PHYS 644 Lecture #3: Galaxies as Collisionless Fluids

From last time, our discussion showed that we cannot think about stars in terms of individual encounters, but need to think about them as a collective collisionless fluid.

Today's goal is to build the formalism for this type of description.

We will define a phase space density $f(\vec{x},\vec{v},t)$

$$f(\vec{x},\vec{v},t)\, d^3\vec{x}\, d^3\vec{v} \equiv \left(\begin{array}{l} \#\text{ of stars located in small volume } d^3\vec{x} \\ \text{centred on } \vec{x} \text{ and velocities in small} \\ \text{range } d^3\vec{v} \text{ centred on } \vec{v} \end{array} \right)$$

In other words, it is a density in 6D phase space given by coordinates $\vec{w} \equiv (\vec{x},\vec{v})$.

(From stat. mech. you may be more used to a phase space density in 6N-dimensional space, where each particle gets its own set of axes. Our current density can be obtained by integrating over N-1 particles to leave just one.)

We can think about a swarm / blob of particles moving through phase space, flowing with velocity

$$\dot{\vec{w}} = (\dot{\vec{x}},\dot{\vec{v}}) = (\vec{v}, -\vec{\nabla}\Phi)$$

Grav. potential, not grav. potential energy, ie this is energy per unit mass.

Given Newton's Laws, if we know $f(\vec{x},\vec{v},t)$ and some t, we can know $f(\vec{x},\vec{v},t)$ at a later time.

$\Rightarrow$ $\boxed{\text{Phase space intuition slides}}$

With our mental picture of stars being collisionless, and also assuming that no stars are formed or die, we can reasonably expect the phase space distribution to evolve smoothly and respect the continuity equation:

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \frac{\partial (f\dot{w}_\alpha)}{\partial w_\alpha} = 0.$$

$f\dot{\vec{w}}$ being like a "phase space current"

Simplifying gives:

$$\frac{\partial f}{\partial t} + \sum_\alpha \left(f \frac{\partial \dot{w}_\alpha}{\partial w_\alpha} + \dot{w}_\alpha \frac{\partial f}{\partial w_\alpha} \right) = 0.$$

The flow described by $\dot{\vec{w}}$ is quite special because

$$\sum_\alpha \frac{\partial \dot{w}_\alpha}{\partial w_\alpha} = \sum_{i=1}^{3} \left(\underbrace{\frac{\partial v_i}{\partial x_i}}_{=0} + \frac{\partial \dot{v}_i}{\partial v_i} \right) = \sum_{i=1}^{3} -\frac{\partial}{\partial v_i}\left(\frac{\partial \Phi}{\partial x_i}\right) = 0$$

$=0 \because \vec{v}$ and $\vec{x}$ independent

$\dot{\vec{v}} = -\vec{\nabla}\Phi$

$\vec{\nabla}\Phi$ not velocity-dependent

This gives us the collisionless Boltzmann equation:

$$\frac{\partial f}{\partial t} + \sum_\alpha \dot{w}_\alpha \frac{\partial f}{\partial w_\alpha} = 0$$

or

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left(v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} \right) = 0$$

or

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f - \vec{\nabla}\Phi \cdot \vec{\nabla}_v f = 0$$

The collisionless Boltzmann equation can also be recast in terms of the Lagrangian (or convective) derivative:

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \sum_{\alpha} \dot{w}_{\alpha} \frac{\partial}{\partial w_{\alpha}}$$

This is the rate of change for an observer that's flowing along a trajectory in phase space. It says that I can get a change either from some ~~~~ explicit time dependence (1st term) or because I've moved to a different part of phase space (2nd term).

Our Boltzmann equation is then just $\frac{df}{dt} = 0$.

In other words, an observer moving through phase space with a star at $\vec{w}$ would not see the phase space density around them change. The flow of stars in phase space is like that of an incompressible fluid.

Special case of Liouville's theorem!

$\Rightarrow$ $\boxed{\text{Slide examples of incompressibility.}}$

While phase space can be useful theoretically, we don't directly observe it. To extract this information, we take moments of the phase space distribution.

0th moment: $n(\vec{x}) = \int_{-\infty}^{+\infty} f(\vec{x}, \vec{v}, t)\, d^3\vec{v}$

Spatial # density of stars

1st moment: $$\langle \vec{v}(\vec{x}) \rangle = \frac{1}{n} \int_{-\infty}^{+\infty} f(\vec{x}, \vec{v}, t)\, \vec{v}\, d^3\vec{v}$$

Average velocity.

2nd moment: $$\langle v_i(\vec{x})\, v_j(\vec{x}) \rangle = \frac{1}{n} \int_{-\infty}^{\infty} f(\vec{x}, \vec{v}, t)\, v_i v_j\, d^3\vec{v}$$

$$\equiv \langle v_i(\vec{x}) \rangle \langle v_j(\vec{x}) \rangle + \sigma_{ij}^2$$

Velocity dispersion tensor

To get equations for these quantities, we can take the corresponding moments of the Boltzmann eqn.

0th moment

$$0 = \int d^3\vec{v} \left[\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f - (\vec{\nabla}_x \Phi) \cdot (\vec{\nabla}_v f) \right]$$

Integration by parts

$$= \frac{\partial n}{\partial t} + \vec{\nabla}_x \cdot \underbrace{\int d^3\vec{v}\, \vec{v} f}_{n\langle \vec{v} \rangle} - \underbrace{\sum_{i=1}^{3} f\, \nabla_{x,i} \Phi \Big|_{-\infty}^{+\infty}}_{=0 \because f \to 0 \text{ as } \vec{v} \to \infty} + \underbrace{\int d^3\vec{v}\, f \sum_{i=1}^{3} \vec{\nabla}_{x,i} \vec{\nabla}_{v,i} \Phi}_{=0 \because \vec{\nabla}_v \Phi = 0}$$

$$\Rightarrow \boxed{\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n \langle \vec{v} \rangle) = 0}$$ Continuity equation!

Conservation of stars: change in stuff = flux of stuff out of a region

1st moment

$$0 = \int_{-\infty}^{+\infty} d^3\vec{v}\ \vec{v}\left[\frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla}_x f - (\vec{\nabla}_x \Phi)\cdot\vec{\nabla}_v f\right]$$

$$\qquad\quad (A) \qquad\quad (B) \qquad\qquad\quad (C)$$

Term (A): $\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} d^3\vec{v}\ \vec{v} f = \partial_t(n\langle\vec{v}\rangle)$

$$= n\,\partial_t\langle\vec{v}\rangle + \langle\vec{v}\rangle\partial_t n = n\,\partial_t\langle\vec{v}\rangle - \langle\vec{v}\rangle\vec{\nabla}\cdot(n\langle\vec{v}\rangle)$$

Continuity equation

Term (B): Just like before, $\vec{\nabla}_x$ sails outside the $\vec{v}$ integral and we get

$$\sum_i \vec{\nabla}_{x_i}\left[\int d^3\vec{v}\ v_i v_j f\right] = \sum_i \vec{\nabla}_{x,i}\left[n(\sigma_{ij}^2 + \langle v_i\rangle\langle v_j\rangle)\right]$$

Term (C): Integrate by parts and the surface term vanishes because $f \to 0$ as $\vec{v} \to \infty$ still. But this time the $\vec{\nabla}_v$ "sees" the extra copy of $\vec{v}$, so the other term survives this time.

$$\int d^3\vec{v}\ f \sum_i \partial_{v,i}(v_j\,\partial_{x,i}\Phi)$$

$$= \int d^3\vec{v}\ f \sum_i \delta_{ij}\,\partial_{x,i}\Phi = n\,\vec{\nabla}_{x,j}\Phi$$

Our original equation was (A) + (B) + (C) = 0, so . . .

$$n\,\partial_t\langle v_j\rangle + \overbrace{\sum_i \vec{\nabla}_{x,i}\left(n\langle v_i\rangle\langle v_j\rangle\right) - \langle v_j\rangle\sum_i \vec{\nabla}_{x,i}\left(n\langle v_i\rangle\right)}^{\text{Combine using product rule}}$$

$$+\, n\,\vec{\nabla}_{x_j}\Phi + \vec{\nabla}_{x,i}\left(n\sigma_{ij}^2\right) = 0$$

$$\Rightarrow \boxed{\partial_t\langle v_j\rangle + \sum_i \langle v_i\rangle\vec{\nabla}_{x,i}\langle v_j\rangle = -\vec{\nabla}_{x,j}\Phi - \sum_i \frac{\vec{\nabla}_{x,i}\left(n\sigma_{ij}^2\right)}{n}}$$

Bulk acceleration | Velocity shear | Grav. force | Like a pressure!

This is known as the Jeans equation, and it's equivalent to the Euler equation in fluid dynamics. It's the "F = ma" of a fluid, except here we have a "fluid" of stars.

This is a hugely significant result — the final term looks exactly like a differential pressure $\vec{\nabla}P$ term! Somehow, a collisionless, gravitating fluid behaves mathematically just like a fluid with pressure if it has a non-zero velocity dispersion!

This is the justification for thinking of the singular isothermal sphere as having a pressure on your homework.

In some ways this is a shocking result, but with a little more thought its not crazy. Remember that the distribution function behaves like an incompressible fluid. If I squash the "fluid" in the $d^3\vec{x}$ direction, it responds by spreading in the $d^3\vec{v}$ direction & there is a greater spread of velocities in the system. When particles move with different velocities, over time they will end up in different places — more spread out from each other

↳ Ironically, squashing the "fluid" caused it to fight back. This is ~~als~~ exactly what we would colloquially call pressure.

Stability analysis

What does Jeans say about stability, eg. of a gas trying to collapse? Does it collapse or does it bounce back?

Assume the gas consists of particles of mass m, and multiply throughout by m so that $n \to mn = \rho$. Also assume that the velocity dispersion is diagonal and that $\rho\sigma_{ij}^2 \equiv P$, so that our analysis applies equally well to a "gas" with "pressure" due to velocity dispersion or a real gas with thermal pressure.

All gradients from now on are spatial, so drop subscript $\vec{\nabla}_x \to \vec{\nabla}$

$$\partial_t \rho + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$

$$\partial_t \vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}\Phi - \frac{\vec{\nabla}P}{\rho}$$

$$\rho = \rho_0 + \varepsilon\rho_1$$
$$\vec{v} = \vec{v}_0 + \varepsilon\vec{v}_1$$
$$P = P_0 + \varepsilon P_1$$
$$\Phi = \Phi_0 + \varepsilon\Phi_1$$

Small perturbations to see what happens

We assume a static background where $\rho_0 = \text{const.}$, $\vec{v}_0 = 0$, $\Phi_0 = \text{const.}$

Insert our perturbative guess and group in powers of ε:

ε^0 terms: cancel out

ε^1 terms: $\partial_t \rho_1 + \rho_0 \vec{\nabla}\cdot\vec{v}_1 = 0$ and

$$\partial_t \vec{v}_1 = -\vec{\nabla}\Phi_1 - \frac{\vec{\nabla}P_1}{\rho_0}$$

$$\vec{\nabla}P = \frac{\partial P}{\partial \rho}\frac{\partial \rho}{\partial \vec{x}} = c_s^2 \vec{\nabla}\rho$$

c_s: sound speed

Combining these two equations:

$$\partial_t(\text{LHS}) = \partial_t^2 \rho_1 + \rho_0 \vec{\nabla}\cdot\partial_t\vec{v}_1 = \partial_t^2\rho_1 - \rho_0\nabla^2\Phi_1 - c_s^2\nabla^2\rho_1$$

This is not as ~~innocent~~ innocent an assumption as it appears and is actually called the "Jeans Swindle". We are implicitly requiring that $\nabla^2 \Phi_0 = 4\pi G \rho_0$, which cannot ~~hold~~ be true for Φ_0 and ρ_0 both constant unless $\rho_0 = 0$

To simplify, use a Poisson eqn (but perturbed): $\nabla^2 \Phi_1 = 4\pi G \rho_1$

This gives $\boxed{\partial_t^2 \rho_1 - 4\pi G \rho_0 \rho_1 - c_s^2 \nabla^2 \rho_1 = 0}$

This is a wave equation for density perturbations. Plug in a wave ansatz:

$$\rho_1 = \rho_1(0)\, e^{i(wt \pm kx)}, \text{ where } k \equiv 2\pi/\lambda$$

This gives a dispersion relation: $\omega^2 = c_s^2 k^2 - 4\pi G \rho_0$

Note: these ω are "omegas", not the w from earlier in lecture!

For small λ, $c_s k^2 > 4\pi G \rho_0$, so ω is real and $e^{i\omega t}$ is oscillatory

$\Rightarrow$ Stable

For large λ, $c_s k^2 < 4\pi G \rho_0$, so ω is imaginary and $e^{i\omega t}$ is exponential

$\Rightarrow$ Collapse

The boundary case is $k_J^2 = \dfrac{4\pi G \rho_0}{c_s^2}$, $\lambda_J = \dfrac{2\pi}{k_J}$ "Jeans length"

This is the dividing line between a "blob" being large enough to collapse or not.

Intuition: $\lambda_J \sim c_s / \sqrt{G\rho}$

On homework, you will show that collapse (free fall) timescale is $1/\sqrt{G\rho}$. Meanwhile sound / pressure crossing time is $\sim \dfrac{r}{c_s}$.

If object collapse time is faster than pressure can propagate across object, then the system cannot respond in time

$\Rightarrow$ collapse!

Phase mixing and violent relaxation

We've seen that under the right circumstances, a gas cloud can collapse — but how does it then relax into virial equilibrium? After all, when we calculated the relaxation time ~~[illegible]~~ due to stellar interactions (last lecture) we got something absurdly long.

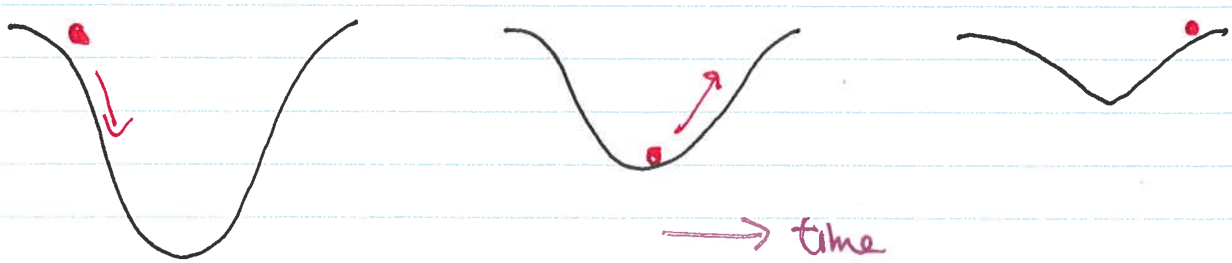
Two important mechanisms are phase mixing and violent relaxation

⇒ $\boxed{\text{Pendulum phase mixing example}}$

What one can see from the pictures is that the phases get more mixed with time and rather than most particles being in a very specific, coherent part of phase space, it's much more likely to find particles in a "typical" trajectory that's allowed, whatever that ensemble of allowed trajectory is — the system has relaxed.

Or we can think of a phase-mixed situation as one where initial conditions have been forgotten

Additionally, the idea of violent relaxation is that during a galaxy formation scenario, the ~~p~~ gravitational potentials ~~[illegible]~~ are evolving with time. This can cause particles to gain or lose energy.



This particle gains energy because it fell into a deep potential well but only had to climb out of a shallow one.

In a typical phase diagram (like our pendulum example), the total energy of a particle is proportional to the ellipse size, so if they can gain or lose energy, we broaden the energy distribution and fill out phase