PHYS644 Problem Set 7

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Problem 1: Warm Dark Matter (WDM)

Problem 1A:

We have our WDM species X, and three three other particles A, B, and C — with associated degeneracy factors g_i where $i = \text{WDM}, A, B, C$. In our case $g_{\text{WDM}} = 2$ - they are Fermions.

We assume X has temperature T_X and has decoupled just before the first annihilation event which heats up the others. A is the first to annihilate and so on, and heats up the rest to $T_{\text{After A}}$ and so on.

We are asked to find an expression to relate T_X and $T_{\text{After A}}$.

We can use the conservation of commoving entropy density $s \propto g_i T^3 a^3$. I assume all species are relativistic, We can write:

$$(g_A + g_b + g_c)T_X^3 a^3 = (g_b + g_c)T_{\text{After A}}^3 a^3 \tag{1}$$

the a's cancel as they are at the same time, This leads to:

$$\boxed{T_{\text{After A,}}^3 = (\frac{g_A + g_b + g_c}{g_b + g_c})T_x^3} \tag{2}$$

Problem 1B:

We use the same reasoning as before for problem A, and state

$$\boxed{T_{\text{After A,B}}^3 = (\frac{g_b + g_c}{g_c})T_{\text{After A}}^3} \tag{3}$$

Problem 1B:

Now we just substitute in our two parts.

$$T_{\text{After A,B}}^3 = (\frac{g_b + g_c}{g_c})(\frac{g_A + g_b + g_c}{g_b + g_c})T_x^3 \tag{4}$$

$$T_{\text{After A,B}}^3 = (\frac{g_A + g_b + g_c}{g_c})T_x^3 \tag{5}$$

It doesn't matter if we did the annihilations sequentially A, and then B or all at once. Only the total degeneracy g before annihilations vs after matters.

Problem 1D:

Problem 1D: Part i

We can use the trick we learned from parts A, B, C[1].

[1] almost a baby name ABCDE!

If X decouples at temperature T_f — where it no longer shares entropy, and then the thermal bath evolves from having degrees of freedom $g_*(T_f) \Rightarrow g_*(T_{\text{new}})$.

We can write the following:

$$g_*(T_f)T^3_{\text{before}} = g_*(T_{\text{new}})T^3_{\text{after}} \tag{6}$$

or more like the handout, where we can write $T_X = T_{\text{before}}$, and

$$\boxed{\frac{T_x}{T_{\text{new}}} = \left(\frac{g_*(T_{\text{new}})}{g_*(T_f)}\right)^{1/3}} \tag{7}$$

Problem 1D Part II

If T_{new} is the point in time just before neutrinos decouple, we can use the relationship from class.

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma \Rightarrow T_\gamma = \left(\frac{11}{4}\right)^{1/3} T_{\text{new}} \tag{8}$$

As just before e^+e^- annihilation, photons, electrons, and positrons, and neutrinos where at the same T.

Now we just toss that into our solution for part I.

$$\frac{T_X}{T_\gamma} = \frac{T_X}{T_{\text{new}}}\frac{T_{\text{new}}}{T_\gamma} = \left(\frac{g_*(T_{\text{new}})}{g_*(T_f)}\right)^{1/3}\left(\frac{4}{11}\right)^{1/3} \tag{9}$$

Cubing both sides

$$\boxed{\left(\frac{T_X}{T_\gamma}\right)^3 = \left(\frac{g_*(T_{\text{new}})}{g_*(T_f)}\right)\left(\frac{4}{11}\right)} \tag{10}$$

We can see that $g_*(T_{\text{new}}) = 10.75$.

Problem 1D Part III

This is an implicit part of the problem. Calculating $g_*(T_{\text{new}}) = 10.75$ by hand.

$$g_* = g_{\text{Bosons}} + \frac{7}{8}g_{\text{Fermions}} \tag{11}$$

We have photons, which are bosons with a factor of 2, and electrons and positrons which are each fermions with a factor of 2 each. and three neutrino species each with a factor of 2 and are fermions.

$$\boxed{g_*(T_{\text{new}}) = 2 + \frac{7}{8}(4+6) = 10.75} \tag{12}$$

Problem 1E:

if X is the dark matter and that it is non-relativistic today, its number density is something we can write down now.

We know:

$$\rho_{X,0} = m_X n_{X,\mathrm{relic}} \tag{13}$$

$$\tag{14}$$

$$\Omega_{\mathrm{DM}} = \frac{\rho_{X,0}}{\rho_{\mathrm{crit},0}} \tag{15}$$

$$\tag{16}$$

$$\rho_{\mathrm{crit}} = \frac{3H_0^2}{8\pi G} = 1.87 \times 10^{-26} h \,\mathrm{kgm}^{-3} \tag{17}$$

So we can write:

$$n_{x,\mathrm{relic}} = \frac{\rho_{X,0}}{m_X} = \frac{\Omega_{\mathrm{DM}}}{m_X}\rho_{\mathrm{crit},0} = \frac{\Omega_{\mathrm{DM}}}{m_X}\frac{3H_0^2}{8\pi G} \tag{18}$$

Big Omega DM is unit-less, and n should have units of number density, we need to restore our missing units from natural units. We can use $H = 100\,\mathrm{Km/s/Mpc}$, and h as the dimensionless correction. We know 1eV =... let's just have `astropy` do this for us.

The factor on the right comes out to be $8.095897552394895 \times 10^{-11}$ when using H instead of H_0.

This gives us

$$\boxed{n_{X,\mathrm{relic}} \simeq 8 \times 10^{-11}\ \mathrm{eV}^3 \left(\frac{1\ \mathrm{eV}}{m_X}\right) \Omega_{\mathrm{DM}} h^2.} \tag{19}$$

Problem 1F:

Recall that y is a conserved quantity

$$y = \frac{n_X}{s_{SM}} \tag{20}$$

where s_{SM} is the entropy of all standard model particles not in the thermal bath that are not apart of our proposed dark matter. y is independent of redshift as both the numerator and denominator scale as a^{-3}.

The entropy density s in natural units is:

$$s = \frac{2\pi^2}{45} g_{*s}(T) T^3 \tag{21}$$

The relativistic relics today are photons and cosmic neutrinos:

$$g_{*s_0} = 2 + \frac{7}{8}\left(6(\frac{T_\nu}{T\gamma})^3\right) = \frac{43}{11} \approx 3.9091 \tag{22}$$

Recall that we know $\frac{T_\nu}{T_\gamma}$, and that there are 3 species of neutrinos each with a factor of 2. Therefore:

$$s_{SM,0} \approx 2.22 \times 10^{-11}\,\mathrm{eV}^3 \tag{23}$$

Now we can just use this result and our previous result.

$$\boxed{y = \frac{n_X}{s_{SM}} = \frac{8 \times 10^{-11}\ \mathrm{eV}^3 \left(\frac{1\ \mathrm{eV}}{m_X}\right) \Omega_{\mathrm{DM}} h^2}{2.22 \times 10^{-11}\,\mathrm{eV}^3} = 3.7 \left(\frac{1\ \mathrm{eV}}{m_X}\right) \Omega_{\mathrm{DM}} h^2} \tag{24}$$

I actually get 3.6 but I probably round differently.

Problem 1G

:

Now we want to evaluate y at the moment when X froze out IE at $T = T_f$.

The X particles are still in equilibrium, and for a relativistic fermion the equilibrium number density is:

$$n_{eq}(T_f) = \frac{3\zeta(3)}{4\pi^2} g_{DM} T_f^3 \tag{25}$$

and we use $g_{DM} = 2$ as we have done for all of the problem. The SM entropy at T_f — excluding X is given by:

$$s_{SM} = \frac{2\pi^2}{45} g_*(T_f) T_f^3 \tag{26}$$

where g_{*T_f} is the effective multiplicity at T_f of the standard model particles. We can evaluate y now at T_f.

$$\boxed{y_f = \frac{n_x}{s_{SM}} = \frac{\frac{3\zeta(3)}{4\pi^2} g_{DM} T_f^3}{\frac{2\pi^2}{45} g_*(T_f) T_f^3} = \frac{135\zeta(3) g_{DM}}{8\pi^4 g_{*T_f}}} \tag{27}$$

Or numerically as:

$$\boxed{y_f \approx \frac{0.417}{g_* T_F}} \tag{28}$$

Problem 1H:

Now we set our two expressions for $y = y$, and get the given equation at the beginning.

$$\frac{0.417}{g_* T_F} = 3.7 \left(\frac{1\ \mathrm{eV}}{m_X}\right) \Omega_{\mathrm{DM}} h^2 \tag{29}$$

This leads to:

$$\Omega_{\mathrm{DM}} h^2 \approx 0.1127 \frac{m_X}{1\mathrm{eV}} \frac{1}{g_*(T_f)} \tag{30}$$

We can replace $g_*(T_f)$ with the relationship between $\frac{T_X}{T_\gamma}$ and $g_*(T_f)$ from Problem 1D.

$$g_{*T_f} = \frac{43}{11} (\frac{T_\gamma}{T_x})^3 \tag{31}$$

Now we can replace our $g_* T_f$.

$$\boxed{\Omega_{\rm DM} h^2 \approx 0.1127(\frac{11}{43})\frac{m_X}{1\rm eV}(\frac{T_X}{T_\gamma})^3} \tag{32}$$

Scary - my numerical factors are different, but alas relax they evaluate out to the same $0.1127(\frac{11}{43}) \approx 0.0288$, and $\frac{1}{94}\frac{11}{4} = 0.02925$ which agree to like two sig figs.

Problem 1I:

LMAO I read this as H part i at first. Low key this problem is annoying because we have to do another sub like in Problem 1H but not. Anyway we set $m_x = 5\,\mathrm{eV}$ for the limiting value.

$$\Omega_{\rm DM} h^2 = \frac{m_x}{94\,\mathrm{eV}}\frac{10.75}{g_{*T_f}} \tag{33}$$

Again this is our solution for H but with a different numerical approx, and now we solve for g_{*T_f}.

$$\boxed{g_{*T_f} = \frac{m_x}{94\,\mathrm{eV}}\frac{10.75}{\Omega_{\rm DM} h^2}} \tag{34}$$

Planck 2018 (thanks `astropy`) says $\Omega_{DM} h^2 = 0.12$, I get.

$$\boxed{g_{*T_f} = 4.7 \times 10^3} \tag{35}$$

Comparing with the plot... $g_* \sim 10^4$ is not on the plot... so WMP seems rather **unplausible given the standard model particles. Unless** there are additional particles discovered.

I feel happy? This is a science question that we can answer —- given the standard model, we can rule out WMP. This is a fun "OOM" problem, and inches us closer to understanding the nature of DM.

Problem 2: Redshift of Reionization

I love optical depth!

Problem 2A:

We can write $n_e = n_{e,0}(1+z)^3$, and $H = H_0\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_\Lambda}$. We are told the values of these Ω_i.

We can write the optical depth as:

$$\tau = \int c n_e(z)\sigma_t \frac{dt}{dz}\,dz \tag{36}$$

Where σ_t is the Thomson-cross section of a free electron - the relevant scattering across section, the integrand $c\sigma_t n_e dt$gives the incremental optical-depth in the correct units.

We can write $n_e = X_e n_B$, relating it to the number of Baryons.

$$\tau = \int c X_e n_{B,0}(1+z)^3\sigma_t \frac{dt}{dz}\,dz \tag{37}$$

We know from class that:

$$\frac{dt}{dz} = -\frac{1}{H(z)(1+z)} \tag{38}$$

We can then write the integral as:

$$\tau = -\frac{c X_e n_{B,0}\sigma_t}{H_0}\int_{z=cmb}^{0}(1+z)^2\frac{1}{\sqrt{0.3(1+z)^3+0.7}}\,dz \tag{39}$$

Now we are ready for `sympy` I get:

$$\int\limits_{z_{reion}}^{0}\frac{(z+1)^2}{\sqrt{0.3\,(z+1)^3+0.7}}\,dz = \frac{20}{9} - \frac{2\sqrt{30\,(z_{reion}+1)^3+70}}{9} \tag{40}$$

so our equation for τ is:

$$\boxed{\tau = -\frac{c X_e n_{B,0}\sigma_t}{H_0}\frac{20}{9}\left(1-\sqrt{0.3(1+z_{reion})^3+0.7}\right)} \tag{41}$$

We know from class that $c n_{B,0}\sigma_t = 5\times 10^{-21}\,\mathrm{s}^{-1}$, and lets use `astropy`'s value of $H_0 = 67.66\,\mathrm{km/s/mpc}$. and lets use $X_e \approx 0.8$.

I get:

$$\boxed{\tau = -0.0040538346\left(1-\sqrt{0.3(1+z_{reion})^3+0.7}\right)} \tag{42}$$

Problem 2B

Plank says $\tau = 0.0540.008.$, lets use $\tau = 0.054$ and solve for z_{reion}.

I solve this numerically and get $\boxed{z_{reion} \approx 7.79}$

This means that roughly z_{reion} happend at a redshift of 8. $=$