PHYS644 Problem Set 10

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Problem 1: Peculiar Velocities

Problem 1A:

We are going to use a subscript to indicate the Fourier transforms because that's easier for me typing. We are going to start from perturbed continuity equation:

$$\dot{\delta}(\mathbf{r},t) + \nabla\cdot\delta\mathbf{v}(\mathbf{r},t) + H\,\mathbf{r}\cdot\nabla\delta(\mathbf{r},t) = 0, \tag{1}$$

From our Fourier convention we have:

$$\delta(\mathbf{r},t) = \int \frac{d^3k}{(2\pi)^3}\,\delta_{\mathbf{k}}(t)\,e^{i\mathbf{k}\cdot\mathbf{r}/a(t)}, \tag{2}$$

$$\delta\mathbf{v}(\mathbf{r},t) = \int \frac{d^3k}{(2\pi)^3}\,\mathbf{v}_{\mathbf{k}}(t)\,e^{i\mathbf{k}\cdot\mathbf{r}/a(t)}. \tag{3}$$

Now we are going to take a time derivative of $\delta(\mathbf{r},t)$, recall $H = \frac{\dot{a}}{a}$

$$\dot{\delta}(\mathbf{r},t) = \int \frac{d^3k}{(2\pi)^3}\left[\dot{\delta}_{\mathbf{k}}(t) - i\frac{H}{a}(\mathbf{k}\cdot\mathbf{r})\,\delta_{\mathbf{k}}(t)\right]e^{i\mathbf{k}\cdot\mathbf{r}/a(t)}. \tag{4}$$

Now we are going to create the divergence term:

$$\nabla\cdot\delta\mathbf{v} = \int \frac{d^3k}{(2\pi)^3}\,i\frac{\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}}(t)}{a}\,e^{i\mathbf{k}\cdot\mathbf{r}/a}. \tag{5}$$

Now the next term:

$$H\,\mathbf{r}\cdot\nabla\delta = \int \frac{d^3k}{(2\pi)^3}\,H\left(i\frac{\mathbf{k}\cdot\mathbf{r}}{a}\right)\delta_{\mathbf{k}}(t)\,e^{i\mathbf{k}\cdot\mathbf{r}/a}. \tag{6}$$

The terms proportional to $(\mathbf{k}\cdot\mathbf{r})\delta_{\mathbf{k}}$ cancel, leaving us with:

$$\int \frac{d^3k}{(2\pi)^3}\left[\dot{\delta}_{\mathbf{k}}(t) + i\frac{\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}}(t)}{a}\right]e^{i\mathbf{k}\cdot\mathbf{r}/a} = 0. \tag{7}$$

Since each Fourier mode is orthogonal, the term in the bracket must vanish, leaving us with

$$\boxed{\dot{\delta}_{\mathbf{k}}(t) + \frac{i}{a}\,\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}}(t) = 0}. \tag{8}$$

Problem 1B

Now we are going to do the same for the Euler equation. Yay. This time , we start from the Euler equation.

$$\dot{\delta\mathbf{v}} + (H\mathbf{r}\cdot\nabla)\,\delta\mathbf{v} + H\,\delta\mathbf{v} = -\nabla\delta\Phi - v_s^2\nabla\delta, \tag{9}$$

where $v_s^2 = \partial p/\partial \rho$.

As before the Fourier transforms are

$$\delta(\mathbf{r}, t) = \int \frac{d^3k}{(2\pi)^3}\, \delta_{\mathbf{k}}(t)\, e^{i\mathbf{k}\cdot\mathbf{r}/a(t)}, \tag{10}$$

$$\delta\mathbf{v}(\mathbf{r}, t) = \int \frac{d^3k}{(2\pi)^3}\, \mathbf{v}_{\mathbf{k}}(t)\, e^{i\mathbf{k}\cdot\mathbf{r}/a(t)}. \tag{11}$$

This time we have

$$\delta\Phi(\mathbf{r}, t) = \int \frac{d^3k}{(2\pi)^3}\, \delta\Phi_{\mathbf{k}}(t)\, e^{i\mathbf{k}\cdot\mathbf{r}/a(t)}. \tag{12}$$

We take the time derivative as before,

$$\frac{\partial}{\partial t}\Big[\mathbf{v}_{\mathbf{k}}(t)\, e^{i\mathbf{k}\cdot\mathbf{r}/a(t)}\Big] = \dot{\mathbf{v}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}/a} - i\frac{H}{a}(\mathbf{k}\cdot\mathbf{r})\,\mathbf{v}_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{r}/a}. \tag{13}$$

Now the term with H thingy:

$$(H\mathbf{r}\cdot\nabla)\delta\mathbf{v} = \int \frac{d^3k}{(2\pi)^3}\, H\Big(i\frac{\mathbf{k}\cdot\mathbf{r}}{a}\Big)\mathbf{v}_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{r}/a}, \tag{14}$$

This cancels the $-i\frac{H}{a}(\mathbf{k}\cdot\mathbf{r})\mathbf{v}_{\mathbf{k}}$ in the time derivative when the two are summed.

The $H\delta\mathbf{v}$ term this term is not hard:

$$H\,\delta\mathbf{v}(\mathbf{r}, t) = \int \frac{d^3k}{(2\pi)^3}\, H\,\mathbf{v}_{\mathbf{k}}(t)\, e^{i\mathbf{k}\cdot\mathbf{r}/a}. \tag{15}$$

The gradient parts (with $(i\mathbf{k}/a)$):

$$\nabla\delta\Phi(\mathbf{r}, t) = \int \frac{d^3k}{(2\pi)^3}\, i\frac{\mathbf{k}}{a(t)}\, \delta\Phi_{\mathbf{k}}(t)\, e^{i\mathbf{k}\cdot\mathbf{r}/a(t)}, \tag{16}$$

$$\nabla\delta(\mathbf{r}, t) = \int \frac{d^3k}{(2\pi)^3}\, i\frac{\mathbf{k}}{a(t)}\, \delta_{\mathbf{k}}(t)\, e^{i\mathbf{k}\cdot\mathbf{r}/a(t)}. \tag{17}$$

Now we put it all together and pray. The $(\mathbf{k}\cdot\mathbf{r})$ parts cancel, the inside of the integral parts gives:

$$\boxed{\dot{\mathbf{v}}_{\mathbf{k}} + H\mathbf{v}_{\mathbf{k}} + i\frac{\mathbf{k}}{a}\big(\delta\Phi_{\mathbf{k}} + v_s^2\delta_{\mathbf{k}}\big) = \mathbf{0}}. \tag{18}$$

Problem 1C

We start from our solution to 1B. Now we want to decompose into velocity components into components parallel and perpendicular to $\mathbf{k}$

$$\mathbf{v}_{\mathbf{k}} = v_{\mathbf{k}\parallel}\cdot\hat{\mathbf{e}}_{\parallel} + v_{\mathbf{k}\perp}\cdot\hat{\mathbf{e}}_{\perp} \tag{19}$$

I guess I don't really need to write v perp and parallel because the dot product does it for me but I already wrote it.

The unit vector is defined as:

$$\hat{\mathbf{e}}_{\parallel} = \mathbf{k}/|\mathbf{k}| \tag{20}$$

Now we dot with the perpendicular direction, this one is simple

$$\hat{\mathbf{e}}_\perp \cdot (\dot{\mathbf{v}}_{\mathbf{k}} + H\mathbf{v}_{\mathbf{k}} + i\frac{\mathbf{k}}{a}(\delta\Phi_{\mathbf{k}} + v_s^2\delta_{\mathbf{k}})) = \dot{v}_{\mathbf{k}\perp} + Hv_{\mathbf{k}_\perp} = 0 \tag{21}$$

This is now our ODE.

$$\frac{dv_{\mathbf{k}\perp}}{dt} = -H(t)\, v_{\mathbf{k}\perp}. \tag{22}$$

$$\frac{dv_{\mathbf{k}\perp}}{v_{\mathbf{k}\perp}} = -H(t)\, dt. \tag{23}$$

$$\ln v_{\mathbf{k}\perp}(t) = -\int^t H(t)\, dt \tag{24}$$

Recall that $\int H(t)\, dt = \ln a(t)$,

$$\boxed{v_{\mathbf{k}\perp}(t) = \frac{v_{\mathbf{k}\perp}(t_0)}{a(t)}}. \tag{25}$$

as the Universe expands, the perpendicular component of the peculiar velocity decays as $1/a(t)$.

Problem 1D

Starting from the solution to problem 1A

$$\dot{\delta}_{\mathbf{k}}(t) + \frac{i}{a}\, \mathbf{k}\cdot\mathbf{v}_{\mathbf{k}}(t) = 0 \tag{26}$$

At late times the perpendicular component of the peculiar velocity decays as $1/a(t)$, so can neglect it and instead focus on the parallel components.

$$\dot{\delta}_{\mathbf{k}} + \frac{i}{a}\, k\, v_{\mathbf{k}\parallel} = 0. \tag{27}$$

Solve for the parallel velocity component:

$$v_{\mathbf{k}\parallel} = -\,\frac{a}{ik}\, \dot{\delta}_{\mathbf{k}}. \tag{28}$$

This is our ODE!, to solve it we will invoke a growth factor D

$$\delta_{\mathbf{k}}(a) = D_1(a)\, \delta_{\mathbf{k}}(0). \tag{29}$$

Differentiate:

$$\dot{\delta}_{\mathbf{k}} = \frac{\dot{D}_1}{D_1}\, \delta_{\mathbf{k}} = Hf\, \delta_{\mathbf{k}}, \tag{30}$$

f here is the linear growth rate, now we put it into our equation:

$$v_{\mathbf{k}\parallel} = i\, \frac{aHf}{k}\, \delta_{\mathbf{k}}. \tag{31}$$

Now we put it back into vector form to get our boxed solution. $\hat{\mathbf{k}} = k/|k|$.

$$\boxed{\mathbf{v}_{\mathbf{k}} = i\, \frac{aHf}{k^2}\, \mathbf{k}\, \delta_{\mathbf{k}}}. \tag{32}$$

Problem 2: Matter Power Spectrum

For this problem we navigate to: `https://mybinder.org/v2/gh/acliu/CAMB/master?filepath=pycamb%2Fdocs%2FCRAQ.ipynb`

Problem 2A

We are asked to plot the matter power spectrum $P(k)$ for various z, and comment.

This is done for us by the notebook, see figure 1. We see that as z increases from 0 to 2 that the entire $P(k)$ is downward shifted by a approx constant factor

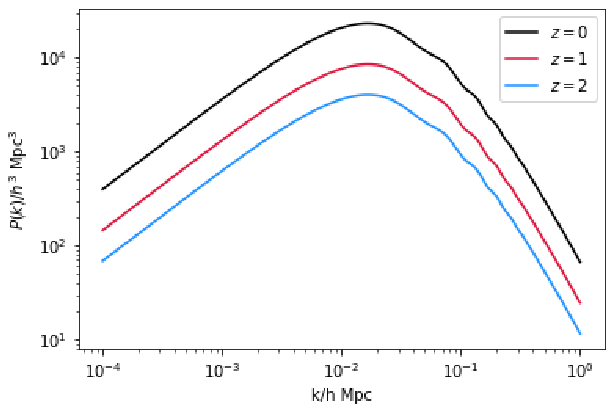

Figure 1: I'm not writing a caption

Problem 2B

The primordial power spectrum is predicted by theories of the very early Universe to be of the form $P_*(k) = A_s k^{n_s}$., We are asked to vary A_s and see what happens to today's power spectrum.

This is shown in figure 2. We see that varying A_s also causes a similar shift in the plot - they are degenerate. We talked about this in class briefly.

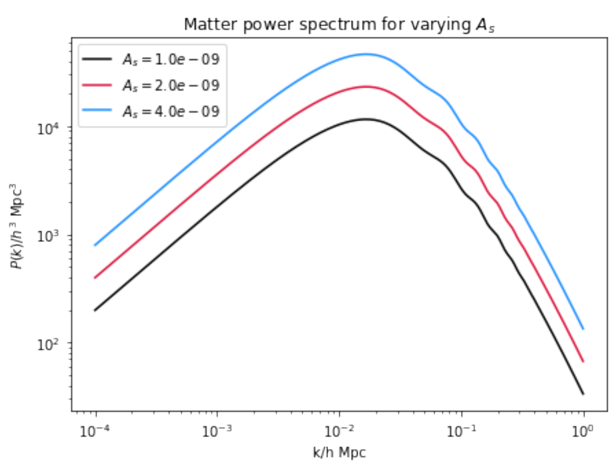

Figure 2: I'm not writing a caption

Problem 2C

See figure 3. We are now asked to plot with varying n_s, it is clear that this controls the slope of the power spectrum in log-log space $n_s > 1$ the spectrum tilts upward at large k, and $n_s < 1$ the spectrum tilts downward at large k, and $n_s = 1$ we have "scale invariant" spectrum.

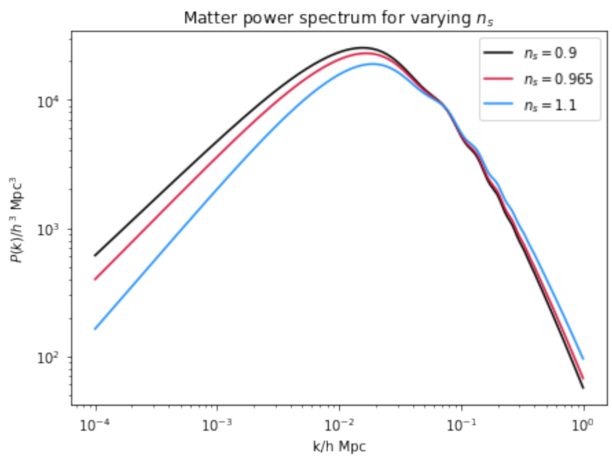

Figure 3: I'm not writing a caption

Problem 2D

More baryons means larger baryon acoustic oscillations (BAO wiggles), on the right hand side of the $P(k)$ plot.

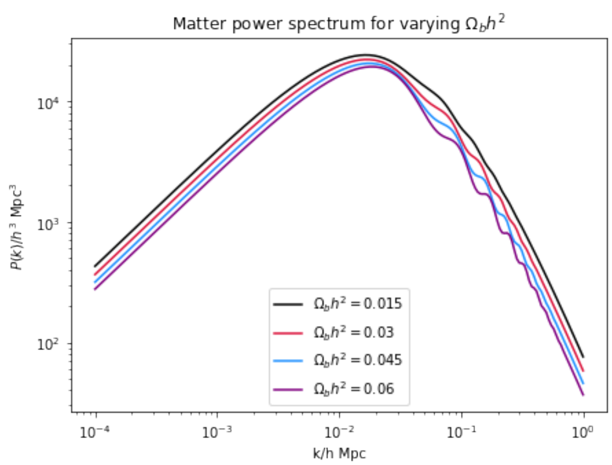

Figure 4: I'm not writing a caption