PHYS644 Problem Set 6

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Problem 1: Photons in Today's Universe

We are told that the CMB $T_0 = 2.725\,\mathrm{K}$ (Kelvin isn't a degree!). We are asked to calculate the number density of photons in the universe today AND the energy density of photons - $\Omega_{\gamma,0}H^2$.

From lecture notes we know for bosons (photons are bosons)

$$n_\gamma = \frac{g\zeta(3)T^3}{\pi^2} \tag{1}$$

In natural units of $k_b = c = \hbar = 1$, g is the number of internal degrees of freedom per particle in case $g = 2$.

To add back in units, we know that n should have units of 1/volume^3,

$$\mathrm{cm}^{-3} = T^3 \tag{2}$$

but the right hand side has units of temperature. k_bT gets us units of energy, and then we need to get energy to volume, recall that $\hbar c$ has units of energy * length.

$$\boxed{n_\gamma = \frac{g\zeta(3)}{\pi^2}(\frac{k_bT}{\hbar c})^3} \tag{3}$$

Now we can plug in our value of T.

$\boxed{n_\gamma = 411\,\mathrm{cm}^{-3}}$.

Problem 1b: $\Omega_{\gamma,0}h^2$

From lecture notes we know:

the energy density of photons is

$$\mu_\gamma = aT^4 \tag{4}$$

With $a = 4\frac{\sigma}{c}$ where σ is the stefan-boltzman constant, but in natural units. This is in energy density, to convert to mass density we divide by c^2.

$$\rho_{\gamma,0} = \frac{aT^4}{c^2} \tag{5}$$

and $\rho_{\mathrm{crit},0}$ is given by:

$$\rho_{\mathrm{crit},0} = \frac{3H^2h^2}{8\pi G} \tag{6}$$

and $\Omega_{\gamma,0}$ is given by:

$$\Omega_{\gamma,0} = \frac{\rho_{\gamma,0}}{\rho_{\mathrm{crit},0}} \tag{7}$$

So $\Omega_{\gamma,0}h^2$ is given by:

$$\Omega_{\gamma,0}h^2 = \frac{\rho_{\gamma,0}}{\rho_{crit,0}} = \frac{8\pi G}{3H^2c^2}aT_0^4 \tag{8}$$

$\boxed{\Omega_{\gamma,0}h^2 = 2.47 \times 10^{-5}}$

Problem 2: Acceleration Redshift

z_{acc} happens when the universe expansion just starts (postively) accelerating IE when $\ddot{a} = 0$.

The second Friedmann equation becomes

$$\frac{\ddot{a}}{a} = 0 = -\frac{4\pi G}{3}(\rho + 3p) \tag{9}$$

Therefor our condition is when $\rho + 3p = 0$. We are absorbing Λ into its own effect ρ and p. Baryonic matter gives $p = 0$, and dark energy has $p = w\rho_{DE}$.

Now we have

$$0 = \rho_m + \rho_{DE}(1 + 3w) \Rightarrow \rho_m = -\rho_{DE}(1 + 3w) \tag{10}$$

recall that

$$\rho_{\text{crit},0} = \frac{3H_0^2}{8\pi G} \tag{11}$$

$$\Omega_{m,0} = \frac{\rho_{m,0}}{\rho_{\text{crit},0}} \tag{12}$$

$$\Omega_\Lambda = \Omega_{DE,0} = \frac{\rho_{DE,0}}{\rho_{\text{crit},0}} \tag{13}$$

We now insert these into our condition, we convert to values at z recall that $a = 1/(1 + z)$

$$\Omega_{m,0}\, \rho_{\text{crit},0}\, (1 + z)^3 = -(1 + 3w)\, \Omega_\Lambda\, \rho_{\text{crit},0}\, (1 + z)^{3(1+w)}. \tag{14}$$

Now we solve for z — this will be z_{acc}.

$$(1 + z)^{-3w} = -(1 + 3w)\, \frac{\Omega_\Lambda}{\Omega_{m,0}} \tag{15}$$

$$\boxed{z = [-(1 + 3w)\, \frac{\Omega_\Lambda}{\Omega_{m,0}}]^{-1/3w} - 1} \tag{16}$$

We learned from class (and from $a(t) = 1/(1 + z)$ that $z = 0$ is now, and $z = -1$ is $t = \infty$. Aka $z > 0$ if we are looking backwards in time. We see that the first term on the left is always positive when $w < -\frac{1}{3}$.

Problem 3: Phantom Dark Energy

Problem 3A:

We know normal matter scales like $\rho_m \propto a^{-3}$, and a fluid with constant equation-of-state w_p goes like $\rho_p \propto a^{-(3(1+w_p))}$. We can use $\rho_{i,0} \propto \Omega_{i,0}$ and write:

$$\Omega_{m,0}a^{-3} = \Omega_{p,0}a^{-(3(1+w_p))} \tag{17}$$

And solve for the scaling of a:

$$a^{3w_p} = \frac{\Omega_{p,0}}{\Omega_{m,0}} \tag{18}$$

And the scale factor when they are equal is then

$$\boxed{a_{mp} = (\frac{\Omega_{p,0}}{\Omega_{m,0}})^{1/(3w_p)}} \tag{19}$$

Problem 3B:

For a spatially flat universe we have:

$$(\frac{\dot{a}}{a})^2 = H^2 = H_0^2(\Omega_{m,0}a^{-3} + \Omega_{p,0}a^{-3(1+w_p)}) \tag{20}$$

In the regime $a >> a_{mp}$ the phantom energy term dominates ($w_p < -1$).

$$(\frac{\dot{a}}{a})^2 = H^2 \approx H_0^2\Omega_{p,0}a^{-3(1+w_p)} \tag{21}$$

Now we can solve this ODE,

$$\dot{a} = H_0\Omega_{p,0}^{1/2}a^{1-\frac{3}{2}(1+w_p)} \tag{22}$$

$$t_{\text{rip}} - t_0 = \frac{1}{H_0\Omega_{p,0}^{1/2}}\int_1^\infty a^{(1+3w_p)/2}da \tag{23}$$

the power in the integral is < -1 and the integral converges. We dot even need wolfram alpha for this! Is is a power intergral! The integral is $\frac{2}{3|1+w_p|}$.

Pluggin it in we have:

$$t_{\text{rip}} - t_0 = \frac{2}{H_0\Omega_{p,0}^{1/2}3|1+w_p|} \tag{24}$$

For a flat universe, $\Omega_{p,0} = 1 - \Omega_{m,0}$ so we can write:

$$\boxed{H_0(t_{\text{rip}} - t_0) = \frac{2}{3|1+w_p|}(1-\Omega_{m,0})^{-1/2}} \tag{25}$$

0.1 Problem 3C:

Time until Big Rip: $\boxed{t_{\text{rip}} = 110\,\text{Gyr}}$.

Problem 4: Distance Measures

Problem 4A:

We are told that $\Omega_{m,0} = 0.3$, $\Omega_{r,0} = 8.5 \times 10^{-5}$, $\Omega_\Lambda = 1 - \Omega_{r,0} - \Omega_{m,0}$

We know that the physical distance R is given by

$$R(t) = a(t)\chi \tag{26}$$

and that $a(t)$ is the scale factor and χ is the commoving distance which is fixed if objects are just drifting due to $a(t)$. χ

Co-moving distance χ is also written as D_c, We know that the co-moving distance is

$$D_c(z) = c\int_0^z H(z)^{-1}\,dz \tag{27}$$

and we can write $H(z)$ as

$$H(z) = H_0\sqrt{\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_k(1+z)^2 + \Omega_\Lambda} \tag{28}$$

This is the corrected for z version! In our case $\Omega_k = 0$ because like the Earth[1] - the universe is flat. We now have:

$$D_c(z) = c\int_0^z \frac{1}{H_0\sqrt{\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_\Lambda}}\,dz \tag{29}$$

While I could solve this analytically, we are asked to plot so why bother?

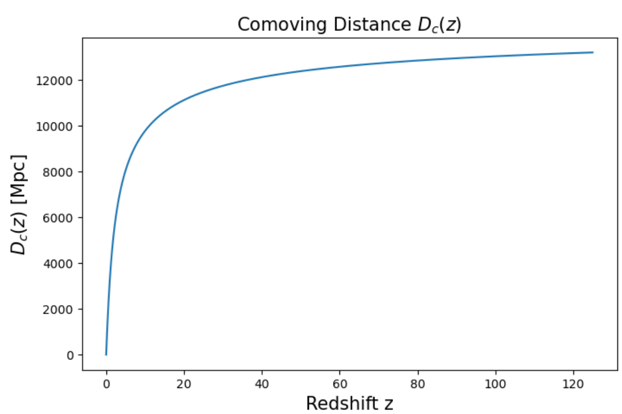

Figure 1: I'm not writing a caption

Here is our plot figure 1! Most of the evolution happens at "low" redshift and then it looks like we start to approach an asymptote as we go on.

Problem 4B:

We do the same thing now until the egde of the observable universe! This is $z = \infty$

Here is our plot figure 2. We can see we run into a numerical issue at high z, but that D_c asymptotically approaches roughly a constant value.

[1] This is a meme.

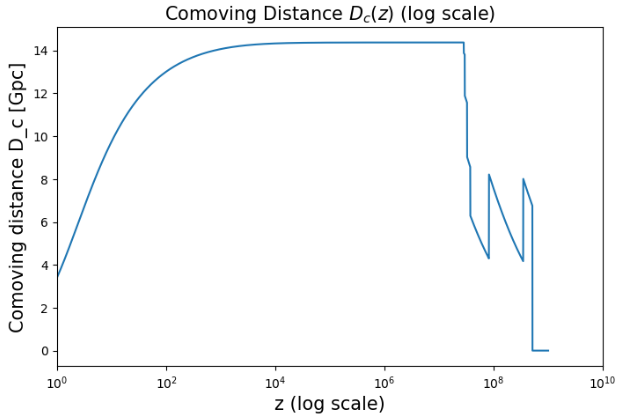

Figure 2: no caption!

Problem 4C

From figure 2 we can read the asymtotically value as $\sim 14\,\mathrm{Gpc}$. So half of this value is $\sim 7\,\mathrm{Gpc}$ — by my eye that is around $z = 10$ ish. (Its actually easier to see on 4A!)

Problem 5: Chemical Potential of Electrons and Positrons

This reminds me of the good times back in advanced statistical mechanics! If only I remembered more from the class :(.

Problem 5A

We are asked to calculate the difference between the number densities of electrons and positrons, $n_- - n_+$, in the relativistic limit $m_e << T$.

From phase-space density we know:

$$f(P) = \frac{1}{\exp((E-\mu)/T) \pm 1} \tag{30}$$

Where $+$ is for Fermions, and $-$ is for Bosons. We use the $+$ as electrons are Fermions.

The number density is

$$n = \frac{g}{(2\pi)^3} \int f(P) dP^3 \tag{31}$$

These are also given in the lecture notes for lecture 10.

Switching to spherical coordinates and into momentum space, $dp^3 = p^2 dp d\Omega$, Ω is the solid angle! so the integral over it $\int \Omega = 4\pi$ as always.

we can write:

$$n = \frac{g}{(2\pi)^3} 4\pi \int_0^\infty p^2 f(p) dp = \frac{g}{2\pi^2} \int_0^\infty p^2 f(p) dp \tag{32}$$

In our natural units $E = \sqrt{P^2 + m^2}$, and in the relativistic limit this becomes $E \approx |p|$.

Let's define the electron $\mu_e = \mu$, and positrons as $-\mu_e = -\mu$. We can then write:

$$n_\mp = \frac{g}{2\pi^2} \int_0^\infty \frac{p^2}{\exp((p \mp \mu)/T) + 1} dp \tag{33}$$

where $-$ is for electrons and $+$ is for positrons. Writing the Δ now:

$$n_- - n_+ = \frac{g}{2\pi^2} \left[\int_0^\infty \frac{p^2}{e^{(p-\mu)/T} + 1} \, dp - \int_0^\infty \frac{p^2}{e^{(p+\mu)/T} + 1} \, dp \right]. \tag{34}$$

In the first integral, we set $p = T(u + x)$ with $x \equiv \mu/T$. Following the hint. Then $dp = T\,du$ and when $p = 0$, $u = -x$. The first integral becomes:

$$\int_{-x}^\infty \frac{(T(u+x))^2}{e^u + 1} T\,du = T^3 \int_{-x}^\infty \frac{(u+x)^2}{e^u + 1} \, du. \tag{35}$$

In the second integral, set $p = T(u - x)$ again forom the hint, giving:

$$\int_{x}^\infty \frac{(T(u-x))^2}{e^u + 1} T\,du = T^3 \int_{x}^\infty \frac{(u-x)^2}{e^u + 1} \, du. \tag{36}$$

Substituting both results into the yields:

$$\boxed{n_- - n_+ = \frac{gT^3}{2\pi^2} \left[\int_{-x}^\infty \frac{(u+x)^2}{e^u + 1} \, du - \int_{x}^\infty \frac{(u-x)^2}{e^u + 1} \, du \right]} \tag{37}$$

From the problem set, we know that the bracketed integrals become $(\pi^2 x + x^3)/3$.

Problem 5B

The electrical neutrality of the universe implies that the number of protons n_p (or nearly equivalently, the number of baryons n_B) is equal to $n_- - n_+$. Use this fact to estimate μ_e/T. It may be useful to express n_B as ηn_γ, where η is the baryon-to-photon ratio, which is approximately 6×10^{-10}.

Starting from our solution to problem 5B + the [] term.

$$n_- - n_+ = \frac{gT^3}{2\pi^2}(\pi^2 x + x^3)/3 = n_p \sim n_b = \eta n_\gamma \tag{38}$$

We know $n_\gamma = \frac{2\zeta(3)T^3}{\pi^2}$ from equation 1. (with $g = 2$, and the $T = T_0$ of the CMB).
This gives us

$$\frac{gT^3}{2\pi^2}(\pi^2 x + x^3)/3 = \eta \frac{2\zeta(3)T^3}{\pi^2} \tag{39}$$

The T, and π cancel, electrons also have $g = 2$, so the 2 cancels.

$$\pi^2 x + x^3 = 6\eta\zeta(3) \tag{40}$$

Because $\eta << 1$ we can neglect the x^3 term, and we have:

$$\boxed{x = \frac{6\eta\zeta(3)}{\pi^2} = \mu_e/T \approx 4.3 \times 10^{-10}} \tag{41}$$