease and the particles don't get destroyed any more. The curve of Y against x then levels off.

At exactly what Y (what "relic abundance") this happens depends on $\langle \sigma v \rangle$.

$\Longrightarrow$ $\boxed{\text{Slide Q}}$

Let's conform our intuition with math. We need to set up the differential eqn:

$$\dot{n} + 3Hn = (\text{interactions / collisions})$$

Recall that we are modelling a process $X + \bar{X} \to$ other things.

The rate $\Gamma = n\langle\sigma v\rangle$ ~~for~~ is for n particles hitting one target. But I have n possible targets ie this is ~~an~~ n^2 process. The total depletion rate is therefore $n^2\langle\sigma v\rangle$

$$\Rightarrow \quad \dot{n} + 3Hn = \underbrace{-n^2\langle\sigma v\rangle}_{\text{depletion of } X} + \underbrace{q\langle\sigma v\rangle}_{\text{replenishing of } X \text{ by reverse process}}$$

an unknown

The second term is independent of n (which is the density of X)
We don't know what "q" is, but there is a ~~hand~~ handy trick. In equilibrium, the LHS is zero, so

$$0 = -n_{eq}^2\langle\sigma v\rangle + q\langle\sigma v\rangle \Rightarrow q = n_{eq}^2$$

and therefore $\boxed{\dot{n} + 3Hn = -\langle\sigma v\rangle(n^2 - n_{eq}^2)}$

("Ricatti differential equation")

Make the following variable transformation to non-dimensionalize it:

$$Y \equiv \frac{n}{s}, \quad x \equiv \frac{m}{T}, \quad \lambda \equiv \frac{2\pi^2}{45} g_{*s} \frac{m^3\langle\sigma v\rangle}{H(T=m)}$$

Hubble param evaluated when ~~non~~ relativistic to non-rel. transition.

$$\Rightarrow \quad \boxed{\frac{dY}{dx} = -\frac{\lambda}{x^2}(Y^2 - Y_{eq}^2)}$$

This cannot be solved analytically. But after freezeout, we know that $Y \gg Y_{eq}$ (remember the exponential drop-off!)

Abundance in infinite far future

Then $\frac{dY}{dx} = -\frac{\lambda}{x^2} Y^2 \Rightarrow \int_{Y_{freezeout}}^{Y_\infty} \frac{dY}{Y^2} = -\lambda \int_{x_{freezeout}}^{\infty} \frac{dx}{x^2}$

$$\Rightarrow \frac{1}{Y_\infty} - \frac{1}{Y_{freezeout}} = \frac{\lambda}{x_{freezeout}}$$

Assuming $Y_{freezeout} \gg Y_\infty$, we get $\boxed{Y_\infty \approx \frac{x_{freezeout}}{\lambda}}$

Recalling that $\lambda \propto \langle \sigma v \rangle$, we see indeed that the higher the cross-section for annihilation, the lower the relic abundance.

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We now have all the pieces! Let's look @ two applications, one involving cold relics and one involving hot relics

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WIMP Dark Matter (cold relic application)

Suppose we hypothesize that dark matter is a cold relic whose abundance was set by

$$X + X \rightleftharpoons SM + SM$$

(annihilation to Standard Model particles)

We know that the relic abundance is set by $\langle \sigma v \rangle$, so we might ask the following — if we set the relic abundance to be the observationally known $\Omega_{DM} \approx 5\Omega_b$, do we get something reasonable to $\langle \sigma v \rangle$, or something ridiculous?

Warning: serious order-of-magnitude estimation ahead!

First, remember that at freezeout, $\Gamma = n\langle\sigma v\rangle \sim H$, so

Subscript "f" for "evaluated at freezeout"

$$n_f\langle\sigma v\rangle \sim \underbrace{\frac{T_f^2}{M_{pl}}}$$

From expression for H during radiation domination.

I can relate n_f to some numbers that I know better. I know that at matter-radiation equality ("mre") the energy densities of matter and photons were equal

$$\Rightarrow \quad \rho_m = \rho_r \quad \Rightarrow \quad m_{DM}\, \underbrace{n_{mre}}_{\text{DM \# density @ mre}} \approx \underbrace{\frac{\pi^2 g_*}{30}}_{=1 \ \because\ g_* \simeq 3 \text{ @ mre}} T_{mre}^4$$

$$\Rightarrow \quad n_{mre} \approx \frac{T_{mre}^4}{m_{DM}}.$$

But since freezeout, all that the DM has been doing is diluting as a^{-3}, which means

$$n_f = n_{mre}\left(\frac{a_{mre}}{a_f}\right)^3 \underset{T \propto \frac{1}{a}}{=} n_{mre}\left(\frac{T_f}{T_{mre}}\right)^3 \overset{\text{Insert}}{=} \frac{T_{mre}\, T_f^3}{m_{DM}}$$

This gives $\langle\sigma v\rangle \sim \underbrace{\left(\frac{m_{DM}}{T_f}\right)}_{\approx 1,\ \text{assume barely non-relativistic @ freezeout}} \dfrac{1}{m_{Pl} \times T_{mre}}$

Plug in $M_{Pl} \sim 10^{19}$ GeV and $T_{mre} \sim 1$ eV and get

$$\boxed{\langle\sigma v\rangle \sim 10^{-27}\ \mathrm{cm^3/s}}$$

In natural units where $\hbar = c = 1$, $[\langle \sigma v \rangle] = \text{mass}^{-2}$.
If $\langle \sigma v \rangle$ doesn't depend on velocity ("s-wave" scattering) then by dimensional analysis,

$$\langle \sigma v \rangle \sim \frac{\alpha^2}{m_{DM}^2} \leftarrow \text{fine structure constant}$$

In electroweak theory, $\alpha \sim 10^{-2}$, so to get $\langle \sigma v \rangle \sim 10^{-27}\ \text{cm}^3/\text{s}$

$$\Rightarrow \boxed{m_{DM} \sim \text{few} \times 100\ \text{GeV}}$$

This is the typical energy scale of a weak interaction!

This was called the "WIMP miracle": if we hypothesize that a "weakly interacting massive particle" interacts with the Standard Model via cross-sections roughly expected for the weak nuclear force, then the DM abundance Ω_{DM} comes out right!