PHYS514 Lecture 1 Problems

Maxwell A. Fine: SN 261274202
maxwell.fine@mail.mcgill.ca
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Problem 1: Relativistic Boosting

Problem 1A

We are asked to show that $\delta^{(3)}(\vec{x} - \vec{x_1}(t'))/E$ is invariant under a boost in the y direction. δ here is the 3D Dirac-delta function. We are told that the test mass is at rest in the unprimed frame.

A boost in one direction in frame $'$ is simply the test mass moving in that (opposite) direction. A boost along one direction is given by the matrix:

$$\Lambda = \gamma \begin{pmatrix} 1 & +\beta \\ +\beta & 1 \end{pmatrix} \tag{1}$$

Here $\beta = v/c$, and $\gamma = 1/\sqrt{1-\beta^2}$.

Idea: We compute the boost using the matrix and then show it is $\delta^{(3)}(\vec{x}' - \vec{x_1}')/E'$.

First we want to apply the boost, which is the following.

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} \tag{2}$$

Where $x^{\mu} = \begin{pmatrix} t \\ y \\ x \\ z \end{pmatrix}$. This does not have a minus sign, and we write y as the second dimension for convenience as it is the boosted dimension. Now the full boost matrix is:

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & +\beta\gamma & 0 & 0 \\ +\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

In the primed frame we have the following:

$$t' = \gamma(t + \beta y), \tag{4}$$
$$y' = \gamma(y + \beta t), \tag{5}$$
$$x' = x, \tag{6}$$
$$z' = z, \tag{7}$$
$$\tag{8}$$

Great. Now we also need to see how the transformation effects E. Recall that E is the zeroth component of the energy-momentum vector. $p^{\mu} = \begin{pmatrix} E \\ p_y \\ p_x \\ p_z \end{pmatrix}$. Under the boost matrix we have:

$$E' = \gamma(E + \beta p_y), \tag{9}$$
$$p'_y = \gamma(p_y + \beta E), \tag{10}$$
$$p_x = p_x, \tag{11}$$
$$p_z = p_z, \tag{12}$$
$$\tag{13}$$

We can now construct the primed frame:
Recall:

$$\boxed{\delta^{(3)}\left(\vec{x}' - \vec{x}'_1\right) = \frac{\delta^{(3)}\left(\vec{x} - \vec{x}_1\right)}{\left|\det \frac{\partial \vec{x}'}{\partial \vec{x}}\right|}} \tag{14}$$

Finding the Jacobian of Λ. This Jacobian is with respect to the spatial coordinates at fixed time in the primed frame. (IE we plug in t' for t in the coordinate transformations.

$$\text{Jacobian matrix:} \quad \frac{\partial(x', y', z')}{\partial(x, y, z)}|_{t'=\text{fixed}} = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} \\ \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} \end{pmatrix}. \tag{15}$$

In our case $|\det J| = 1/\gamma$.

$$\delta^{(3)}(\vec{x}' - \vec{x'_1}(t')')/E \Rightarrow \frac{\delta^{(3)}\left(\vec{x} - \vec{x}_1\right)}{|\det J_{t'=\text{fixed}}|\,\gamma(E + \beta p_y)} \tag{16}$$

$$\frac{\delta^{(3)}\left(\vec{x} - \vec{x}_1\right)}{(E + \beta p_y)} \tag{17}$$

Since the particle is at rest in the unprimed frame, $p_y = 0$:

$$\boxed{\frac{\delta^{(3)}\left(\vec{x} - \vec{x}_1\right)}{E}} \tag{18}$$

Problem 1B

This time, we are assuming that the test mass is moving in the unprimed frame that $\vec{x}_1 = \vec{x}_1(t)$. We are told that $y_1 = v_1 t$. By transforming y, t (but considering v_1 to be a fixed param) we are asked to find the more general transormation of the delta function, and energy. Hint: How does v_1 relate to p^μ in the unprimed frame?

This time we have: $p^\mu = \begin{pmatrix} E = \gamma_1 m \\ p_y = \gamma_1 m v_1 \\ p_x = 0 \\ p_z = 0 \end{pmatrix}$. With $\gamma_1 = 1/\sqrt{1 - (v_1)^2}$ In natural units. We already know what the primed frame $x^{'\mu}$, and $p^{'\mu}$ are from problem 1.1.

In the boosted frame we have: This time we have: $p'^\mu = \begin{pmatrix} E' = \gamma(E + \beta p_y) = \gamma\gamma_1 m(1 + \beta v_1) \\ p_y = \gamma(p_y + \beta E) = \gamma\gamma_1 m(v_1 + \beta) \\ p_x = 0 \\ p_z = 0 \end{pmatrix}$

And for the space-time coordinates we have: $x'^{\mu} = \begin{pmatrix} t' = \gamma(t + \beta y_1) = \gamma t(1 + \beta v_1) \\ y' = \gamma(y + \beta t) = \gamma t(v_1 + \beta) \\ x' = x \\ z' = z \end{pmatrix}$

We want to write $y'(t')$, we know $y' = \gamma t(v_1 + \beta)$, and can plug in our $t = \frac{t'}{\gamma(1+\beta v_1)}$.

$$y' = \frac{t'(v_1 + \beta)}{(1 + \beta v_1)} \tag{19}$$

Now we can start to transform the delta function. We need $\frac{dy'}{dy}|_{t'=\text{fixed}}$. $y' = \gamma(y + \beta t) = \gamma(y + \beta(\frac{t'}{\gamma} - \beta y))$, so we get $\frac{dy'}{dy}|_{t'=\text{fixed}} = \gamma(1 - \beta^2) = \frac{1}{\gamma}$

$$\boxed{\frac{\delta^{(3)}(\vec{x}' - \vec{x_1}'(t'))}{E'} = \frac{\delta^{(3)}(\vec{x} - \vec{x_1}(t))}{\gamma_1 m(1 + \beta v_1)}} \tag{20}$$

This is the generalized answer.

Problem 1C:

Show that the argument of the transformed delta function tells you how the originally velocity v_1 gets transformed when going to the new frame. Correct any missing factors of c in case you have been using $c = 1$ in your 4-vectors. This is why you do not transform v_1 inside the delta function: it gets done for you by transforming y and t.

I think we can do this from our y' formula we have written down equation 19.

$$y' = \frac{t'(v_1 + \beta)}{(1 + \beta v_1)} \tag{21}$$

It should be the case that $\frac{dy'}{dt'} = v_1'$, therefore:

$$v_1' = \frac{(v_1 + \beta c)}{(1 + (\beta/c)v_1)} \tag{22}$$

the transformation of y,and t automatically gives us the correct transformed velocity.

Problem 2: stress-energy tensor

Problem 2A:

Compute the stress-energy tensor of the particle in problem 1A, before and after the boost, using:

$$\boxed{T^{\mu\nu} = \frac{p^\mu p^\nu}{p^0}\delta^{(3)}(\vec{x} - \vec{x_1}(t))} \tag{23}$$

Assume the boosted velocity is $-v$. Express in terms of m, β, γ. Ignore the dimensions where the particle is stationary.

From 1A we have: $\vec{x_1} = 0$, $p^\mu = E$, in units of $c = 1$, so $E = m$ in the rest frame.

We have:

$$T^{ij} = \frac{p^i p^j}{p^0}\delta^{(3)}(\vec{x} - \vec{x_1}(t)) \tag{24}$$

$p^i p^j$ Gives the momentum flux / stress along each direction. More detailed T^{ij} represents the momentum flux (transport) in the i-direction across a surface normal to the j-direction. T^{ij} is a symmetrical matrix.

$p^i p^j$ is only non-zero for when $i = j = 0$ term. So for our case we have in the rest frame:

$$\boxed{T^{\mu\nu} = \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}\delta^{(3)}(\vec{x})} \tag{25}$$

Now, we can apply the boost matrix Λ. As the stress-energy tensor transforms as a rank 2 tensor. Recall that it is a symmetrical matrix.

$$T^{\mu\nu'} = \Lambda^\mu_\alpha \Lambda^\nu_\beta T^{\alpha\beta} \tag{26}$$

Lets go component by component:

$$T'^{00} = \Lambda^0_0 \Lambda^0_0 T^{00} = \gamma^2 m \delta^{(3)(\vec{x})} \tag{27}$$

$T^{\mu\nu}$ only has the double zero index term, which makes life nice.

Next term.

$$T'^{01} = \Lambda^0_0 \Lambda^1_0 T^{00} = \gamma^2 \beta m \delta^{(3)(\vec{x})} \tag{28}$$

By symmetry we have $T'^{10} = \gamma^2 \beta m \delta^{(3)(\vec{x})}$.

Next term.

$$T'^{11} = \Lambda^1_0 \Lambda^1_0 T^{00} = \gamma^2 \beta^2 m \delta^{(3)(\vec{x})} \tag{29}$$

But we are not done yet, we need to transform the δ into the primed frame! We add a factor of $1/\gamma$.

Then our final solution is:

$$\boxed{T'^{\mu\nu} = \gamma m \begin{pmatrix} 1 & \beta \\ \beta & \beta^2 \end{pmatrix}\delta^{(3)}(\vec{x'})} \tag{30}$$

Done!

Problem 2B:

Show that we get the same result by using:

$$T^{\alpha\beta} = (\Lambda T \Lambda^T)^{\alpha\beta} \tag{31}$$

(feed into lin alg calculator, checked!)

Problem 3: Stress-Energy Continued

Consider the stress-energy tensor from a very large collection of particles moving in random directions, i.e., a perfect fluid.

Problem 3A:

(A) Show that by summing over the directions of motion, the off diagonal elements average to zero.
Lets start with the equation for the stress-energy tensor for a single particle.

$$T^{\mu\nu} = \frac{p^\mu p^\nu}{p^0}\delta^{(3)}(\vec{x} - \vec{x_1}(t)) \tag{32}$$

For many (n) particles, we have

$$T^{\mu\nu}_{\text{total}} = \sum_n \frac{p^\mu_n p^\nu_n}{p^0_n}\delta^{(3)}(\vec{x} - \vec{x_n}(t)) \tag{33}$$

I am keeping the $\sum$ for now, as I am learning Einstein notation. The off diagonal terms are given by $\mu \neq \nu$. For random directions, $p^\mu_n p^\nu_n$ should be uncorrelated IE isotropic. If this is the case then the average $\langle p^\mu p^\nu \rangle = 0, \quad \mu \neq \nu$, this makes sense as the diagonal elements correspond to the pressure IE $T^{\mu\mu} =$ Pressure.

Problem 3B:

Energy-density is $\rho = T^{0,0}$, and pressure $P = \frac{1}{3}\sum_{i=1}^{3} T^{ii}$
We can derive this from:

$$T^{\mu\nu}_{\text{total}} = \sum_n \frac{p^\mu_n p^\nu_n}{p^0_n}\delta^{(3)}(\vec{x} - \vec{x_n}(t)) \tag{34}$$

The 00 term is $T^{00}_{\text{total}} = \sum_n p^0_n \delta^{(3)}(\vec{x} - \vec{x_n}(t))$, we can replace $p^0 = E/c$, multiply by a small volume ΔV, and integrate using the delta function.

$$\int_{\Delta V} T^{00}_{\text{total}} d^3x = \int_{\Delta V} \sum_n (E_n/c)\delta^{(3)}(\vec{x} - \vec{x_n}(t))d^3x = \sum_{n\in\Delta V} E_n/c \tag{35}$$

Multiply both sides by C:

$$c\int_{\Delta V} T^{00}_{\text{total}} d^3x = \sum_{n\in\Delta V} E_n \tag{36}$$

For this to be true, cT^{00} must be the energy-density.

Problem 3C:

Energy density for massless particles. For a perfect fluid, only the diagonal elements survive and we have:

$$T^{\mu\mu}_{\text{total}} = \sum_n \frac{p^\mu_n p^\mu_n}{p^0_n}\delta^{(3)}(\vec{x} - \vec{x_n}(t)) = \frac{\langle p^\mu p^\mu \rangle}{p^0_n}\rho_{\text{number}} \tag{37}$$

Where ρ_{number} is the number density. This is the momentum flux in the μ direction across a surface normal to μ.

Pressure has units of force / area $\Rightarrow$ momentum flux. What units does $T^{\mu\nu}$ have?

$$[p_i] = \mathrm{kg\,m\,s^{-1}}, \tag{38}$$
$$[p^0] = [E/c] = \mathrm{kg\,m\,s^{-3}}, \tag{39}$$
$$\delta^{(3)}(\vec{x}) = \mathrm{m^{-3}} \tag{40}$$

Therefore, $T^{\mu\nu}$ has units of $[\mathrm{kg\,m^{-2}}s^{-1}]$. The units of pressure.

For massless particles $E = pc$, and $p^0 = p$. $\rho = cT^{00} = n < E >$. For the diagonal entries, we have: $T^{ii} = n < p_i^2/p >=$, we can argue isotropy that $\frac{1}{3} < p^2 >=< p_i^2 >$, then for $T^{ii} = \frac{n}{3} < p >$ = pressure, and bam we are done. Now we can say $T^{00} = \frac{n}{3} < p >= n < \rho >$. [1]

[1] Why do we write E, when it is ρ?

Problem 4: The Minkowski metric $\eta^{\mu\nu}$

Problem 4A:

Show that $\eta^{\mu\nu}$ is invariant under a boost in the y-direction. This means that:

$$\eta'^{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1) \tag{41}$$

This transforms under the boost as a 2nd rank tensor.

$$\eta'^{\mu\nu} = \Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}\eta^{\alpha\beta} \tag{42}$$

As this is a boost in 1D, we only need to keep time, and the y direction.

$$\eta'^{00} = \gamma^2 - (\beta\gamma)^2 = \gamma^2(1-\beta) = 1 \tag{43}$$

and

$$\eta'^{01} = \eta'^{10} = \Lambda^0_0\Lambda^1_0\eta^{00} + \Lambda^0_1\Lambda^1_1\eta^{01} = \gamma^2\beta - \gamma^2\beta = 0 \tag{44}$$

and finally,

$$\eta'^{11} = \Lambda^1_{\alpha}\Lambda^1_{\beta}\eta^{\alpha\beta} = \Lambda^1_0\Lambda^1_0\eta^{00} + \Lambda^1_1\Lambda^1_1\eta^{11} = (\beta\gamma)^2 - \gamma^2 = -1 \tag{45}$$

and we have:

$$\boxed{\eta'^{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)} \tag{46}$$

Problem 4B:

In (1+1) dimensions (one space plus one time), the Levi-Civita tensor $\varepsilon^{\mu\nu}$ is antisymmetric, with off-diagonal entries $\pm$ 1. Show that this also remains invariant under a such a Lorentz transformation.

Sure, we do the same thing as before! this time our matrix looks like

$$\varepsilon^{01} = +1, \qquad \varepsilon^{10} = -1, \qquad \varepsilon^{00} = \varepsilon^{11} = 0. \tag{47}$$

This should transform like a rank 2 tensor:

$$\varepsilon'^{\mu\nu} = \Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}\varepsilon^{\alpha\beta} = \varepsilon^{\mu\nu} \tag{48}$$

Lets go find each component.

$$\varepsilon'^{00} = \Lambda^0_{\alpha}\Lambda^0_{\beta}\varepsilon^{\alpha\beta} = \Lambda^0_1\Lambda^0_0\varepsilon^{10} + \Lambda^0_0\Lambda^0_1\varepsilon^{01} = -\beta\gamma^2 + \beta\gamma^2 = 0 \tag{49}$$

$$\varepsilon'^{10} = \Lambda^0_{\alpha}\Lambda^1_{\beta}\varepsilon^{\alpha\beta} = \Lambda^0_0\Lambda^1_1\varepsilon^{01} + \Lambda^0_1\Lambda^1_0\varepsilon^{10} = \gamma^2 - (\beta\gamma)^2 = -1 \tag{50}$$

$$\varepsilon'^{01} = \Lambda^0_{\alpha}\Lambda^1_{\beta}\varepsilon^{\alpha\beta} = \Lambda^0_0\Lambda^1_1\varepsilon^{01} + \Lambda^0_1\Lambda^1_0\varepsilon^{10} = \gamma^2(1-\beta^2) = +1 \tag{51}$$

$$\boxed{\varepsilon'^{\mu\nu} = \Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}\varepsilon^{\alpha\beta} = \varepsilon^{\mu\nu}} \tag{52}$$

Problem 4C

Now we are asked to show th same but for the (3+1) version.

$$\epsilon^{\alpha\beta\mu\nu} = \begin{cases} +1 & \text{if } (\alpha\beta\mu\nu) \text{ is an even permutation of } (0,1,2,3), \\ -1 & \text{if } (\alpha\beta\mu\nu) \text{ is an odd permutation of } (0,1,2,3), \\ 0 & \text{if any two indices are equal.} \end{cases} \tag{53}$$

Under a Lorentz transformation $\Lambda^{\mu}{}_{\nu}$, it transforms as a rank-4 tensor:

$$\epsilon'^{\alpha\beta\mu\nu} = \Lambda^{\alpha}{}_{\rho}\Lambda^{\beta}{}_{\sigma}\Lambda^{\mu}{}_{\kappa}\Lambda^{\nu}{}_{\lambda}\,\epsilon^{\rho\sigma\kappa\lambda}. \tag{54}$$

For a boost in the y-direction, the Lorentz matrix is[2]

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{55}$$

Let us explicitly compute one component, e.g. ϵ'^{0123}:

$$\begin{aligned} \epsilon'^{0123} &= \Lambda^{0}{}_{\rho}\Lambda^{1}{}_{\sigma}\Lambda^{2}{}_{\kappa}\Lambda^{3}{}_{\lambda}\,\epsilon^{\rho\sigma\kappa\lambda} && (56) \\ &= \Lambda^{0}{}_{0}\Lambda^{1}{}_{1}\Lambda^{2}{}_{2}\Lambda^{3}{}_{3}\,\epsilon^{0123} \quad \text{(all other terms are 0 since } \epsilon \text{ is zero if indices repeat)} && (57) \\ &= \gamma\cdot 1\cdot 1\cdot 1\cdot 1 && (58) \\ &= 1. && (59) \end{aligned}$$

Similarly, all other nonzero components transform to themselves. For example,

$$\epsilon'^{0213} = \Lambda^{0}{}_{0}\Lambda^{2}{}_{2}\Lambda^{1}{}_{1}\Lambda^{3}{}_{3}\,\epsilon^{0213} = 1\cdot 1\cdot 1\cdot 1\cdot(-1) = -1. \tag{60}$$

All components with repeated indices remain zero. Therefore, we have

$$\boxed{\epsilon'^{\alpha\beta\mu\nu} = \epsilon^{\alpha\beta\mu\nu}}. \tag{61}$$

This shows that the 4D Levi-Civita tensor is explicitly invariant under a Lorentz boost in the y-direction, just like in the 2D case.

[2] This time, I am using the standard t, x,y,z ordering.

Problem 5: Newtonian potentials

Problem 5A:

We are asked to compute the value of the Newtonian potential on the surface of Neptune, and make it dimensionless using appropriate powers of c. Then show that the corresponding metric perturbation is indeed $<< 1$.

We know that the Newtonian potential is:

$$\Phi = -\frac{GM}{|\mathbf{r}|} \tag{62}$$

Dimensionally it has units of:

$$[\Phi] = [-\frac{GM}{|\mathbf{r}|}] = [gh] = [\,\mathrm{m}^2\mathrm{s}^{-2}] \tag{63}$$

To get a dimensionally value, we can multiply by c^{-2} which has units of $\mathrm{s}^2\mathrm{m}^{-2}$

Our Dimensional-less Φ is:

$$\boxed{\Phi = -\frac{GM}{c^2|\mathbf{r}|}} \tag{64}$$

I am using Φ to mean both the one with units and without units.

From class recall, the dimensionless perturbation of the Minkowski metric (weak field) is $g_{00} = -(1 + 2\phi)$:

$$h_{00} = g_{00} - \eta_{00} = -2\Phi \tag{65}$$

This is only valid when $|h_{00}| << 1$. This is linearized gravity.

In our case for Neptune we have:

$$\boxed{|h_{00}| 6 \times 10^{-9} << 1} \tag{66}$$

See the associated jupyter-notebook located at https://github.com/afinemax/mcgill$_phys514_general_relativity$ for the

Problem 5B:

For this problem we are now asked to calculate h_{00} for the surface of a Neutron star.

This time we have:

$$\boxed{|h_{00}| \sim 0.4} \tag{67}$$

This is of order 1, meaning that our aproximation is starting to break down.

Problem 5C

We are asked how much slower do clocks run on the surface of the neutron star than on Neptune?

The notes say: clocks run slower in a gravitational field, by the factor $\sqrt{1 + 2\phi}$, so we can just take the ratio of the two.

Clocks on the neutron star run at $\boxed{0.76}$ times the rate of clocks on Neptune.