PHYS 644 Lecture #10: Principles of Early Universe Thermodynamics

To an excellent approximation, we have so far only dealt with gravity — we've asked how different forms of energy density gravitate and affect our Universe's expansion

But particles have all sorts of interesting interactions, especially in the hot cauldron of our early Universe!

A few examples we'll study:

photon

$$e^+ + e^- \rightleftharpoons 2\gamma \qquad \text{(annihilation / pair production)}$$

$$n \rightarrow p + e^- + \bar{\nu}_e \qquad \text{(beta decay)}$$

$$e^+ + e^- \rightleftharpoons \nu_e + \bar{\nu}_e \qquad \text{(example weak nuclear interaction)}$$

Unknown dark matter particle!

$$DM + DM \rightleftharpoons X + X \qquad \text{(DM annihilation)}$$

Some Standard Model particle (not photons! We know DM does not couple to electromagnetism)

Layering this onto our steadily expanding universe gives rise to lots of interesting phenomena.

$\Rightarrow$ **Show slide with timeline of phase transitions etc.**

We will be focusing on the early universe in the next few lectures, so it's useful to remember some key results from a radiation dominated universe (which is the regime we're in)

$$\rho_r \propto T^4, \text{ so } H = \left(\frac{8\pi \rho_R}{3 m_{Pl}^2} \right)^{1/2} \propto T^2 .$$

Recall that $G \equiv 1/m_{Pl}^2$ in these units. Expressing things in mass/energy useful for particle + nuclear processes

Also recall that $a(t) \propto t^{1/2}$, which means that

$H = \dot{a}/a = 1/2t$ and therefore $t \propto T^{-2}$.

You might wonder... temperature of what? I'm intentionally vague here because lots of species are in thermal equilibrium. Some important exceptions later!

The temperature is then like another way to keep time. This handy because all of the particle interactions have rates that depend on temperature.

This relation also is consistent with $T \propto 1/a$, which we had previously shown for a blackbody distribution of photons.

Before we dive into the details of what actually happens, I want to establish some basic principles.

(1) Phase Space Density is No Longer Collisionless

Yes! The phase space density is back! However, there are three differences ~~one cosmetic, two~~ that I want to highlight.

First, we are no longer dealing with a fake "gas" of stars, but a real gas of particles in a high-energy environment.

Go from velocity to momentum in distribution $f(\vec{x},\vec{v},t) \rightarrow f(\vec{x},\vec{p},t)$ (more natural for relativistic particles)

Also, from quantum stat. mech, have that each state/particle takes up a phase space volume of h^3.

Putting this together, we now have

Why no $\vec{x}$? Homogeneity!

$$\frac{d^6N}{d^3p\, d^3x} = \frac{g}{(2\pi)^3} f(\vec{p},t)$$

Recall $\hbar \equiv 1$, so $h = 2\pi$ since $\hbar \equiv \frac{h}{2\pi}$

What is g? This is the degeneracy factor to account for the way that particles have internal degrees of freedom like spin) which allow multiple particles to occupy the same position-momentum state and still be in a different quantum state

$\Rightarrow$ $\boxed{\text{Show table of degeneracy factors}}$

Recall that when we did the "gas" of stars, we defined the convective derivative:

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \cancel{\vec{v}\cdot\frac{\partial}{\partial \vec{x}} + \dot{\vec{p}}\cdot\frac{\partial}{\partial \vec{p}}}\ \vec{v}\cdot\frac{\partial}{\partial \vec{x}} + \frac{\partial \vec{p}}{\partial t}\cdot\frac{\partial}{\partial \vec{p}}$$

and what we said was $\left(\frac{df}{dt}\right) = 0$. This was a statement that as I follow some particles through a trajectory in phase space, the density f is preserved. Said differently, we had a collisionless system with only smooth evolution. No particles abruptly created or destroyed.

Recall also that when we took the 0th moment of the equation, we got the continuity equation:

$$\frac{\partial n}{\partial t} + \vec{\nabla}\cdot\left(n\langle\vec{v}\rangle\right) = 0 \quad \Longrightarrow \quad \frac{\partial n}{\partial t} = 0$$

$\langle\vec{v}\rangle$ ∵ no net flow in a direction in isotropic universe.

You probably see one of the problems. This gives n = constant, appropriate for a static universe, not an expanding universe.

The issue was the convective derivative, which was not written in the context of a dynamic spacetime. We could do this using GR, or there's an easier argument...

If no particles created or destroyed, then $na^3 = \text{constant}$

$$\Rightarrow \frac{d}{dt}(na^3) = 0 \Rightarrow a^3\frac{dn}{dt} + 3a^2 n\frac{da}{dt}$$

$$\Rightarrow \dot{n} + 3Hn = 0.$$

Compared to before when $\dot{n}=0$, there is an extra Hubble dilution.

There is another problem! With interactions like $e^+ + e^- \rightleftharpoons 2\gamma$, particles can be created or destroyed.

$$\Rightarrow \dot{n} + 3Hn = (\text{interaction terms})$$ Examples later!

This makes things more complicated, but in other respects the problem is simpler. With stars in a galaxy, the distribution function f could take a complicated form. Here, we're dealing with particle interactions, so the rules of quantum statistical mechanics apply:

$$f(p) = \frac{1}{\exp\left[\frac{E-\mu}{T}\right] \pm 1}$$

[$+$: Fermi-Dirac dist for fermions.
$-$: Bose-Einstein dist for bosons.]

if in thermal equilibrium.

Here, $E = \sqrt{p^2+m^2}$ is the total relativistic energy of a particle and μ = chemical potential.

Recall that $\mu = \left(\frac{\partial U}{\partial N}\right)_{S,V}$ because $dU = TdS + \mu dN - PdV$.

so the chemical potential is the change in energy of a system to add a particle.

Photons automatically have $\mu_\gamma = 0$ because I can have processes like radiation when and e^- is accelerated by a proton p^+:

$$e^- + p^+ \rightleftharpoons e^- + p^+ + \gamma$$

Which means that photons can be created "for free" without changing the energy of the system $U \Rightarrow \mu_\gamma = 0$.

On homework, will show that a lot of the time, we can set $\mu = 0$ for other species too. (More specifically, $\mu \ll T$ at early times).

In some sense, then, as long as we can assume thermal equilibrium, our job is much simpler because f is determined. Once we have that, then the prescription is the same as before:

$$n = \frac{g}{(2\pi)^3} \int f(\vec{p})\, d^3p \qquad \text{(number density)}$$

and $$\rho = \frac{g}{(2\pi)^3} \int f(p)\, \underbrace{E}_{=\sqrt{m^2+p^2}}\, d^3p \qquad \text{(energy density)}$$

② Particles Can be Relativistic Sometimes and Non-Relativistic at Other Times

If a particle is in thermal equilibrium in the hot particle soup of the early universe, then at early times when $T \gg m$, it will be relativistic. At later times when our Universe has cooled more and $T \ll m$, it will be non-relativistic

(Don't forget this is secretly $k_B T$ and mc^2!)

There's another important effect that happens as T drops. Consider something like

$$e^+ + e^- \rightleftharpoons 2\gamma.$$

When $T \gg m_e$, there is lots of ambient energy around and its "easy" to do the backward reaction (pair production). But when $T \ll m_e$, the forward reaction (annihilation) dominates and we "lose" the species.

$\Rightarrow$ Not only do particle species get less relativistic as time goes on, but we also lose species.

With this in mind .... $\boxed{\text{Slide Q on H and } g_* \text{ figure}}$

What is this effective number of relativistic degrees of freedom g_* quantity?

It is a convenient bookkeeping device when particles are relativistic. Consider:

$$n = \frac{g}{2\pi^2}\int_0^\infty \frac{p^2\,dp}{\exp\left(\frac{\sqrt{p^2+m^2}}{T}\right) \pm 1} \longrightarrow \frac{gT^3}{2\pi^2}\int_0^\infty \frac{y^2\,dy}{e^y \pm 1}$$

$$\rho = \frac{g}{2\pi^2}\int_0^\infty \frac{p^2\sqrt{p^2+m^2}}{\exp\left(\frac{\sqrt{p^2+m^2}}{T}\right) \pm 1} \longrightarrow \frac{gT^4}{2\pi^2}\int_0^\infty \frac{y^3\,dy}{e^y \pm 1}$$

$T \gg m$ and $y \equiv p/T$

These integrals can actually be evaluated!

$$\rho = \frac{\pi^2}{30} g T^4 \times \begin{cases} 1 & \text{(bosons)} \\ 7/8 & \text{(fermions).} \end{cases}$$

(Generalization of Stefan-Boltzmann Law!)

This is for one particle species. If we have multiple types of particles then we have

$$\rho_r \equiv \sum_i \rho_i = \frac{\pi^2}{30} g_* T^4 \quad \text{where} \quad \boxed{g_* \equiv \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \sum_{\text{fermions}} \left(\frac{7}{8}\right) g_i \left(\frac{T_i}{T}\right)^4}$$

We see that g_* lets us account for all relativistic species without doing each one. It's an "effective" number because fermions count less with the 7/8 factor. And notice how we are allowing for each species to be at a different temperature, because at some point some of them may fall out of equilibrium.

We can similarly ~~say that~~ do the integral for n:

Riemann Zeta fct.

$$n = \frac{\zeta(3)}{\pi^2} g T^3 \times \begin{cases} 1 & \text{(bosons)} \\ 3/4 & \text{(fermions)} \end{cases}$$

Now, another useful quantity is the entropy S and the closely related entropy density $s \equiv \frac{S}{V}$

From thermodynamics, $s = \frac{S}{V} = \frac{\rho + P}{T}$ ← Pressure! Sorry, p is momentum in this lecture.

For relativistic particles, $P = \frac{1}{3}\rho$ ← Recall $w_{\text{radiation}} = 1/3$.

Thus $s = \sum_i \frac{\rho_i + P_i}{T_i} = \frac{4}{3} \sum_i \frac{\rho_i}{T_i} = \frac{2\pi^2}{45} g_{*s} T^3$

where $g_{*S} = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{\text{fermions}} g_i \left(\frac{T_i}{T}\right)^3.$

$\Rightarrow$ $\boxed{\text{Slide Q on } g_* \text{ vs } g_{*S}}$

Because $S = sV$ is conserved, we have $\boxed{\underbrace{g_{*S}(T)\, T^3}_{\propto s}\, \underbrace{a^3}_{\propto V} = \text{const.}}$

We will actually use this to help us work out how temperatures change as species fall out of equilibrium.

A useful quantity will be the number-density-to-entropy-density-ratio for a species:

$$Y_i \equiv \frac{n_i}{s}$$

Since $s \propto a^{-3}$, $Y_i \propto n_i a^3$ so it's a measure of the # of particles of species i, in a comoving volume.

If none of ~~this species~~ the particles of this species are created or destroyed, Y_i is constant.

With these results, it's also possible to go back to the beginning of the lecture and to put some numbers in:

$$H \simeq 1.66\, g_*^{1/2} \frac{T^2}{m_{pl}} \quad \text{and} \quad \left(\frac{t}{1\,\text{sec}}\right) \simeq 2.42\, g_*^{-1/2} \left(\frac{T}{\text{MeV}}\right)^{-2}.$$