PHY644 Problem set 2

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Problem 1: Free-fall Time

We are asked to derive the true free fall time t_{ff} of presser-less dust ball of uniform density ρ collapsing.

The total mass of the sphere is :

$$M = \frac{4}{3}\pi\rho r_0^3 \tag{1}$$

where M is the total mass, and r_0 is the initial radius (max radius). The t_{ff} is the time it takes for a test mass on the surface to fall to the centre.

I am assuming that energy conservation holds, for a test mass at the edge of the surface

$$E = \frac{1}{2}v_0^2 - \frac{GM}{r_0} = \frac{1}{2}v(r)^2 - \frac{GM}{r} \tag{2}$$

where E is a constant, and this is the per unit mass energy. We take v_0 to be 0.

We can rearrange for $v(r)$:

$$v(r)^2 = 2GM(\frac{1}{r} - \frac{1}{r_0}) \tag{3}$$

now we have a first order differential equation:

$$\frac{dr}{dt} = -[2GM(\frac{1}{r} - \frac{1}{r_0})]^{0.5} \tag{4}$$

with the same initial conditions, the $-$ comes from falling inwards.

$$-[2GM(\frac{1}{r} - \frac{1}{r_0})]^{-0.5}dr = dt \tag{5}$$

The integral bounds are from r_0 to 0 on the left hand side and from 0 to t_{ff} on the right hand side

$$\int_{r_0}^{0} -\left[2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)\right]^{-1/2} dr = \int_0^{t_{ff}} dt \tag{6}$$

$$\int_0^{r_0} \left[2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)\right]^{-1/2} dr = t_{ff} \tag{7}$$

~~Now we use our integral table aka wolfram alpha (it looks like a u and then trig sub).~~

Before we can use an integral table, we need to simplify more, let $u = \frac{r}{r_0}$, $du = \frac{1}{r_0}dr$.

$$\int_0^1 [2GM(\frac{1-u}{ur_0})]^{-1/2} dr = t_{ff} \tag{8}$$