PHY644 Notes

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Contents

1 Galaxies As Collisionless Fluids

We cannot think of stars as individual elements, but rather the collection of stars a s **collision-less fluid**.

1.1 Phase Space Density

We can define a Phase Space Density,

$$f(\vec{x}, \vec{v}, t)\, d^3\vec{x}\, d^3\vec{v} \tag{1}$$

This is the number of stars in volume $d^3\vec{x}$, centred on $\vec{x}$., and velocities in small range $d^3\vec{v}$ centred on $\vec{v}$.aka density in 6D phase space given by $\vec{w} = (\vec{x}, \vec{v})$

We can think of a swarm (or blob) of stars (particles) moving in this phase space with velocity:

$$\dot{\vec{w}} = (\dot{\vec{x}}, \dot{\vec{v}}) = (\vec{v}, -\nabla\Phi) \tag{2}$$

Here Φ is the gravitational potential not gravitational potential energy (IE it is per unit mass).

1.2 Fluid Mechanics for Stars

Let's apply Newtonion Mechanics (Fluid mechanics). We assume that stars are collision-less, and that no stars form or die.

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \frac{\partial (f\dot{w_\alpha})}{\partial w_\alpha} = 0 \tag{3}$$

$f\dot{w_\alpha}$ is the phase space current. Using the product rule, we can express the term with the sum as:

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} f\frac{\partial \dot{w_\alpha}}{\partial w_\alpha} + \dot{w_\alpha}\frac{\partial f}{\partial w_\alpha} = 0 \tag{4}$$

The flow described by $\dot{w}$ is special because:

$$\sum_{\alpha}^{6} \frac{\partial \dot{w_\alpha}}{\partial w_\alpha} = \sum_{i}^{3}(\frac{\partial v_i}{\partial x_i} + \frac{\partial \dot{v_i}}{\partial v_i}) = \sum_{i}^{3}(0 + \frac{-\partial}{\partial v_i}\frac{d\Phi}{\partial x_i}) = 0 \tag{5}$$

Because v_i is independent of x_i, and we can write $\dot{v_i}$ as the gradient of the potential, and then the partial derivative of that is equal to 0 because it is not dependent on v_i.

$$\boxed{\sum_{i}^{3}(0 + \frac{-\partial}{\partial v_i}\frac{d\Phi}{\partial x_i}) = 0} \tag{6}$$

Throwing this back into equation 4, we have the colisionless Boltzman equation:

$$\boxed{\frac{\partial f}{\partial t} + \sum_{\alpha}^{6} \dot{w_\alpha}\frac{\partial f}{\partial w_\alpha} = 0} \tag{7}$$

There are a few equivalent variations of this using the above equations.

1.3 Co-moving fluid mechanic Equation

Recasting into Lagrangian or curvature forms.

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{\alpha}^{6} \dot{w}_{\alpha} \frac{\partial}{\partial w_{\alpha}} \tag{8}$$

This is the form of the equation for someone (or a reference frame) flowing along a trajectory in phase space. It says that I can get a change either from some explicit time dependence (1st term) or because it moved to a different part of phase space (2nd term).

The Boltzman equation is then just $\boxed{\frac{df}{dt} = 0}$. In other words, for an observer moving along with a star's path $\dot{w}$ would not see the local phase space density change.

1.4 Moments of Phase space

We do not observe phase space directly, but we can observe the moments of phase space.

1.4.1 0th Moment

The zeroth moment - gives the spacial number density of stars:

$$n(\vec{x}) = \int_{-\infty}^{+\infty} f(\vec{x}, \vec{v}, t) d^3\vec{v} \tag{9}$$

The first moment - gives the average velocity:

$$\mathbb{E}[\vec{v}(\vec{x})] = \frac{1}{n} \int_{-\infty}^{+\infty} \vec{v} f(\vec{x}, \vec{v}, t) d^3\vec{v} \tag{10}$$

The second moment - is related to the velocity dispersion tensor:

$$\langle v_i(\vec{x}) v_j(\vec{x}) \rangle = \frac{1}{n} \int_{-\infty}^{+\infty} (v_i v_j) f(\vec{x}, \vec{v}, t) d^3\vec{v} = \langle v_i(\vec{x}) \rangle \langle v_j(\vec{x}) \rangle + \sigma_{ij}^2 \tag{11}$$

The σ_{ij}^2 is the velocity dispersion.

1.5 Getting Useful forms

To actually get the equations of this, we can take the moments of the Boltsman equation.
For the 0th moment:

$$0 = \int_{-\infty}^{+\infty} [\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f - (\vec{\nabla}_x \Phi) \cdot (\vec{\nabla}_v f)] d^3\vec{v} \tag{12}$$

Use Integration by parts...
The 0th moment is the continuity equation:

$$\boxed{\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n(\langle \vec{v} \rangle)) = 0} \tag{13}$$

This says that stars are conserved.

1.5.1 1st moment: The Jeans Equation

$$0 = \int_{-\infty}^{+\infty} [\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f - (\vec{\nabla}_x \Phi) \cdot (\vec{\nabla}_v f)] \vec{v} d^3\vec{v} \tag{14}$$

break into terms, and use integration by parts!

$$\boxed{\partial_t \langle \vec{v_j} \rangle + \sum_i \langle \vec{v_i} \rangle \vec{\nabla}_{x,i} \langle \vec{v_j} \rangle = -\vec{\nabla}_{x,j}\Phi - \sum_i \frac{\vec{\nabla}_{x,i}(n\sigma_{ij}^2)}{n}} \tag{15}$$

From left to right, we label the terms as - Bulk accretion, velocity sheer, grav force, and pressure. This is the Jeans equation and is the equivalent of the Euler equation in classical fluid mechanics - the $F = MA$ of fluid dynamics.

The right hand term - the pressure, means that a collisionless fluid will still have a pressure as long as the velocity dispersion is non-zero.

Squashing this fluid increases the velocity dispersion increasing the pressure for it to spread back out.

1.6 Stability Analysis

The Jeans equation tells us some information about the stability of a gas trying to collapse (if it collapses or if it bounces back).

Assume the gas consists of particles of mass m, we can turn $n \Rightarrow \rho$ by $\rho = nm$. We assume the velocity dispersion is diagonal and that $\rho\sigma^2 = P$ gives the pressure. This means that our analysis applies to normal gases as well.

All gradients are spacial so we can drop the x $\vec{\nabla}_x \Rightarrow \vec{\nabla}$.

For a small perturbation we assume a static background (the subscripts with 0) and use:

- $\rho = \rho_0 + \epsilon\rho_1$
- $\vec{v} = \vec{v}_0 + \epsilon\vec{v}_1$
- $P = P_0 + \epsilon P_1$
- $\Phi = \Phi_0 + \epsilon\Phi_1$

Now we insert into the continuity equation, and the Jeans equation and group powers of ϵ. The ϵ^0 terms cancel. The ϵ^1 terms give.

$$\partial_t \rho_1 + \rho_0 \vec{\nabla} \cdot \vec{v}_1 = 0 \tag{16}$$

$$\partial_t \vec{v}_1 = -\vec{\nabla}\Phi - \frac{\vec{\nabla} P_1}{\rho} \tag{17}$$

Recall that sound speed is given by:

$$\boxed{\vec{\nabla} P = \frac{\partial P}{\partial \rho} \frac{\partial \rho}{\partial \vec{x}} = c_s^2 \vec{\nabla}\rho} \tag{18}$$

Jeans swindle: We substitute in the c_s version, and use Poission's equation for gravity for $\nabla^2\Phi_0 = 4\pi G\rho_0$ which cannot be true for Φ_0 and ρ_0 both constant when $\rho_0 = 0$.

The brings us to a wave equation of the form:

$$\partial_t^2 \rho_1 - 4\pi G \rho_0 \rho_1 - c_s^2 \nabla^2 \rho_1 = 0 \tag{19}$$

The general solution is:

$$\rho_1 = \rho_1(0) e^{i(\omega t \pm K x)} \tag{20}$$

where ω is the angular velocity, and $k = 2\pi/\lambda$.

2 Geometry of Our Universe

Now we begin cosmology! In Galaxy evolution we cared about individual objects, but here we care about population statistics. It is like being a public health official vs a doctor.

Quick note: this section uses natural units $\hbar = c = k_b = 1$. This is different from geometrical units where $G = c = 1$.

$$M_{\text{planck}} = \sqrt{\frac{\hbar c}{G}} \tag{21}$$

In Natural units, this becomes $G^{-1/2}$.

2.1 The Cosmological Principle

Describing the spacetime geometry of our universe is a difficult task, but we can make three simplifications we make and test empirically.

The Cosmological Principle: the Universe is **statistically homogeneous** and **isotropic**. Homogeneity implies that the Universe has the same average properties at every point in space, whereas isotropy implies that the Universe looks the same in all directions from any given vantage point. Together, these symmetries constrain the possible forms of the spacetime metric, leading naturally to the **Friedmann–Lemaître–Robertson–Walker (FLRW)** metric as the general solution for a universe governed by general relativity.

The cosmological principle does not hold for time, because the universe is expanding — time marches forward.

2.2 Hubble Law

Hubble Law: Hubble discovered that the redshift recession velocity of distance galaxies follows the relationship:

$$v = H_0 d \tag{22}$$

$H_0 \sim 70\,\text{kms}^{-1}\text{Mpc}$ The subscript 0 means today value in the present universe.

With the current Hubble tension it is easier to write $H_0 = 100\,\text{kms}^{-1}\text{Mpc}$, and then use $h = 0.7$ to remain noncommittal.

Another way to think about the expansion of the universe is as a **scaling factor**, and that if we measure the universe with some meter stick that the scaling on the meter stick is changing by some function $a(t)$.

By convention we say $a_0 = a(t_0) = 1$ — that today's time is the age of the universe.

Scaling Factors: It's annoying to describe distance if it is constantly changing so we go instead with comoving distances:

$$d_{\text{physical}} = a(t)\chi_{cosmo} \tag{23}$$

Where d_{physical} is what we measure on our meter stick, and χ_{cosmo} "takes out the expansion". If two objects have a changing cosmological distance then there motion is not just due to the expansion of the universe. Remember that $a_0 = a(t_0) = 1$ — that today's time is the age of the universe.

Let's differentiate both sides:

$$\vec{v}(t) = \frac{d\vec{d}_{\text{physical}}}{dt} = \frac{da}{dt}\vec{\chi}_{cosmo} = \frac{da}{dt}\frac{1}{a}\vec{d}_{\text{physical}}(t) \tag{24}$$

This is the Hubble Law! $\vec{v} \propto d$ but at a different time, thus Hubble's law holds for all time but with different values of H_0.

$$\boxed{H(t) = H = \frac{\dot{a}}{a}} \tag{25}$$

This is called the **Hubble parameter** H_0 is today's value.

In general $H(t)$ is some complicated function of time. However, an interesting special case is a universe where the expansion just makes galaxies coast along at a constant speed.

In this case $\dot{a} = const \Rightarrow H(t) = \frac{const}{a} = \frac{H_0}{a}$. The last substitution is because $H(t_0) = H_0$ as $a_0 = 1$. Because $H = \frac{\dot{a}}{a}$ always we can say $\dot{a} = H_0 \Rightarrow \int_0^1 da = H_0 \int_0^{t_0} dt$.

This is how we take the inverse of H_0 as the age of the universe! **This is the Hubble Time $T_h = \frac{1}{H_0} \approx 14.5\,\text{Gyr}$.**

2.3 Allowed Expansion Rates

While objects cannot travel faster then c, space can expand faster then c, this is known as **super luminal expansion** and defines the observable universe.

Using a comoving coordinates makes this problem easy as we are factoring out the expansion.

In time dt light moves a physical distance cdt, so the comoving distance is $\frac{cdt}{a}$ and for a finite time integral we have $\chi = \int \frac{1}{a}dt$ (recall $c = 1$).

Basically its $c \to \frac{c}{a}$. In GR, space itself can expand or contract.

Particle (Causal) Horizon: This is the furthest distance unimpeded light can have travelled from $t = 0$ to $t = t$

$$\chi = \int_0^t \frac{1}{a}dt' \tag{26}$$

To get the physical distance we multiply by $a(t)$.

$$d_p = a(t)\chi = a(t)\int_0^t \frac{1}{a}dt' \tag{27}$$

In the case of subluminal expansion: we have $a(t) \sim t^p$ where $0 < p < 1$ meaning that $\ddot{a} < 0$ (slowing down).

Given enough time, light will see the entire universe.

in the case of superluminal expansion we have $a(t) \sim t^p$ where $p > 1$ meaning that $\ddot{a} > 0$ (getting faster).

You only see finite χ regardless of how long you wait. Light cannot catch up with the expansion of the universe. We live in this type of universe presently.

Intuition: $p = 1$ where $a \propto t$ is the dividing line because the amount of distance light can travel is ct so if $a \propto t$, the expansion balances this effect.

2.4 Friedmann–Lemaître–Robertson–Walker (FLRW) metric

Friedmann–Lemaître–Robertson–Walker (FLRW) metric: This is the metric of space-time for our universe as a whole, result of the cosmological principle.

$$ds^2 = -dt^2 + a(t)^2[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2] \tag{28}$$

Where $d\Omega^2 = d\theta^2 + sm^2\theta d\phi^2$, and K is the spatial curative and it can be positive, zero, or negative. (Curvature of space, not of space-time).

2.5 Observables

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With the FLRW metric we can make predictions, and observe. one of the primary ones is the redshift.

$$z = \frac{\lambda_{obs} - \lambda_0}{\lambda_0} \tag{29}$$

The elongation of wavelengths can be shown to be proportional to $a(t)$ therefore:

$$\frac{\lambda_{obs}}{\lambda_0} = \frac{a(t_0)}{a(t)} \tag{30}$$

$a(t_0) = 1$

$$1 + z = \frac{1}{a} \tag{31}$$

redshift is both a measure of distance, and time.

3 Dynamics of Our Universe

In the last section we talked about $a(t)$ in kinematic terms, and this time we will talk about in dynamic terms — what causes it move the way it does. These are given by the two Friedman equations.

3.1 The Friedmann Equations

First Friedmann equation:

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\,\rho}{3} - \frac{k}{a^2},$$

where ρ is the energy (mass–energy) density (units: energy per volume, e.g. $Jm(^{-3})$; in $c = 1$ units ρ carries same units as mass density).

Second (acceleration) Friedmann equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\,(\rho + 3p)\,,$$

where p is the pressure. (Again written in $c = 1$ units; with a cosmological constant Λ add $+\Lambda/3$ on the right.) Another usefull equation is $\dot{\rho} = -3(\rho + p)(\frac{\dot{a}}{a})$

The first equation is analogous to an energy equation in Newtonian mechanics.

If you ignore the pressure term in the second equation, it looks similar to "$F = Ma$". The negative sign in the second equation says that matter causes the expansion to decrease.

The second equation says that pressure is source of gravity as well, not just density —- this is a property from general relativity.

3.1.1 Deriving the first Friedman equation from Newton's Cosmology

We can derive the first Friedman equation from Newtonian thinking. Imagine a homogeneous universe filled with matter.

Draw a little circle around a test mass - the test mass is on the shell.

we have:

$$m\frac{d^2r}{dt^2} = -\frac{GM_{\text{enc}}m}{R(t)^2} \Rightarrow \ddot{R} = -\frac{GM_{\text{enc}}}{R^2} \tag{32}$$

Multiply both sides by $\dot{R}$ and integrate $\int \dot{R}\ddot{R}\,dt = \frac{\dot{R}^2}{2} + c$.

$$\frac{1}{2}\dot{R}^2 - \frac{GM_{\text{enc}}}{R} = \text{const} \equiv \frac{-R_0^2 k}{2}. \tag{33}$$

The first term on the left hand side is the kinetic energy per mass, and the second term on the left hand side is the Gravitational Potential Energy per unit mass. The constant we chose is well chosen in advance.

$$M_{\text{enc}} = \frac{4}{3}\pi\rho(t)R(t)^3 \tag{34}$$

and we can let $R(t) = a(t)R_0$ filling in we have:

$$\frac{1}{2}(\dot{a}(t)^2 R_0^2) - \frac{4G\pi\rho(t)a(t)^2R_0^2}{3} = \text{const} \equiv \frac{-R_0^2 k}{2}. \tag{35}$$

finally we have

$$\boxed{(\frac{\dot{a}}{a})^2 = \frac{8}{3}\pi G\rho - \frac{\kappa}{a^2}} \tag{36}$$

This is the first Friedman equation!

3.1.2 Getting the second Friedman Equation

To get the second Friedman equation we appeal to thermodynamics. The first law of thermodynamics says:

$$dE = dQ - pdV \tag{37}$$

where dE is the change in energy of the system, dQ is heat added to the system, and $-pdV$ is the work done by the system on its surroundings. Due to our assumption of the cosmology principle $dQ = 0$ as if any small parcel of gas was non-zero it would be special.

We can write $dE = pdV$, but we can also write $dE = d(\rho a^3)$ as this is what is changing where ρ is the energy density. This is the co-moving volume so $v = a^3$.

Rewriting we have:

$$a^3 d\rho + 3a^2 \rho da = -3a^2 p da \tag{38}$$

rearranging we have:

$$\boxed{\dot{\rho} = -3(\rho + p)(\frac{\dot{a}}{a})} \tag{39}$$

Differentiating the first Frieddman equation we have $2a\dot{a} = \frac{8}{3}\pi G(2a\dot{a}\rho + a^2\dot{\rho})$ we use this and wearrange:

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G(\rho + 3p)} \tag{40}$$

3.2 The Parameters of the Friedman equations

We have derived the equations that govern the dynamics of $a(t)$, and they depend on the stuff in the universe ρ, p etc.

Here are the common parameters.

$$p = w\rho \tag{41}$$

w is known as the equation of state parameter and is dimensionless, this works because ρ and p have the same units, its similar to an ideal gas.

plugging this into the equation for $\dot{\rho}$ we have:

$$\dot{\rho} = -3(1 + w)\rho(\frac{\dot{a}}{a}) \Rightarrow \boxed{\rho \sim a^{-3(1+w)}} \tag{42}$$

In a flat universe $k = 0$, we have

$$\boxed{(\frac{\ddot{a}}{a})^2 \propto a^{-3(1+w)} \Rightarrow a(t) \propto t^{\frac{2}{3(1+w)}}} \tag{43}$$

3.2.1 Normal Matter

For normal matter , and "dust" $w = 0$. If $k_b T << mc^2$ then the rest mass energy dominates, and we can safely neglect pressure.

For $p = 0$, $w = 0$, $\rho_m \sim a^{-3}$.

3.2.2 Vacuum Energy

Vacuum Energy $w = -1$: The energy of a vacuum just comes from the vacumm... the density is a const

$$\rho_\Lambda = \text{const} \Rightarrow w = -1 \tag{44}$$

from $\rho \sim a^{-3(1+w)}$ so $a \sim 1$ when $w = -1$.

in this case $a(t) \propto e^{H_0 t}$. This is an accelerated expansion.

w needs to be $w < -\frac{1}{3}$ for an accelerated expansion.

3.2.3 An Empty Universe

Here $\rho = 0$, and $(\frac{\dot{a}}{a})^2 = -\frac{1}{2}k \Rightarrow \dot{a} = \text{const}$

aka

$$a(t) \propto t \tag{45}$$

3.2.4 Light Domination or Radiation

In this case $w = \frac{1}{3}$ this comes from thermodynamics.

$P = \frac{4\sigma}{3c}T^4$ and $\rho = 4\sigma T^4$. so $w = \frac{1}{3}$ From the Stefan-Boltzman equations. This assumes thermal equilibrium

in this case

$$\boxed{a \propto t^{1/2}} \tag{46}$$

$$\boxed{\rho_r \propto a^4} \tag{47}$$

Another way to think of $\rho_r \propto a^4$ this is you get the the a^{-3} from the normal volume dilution, and an extra a^{-1} from $E = \frac{hc}{\lambda}$ and λ is changing as well due to redshift.

3.3 Subtle Parts to Highlight

3.3.1 CMB in thermal equilibrium

When talking about radiation, we assumed it was in thermal equilibrium so we could invoke the Stefan-Boltzman equation.

The CMB looks like it is in thermal equilibrium but it is not, but it still follows the Black-Body radiation curve so its alright, but with a modified temperature $\boxed{T_f = \frac{a_i}{a_f} T_i}$.

$$T \propto \frac{1}{a} \tag{48}$$

3.3.2 Normal Matter is pressure-less

We saw that Galaxies have an effective pressure — from the peculiar motions of the stars, even if it is not from the thermal motion. Does normal matter not have a pressure?

The interesting thing about peculiar motions is that they decay. As the universe expands their peculiar velocities get damped to 0.