PHY644 Problem set 2

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Problem 1: Free-fall Time

We are asked to derive the true free fall time t_{ff} of presser-less dust ball of uniform density ρ collapsing.

The total mass of the sphere is :

$$M = \frac{4}{3}\pi\rho r_0^3 \tag{1}$$

where M is the total mass, and r_0 is the initial radius (max radius). The t_{ff} is the time it takes for a test mass on the surface to fall to the centre.

We know from Guass's law for gravity that this problem is the equivalent of asking how long it takes for a test mass to fall into the body it is orbiting.

From Kepler's 3rd law we know that:

$$\frac{P^2}{a^3} = \frac{4\pi^2}{G(M+u)} \tag{2}$$

Where P is the period of the orbit, a is the semi-major axis, M mass of the large body and u is our test mass. $M >> u$, so we can neglect u. This is true regardless of the eccentricity (e) of the orbit, for the special case of $e = 1$ (meaning $a = \frac{1}{2}r_0$), the orbit is a line. The t_{ff} is then interpreted as $\frac{1}{2}P$. Substituting into equation 2:

$$\frac{t_{ff}^2}{r_0^3} = \frac{\pi^2}{8GM} \tag{3}$$

$$t_{ff}^2 = \frac{\pi^2 r_0^3}{8GM} \tag{4}$$

$$t_{ff} = (\frac{\pi^2 r_0^3}{8GM})^{\frac{1}{2}} \tag{5}$$

Now we replace M with equation 1.

$$t_{ff} = (\frac{\pi^2 r_0^3}{8G\frac{4}{3}\pi\rho r_0^3})^{\frac{1}{2}} \tag{6}$$

Lots of things cancel!

$$t_{ff} = (\frac{3\pi}{32G\ \rho})^{\frac{1}{2}} \tag{7}$$

We can factor out a $\frac{1}{\sqrt{16}}$

$$\boxed{t_{ff} = \frac{1}{4}\sqrt{\frac{3\pi}{2G\ \rho}}} \tag{8}$$

The hard way is using conservation of energy, which I will also do. I am assuming that energy conservation holds, for a test mass at the edge of the surface

$$E = \frac{1}{2}v_0^2 - \frac{GM}{r_0} = \frac{1}{2}v(r)^2 - \frac{GM}{r} \tag{9}$$

where E is a constant, and this is the per unit mass energy. We take v_0 to be 0.

We can rearrange for $v(r)$:

$$v(r)^2 = 2GM(\frac{1}{r} - \frac{1}{r_0}) \tag{10}$$

now we have a first order differential equation:

$$\frac{dr}{dt} = -[2GM(\frac{1}{r} - \frac{1}{r_0})]^{0.5} \tag{11}$$

with the same initial conditions, the $-$ comes from falling inwards.

$$-[2GM(\frac{1}{r} - \frac{1}{r_0})]^{-0.5}dr = dt \tag{12}$$

The integral bounds are from r_0 to 0 on the left hand side and from 0 to t_{ff} on the right hand side

$$\int_{r_0}^{0} -\left[2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)\right]^{-1/2} dr = \int_0^{t_{ff}} dt \tag{13}$$

$$\int_0^{r_0} \left[2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)\right]^{-1/2} dr = t_{ff} \tag{14}$$

~~Now we use our integral table aka wolfram alpha (it looks like a u and then trig sub).~~

Before we can use an integral table, we need to simplify more, let $u = \frac{r}{r_0}$, $du = \frac{1}{r_0}dr$.

$$\int_0^1 [2GM(\frac{1-u}{ur_0})]^{-1/2}\frac{1}{r_0}du = t_{ff} \tag{15}$$

Problem

Problem 3: Singular Isothermal Spheres

Suppose that the probability distribution of velocities of particles (e.g., stars) in a galaxy are given by a Maxwell-Boltzmann distribution, where the velocities are Gaussian:

$$p(\mathbf{v}) = \frac{1}{(2\pi\sigma^2)^{3/2}} \exp\left(-\frac{|\mathbf{v}|^2}{2\sigma^2}\right), \tag{16}$$

Here, $\mathbf{v}$ represents the 3D velocity vector (v_x, v_y, v_z), and $|\mathbf{v}|$ is its magnitude v.

Problem 3A:

We are asked to show that the standard deviation of Equation 16 is σ.

The symmetrical 3D Gaussian factorizes:

$$p(\mathbf{v}) = \frac{1}{(2\pi\sigma^2)^{3/2}} \exp\left(-\frac{{v_x}^2 + {v_y}^2 + {v_z}^2}{2\sigma^2}\right), \tag{17}$$

In any single dimension the Maxwell-Boltsman equation is therefor:

$$p(v) = \frac{1}{(2\pi\sigma^2)^{3/2}} \exp\left(-\frac{v^2}{2\sigma^2}\right), \tag{18}$$

The variance is defined as $\mathrm{Var}(v) = \mathbb{E}[v^2] - (\mathbb{E}[v])^2$. (Variance is the square of the STD). In our cause $\mathbb{E}[v] = 0$ due to symmetry.

$\mathbb{E}[x^2]$ is given by:

$$\mathbb{E}[v^2] = \int_{-\infty}^{\infty} v^2 p(v) dv \tag{19}$$

Substituting in our $p(v$

$$\mathbb{E}[v^2] = \int_{-\infty}^{\infty} v^2 \frac{1}{(2\pi\sigma^2)^{3/2}} \exp\left(-\frac{v^2}{2\sigma^2}\right) dv \tag{20}$$

Integrals of this type are known as Gaussians, and their solution is well known.

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2}\, dx = \frac{\sqrt{\pi}}{2a^{3/2}}, \quad a > 0 \tag{21}$$

In our case, $a = \frac{1}{2\sigma^2}$.

Clearly this evaluates to $\boxed{\mathbb{E}[v^2] = \sigma^2}$.

Problem 3B:

Write down a differential equation for hydrostatic equilibrium of a spherically symmetric system in terms of the radial coordinate r, the density ρ and pressure P (both of which can be functions of r), G, and M_r (the total mass enclosed internal to radius r).

Hydrostatic equilibrium follows:

$$\boxed{\frac{dP}{dr} = -\rho(r)\frac{GM(r)}{r^2}} \tag{22}$$

where M is the enclosed mass.

(we are not asked to *solve* the differential equation, only to write it)

Problem 3C:

For an isothermal, isotropic velocity distribution the pressure is

$$P = \rho\sigma^2 \tag{23}$$

We are asked now to solve the differential equation :(. Rewriting Equation 22, and using $\frac{dM}{dr} = 4\pi r^2 \rho$, (its easier with the ln according to some online textbook I found)

$$\frac{\sigma^2}{r^2}\frac{d}{dr}(r^2\frac{d\ln\rho}{dr}) = -4\pi G\rho(r) \tag{24}$$

Ansatz for a power-law solution:

$$\rho(r) = \frac{A}{r^n}, \tag{25}$$

A is a constant with appropriate units, and I require $A > 0$, $n > 0$.

This means that:

$$\frac{d\ln\rho}{dr} = \frac{-n}{r} \tag{26}$$

The full left hand side (LHS) of Equation 24 is:

$$\text{LHS} = \frac{-\sigma n}{r^2} \tag{27}$$

The equation is now:

$$\frac{-\sigma n}{r^2} = -4\pi G\frac{A}{r^n} \tag{28}$$

If this holds, then $n = 2$, and $A = \frac{\sigma^2}{2\pi G}$.

and therefor:

$$\boxed{\rho(r) = \frac{\sigma^2}{2\pi G}\frac{1}{r^2}}, \tag{29}$$

which is singular at $r = 0$.

Problem 3D:

We are asked to find the circular velocity v_c of a test particle placed in a circular orbit.

The circular orbit is given by:

$$v_c = \frac{GM(r)}{r} \tag{30}$$

where $M(r)$ is the enclosed mass.

$M(r)$ is given by:

$$M(r) = 4\pi\int_0^r \rho(r)\, r^2\, dr. \tag{31}$$

We know $\rho(r)$, its our answer to problem 3C equation 29.

So the circular orbit is given by:

$$v_c = \frac{G}{r}4\pi\int_0^r \frac{\sigma^2}{2\pi G}\frac{1}{r^2}\, r^2\, dr \tag{32}$$

This looks scary but its really easy, just with a lot of constants, solving we have the solution.

$$\boxed{v_c = \sigma^2} \tag{33}$$

Woah! This is actually a super cool result!

Problem 3E:

A more realistic $\rho(r)$ is Navarro-Frenk-White (NFW) given by:

$$\rho(r) = \frac{\rho_0}{(\frac{r}{r_s})(1 + \frac{r}{r_s})^2} \tag{34}$$

where ρ_0 is an overall normalization and r_s is known as the scale radius.

The question is how does the steepness of the NFW profile compare to that of the singular isothermal sphere at small r? At large r?. This is pretty straight forward comparison of $\frac{r}{r_S}$, and a derivative.