PHYS 644 Lecture #7: Geometry of our Universe

And now we begin cosmology! Unlike with our study of galaxies where we cared about individual objects, these things will now just be samples from a statistical distribution. It's like being a public health official vs. a doctor.

Quick note: for the next portion of the course, we will use natural units where $\hbar = c = k_B = 1$.

Notably, we are not working in geometric units where $G=1$. Here, $G \neq 1$, and in fact

Lots of particle cosmologists prefer

$$M_{\text{planck}} \equiv \sqrt{\frac{\hbar c}{G}} = G^{-1/2}.$$

Most physical cosmologists prefer to write equations in terms of

Cosmological Principle

Describing the spacetime geometry of our Universe is a daunting task, but luckily there are simplifications we can make (which we can later test empirically). For example...

Cosmological principle: our Universe is homogeneous (no special locations) and isotropic (no special directions)

Sometimes people like to say that this is only true on large scales, but I prefer instead to make the more powerful statement that our Universe is statistically homogeneous and isotropic. Even on small scales, the statistics of our Universe is the same everywhere.

$\Rightarrow$ $\boxed{\text{Slide Qs on homogeneity + isotropy + periodic boundary conditions}}$

While there are no special locations or directions in space, this is not true for time because our Universe is expanding. We know this because...

Hubble Law

Hubble discovered galaxies all receding from us, with the recession velocity v proportional to the distance to the galaxy d:

$$\boxed{v = H_0 d}$$

$\Rightarrow$ Show slide with Hubble's data

The constant H_0 is the Hubble constant and is roughly

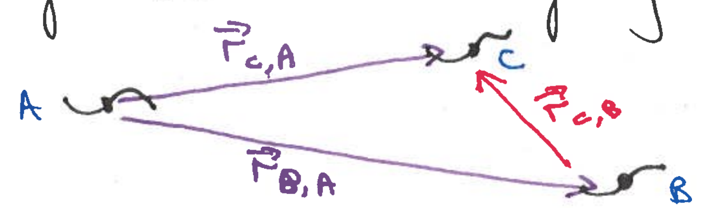
$$H_0 \simeq 70 \ \frac{\text{km/s}}{\text{Mpc}}.$$

Cosmology convention: subscript "0" means value in today's universe

I'm not giving you more significant figures because there is currently a statistically significant disagreement between different methods to measure H_0 known as the Hubble tension.

Often we will write $H_0 = 100\,h \ \frac{\text{km/s}}{\text{Mpc}}$ where $h \simeq 0.7$. Then we can remain non-committal and just express everything in terms of h.

The Hubble Law has an important property that is often underappreciated. Consider three galaxies, A, B, and C, and imagine that we are at galaxy A:



where $\vec{r}_{i,j} \equiv$ position of galaxy i relative to galaxy j.

From our perspective, we have $\vec{v}_{B,A} = H_0 \vec{r}_{B,A}$

$$\vec{v}_{C,A} = H_0 \vec{r}_{C,A}$$

Subtracting these gives $\underbrace{(\vec{v}_{C,A} - \vec{v}_{B,A})}_{\text{This is } \vec{v}_{C,B}} = H_0 (\underbrace{\vec{r}_{C,A} - \vec{r}_{B,A}}_{\text{This is } \vec{r}_{C,B}\,!})$

Which means that $\vec{v}_{C,B} = H_0 \vec{r}_{C,B}$. But this is precisely the Hubble Law that would be written down by an observer at galaxy B! It's what we would expect based on the cosmological principle — neither us nor galaxy B should be special — but it is good to see that consistency with the cosmological principle is built into the Hubble Law.

Scale Factor

Another way to describe our Universe's expansion is to talk about the scale factor. As the distance between galaxies increases, we can say that all the distances are proportional to some function $a(t)$, which is the scale factor. Note again the role played by the cosmological principle — it's what allows us to describe the expansion as a single fct. of time.

By convention, we say $a_0 \equiv a(t_0) = 1$ today's time is age of our Universe.

Now, it can be annoying to describe distances if they're constantly changing, so we define

$$d_{\text{physical}} = a(t) \cdot \chi_{\text{comoving}}$$

Physical distance measured by rulers

"comoving distance"

The comoving distance "takes out the expansion", so if two objects have a changing comoving distance between them, then their motion is not just due to expansion. Alternatively, if we imagine some distance that's just expanding with our Universe, then the comoving distance is the physical distance that we would measure today, since $a_0 = 1$.

$\Rightarrow$ $\boxed{\text{Show visualization}}$

Now let me differentiate both sides:

Reinsert $x = \frac{d_{physical}}{a}$

$$\vec{V}(t) \equiv \frac{d\,\vec{d}_{physical}}{dt} = \frac{da}{dt}\vec{x}_{comoving} = \frac{da}{dt}\frac{1}{a}\vec{d}_{physical}(t)$$

This is nothing other than the Hubble Law $\vec{V} \propto \vec{d}_{physical}$ but at a different time! Thus, the Hubble Law holds at all time provided we use a different proportionality constant:

This is why we used H_0 as our notation for today's Hubble parameter!

$$\boxed{H(t) \equiv H \equiv \frac{\dot{a}}{a} \qquad \text{(Hubble parameter)}}$$

In general, this is a complicated function of time. However, an interesting special case is when we have a universe where the expansion just makes galaxies coast along at a constant speed.

Then $\dot{a} = \text{const.}$ $\Rightarrow$ $H(t) = \frac{\text{const.}}{a} = \frac{H_0}{a}$

$\because$ need $H(t_0) = H_0$ since $a_0 = 1$

Now, $H \equiv \dot{a}/a$ always, so this means $\dot{a} = H_0$ and

$$\int_0^1 da = H_0 \int_0^{t_0} dt \Rightarrow 1 = H_0 t_0 \Rightarrow t_0 = \frac{1}{H_0} \text{ in this universe.}$$

Even outside of this weird universe, this is a useful quantity to define:

$$\boxed{t_H \equiv \frac{1}{H_0} \simeq 14.5 \text{ Gyr} \quad \text{(Hubble time)}}$$

For this "coasting universe" it would be the age of the universe. But the Hubble time is a good rough estimate even for our real Universe.

What kinds of expansion are allowed?

⇒ Slide Q on superluminal vs subluminal expansion.

Comoving coordinates can be useful for seeing if our expansion is superluminal or not. If I used physical coordinates I'd need to see if light can catch up with expansion. In comoving coords, the expansion is factored out, so I'm actually seeing if the (comoving) distance travelled by light goes up or down with time!

In time dt, light moves a physical distance $c\,dt$, so the comoving distance travelled is $\frac{c\,dt}{a}$, and over a finite time interval we have:

c=1 here!

$$\chi = \int \frac{dt'}{a}$$

It's like $c \to c/a$. In GR, the space itself can expand or contract, so even though the light might pass me at c, its "ability to cover space" can be different.

Define $\chi_p \equiv \int_0^t \frac{dt'}{a}$ as the particle (causal) horizon

This is the farthest distance that (unimpeded) light can have travelled from $t=0$ to t. It will be useful later in the course when we want to talk about how far away two things can be and have causal contact.

To get the physical — as opposed to comoving version — we multiply by $a(t)$ to get

$$d_p = a(t)\int_0^t \frac{dt'}{a(t')} .$$

Let's consider two cases for the expansion:

① Subluminal expansion: $a(t) \sim t^p$ where $0 < p < 1$
$\Rightarrow \ddot{a} < 0$ (decelerating)

$$\chi_p \propto \int_0^t \frac{dt'}{t'^p} = \left.\frac{t'^{\,1-p}}{1-p}\right|_0^t = \frac{t^{1-p}}{1-p}.$$

Eg $p = \frac{2}{3}$: $\chi_p \propto \frac{t^{1/3}}{(1/3)} = 3t^{1/3}$

Diverges as $t \to \infty$! Light wins and given enough time, can see the whole universe!

② Superluminal expansion:

$a(t) \sim t^p$ where $p > 1$
$\Rightarrow \ddot{a} > 0$ (accelerating)

$$\chi_p \propto \int_{t_0}^{t'} \frac{dt'}{t'^p} = \frac{1}{(p-1)}\frac{1}{t_0^{p-1}} - \frac{1}{(p-1)}\frac{1}{t^{p-1}}$$

As $t \to \infty$, $\chi_p \to \frac{1}{(p-1)}\frac{1}{t_0^{p-1}}$.

Only see finite range of χ no matter how long I wait!

The light cannot catch up with expansion for this case.

[S]adly, we [a]re now [i]n an [a]ccelerating universe, so we are [i]n this case [a]nd cannot see much more just by living billions more years!

Intuition: $p=1$ where $a \propto t$ is the dividing line because the amount of distance light can travel is ct, so if $a \propto t$, the expansion balances this effect.

Friedman – Lemaître – Robertson – Walker metric

Let's bring together space and time. We've talked about how to describe the expansion rate, and combining this with the cosmological principle, it turns out that GR only allows three possible forms for the spacetime metric:

This describes the 3D curvature of space, not 4D curvature of spacetime

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right] \quad \text{(FLRW metric)}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$ and κ is the spatial curvature, and it can be positive, zero, or negative

$\Rightarrow$ $\boxed{\text{Show visualization slides}}$

With some algebraic tricks, this can also be written as

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + S_\kappa(\chi)^2 d\Omega^2\right]$$

where
$$S_\kappa(\chi) = \begin{cases} R_0 \sinh\left(\frac{\chi}{R_0}\right) & \text{for } \kappa < 0 \\ \chi & \text{for } \kappa = 0 \\ R_0 \sin\left(\frac{\chi}{R_0}\right) & \text{for } \kappa > 0. \end{cases}$$

with R_0 being the radius of curvature of our Universe.

※ People will often also define $k \equiv \kappa R_0^2$ and $k = +1, 0, \text{or } -1$.

Observables

With the FLRW metric, we can start to compute some observable quantities, because what we have so far (comoving distances, scale factors, ...) aren't very observable quantities.

First, we have redshift : $$z \equiv \frac{\lambda_{obs} - \lambda_{emitted}}{\lambda_{emitted}}$$ of some spectral line from a galaxy.

The elongation of wavelengths can be shown to be proportional to the scale factor, which means

$$\frac{\lambda_{obs}}{\lambda_{em}} = \frac{a(t_0)^{=1 \text{ today}}}{a(t)_{\text{earlier time}}} \Rightarrow \boxed{1+z = \frac{1}{a}}$$

The redshift is a directly observable quantity, unlike $a(t)$. And since different $a(t)$ also correspond to different times, we also use redshift to keep time – earlier being higher z.

And since it takes time for light to get to us from distant objects – and the further away the longer the time taken – we see that redshift is also a measure of distance