PHYS644 Problem Set 5

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Problem 1: Don't Be a Hypocrite

We are asked to convert some expressions from natural units, where $\hbar = c = k_B = 1$, to proper SI units. Important ρ is the energy density IE energy per volume, not mass density.

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{\kappa}{a^2}, \tag{1}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p), \tag{2}$$

$$p = w\rho, \tag{3}$$

$$\dot{\rho} = -3(\rho + p)\left(\frac{\dot{a}}{a}\right). \tag{4}$$

Recall that $k = \kappa R_0^2$, where $k = 0, -1, +1$ and R_0 is the radius of curvature of our Universe.

Problem 1.1

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{\kappa}{a^2} \tag{5}$$

The Two terms on the left hand side both have the same units of $1/t^2$ where t is time. (a is dimensionless, and H has the same units as H_0). This means that the term on the right hand side must also have units of $1/t^2$. Let's use check the units.

The right side as written has units of:

$$G\rho - R_0^{-2} \tag{6}$$

let's crack open ρ, and big G. I will use R for units of distance. $\rho = KgR^{-1}t^{-2}$, and $G = R^3Kg^{-1}t^{-2}$.

$$(R^3Kg^{-1}t^{-2})(KgR^{-1}t^{-2}) - R^{-2} \Rightarrow R^2t^{-4} - R^{-2} \tag{7}$$

It looks like a factor of $c^{-2} = t^2R^{-2}$, and $c^2 = R^2t^{-2}$ will give us the correct units.

Therefore in SI units:

$$\boxed{H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3c^2} - c^2\frac{\kappa}{a^2}} \tag{8}$$

Problem 1.2

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{9}$$

Switching to units ish we have:

$$1/t^2 = G\rho + Gp \tag{10}$$

so the right hand side has to have $1/t^2$, $p = KgR^{-1}t^{-2}$

$$G\rho + Gp \Rightarrow (R^3Kg^{-1}t^{-2})(KgR^{-1}t^{-2}) + (R^3Kg^{-1}t^{-2})(KgR^{-1}t^{-2}) \tag{11}$$

Simplifying:

$$(R^3Kg^{-1}t^{-2})(KgR^{-1}t^{-2}) + (R^3Kg^{-1}t^{-2})(KgR^{-1}t^{-2}) = R^2t^{-4} + R^2t^{-4} \tag{12}$$

This looks like another factor of $c^{-2} = t^2R^{-2}$ and we get $1/t^2$ on the terms on the right hand side.

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho + 3p)} \tag{13}$$

Problem 1.3

$$p = w\rho \tag{14}$$

$$KgR^{-1}t^{-2} = KgR^{-1}t^{-2} \tag{15}$$

Energy density has the same units as pressure, and w is dimensionless.

$$\boxed{p = w\rho} \tag{16}$$

Problem 1.4

$$\dot{\rho} = -3(\rho + p)\left(\frac{\dot{a}}{a}\right) \tag{17}$$

Moving into unit ish we have

$$KgR^{-1}t^{-3} = KgR^{-1}t^{-3} + KgR^{-1}t^{-3} \tag{18}$$

This is also already kosher.

$$\boxed{\dot{\rho} = -3(\rho + p)\left(\frac{\dot{a}}{a}\right)} \tag{19}$$

Problem 2: Generalized Hubble Laws?

Problem 2A: Hubble law satisfies velocity addition

$$\mathbf{v}_{B,A} = H(t)\,\mathbf{r}_{B,A}. \tag{20}$$

Using the relation $\mathbf{r}_{C,B} = \mathbf{r}_{C,A} - \mathbf{r}_{B,A}$, we can write

$$\mathbf{v}_{C,B} = H(t)\,\mathbf{r}_{C,B} = H(t)(\mathbf{r}_{C,A} - \mathbf{r}_{B,A}) = H(t)\,\mathbf{r}_{C,A} - H(t)\,\mathbf{r}_{B,A} = \mathbf{v}_{C,A} - \mathbf{v}_{B,A}. \tag{21}$$

Thus the Hubble law is consistent with the Newtonian velocity addition.

Problem 2B: A Linear Equation

Here we are asked to see if the Hubble law is unique in that — is there a more general form of the equation which satisfies Newtonian vector addition (and hence the Cosmological principle).

In order for the vector addition rules to work, Hubble's law has to be a linear equation in R, (it can still vary in time).

we can write this as:

$$f(\mathbf{r}_{C,B}, t) = f(\mathbf{r}_{C,A}, t) - f(\mathbf{r}_{B,A}, t) \tag{22}$$

for arbitrary vectors $\mathbf{r}_{C,A}, \mathbf{r}_{B,A}$.

This is a functional equation of the form

$$f(\mathbf{x} - \mathbf{y}, t) = f(\mathbf{x}, t) - f(\mathbf{y}, t), \tag{23}$$

whose general solution is linear in $\mathbf{x}$. Therefore, f must take the form

$$f(\mathbf{r}, t) = H(t)\,\mathbf{r}, \tag{24}$$

where $H(t)$ can be any function of time. Any non-linear dependence on R stuff will violate our vector addition rules.

Problem 3: Cosmology and Cooking

In an *Einstein-de Sitter* universe, we want the rasin-bread version of Hubble's law:

$$v = H_0 D \tag{25}$$

that the recession velocity, equals the h_0 constant times distance.

We also know that

$$H(t) = \frac{\dot{a}}{a} \tag{26}$$

Where H is the expansion rate as a function of time t, and a is the relative scaling factor. In our case the rasins are only moving

For this to make sense we have to think of the problem as while the bread is baking. While the bread is baking it is expanding, after it is baked it is frozen and before it is also frozen.

Lets say $D_0 = \text{now} = t_0$. I can measure the expansion rate as the average distance of the raisins travel during baking. IE lets assume the dough rises $3x$ from its initial state at the end of cooking for 1 hour, and the initial density of raisins was $\rho_n = 1\,\mathrm{cm}^{-3}$ 1 per cubic cm as I like raisins. After the 1 hour of cooking the new volume is $3x$ the original so the density is 1/3 of the orginal.

We can normalize a at our will, we go by convention that $a_0 = 1$

If the final volume is $3x$ the initial volume then

$$(\frac{a_0}{a_i})^3 = 3 \Rightarrow a_i = 3^{-1/3} \sim 0.69 \tag{27}$$

If we assume that the expansion rate is constant then we have $a(t) = a_i e^{Ht}$ - the solution to Equation 26).

After 1 hour we have $\frac{1}{a_i} = e^H$. Now we can solve for H.

$$\boxed{H = \ln(3^{1/3}) \sim .37\,\mathrm{Hr}^{-1}} \tag{28}$$

Figure 1: I'm not writing a caption

I think inverse hours are nice units for this, as we can write $v = HD$ with v in cm/hr if we measure D in cm.

The the Hubble time here, $T_h = 1/H_0 \sim 2.7\,\mathrm{Hr}$. Which we can interpret as the the time it takes for the raisins to go from infinite density, to the current density at now (or say 1 hour of baking as I used). This is longer then 1 hour of baking, because our raisin bread **started at a finite density.**

Extra credit

See Figure 1 for the cookies I made. The proff said in an email it would be 5% extra creddit.

Problem 4: Einstein-de Sitter Universe

The first Friedmann equation is given by

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda}{3} \tag{29}$$

where $a(t)$ is the scale factor, ρ is the energy density of the universe, k is the spatial curvature constant ($k = 0, +1, -1$ for flat, closed, and open universes, respectively), Λ is the cosmological constant, G is the gravitational constant.

For an **Einstein–de Sitter universe**, we have:

$\rho \propto a^{-3}$ (matter-dominated).

Hence, the Friedmann equation simplifies to

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \tag{30}$$

and since

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^3, \tag{31}$$

we can use this relation to solve for $a(t)$, $a_0 = 1$ (today).

$$\rho = \rho_0 a^{-3}, \tag{32}$$

Now we plug this into our reduced Friedman equation 30

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_0 a^{-3} \tag{33}$$

Now we have an simple in ODE in terms of a, and we can solve it, we just take the sqrt of both sides and move over an a and integrate.

$$\dot{a} = a^{-1/2}\sqrt{\frac{8\pi G}{3}\rho_0} \tag{34}$$

$$a^{1/2}\,da = \sqrt{\frac{8\pi G}{3}\rho_0}\,dt \tag{35}$$

$$\frac{2}{3}a^{3/2} = t\sqrt{\frac{8\pi G}{3}\rho_0} \tag{36}$$

our scale factor then evolves as:

$$\boxed{a(t) = [\frac{3}{2}t\sqrt{\frac{8\pi G\rho_0}{3}}]^{2/3}} \tag{37}$$

The age of the universe is given by $a(t_0) = 1$, and the Hubble param by $H = \frac{\dot{a}}{a}$ (using $a = 1$ for today's value).

$$H_0 = \sqrt{\frac{8\pi G\rho_0}{3}} \tag{38}$$

Now for the age of the universe:

$$1 = \frac{3}{2}t\sqrt{\frac{8\pi G\rho_0}{3}} \tag{39}$$

$$\boxed{t = \frac{2}{3}[\frac{8\pi G\rho_0}{3}]^{-1/2} = \frac{2}{3}H_0^{-1}} \tag{40}$$

Aka the age of the universe is two thirds of the Hubble time.

Problem 5: Happy Birthday!

This is a depressing question, nothing like getting older then being asked to calculate the redshift (z) the year I am from the year 1999, my age is 26. I use $H_0 = 67.66\,\mathrm{km/s/mpc}$ as is used in `astropy`'s `Planck18.H0`.

Problem 5A

It is H_0. Because it has units of inverse time, and my age is in units of time. Redshift z is dimensionless, so to first order $z \sim H_0 t$? where t is my age.

$$\boxed{z \approx H_0 t = 1.8 \times 10^{-9}} \tag{41}$$

Problem 5B

In a dark-energy dominated universe, the scale factor evolves as

$$a(t) \propto e^{H_0 t} \quad \text{or equivalently} \quad a(t) = a_i e^{H_0 t}.$$

The cosmological redshift is defined as

$$z = \frac{\lambda_{\text{obs}} - \lambda_0}{\lambda_0}. \tag{42}$$

From class we know

$$\frac{\lambda_{\text{obs}}}{\lambda_0} = \frac{a(t_0)}{a(t)}, \tag{43}$$

where t_0 is the present time. Taking $a(t_0) = 1$, this becomes

$$1 + z = \frac{1}{a(t)} \quad \Rightarrow \quad z = \frac{1}{a(t)} - 1. \tag{44}$$

For a person born in 1999, with the current year being 2025, my age is $t = -26\,\text{yr}$. Using $a(t) = e^{-H_0 t}$ (time is counted backgrounds hence the).

$$\boxed{z = \frac{1}{e^{H_0 t}} - 1 = e^{-H_0 |t|} - 1 = 1.8 \times 10^{-9}}. \tag{45}$$

For small $H_0 t \ll 1$, we can expand the exponential as $e^{-h_0|t|} \approx 1 - h_0 t$:

$$\boxed{z \approx H_0 t,} \tag{46}$$

which agrees with the dimensional analysis estimate.