PHY644 Problem set 1

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September 4, 2025

Problem 1: Redshifts

This problem concerns determining redshifts using photometric techniques with filters to identify the Lyman break. At the rest-frame Lyman limit, $\lambda_0 = 912\,\text{Å}$[1], the flux drops to nearly zero due to absorption by stellar atmospheres and the interstellar medium (ISM) causing a step like feature. This is the basis of the "dropout technique." However, at high z this feature can be confused with absorption by the intergalactic medium (IGM), which affects the spectrum over the range $912\,\text{Å} < \lambda_0 < 1216\,\text{Å}$. The Lyman series corresponds to electronic transitions in hydrogen where electrons fall to the ground state ($n = 1$).

The photometric technique is inherently less precise than spectroscopic redshifts, which are obtained by taking a full spectrum and fitting actual spectral lines. However, it is much cheaper and faster. This method relies on fitting template spectra to the observed data through the filters used.

A.

We estimate a photometric redshift using relative fluxes (magnitudes) from different filters (Figure). By comparing the template spectrum and filter responses in the video, we find $z \sim 4.6$ (Figure 2).

Filter	Δ mag
b	No Flux
v	1.5
i	0.1
z	0.0

Figure 1: Photometric magnitude differences by filter.

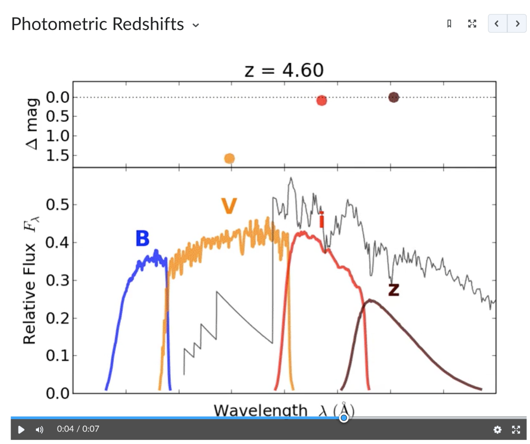

Figure 2: Screen shot of the photometric z simulator, from `https://mycourses2.mcgill.ca/d2l/le/content/802628/viewContent/8637994/View`

[1] where λ_0 is the rest frame wavelength.

B.

This time we are asked to look at Figure 1 of the homework (not reproduced), and and reflect on what redshifts can be most cleanly identified with the Lyman dropout technique before it gets confused with absorption from the IGM.

Using the Lyman break at $\lambda_0 = 1216\,\text{Å}$, and eyeballing the half max response of the filters in the figure, the redshift ranges at which galaxies can be most cleanly identified with the Lyman Break technique are:

$$\begin{aligned}
&\text{U-dropouts:} && z \approx 1.47 \ - \ 2.29 \\
&\text{B-dropouts:} && z \approx 3.11 \quad \text{(B and V overlap at} \sim 5000\ \text{Å)} \\
&\text{V-dropouts:} && z \approx 4.76
\end{aligned}$$

The idea being the "spectrum" is in one image but not the next. In practice, with only the U, B, V, I filters, you can use the dropout technique up to

$$z \sim 4.76 \quad (\text{V} \rightarrow \text{I dropout}).$$

The Lyman break remains inside the I band until

$$z_{\text{max}} \approx \frac{9000}{1216} - 1 \approx 6.40,$$

but a redder filter than I would be required to know where the dropout happens.

Problem 2: The Plummer Potential

The Plummer gravitational potential is:

$$\Phi(r) = \frac{-GM}{(r^2 + r_0^2)^{1/2}} \tag{1}$$

where r is the distance from the center, M is the total mass of the galaxy cluster, r_0 is the characteristic radius, and G is Newton's gravitational constant.

Problem 2A

We are asked to derive $\rho(r)$ — the mass density of the Plummer potential. My idea here is to use Gauss's law for gravity in differential form! It looks like this (It's in Griffith's EM):

$$\nabla \cdot \mathbf{g} = -4\pi G\rho \tag{2}$$

Recall that $\mathbf{g}(r) = -\nabla\Phi(r)$, putting this into Equation 2 we have:

$$\nabla \cdot (-\nabla\Phi) = -4\pi G\rho \quad \Rightarrow \quad \nabla^2\Phi = 4\pi G\rho \tag{3}$$

Recall for a spherically symmetric potential, the Laplacian in spherical coordinates is:

$$\nabla^2\Phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) \tag{4}$$

Now we just need to take a few derivatives and rearrange! I wonder if there is a faster way. Anyway lets start with $\frac{d\Phi}{dr}$:

$$\frac{d\Phi}{dr} = \frac{GMr}{(r^2+r_0^2)^{3/2}} \tag{5}$$

The term in () in Equation 4 is then:

$$\Rightarrow \frac{GMr^3}{(r^2+r_0^2)^{3/2}} \tag{6}$$

Taking the next derivative we get:

$$\Rightarrow GM\frac{3r^2r_0^2}{(r^2+r_0^2)^{5/2}} \tag{7}$$

Tossing in the factor of $1/r^2$, the Laplacian aka Equation 4 is:

$$\nabla^2\Phi = \frac{3GMr_0^2}{(r^2+r_0^2)^{5/2}} \tag{8}$$

Finally, using Equation 3, we solve for $\rho(r)$ - the mass density profile of the Plummer potential.

$$\boxed{\rho(r) = \frac{3Mr_0^2}{4\pi(r^2+r_0^2)^{5/2}}} \tag{9}$$

Factoring the r_0 we get the form from the problem set.

$$\boxed{\rho(r) = \frac{3M}{4\pi r_0^3\left(1+\left(\frac{r}{r_0}\right)^2\right)^{5/2}}} \tag{10}$$

Problem 2B

This problem asks us to find the surface mass density projected onto the sky. After consultation with the TA, this means that we integrate the 3D mass density along the line-of-sight to get a 2D (projected) mass density.

$$\Sigma(R) = \int_{-\infty}^{\infty} \rho\big(R^2+z^2\big)\ dz \tag{11}$$

here z is not redshift, but the line-of-sight distance.

We can relate r, R, and z as $r^2 = R^2 + z^2$. Big R is the projected is the projected radius from the centre of the cluster.

Expressing $\rho\big(R^2+z^2\big)$:

$$\rho = \frac{3M}{4\pi r_0^3}(1+\frac{R^2+z^2}{r_0^2})^{-5/2} \tag{12}$$

Substituting into the integral:

$$\Sigma(R) = \frac{3M}{2\pi r_0^3}\int_0^{\infty}\left(1+\frac{R^2+z^2}{r_0^2}\right)^{-\frac{5}{2}}\ dz \tag{13}$$

Using trusty wolfram alpha as our integral table:

$$\Sigma(R) = \frac{3M}{2\pi r_0^3}\frac{2r_0^5}{3(r_0^2+R^2)^2} \tag{14}$$

Simplifying:

$$\boxed{\Sigma(R) = \frac{M}{\pi r_0^2}\frac{1}{\left[1+\left(\frac{R}{r_0}\right)^2\right]^2}} \tag{15}$$

Problem 2C

This is really 3 problems, but they are fairly simple. We are told to assume the globular cluster is made of identical stars and that the cluster contains no dark matter, and use units of r_0.

2C I

We are asked to compute the Core radius r_c defined as where the surface density falls to half its central value. This is simply the condition of $\Sigma(r_c) = \frac{1}{2}\Sigma(0)$.

$$\frac{M}{\pi r_0^2}\frac{1}{\left[1+\left(\frac{r_c}{r_0}\right)^2\right]^2} = \frac{1}{2}\frac{M}{\pi r_0^2}\frac{1}{\left[1+\left(\frac{0}{r_0}\right)^2\right]^2} \tag{16}$$

Lots of things cancel!

$$\Rightarrow \frac{1}{\left[1+\left(\frac{r_c}{r_0}\right)^2\right]^2} = \frac{1}{2} \tag{17}$$

$$\Rightarrow \left[1+\left(\frac{r_c}{r_0}\right)^2\right]^2 = 2 \tag{18}$$

$$\Rightarrow 1+\left(\frac{r_c^2}{r_0^2}\right) = \sqrt{2} \tag{19}$$

$$\boxed{r_c = r_0((\sqrt{2}-1))^{0.5}} \approx 0.64 r_0 \tag{20}$$

Problem 2C II

Now we calculate the Half-light radius, within which half of the total light is projected. I'm going to call this $H_{0.5}$ (H for half?). (we are assuming 0 extinction as well).

Assuming globular cluster is made of identical stars and that the cluster contains no dark matter, and use units of r_0. We can relate the $\Sigma(R)$ to surface brightness density $B(R)$.

The total mass M is given by $M = Nm$, where N is the number of stars and m is the mass of the stars (in our case a constant). We also know that mass-to-luminosity ratio is a constant either $k = \frac{m}{l}$ per stare or $K = \frac{M}{L}$ for the cluster.

Multiplying Σ by kL should give us the luminosity density. Aka we just replace M with KL in Σ or σ. Then we integrate to $H_{0.5}$.

$$B(R) = \frac{KL}{\pi r_0^2} \frac{1}{\left[1 + \left(\frac{R}{r_0}\right)^2\right]^2} \tag{21}$$

Now we integrate $B(R)$ 0 to 2 π, and over R. $da = RdRd\theta$ - the surface of our projected circle blob.

$$\frac{K}{2} = \frac{2KL}{r_0^2} \int_0^{H_{0.5}} R \frac{1}{\left[1 + \left(\frac{R}{r_0}\right)^2\right]^2} \, dR \tag{22}$$

Using wolfram alpha:

$$\frac{K}{2} = \frac{2K}{r_0^2} \frac{H_{0.5}^2 r_0^2}{2(r_0^2 + H_{0.5}^2)} \tag{23}$$

Now lets go cancel a lot of things

$$\Rightarrow \frac{1}{2} = \frac{H_{0.5}^2}{(r_0^2 + H_{0.5}^2)} \tag{24}$$

$$\Rightarrow r_0^2 + H_{0.5}^2 = 2H_{0.5}^2 \tag{25}$$

$$\Rightarrow r_0^2 = 2H_{0.5}^2 - H_{0.5}^2 = H_{0.5}^2 \tag{26}$$

$$\boxed{H_{0.5} = r_0} \tag{27}$$

The Half-Right radius is the r_0 characteristic radius!

Problem 2C III

Half-mass radius, the radius of the 3D sphere which contains half the cluster's total mass. This is very similar to the previous part, we use the same integral method but on $\rho(r)$, and relate it to $\frac{M}{2}$. Let's use r_h for the half-mass radius:

$$\frac{M}{2} = \int_0^{r_h} 4\pi r^2 \rho(r) \, dr, \tag{28}$$

the 4π comes from being spherically symmetrical.

Plugging in our $\rho(r)$:

$$\frac{M}{2} = \int_0^{r_h} 4\pi r^2 \frac{3Mr_0^2}{4\pi(r^2 + r_0^2)^{5/2}} \, dr, \tag{29}$$

Moving stuff to make it easier to use wolfram alpha:

$$\frac{1}{2} = 3 \int_0^{r_h} r^2 \frac{r_0^2}{(r^2 + r_0^2)^{5/2}} \, dr, \tag{30}$$

Now using wolfram alpha:

$$\frac{1}{2} = 1 - \frac{1}{\left(1 + \frac{r_h^2}{r_0^2}\right)^{3/2}} \tag{31}$$

Simplifying:

$$\boxed{r_h = \frac{r_0}{\sqrt{2^{2/3} - 1}} \approx 0.76 r_0} \tag{32}$$

Problem 3: Surface Brightness

Ah this is a classic, and counter intuitive derivation in any atmospheres & radiative transfers class. (Which I have taken). The answer is that there is a $1/r^2$ factor from the flux, there is also a factor in angular size which precisely cancel.

I'm gonna do the full derivation + a diagram because I enjoy this problem.

the specific surface brightness aka specific intensity is the energy emitted per unit time per unit area per unit frequency per unit solid angle:

$$I_\nu = \frac{dE}{(\cos\theta dA)dtd\nu d\Omega} \tag{33}$$

Where I_ν is the specific intensity, dE is energy emitted, $\cos\theta dA$ is the projected 2D area (see Figure 3for geometry), dt is the per time, and $d\nu$ is per frequency, and $d\Omega$ is our solid angle.

Let's have two people (or the sun and a small solar panel if you will), the question is if they agree on the specific intensity:

$$I_\nu \stackrel{?}{=} I_\nu^{'} \tag{34}$$

$$\frac{dE}{(\cos\theta dA)dtd\nu d\Omega} \stackrel{?}{=} \frac{dE^{'}}{(\cos\theta' dA')dt'd\nu' d\Omega'} \tag{35}$$

Due to symmetry, the solid angle, and θ's cancel Assuming classical physics, dt, and $d\nu$ are also the same for both people[2].

$$\boxed{dE = dE^{'}} \tag{36}$$

therefor, specific intensity is conserved (does not vary with distance).

Problem 4: The Tully-Fisher Relation

the Tully-Fisher relation, which states that the luminosity L of a spiral galaxy is roughly proportional to the circular velocity v_c of stars to the fourth power:

$$L \propto v_c^4. \tag{37}$$

This used to be used to infer distances on the cosmic distance ladder until better techniques arised, the Tully-Fisher relation has a $\sim 10\%$ scatter.

Problem 4A

The Virial Theorem states $2K + V = 0$, where K, and V are the kinetic and potential energy respectfully.

In our case we can write:

$$Mv_c^2 + \frac{\alpha GM^2}{R} = 0 \tag{38}$$

here, M is the mass the galaxy, and R is the distance from the centre, α is some correction factor.

We can rewrite this as:

$$v_c^2 \propto \frac{M}{R} \tag{39}$$

[2] $d\Omega = \frac{cos\theta^{'} da^{'}}{r^2}$, and $d\Omega^{'} = \frac{cos\theta da}{r^2}$, where r is the distance between the two people.

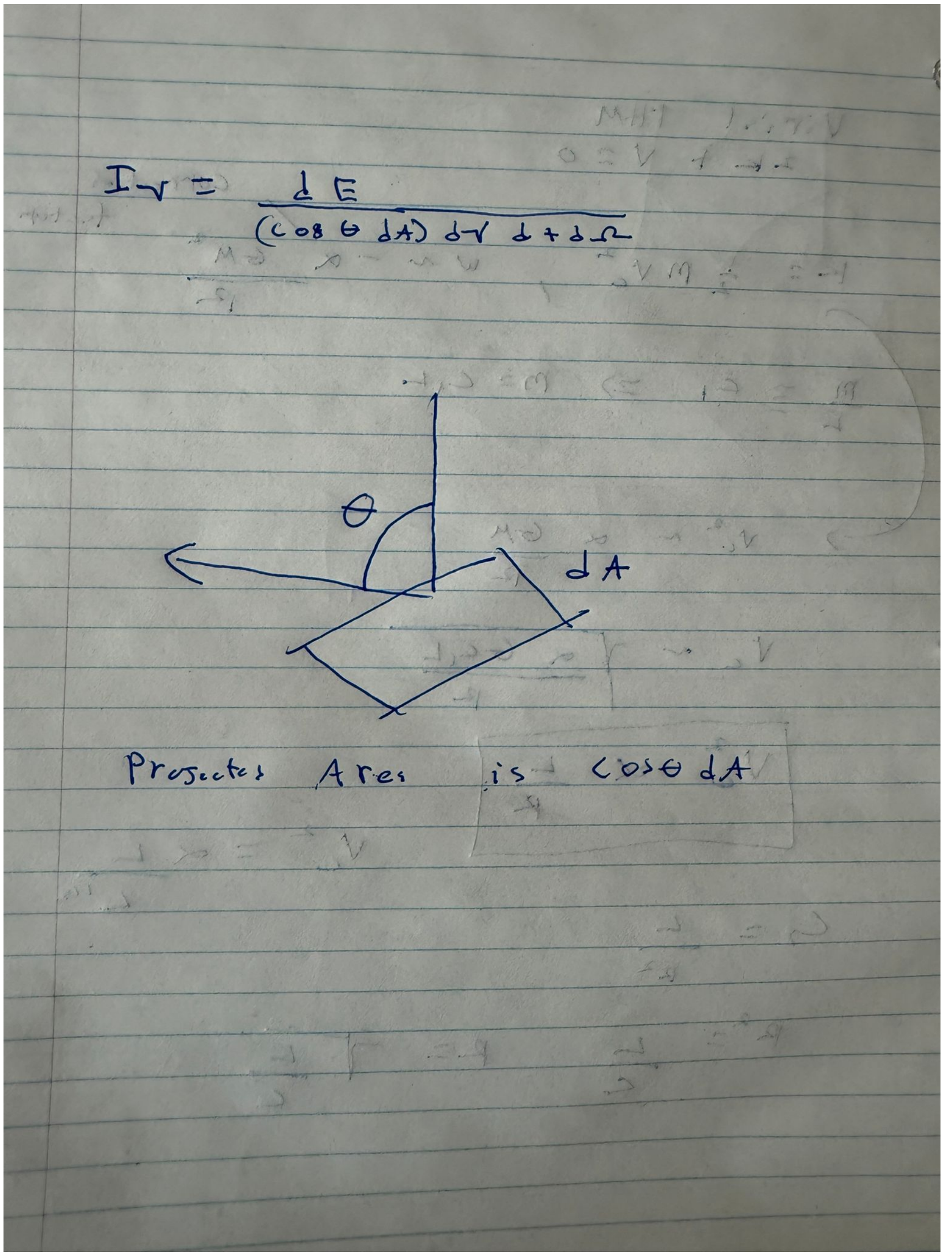

Figure 3: Geometry of specific intensity.

For the case of a constant mass-luminosity ratio ($M/L = c_1$), we can replace M in terms of L.

$$\boxed{v_c^2 \propto \frac{L}{R}} \tag{40}$$

Problem 4B

Here we now also assume constant surface brightness $c_2 = \frac{L}{A}$, where A is area. We can write $A = \alpha R^2$, where again α is some correction factor. Using our solution to problem 2A equation 40, we can write:

$$R \propto L^{1/2} \tag{41}$$

and subsisting into equation 40,

$$v_c^2 \propto L^{1/2} \tag{42}$$

rewriting:

$$\boxed{L \propto v_c^4} \tag{43}$$

Problem 4C

Assumptions made in this problem were 1) Virial Theorem holds, 2) constant Mass-to-Luminosity ratio, 3) constant surface brightness, 4) no darkmatter.

1) Virtualisation holds for bound systems like some atoms, globular cluster, (closed) planetary orbits. and for galaxies by the defintion given in class.

2) is unrealistic unless there is no active star-formation. We know that for main sequence stars there $L \sim M^\alpha$, with $\alpha = 3.5$ - a power-law not a constant.

3) unrealistic, galaxies are brighter at their center and dimmer at their edges

4) We never invoked the existence of Dark matter into our analysis, at least to my understanding the virtualisation I used is for normal matter.