PHY644 Problem set 2

Maxwell A. Fine: SN 261274202
maxwell.fine@mail.mcgill.ca
September 11, 2025

Problem 1: Free-fall Time

We are asked to derive the true free fall time t_{ff} of presser-less dust ball of uniform density ρ collapsing.

The total mass of the sphere is :

$$M = \frac{4}{3}\pi\rho r_0^3 \tag{1}$$

where M is the total mass, and r_0 is the initial radius (max radius). The t_{ff} is the time it takes for a test mass on the surface to fall to the centre.

We know from Guass's law for gravity that this problem is the equivalent of asking how long it takes for a test mass to fall into the body it is orbiting.

From Kepler's 3rd law we know that:

$$\frac{P^2}{a^3} = \frac{4\pi^2}{G(M+u)} \tag{2}$$

Where P is the period of the orbit, a is the semi-major axis, M mass of the large body and u is our test mass. $M >> u$, so we can neglect u. This is true regardless of the eccentricity (e) of the orbit, for the special case of $e = 1$ (meaning $a = \frac{1}{2}r_0$), the orbit is a line. The t_{ff} is then interpreted as $\frac{1}{2}P$. Substituting into equation 2:

$$\frac{t_{ff}^2}{r_0^3} = \frac{\pi^2}{8GM} \tag{3}$$

$$t_{ff}^2 = \frac{\pi^2 r_0^3}{8GM} \tag{4}$$

$$t_{ff} = (\frac{\pi^2 r_0^3}{8GM})^{\frac{1}{2}} \tag{5}$$

Now we replace M with equation 1.

$$t_{ff} = (\frac{\pi^2 r_0^3}{8G\frac{4}{3}\pi\rho r_0^3})^{\frac{1}{2}} \tag{6}$$

Lots of things cancel!

$$t_{ff} = (\frac{3\pi}{32G\ \rho})^{\frac{1}{2}} \tag{7}$$

We can factor out a $\frac{1}{\sqrt{16}}$

$$\boxed{t_{ff} = \frac{1}{4}\sqrt{\frac{3\pi}{2G\ \rho}}} \tag{8}$$

The hard way is using conservation of energy, which I will also do. I am assuming that energy conservation holds, for a test mass at the edge of the surface

$$E = \frac{1}{2}v_0^2 - \frac{GM}{r_0} = \frac{1}{2}v(r)^2 - \frac{GM}{r} \tag{9}$$

where E is a constant, and this is the per unit mass energy. We take v_0 to be 0.

We can rearrange for $v(r)$:

$$v(r)^2 = 2GM(\frac{1}{r} - \frac{1}{r_0}) \tag{10}$$

now we have a first order differential equation:

$$\frac{dr}{dt} = -[2GM(\frac{1}{r} - \frac{1}{r_0})]^{0.5} \tag{11}$$

with the same initial conditions, the $-$ comes from falling inwards.

$$-[2GM(\frac{1}{r} - \frac{1}{r_0})]^{-0.5}dr = dt \tag{12}$$

The integral bounds are from r_0 to 0 on the left hand side and from 0 to t_{ff} on the right hand side

$$\int_{r_0}^{0} -\left[2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)\right]^{-1/2} dr = \int_0^{t_{ff}} dt \tag{13}$$

$$\int_0^{r_0} \left[2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)\right]^{-1/2} dr = t_{ff} \tag{14}$$

~~Now we use our integral table aka wolfram alpha (it looks like a u and then trig sub).~~

Before we can use an integral table, we need to simplify more, let $u = \frac{r}{r_0}$, $du = \frac{1}{r_0}dr$.

$$\int_0^1 [2GM(\frac{1-u}{ur_0})]^{-1/2}\frac{1}{r_0}du = t_{ff} \tag{15}$$

Problem 2: Virial Theorem

We are asked to show that the Virial theorem is:

$$U + 2T = \frac{1}{2}\frac{d^2I}{dt^2}, \tag{16}$$

where U is the potential energy, T is the kinetic energy, and I is the moment of inertia $I = \sum_i m_i r_i^2$, where r is the position, and m is the mass of each particle.

We are given a hint to use G:

$$G = \sum_i \boldsymbol{P}_i \cdot \boldsymbol{r}_i, \tag{17}$$

where $\boldsymbol{P}$ is the momentum of each particle.

G, and I are related:

$$G = \sum_i \boldsymbol{P}_i \cdot \boldsymbol{r}_i = \sum_i m_i \boldsymbol{v}_i \cdot r_i = \frac{1}{2}\sum_i m_i \frac{d}{dt}(r_i^2) = \frac{1}{2}\frac{dI}{dt} \tag{18}$$

Now lets take a second $\frac{d}{dt}$ of G, we need to find the Left hand side of the following equation and we are done.

$$\frac{dG}{dt} = \frac{d}{dt}(\frac{1}{2}\frac{dI}{dt}) \tag{19}$$

Let's start:

$$\frac{dG}{dt} = \frac{d}{dt}\sum_i m_i \boldsymbol{v}_i \cdot r_i = \sum_i F_i \cdot r_i + \sum_i m_i v_i^2 = \sum_i F_i \cdot r_i + 2T \tag{20}$$

$\sum_i F_i \cdot r_i = U$ is the definition of potential energy!
therefor:

$$\boxed{U + 2T = \frac{1}{2}\frac{d^2 I}{dt^2}} \tag{21}$$

Problem 3: Singular Isothermal Spheres

Suppose that the probability distribution of velocities of particles (e.g., stars) in a galaxy are given by a Maxwell-Boltzmann distribution, where the velocities are Gaussian:

$$p(\mathbf{v}) = \frac{1}{(2\pi\sigma^2)^{3/2}} \exp\left(-\frac{|\mathbf{v}|^2}{2\sigma^2}\right), \tag{22}$$

Here, $\mathbf{v}$ represents the 3D velocity vector (v_x, v_y, v_z), and $|\mathbf{v}|$ is its magnitude v.

Problem 3A:

We are asked to show that the standard deviation of Equation 22 is σ.

The symmetrical 3D Gaussian factorizes:

$$p(\mathbf{v}) = \frac{1}{(2\pi\sigma^2)^{3/2}} \exp\left(-\frac{{v_x}^2 + {v_y}^2 + {v_z}^2}{2\sigma^2}\right), \tag{23}$$

In any single dimension the Maxwell-Boltsman equation is therefor:

$$p(v) = \frac{1}{(2\pi\sigma^2)^{3/2}} \exp\left(-\frac{v^2}{2\sigma^2}\right), \tag{24}$$

The variance is defined as $\mathrm{Var}(v) = \mathbb{E}[v^2] - (\mathbb{E}[v])^2$. (Variance is the square of the STD). In our cause $\mathbb{E}[v] = 0$ due to symmetry.

$\mathbb{E}[x^2]$ is given by:

$$\mathbb{E}[v^2] = \int_{-\infty}^{\infty} v^2 p(v) dv \tag{25}$$

Substituting in our $p(v$

$$\mathbb{E}[v^2] = \int_{-\infty}^{\infty} v^2 \frac{1}{(2\pi\sigma^2)^{3/2}} \exp\left(-\frac{v^2}{2\sigma^2}\right) dv \tag{26}$$

Integrals of this type are known as Gaussians, and their solution is well known.

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2}\, dx = \frac{\sqrt{\pi}}{2a^{3/2}}, \quad a > 0 \tag{27}$$

In our case, $a = \frac{1}{2\sigma^2}$.

Clearly this evaluates to $\boxed{\mathbb{E}[v^2] = \sigma^2}$.

Problem 3B:

Write down a differential equation for hydrostatic equilibrium of a spherically symmetric system in terms of the radial coordinate r, the density ρ and pressure P (both of which can be functions of r), G, and M_r (the total mass enclosed internal to radius r).

From dimensional analysis and Newton's law of gravity, Hydrostatic equilibrium follows:

$$\boxed{\frac{dP}{dr} = -\rho(r)\frac{GM(r)}{r^2}} \tag{28}$$

where M is the enclosed mass.

(we are not asked to *solve* the differential equation, only to write it)

Problem 3C:

For an isothermal, isotropic velocity distribution the pressure is

$$P = \rho\sigma^2 \tag{29}$$

We are asked now to solve the differential equation :(. Rewriting Equation 28, and using $\frac{dM}{dr} = 4\pi r^2\rho$, (its easier with the ln according to some online textbook I found)

$$\frac{\sigma^2}{r^2}\frac{d}{dr}(r^2\frac{d\ln\rho}{dr}) = -4\pi G\rho(r) \tag{30}$$

Ansatz for a power-law solution:

$$\rho(r) = \frac{A}{r^n}, \tag{31}$$

A is a constant with appropriate units, and I require $A > 0$, $n > 0$.

This means that:

$$\frac{d\ln\rho}{dr} = \frac{-n}{r} \tag{32}$$

The full left hand side (LHS) of Equation 30 is:

$$\text{LHS} = \frac{-\sigma n}{r^2} \tag{33}$$

The equation is now:

$$\frac{-\sigma^2 n}{r^2} = -4\pi G\frac{A}{r^n} \tag{34}$$

If this holds, then $n = 2$, and $A = \frac{\sigma^2}{2\pi G}$.

and therefor:

$$\boxed{\rho(r) = \frac{\sigma^2}{2\pi G}\frac{1}{r^2}}, \tag{35}$$

which is singular at $r = 0$.

Problem 3D:

We are asked to find the circular velocity v_c of a test particle placed in a circular orbit.

The circular orbit is given by:

$$v_c^2 = \frac{GM(r)}{r} \tag{36}$$

where $M(r)$ is the enclosed mass.

$M(r)$ is given by:

$$M(r) = 4\pi \int_0^r \rho(r)\, r^2\, dr. \tag{37}$$

We know $\rho(r)$, its our answer to problem 3C equation 35.

So the circular orbit is given by:

$$v_c^2 = \frac{G}{r} 4\pi \int_0^r \frac{\sigma^2}{2\pi G} \frac{1}{r^2}\, r^2\, dr \tag{38}$$

This looks scary but its really easy, just with a lot of constants, solving we have the solution.

$$\boxed{v_c = \sqrt{2}\sigma} \tag{39}$$

Woah! This is actually a super cool result!

Problem 3E:

A more realistic $\rho(r)$ is Navarro-Frenk-White (NFW) given by:

$$\rho(r) = \frac{\rho_0}{(\frac{r}{r_s})(1 + \frac{r}{r_s})^2} \tag{40}$$

where ρ_0 is an overall normalization and r_s is known as the scale radius.

The question is how does the steepness of the NFW profile compare to that of the singular isothermal sphere at small r? At large r?. This is pretty straight forward comparison of $\frac{r}{r_S}$.

At small r, $\frac{r}{r_s} << 1$, and so $(1 + \frac{r}{r_s})^2 \approx 1$ NFW becomes:

$$\rho(r) = \frac{\rho_0 r_s}{r} \tag{41}$$

Here the scaling is $\propto \frac{1}{r}$, which is **shallower** then the isothermal $\frac{1}{r^2}$.

On the other end, at large r, $\frac{r}{r_s} >> 1$ so $(1 + \frac{r}{r_s})^2 \approx \frac{r^2}{r_s^2}$. So NFW becomes:

$$\rho(r) = \frac{\rho_0 r_s^3}{r^3} \tag{42}$$

Here the scaling is $\propto \frac{1}{r^3}$, which is **steeper** then the isothermal $\frac{1}{r^3}$.

Problem 4: Applications of Dynamical Friction

The Chandrasekhar dynamical friction formula tells us that for a large mass M moving through a sea of smaller masses each of mass m, the drag force F_{df} is given by

$$F_{\mathrm{df}} = M\frac{dv_M}{dt} = -16\pi^2(\ln\Lambda)G^2 M m n_0 (M+m) \int_0^{v_M} p(v_m) v_m^2\, dv_m\, \frac{\mathbf{v}_M}{v_M^3}, \tag{43}$$

where $\ln\Lambda$ is the Coulomb logarithm, n_0 is the local density of small masses, v_m their velocity, and G is the gravitational constant.

Problem 4A:

We are asked to assume singular isothermal system, and that $M >> m$, and to reduce the dynamical friction equation into a given final solution.

I think a good start is to introduce the er$f$ function, which is defined as $\text{erf}(z) = \int_0^z e^{-t^2}\, dt$. We have that integral on the RHS that is giving made ERF vibes.

Lets express this part in an erf:

$$\int_0^{v_m} 4\pi p(v_M) v_m^2 \, dv_m \tag{44}$$

Introducing a new variable $u = \frac{v}{\sigma\sqrt{2}}$, we can express this as:

$$\frac{4}{\sqrt{\pi}} \int_0^{u_m} u^2 e^{-u^2}\, du \tag{45}$$

Almost the error function but we have a u^2 still, lets integrate by parts to get it there.

$$\int_0^{u_m} u^2 e^{-u^2}\, du = \frac{1}{2}\int_0^{u_m} e^{-u^2}\, du - \frac{u_m}{2} e^{-u^2} = \frac{\sqrt{\pi}}{2}\text{erf}(u_m) - \frac{u_m}{2} e^{-u^2} \tag{46}$$

putting it all together now:

$$\int_0^{v_M} p(v)\, 4\pi v^2 \, dv = \text{erf}\left(\frac{v_M}{\sqrt{2}\,\sigma}\right) - \frac{2 v_M}{\sqrt{\pi}\,\sqrt{2}\,\sigma}\, \exp\left(-\frac{v_M^2}{2\sigma^2}\right). \tag{47}$$

We can make this easier on the eyes by using our result from 3D equation 39 $v_c = \sigma\sqrt{2}$:

$$\int_0^{v_M} p(v)\, 4\pi v^2 \, dv = \text{erf}(1) - \frac{2}{\sqrt{\pi}} e^{-1}. \tag{48}$$

oki, now we can insert this into starting equation 43.

$$F_{\text{df}} = -4\pi(\ln\Lambda) G^2 M m n_0 (M+m)(\text{erf}(1) - \frac{2}{\sqrt{\pi}} e^{-1})\, \frac{\mathbf{v}_M}{v_M^3}, \tag{49}$$

In our case of $M >> m$, $M + m \approx M$, and $\rho = mn_0$ which we know! Further for the singular isothermal case, $\rho = \frac{v_c^2}{4\pi G r^2}$.

$$F_{\text{df}} = -4\pi(\ln\Lambda) G^2 M^2 m n_0 [\text{erf}(1) - \frac{2}{\sqrt{\pi}} e^{-1}]\, \frac{\mathbf{v}_M}{v_M^3}, \tag{50}$$

Lets sub in $\rho(r)$, and v_c

$$\boxed{F_{\text{df}} = \frac{-(\ln\Lambda) G M^2}{r^2} [\text{erf}(1) - \frac{2}{\sqrt{\pi}} e^{-1}]\, \hat{\mathbf{v}}_M} \tag{51}$$

Problem 4B:

Given this, derive the following differential equation for the radial distance r between the object and the centre of the galaxy:

The angular momentum is given by $L = M v_c r$, we can take the derivative of this and relate to rF_{df} the torq τ.

$$\tau = \frac{dL}{dt} = Mr\frac{dv_c}{dt} + Mv_c\frac{dr}{dt} = Mv_c\frac{dr}{dt} \tag{52}$$

now we just set $Mv_c\frac{dr}{dt} = rF_{\mathrm{df}}$, and

$$\boxed{r\frac{dr}{dt} = -\frac{GM(\ln\Lambda)}{v_c}[\mathrm{erf}(1) - \frac{2}{\sqrt{\pi}}e^{-1}]} \tag{53}$$

Problem 4C:

Now we solve this differential equation, this is actually easy as the RHS in our solution for 4B equation 53 is a constant in time.

$$r\frac{dr}{dt} = -\frac{GM(\ln\Lambda)}{v_c}[\mathrm{erf}(1) - \frac{2}{\sqrt{\pi}}e^{-1}] = k \tag{54}$$

where k is a constant

$$r\frac{dr}{dt} = k \Rightarrow rdr = kdt \tag{55}$$

Integrating from r_i to r_f, and from 0 to t, we have:

$$\frac{1}{2}(r_f^2 - r_i^2) = kt_f \tag{56}$$

The rest is just algebra to move around.

We can write $r(t)$, and t_{crash}, by solving for r_f, and t (with $r_f = 0$).

$$r(t) = [(-\frac{2GM(\ln\Lambda)}{v_c}[\mathrm{erf}(1) - \frac{2}{\sqrt{\pi}}e^{-1}])t + r_i]^{\frac{1}{2}} \tag{57}$$

and

$$\boxed{t_{\mathrm{crash}} = \frac{r_i^2 v_c}{2GM(\ln\Lambda)}[\mathrm{erf}(1) - \frac{2}{\sqrt{\pi}}e^{-1}]^{-1}} \tag{58}$$

This at least has the correct units.

Problem 4E:

Problem 5: Galactic Cannibalism

Problem 5A:

Here we are asked to define the feeding zone of typical galaxies by using our solution to problem 4. We can start by using equation 57 from problem 4, and using $r_f = 0$, and $t = t_h$, and then solve for r_i.

$$0 = [(-\frac{2GM(\ln\Lambda)}{v_c}[\mathrm{erf}(1) - \frac{2}{\sqrt{\pi}}e^{-1}])t_h + r_{\max}]^{\frac{1}{2}} \tag{59}$$

$$\boxed{r_{\max} = -\frac{2GM(\ln\Lambda)}{v_c}[\mathrm{erf}(1) - \frac{2}{\sqrt{\pi}}e^{-1}]t_h} \tag{60}$$

Problem 5B:

Assuming a mass-luminosity ratio of $\Upsilon = \frac{M}{L}$, we can write $M = \Upsilon L$.

The given equation is:

$$n(r, L)dL = \Phi(L)(\frac{r_0}{r})^{1.8}dL. \tag{61}$$

we are asked to find $N(L)dL$, this is simply the integral over space dr from 0 to $r_{\max}$

$$n(L)\,dL = 4\pi\Phi(L)\int_0^{r_{\max}} \left(\frac{r_0}{r}\right)^{1.8} r^2\,dr\,dL \tag{62}$$

the factors of r^2, and 4π comes from integrating over the unit sphere as we are in $3d$

Wolfram alpha says that:

$$\int_0^B (\frac{a}{x})^{1.8}x^2\,dx = \frac{5}{6}ax^2(\frac{a}{x})^{4/5}|_0^B = \frac{5}{6}B^3(\frac{a}{B})^{1.8} \tag{63}$$

In our case $B = r_{\max}$, and $a = r_0$. So we have:

$$\boxed{n(L)\,dL = \frac{10\pi\Phi(L)}{3}r_{\max}^3(\frac{r_0}{r_{\max}})^{1.8}} \tag{64}$$

This is our expression, we can substitute in $r_{\max}$ in terms of L to get:

$$\boxed{n(L)\,dL = \frac{10\pi\Phi(L)}{3}(-\frac{2G\Upsilon L(\ln\Lambda)}{v_c}[\text{erf}(1) - \frac{2}{\sqrt{\pi}}e^{-1}]t_h)^3\,(\frac{r_0}{-\frac{2G\Upsilon L(\ln\Lambda)}{v_c}[\text{erf}(1) - \frac{2}{\sqrt{\pi}}e^{-1}]t_h})^{1.8}} \tag{65}$$

This looks god-awful.

Problem 5C:

We are asked to Obtain an expression for the total luminosity that has been consumed, assuming a Schechter form for the luminosity function.