PHYS644 Final Problem Set

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Problem 1: Peculiar Velocities

Problem 2: Useful expressions for low-redshift cosmology ($z << 1$)

Problem 2A:

We are asked to find an expression for the commoving distance d as a function of redshift z, and then to comment on if it matters what type of distance.

We can start from the equation for commoving distance in natural units is.

$$d(z) = \int_0^z \frac{1}{H(z)}\, dz \tag{1}$$

For the case that $z << 1$, The Hubble parameter is a constant, $H(z) \approx H_0$. So we can take it out of the integral, and then the integral is trivial.

$$\boxed{d(z) = \frac{1}{H_0} \int_0^z dz = \frac{z}{H_0}} \tag{2}$$

Does the requested kind of distance matter? $\boxed{\text{No.}}$ The luminosity distance and angular-diameter distance are different from the comoving distance; however, They differ by factors of $(1+z)$. But to first order those are now 1.

Problem 2B:

Now we are asked to write the recession velocity at redshift z.

$$\boxed{v = H_0 d = v = H_0 \frac{z}{H_0} = z} \tag{3}$$

This is pretty simple, we can just directly substitute into our earlier expression.

Problem 2C:

We are asked to show that H_0^{-1} is close to a round number when expressed in units of $h^{-1}\,\text{Mpc}$. h here is dimensionless, by definition is $H_0 = 100h\text{kms}^1\text{Mpc}^1$. Given that it says this will be usefull for expressions like in problem 1A, it is assumed that this expression has a factor of c possibly mixed in.

Using $c \approx 3 \times 10^5\,\text{km}\,\text{s}^{-1}$, we obtain

$$\frac{c}{H_0} = \frac{3 \times 10^5}{100h}\,\text{Mpc} = \frac{3000}{h}\,\text{Mpc} = 3000\,h^{-1}\,\text{Mpc}.$$

To one significant digit, H_0^{-1} to $\boxed{c/H_0 \approx 3 \times 10^3\,h^{-1}\,\text{Mpc}}$.

Problem 2D:

We are asked to now express the comoving volume per solid angle per redshift interval. The comoving volume element per unit solid angle and redshift is given by:

$$\frac{dV}{d\Omega\,dz} = d^2(z)\,\frac{dd}{dz}. \tag{4}$$

This comes from $dV = d^2 d\Omega dd$ which is the area multiplied by the thickness of our solid angle "slice", and divide by $d\Omega$, and change from dd to dz using the chain rule $dd = \frac{dd}{dz}dz$

Since we are in $z << 1$, we can use the low-redshift approximation.

$$d(z) \approx \frac{z}{H_0}, \tag{5}$$

$$\frac{dd}{dz} \approx \frac{1}{H_0}. \tag{6}$$

We can substitute these into the volume element expression above giving:

$$\frac{dV}{d\Omega\, dz} \approx \left(\frac{z}{H_0}\right)^2 \frac{1}{H_0} = \frac{z^2}{H_0^3}. \tag{7}$$

Rewriting:

$$\boxed{\frac{dV}{d\Omega\, dz} \approx \frac{z^2}{H_0^3}} \tag{8}$$

Problem 2E:

We are asked to compute the following quantities without a calculator or a computer[1].

We are using in natural units $H^{-1} \approx c/H_0 \approx 3 \times 10^3\, h^{-1}\,\mathrm{Mpc}$.

- The comoving distance to $z = 0.1$. Answer: $\frac{z}{H_0} = 0.1 * 3 \times 10^3\, h^{-1}\,\mathrm{Mpc} = \boxed{3 \times 10^2\, h^{-1}\,\mathrm{Mpc}}$.
- The comoving volume out to $z = 0.1$. Answer: $\frac{z^2}{H_0^3} = 0.1^2 * (3 \times 10^3\, h^{-1}\,\mathrm{Mpc})^3 = \boxed{3 \times 10^1\, (h^{-1}\,\mathrm{Mpc})^3}$. I am assuming it is asking about the comoving volume per solid angle per redshift interval, as that makes more sense. I'm happy to be wrong.
- What redshift does a galaxy with a peculiar velocity of $v_p \approx 300\,\mathrm{km/s}$ have this peculiar velocity be 10% of its recession velocity? How far away is such a galaxy? Answer: $v_p = 0.1 v_{rec}$, then $\boxed{v_{rec} = 3000\,\mathrm{km/s}}$. Then $\boxed{z \approx \frac{v_{rec}}{c} \approx 0.01}$

[1] There is no rule against using slide rules!!

Problem 3: CMB power spectrum in a different universe

Problem 4: Redshift Space Distortions

From Problem Set 10 that in linear theory there is a tight relation between the peculiar velocities of matter and their overdensity. In particular, we found that:

$$\tilde{\mathbf{V}} = \frac{ifaH}{k}\,\hat{\mathbf{k}}\,\tilde{\delta} \tag{9}$$

where $f = \frac{d\ln D_1}{d\ln a}$.

We assume that if a galaxy has redshift z, we assume it is at position:

$$\mathbf{r}_s = \frac{z}{H_0}(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{10}$$

This is known as redshift space, and is slightly wrong because galaxies have peculiar velocities.

Problem 4A:

Because it is sperhically symmetric, and shell of matter — The overdensity produces radial infall. every particle on the shell has the same peculiar velocity pointing toward the center (really the same magnitude). As we are presumably outside the shell looking at it from a distance, one side would look redder and the other side bluer then our expectations of $\mathbf{r} - s$ as one side is slightly going faster towards us and the other slower. So I think it means we would observer it as slightly denser?

Problem 4B

The observed redshift for non-relativistic peculiar velocities is

$$1 + z_{\text{obs}} = (1 + z_{\text{cos}})\left(1 + \frac{v_{\parallel}}{c}\right), \tag{11}$$

where z_{cos} is the cosmological redshift and z_{obs} is the redshift we measure, which is affected by the galaxy's peculiar velocity[2].

To first order we can say:

$$z_{\text{obs}} = z_{\text{cos}} + \frac{v_{\parallel}}{c} \tag{12}$$

where we throw out the terms with $z_{\text{cos}}\frac{v_{\parallel}}{c}$. Of course, $v_{\parallel} = \mathbf{v}\cdot\hat{\mathbf{r}}$.

We can write $z_{\text{cos}} = \frac{H_0 r}{c}$, the implied $\mathbf{r}_s$ is at:

$$\mathbf{r}_s = \frac{cz_{\text{obs}}}{H_0}\hat{\mathbf{r}} \tag{13}$$

Plugging in our z_{obs}, we get:

$$\mathbf{r}_s = \frac{c(z_{\text{cos}} + \frac{v_{\parallel}}{c})}{H_0}\,\hat{\mathbf{r}} = \frac{cz_{\text{cos}}}{H_0}\hat{\mathbf{r}} + \frac{v_{\parallel}}{H_0}\hat{\mathbf{r}} = \mathbf{r} + \frac{(\mathbf{v}\cdot\hat{\mathbf{r}})}{H_0}\hat{\mathbf{r}} \tag{14}$$

$$\boxed{\mathbf{r}_s = \mathbf{r} + \frac{(\mathbf{v}\cdot\hat{\mathbf{r}})}{H_0}\hat{\mathbf{r}}} \tag{15}$$

[2] Redshifts add like this $1 + z_{\text{tot}} = (1 + z_1)(1 + z_2)$...

Problem 4C

We know the number of galaxies is a conserved quantity,

$$n_s(\mathbf{r}_s)\, d^3r_s = n(\mathbf{r})\, d^3r. \tag{16}$$

We know from 4B that the redshift-space map only changes the radial coordinate, and does not affect the angular coordinates. **Peculiar velocities do not affect the angular coordinates.** This means we can factor out the angular parts.

We can write the 3D volume element in spherical coordinates:

$$d^3r = r^2 \sin\theta\, dr\, d\theta\, d\phi, \qquad d^3r_s = r_s^2 \sin\theta\, dr_s\, d\theta\, d\phi. \tag{17}$$

Now we can write

$$\frac{d^3r_s}{d^3r} = \frac{r_s^2}{r^2}\frac{dr_s}{dr}. \tag{18}$$

We define

$$\mathbf{r}_s = \mathbf{r} + \aleph(\mathbf{r}, \theta, \phi), \tag{19}$$

with

$$\aleph(\mathbf{r}) = \frac{(\mathbf{v}\cdot\hat{\mathbf{r}})}{H_0}\hat{\mathbf{r}}. \tag{20}$$

Now, the idea is to simply take derivatives of this and then construct our differential equation. Taking the derivative gives:

$$\frac{dr_s}{dr} = 1 + \frac{\partial}{\partial r}\left(\frac{\mathbf{v}\cdot\hat{\mathbf{r}}}{H_0}\right), \tag{21}$$

and the ratio of squared radii is

$$\frac{r_s^2}{r^2} = \left(1 + \frac{\aleph}{r}\right)^2 = \left(1 + \frac{\mathbf{v}\cdot\hat{\mathbf{r}}}{H_0 r}\right)^2. \tag{22}$$

Putting this all together in the number-conservation equation, we find

$$n_s(\mathbf{r}_s) = n(\mathbf{r})\frac{d^3r}{d^3r_s} = n(\mathbf{r})\left[1 + \frac{\partial}{\partial r}\left(\frac{\mathbf{v}\cdot\hat{\mathbf{r}}}{H_0}\right)\right]^{-1}\left[1 + \frac{\mathbf{v}\cdot\hat{\mathbf{r}}}{H_0 r}\right]^{-2}. \tag{23}$$

$$\boxed{n_s(\mathbf{r}_s) = n(\mathbf{r})\left(1 + \frac{\partial}{\partial r}\frac{\mathbf{v}\cdot\hat{\mathbf{r}}}{H_0}\right)^{-1}\left(1 + \frac{\mathbf{v}\cdot\hat{\mathbf{r}}}{H_0 r}\right)^{-2}} \tag{24}$$

Problem 4D

From the hint, we can neglect the second bracketed term and start with:

$$n_s(\mathbf{r}_s) = n(\mathbf{r})\left(1 + \frac{\partial}{\partial r}\frac{\mathbf{v}\cdot\hat{\mathbf{r}}}{H_0}\right)^{-1} \tag{25}$$

Since we are assuming a homogeneous region (galaxies have same mass etc), the real-space number density $n(\mathbf{r})$ can be written as:

$$n(\mathbf{r}) = \bar{n}[1 + \delta(r)] \tag{26}$$

Where $\bar{n}$ is the average, and $\delta(r)$ is the overdensity $\delta(r) = \frac{n(r)-\bar{n}}{\bar{n}}$. Since we are assuming the peculiar velocities are small, $v << 1$, we can use $(1+a)^{-1} \sim 1 - A$, with $A = \frac{\partial}{\partial r}\frac{\mathbf{v}\cdot\hat{\mathbf{r}}}{H_0}$ [3].

This leads to:

$$n_s(\mathbf{r}_s) = \bar{n}[1 + \delta(\mathbf{r})]\left[1 - \frac{\partial}{\partial r}\left(\frac{v \cdot \hat{r}}{H_0}\right)\right] \tag{27}$$

We are going to multiply this out, drop some higher order terms, and then I think we do is swap our n_s for $\delta_s(r)$, and we should get the final expression. Let's try that.

$$n_s(\mathbf{r}_s) \simeq \bar{n}\Big[1 + \delta(\mathbf{r}) - \frac{\partial}{\partial r}\left(\frac{v \cdot \hat{r}}{H_0}\right)\Big] \tag{28}$$

and,

$$\boxed{\delta_s(\mathbf{r}) = \delta(\mathbf{r}) - \frac{\partial}{\partial r}\left(\frac{v(\mathbf{r}) \cdot \hat{r}}{H_0}\right)} \tag{29}$$

Both sides are dimensionless.

Problem 4E:

We start from our boxed solution to problem 4D.

$$\delta_s(\mathbf{r}) = \delta(\mathbf{r}) - \frac{\partial}{\partial r}\left(\frac{v(\mathbf{r}) \cdot \hat{r}}{H_0}\right) \tag{30}$$

In the flat sky approximation, we can replace the spherical radial derivative with a simple 1D derivative in z, and replace r with z.

$$\delta_s(\mathbf{z}) = \delta(\mathbf{z}) - \frac{\partial}{\partial z}\left(\frac{v(\mathbf{z}) \cdot \hat{z}}{H_0}\right) \tag{31}$$

Now we take the Fourier transform of each term. The hint from the previous problem tells us a trick though. O wait what is f? Ah recall that $f = \frac{d\ln D_1}{d\ln a}$ is the linear growth rate, and was mentioned in HW 10.

$$\tilde{\delta}_s(\mathbf{k}) = \tilde{\delta}(\mathbf{k}) - ik_z\left(\frac{\widetilde{\mathbf{v}\cdot\hat{z}}}{H_0}\right)(\mathbf{k}) \tag{32}$$

Now hmm. We can use the Fourier transform of the continuity equation [4].

The continuity is:

$$\frac{\partial \delta}{\partial t} + \nabla \cdot \mathbf{v} = 0 \tag{33}$$

In Fourier space, for linear growth factors this becomes $\nabla \Rightarrow i\mathbf{k}$:

$$ik \cdot \tilde{v}(k) = -Hf\tilde{\delta}(\mathbf{k}) \tag{34}$$

[3] You might be worried that v being small is not the same as A being small, but with more thought the hint/ start of the problem it is ok.

[4] I know this because it is on my reference sheet

For the case of irrational flow ($\tilde{\mathbf{v}}$) is parallel to $\mathbf{k}$, we can say[5]:

$$\tilde{\mathbf{v}}(\mathbf{k}) = ifH_0 \frac{\mathbf{k}}{k^2}\tilde{\delta}(\mathbf{k}) \tag{35}$$

This lets us write:

$$\widetilde{\left(\frac{\mathbf{v}\cdot\hat{z}}{H_0}\right)}(\mathbf{k}) = if\frac{\mathbf{k}\cdot\hat{z}}{k^2}\tilde{\delta}(\mathbf{k}) \tag{36}$$

I just found out about the `wildetilde`, anyway I think we are making forward progress finally. Now we can write

$$\tilde{\delta}_s(\mathbf{k}) = \tilde{\delta}(\mathbf{k}) - ik_z[if\frac{\mathbf{k}\cdot\hat{z}}{k^2}\tilde{\delta}(\mathbf{k})] = \tilde{\delta}(\mathbf{k}) + f\frac{k_z^2}{k^2}\tilde{\delta}(\mathbf{k}). \tag{37}$$

From the hint, we say $\mu_k = \hat{k}\cdot\hat{z} = k_z/k$, then we can say $k_z^2/k^2 = \mu_k^2$ and we get the boxed result.

$$\boxed{\tilde{\delta}_s(\mathbf{k}) = (1 + f\mu_k^2)\tilde{\delta}(\mathbf{k})} \tag{38}$$

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Problem 4F) This is a long problem, there is another part after this! Starting from the boxed solution to the last part.

$$\tilde{\delta}_s(\mathbf{k}) = (1 + f\mu_k^2)\tilde{\delta}(\mathbf{k}) \tag{39}$$

$f > 0$ acts as the growth rate, and μ_k is the cosine of the angle between the mode and the line of sight.

as $1 + f\mu_k^2 \geq 1$ the observed overdensity in redshift-space is larger than the real-space over density. The boost is larger for modes along the line-of-sight where $\mu = 1$. this is consistent with my problem 4A comment.

Problem 4G

We start from a given, the return of the power spectrum!

$$P_s(\mathbf{k}) = P_k(1 + f\mu_{\mathbf{k}}^2)^2 \tag{40}$$

Conceptually, we just said from problem 4F that over densities result in larger than the real over densitiy in redshift space when we observe it. So it makes sense that the power spectrum when we average over all $\mathbf{k}$ is also biased to be an over estimate.

When we assume spherical isotropy we are saying:

$$< P_s(k) >= \frac{1}{4\pi}\int P_s(\mathbf{k})\, d\Omega_k \tag{41}$$

We can actually just do this with our above P_s expression, and see what it looks like.

$$< P_s(k) >= P(k)\frac{1}{4\pi}\int (1 + f\mu_{\mathbf{k}}^2)^2\, d\Omega_k \tag{42}$$

[5] Also on my ref sheet, also recall that $\nabla \Rightarrow i\mathbf{k}$

Looks like an easy integral? We foil out the inside part and then do each term.

$$< P_s(k) >= P(k)\frac{1}{4\pi}\int(1 + 2f\mu_{\mathbf{k}}^2 + f^2\mu_{\mathbf{k}}^4)\, d\Omega_k \tag{43}$$

Recall that $\mu = \cos\theta = \hat{k}\cdot\hat{z}$, we know that the isotropic averages for $< \mu^2 >= \frac{1}{3}$, and $< \mu^4 >= \frac{1}{5}$ (from a table).

This means that the above integral is:

$$\boxed{< P_s(k) >= P(k)(1 + f\frac{2}{3} + f^2\frac{1}{5})} \tag{44}$$

This one was actually not as hard as some of the previous steps! We see that $< P_s(k) >\geq P(k)$

Problem 5: Fitting a straight line with x and y errors

Problem 5A:

Recall $p(y) = \int p(x, y)\, dy$.

Our model is $y = mx$, and we say there is some Gaussian error on $y_0 \sim N(mx_0, \sigma_y^2)$. The likelihood is then the Gaussian distribution:

$p(y_0|m) = \frac{1}{\sqrt{2\pi\sigma_y^2}} \exp(-\frac{(y_0 - mx_0)^2}{2\sigma_y^2})$. We are told to assume a uniform prior $p(m) = \text{const}$.

Using Bayes' theorem we can write:

$$p(m|y_0) \propto p(y_0|m)p(m) \propto p(y_0|m) \tag{45}$$

Since, $p(m)$ is a constant we can ignore it. This means the posterior is:

$$\boxed{p(y_0|m) = \frac{1}{\sqrt{2\pi\sigma_y^2}} \exp(-\frac{(y_0 - mx_0)^2}{2\sigma_y^2})} \tag{46}$$

Assuming a "over a very large range" $\Rightarrow (-\infty, +\infty)$ we can easily use the known integral for the Gaussian to compare the width of the distribution, but we need to write in terms of m.

Thankfully it isnt that bad, we have $(y_0 - mx_0)^2 = x_0^2(m - \frac{y_0}{x_0})^2$

This gives:

$$p(m|y_0) \propto \exp(-\frac{x_0^2}{2\sigma_y^2}(m - \frac{y_0}{x_0})^2) = N(m, |\frac{y_0}{x_0}, \frac{\sigma_y^2}{x_0^2}) \tag{47}$$

In other words, $\bar{m} = \frac{y_0}{x_0}$, and $\sigma_m^2 = \frac{\sigma_y^2}{x_0^2}$. Googling the comparison, says that this is the same as traditional least squares fitting.

Problem 5B:

When there are uncertainties in x_0, we want to find $p(m|x_0, y_0)$. Bayes thm says:

$$p(m|x_0, y_0) \propto p(x_0, y_0|m)p(m) \tag{48}$$

This time the likelihood $p(x_0, y_0|m)$ is harder to write down, the $y = mx$ works for the true value of x, y. Using Bayes' thm 2x more gives:

$$p(m|x_0, y_0) \propto \int p(x_0, y_0|x, y, m)p(x, y|m)p(m)dxdy \propto \int p(x_0, y_0|x, y)p(y|x, m)p(x)p(m)dxdy \tag{49}$$

Basically we break them up some more, assuming uncorrelated errors. This is all apart of the given. Since we are assuming uncorrelated errors we use a 2D Gaussian for the $p(x_0, y_0|x, y)$.

$$p(x_0, y_0|x, y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp(-\frac{(x_0 - x)^2}{2\sigma_x^2} - \frac{(y_0 - y)^2}{2\sigma_y^2}) \tag{50}$$

Assuming the priors for $p(x)$, $p(m)$ are uniform. We have the true relationship $y = mx$, so we can also write $p(y|m, x) = \delta(y - mx)$ here δ is the Dirac delta function, and this works because we are saying it is for the true values aka deterministic. We can now substitute these two expressions into the integral, but we can use the Dirac to just do the integral...

$$p(m|x_0, y_0) = \int p(x_0, y_0|x, y = mx)p(x)p(m)dx \tag{51}$$

We can drop out the $p(m)$ which we assume has a uniform prior, We are also dropping $p(x)$, actually writing the Gaussian:

$$\boxed{p(m|x_0, y_0) = \int_{-\infty}^{+\infty} \frac{1}{2\pi\sigma_x\sigma_y} \exp(-\frac{(x_0 - x)^2}{2\sigma_x^2} - \frac{(y_0 - mx)^2}{2\sigma_y^2})p(x)dx} \tag{52}$$

This is now an integral over a 2D Gaussian. At first glance you might try reading off the 1D integral, but that is wrong as we replaced $y = mx$ in the y Gaussian.

Problem 5C:

Let's assume 2D independent Gaussian, then each dimension is it's own 1D Gaussian. and assuming broad uniform priors on $p(x), p(m)$ we can neglect them.

We get the joint posterior:

$$\boxed{p(m, x|x_0, y_0) \propto \exp(-\frac{(x_0 - x)^2}{2\sigma_x^2} - \frac{(y_0 - mx)^2}{2\sigma_y^2})} \tag{53}$$

Problem 5D:

See notebook for this problem located Here.

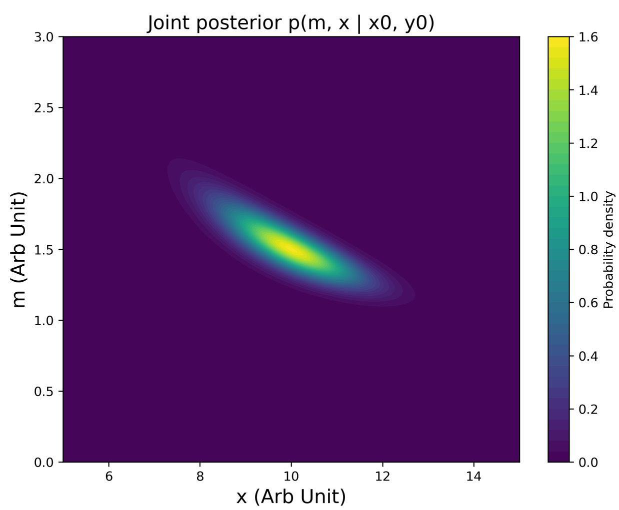

Figure 1: I'm not writing a caption

We can tell that there is a bit of degeneracy between x, m due to the angled shape.

Problem 5E

Again, see notebookHere.

We see that when there is no σ_x, the the PDF is much tighter / denser.

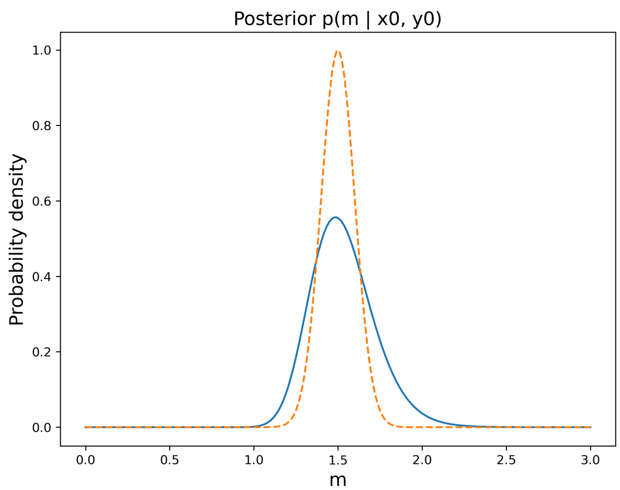

Figure 2: I am not writing a caption