PHYS644 Problem set 3

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Problem 1: Density of a Self-Gravitating Disk

Here we consider an infinite disk of stars of identical mass, m_*, in the xy plane. Assume the stars are in equilibrium (their phase space is in steady state).

0.1 Problem 1A

The Jeans equation from class is:

$$\partial_t \langle \vec{v_j} \rangle + \sum_i \langle \vec{v_i} \rangle \vec{\nabla}_{x,i} \langle \vec{v_j} \rangle = -\vec{\nabla}_{x,j}\Phi - \sum_i \frac{\vec{\nabla}_{x,i}(n\sigma_{ij}^2)}{n} \tag{1}$$

From left to right, we label the terms as - Bulk accretion, velocity sheer, grav force, and pressure. We are asked to find $n(z)$ in terms of the velocity dispersion in the $\hat{z}$ direction σ_z^2, Φ, and midplane density $n(0)$.

Since we are in a steady state, $\partial_t \langle \vec{v_z} \rangle = 0$

In $\hat{z}$ we have:

$$0 = -\frac{d\Phi}{dz} - \frac{1}{n}\frac{d(n\sigma_z^2)}{dz} \tag{2}$$

This looks like a straight forward differential equation, let's attack it.

$$\frac{d(n\sigma_z^2)}{dz} = -n\frac{d\Phi}{dz} \tag{3}$$

$$\frac{1}{n\sigma_z^2}\frac{d(n\sigma_z^2)}{dz} = -\frac{1}{\sigma_z^2}\frac{d\Phi}{dz} \tag{4}$$

Switching to $\ln n\sigma_z^2$:

$$\frac{d\ln{(n\sigma_z^2)}}{dz} = -\frac{1}{\sigma_z^2}\frac{d\Phi}{dz} \tag{5}$$

Now we integrate both sides from 0 to z.

$$\boxed{\ln(\frac{n(z)\sigma_z^2}{n(0)\sigma_0^2}) = -\int_0^\infty \frac{1}{\sigma_z^2}\frac{d\Phi}{dz}dz} \tag{6}$$

We can rearrange and solve for $n(z)$ but it look a bit ugly

$$\boxed{n(z) = n(0)\frac{\sigma_z^2(0)}{\sigma_z^2(z)}\exp(-\int_0^\infty \frac{1}{\sigma_z^2}\frac{d\Phi}{dz}dz)} \tag{7}$$

Problem 1B

In the case of an isothermal gas, and assuming $\sigma_z^2 = C$ a constant in z, and setting $\Phi(0) = 0$.
The right hand side in 7 simplifies:

$$\boxed{n(z) = n(0)e^{-\frac{\Phi(z)}{\sigma_z^2}}} \tag{8}$$

Interpreting this as a thermal equilibrium (Boltzmann) distribution for a "gas" of particles of mass m_*, the velocity dispersion plays the role of the thermal kinetic energy per unit mass. The effective temperature T is given by:

$$\frac{1}{2}m_*\langle v^2\rangle \sim \frac{1}{2}m_*\sigma_z^2 \sim \frac{1}{2}k_bT \tag{9}$$

So the temperature of the gas is given by:

$$\boxed{T = \frac{m_*\sigma_z^2}{k_b}} \tag{10}$$

Problem 1C

Now use the Poisson equation to solve for $\Phi(z)$.

$$\nabla^2\Phi = 4\pi G\rho \tag{11}$$

With our given by $\rho = m_*n(\vec{R})$, since the system is uniform in the xy plane

$$\frac{d^2\Phi}{dz^2} = 4\pi Gm_*n(z) \tag{12}$$

Let's attack this!

$$\frac{d^2\Phi}{dz^2} = 4\pi Gm_*n_0e^{-\frac{\Phi(z)}{\sigma_z^2}} \tag{13}$$

Let's redefine the part in the exponent to be:

$$\aleph = \frac{\Phi}{\sigma_z^2} \tag{14}$$

$$\frac{d^2\aleph}{dz^2} = \frac{4\pi Gm_*n_0}{\sigma_z^2}e^{-\aleph} \tag{15}$$

We recognize the scale height as $\boxed{h^2 = \frac{\sigma_z^2}{2\pi Gm_*n_0}}$

$$\frac{d^2\aleph}{dz^2} = \frac{2}{h^2}e^{-\aleph} \tag{16}$$

$$\frac{d^2\aleph}{dz^2}\frac{d\aleph}{dz} = \frac{2}{h^2}e^{-\aleph}\frac{d\aleph}{dz} \tag{17}$$

Integrate:

$$\frac{1}{2}(\frac{d\aleph}{dz})^2 = -\frac{2}{h^2}e^{-\aleph} + c \tag{18}$$

at $z = 0$, we expect $\frac{d\aleph}{dz} = 0$ due to symmetry, and we have $\aleph(0) = 0$.
Therefore:

$$0 = -\frac{2}{h^2} + c \Rightarrow c = \frac{2}{h^2} \tag{19}$$

Throwing back into equation 18 we have:

$$\frac{d\aleph}{dz} = \frac{2}{h}\sqrt{1 - e^{-\aleph}} \tag{20}$$

The anti-derivative of this is sech (from a table).

$$\boxed{n(z) = n_0 \text{sech}^2(\frac{z}{2h})} \tag{21}$$

with $\boxed{h^2 = \frac{\sigma_z^2}{2\pi G m_* n_0}}$