PHYS644 Problem Set 9

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Problem 1: A more practical expression for the power spectrum

Problem 1A:

Let's start by recalling the definition of the forward Fourier transform in this notation.

$$\tilde{\delta}_{obs}(k) = \int d^3\mathbf{r}\delta_{obs}(\mathbf{r})\exp(-i\mathbf{k}\cdot\mathbf{r}) \tag{1}$$

The inverse transform is:

$$\delta(\mathbf{r}) = \frac{1}{8\pi^3}\int d^3\tilde{k}\tilde{\delta}(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r}) \tag{2}$$

With our value of $\delta_{obs}(\mathbf{r}) = \gamma(\mathbf{r})\delta(\mathbf{r})$. We can write the inverse Fourier[1] transform of $\delta(\mathbf{r})$, and then plug into the forward one.

Step 1:

$$\delta_{obs}(\mathbf{r}) = \frac{\gamma(\mathbf{r})}{8\pi^3}\int d^3\tilde{k}'\tilde{\delta}(\mathbf{k}')\exp(i\mathbf{k}'\cdot\mathbf{r}) \tag{3}$$

Now we put this entire expression into the forward transform for $\delta(\mathbf{r})$.

$$\tilde{\delta}_{obs}(k) = \int d^3\mathbf{r}\frac{\gamma(\mathbf{r})}{8\pi^3}\int d^3\tilde{k}'\tilde{\delta}(\mathbf{k}')\exp(i\mathbf{k}'\cdot\mathbf{r})\exp(-i\mathbf{k}\cdot\mathbf{r}) \tag{4}$$

Now we want to combine the exponential s terms, and recognize the Fourier transform of $\gamma(\mathbf{r})$. The Forward transform is $\tilde{\gamma}(\aleph) = \int d^3\mathbf{r}\gamma(\mathbf{r})\exp(-i\aleph\cdot r)$.

$$\tilde{\delta}_{obs}(k) = \int d^3\mathbf{r}\frac{\gamma(\mathbf{r})}{8\pi^3}\int d^3\tilde{k}'\tilde{\delta}(\mathbf{k}')\exp(-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}) \tag{5}$$

In our case $\aleph = k - k'$.

$$\boxed{\tilde{\delta}_{obs}(k) = \int\frac{\tilde{\gamma}(\mathbf{k}-\mathbf{k}')}{8\pi^3}d^3\tilde{k}'\tilde{\delta}(\mathbf{k}')} \tag{6}$$

I think this is a convolution?

Problem 1B:

We want to compute:

$$\langle|\tilde{\delta}_{\rm obs}(\mathbf{k})|^2\rangle = \langle\tilde{\delta}_{\rm obs}(\mathbf{k})\tilde{\delta}^*_{\rm obs}(\mathbf{k})\rangle \tag{7}$$

This is the definition, and the * is the complex conjugate because we are dealing with complex valued stuff. We know from Problem 1A what $\tilde{\delta}_{\rm obs}$ is, so we can write:

[1] Fourier is a persons name and should be capitalized.

$$\langle|\tilde{\delta}_{\text{obs}}(\mathbf{k})|^2\rangle = \int \frac{\tilde{\gamma}(\mathbf{k}-\mathbf{k}_1)}{8\pi^3} d^3\tilde{k}'\tilde{\delta}(\mathbf{k}_1) \int \frac{\tilde{\gamma}^*(\mathbf{k}-\mathbf{k}_2)}{8\pi^3} d^3\tilde{k}'\tilde{\delta}^*(\mathbf{k}_2) \tag{8}$$

From Eqn 1 in the handout / the definition of the power spectrum we know

$$\langle\tilde{\delta}(\mathbf{k}_1)\tilde{\delta}^*(\mathbf{k}_2)\rangle = (2\pi)^3\,\delta^D(\mathbf{k}_1-\mathbf{k}_2)\,P(\mathbf{k}_1) \tag{9}$$

Why do we write the Dirac with a top exponet? This is horrible, I might switch to a subscript. Anyway, we can use this to replace our $\delta(k)$'s in the integral form.

$$\langle|\tilde{\delta}_{\text{obs}}(\mathbf{k})|^2\rangle = \int \frac{\tilde{\gamma}(\mathbf{k}-\mathbf{k}_1)}{8\pi^3} d^3\tilde{k}' \int \tilde{\gamma}^*(\mathbf{k}-\mathbf{k}_2) d^3\tilde{k}'\,\delta^D(\mathbf{k}_1-\mathbf{k}_2)\,P(\mathbf{k}_1) \tag{10}$$

One of the for factors also cancels!, Now we can use the Dirac function to reduce one of the integrals.

$$\langle|\tilde{\delta}_{\text{obs}}(\mathbf{k})|^2\rangle = \int \frac{|\tilde{\gamma}(\mathbf{k}-\mathbf{k}_1)|^2}{8\pi^3} d^3\tilde{k}' P(\mathbf{k}_1) \tag{11}$$

And $k_1 = k'$

$$\boxed{\langle|\tilde{\delta}_{\text{obs}}(\mathbf{k})|^2\rangle = \int \frac{|\tilde{\gamma}(\mathbf{k}-\mathbf{k}')|^2}{8\pi^3} d^3\tilde{k}' P(\mathbf{k}')} \tag{12}$$

Problem 1C:

Oki, so the first half of this is easy, we say $\gamma \approx$ a Dirac function, for the second half we will have to look up Parseval's theorem.

We can pull the P factor out of the integral.

$$\langle|\tilde{\delta}_{\text{obs}}(\mathbf{k})|^2\rangle = P(\mathbf{k}) \int \frac{|\tilde{\gamma}(\mathbf{k}-\mathbf{k}')|^2}{8\pi^3} d^3\tilde{k}' \tag{13}$$

Now our hope lies in Parseval. Parseval's theorem says:

$$\int d^3r|f(\mathbf{r})|^2 = \int \frac{d^3k}{(2\pi)^3}|\tilde{f}(\mathbf{k})|^2 \tag{14}$$

We can change the γ from Fourier space - the tilde to normal space with an r, and inside the survey space it is equal to 1.

$$\int d^3r|\gamma(\mathbf{r})|^2 = V \tag{15}$$

So we can say:

$$\langle|\tilde{\delta}_{\text{obs}}(\mathbf{k})|^2\rangle = P(\mathbf{k})V \tag{16}$$

Or matching the form on the handout

$$\boxed{P(\mathbf{k}) = \frac{\langle|\tilde{\delta}_{\text{obs}}(\mathbf{k})|^2\rangle}{V}} \tag{17}$$

Problem 2: Definition of σ_8

Problem 2A:

Lets start with:

$$y = \int d^3r w_R(\mathbf{r})\delta(\mathbf{r}) \tag{18}$$

Lets take the backwards Fourier transform $\delta(\mathbf{r})$, and plug it in.

$$y = \int \frac{d^3k}{(2\pi)^3}\tilde{\delta}(\mathbf{k}) \int d^3r w_R(\mathbf{r}) \exp(i\mathbf{k} \ \cdot \mathbf{r}) \tag{19}$$

We recognize the Fourier transform of w_R!

$$y = \int \frac{d^3k}{(2\pi)^3}\tilde{w}_R(k)\tilde{\delta}(\mathbf{k}) \tag{20}$$

Now we compute the variance! We can use the powerspectrum trick again from problem 1. Since this is a real valued thing, the complex conjugate is equal to the original.

$$\langle y^2 \rangle = \langle \int \frac{d^3k}{(2\pi)^3}\tilde{w}_R(k)\tilde{\delta}(\mathbf{k}) \int \frac{d^3k'}{(2\pi)^3}\tilde{w}_R(k')\tilde{\delta}(\mathbf{k}')\rangle \tag{21}$$

$$\langle y^2 \rangle = \int\int \frac{d^3k}{(2\pi)^3}\tilde{w}_R(k)\frac{d^3k'}{(2\pi)^3}\tilde{w}_R(k')\langle\tilde{\delta}(\mathbf{k})\tilde{\delta}(\mathbf{k}')\rangle \tag{22}$$

Now we insert the powerspectrum trick from the handout.

$$\langle y^2 \rangle = \int\int \frac{d^3k}{(2\pi)^3}\tilde{w}_R(k)\frac{d^3k'}{(2\pi)^3}\tilde{w}_R(k')(2\pi)^3\delta^D(k-k')P(k) \tag{23}$$

$$\boxed{\sigma_R^2 = \langle y^2 \rangle = \int \frac{d^3k}{(2\pi)^3}|\tilde{w}_R(\mathbf{k})|^2 P(k)} \tag{24}$$

Problem 2B:

$$w_R(\mathbf{r}) = \frac{3}{4\pi R^3}\Theta(1-\frac{r}{R}) \tag{25}$$

$$\tilde{w}_R(k) = \int d^3r w_R(\mathbf{r})\exp(-i\mathbf{k}\cdot r) \tag{26}$$

We are in 3D, so can can (attempt) some of the integrals in 3D spherical, this takes us to:

$$\tilde{w}_R(k) = \int d^3r \exp(-i\mathbf{k}\cdot r) \tag{27}$$

Our $d^3r = r^2drd\Omega$, we can $\mathbf{k}\cdot r = kr\cos\theta$ as

$$\tilde{w}_R(k) = \frac{3}{4\pi R^3}\int_{r<R} d\Omega dr \exp(-ikr\cos\theta) \tag{28}$$

From an integral table:

$$\int d\Omega \exp\left(-ik\cos\theta\right) = 2\pi \int_{-1}^{1} du \exp\left(-ikru\right) = \frac{4\pi\sin(kr)}{kr} \tag{29}$$

Now we are left with the $r^2 dr$ integral

$$\tilde{w}_R(k) = \frac{3}{R^3}\int_0^R r^2 \frac{\sin(kr)}{kr} dr = \frac{3}{kR^3}\int_0^R r\sin(kr)dr \tag{30}$$

and integral table gives us $\int r\sin(kr)dr = -\frac{r\cos(kr)}{k} + \frac{\sin(kr)}{k^2}$.
Using this we can say:

$$w_R(\mathbf{k}) = \frac{3}{k^3R^3}\frac{\sin(kR) - kR\cos(kR)}{} \tag{31}$$

But we are only halfway there, we need to to now invoke the power spectrum trick used in the other problems.

$$\sigma_r^2 = \langle y^2 \rangle = \int \frac{d^3k}{(2\pi)^3} |\tilde{w}_R(k)|^2 P(k). \tag{32}$$

Due to isotropy, $d^3k = 4\pi k^2 dk$ and we can now say:

$$\sigma_r^2 = \frac{1}{2\pi^2}\int_0^\infty k^2 \left[\frac{3\big(\sin(kR) - kR\cos(kR)\big)}{(kR)^3}\right]^2 P(k)dk. \tag{33}$$

Taking the square root yields

$$\boxed{\sigma_R = \left(\frac{4\pi}{(2\pi)^3}\int_0^\infty \left[\frac{3\big(\sin(kR) - kR\cos(kR)\big)}{(kR)^3}\right]^2 P(k)k^2dk\right)^{1/2},} \tag{34}$$

I think I have used the correct convention for the forward transform throughout, but no guarantees.