PHYS644 Problem set 3

Maxwell A. Fine: SN 261274202
maxwell.fine@mail.mcgill.ca
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Problem 1: Density of a Self-Gravitating Disk

Here we consider an infinite disk of stars of identical mass, m_*, in the xy plane. Assume the stars are in equilibrium (their phase space is in steady state).

Problem 1A

The Jeans equation from class is:

$$\partial_t \langle \vec{v_j} \rangle + \sum_i \langle \vec{v_i} \rangle \vec{\nabla}_{x,i} \langle \vec{v_j} \rangle = -\vec{\nabla}_{x,j} \Phi - \sum_i \frac{\vec{\nabla}_{x,i}(n\sigma_{ij}^2)}{n} \tag{1}$$

From left to right, we label the terms as - Bulk accretion, velocity sheer, grav force, and pressure. We are asked to find $n(z)$ in terms of the velocity dispersion in the $\hat{z}$ direction σ_z^2, Φ, and midplane density $n(0)$.

Since we are in a steady state, $\partial_t \langle \vec{v_z} \rangle = 0$

In $\hat{z}$ we have:

$$0 = -\frac{d\Phi}{dz} - \frac{1}{n} \frac{d(n\sigma_z^2)}{dz} \tag{2}$$

This looks like a straight forward differential equation, let's attack it.

$$\frac{d(n\sigma_z^2)}{dz} = -n \frac{d\Phi}{dz} \tag{3}$$

$$\frac{1}{n\sigma_z^2} \frac{d(n\sigma_z^2)}{dz} = -\frac{1}{\sigma_z^2} \frac{d\Phi}{dz} \tag{4}$$

Switching to $\ln n\sigma_z^2$:

$$\frac{d \ln (n\sigma_z^2)}{dz} = -\frac{1}{\sigma_z^2} \frac{d\Phi}{dz} \tag{5}$$

Now we integrate both sides from 0 to z.

$$\boxed{\ln(\frac{n(z)\sigma_z^2}{n(0)\sigma_0^2}) = -\int_0^\infty \frac{1}{\sigma_z^2} \frac{d\Phi}{dz} dz} \tag{6}$$

We can rearrange and solve for $n(z)$ but it look a bit ugly

$$\boxed{n(z) = n(0) \frac{\sigma_z^2(0)}{\sigma_z^2(z)} \exp(-\int_0^\infty \frac{1}{\sigma_z^2} \frac{d\Phi}{dz} dz)} \tag{7}$$

Problem 1B

In the case of an isothermal gas, and assuming $\sigma_z^2 = C$ a constant in z, and setting $\Phi(0) = 0$.

The right hand side in 7 simplifies:

$$\boxed{n(z) = n(0)e^{-\frac{\Phi(z)}{\sigma_z^2}}} \tag{8}$$

Interpreting this as a thermal equilibrium (Boltzmann) distribution for a "gas" of particles of mass m_*, the velocity dispersion plays the role of the thermal kinetic energy per unit mass. The effective temperature T is given by:

$$\frac{1}{2}m_*\langle v^2\rangle \sim \frac{1}{2}m_*\sigma_z^2 \sim \frac{1}{2}k_bT \tag{9}$$

So the temperature of the gas is given by:

$$\boxed{T = \frac{m_*\sigma_z^2}{k_b}} \tag{10}$$

Problem 1C

Now use the Poisson equation to solve for $\Phi(z)$.

$$\nabla^2\Phi = 4\pi G\rho \tag{11}$$

With our given by $\rho = m_* n(\vec{R})$, since the system is uniform in the xy plane

$$\frac{d^2\Phi}{dz^2} = 4\pi G m_* n(z) \tag{12}$$

Let's attack this!

$$\frac{d^2\Phi}{dz^2} = 4\pi G m_* n_0 e^{-\frac{\Phi(z)}{\sigma_z^2}} \tag{13}$$

Let's redefine the part in the exponent to be:

$$\aleph = \frac{\Phi}{\sigma_z^2} \tag{14}$$

$$\frac{d^2\aleph}{dz^2} = \frac{4\pi G m_* n_0}{\sigma_z^2}e^{-\aleph} \tag{15}$$

We recognize the scale height as $\boxed{h^2 = \frac{\sigma_z^2}{2\pi G m_* n_0}}$

$$\frac{d^2\aleph}{dz^2} = \frac{2}{h^2}e^{-\aleph} \tag{16}$$

$$\frac{d^2\aleph}{dz^2}\frac{d\aleph}{dz} = \frac{2}{h^2}e^{-\aleph}\frac{d\aleph}{dz} \tag{17}$$

Integrate:

$$\frac{1}{2}(\frac{d\aleph}{dz})^2 = -\frac{2}{h^2}e^{-\aleph} + c \tag{18}$$

at $z = 0$, we expect $\frac{d\aleph}{dz} = 0$ due to symmetry, and we have $\aleph(0) = 0$.
Therefore:

$$0 = -\frac{2}{h^2} + c \Rightarrow c = \frac{2}{h^2} \tag{19}$$

Throwing back into equation 18 we have:

$$\frac{d\aleph}{dz} = \frac{2}{h}\sqrt{1 - e^{-\aleph}} \tag{20}$$

The anti-derivative of this is sech (from a table).

$$\boxed{n(z) = n_0 \text{sech}^2(\frac{z}{2h})} \tag{21}$$

with $\boxed{h^2 = \frac{\sigma_z^2}{2\pi G m_* n_0}}$

Problem 1D

Let $h = 300 \quad \text{pc}$ and $\sigma_z^2 = 20 \quad \text{km/s}$. We are asked to calculate the surface density $\Sigma = \int \rho(z)dz$

1 Problem 2: Spherical Tophat Model

Problem 2A:

We are asked to solve this ODE for $\rho = f(t)$:

$$\frac{1}{2}(\frac{dr}{dt})^2 - \frac{GM}{r} = \epsilon \tag{22}$$

with $\epsilon = \frac{E}{m}$, in our universe a good model is that $\epsilon = 0$ — it has just enough energy to escape.

$$\frac{1}{2}(\frac{dr}{dt})^2 - \frac{GM}{r} = 0 \tag{23}$$

$$r(\frac{dr}{dt})^2 = 2GM \tag{24}$$

$$r^{1/2}\frac{dr}{dt} = (2GM)^{1/2} \tag{25}$$

$$r^{1/2}dr = (2GM)^{1/2}dt \tag{26}$$

Integrate both sides from 0 to r, and 0 to t

$$\frac{2}{3}r^{\frac{3}{2}} = [2GM]^{\frac{1}{2}}t \tag{27}$$

$$r^3 = \frac{9}{2}GMt^2 \tag{28}$$

We now replace $M = \frac{4}{3}\pi\rho r^3$

$$r^3 = \frac{9}{2}G\frac{4}{3}\pi\rho r^3 t^2 \tag{29}$$

$$1 = 6G\pi\rho t^2 \tag{30}$$

$$\boxed{\rho = \frac{1}{6G\pi t^2}} \tag{31}$$

Problem 2B

Negative.

Reason: $\epsilon = \frac{E}{m}$ is the total energy per unit mass. In the background borderline case we had $\epsilon = 0$ (just enough energy to escape to infinity). An overdensity that will eventually stop expanding and collapse because it has slightly more attraction, it is gravitationally bound, so its total energy is negative. You can see this from the energy equation: at turnaround $\dot{r} = 0$ so

$$\epsilon = -\frac{GM}{r_{\max}} < 0, \tag{32}$$

hence ϵ must be negative for a region that will turn around and collapse into a halo.

Problem 2C

Start from the energy equation for a spherical shell and assume the shell is bound so $\varepsilon < 0$. Write $\varepsilon = -|\varepsilon|$. Then

$$\frac{1}{2}\dot{r}^2 - \frac{GM}{r} = \varepsilon = -|\varepsilon| \quad \Longrightarrow \quad \dot{r}^2 = \frac{2GM}{r} - 2|\varepsilon|.$$

Define the constant

$$A \equiv \frac{GM}{2|\varepsilon|}$$

and introduce the parameter η by the ansatz

$$r(\eta) = A(1 - \cos\eta).$$

Differentiate with respect to η:

$$\frac{dr}{d\eta} = A\sin\eta.$$

Using $GM = 2|\varepsilon|A$ and $r = A(1-\cos\eta)$ in the expression for $\dot{r}^2$ gives

$$\dot{r}^2 = 2|\varepsilon|\left(\frac{2A}{r} - 1\right) = 2|\varepsilon|\frac{1+\cos\eta}{1-\cos\eta}.$$

Hence

$$\dot{r} = \sqrt{2|\varepsilon|}\sqrt{\frac{1+\cos\eta}{1-\cos\eta}}.$$

Now compute $dt/d\eta$ from $dt/d\eta = (dr/d\eta)/\dot{r}$:

$$\frac{dt}{d\eta} = \frac{A\sin\eta}{\sqrt{2|\varepsilon|}\sqrt{\frac{1+\cos\eta}{1-\cos\eta}}} = \frac{A}{\sqrt{2|\varepsilon|}}\sin\eta\sqrt{\frac{1-\cos\eta}{1+\cos\eta}}.$$

Using the trigonometric identity $\sin\eta\sqrt{\dfrac{1-\cos\eta}{1+\cos\eta}} = 1-\cos\eta$, we obtain

$$\frac{dt}{d\eta} = \frac{A(1-\cos\eta)}{\sqrt{2|\varepsilon|}}.$$

Integrate from $\eta = 0$ (where $t = 0$) to general η:

$$t(\eta) = \frac{A}{\sqrt{2|\varepsilon|}}(\eta - \sin\eta).$$

Finally substitute back $A = GM/(2|\varepsilon|)$ to express the solution in terms of GM and $|\varepsilon|$:

$$\boxed{r(\eta) = \frac{GM}{2|\varepsilon|}\,(1-\cos\eta), \qquad t(\eta) = \frac{GM}{(2|\varepsilon|)^{3/2}}\,(\eta - \sin\eta)\,.}$$

These are the standard cycloidal parametric equations for a bound shell with $r(0) = 0,\ t(0) = 0$, turnaround at $\eta = \pi$ (where $r_{\max} = 2A = GM/|\varepsilon|$), and recollapse at $\eta = 2\pi$.