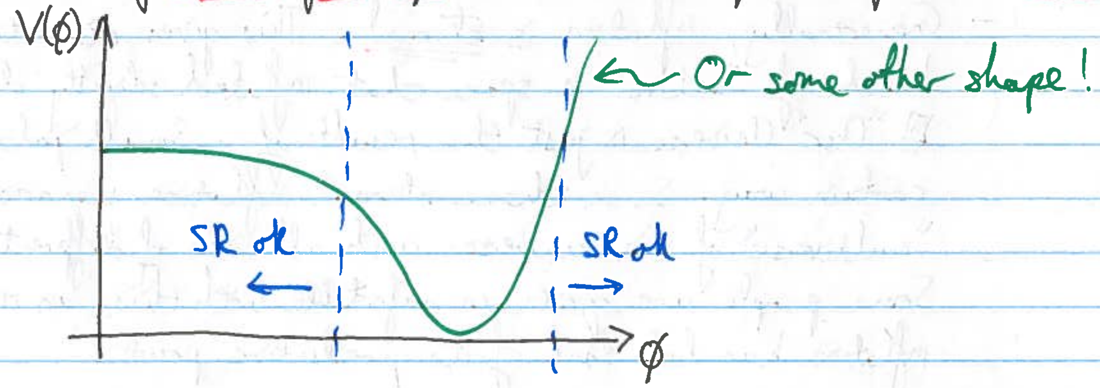
Initial Fluctuations

Recap from before:

- Motivated by the flatness, horizon, monopole problems, we postulated an early phase of accelerated, exponential inflation
- We found that we could drive this sort of expansion by some sort of scalar field ϕ, ~~that~~ which feels a potential $V(\phi)$



- The inflaton field (as we like to call it) has an equation of motion that governs how the value of the field evolves with time:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0.$$

Acts like a rolling ball on $V(\phi)$.

- Exp. acceleration
- At least 60 e-folds of this expansion

- To get the inflaton to have the right properties to give us what we want, the potential needs to satisfy the slow-roll conditions

$$\varepsilon \equiv \frac{1}{16\pi}\left(\frac{\partial V/\partial\phi}{V}\right)^2 \quad \text{and} \quad |\eta| \equiv \frac{1}{8\pi}\frac{|\partial^2 V/\partial\phi^2|}{V}$$

The slow-roll conditions are that $\varepsilon \ll 1$ and $|\eta| \ll 1$.

- In the potentials we envision, certain parts of the potential satisfy the slow-roll conditions, while others do not. (see blue delineations in picture above). We imagine that the inflaton starts in one of these regions, inflation happens, and continues until we roll down to a non-SR region. Then the inflaton couples to the rest of the matter fields and reheats our Universe, populating it with the particles that we know and love.

- Generically, inflation is eternal. This gives rise to a whole bunch of "sectors" in space that inflated slightly differently. "Our" Universe is just the result of a small patch inflating a certain way. So in these eternal inflation scenarios, we get a "multiverse" of mini universes with all sorts of different properties. Some people are quite uncomfortable with this, as it feels like inflation has lost some of its predictive power.

So why is inflation our "default" model of the very early universe? In my view it's because it provides a mechanism for generating the spatial fluctuations in density that we see in our Universe.

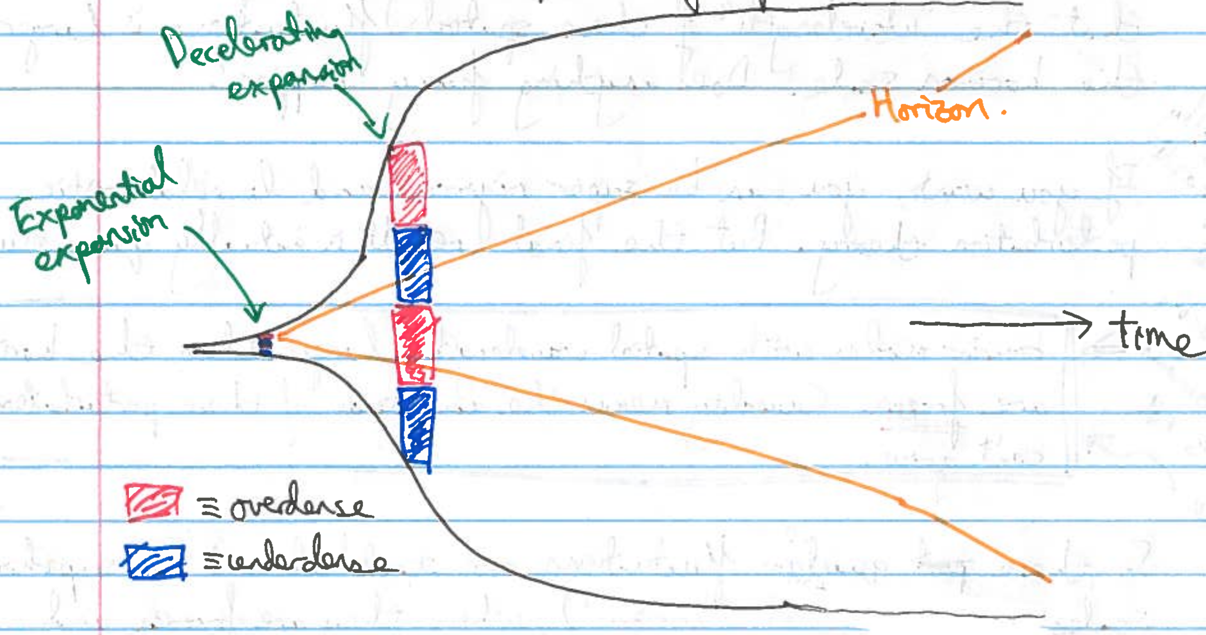
A Cartoon View of What Happens

So far, we have looked at the inflaton field as a ~~[illegible]~~ classical field. But of course it's fundamentally quantum in nature.

> What happens ~~[illegible]~~ is that quantum fluctuations in the inflaton field — usually microscopic — get stretched out into cosmological scales. These quantum fluctuations seed the initial conditions (the ~~[illegible]~~ tiny inhomogeneities) that give rise to large scale structure and galaxies.

Show structure formation movie + CMB beach ball

Recall that our picture of inflation looks like this:



To solve the horizon problem, recall that opposite sides of our Universe are initially in causal contact, get yanked out of causal contact, and then as time goes on, more and more distant parts of our Universe comes into causal contact again, as light has more and more time to traverse our Universe. (This is what the orange "Horizon" line is meant to represent).

Suppose I have some quantum fluctuations  that get stretched out to cosmological scales . What happens to them afterwards?

Normally, if I have something like , the amplitude of this fluctuation will grow. Just like in the movie, where initially overdense parts have stronger gravity, attracting more stuff and becoming even more dense etc., in a runaway process, But be careful! We are stretching these perturbations so that the "wavelength" of these spatial fluctuations is larger than the horizon scale. Does anything funny happen?

If you want, you can be super rigorous and do relativistic perturbation theory. But the final result is actually quite intuitive:

This is actually a gauge-dependent statement and in GR it's possible to pick coordinates where this is not the case

$\boxed{\text{Fourier modes with spatial wavelengths larger than the horizon are frozen. Causality means the amplitude of these perturbations can't grow.}}$

So these ~~pert~~ quantum fluctuations are stretched out to superhorizon scales (they "exit the horizon"), where they are frozen and gravitational collapse can't happen. Eventually, the size of the horizon gets large enough to match the scale of the fluctuations (they "reenter the horizon") and only then does gravitational clustering begin for that mode.

Let's take this a step further:

In other words, small scale modes are sensitive to the end of inflation; large scale modes to the beginning

- Quantum fluctuations in the inflaton field that occurred early on in the inflationary expansion exited the horizon early and got stretched to the largest scales (because inflation is still going strong ⇒ lots of time to stretch). These are the modes that reenter the horizon the latest.
- Quantum fluctuations that occurred later in the inflationary process exited the horizon later and got stretched less. They are therefore fluctuations on smaller scales. These are the modes that reenter the horizon the earliest.

This is the qualitative picture. To make it quantitative, we need to make a quick mathematical detour ....

Describing fluctuations — 2-point statistics.

Suppose I have some field $\rho(\vec{r})$. I can define the overdensity of this field:

$$\delta(\vec{r}) \equiv \frac{\rho(\vec{r}) - \bar{\rho}}{\bar{\rho}}$$

Average value of this field.

This could be anything, but here I picked the notation ρ because we are often concerned with the matter density. (It could be the inflaton field, the gravitational potential, ...)

Now, no theory of our Universe predicts exactly what $\delta(\vec{r})$ is, because our Universe is a random place. (We can't predict how the quantum fluctuations in inflation would go, for example, because otherwise they wouldn't be random). All we can do is predict the statistical properties of the field.

What statistical properties might be useful?

"Ensemble average": Imagine having universe generation machine and being able to produce an infinite number of realizations of our Universe to average them together.

Mean: $\langle \delta(\vec{r}) \rangle$. Useless because this is zero by construction, since it is the overdensity field.

Variance: $\langle \delta^2(\vec{r}) \rangle$. Better! Tells you the amplitude of fluctuations. Are they $O(10^{-5})$ or $O(1)$?

Covariance: $\langle \delta(\vec{r}_1)\, \delta(\vec{r}_2) \rangle$. Even better. Tells you how different points are correlated, which teaches you about the underlying physics.

$\boxed{\text{Show three slides of Qs} \to \text{which universe has gravity? / Which is the older universe? / Which universe has more of its mass in neutrinos?}}$

Now, $\xi(\vec{r}_1, \vec{r}_2)$ is often called the correlation function.
It can be simplified.

$$\xi(\vec{r}_1, \vec{r}_2) \equiv \langle \delta(\vec{r}_1)\, \delta(\vec{r}_2) \rangle$$

$$\xrightarrow{1} \xi(\vec{r}_1 - \vec{r}_2) \xrightarrow{2} \xi(|\vec{r}_1 - \vec{r}_2|) \equiv \xi(r)$$

1 because of homogeneity. The universe doesn't care what we call our origin. There is statistical translation invariance. It should only care about the difference in coordinates.

2 because of isotropy. No preferred direction.

The correlation function is what we call the 2-pt fct. in configuration space $\Rightarrow$ $\boxed{\text{Show city lights + galaxy correl. fct.}}$

We can also compute a 2-pt. fct in Fourier space:

$$\tilde{\delta}(\vec{k}) \equiv \int d^3\vec{r}\, e^{-i\vec{k}\cdot\vec{r}}\, \delta(\vec{r}) \quad \text{and} \quad \delta(\vec{r}) = \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}}\, \tilde{\delta}(\vec{k}).$$

$$\langle \tilde{\delta}(\vec{k}_1)\, \tilde{\delta}(\vec{k}_2)^* \rangle = \left\langle \int d^3\vec{r}_1\, e^{-i\vec{k}_1\cdot\vec{r}_1}\, \delta(\vec{r}_1) \int d^3\vec{r}_2\, e^{i\vec{k}_2\cdot\vec{r}_2}\, \delta(\vec{r}_2) \right\rangle$$

$$= \int d^3\vec{r}_1\, d^3\vec{r}_2\, e^{-i\vec{k}_1\cdot\vec{r}_1} e^{i\vec{k}_2\cdot\vec{r}_2} \underbrace{\langle \delta(\vec{r}_1)\, \delta(\vec{r}_2) \rangle}_{=\xi(\vec{r}_1 - \vec{r}_2)}$$

Now do a change of coordinates: Let $\vec{r}_2 = \vec{r}_1 + \vec{x}$.

Then $$\langle \tilde{\delta}(\vec{k}_1)\tilde{\delta}(\vec{k}_2)^* \rangle = \int d^3\vec{r}_1\, d^3\vec{x}\; e^{-i\vec{k}_1\cdot\vec{r}_1}\, e^{i\vec{k}_2\cdot(\vec{r}_1+\vec{x})}\, \xi(\vec{x})$$

Dirac delta fct.

$$= \int d^3\vec{x}\; (2\pi)^3\, \delta^D(\vec{k}_1 - \vec{k}_2)\, \xi(\vec{x})\, e^{i\vec{k}_2\cdot\vec{x}}$$

$$= (2\pi)^3\, \delta^D(\vec{k}_1 - \vec{k}_2) \underbrace{\int d^3\vec{x}\; e^{i\vec{k}\cdot\vec{x}}\, \xi(\vec{x})}$$

Define this, which is the Fourier transform of the correlation fct. to be $P(\vec{k})$, the power spectrum

So $\langle \tilde{\delta}(\vec{k}_1)\tilde{\delta}(\vec{k}_2)^* \rangle = (2\pi)^3 \delta^D(\vec{k}_1 - \vec{k}_2)\, P(\vec{k}_1)$

If we assume isotropy as well, then $P(\vec{k}) \rightarrow P(|\vec{k}|)$.

The power spectrum and the correlation function are two equivalent ways to look @ fluctuations.

- The correlation function doesn't depend on the "origin" of $\vec{r}_1$ and $\vec{r}_2$. But different positions are correlated.
- The power spectrum does depend on the size of k. But different Fourier modes are not correlated.

The power spectrum can be thought of as the variance of fluctuations on different k-scales.

Here's another way to look at it. Suppose I wanted to compute the contributions to the variance in configuration space from Fourier modes of various length scales:

$$\langle \delta^2(\vec{r}) \rangle = \left\langle \int \frac{d^3\vec{k}_1}{(2\pi)^3}\, e^{i\vec{k}_1\cdot\vec{r}}\, \tilde{\delta}(\vec{k}_1) \int \frac{d^3\vec{k}_2}{(2\pi)^3}\, e^{-i\vec{k}_2\cdot\vec{r}}\, \tilde{\delta}^*(\vec{k}_2) \right\rangle$$

$$= \int \frac{d^3\vec{k}_1}{(2\pi)^3} \frac{d^3\vec{k}_2}{(2\pi)^3} e^{i\vec{k}_1\cdot\vec{r}} e^{-i\vec{k}_2\cdot\vec{r}} \underbrace{\langle \tilde{\delta}(\vec{k}_1)\tilde{\delta}^*(\vec{k}_2)\rangle}_{\equiv (2\pi)^3 \delta^D(\vec{k}_1-\vec{k}_2) P(\vec{k}_1)}$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} P(k)$$ Assuming isotropy.

$$= \int \frac{dk\, 4\pi k^2}{(2\pi)^3} P(k) = \int d\ln k \; \frac{k^3 P(k)}{2\pi^2}.$$

We often define $\Delta^2(k) \equiv \frac{k^3 P(k)}{2\pi^2}$ as the "dimensionless power spectrum"

(You can tell that it's dimensionless because $\langle\delta^2\rangle$ is dimensionless and so is $d\ln k$).

From this integral, we see also that $\Delta^2(k)$ is the contribution to the variance of fluctuations $\langle\delta^2\rangle$ per logarithmic interval in k.