PHY644 Notes

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Fall 2025

Contents

1 Galaxies As Collisionless Fluids

We cannot think of stars as individual elements, but rather the collection of stars a s **collision-less fluid**.

1.1 Phase Space Density

We can define a Phase Space Density,

$$f(\vec{x}, \vec{v}, t)\, d^3\vec{x}\, d^3\vec{v} \tag{1}$$

This is the number of stars in volume $d^3\vec{x}$, centred on $\vec{x}$., and velocities in small range $d^3\vec{v}$ centred on $\vec{v}$.aka density in 6D phase space given by $\vec{w} = (\vec{x}, \vec{v})$

We can think of a swarm (or blob) of stars (particles) moving in this phase space with velocity:

$$\dot{\vec{w}} = (\dot{\vec{x}}, \dot{\vec{v}}) = (\vec{v}, -\nabla\Phi) \tag{2}$$

Here Φ is the gravitational potential not gravitational potential energy (IE it is per unit mass).

1.2 Fluid Mechanics for Stars

Let's apply Newtonion Mechanics (Fluid mechanics). We assume that stars are collision-less, and that no stars form or die.

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \frac{\partial (f \dot{w_\alpha})}{\partial w_\alpha} = 0 \tag{3}$$

$f\dot{w_\alpha}$ is the phase space current. Using the product rule, we can express the term with the sum as:

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} f \frac{\partial \dot{w_\alpha}}{\partial w_\alpha} + \dot{w_\alpha} \frac{\partial f}{\partial w_\alpha} = 0 \tag{4}$$

The flow described by $\dot{w}$ is special because:

$$\sum_{\alpha}^{6} \frac{\partial \dot{w_\alpha}}{\partial w_\alpha} = \sum_{i}^{3} (\frac{\partial v_i}{\partial x_i} + \frac{\partial \dot{v_i}}{\partial v_i}) = \sum_{i}^{3} (0 + \frac{-\partial}{\partial v_i} \frac{d\Phi}{\partial x_i}) = 0 \tag{5}$$

Because v_i is independent of x_i, and we can write $\dot{v_i}$ as the gradient of the potential, and then the partial derivative of that is equal to 0 because it is not dependent on v_i.

$$\boxed{\sum_{i}^{3} (0 + \frac{-\partial}{\partial v_i} \frac{d\Phi}{\partial x_i}) = 0} \tag{6}$$

Throwing this back into equation 4, we have the colisionless Boltzman equation:

$$\boxed{\frac{\partial f}{\partial t} + \sum_{\alpha}^{6} \dot{w_\alpha} \frac{\partial f}{\partial w_\alpha} = 0} \tag{7}$$

There are a few equivalent variations of this using the above equations.

1.3 Co-moving fluid mechanic Equation

Recasting into Lagrangian or curvature forms.

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{\alpha}^{6} \dot{w}_{\alpha} \frac{\partial}{\partial w_{\alpha}} \tag{8}$$

This is the form of the equation for someone (or a reference frame) flowing along a trajectory in phase space. It says that I can get a change either from some explicit time dependence (1st term) or because it moved to a different part of phase space (2nd term).

The Boltzman equation is then just $\boxed{\frac{df}{dt} = 0}$. In other words, for an observer moving along with a star's path $\dot{w}$ would not see the local phase space density change.

1.4 Moments of Phase space

We do not observe phase space directly, but we can observe the moments of phase space.

1.4.1 0th Moment

The zeroth moment - gives the spacial number density of stars:

$$n(\vec{x}) = \int_{-\infty}^{+\infty} f(\vec{x}, \vec{v}, t) d^3\vec{v} \tag{9}$$

The first moment - gives the average velocity:

$$\mathbb{E}[\vec{v}(\vec{x})] = \frac{1}{n} \int_{-\infty}^{+\infty} \vec{v} f(\vec{x}, \vec{v}, t) d^3\vec{v} \tag{10}$$

The second moment - is related to the velocity dispersion tensor:

$$\langle v_i(\vec{x}) v_j(\vec{x}) \rangle = \frac{1}{n} \int_{-\infty}^{+\infty} (v_i v_j) f(\vec{x}, \vec{v}, t) d^3\vec{v} = \langle v_i(\vec{x}) \rangle \langle v_j(\vec{x}) \rangle + \sigma_{ij}^2 \tag{11}$$

The σ_{ij}^2 is the velocity dispersion.

1.5 Getting Useful forms

To actually get the equations of this, we can take the moments of the Boltsman equation.

For the 0th moment:

$$0 = \int_{-\infty}^{+\infty} [\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f - (\vec{\nabla}_x \Phi) \cdot (\vec{\nabla}_v f)] d^3\vec{v} \tag{12}$$

Use Integration by parts...

The 0th moment is the continuity equation:

$$\boxed{\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n(\langle \vec{v} \rangle)) = 0} \tag{13}$$

This says that stars are conserved.

1.5.1 1st moment: The Jeans Equation

$$0 = \int_{-\infty}^{+\infty} [\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f - (\vec{\nabla}_x \Phi) \cdot (\vec{\nabla}_v f)] \vec{v} d^3 \vec{v} \tag{14}$$

break into terms, and use integration by parts!

$$\boxed{\partial_t \langle \vec{v_j} \rangle + \sum_i \langle \vec{v_i} \rangle \vec{\nabla}_{x,i} \langle \vec{v_j} \rangle = -\vec{\nabla}_{x,j} \Phi - \sum_i \frac{\vec{\nabla}_{x,i}(n\sigma_{ij}^2)}{n}} \tag{15}$$

From left to right, we label the terms as - Bulk accretion, velocity sheer, grav force, and pressure. This is the Jeans equation and is the equivalent of the Euler equation in classical fluid mechanics - the $F = MA$ of fluid dynamics.

The right hand term - the pressure, means that a collisionless fluid will still have a pressure as long as the velocity dispersion is non-zero.

Squashing this fluid increases the velocity dispersion increasing the pressure for it to spread back out.

1.6 Stability Analysis

The Jeans equation tells us some information about the stability of a gas trying to collapse (if it collapses or if it bounces back).

Assume the gas consists of particles of mass m, we can turn $n \Rightarrow \rho$ by $\rho = nm$. We assume the velocity dispersion is diagonal and that $\rho\sigma^2 = P$ gives the pressure. This means that our analysis applies to normal gases as well.

All gradients are spacial so we can drop the x $\vec{\nabla}_x \Rightarrow \vec{\nabla}$.

For a small perturbation we assume a static background (the subscripts with 0) and use:

- $\rho = \rho_0 + \epsilon \rho_1$
- $\vec{v} = \vec{v}_0 + \epsilon \vec{v}_1$
- $P = P_0 + \epsilon P_1$
- $\Phi = \Phi_0 + \epsilon \Phi_1$

Now we insert into the continuity equation, and the Jeans equation and group powers of ϵ. The ϵ^0 terms cancel. The ϵ^1 terms give.

$$\partial_t \rho_1 + \rho_0 \vec{\nabla} \cdot \vec{v}_1 = 0 \tag{16}$$

$$\partial_t \vec{v}_1 = -\vec{\nabla}\Phi - \frac{\vec{\nabla} P_1}{\rho} \tag{17}$$

Recall that sound speed is given by:

$$\boxed{\vec{\nabla} P = \frac{\partial P}{\partial \rho} \frac{\partial \rho}{\partial \vec{x}} = c_s^2 \vec{\nabla} \rho} \tag{18}$$

Jeans swindle: We substitute in the c_s version, and use Poission's equation for gravity for $\nabla^2 \Phi_0 = 4\pi G \rho_0$ which cannot be true for Φ_0 and ρ_0 both constant when $\rho_0 = 0$.

The brings us to a wave equation of the form:

$$\partial_t^2 \rho_1 - 4\pi G \rho_0 \rho_1 - c_s^2 \nabla^2 \rho_1 = 0 \tag{19}$$

The general solution is:

$$\rho_1 = \rho_1(0) e^{i(\omega t \pm K x)} \tag{20}$$

where ω is the angular velocity, and $k = 2\pi/\lambda$.

2 Geometry of Our Universe

Now we begin cosmology! In Galaxy evolution we cared about individual objects, but here we care about population statistics. It is like being a public health official vs a doctor.

Quick note: this section uses natural units $\hbar = c = k_b = 1$. This is different from geometrical units where $G = c = 1$.

$$M_{\text{planck}} = \sqrt{\frac{\hbar c}{G}} \tag{21}$$

In Natural units, this becomes $G^{-1/2}$.

2.1 The Cosmological Principle

Describing the spacetime geometry of our universe is a difficult task, but we can make three simplifications we make and test empirically.

The Cosmological Principle: the Universe is **statistically homogeneous** and **isotropic**. Homogeneity implies that the Universe has the same average properties at every point in space, whereas isotropy implies that the Universe looks the same in all directions from any given vantage point. Together, these symmetries constrain the possible forms of the spacetime metric, leading naturally to the **Friedmann–Lemaître–Robertson–Walker (FLRW)** metric as the general solution for a universe governed by general relativity.

The cosmological principle does not hold for time, because the universe is expanding — time marches forward.

2.2 Hubble Law

Hubble Law: Hubble discovered that the redshift recession velocity of distance galaxies follows the relationship:

$$v = H_0 d \tag{22}$$

$H_0 \sim 70\,\text{kms}^{-1}\text{Mpc}$ The subscript 0 means today value in the present universe.

With the current Hubble tension it is easier to write $H_0 = 100\,\text{kms}^{-1}\text{Mpc}$, and then use $h = 0.7$ to remain noncommittal.

Another way to think about the expansion of the universe is as a **scaling factor**, and that if we measure the universe with some meter stick that the scaling on the meter stick is changing by some function $a(t)$.

By convention we say $a_0 = a(t_0) = 1$ — that today's time is the age of the universe.

Scaling Factors: It's annoying to describe distance if it is constantly changing so we go instead with comoving distances:

$$d_{\text{physical}} = a(t)\chi_{cosmo} \tag{23}$$

Where d_{physical} is what we measure on our meter stick, and χ_{cosmo} "takes out the expansion". If two objects have a changing cosmological distance then there motion is not just due to the expansion of the universe. Remember that $a_0 = a(t_0) = 1$ — that today's time is the age of the universe.

Let's differentiate both sides:

$$\vec{v}(t) = \frac{d\vec{d}_{\text{physical}}}{dt} = \frac{da}{dt}\vec{\chi}_{cosmo} = \frac{da}{dt}\frac{1}{a}\vec{d}_{\text{physical}}(t) \tag{24}$$

This is the Hubble Law! $\vec{v} \propto d$ but at a different time, thus Hubble's law holds for all time but with different values of H_0.

$$\boxed{H(t) = H = \frac{\dot{a}}{a}} \tag{25}$$

This is called the **Hubble parameter** H_0 is today's value.

In general $H(t)$ is some complicated function of time. However, an interesting special case is a universe where the expansion just makes galaxies coast along at a constant speed.

In this case $\dot{a} = const \Rightarrow H(t) = \frac{const}{a} = \frac{H_0}{a}$. The last substitution is because $H(t_0) = H_0$ as $a_0 = 1$. Because $H = \frac{\dot{a}}{a}$ always we can say $\dot{a} = H_0 \Rightarrow \int_0^1 da = H_0 \int_0^{t_0} dt$.

This is how we take the inverse of H_0 as the age of the universe! **This is the Hubble Time $T_h = \frac{1}{H_0} \approx 14.5\,\text{Gyr}$.**

2.3 Allowed Expansion Rates

While objects cannot travel faster then c, space can expand faster then c, this is known as **super luminal expansion** and defines the observable universe.

Using a comoving coordinates makes this problem easy as we are factoring out the expansion.

In time dt light moves a physical distance cdt, so the comoving distance is $\frac{cdt}{a}$ and for a finite time integral we have $\chi = \int \frac{1}{a}dt$ (recall $c = 1$).

Basically its $c \to \frac{c}{a}$. In GR, space itself can expand or contract.

Particle (Causal) Horizon: This is the furthest distance unimpeded light can have travelled from $t = 0$ to $t = t$

$$\chi = \int_0^t \frac{1}{a}dt' \tag{26}$$

To get the physical distance we multiply by $a(t)$.

$$d_p = a(t)\chi = a(t)\int_0^t \frac{1}{a}dt' \tag{27}$$

In the case of subluminal expansion: we have $a(t) \sim t^p$ where $0 < p < 1$ meaning that $\ddot{a} < 0$ (slowing down).

Given enough time, light will see the entire universe.

in the case of superluminal expansion we have $a(t) \sim t^p$ where $p > 1$ meaning that $\ddot{a} > 0$ (getting faster).

You only see finite χ regardless of how long you wait. Light cannot catch up with the expansion of the universe. We live in this type of universe presently.

Intuition: $p = 1$ where $a \propto t$ is the dividing line because the amount of distance light can travel is ct so if $a \propto t$, the expansion balances this effect.

2.4 Friedmann–Lemaître–Robertson–Walker (FLRW) metric

Friedmann–Lemaître–Robertson–Walker (FLRW) metric: This is the metric of space-time for our universe as a whole, result of the cosmological principle.

$$ds^2 = -dt^2 + a(t)^2[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2] \tag{28}$$

Where $d\Omega^2 = d\theta^2 + sm^2\theta d\phi^2$, and K is the spatial curative and it can be positive, zero, or negative. (Curvature of space, not of space-time).

2.5 Observables

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With the FLRW metric we can make predictions, and observe. one of the primary ones is the redshift.

$$z = \frac{\lambda_{obs} - \lambda_0}{\lambda_0} \tag{29}$$

The elongation of wavelengths can be shown to be proportional to $a(t)$ therefore:

$$\frac{\lambda_{obs}}{\lambda_0} = \frac{a(t_0)}{a(t)} \tag{30}$$

$a(t_0) = 1$

$$1 + z = \frac{1}{a} \tag{31}$$

redshift is both a measure of distance, and time.

3 Dynamics of Our Universe

In the last section we talked about $a(t)$ in kinematic terms, and this time we will talk about in dynamic terms — what causes it move the way it does. These are given by the two Friedman equations.

3.1 The Friedmann Equations

First Friedmann equation:

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\,\rho}{3} - \frac{k}{a^2},$$

where ρ is the energy (mass–energy) density (units: energy per volume, e.g. $Jm(^{-3})$; in $c = 1$ units ρ carries same units as mass density).

Second (acceleration) Friedmann equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right),$$

where p is the pressure. (Again written in $c = 1$ units; with a cosmological constant Λ add $+\Lambda/3$ on the right.) Another usefull equation is $\dot{\rho} = -3(\rho + p)(\frac{\dot{a}}{a})$

The first equation is analogous to an energy equation in Newtonian mechanics.

If you ignore the pressure term in the second equation, it looks similar to "$F = Ma$". The negative sign in the second equation says that matter causes the expansion to decrease.

The second equation says that pressure is source of gravity as well, not just density —- this is a property from general relativity.

3.1.1 Deriving the first Friedman equation from Newton's Cosmology

We can derive the first Friedman equation from Newtonian thinking. Imagine a homogeneous universe filled with matter.

Draw a little circle around a test mass - the test mass is on the shell.

we have:

$$m\frac{d^2r}{dt^2} = -\frac{GM_{\text{enc}}m}{R(t)^2} \Rightarrow \ddot{R} = -\frac{GM_{\text{enc}}}{R^2} \tag{32}$$

Multiply both sides by $\dot{R}$ and integrate $\int \dot{R}\ddot{R}\,dt = \frac{\dot{R}^2}{2} + c$.

$$\frac{1}{2}\dot{R}^2 - \frac{GM_{\text{enc}}}{R} = \text{const} \equiv \frac{-R_0^2 k}{2}. \tag{33}$$

The first term on the left hand side is the kinetic energy per mass, and the second term on the left hand side is the Gravitational Potential Energy per unit mass. The constant we chose is well chosen in advance.

$$M_{\text{enc}} = \frac{4}{3}\pi\rho(t)R(t)^3 \tag{34}$$

and we can let $R(t) = a(t)R_0$ filling in we have:

$$\frac{1}{2}(\dot{a}(t)^2 R_0^2) - \frac{4G\pi\rho(t)a(t)^2 R_0^2}{3} = \text{const} \equiv \frac{-R_0^2 k}{2}. \tag{35}$$

finally we have

$$\boxed{(\frac{\dot{a}}{a})^2 = \frac{8}{3}\pi G\rho - \frac{\kappa}{a^2}} \tag{36}$$

This is the first Friedman equation!

3.1.2 Getting the second Friedman Equation

To get the second Friedman equation we appeal to thermodynamics. The first law of thermodynamics says:

$$dE = dQ - pdV \tag{37}$$

where dE is the change in energy of the system, dQ is heat added to the system, and $-pdV$ is the work done by the system on its surroundings. Due to our assumption of the cosmology principle $dQ = 0$ as if any small parcel of gas was non-zero it would be special.

We can write $dE = pdV$, but we can also write $dE = d(\rho a^3)$ as this is what is changing where ρ is the energy density. This is the co-moving volume so $v = a^3$.

Rewriting we have:

$$a^3 d\rho + 3a^2\rho da = -3a^2 pda \tag{38}$$

rearranging we have:

$$\boxed{\dot{\rho} = -3(\rho + p)(\frac{\dot{a}}{a})} \tag{39}$$

Differentiating the first Frieddman equation we have $2a\dot{a} = \frac{8}{3}\pi G(2a\dot{a}\rho + a^2\dot{\rho})$ we use this and wearrange:

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G(\rho + 3p)} \tag{40}$$

3.2 The Parameters of the Friedman equations

We have derived the equations that govern the dynamics of $a(t)$, and they depend on the stuff in the universe ρ, p etc.

Here are the common parameters.

$$p = w\rho \tag{41}$$

w is known as the equation of state parameter and is dimensionless, this works because ρ and p have the same units, its similar to an ideal gas.

plugging this into the equation for $\dot{\rho}$ we have:

$$\dot{\rho} = -3(1 + w)\rho(\frac{\dot{a}}{a}) \Rightarrow \boxed{\rho \sim a^{-3(1+w)}} \tag{42}$$

In a flat universe $k = 0$, we have

$$\boxed{(\frac{\ddot{a}}{a})^2 \propto a^{-3(1+w)} \Rightarrow a(t) \propto t^{\frac{2}{3(1+w)}}} \tag{43}$$

3.2.1 Normal Matter

For normal matter , and "dust" $w = 0$. If $k_b T << mc^2$ then the rest mass energy dominates, and we can safely neglect pressure.

For $p = 0$, $w = 0$, $\rho_m \sim a^{-3}$.

3.2.2 Vacuum Energy

Vacuum Energy $w = -1$: The energy of a vacuum just comes from the vacumm... the density is a const

$$\rho_\Lambda = \text{const} \Rightarrow w = -1 \tag{44}$$

from $\rho \sim a^{-3(1+w)}$ so $a \sim 1$ when $w = -1$.

in this case $a(t) \propto e^{H_0 t}$. This is an accelerated expansion.

w needs to be $w < -\frac{1}{3}$ for an accelerated expansion.

3.2.3 An Empty Universe

Here $\rho = 0$, and $(\frac{\dot{a}}{a})^2 = -\frac{1}{2}k \Rightarrow \dot{a} = \text{const}$

aka

$$a(t) \propto t \tag{45}$$

3.2.4 Light Domination or Radiation

In this case $w = \frac{1}{3}$ this comes from thermodynamics.

$P = \frac{4\sigma}{3c}T^4$ and $\rho = 4\sigma T^4$. so $w = \frac{1}{3}$ From the Stefan-Boltzman equations. This assumes thermal equilibrium

in this case

$$\boxed{a \propto t^{1/2}} \tag{46}$$

$$\boxed{\rho_r \propto a^4} \tag{47}$$

Another way to think of $\rho_r \propto a^4$ this is you get the the a^{-3} from the normal volume dilution, and an extra a^{-1} from $E = \frac{hc}{\lambda}$ and λ is changing as well due to redshift.

3.2.5 de sitter space

de sitter space: An universe with constant energy density per space

3.3 Subtle Parts to Highlight

3.3.1 CMB in thermal equilibrium

When talking about radiation, we assumed it was in thermal equilibrium so we could invoke the Stefan-Boltzman equation.

The CMB looks like it is in thermal equilibrium but it is not, but it still follows the Black-Body radiation curve so its alright, but with a modified temperature $\boxed{T_f = \frac{a_i}{a_f} T_i}$.

$$T \propto \frac{1}{a} \tag{48}$$

3.3.2 Normal Matter is pressure-less

We saw that Galaxies have an effective pressure — from the peculiar motions of the stars, even if it is not from the thermal motion. Does normal matter not have a pressure?

The interesting thing about peculiar motions is that they decay. As the universe expands their peculiar velocities get damped to 0.

4 Cosmological Parameters & Distances

From Sec. 3 we know how the universe evolves for single-component cases, but in reality everything is mixed!

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{\text{radiation},0}\left(\frac{1}{a^4}\right) + \frac{8\pi G}{3}\rho_{\text{mass},0}\left(\frac{1}{a^3}\right) - \frac{\kappa}{a^2} + \frac{8\pi G}{3}\rho_{\text{vacuum}} \tag{49}$$

Subscript zero means "value today."

In reality it is a bit messy, but we can make it easier by neglecting the terms that do not dominate. Our universe evolves from **radiation**-dominated to **matter**-dominated to **curvature**-dominated (although our universe appears to have $\kappa = 0$), and finally to **dark energy**-dominated.

de Sitter Space: A universe with constant energy density per space; it is a good approximation to the universe of today, but in reality $\rho_m \neq 0$.

4.1 Ratios: Big Ω

We can write the mixed-component universe today as:

$$H_0^2 = \frac{8\pi G}{3}\left(\rho_{r,0} + \rho_{m,0} + \rho_\Lambda\right) - \frac{\kappa}{a_0^2}. \tag{50}$$

Recall that the subscript 0 means "evaluated today." Next we divide by H_0^2:

$$1 = \frac{8\pi G}{3H_0^2}\rho_{r,0} + \frac{8\pi G}{3H_0^2}\rho_{m,0} + \frac{8\pi G}{3H_0^2}\rho_\Lambda - \frac{\kappa}{H_0^2 a_0^2} \tag{51}$$

$$1 = \Omega_{r,0} + \Omega_{m,0} + \Omega_{\Lambda,0} + \Omega_{k,0} \tag{52}$$

The value with physical meaning for mass is $\rho_{m,0} \propto \Omega_{m,0}H_0^2$. This is a sum rule that must add up to one; in other words, each term is expressed as a ratio to the critical density.

Critical Density: The critical density is

$$\rho_{\text{crit}} = \frac{3H_0^2}{8\pi G}, \tag{53}$$

and is approximately one atom per cubic meter. In general, for the ith component of matter or energy, we have:

$$\Omega_{i,0} = \frac{\rho_{i,0}}{\rho_{\text{crit}}}. \tag{54}$$

The critical density is interesting because if we add up the different sources of energy density in the universe and we get ρ_{crit}, then $\Omega_k = 0$, implying a spatially flat universe ($k = 0$). Otherwise, the universe is open or closed depending on the sign of Ω_k.

We can evaluate the Ωs for non-present times by looking at how the energy density dilutes with scale factor a. For example, note how $\rho_{\text{radiation}} = \rho_{\text{radiation},0}/a^4$, so $\Omega_\gamma = \Omega_{\gamma,0}/a^4$. You might see Ω_{tot}, which means everything except the $\Omega_{k,0}$ term.

4.2 Parameters of Our Universe

Here is what we currently believe the values are, mostly from the *Planck* satellite:

Parameter	**Symbol**	**Current Beliefs**
Spatial Curvature	$\Omega_{k,0}$	0.001 ± 0.002
Matter Density	$\Omega_{m,0}\, h^2$	0.14240 ± 0.00087
Baryon Density	$\Omega_{b,0}\, h^2$	0.02242 ± 0.00014
Hubble Constant	H_0	$(67.9 \pm 0.7)\ \mathrm{km\, s^{-1}\, Mpc^{-1}}$
Amplitude of Matter Fluctuations	A_S	$(2.10 \pm 0.03) \times 10^{-9}$
Scalar Spectral Index	n_S	0.966 ± 0.005
CMB Optical Depth	τ	0.0561 ± 0.0071
CMB Temperature	$T_{\mathrm{CMB},0}$	2.7255 ± 0.0006 K
Dark Energy Equation of State	w	-1.04 ± 0.06

Notes from this table:

- Our universe is flat
- The value with physical meaning for mass is $\rho_{m,0} \propto \Omega_{m,0} H_0^2$.
- Ω_m includes all types of matter including cold dark matter
- Ω_b (Baryonic matter) is $\sim \frac{1}{6}$ or $\sim \frac{1}{7}\ \Omega_m$.
- these parameters describe the lumpiness of the matter distribution
- small, but non-zero prob that a CMB photon is scattered on the way to our telescope

Things We Haven't Detected Yet

Parameter	**Symbol**	**Current Beliefs**
Sum of Neutrino Masses	$\sum m_\nu$	< 0.12 eV (95% credibility)
Effective Number of Neutrino Species	N_{eff}	$2.99^{+0.34}_{-0.33}$ (95% credibility)
Tensor-to-Scalar Ratio	r	< 0.106 (95% credibility)

Neutrino masses affect cosmology and can be constrained through observations of the cosmic microwave background and large-scale structure. Gravitational waves can also leave imprints on the CMB polarization.

4.3 Notes from the parameters

- Our universe is flat
- The value with physical meaning for mass is $\rho_{m,0} \propto \Omega_{m,0} H_0^2$.
- Ω_m includes all types of matter including cold dark matter
- Ω_b (Baryonic matter) is $\sim \frac{1}{6}$ or $\sim \frac{1}{7}\ \Omega_m$.
- these parameters describe the lumpiness of the matter distribution
- small, but non-zero prob that a CMB photon is scattered on the way to our telescope

4.3.1 A digression on ΛCDM

Sometimes people look at matter density, baryon density, the hubble constant, A_S, n_S, τ and say "Our universe is well-fit by a 6-parameter ΛCDM model". Λ is the vacuum energy (what we have been investigating with $w = -1$.)

We have freedom to choose the 6 parameters. For example, where is Λ? Where are those 6 parameters? Well, we can choose them.

$1 = \Omega_{m,0} + \Omega_\Lambda + \Omega_k$

$\Omega_\Lambda = 1 - \Omega_{m,0}$ $(Omega_k = 0)$

$\Omega_\Lambda = 1 - \frac{\Omega_{m,0} h_0^2}{h_0^2}$

Additionally, why 6 parameters? It's only 6 parameters because we chose them. For example, we assumed $\Omega_k = 0$, or might assume a specific value of T_{CMB} because we have measured it well. You may go beyond vanilla ΛCDM to add add more parameters. For example, you may assume a nonzero neutrino mass now that we know it from experiments.

For example, changing neutrino mass might affect how much they get caught by black holes, and therefore change black hole growth predictions slightly.

4.3.2 Parameters we can derive

CMB Photon Density:
$\rho_\gamma = 4\sigma T^4_{CMB,0}$
$\Omega_{\gamma,0} h^2 = 2.47 \cdot 10^{-5}$

Neutrino Density: $\Omega_{\nu,0} h^2 = 1.68 \cdot 10^{-5}$

Radiation Density: CMB photon density + neutrino density
$\Omega_r = \Omega_\gamma + \Omega_\nu$
$\Omega_{r,0} = 8.5 * 10^{-5}$

This is small today, but when a is small in the early universe, $1/a^4$ wins over $1/a^3$.

So at one point early on, the radiation density was higher than matter density. We can work out the redshift when those densities were equal, z_{eq} recall that $1 + z = 1/a$)

4.3.3 The age of our Universe

We have to take into account the evolution of each component.

$$H^2(z) = \frac{8\pi G}{3}(\rho_m (1+z)^3 + \rho_r (1+z)^4 + \rho_\Lambda - \kappa(1+z)^2) \tag{55}$$

$$H^2(z) = \frac{8\pi G}{3}(H_0^2 \Omega_m (1+z)^3 + H_0^2 \Omega_r (1+z)^4 + H_0^2 \Omega_\Lambda - \kappa(1+z)^2) \tag{56}$$

Do to the evolution of the universe the approx $t \sim 1/H_0$ works better then we might naively expect.

4.4 Distances

4.4.1 Comoving Distance

We see a thing at redshift z. How far is it now?

Use the FLRW metric

$$ds^2 = -dt^2 + a^2 dr^2/(1 - \kappa r^2 \tag{57}$$

From relativity photons travel on paths $ds^2 = 0$

$$a^2 dr^2/(1 - \kappa r^2) = dt^2 \tag{58}$$

Assume $\kappa = 0$

$$dr = dt/a(t) \tag{59}$$

and integrate

$$r = \int \frac{dt}{a(t)} = \int \frac{da}{a^2(t)H(z)} = -\int_0^z \frac{dz'}{H(z')} \tag{60}$$

this is the same as

Note that times in cosmology are proportional to c/H_0 and times seem to be proportional to cH_0

4.4.2 Luminosity Distance

One way we might define a distance is the luminosity Distance:

luminosity Distance: We know the flux follows an inverse square law: $f = \frac{L}{4\pi d_L^2}$ If we know the observed flux F and the intrinsic luminosity L, the luminosity distance d_L is the distance that makes this true.

What is the value of dL as a function of z?

This is tricky for 3 reasons:

1. As photons spread out from a source, they spread out over an area $A = 4\pi S_\kappa(\chi)^2$, where that S_κ is from the FLRW metric's second form (if flat universe, $= R_0\chi$) 2. Photons redshift. they loose energy by a factor of $(1 + z)$ when received. 3. Distances stretch by a factor of (1+z) photons take longer to arrive.

(skipped)

5 Principles of Early Universe Thermodynamics

To an excellent approximation, we have so far only only dealt with with **gravity** — we have asked how different forms of energy density **gravitate** and effect our universe's expansion.

But particles have interesting interactions of their own — especially in the hot cauldron of the early universe!

In the early universe, particles and antiparticles were in thermal equilibrium, constantly being created and annihilated through various fundamental interactions. Some representative processes include:

$$e^+ + e^- \rightleftharpoons 2\gamma \qquad \text{(Annihilation / Pair Production)}$$

$$n \rightarrow p + e^- + \bar{\nu}_e \qquad \text{(Beta Decay)}$$

$$e^+ + e^- \rightleftharpoons \nu_e + \bar{\nu}_e \qquad \text{(Weak Nuclear Interaction)}$$

$$\mathrm{DM} + \mathrm{DM} \rightleftharpoons X + X \qquad \text{(Dark Matter Annihilation)}$$

Here, X represents standard model particles (excluding photons), since dark matter does not couple directly to light.

The early universe is **radiation dominate**d so recall

$$\rho_r \propto T^4 \tag{61}$$

$$H = \left(\frac{8\pi\rho_r}{3m_{pl}^2}\right)^{1/2} \propto T^2 \tag{62}$$

$$a(t) \propto t^{1/2} \tag{63}$$

$$t \propto T^{-2} \tag{64}$$

Recall that in Natural units, $G = \frac{1}{m_{pl}^2}$. Temperature T is another way to keep time. This is handy because particles have reaction rates that depend on temperature. The relationship for Temp, and a is consistent with a Black-body radiation curve which we defined earlier.

5.1 Basic Principles

1. Phase Space Density Is No Longer Collisionless

We are no longer dealing with a "gas" of stars, but a real gas of particles in a high-energy environment. We will be using momentum instead of velocity for the phase space — which is more natural for relativistic particles.

From quantum statistical mechanics we have that each state / particle takes up a volume of h^3. In natural units $\hbar = 1$ so $h = 2\pi$.

Putting this together we have:

$$\frac{d^6N}{d^3p d^3x} = \frac{g}{(2\pi)^3} f(\vec{p}, t) \tag{65}$$

g here is called the **degeneracy factor** to account for the internal degrees of freedom of a particles, like spin which allow for multiple particles to occupy the same position-momentum state and still be in a different quantum state.

Recall the phase-space current equation 3 defined as

$$\frac{1}{dt} = \frac{\partial}{\partial t} + \vec{v}\frac{d}{d\vec{x}} + \frac{d\vec{p}}{dt}\frac{d}{d\vec{p}} \tag{66}$$

and we said that $\dot{f} = 0$ — that says that as we follow some particles along a phase-space trajectory that the density f is preserved. This means we had a **collisionless** system with only smooth evolution. **No particles were created or destroyed**.

Recall also the 0th moment – the continuity equation.

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\langle\vec{v}\rangle) = 0 \Rightarrow \frac{\partial n}{\partial t} = 0 \tag{67}$$

In our case, $\langle\vec{v}\rangle$ is 0 due to the cosmological principle.

Problem This gives $n =$ constant — appropriate for a static universe, but irrational for an expanding universe.

The issue is our time derivative equation! There is a correction for GR which is hard... the easier solution is to say

$$na^3 = \text{constant} \tag{68}$$

If no particles are created or destroyed then the particle density is proportional to scale volume. This implies:

$$\frac{d}{dt}(na^3) = 0 = a^3\frac{dn}{dt} + n3a^2\frac{da}{dt} \Rightarrow \dot{n} + 3Hn = 0 \tag{69}$$

Compared to a static universe / before — there is an extra Hubble dilution term.

2. There is another issue — Particles are created, and destroyed.

$$\dot{n} + 3Hn = \text{Reaction Terms} \tag{70}$$

We apply quantum statistical mechanics, if in thermal equilibrium.

$$f(p) = \frac{1}{\exp[\frac{E-\mu}{T}] \pm 1} \tag{71}$$

Positive for the Fermi-Dirac distribution used for Fermions, Negative for Bose-Einstein distribution for Bosons. The Classical Boltzmann distribution is when the 1 goes to 0. It gives the occupation number the average number of particles per quantum state of momentum in thermal equilibrium for systems of indistinguishable particles obeying quantum statistics.

E is the total relativistic energy $E = \sqrt{p^2 + m^2}$, and μ is the chemical potential.

Recall that:

$$du = Tds + \mu dN - PdV \tag{72}$$

Holding s, and v constant, $\mu = (\frac{du}{dN})_{s,v}$ — the chemical potential is the change in energy of a system to add a particle.

Photons automatically have $\mu_\gamma = 0$ because photons can be created for free without changing the energy of the system.

$$e^- + p^+ \rightleftharpoons e^- + p + +\gamma \tag{73}$$

.

A lot of the time we can set $\mu \approx 0$ at early times when $\mu << T$.
Once we have f the prescription is the same as before.
Number density is given by:

$$n = \frac{g}{(2\pi)^3} \int f(\vec{p}) d^3p \tag{74}$$

and the energy density is given by

$$\rho = \frac{g}{(2\pi)^3} \int f(\vec{p}) E d^3p \tag{75}$$

Recall that $E = \sqrt{p^2 + m^2}$.

5.1.1 Relativistic and non-Relativistic particles

Particles can be relativistic sometimes and non-relativistic at other times.

If a particle is in thermal equilibrium in the hot particle soup of the early universe then at early times when $T >> M$ it will be relativistic. At later times when the universe has cooled more $T << M$, and it is now non-relativistic. Remember these equations have a hidden factor of k_b and mc^2.

There is another important effect that happens as T drops consider something like:

$$e^+ e^- \rightleftharpoons 2\gamma \tag{76}$$

When $T >> M$ it is easy to do the backward reaction (pair production), and when $T << M$ the forward reaction (annihilation) dominates and we “lose” the species.

Particle species not only become less relativistic as time goes on, but we lose species.

In the relativistic case we have:
Number density is given by:

$$n = \frac{g}{(2\pi)^3} \int_0^\infty f(\vec{p}) d^3p = \frac{gT^3}{(2\pi)^3} \int_0^\infty \frac{y^2}{e^y \pm 1} dy \tag{77}$$

and the energy density is given by

$$\rho = \frac{g}{(2\pi)^3} \int_0^\infty f(\vec{p}) E d^3p = \frac{gT^4}{(2\pi)^3} \int_0^\infty \frac{y^3}{e^y \pm 1} dy \tag{78}$$

with $T >> m$, and $y = p/T$. These integrals can be solved analytically.
The generalized Stefan-Boltzman law becomes:

$$\rho = \frac{\pi^2 g T^4}{30} \begin{cases} 1, & \text{for bosons} \\ \frac{7}{8}, & \text{for fermions} \end{cases} \tag{79}$$

This is for one particle species, for multiple types of particles we have:

$$\rho_r = \sum_i \rho_i = \frac{\pi^2 g_* T^4}{30} \tag{80}$$

where g_* is given by:

$$= \sum_{\text{Bosons}} g_i (\frac{T_i}{T})^4 + \sum_{\text{Fermions}} (\frac{7}{8}) g_i (\frac{T_i}{T})^4 \tag{81}$$

g_* lets us account for all relativistic species without doing each one by hand — its an effective number. We average Fermions, and Bosons and each species can be at a different temperature. At some point some of them may fall out of equilibrium.

We can similarly do the integral for n:

$$n = \frac{\zeta(3)\, g T^3}{\pi^2} \begin{cases} 1, & \text{for bosons} \\ \frac{3}{4}, & \text{for fermions} \end{cases} \tag{82}$$

Other useful quantities are the entropy S, and the entropy density $s = \frac{S}{V}$.

From thermodynamics:

$$s = \frac{S}{V} = \frac{\rho + P}{T} \tag{83}$$

This P is pressure! Not momentum. For relativistic particles $P = \frac{1}{3}\rho$ recall that $w_r = \frac{1}{3}$.

thus:

$$s = \sum_i \frac{\rho_i + P_i}{T_i} = \frac{4}{3} \sum_i \frac{\rho_i}{T_i} = \frac{2\pi^2}{45} g_{*s} T^3 \tag{84}$$

where g_{*s} is:

$$g_{*s} = \sum_{\text{Bosons}} g_i (\frac{T_i}{T})^3 + \sum_{\text{Fermions}} (\frac{7}{8}) g_i (\frac{T_i}{T})^3 \tag{85}$$

Because $S = sV$ is conserved, $g_{*s}(T)T^3 a^3 = \text{constant}$. the $a^3 \propto V$, and $g_{*s}(T)T^3 \propto S$. We will use this to figure out how the temperature changes as species fall out of equilibrium.

Another useful quantity is the number density to entropy density ratio for a species.

$$Y_i = \frac{n_i}{s_i} \tag{86}$$

Since $s \propto a^{-3}$, $Y_i \propto n_i a^3$ — it is the measure of the number of particles of a species i in a **comoving volume**. If no particles for this species are created or destroyed Y_i is a constant.

6 Relics

We are still in the early universe, where radiation dominates.

Particles can have interactions that are "on" or "off" at different times.

Interactions are how species maintain thermal-equilibrium, and are important for equations like $\dot{n} + 3Hn =$ (interactions).

Suppose we have a sea of particles and we ask how likely it is to interact with a particle — this leads into the general arguments for cross sectional area, and mean free path. Think of a column of particles of cross sectional area A, and density n. The target particle has a cross-sectional area ρ. To get a flux of particles through the cylinder we can multiply nv where v is the velocity of the particles. nv has units of $\frac{\#}{\text{area}\times\text{time}}$. To convert this flux into a rate of hits we multiply by the cross sectional area ρ of the target.

$$\Gamma = n\langle\sigma v\rangle \tag{87}$$

The angle brackets represent a thermal average.

If a reaction rate is high then it is important! We can compare it to H the cosmological parameter that has units of inverse time. You can also think of H as the time it takes for particles to escape each other due to the expansion of the universe. If we take the inverse it is a comparison of the age of the universe at that time, vs the time per interaction.

Keep in mind that H is itself time-dependent, so the reaction rates become important or less important with time as well.

Relic Abundance: Relic abundance depends on when interactions stop, and weather a particle was relativistic or non-relativistic at the time. By relic abundance, we mean how many particles are left over when annihilation stops.

When this happens is, and if a particle was relativistic or not is important. This is also known as **freeze-out.**.

6.1 Hot Relics

Hot Relics: Particles that froze-out while relativistic — meaning that Γ dropped below H, while $T >> m$.

Relativistic particles tend to be quite abundant just before freeze-out, the interactions are still "on", so the particles are in thermal-equilibrium with the rest of the universe. This means that we can use the thermal equilibrium equation from the last section.

$$n \propto gT^3 \tag{88}$$

Annihilation is happening, but there are enough reverse process reactions (pair production) that the particles are replenished.

At some point Γ drops below H and the interactions stop — so the number of particles na^3 becomes a constant.

Explicitly:

$$n_{\text{freezeout}} = \frac{\zeta(3)\, g\, T^3_{\text{freezeout}}}{\pi^2} \times \begin{cases} 1, & \text{for Bosons} \\ \frac{3}{4}, & \text{for Fermions} \end{cases} \tag{89}$$

Then n is:

$$n = n_{\text{freezeout}}(\frac{a_{\text{freezeout}}}{a})^3 \tag{90}$$

Recalling that $T \sim \frac{1}{a}$

$$n = n_{\text{freezeout}}(\frac{T}{T_{\text{freezeout}}})^3 \tag{91}$$

After freeze-out, the particle species no longer in thermal-equilibrium with the rest of universe, so the T is not the same. This is known as being **decoupled** .

6.2 Cold Relics

Cold Relics: A Cold Relics is when freeze-out - $\Gamma < H$ happens when the species is **non-relativistic** — $T_{\text{freezeout}} << m$. For cold relics plots of $Y = \frac{n}{s}$ — the number density to entropy density vs $\frac{m}{T}$.

If a species of particle is still in thermal-equilibrium when it becomes non-relativistic we can use the non-relativistic limit of the quantum statistical mechanics equation. We still generally assume $\mu = 0$.

$$n = g(\frac{mT}{2\pi})^{3/2}\exp[-\frac{m}{T}] \tag{92}$$

and

$$\rho \approx mn \tag{93}$$

The energy density is dominated by the rest mass. In this cold-regime annihilation dominates, and the **abundance is exponentially suppressed**. At some point $\Gamma << H$ and the interactions cease so the particles do not get destroyed anymore. At what Y that interactions stop depends on $\langle\sigma v\rangle$.

The rate Γ is for n particles hitting a single target, but there are really n possible targets — the collision rate is $\propto n^2$.

Writing out the equation:

$$\dot{n} + 3Hn = -n^2\langle\sigma v\rangle + q\langle\sigma v\rangle \tag{94}$$

The first bracket term on the RHS is the depletion of species x, and the second is the replenishing of the species with an unknown factor of q. The second term does not depend on n.

We do not know what q is, but we know in thermal-equilibrium that the LHS = 0, then we can say:

$$0 - n_{eq}^2\langle\sigma v\rangle + q\langle\sigma v\rangle \Rightarrow q = n_{eq}^2 \tag{95}$$

and therefore:

$$\dot{n} + 3Hn = (n_{eq}^2 - n^2)\langle\sigma v\rangle \tag{96}$$

This is known as the Ricatti differential equation. Which is solved by doing the following substitions.

$$Y = \frac{n}{s}, \tag{97}$$
$$x = \frac{m}{T}, \tag{98}$$
$$\lambda = \frac{2\pi^2}{45}\, g_{*s}\, \frac{m^3\langle\sigma v\rangle}{H(T=m)}. \tag{99}$$

Where the Hubble parameter is evaluated at the transition to non-relativistic.

$$\frac{dY}{dx} = -\frac{\lambda}{x}(Y^2 - Y_{eq}^2) \tag{100}$$

This cannot be solved analytically. But after freezeout we know that $Y >> Y_{eq}$ — as there is an exponetial drop off.

This gives us:

$$Y_\infty \approx \frac{x_{\text{freezeout}}}{\lambda} \tag{101}$$

Recall that $\lambda \propto \langle\sigma v\rangle$, we see that the higher the cross-section area for annihilation, the lower the relic abundance.

6.3 WIMP Dark Matter (Cold Relic Application)

Weakly Interacting Massive Particles (WIMP) is a theory of dark matter. Suppose it follows a relation like:

$$X + X \rightleftharpoons SM + SM \tag{102}$$

Where SM is a standard model particle. We know that the relic abundance is set by $\langle\sigma v\rangle$. We might ask the following if we set the relic abundance to be the observationally known $\Omega_{DM} \approx 5\Omega_b$ do we get something reasonable for $\langle\sigma v\rangle$, or something ridiculous? This is an Order-of-magnitude problem.

At freeze out $\Gamma = \langle\sigma v\rangle \sim H$ so we can say:

$$n_f\langle\sigma v\rangle \sim \frac{T_f^2}{m_{pl}} \tag{103}$$

The m_{pl} comes from the expression for H during radiation domination. The subscript f stands for at freeze-out.

We know that at matter-radiation equality (MRE) the energy density of matter and photons were equal.

$$\rho_m = \rho_r \Rightarrow m_{DM}n_{MRE} \approx \frac{\pi^2 g_*}{30}T^4 \tag{104}$$

The left hand side after the arrow is the DM number density at MRE, and the pre-factor on the RHS is ≈ 1, $g_* \approx 3$.

$$n_{MRE} \approx \frac{T^4}{m_{DM}} \tag{105}$$

Since freezout, n dilutes as a^{-3}.

(skipped)

7 Neutrinos and Dark Matter

Last time there was an example for Cold Relics — the hypothetical WIMP dark matter, this time lets start with a Hot relic example

7.1 Neutrinos (A Hot Relic Application)

At $t \lesssim 1\,\mathrm{s}$, $T \gtrsim 1\,\mathrm{MeV}$ $(10^{10}k)$ neutrino interactions are active, here we use neutrino to refer to neutrinos and anti-neutrinos.

$$e^- + \nu_e \rightleftharpoons e^- + \nu_e \tag{106}$$
$$e^+ + e^- \rightleftharpoons \nu_e + \bar{\nu}_e \tag{107}$$
$$...\text{etc} \tag{108}$$

The bar represents a photon from an annihilation process. These types of processes are also felt by muons $\mu^{\pm}$ talking to muon neutrinos ν_μ, and $\bar{\nu}_\mu$. But $m_\mu \sim 105\,\mathrm{MeV}$, these reactions long since froze out / annilhated.

To figure out when the neutrino interactions turn off — and decouple from being in thermal equilibrium from everything else, we compare Γ and H. This means we need to know σ for these interactions.

We can use:

$$n = n_{\text{freeze}}(\frac{a_{\text{freezeout}}}{a})^3 \tag{109}$$

and recall that $a \sim \frac{1}{T}$.

$$\Gamma = \ n\langle\sigma v\rangle \approx T^3 G_f^2 T^2 = G_f^2 T^5 \tag{110}$$

$$\boxed{\Gamma \sim G^2 T^N} \tag{111}$$

This is because $v \approx c = 1$, and $n_\nu \propto T^3$ because it is relativistic. I do not know what G_f means but it is a G-fermi constant that comes from particle physics, and dimensional analysis gives us the power.

For a radiation dominated universe like we are in, $H \sim \frac{T^2}{\text{Mpc}}$. Then we can write:

$$\frac{\Gamma}{H} = \frac{G_f^2 T^5}{\frac{T^2}{\text{Mpc}}} \approx (\frac{T}{\text{MeV}})^3 \tag{112}$$

This means that neutrinos decouple around $T = 1\,\mathrm{MeV}$. Once they decouple their number density goes down as the $n \sim a^{-3}$. Their distribution function is a snapshot of what it looked like at decoupling, but the T also goes down as $T \sim 1/a$. So

$$T_\nu(a) = T_{\nu,\text{decouple}}(\frac{a_{\text{decouple}}}{a}) \tag{113}$$

However if we evaluate $T\nu$ at $a = 1$ (Today) we do not get the CMB temp, because soon after neutrinos decuople, we have...

Electron-Positron Annihilation

Even though by $T \sim 1\,\mathrm{MeV}$ there are no more electron - neutrino interactions, there are still electron interactions because electrons are relativistic, and feel electro-magnetism (They react to make photons).

$m_e \approx 0.5\,\mathrm{MeV}$. so no long after neutrino decoupling they become non-relativistic and are exponentially suppressed as annihilation dominates. This causes their entropy to dumped into the photons — heating them up. Let's work out the new photon temperature.

recall the entropy density s

$$s = \frac{2\pi^2}{45} g_{*,s} T^3 \tag{114}$$

and that $sa^3 \propto g_{*,s} a^3 T^3$ is a conserved quantity — because total entropy is conserved. If $g_{*,s}$ chances due to positron - electron annihilation, then T must change.

Right before they annihilate we can write $g_{*,s}$, and compare it to after, they are at the same temperature for this. Neutrinos already decoupled so its only the electrons, positrons, and photons that we count.

Before $g_{*,s} = \frac{11}{2}$, and after we have $g_{*,s} = 2$

Therefore we can write:

$$T_{\gamma,\mathrm{after}} = (\frac{11}{4})^{1/3} T_{\gamma,\mathrm{before}} \tag{115}$$

Then we can relate this back to the neutrino temperature, The neutrino T_ν was scaling at the same rate as T_γ before the annihilation of positrons, and electrons, and $T_\nu = T_{\gamma,\mathrm{before}}$. And $T_\nu \sim 1.96k$, as $T_{\gamma,\mathrm{after}} = 2.75k$

Entropy conservation is actually:

$$T_\gamma \propto \frac{1}{g_{*,s}^{1/3} a} \tag{116}$$

This is important when $g_{*,s}$ changes! which happens in the early universe.

For neutrinos — we have a hot relic that still exists today, and have figured out its background temperature today. We know that there are three types of neutrino, and their anti-particle represented by a bar.

(skipped) After electron - positron annilhation the effective number of degrees of freedom aka $g_* \approx 3$

We typically fit for the number of degrees of freedom in cosmological models to determine if there are unknown particles.

We also know that neutrinos have mass, but not exactly what it is, we have an upper limit.

(skipped)

Roughly speaking, Neutrinos are light and travel close to the speed of light — and do not interact much. They stream out of over-densities in the universe inhibiting large structure formations.

7.2 Dark Matter

So far we have discussed two examples of relics, neutrinos (hot relics), and WIMPs (cold relics) — could either be Dark Matter?

neutrino relics cannot be dark matter, they would wash out too much dark matter and have other issues (see hw 7)

We know this about Dark Matter

- Ω_m, Ω_b imply a limit on Ω_{DM}
- Locally $\rho_{DM} \sim 0.4 \pm 0.1\,\mathrm{GeV/cm^3}$
- Should be stable $\Gamma >> H_0^{-1}$

- upper bounds on interaction rates
- upper bound on DM velocity streaming (Like neutrinos)

These combine to give us some interesting constraints on m_{DM} the mass of a dark matter particle, and upper and lower bound on mass. This gives us about 90 orders of magnitude on the dark matter particle mass.

8 Big Bang Nuclear Synthesis (BBN)

So far we have talked about some of the key events that have happened in our universe at $t \sim 1\,\text{s}$, $T \sim$ 1MeV, and $T \sim 10^{10}k$

We now press on with the story and talk about **Big Bang Nuclear Synthesis (BBN)**. This is where the hot cauldron of the universe makes the first (stable) atoms, and produce a lot of ^{4}He, and small amounts of elements up to ^{7}Li.

Context. BBN begins around $T \sim 0.1\,\text{MeV}$, so not long after e^+/e^- annihilation. At this point $g_* \approx 3.38$ (photons and neutrinos), which means that $t \approx 132\text{s}(\frac{0.1\,\text{MeV}}{T})^2$. This means that BBN starts a few seconds to a few minutes after the big bang.

8.1 Neutrinos

Neutrons are the main character. The goal is to work out the elemental abundances, conceptually the most important thing is to keep in mind in BBN is to "follow the neutrinos".

Neutrons are slightly more massive than protons $Q = m_n - m_p = 1.29\,\text{MeV}$ this disfavors them. Neutrons and Protons can convert into each other via weak interactions

$$n + \nu_e \rightleftharpoons p + e^- \tag{117}$$
$$n + e^+ \rightleftharpoons p + \bar{\nu}_e \tag{118}$$

To make heavy nuclei, we need both neutrons and protons. A nucleus made of just protons or neutrons is unstable and decays. Here we consider an nucleus stable if it is long-lived compared to the age of the universe at this time — a few seconds to a few minutes.

- **Hydrogen Isotopes:**
  - Proton(s) $^1_1$H: stable
  - Deuterium $^2_1$H: stable
  - Tritium $^3_1$H: $t_{1/2} = 12\,\text{yr}$
- **Helium Isotopes:**
  - Helium-3 $^3_2$He: stable
  - Helium-4 $^4_2$He: stable
- **Lithium Isotopes:**
  - Lithium-7 $^7_3$Li: stable
- **Beryllium Isotopes:**
  - Beryllium-7 $^7_4$Be: $t_{1/2} = 52\,\text{days}$

Starting prior to BBN — say $T \gtrsim 10\,\text{MeV}$, the neutrons and protons are in equilibrium via their interactions with the weak force. Recall that $m_p, m_n \sim 1\,\text{GeV}$, and **they are non-relativistic** at this point. We use the non-relativistic limit of the equilibrium equations.

$$n_p = g_p \left(\frac{m_p T}{2\pi} \right)^{3/2} \exp \left[-\frac{m_p - \mu_p}{T} \right], \tag{119}$$

$$n_n = g_n \left(\frac{m_n T}{2\pi} \right)^{3/2} \exp \left[-\frac{m_n - \mu_n}{T} \right]. \tag{120}$$

We can then write the ratio of $\frac{n_n}{n_p}$

$$\frac{n_n}{n_p} = \frac{g_n \left(\frac{m_n T}{2\pi} \right)^{3/2} \exp \left[-\frac{m_n - \mu_n}{T} \right]}{g_p \left(\frac{m_p T}{2\pi} \right)^{3/2} \exp \left[-\frac{m_p - \mu_p}{T} \right]} = (\frac{m_n}{m_p})^{3/2} \exp -\frac{Q}{T} \tag{121}$$

Where $Q = m_n - m_p$. Informally the neutron and proton are like two states of a nucleon, with a Boltzmann factor governing the relative number between the two, IE this changes as a function of T.

We see that as the universe cools to $T \sim 1\,\text{MeV}$, and then $T \sim 0.1\,\text{MeV}$ that there are more protons than neutrons. Like all the interactions we have studied so far, this competes with H.

Neutron freeze-out happens at around $T \sim 0.8\,\text{MeV}$. and results in $\frac{n_n}{n_p} = 0.2$ at freeze out. So when the interactions stop, only $\sim 1/6$ nucleons are neutrinos, but worse free neutrons are not stable and they decay via **Beta Decay**

$$n \to p + e^- + \bar{\nu}_e. \tag{122}$$

This has a decay lifetime of $\tau_n \sim 15\,\text{min}$, and **is** relevant to BBN. This number is not easy to study because neutrons are neutral, and is a key uncertainty in BBN codes.

Neutrons then determine the outcome of BBN, to a good approximation all neutrons end up in ^{4}He, and this is how we predict He abundances.

BBN is a race against time, all the nuclear reactions need to happen before the neutrons decay. Now we need to look at the nuclear reactions.

8.2 Steps in BNN

There are a handful of important reactions in BBN. Deuterium production is what kicks off BBN,

$$p + n \rightleftharpoons D + \gamma. \tag{123}$$

This gives a lightly bound nucleus with binding energy of $B_D = (m_n + m_p - m_p) = 2.22\,\text{MeV}$

Using similar non-relativistic equilibrium distribution math we have:

$$\frac{n_D}{n_p n_n} = \frac{g_D}{g_p g_n} (\frac{m_D}{m_p m_n})^{3/2} (\frac{T}{2\pi})^{-3/2} \exp(\frac{B_D}{T}) \tag{124}$$

We can get rid of the gs because $g_D = 3$ it is a spin 1 particle, and $g_p = g_n = 2$, and $m_n \approx m_p \approx \frac{m_d}{2}$.

This means we can write it as:

$$\frac{n_D}{n_p n_n} \approx 6(\frac{m_n T}{\pi})^{-3/2} \exp(\frac{B_D}{T}) \tag{125}$$

It is custom to write express this in terms of the **Baryon-to-photon ratio** η

$$\eta = \frac{n_B}{n_\gamma} = 6.12 \times 10^{-10} \tag{126}$$

It is tiny because matter, antimatter asymmetry is tiny.

After neutron freeze out, 5/6 of Baryons are protons therefore

$$n_p \approx 0.83 n_B \approx 0.83 \eta n_\gamma \approx 0.2 n_\gamma \eta T^3 \tag{127}$$

And we can write the $\frac{n_D}{n_n}$ as :

$$\frac{n_D}{n_n} \approx 6.7\eta(\frac{T}{m_N})^{3/2} \exp(\frac{B_D}{T}) \tag{128}$$

This ratio starts out << 1 at the beginning of BBN where there is barely any D, it becomes >> 1 as BBN proceeds and the number of free neutrons n_n plummets as neutrons get incorporated into nuclei.

When does a significant amount of D get produced? This is basically the same as asking when ^{4}He is produced because the later reactions are very quick.

A reasonable metric is when $\frac{n_D}{n_n} \sim 1$, with our expressions this happens at $T_{\text{BBN}} \sim 0.07\,\text{MeV}$ and $t_{\text{BBN}} \sim 300\,\text{s}$ This gives us when T_{BBN} that we stated at the opening of this section.

This happens at later times / cooler temp then one would naively expect from looking at nuclear reactions with $B_D \sim 2\,\text{MeV}$. The reason is the **very small** η. There are so many photons around that the reverse reaction — breaking up of D is favoured.

A more detailed calculation gives $t_{\text{BBN}} \sim 200\,\text{s}$. With the time of BBN, we can figure out how many neutrons are around (after decays) to be bound up in nuclei. The Neutron fraction is given by:

$$X_n(t) = \frac{n_n}{n_p + n_n} = X_n(t_{\text{nfreeze}}) \exp(-\frac{t_{\text{BNN}}}{\tau_N}) \tag{129}$$

$$X_n(t) = (\frac{1}{6}) \exp(\frac{-200}{890}) \approx 0.13 \tag{130}$$

Again, to an excellent approximation all of this ends up in ^{4}He, and it is conventional to express this amount as a mass fraction.

The primordial fraction Y_P:

$$Y_p = \frac{\rho_{\text{He}}}{\rho_B} = \frac{m_{\text{He}} n_{\text{He}}}{m_H n_H + m_{\text{He}} n_{\text{He}}} \approx \frac{4 n_{\text{He}}}{n_H + 4 n_{\text{He}}} = \frac{4 n_{\text{He}}}{n_p + n_n} \tag{131}$$

We used $m_{\text{He}} \approx 4 m_{\text{H}}$. n_H = protons that remain free, and n_0 = total number of protons.
$n_{He} = \frac{n_n}{2}$ if $\approx$ all neutrons end up in ^{4}He.

$$n_{\text{He}} = \frac{2 n_n}{n_p + n_n} = 2X_n \approx 0.26 \tag{132}$$

The heavier elements exist only in trace amounts, our universe is 75% Hydrogen, and 25% Helium. Hard to make heavier elements because it is not as dense as the core of stars.

8.3 Testing BBN

The only free parameter is η, a good test is to see if η is consistent based on the different elements abundances. This works extremely well.

You can also then use BBN as a probe of standard cosmology turning the problem around.