PHYS644 Problem Set 4

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Problem 1: Exploring the $M_{bh} - \sigma$ relationship

Problem 1A:

i:

Figure 1 shows our initial log-uniform distribution, with a correlation coefficient of $|r| = 0.02$ which is strongly uncorrelated.

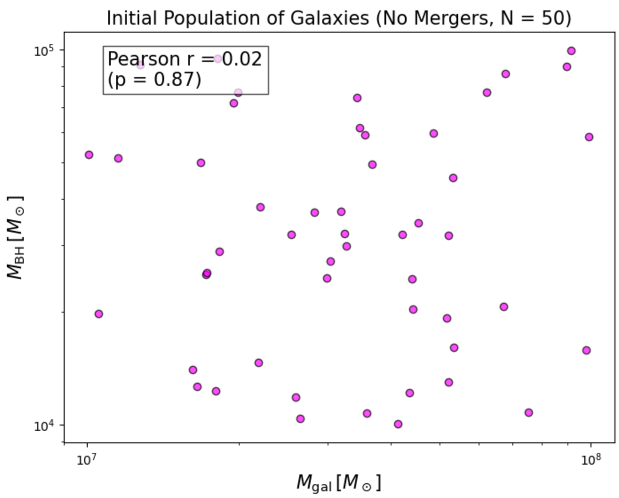


Figure 1: I'm not writing a caption

ii:

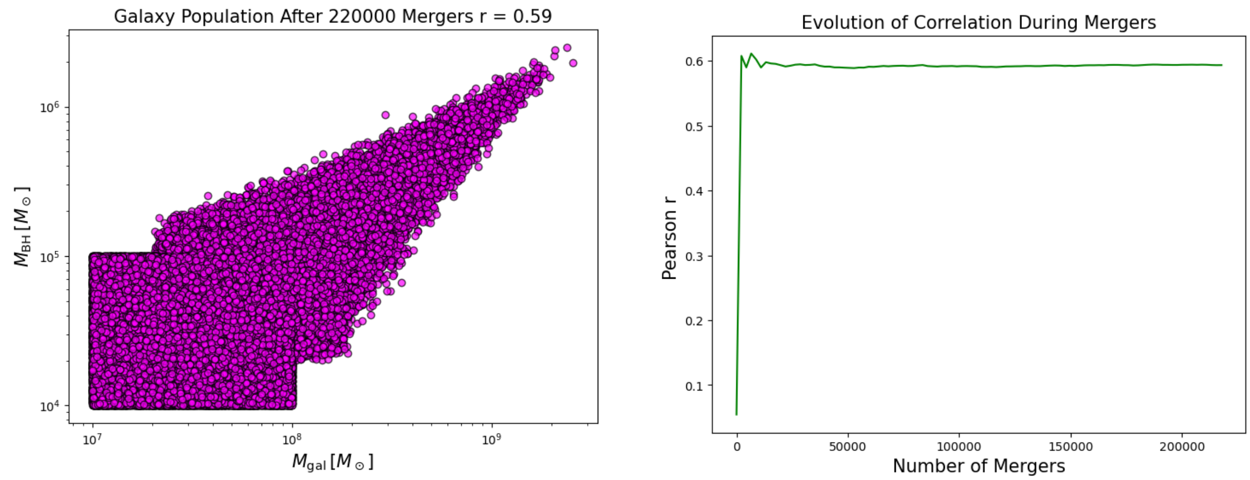
Figure 2a and 2b shows the result of many galaxy mergers , and the corresponding correlation r as function of the number of mergers.

We see that $M_{bh} \propto M_{\rm gal}$ as the number of mergers increases, and becomes tighter. Given the finite number of mergers, only using a small population size , the correlation does not increase all the way up to $r \sim 1$, also the fact that we did not exclude the intial population boundries for calculating r. The sharp rise in r is caused by the small number of initial galaxies compared to the number of mergers, this doesn't strongly effect the outcome because we are merging with replacement. (The better way is to not sample with replacement, and exclude the starting values from the correlation measurements).

Problem 1B:

Suppose that black hole accretion is governed by the relation:

$$\dot{M_{\rm bh}} = D(t) M_{\rm bh}^p \tag{1}$$



(a) I'm not writing a caption

(b) I'm not writing a caption

Figure 2: I'm not writing a caption

Where p is some powerlaw index, $D(t)$ is some function of time that governs the accretion rate. We are asked to show that for two black holes with masses Ma and Mb, we have

$$\frac{d}{dt}\left(\frac{M_a}{M_b}\right) = D(t)\frac{M_a^p}{M_b}\left[1-\left(\frac{M_a}{M_b}\right)^{1-p}\right] \tag{2}$$

We are also asked: What must be true about p for Ma and Mb to converge over time? We can use the product rule to express the LHS in equation 2 as:

$$\frac{d}{dt}\left(\frac{M_a}{M_b}\right) = \frac{dM_a}{dt}M_b^{-1} - M_b^{-2}\frac{dM_b}{dt}M_a \tag{3}$$

We can use 1 to express the derivatives now.

$$\boxed{\frac{dM_a}{dt}M_b^{-1} - M_b^{-2}\frac{dM_b}{dt}M_a = D(t)M_a^pM_b^{-1} - M_b^{-2}M_aD(t)M_b^p = D(t)\frac{M_a^p}{M_b}\left[1-\left(\frac{M_a}{M_b}\right)^{1-p}\right]} \tag{4}$$

For this to converge, $\frac{d}{dt}\left(\frac{M_a}{M_b}\right) < 0$ this means that $\left[1-\left(\frac{M_a}{M_b}\right)^{1-p}\right] < 0$. The values of p will depend on if $M_a > M_b$ or the inverse, assuming $M_a > M_b$ we have $\boxed{p < 1}$.

Problem 1B ii:

What's an appropriate value for p in Equation 1?

Compared with the Bondi accretion rate:

$$\dot{M}_{\rm bh} = \alpha\frac{4\pi G^2 M_{\rm bh}^2\rho}{(c_s^2+V_\infty^2)^{3/2}}. \tag{5}$$

From comparing the two, we see that $p = 2$, which is odd because we said for convergence $p < 1$. This perhaps means that they do not converge? The question tells us to think of stuff, and that observationally $p \sim 1/6$.

Problem 2: The Bathtub Model

Problem 2A:

Here we consider a simplified bathtub model, where we think of a galaxy as a factory that accepts inflowing gas, forms stars, and then expels some of this gas in outflows. An equation describing the "conservation of gas" can be written as:

$$\dot{m}_g = f_b \dot{m}_h - \dot{m}_w \tag{6}$$

Where m_g is the gas content in the galaxy, m_h is the halo mass, and m_w is the max expelled, f_b is some conversation factor from total matter to baryonic matter.

$\boxed{\text{I am convinced}}$. The total mass vs time is given by the mass into the system minus the mass leaving the system.

Problem 2B:

Assume that:

$$\dot{m}_w = \eta \dot{m}_* \tag{7}$$

That the mass expelled is proportional to the star formation rate, η is known as the *mass loading factor* . Also assume $\dot{m}_* = \frac{m_g}{\tau_*}$, $\tau_{ff} = \epsilon_{sf}\tau_*$ with ϵ_{sf} being the efficiency of star formation, τ_* being a char time for star formation, and τ_{ff} is the free fall time.

pluggin in, we can write the conservation of gas as:

$$\frac{dm_g}{dt} + \frac{\eta\epsilon_{sf}}{\tau_{ff}} m_g = f_b \dot{m}_h \tag{8}$$

we are asked to solve for $m_g(t)$ assuming η, ϵ_{sf}, τ_{ff}, f_b, and $\dot{m}_h$ are constants

This is a first-order linear equation, I remember learning how to solve these in my old (and missed) ODE book. This is a classic solve with integrating factor problem.

$$\frac{dm_g}{dt} + P m_g = Q \tag{9}$$

with $P = \frac{\eta\epsilon_{sf}}{\tau_{ff}}$, and $Q = f_b \dot{m}_h$. Our integrating factor is $u = e^{\int p dt} = e^{pt} = e^{t\frac{\eta\epsilon_{sf}}{\tau_{ff}}}$.

The general solution is given by:

$$m_g(t) = \frac{\int Qu\, dt + c}{u} \tag{10}$$

But we will solve it step by step

Now we multiply 9 by u

$$u\frac{dm_g}{dt} + uPm_g = uQ \Rightarrow e^{pt}\frac{dm_g}{dt} + e^{pt}Pm_g = e^{pt}Q \tag{11}$$

$$\frac{d}{dt}(m_g e^{pt}) = e^{pt} Q \tag{12}$$

Now we can solve this easily,

$$m_g e^{pt} = \int e^{pt} Q\, dt = \frac{Q}{P} e^{pt} + c \tag{13}$$

where c is from the constant (I haven't picked our boundries yet).

$$m_g e^{pt} = \frac{Q}{P} e^{pt} + c \tag{14}$$

Replacing back what p, and Q are

$$m_g e^{t\frac{\eta\epsilon_{sf}}{\tau_{ff}}} = \frac{f_b \dot{m}_h}{\frac{\eta\epsilon_{sf}}{\tau_{ff}}} e^{t\frac{\eta\epsilon_{sf}}{\tau_{ff}}} + ce^{-pt} \tag{15}$$

and Simplifying we have:

$$\boxed{m_g e^{t\frac{\eta\epsilon_{sf}}{\tau_{ff}}} = f_B \dot{m}_h \frac{\tau_{ff}}{\eta\epsilon_{sf}} e^{t\frac{\eta\epsilon_{sf}}{\tau_{ff}}} + ce^{-pt}} \tag{16}$$

or

$$\boxed{m_g(t) = f_b \dot{m}_h \frac{\tau_{ff}}{\eta\epsilon_{sf}} + ce^{-t\frac{\eta\epsilon_{sf}}{\tau_{ff}}}} \tag{17}$$

where c is an initial condition, and is given by $c = m_g(0) - f_b \dot{m}_h \frac{\tau_{ff}}{\eta\epsilon_{sf}}$

Problem 2C:

we are asked what happens to m_g as time goes forward $t >> \tau_{ff}$, from 18 we simply set it equal to 6 and sub in m_w.

At long times, the decaying term goes to 0, and we have

$$\boxed{m_g(t) = f_b \dot{m}_h \frac{\tau_{ff}}{\eta\epsilon_{sf}}} \tag{18}$$

Which is the steady state gas mass.

Recall that

$$\dot{m}_* = \frac{m_g}{\tau_*} = \frac{m_g \epsilon_{\text{sf}}}{\tau_{ff}} \tag{19}$$

$$\dot{m}_* = f_b \dot{m}_h \frac{\tau_{ff}}{\eta\epsilon_{sf}} \frac{\epsilon_{\text{sf}}}{\tau_{ff}} \tag{20}$$

$$\boxed{\dot{m}_* = \frac{f_b \dot{m}_h}{\eta}} \tag{21}$$

The star formation rate is independent of ϵ_{sf}.

Problem 2D

A conceptual way to view this is that for the Galaxy as a whole, mass is conserved, i.e. $\dot{m}g = 0$. Substituting this condition into equation 6 yields the same expression for the star formation rate, $\dot{m}*$, that we derived earlier.

In plain terms, the star formation rate in a steady-state galaxy is determined by the balance between baryonic matter flowing in and mass lost through outflows. Any additional gas that enters the galaxy beyond what is r must be expelled as winds, driven by the formation of stars.