Inflation Part 2

Last time we talked about some problems with the Hot Big Bang model, and how a period of accelerated expansion early on ($t \ll 1$ sec) could solve these problems.

Monopole problem — rapid expansion dilutes concentration to negligible levels.

Homogeneity problem — our Universe is homogeneous because everything we see was stretched out from a tiny patch in the pre-inflationary universe.

Flatness problem — our Universe is flat because it was so stretched out, driving the curvature to zero.

Of course, we cannot simply declare that our Universe has this accelerated expansion just because it's convenient for us! We need a mechanism for this, and that's what we'll look at today.

Let's look at the requirements on this model:

i) Must produce accelerated expansion.

ii) Must ~~go~~ expand our Universe by enough.

iii) Must end and connect back to a radiation-dominated Universe by "1 second". (Not precise)

iv) Must repopulate universe with particles, since any pre-inflation relic gets diluted to nothing along with the monopoles!

Plug in the numbers and you will find that $a(t)$ needs to grow by e^{60} to solve our problems!

Put another way, Friedman's Eqn. says that

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2}$$

We know what we want on the LHS. Now we need to come up with an appropriate "ρ" on the RHS to make this all work.

So what might fit the bill? We know, for example, that today our Universe is accelerating because of dark energy, which is consistent with a constant vacuum energy. What that work?

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \rho_\Lambda}{3} \quad \leftarrow \text{const.}$$

(omitting the curvature term because inflation will make it unimportant anyway)

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi G \rho_\Lambda}{3}}$$

$$\Rightarrow a(t) \propto \exp\left(\sqrt{\frac{8\pi G \rho_\Lambda}{3}}\, t\right).$$

This fits requirement i) — it's an accelerated expansion — but it doesn't work for requirements ii) and iii). At least with the value of ρ_Λ we have in our Universe, the expansion is just not quick enough. We don't expand by exp(60) in less than a second! Also, a constant vacuum energy is ... well ... constant. This exponential growth needs to stop at some point and latch onto a radiation-dominated universe.

Inflation via a scalar field

We can get something that works if we have inflation be driven by a scalar field.

What's a scalar field? Some field $\phi(\vec{r}, t)$ that has a scalar value in every point in space $\vec{r}$ and time t.

The Higgs field is an example of a scalar field.

Could the Higgs field itself be the inflaton field? Turns out it cannot, at least if you plug in the "vanilla" properties of the Higgs. It predicts the wrong level of density fluctuations

Which scalar field drives inflation? We simply postulate that there is some field called the inflaton, but of course this is just a name.

Whatever the properties of the inflaton, it needs to satisfy the requirements that we laid out earlier in the lecture. But otherwise there's a lot of freedom regarding the detailed properties of the inflaton field. What these details are like will subtly change the observables.

In this way, inflation is a paradigm, not a theory. It doesn't provide the details of the inflaton field. You provide that, and inflation tells you what the observable consequences are.

What we're going to do now is to see if a scalar field gives us the right expansion dynamics for inflation.

The key is identifying ω, our equation of state parameter in

$$p = \omega \rho$$

because recall from before that different ω values give different $a(t)$ solutions in the Friedman eqn.

For a scalar field ϕ, the stress-energy tensor $T_{\mu\nu}$ is

$$T_{\mu\nu} = \partial_\mu \phi \, \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \, \partial_\beta \phi - V(\phi) \right).$$

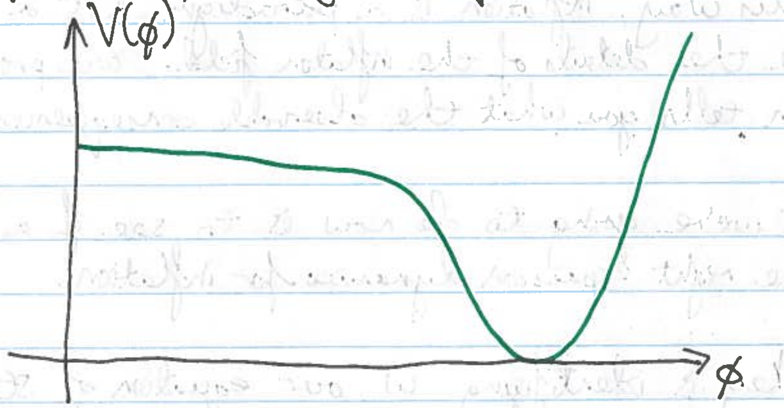
A potential energy

The energy density and the pressure are different components that can be read off the stress-energy tensor.

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \; ; \quad \rho = \overbrace{\frac{1}{2}\dot{\phi}^2}^{\text{Kinetic energy}} + \overbrace{V(\phi)}^{\text{Potential energy}} .$$

So $$w = \frac{p}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}$$

What form does the potential energy of the inflaton take? This is where you can pretty much draw what you want and see what sort of observational consequences we have once we've churned through the inflationary paradigm. It could look something like this:



Now, the value of ϕ can change with time. If we use the Friedman Equation, we have

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V\right).$$

$$2H\dot{H} = \frac{8\pi G}{3}\left(\dot{\phi}\ddot{\phi} + \frac{\partial V}{\partial \phi}\dot{\phi}\right).$$

~~Hee~~ We can then say $H = \left(\frac{\dot{a}}{a}\right)$, so

$$\dot{H} = \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = \frac{\ddot{a}}{a} - H^2$$

From the 2nd Friedman Equation, we have

Inserting ρ and p for a scalar field

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3}(\rho + 3p) = -\frac{8\pi G}{3}(\dot{\phi}^2 - V) \qquad (*)$$

~~which~~ which means $\dot{H} = -\frac{8\pi G}{3}(\dot{\phi}^2 - V) - \frac{8\pi G}{3}\left(\frac{\dot{\phi}^2}{2} + V\right) = -\frac{8\pi G}{3}\left(\frac{3\dot{\phi}^2}{2}\right)$

and therefore $2H\left[-\frac{8\pi G}{3}\left(\frac{3\dot{\phi}^2}{2}\right)\right] = \frac{8\pi G}{3}\left(\dot{\phi}\ddot{\phi} + \frac{\partial V}{\partial \phi}\dot{\phi}\right)$

$$\Rightarrow \boxed{\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0}$$

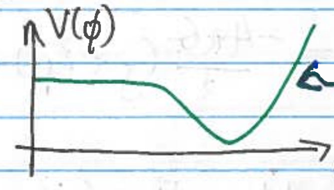
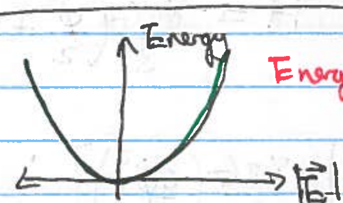
This is the equation of motion for ϕ. It tells us how ϕ evolves with time. This is good – we'll need this because we need time variability to have inflation end.

The dynamics of the inflaton field are very much like that of a ball rolling on a hill. The force is $-\frac{\partial V}{\partial \phi}$, and Hubble expansion causes a friction term.

At this point we've gotten rather abstract. What do rolling balls have to do with our Universe?

Let me offer an analogy with a field that you might be more familiar with – the electric field $\vec{E}$.

Warning: I would really, really like you to treat this as an analogy. It isn't rigorous and a particle physicist would probably be horrified.

Field	Energy	Equation of motion
Inflaton ϕ	 Or something else!	$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0$
Electric field $\vec{E}$	 Energy $= \frac{\lvert\vec{E}\rvert^2}{2\varepsilon_0}$	Maxwell's Eqn's

So I like to imagine one of these pictures in every point in space, with the "ball" moving about the potential as the field changes value.

Alright. Back to the task at hand. What conditions must $V(\phi)$ satisfy for inflation to work?

From (*), one sees that to get a rapidly accelerating universe, the potential energy must dominate.

So I need $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$.

For inflation to persist for long enough to provide 60 e-folds of expansion, I can't have $\dot{\phi}$ change very much or I might violate

So $|\ddot{\phi}|$ must be small compared to the other terms in the equations of motion, $|3H\dot{\phi}|$ and $|\partial V/\partial\phi|$.

If you can invent a $V(\phi)$ that satisfies these conditions, then you will have come up with an inflationary model.

To see this, note that under slow roll, $3H\dot{\phi} \approx -\frac{\partial V}{\partial \phi}$, so saying that $\frac{\partial V}{\partial \phi}$ is small is equivalent to saying $\dot{\phi}$ is small.

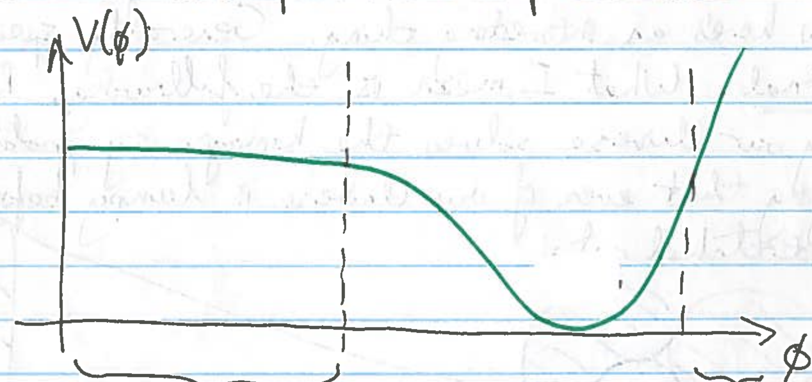
~~It is conventional to encapsulate these in a non-dimensional form~~ We need to satisfy the slow roll conditions:

$$\varepsilon \equiv \frac{1}{16\pi G}\left(\frac{\partial V/\partial\phi}{V}\right)^2 \quad \text{and} \quad |\eta| \equiv \frac{1}{8\pi G}\frac{|\partial^2 V/\partial\phi^2|}{V}$$

Note that some people prefer to define a slightly different set of slow-roll parameters based on H. One can use either version because $H^2 \simeq \frac{8\pi G V}{3}$, so derivatives of V can be related to derivatives of H.

If $\varepsilon, |\eta| \ll 1$, we satisfy the slow roll conditions and inflation happens.

What sorts of potentials satisfy slow roll? It's not an accident that I drew that peculiar shape earlier!



Really flat potentials clearly satisfy the slow-roll conditions.

This sort of region also satisfies the slow-roll conditions!

Eg Consider $V \propto \phi^2$; $\frac{\partial V}{\partial \phi} \propto \phi$; $\frac{\partial^2 V}{\partial \phi^2} = \text{const.}$

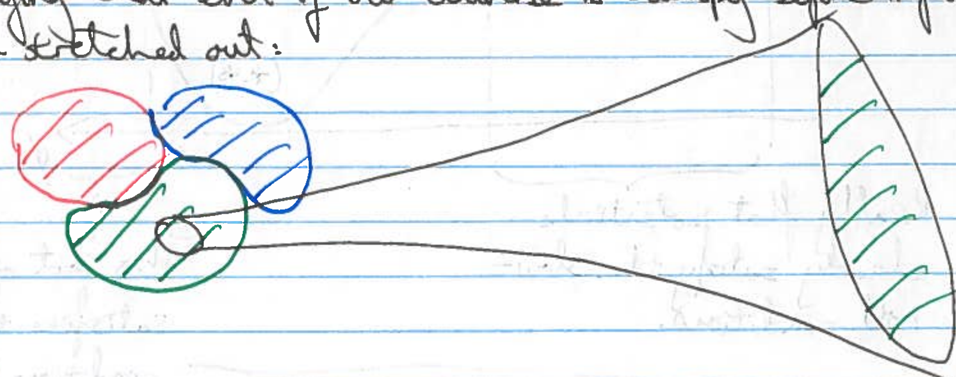
$$\Rightarrow \varepsilon, \eta \propto \frac{1}{\phi^2}$$, and if ϕ is large, ε and η are small.

Alright. So the ball rolls slowly down the potential, and eventually we exit the region that satisfies the slow roll conditions. The ball settles down and oscillates at the bottom of the potential well. At this point, the inflaton field couples to ordinary particle fields and populates those fields (remember that any pre-inflationary relics are diluted away!). This creation of standard particles is known as preheating and is not well understood.

Aside from this last step, which is sketchy we basically have an inflationary model that works!

Eternal inflation

Now here's an interesting thing. Generally speaking, inflation is eternal. What I mean is the following. Recall that the way our Universe solves the homogeneity problem is by saying that even if our Universe is lumpy before inflation, it's all stretched out:



The other patches may also inflate! After all, the values of the scalar field there may satisfy slow roll and so they too might inflate.

In fact, here's an even crazier thing that can happen. Ultimately, ϕ is a quantum field. So even in parts of our Universe where inflation has "ended" and the field is oscillating about the bottom of the well, there is a non-zero probability that the ball can climb uphill, back to where slow-roll works and start inflating again!

You might recall from quantum that climbing up to a potential energy that is in your ~~for~~ "forbidden" region is exponentially suppressed. That's true, so why isn't this effect unimportant?

↳ Because inflation creates exponentially more volume, so even exponentially rare events count!

And if you're creating all this space, you're probably going to have some point in this new space that also goes through this process! It's a runaway, eternal inflation.

What's more, in the regions that get inflated this way, there's no guarantee that they climb back up far enough to get 60 e-folds of inflation. Maybe only 30 e-folds take place. Those parts of the universe won't be that flat.

Some people hate this. They say that inflation has just traded one fine-tuning problem for another. Rather than "why is $|\Omega_k|$ so small?", it's now "why do we live in a patch that inflated just the right way?".

You might say let's just calculate the probability that we live in such a patch! The problem is that the universe is infinite. And when you have an infinite set, you can't really compare the relative sizes of two subsets.

Analogy: Are there more even numbers or odd numbers?

Argument that there are an equal number:

Pair them up

1	3	5	7	9	...
↕	↕	↕	↕	↕	
2	4	6	8	10	...

Argument that there are twice as many even numbers:

1	3	5	7	9	...
↕	↕	↕	↕	↕	
2,4	6,8	10,12	14,16	18,20	...

I can do this because I have an infinite "reserve" on numbers to draw from!

This makes it hard to figure out how likely it is that we live in a patch with the right properties. This is known as the measure problem and it is an open problem.