PHYS644 Problem Set 7

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Problem 1: Warm Dark Matter (WDM)

Problem 1A:

We have our WDM species X, and three three other particles A, B, and C — with associated degeneracy factors g_i where $i = \text{WDM}, A, B, C$. In our case $g_{\text{WDM}} = 2$ - they are Fermions.

We assume X has temperature T_X and has decoupled just before the first annihilation event which heats up the others. A is the first to annihilate and so on, and heats up the rest to $T_{\text{After A}}$ and so on.

We are asked to find an expression to relate T_X and $T_{\text{After A}}$.

We can use the conservation of commoving entropy density $s \propto g_i T^3 a^3$. I assume all species are relativistic, We can write:

$$(g_A + g_b + g_c)T_X^3 a^3 = (g_b + g_c)T_{\text{After A}}^3 a^3 \tag{1}$$

the a's cancel as they are at the same time, This leads to:

$$\boxed{T_{\text{After A,}}^3 = (\frac{g_A + g_b + g_c}{g_b + g_c})T_x^3} \tag{2}$$

Problem 1B:

We use the same reasoning as before for problem A, and state

$$\boxed{T_{\text{After A,B}}^3 = (\frac{g_b + g_c}{g_c})T_{\text{After A}}^3} \tag{3}$$

Problem 1B:

Now we just substitute in our two parts.

$$T_{\text{After A,B}}^3 = (\frac{g_b + g_c}{g_c})(\frac{g_A + g_b + g_c}{g_b + g_c})T_x^3 \tag{4}$$

$$T_{\text{After A,B}}^3 = (\frac{g_A + g_b + g_c}{g_c})T_x^3 \tag{5}$$

It doesn't matter if we did the annihilations sequentially A, and then B or all at once. Only the total degeneracy g before annihilations vs after matters.

Problem 1D:

Problem 1D: Part i

We can use the trick we learned from parts A, B, C[1].

[1] almost a baby name ABCDE!

If X decouples at temperature T_f — where it no longer shares entropy, and then the thermal bath evolves from having degrees of freedom $g_*(T_f) \Rightarrow g_*(T_{\text{new}})$.

We can write the following:

$$g_*(T_f)T^3_{\text{before}} = g_*(T_{\text{new}})T^3_{\text{after}} \tag{6}$$

or more like the handout, where we can write $T_X = T_{\text{before}}$, and

$$\boxed{\frac{T_x}{T_{\text{new}}} = \left(\frac{g_*(T_{\text{new}})}{g_*(T_f)}\right)^{1/3}} \tag{7}$$

Problem 1D Part II

If T_{new} is the point in time just before neutrinos decouple, we can use the relationship from class.

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma \Rightarrow T_\gamma = \left(\frac{11}{4}\right)^{1/3} T_{\text{new}} \tag{8}$$

As just before e^+e^- annihilation, photons, electrons, and positrons, and neutrinos where at the same T.

Now we just toss that into our solution for part I.

$$\frac{T_X}{T_\gamma} = \frac{T_X}{T_{\text{new}}}\frac{T_{\text{new}}}{T_\gamma} = \left(\frac{g_*(T_{\text{new}})}{g_*(T_f)}\right)^{1/3} \left(\frac{4}{11}\right)^{1/3} \tag{9}$$

Cubing both sides

$$\boxed{\left(\frac{T_X}{T_\gamma}\right)^3 = \left(\frac{g_*(T_{\text{new}})}{g_*(T_f)}\right)\left(\frac{4}{11}\right)} \tag{10}$$

We can see that $g_*(T_{\text{new}}) = 10.75$.

Problem 1D Part III

This is an implicit part of the problem. Calculating $g_*(T_{\text{new}}) = 10.75$ by hand.

$$g_* = g_{\text{Bosons}} + \frac{7}{8}g_{\text{Fermions}} \tag{11}$$

We have photons, which are bosons with a factor of 2, and electrons and positrons which are each fermions with a factor of 2 each. and three neutrino species each with a factor of 2 and are fermions.

$$\boxed{g_*(T_{\text{new}}) = 2 + \frac{7}{8}(4+6) = 10.75} \tag{12}$$

Problem 1E:

if X is the dark matter and that it is non-relativistic today, its number density is something we can write down now.

We know:

$$\rho_{X,0} = m_X n_{X,\mathrm{relic}} \tag{13}$$

$$\tag{14}$$

$$\Omega_{\mathrm{DM}} = \frac{\rho_{X,0}}{\rho_{\mathrm{crit},0}} \tag{15}$$

$$\tag{16}$$

$$\rho_{\mathrm{crit}} = \frac{3H_0^2}{8\pi G} = 1.87 \times 10^{-26} h\,\mathrm{kgm}^{-3} \tag{17}$$

So we can write:

$$n_{x,\mathrm{relic}} = \frac{\rho_{X,0}}{m_X} = \frac{\Omega_{\mathrm{DM}}}{m_X}\rho_{\mathrm{crit},0} = \frac{\Omega_{\mathrm{DM}}}{m_X}\frac{3H_0^2}{8\pi G} \tag{18}$$

Big Omega DM is unit-less, and n should have units of number density, we need to restore our missing units from natural units. We can use $H = 100\,\mathrm{Km/s/Mpc}$, and h as the dimensionless correction. We know 1eV =...

Problem 2: Acceleration Redshift

z_{acc} happens when the universe expansion just starts (postively) accelerating IE when $\ddot{a} = 0$.

The second Friedmann equation becomes

$$\frac{\ddot{a}}{a} = 0 = -\frac{4\pi G}{3}(\rho + 3p) \tag{19}$$

Therefor our condition is when $\rho + 3p = 0$. We are absorbing Λ into its own effect ρ and p. Baryonic matter gives $p = 0$, and dark energy has $p = w\rho_{DE}$.

Now we have

$$0 = \rho_m + \rho_{DE}(1 + 3w) \Rightarrow \rho_m = -\rho_{DE}(1 + 3w) \tag{20}$$

recall that

$$\rho_{\text{crit},0} = \frac{3H_0^2}{8\pi G} \tag{21}$$

$$\Omega_{m,0} = \frac{\rho_{m,0}}{\rho_{\text{crit},0}} \tag{22}$$

$$\Omega_\Lambda = \Omega_{DE,0} = \frac{\rho_{DE,0}}{\rho_{\text{crit},0}} \tag{23}$$

We now insert these into our condition, we convert to values at z recall that $a = 1/(1 + z)$

$$\Omega_{m,0}\,\rho_{\text{crit},0}\,(1 + z)^3 = -(1 + 3w)\,\Omega_\Lambda\,\rho_{\text{crit},0}\,(1 + z)^{3(1+w)}. \tag{24}$$

Now we solve for z — this will be z_{acc}.

$$(1 + z)^{-3w} = -(1 + 3w)\,\frac{\Omega_\Lambda}{\Omega_{m,0}} \tag{25}$$

$$\boxed{z = [-(1 + 3w)\,\frac{\Omega_\Lambda}{\Omega_{m,0}}]^{-1/3w} - 1} \tag{26}$$

We learned from class (and from $a(t) = 1/(1 + z)$ that $z = 0$ is now, and $z = -1$ is $t = \infty$. Aka $z > 0$ if we are looking backwards in time. We see that the first term on the left is always positive when $w < -\frac{1}{3}$. =