

PHYS644 Problem set 3

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Problem 1: Density of a Self-Gravitating Disk

Here we consider an infinite disk of stars of identical mass, m_* , in the xy plane. Assume the stars are in equilibrium (their phase space is in steady state).

Problem 1A

The Jeans equation from class is:

$$\partial_t \langle \vec{v}_j \rangle + \sum_i \langle \vec{v}_i \rangle \vec{\nabla}_{x,i} \langle \vec{v}_j \rangle = -\vec{\nabla}_{x,j} \Phi - \sum_i \frac{\vec{\nabla}_{x,i} (n \sigma_{ij}^2)}{n} \quad (1)$$

From left to right, we label the terms as - Bulk accretion, velocity sheer, grav force, and pressure. We are asked to find $n(z)$ in terms of the velocity dispersion in the \hat{z} direction σ_z^2 , Φ , and midplane density $n(0)$.

Since we are in a steady state, $\partial_t \langle \vec{v}_z \rangle = 0$, and the sum on the LHS is also zero (either the average velocity is 0 or the gradient is 0 because of the i, j pairing).

In \hat{z} we have:

$$0 = -\frac{d\Phi}{dz} - \frac{1}{n} \frac{d(n\sigma_z^2)}{dz} \quad (2)$$

This looks like a straight forward differential equation, let's attack it.

$$\frac{d(n\sigma_z^2)}{dz} = -n \frac{d\Phi}{dz} \quad (3)$$

$$\frac{1}{n\sigma_z^2} \frac{d(n\sigma_z^2)}{dz} = -\frac{1}{\sigma_z^2} \frac{d\Phi}{dz} \quad (4)$$

Switching to $\ln n\sigma_z^2$:

$$\frac{d \ln(n\sigma_z^2)}{dz} = -\frac{1}{\sigma_z^2} \frac{d\Phi}{dz} \quad (5)$$

Now we integrate both sides from 0 to z .

$$\ln\left(\frac{n(z)\sigma_z^2}{n(0)\sigma_0^2}\right) = -\int_0^z \frac{1}{\sigma_z^2} \frac{d\Phi}{dz} dz \quad (6)$$

We can rearrange and solve for $n(z)$ but it look a bit ugly

$$n(z) = n(0) \frac{\sigma_z^2(0)}{\sigma_z^2(z)} \exp\left(-\int_0^z \frac{1}{\sigma_z^2} \frac{d\Phi}{dz} dz\right) \quad (7)$$

Problem 1B

In the case of an isothermal gas, and assuming $\sigma_z^2 = C$ a constant in z , and setting $\Phi(0) = 0$.

The right hand side in 7 simplifies:

$$\boxed{n(z) = n(0)e^{-\frac{\Phi(z)}{\sigma_z^2}}} \quad (8)$$

Interpreting this as a thermal equilibrium (Boltzmann) distribution for a “gas” of particles of mass m_* , the velocity dispersion plays the role of the thermal kinetic energy per unit mass. The effective temperature T is given by:

$$\frac{1}{2}m_*\langle v^2 \rangle \sim \frac{1}{2}m_*\sigma_z^2 \sim \frac{1}{2}k_b T \quad (9)$$

So the temperature of the gas is given by:

$$\boxed{T = \frac{m_*\sigma_z^2}{k_b}} \quad (10)$$

Problem 1C

Now use the Poisson equation to solve for $\Phi(z)$.

$$\nabla^2 \Phi = 4\pi G \rho \quad (11)$$

With our given by $\rho = m_* n(\vec{R})$, since the system is uniform in the xy plane

$$\frac{d^2 \Phi}{dz^2} = 4\pi G m_* n(z) \quad (12)$$

Let's attack this!

$$\frac{d^2 \Phi}{dz^2} = 4\pi G m_* n_0 e^{-\frac{\Phi(z)}{\sigma_z^2}} \quad (13)$$

Let's redefine the part in the exponent to be:

$$\aleph = \frac{\Phi}{\sigma_z^2} \quad (14)$$

This is the Hebrew letter Aleph - because they do not have elvish.

$$\frac{d^2 \aleph}{dz^2} = \frac{4\pi G m_* n_0}{\sigma_z^2} e^{-\aleph} \quad (15)$$

We recognize the scale height as $\boxed{h^2 = \frac{\sigma_z^2}{8\pi G m_* n_0}}$

$$\frac{d^2 \aleph}{dz^2} = \frac{1}{2h^2} e^{-\aleph} \quad (16)$$

$$\frac{d^2 \aleph}{dz^2} \frac{d\aleph}{dz} = \frac{1}{2h^2} e^{-\aleph} \frac{d\aleph}{dz} \quad (17)$$

Integrate:

$$\frac{1}{2} \left(\frac{d\aleph}{dz} \right)^2 = -\frac{1}{2h^2} e^{-\aleph} + c \quad (18)$$

at $z = 0$, we expect $\frac{d\aleph}{dz} = 0$ due to symmetry, and we have $\aleph(0) = 0$.

Therefore:

$$0 = -\frac{1}{2h^2} + c \Rightarrow c = \frac{1}{2h^2} \quad (19)$$

Throwing back into equation 18 we have:

$$\frac{d\aleph}{dz} = \frac{1}{h} \sqrt{1 - e^{-\aleph}} \quad (20)$$

The anti-derivative of this is sech (from a table).

$$\boxed{n(z) = n_0 \text{sech}^2\left(\frac{z}{2h}\right)} \quad (21)$$

with $\boxed{h^2 = \frac{\sigma_z^2}{8\pi G m_* n_0}}$

Problem 1D

Let $h = 300$ pc and $\sigma_z = 20$ km/s. We are asked to calculate the surface density $\Sigma = \int \rho(z) dz$

Since we know $n(z)$ is directly proportional to $\rho = m_* n$, we know $\rho(z)$

$$\rho(z) = \rho_0 \text{sech}^2\left(\frac{z}{2h}\right) \quad (22)$$

Now we integrate

$$\Sigma = \int_{-\infty}^{+\infty} \rho_0 \text{sech}^2\left(\frac{z}{2h}\right) dz \quad (23)$$

From wolfram alpha, we know $\int \text{sech}^2 = \tanh$.

$$\Sigma = 4\rho_0 h \tanh\left(\frac{z}{2h}\right) \Big|_0^{+\infty} = 4h\rho_0 \quad (24)$$

We can relate ρ_0 and h to σ_z . $\rho_0 = \frac{\sigma_z^2}{8\pi G h^2}$

$$\boxed{\Sigma = \frac{8\sigma_z^2}{\pi G h}} \quad (25)$$

Now we can fill in our numbers!

$$\boxed{\Sigma = 800} \text{ M}_\odot / \text{pc}^2 \quad \boxed{\Sigma = 0.20} \text{ g} / \text{cm}^2$$

Problem 2: Spherical Tophat Model

Problem 2A:

We are asked to solve this ODE for $\rho = f(t)$:

$$\frac{1}{2} \left(\frac{dr}{dt} \right)^2 - \frac{GM}{r} = \epsilon \quad (26)$$

with $\epsilon = \frac{E}{m}$, in our universe a good model is that $\epsilon = 0$ — it has just enough energy to escape.

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{GM}{r} = 0 \quad (27)$$

$$r\left(\frac{dr}{dt}\right)^2 = 2GM \quad (28)$$

$$r^{1/2}\frac{dr}{dt} = (2GM)^{1/2} \quad (29)$$

$$r^{1/2}dr = (2GM)^{1/2}dt \quad (30)$$

Integrate both sides from 0 to r , and 0 to t

$$\frac{2}{3}r^{\frac{3}{2}} = [2GM]^{\frac{1}{2}}t \quad (31)$$

$$r^3 = \frac{9}{2}GMt^2 \quad (32)$$

We now replace $M = \frac{4}{3}\pi\rho r^3$

$$r^3 = \frac{9}{2}G\frac{4}{3}\pi\rho r^3t^2 \quad (33)$$

$$1 = 6G\pi\rho t^2 \quad (34)$$

$$\boxed{\rho = \frac{1}{6G\pi t^2}} \quad (35)$$

Problem 2B

Negative.

$\epsilon = \frac{E}{m}$ is the total energy per unit mass. In the first case ρ_{bg} case we had $\epsilon = 0$ aka just enough energy to escape to infinity. An overdensity that will cause there is be more gravity therefore, it will eventually stop expanding and collapse because it has slightly more attraction, it is gravitationally bound. Because it is gravitationally bound its total energy is negative.

The energy equation: at turnaround $\dot{r} = 0$ so

$$\epsilon = -\frac{GM}{r_{\max}} < 0, \quad (36)$$

hence ϵ must be negative for a region that will turn around and collapse into a halo.

Problem 2C

These look like cycloid parametric equations, let's go at it. We can start from the same starting point as last time, but with $\epsilon \neq 0$, we want to use $\epsilon < 0$. We say $\epsilon = -|\epsilon|$. The total energy of the system (per unit mass) is a constant which we can use to help solve the ODE.

Then we have, I have switched to using \dot{r} , because it is easier to type.

$$\frac{1}{2}\dot{r}^2 - \frac{GM}{r} = \epsilon = -|\epsilon| \implies \dot{r}^2 = \frac{2GM}{r} - 2|\epsilon|. \quad (37)$$

We define the constant A which just makes things easier to look at.

$$A \equiv \frac{GM}{2|\epsilon|} \quad (38)$$

We have an ansatz of:

$$\begin{aligned} r(\eta) &= A(1 - \cos \eta) \\ t(\eta) &= \frac{GM}{\sqrt{8|\epsilon|^3}}(\eta - \sin \eta) = \frac{A}{\sqrt{2|\epsilon|}}(\eta - \sin \eta) \end{aligned} \quad (39)$$

This is motivated by the given solution, cough - I mean a parametric equation for a cycloid.

Taking the first derivative we have:

$$\frac{dr}{d\eta} = A \sin \eta. \quad (40)$$

Now we substitute in A , and $\frac{dr}{d\eta}$, equation 39 into equation 37

$$\dot{r}^2 = 2|\epsilon| \left(\frac{2}{1 - \cos \eta} - 1 \right) = 2|\epsilon| \frac{1 + \cos \eta}{1 - \cos \eta} \quad (41)$$

We can write \dot{r} :

$$\dot{r} = [2|\epsilon| \frac{1 + \cos \eta}{1 - \cos \eta}]^{1/2} \quad (42)$$

Using the chain rule we can compute:

$$\frac{dt}{d\eta} = \frac{dt}{dr} \frac{dr}{d\eta} = \frac{dr}{d\eta} \dot{r}^{-1} \quad (43)$$

$$\frac{dt}{d\eta} = \frac{A \sin \eta}{[2|\epsilon| \frac{1 + \cos \eta}{1 - \cos \eta}]^{1/2}} = \frac{A}{\sqrt{2|\epsilon|}} \sin \eta \sqrt{\frac{1 - \cos \eta}{1 + \cos \eta}}. \quad (44)$$

Look at the factor in the front! it is the same as our ansatz! We are on the right track, we can use the trigonometric identity (thanks wolfram alpha)

$$\sin \eta \sqrt{\frac{1 - \cos \eta}{1 + \cos \eta}} = 1 - \cos \eta \quad (45)$$

And rewrite as:

$$\frac{dt}{d\eta} = \frac{A}{\sqrt{2|\epsilon|}} (1 - \cos \eta). \quad (46)$$

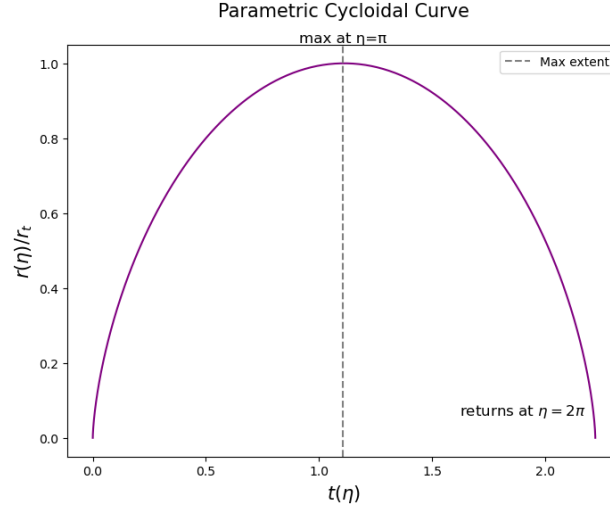


Figure 1: Plot of our parametric cycloid equations with r_{\max} labelled in terms of η , and when it collapses labelled as well. code to reproduce this is on the github repo https://github.com/afinemax/mcgill_phy644_galaxies_cosmology/tree/main/homework/max_fine_phy644_hw3

Integrate from $\eta = 0$, where $t = 0$ to general η :

$$t(\eta) = \frac{A}{\sqrt{2|\epsilon|}}(\eta - \sin \eta) \quad (47)$$

Finally substitute back $A = GM/(2|\epsilon|)$ to express the solution.

$$\begin{aligned} r(\eta) &= \frac{GM}{2|\epsilon|}(1 - \cos \eta) \\ t(\eta) &= \frac{GM}{\sqrt{8|\epsilon|^3}}(\eta - \sin \eta) \end{aligned} \quad (48)$$

These are the standard cycloidal parametric equations for a bound shell with $r(0) = 0$, $t(0) = 0$, turnaround at $\eta = \pi$ (where $r_{\max} = 2A = GM/|\epsilon|$), and recollapse at $\eta = 2\pi$. A plot of this is shown in 1

Problem 2D

We are asked to relate the maximum radius of this blob r_t (turn around), to r_f which is when the blob has collapsed back into a visualized blob.

This is a first-year conservation of energy question in disguise!

At r_t , Kinetic energy $T = 0$, and the potential is:

$$E = T_t + U_t = -\frac{GM^2}{r_t} \quad (49)$$

where the subscript of little t means turn around, E is the total energy.

After it collapses back and is virialized we know:

$$2T_f + U_f = 0 \quad (50)$$

And we know that the energy of the system is a constant E so we can write:

$$E = -\frac{1}{2}U_f + U_f = \frac{1}{2}U_f = U_t \quad (51)$$

$$E = -\frac{2GM^2}{r_t} = -\frac{GM^2}{r_f} \quad (52)$$

Meaning that

$$\boxed{r_f = \frac{1}{2}r_t} \quad (53)$$

Problem 2E

We will use our solutions to problem 2 D, A, and C equations 48, 53, 35.

We know: $r_t = 2A$ and $r_f = A$. The collapse happens when $\eta = 2\pi \Rightarrow t_{\text{col}} = \frac{2\pi A^{3/2}}{\sqrt{GM}}$.

Solving for A^3 we have:

$$A^3 = \frac{GMt_{\text{col}}^2}{4\pi^2} \quad (54)$$

We can write the Mass M as

$$M = \frac{3}{4}\rho_f r_f^3 = \frac{3}{4}\rho_f A^3 \quad (55)$$

Now we set $A^3 = A^3$

$$1 = \frac{G\rho_f t_{\text{col}}^2}{3\pi} \quad (56)$$

now we have ρ_f :

$$\rho_f = \frac{3\pi}{Gt_{\text{col}}^2} \quad (57)$$

Now we use our equation 35 $\rho_{bg} = \frac{1}{6\pi Gt^2}$

$$\frac{\rho_f}{\rho_{bg}} = \frac{\frac{3\pi}{Gt_{\text{col}}^2}}{\frac{1}{6\pi Gt_{\text{col}}^2}} \quad (58)$$

We can do some cancellations:

$$\boxed{\frac{\rho_f}{\rho_{bg}} = \frac{\frac{3\pi}{Gt_{\text{col}}^2}}{\frac{1}{6\pi Gt_{\text{col}}^2}} = 18\pi^2 \approx 200} \quad (59)$$