Initial Fluctuations and Growth

Recap from last time:

- Inflation can provide the seed density fluctuations of our Universe. Quantum fluctuations in the inflaton field get stretched out to cosmological scales.

- Thinking about the perturbations Fourier modes-by-Fourier mode, ~~as inflation stretch~~ once inflation has stretched a particular Fourier mode so that its spatial wavelength is much larger than the horizon scale, that Fourier mode is frozen in amplitude and doesn't undergo gravitational collapse.

- As time goes on, the size of the horizon grows, and eventually the modes "reenter the horizon" — at that point, the fluctuations for that particular Fourier mode can grow in amplitude.

  Large scale (small k) perturbations exited the horizon early and only recently reentered.
  Small scale (large k) perturbations exited the horizon late and ~~only~~ reentered pretty early on.

- 2-pt statistics allow us to describe fluctuations.

  Configuration space: $\xi(r) \equiv \langle \delta(\vec{r}_1)\, \delta(\vec{r}_2) \rangle$

  "Correlation fct." $r \equiv |\vec{r}_1 - \vec{r}_2|$ overdensity $\delta(\vec{r}) \equiv \frac{\rho(\vec{r}) - \bar{\rho}}{\bar{\rho}}$

  Fourier space: $\langle \tilde{\delta}(\vec{k}_1)\, \tilde{\delta}(\vec{k}_2)^* \rangle = (2\pi)^3 \delta^D(\vec{k}_1 - \vec{k}_2)\, P(k_1)$

  $\tilde{\delta}(\vec{k}) = \int d^3\vec{r}\, e^{-i\vec{k}\cdot\vec{r}}\, \delta(\vec{r})$ Different $\vec{k}$ modes uncorrelated "Power Spectrum"

The power spectrum turns out to be the Fourier transform of the correlation function:

$$P(\vec{k}) = \int d^3\vec{x}\; e^{i\vec{k}\cdot\vec{x}}\, \xi(\vec{x}).$$

As we can see from the definition in terms of the 2-pt function of Fourier modes, it also is the variance — i.e. the mean square amplitude — of fluctuations on a particular $\vec{k}$ scale.

Today we finally close the book on inflation by computing some of its predictions regarding the seed fluctuations.

But first I need to build one more piece of mathematical machinery.

Suppose I wanted to compute the contributions to the variance in configuration space from Fourier modes of various length scales:

$$\langle \delta^2(\vec{r}) \rangle = \left\langle \int \frac{d^3\vec{k}_1}{(2\pi)^3} e^{i\vec{k}_1\cdot\vec{r}}\, \tilde{\delta}(\vec{k}_1) \int \frac{d^3\vec{k}_2}{(2\pi)^3} e^{-i\vec{k}_2\cdot\vec{r}}\, \tilde{\delta}(\vec{k}_2)^* \right\rangle$$

$$= \int \frac{d^3\vec{k}_1}{(2\pi)^3} \frac{d^3\vec{k}_2}{(2\pi)^3} e^{i\vec{k}_1\cdot\vec{r}} e^{-i\vec{k}_2\cdot\vec{r}} \underbrace{\langle \tilde{\delta}(\vec{k}_1)\, \tilde{\delta}(\vec{k}_2)^* \rangle}_{\equiv (2\pi)^3 \delta^D(\vec{k}_1 - \vec{k}_2)\, P(\vec{k}_2)}$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} P(\vec{k}) = \int \frac{d^3\vec{k}}{(2\pi)^3} P(k)$$

Assuming isotropy

$$= \int \frac{dk\, 4\pi k^2}{(2\pi)^3} P(k) = \int d\ln k\, \frac{k^3 P(k)}{2\pi^2}$$

This assumes that our field — in this case δ — has no units. Otherwise, $\Delta^2(k)$ has whatever units as the square of the original field might be.

We often define $\Delta^2(k) \equiv \frac{k^3 P(k)}{2\pi^2}$ as the "dimensionless power spectrum"

(You can tell that it's dimensionless because $\langle\delta^2\rangle$ is dimensionless and so is $d\ln k$).

From this integral, we see also that $\Delta^2(k)$ is the contribution to the variance of fluctuations $\langle\delta^2\rangle$ per logarithmic interval in k.

I can also write this mathematically as $\Delta^2(k) = \frac{d\langle\delta^2\rangle}{d\ln k}$.

Density fluctuations from inflation

What we want to do now is to go back and predict the amplitude of density fluctuations.

Quantum fluctuations in inflaton $\xrightarrow[\text{reheating}]{}$ density fluctuations

If we were doing this "properly" using relativistic perturbation theory, the relevant quantity would be the gravitational potential Φ of the perturbations.

So we are trying to compute $\Delta^2_\Phi(k) = \frac{d\langle\Phi^2\rangle}{d\ln k}$

This turns out to be constant ie Δ^2_Φ is independent of k!

We know that it can't be exactly exponential, because then inflation would not end. Hang on to that thought

Why? First, recall that what we have is exponential (or at least approximately exponential) expansion:

$$a(t) \propto \exp(Ht)$$

Now, exponential expansion has no preferred origin in time. If I were to ~~blinfo~~ blindfold you and then take off your blindfold at some random time during inflation, you wouldn't be able to tell whether inflation had just started or had been going on for a while. Translating the origin gives:

$$e^{H(t-t_*)} = \underbrace{e^{-Ht_*}}_{\text{const.}} e^{Ht} \propto e^{Ht}$$ (absorbed extra constant into redefinition of the proportionality factor).

We now combine this with another fact from before — large scale perturbations are sourced by quantum fluctuations early in the inflationary expansion (because it takes time to stretch them out), ~~whe~~ whereas small scale perturbations are sourced late.

But we just said that the behaviour is the same at all times! So the fluctuation amplitude must not depend on the k scale — a scale invariant spectrum where $\Delta^2_\Phi(k) = \text{const.}$

Or to be more precise, since modes freeze once they exit the horizon, the amplitude upon horizon crossing is constant.

Which means all that we have left to do is to determine the amplitude of fluctuations, ie the constant.

How do we do that? Ideally, by doing some quantum field theory. But we don't really want to assume QFT for this class. What do physicists do when there's a piece of physics they don't know how to do / don't want to do?

Dimensional analysis!

One way to think about why there are density fluctuations sourced by inflation is to say that because of quantum fluctuations, different parts of our Universe finish inflating a slightly different times. Different bits ~~re~~ exit the slow-roll part of the inflaton potential at slightly different times. They start reheating at slightly different times. Bits that finish inflation a little earlier have had a little extra time to expand and cool off $\Rightarrow$ these become the parts of our Universe that are a tiny bit below average in energy density ie "cold spots".

So we can write $\delta t = \dfrac{\delta\phi}{\dot\phi}$ inflaton field!

Suppose I want to find the r.m.s. fluctuation amplitude of the gravitational potential, $\sqrt{\Delta^2_\Phi}$. This is a dimensionless number, so we need to combine δt with something that makes it dimensionless. We do have such a thing! The only scale in the problem is H, and this has dimensions of inverse time, so

$$\sqrt{\Delta^2_\Phi} \sim H\,\delta t \sim \frac{H\,\delta\phi}{\dot\phi}.$$

To proceed, I need to know the typical fluctuation amplitude of the inflaton field, $\delta\phi$. This we can do with dimensional analysis too!

We have to careful, though — we've set $\hbar = c = 1$, so we must remember that.

With "natural units", all quantities can be expressed in powers of energy. For example:

$$[t] = [\omega^{-1}] = \left[\frac{\hbar}{E}\right] \to \left[\frac{1}{E}\right]$$

Hubble param.

Since $[H] = [t^{-1}]$, this means $[H] = [E]$ in natural units.

energy

What are the units of $\delta\phi$?

Recall that $\rho \sim \frac{1}{2}\dot{\phi}^2$, where this is the kinetic energy.

$$\left.\begin{aligned} [\text{L.H.S.}] &= \left[\frac{E}{L^3}\right] \underset{c=1}{=} \left[\frac{E}{t^3}\right] = [E^4] \\ [\text{R.H.S.}] &= \left[\frac{\phi^2}{t^2}\right] = \left[E^2\phi^2\right] \end{aligned}\right\} \Rightarrow [\phi] = [E].$$

But since the Hubble parameter is the only scale in the problem, we must have

$$\delta\phi \sim H \xrightarrow[\text{QFT calc}]{\text{more exact}} \delta\phi = \frac{H}{2\pi}$$

This means we have our fluctuation amplitude

$$\boxed{\sqrt{\Delta_\Phi^2} = \frac{H^2}{2\pi\dot{\phi}}}$$

Or said differently, $\boxed{\Delta_\Phi^2(k) = \frac{H^4}{(2\pi\dot{\phi})^2}}$

Note that I have restored the k-dependence here ∵ the full QFT calculation says this result is more exact than we might think

We can simply this into a nice form by remembering that under the slow-roll approx., we have:

$$H^2 \simeq \frac{8\pi G}{3} V(\phi) \quad \text{and} \quad 3H\dot{\phi} = -V'(\phi)$$

$V' \equiv \frac{\partial V}{\partial \phi}$.

Then $\Delta^2_\Phi(k) = \frac{128\pi G^3}{3}\left(\frac{V^3}{V'^2}\right)$

Now suppose I parameterize $\Delta^2_\Phi(k)$ as a power law, such that

$$\Delta^2_\Phi(k) \propto k^{n_s - 1}.$$

The spectral index n_s controls how close to scale invariant we are. We expect $n_s \simeq 1$, but let's see!

$n_s - 1 = \frac{\partial \ln \Delta^2_\Phi}{\partial \ln k}$. How do we compute this derivative?

Recall that we really want to understand what ~~happens~~ the amplitudes of the fluctuations are when modes cross the horizon, because that's when they're frozen in.

So we want to evaluate this at

$$\lambda \sim \frac{a}{k} \sim H^{-1}$$

Physical wavelength

size of horizon $\frac{c}{H}$ when $c=1$

comoving wavenumber

If $H \simeq$ constant, then $\frac{d}{d\ln k} = a\frac{d}{da}$. But $H = \frac{\dot{a}}{a}$ so $da = aHdt$

$$\Rightarrow \quad da = aH\,dt = \frac{aH}{\dot\phi}\,d\phi$$

and thus $$\frac{d}{d\ln k} = \frac{\dot\phi}{H}\frac{d}{d\phi} = -\frac{1}{8\pi G}\frac{V'}{V}\frac{d}{d\phi}.$$

This means $$n_s - 1 = \frac{\partial \ln \Delta^2_\Phi}{\partial \ln k} = -\frac{1}{8\pi G}\frac{V'}{V}\frac{d}{d\phi}\left[\ln\left(\frac{V^3}{V'^2}\right)\right]$$

$$= \underbrace{-\frac{3}{8\pi G}\left(\frac{V'}{V}\right)^2}_{6\varepsilon} + \underbrace{\frac{1}{4\pi G}\frac{V''}{V}}_{2\eta}$$

Slow-roll params!

$$\Rightarrow \quad \boxed{n_s = 1 - 6\varepsilon + 2\eta}$$

Remember that $\Delta^2_\Phi(k) \propto k^{n_s - 1}$, so since $\varepsilon, |\eta| \ll 1$, it is a key prediction of inflation that there is a nearly scale-invariant spectrum of fluctuations.

Note that this predicts slightly less power on small scales, high k. Recall that scale invariance was a consequence of exp. growth. Small scales are generated towards the end of inflation, when inflation was "running out of steam" $\Rightarrow$ weaker perturbations

$$\boxed{\text{Planck 2018 + galaxy surveys: } n_s = 0.9665 \pm 0.0038.}$$

One more thing — here we talked about creating density perturbations, which in turn cause perturbations in the grav. potential Φ.

↳ These types of perturbations (i.e. density) are known as scalar perturbations.

But during inflation, in addition to the inflaton field, the gravitational field is omnipresent too. The metric field itself has quantum fluctuations!

↳ These are tensor perturbations.

The direct perturbations in the scalar field were $\propto H$. Same for the metric perturbations. So:

square ∵ power spectrum

$$\Delta_T^2 \propto GH^2 \propto V$$

This is why people say that if you detect tensor perturbations, you measure the energy scale of inflation

$$\frac{\Delta_T^2}{\Delta_\Phi^2} \propto V\left(\frac{V'^2}{V^3}\right) = \left(\frac{V'}{V}\right)^2 \propto \varepsilon$$

~~⇒ Tensor-to-scalar Ratio $\propto \varepsilon$.~~

Measuring tensor and/or scalar perturbations therefore helps us rule out models of inflation.

⇒ $\boxed{\text{Show Planck slide}}$