PHYS 644 Lecture #8: Dynamics of our Universe

What we did last time was to build some vocabulary for describing the universe and its Kinematics.

$\Rightarrow$ **Review slides**

Friedman Equation

What we did last time was to talk mostly about Kinematics ie describing the motion. Today we want to talk about dynamics ie the causes of motion.

What's the equation of motion that governs $a(t)$ (the only unknown part of the metric)?

Here's the answer, which (if you know GR, can be obtained by plugging the metric into Einstein's Field Equations) comes in the form of two equations known as the Friedman Equations:

1st Friedman Eqn: $$\boxed{H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \rho}{3} - \frac{K}{a^2}}$$

Mass-energy density, expressed as a mass density

2nd Friedman Eqn: $$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)}$$

Pressure of stuff

- If we ignore the pressure term in the 2nd Friedman Eqn, we see that it's a lot like "F=ma" (although remember that in our equation, "a" is the scale factor, which is proportional to distance, not acceleration!) The minus sign says that pressureless matter (more on this later) pulls back gravitationally on everything and causes the expansion of our Universe to decelerate.

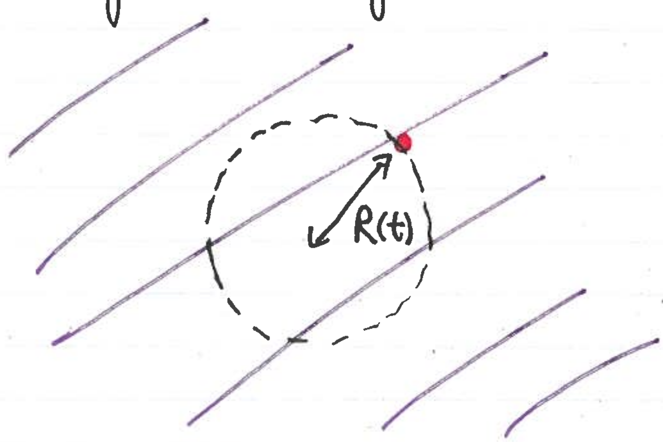
- The equation says that pressure is a source of gravitation too, not just mass! This is actually a general property of GR. Why? Because one of the key lessons of relativity is that phenomena should be describable in whatever reference frame I want. Consider a lump of stuff. In one frame it's a lump of stuff sitting still. In another frame it's some stuff flying at high velocity exerting pressure on things. So the covariant nature of GR forces this upon us.

I know this is not rigorous, but the moral of the story is correct

- The 1st Friedman eqn. is analogous to an energy equation in Newtonian mechanics. Let's see this...

Newtonian Cosmology

Imagine a homogeneous universe filled with matter



Now imagine a test mass. We can say that this particle feels only the gravity from inside the ◌ region. (This is actually a little GR — known as Birkhoff's theorem — slipped in. You might think that this is just Newtonian hollow shells-style arguments, but formally, the relevant integrals diverge in an infinite universe for the Newtonian version).

$$m\frac{d^2R}{dt^2} = -\frac{GM_s m}{R(t)} \Rightarrow \ddot{R} = -\frac{GM_s}{R^2}$$

Multiply both sides by $\dot{R}$ and integrate: $\int \dot{R}\ddot{R}\,dt = \frac{\dot{R}^2}{2} + \text{const.}$

$$\Rightarrow \underbrace{\frac{1}{2}\dot{R}^2}_{\text{K.E. per mass}} - \underbrace{\frac{GM_s}{R}}_{\text{G.P.E. per mass}} = \text{const.} \equiv -\frac{R_0^2 \kappa}{2}$$

Could've called the integration constant anything ... you probably know why I chose this...

Now, $M_s = \frac{4\pi}{3}\rho(t)R(t)^3$ and let $R(t) = a(t)R_0$.

$$\Rightarrow \frac{R_0^2}{2}\dot{a}^2 = \frac{4\pi G R_0^2}{3}\rho(t)a(t)^2 - \frac{R_0^2 \kappa c^2}{2}$$

$$\Rightarrow \boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{\kappa}{a^2}}$$

Yay!

How do we get the 2nd Friedman equation? We can appeal to thermodynamics:

$$dE = \cancel{dQ}^{\,0} - p\,dV$$

We can assume $dQ = 0$. If one imagines a small parcel stuff, there can be no net heat flow into or out of this gas, or else that parcel would be special and we'd violate homogeneity and isotropy.

What's left: $d(\rho a^3) = -p\,d(a^3)$

energy density volume

$$\Rightarrow a^3 d\rho + 3a^2\rho\, da = -3a^2 p\, da$$

$$\Rightarrow \boxed{\dot{\rho} = -3(\rho + p)\left(\frac{\dot{a}}{a}\right)}$$

Insert

Differentiate 1st Friedman equation: $2\dot{a}\ddot{a} = \frac{8\pi G}{3}(2a\dot{a}\rho + a^2\dot{\rho})$

Get $\quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$

Where we are: we have derived a set of relations that tell us how our Universe expands — but only if we know the properties of the stuff in it, via quantities like ρ and p.

Lets do that. First, a common parametrization:

$$\boxed{p = w\rho}$$

"equation of state parameter"

This might seem a little arbitrary, but it's not so crazy. Eg an ideal gas has pressure proportional to mass density.

As our Universe expands, most substances decrease in their energy density ρ. How does this depend on w?

$$\Rightarrow \boxed{\text{Slide Q}}$$

~~If we assume a flat universe for a moment~~

The higher w, the higher the pressure and the harder I need to push when expanding $\Rightarrow$ lose more energy

Plugging this into our equation for $\dot{\rho}$, we get

$$\dot{\rho} = -3(1+w)\rho\left(\frac{\dot{a}}{a}\right) \Rightarrow \frac{d\rho}{\rho} = -3(1+w)\frac{da}{a} \Rightarrow \boxed{\rho \sim \frac{1}{a^{3(1+w)}}}$$

If we assume a flat universe for a moment (ie $K=0$) then

$$\left(\frac{\dot{a}}{a}\right)^2 \propto \frac{1}{a^{3(1+w)}} \Rightarrow \dot{a} \propto a^{-\frac{1}{2}(1+3w)}$$

$$\Rightarrow \boxed{a(t) \propto t^{\frac{2}{3(1+w)}}}$$

What sort of "stuff" do we have in our Universe?

Matter / "Dust" ($w=0$) It doesn't have to literally be dust, but somehow cosmologists call all pressureless matter "dust".

If $k_B T \ll mc^2$, then the rest mass energy dominates the gravitational effect, so we don't have to worry about pressure.

$p=0$, ie $w=0$ $\Rightarrow \boxed{\rho_m \propto \frac{1}{a^3}}$ This is just the dilution of the same amount of stuff as the volume of our Universe expands!

Also $\boxed{a \propto t^{2/3}}$

Vacuum energy ($w=-1$)

The energy of the vacuum just... comes from the vacuum. The density of vacuum energy is therefore constant.

$$\boxed{\rho_\Lambda = \text{const.}} \Rightarrow w=-1 \text{ from } \rho \sim \frac{1}{a^{3(1+w)}}$$

In this case, we have $\left(\frac{\dot{a}}{a}\right)^2 = \text{constant}$

But $\frac{\dot{a}}{a}$ is the Hubble parameter, and if it's constant then it must take on today's value $\Rightarrow \frac{\dot{a}}{a} = H_0 \Rightarrow \boxed{a(t) \propto e^{H_0 t}}$

This is an accelerated expansion. We know that our Universe is expanding at an accelerated rate today. One possiblity is that a pure vacuum energy is the culprit. When people say they want to measure the "dark energy equation of state", they mean measuring w to see if it's just -1, as a test of this hypothesis.

So far, all data are consistent with $w = -1$

Although some DESI galaxy survey results have very mild hints of an evolving w!

Note that while $w = -1$ gives acceleration, this is not a necessary condition.

From the 2nd Friedman eqn we just need $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) > 0$

$$\Rightarrow \rho + 3p < 0 \Rightarrow p < -\frac{1}{3}\rho \Rightarrow \boxed{\text{Need } w < -\tfrac{1}{3} \text{ for acceleration}}$$

Empty universe ($\rho = 0$)

If $\rho = 0$, $\left(\frac{\dot{a}}{a}\right)^2 = -\frac{k}{a^2} \Rightarrow \dot{a} = \text{const.}$

$$\Rightarrow \boxed{a(t) \propto t}$$

Makes sense — with no stuff to pull things back, test particles just coast along at constant velocity.

Or any light degrees of freedom Eg neutrinos at early times

Radiation ($w = 1/3$)

Why is $w = 1/3$ for radiation? Imagine a radiation field in thermal equilibrium. It then follows that we can use the various results for blackbody radiation. In particular:

$$p = \frac{4\sigma}{3c} T^4 \quad \text{and} \quad \rho = 4\sigma T^4 \quad \text{(Stefan-Boltzmann)}$$

So $\boxed{w = \frac{1}{3}}$ and $\boxed{a \propto t^{1/2}}$ and $\boxed{\rho_r \propto \frac{1}{a^4}}$

Another way to see that $\rho_r \propto a^{-4}$ is to say that you get a^{-3} from volume dilution, and then an additional $1/a$ from redshifting, because $E = \frac{hc}{\lambda}$.

In talking about these different components of our Universe, there are two subtle points I want to highlight.

① When talking about radiation, we assumed it was in thermal equilibrium so that we could use blackbody results.

But this is a strange assumption! An example of a radiation field is the CMB. We see CMB photons at almost exactly the same temperature from opposite ends of our Universe! How can they be in thermal equilibrium?

Answer: they're not!

But it still turns out that they follow a blackbody spectrum. Here's why.

Never mind how for now! We'll find out when we study inflation!

Suppose the radiation field started in equilibrium. Then

$$\begin{pmatrix}\text{Energy density} \\ \text{between } \nu_i \text{ and } \nu_i + d\nu_i\end{pmatrix} = \frac{8\pi h}{c^3} \frac{\nu_i^3 \, d\nu_i}{\exp\left(\frac{h\nu_i}{k_B T_i}\right) - 1}$$

"initial"

Divide by $E = h\nu_i$ and multiply by volume dV_i

$$\begin{pmatrix}\#\text{ of photons} \\ \text{between } \nu_i \text{ and } \nu_i + d\nu_i\end{pmatrix} = \frac{8\pi}{c^3} \frac{\nu_i^2 \, d\nu_i \, dV_i}{\exp\left(\frac{h\nu_i}{k_B T_i}\right) - 1}$$

Now, each photon is just minding its own business and redshifting, so $\nu_i = \left(\frac{a_f}{a_i}\right)\nu_f$ ← "final"

This gives
$$\frac{8\pi}{c^3}\frac{\left(\frac{a_f}{a_i}\right)^2 \nu_f^2 \left(\frac{a_f}{a_i}\right) d\nu_f\, dV_i}{\exp\left[\frac{h\nu_f}{k_B\left(\frac{a_i T_i}{a_f}\right)}\right]-1}$$
$= dV_f$!

The distribution remains a blackbody but with a modified blackbody temperature
$$T_f \equiv \frac{a_i T_i}{a_f}.$$

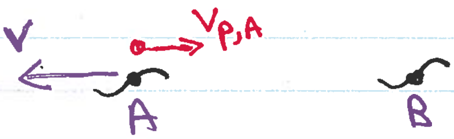
This means that even though it's no longer in thermal eq. it "magically" preserves its blackbody form, and we just need to say that for radiation fields,
$$\boxed{T \propto \frac{1}{a}}$$

(2) We said that matter is pressureless, but there isn't just the thermal pressure. Galaxies, for example, can have peculiar motions apart from the motion due to the Hubble flow. Maybe these kinds of motion can give rise to an effective pressure (like we saw when considering a collisionless fluid).

Here's the interesting thing about peculiar motions — they decay.

Work in the non-relativistic limit where $v = Hd \ll c$.

Consider observers A and B moving with the Hubble flow. A ball is thrown from A to B.



(from B's point of view)

The ball has a peculiar velocity $V_{p,A}$ ~~at~~ when it leaves A. But B is moving relative to A, so by the time the ball reaches B, it has velocity $V_{p,B}$:

$$V_{p,B} = V_{p,A} - V \quad \Rightarrow \quad V_{p,A} - V_{p,B} = V = H\Delta l$$

dist. between A and B.

Hubble Law

Now, the time taken to go from A to B is $\Delta t = \dfrac{\Delta l}{V_{p,A}}$

$$\Rightarrow \quad V_{p,B} - V_{p,A} = -H \Delta t \, V_{p,A}.$$

Taking the $\Delta t \to 0$ limit (so that $A \to B$), we get:

$$\dot{V}_p = -H V_p \quad \text{or} \quad \frac{\dot{V}_p}{V_p} = -\frac{\dot{a}}{a}$$

$$\Rightarrow \int \frac{dV_p}{V_p} = -\int \frac{da}{a} \quad \Rightarrow \quad \boxed{V_p \sim \frac{1}{a}}$$

needs modification for relativistic particles.

Peculiar velocities therefore tend to damp down and matter settles down to following the Hubble flow.