PHY408

Lecture 3: Linear Systems and Convolution

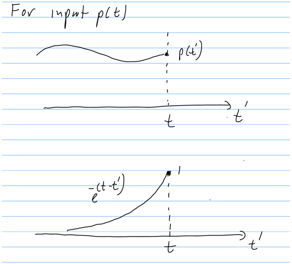
January 25, 2023

Leaky Bucket Problem

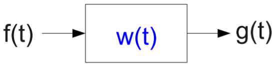
Output $y(t)$ is related to input $p(t)$ by:

$$y(t) = \int_{-\infty}^{t} e^{-k(t-t')} p(t') dt' = \int_{0}^{\infty} e^{-k\tau} p(t-\tau)\, d\tau \tag{1}$$

- at any given time t, current water level $y(t)$ depends on the past precipitation $p(t-\tau)$, $\tau > 0$, but with decreasing weight $e^{-k\tau}$ as τ increases.



Convolution



Convolution of $w(t)$ and $f(t)$ is defined as

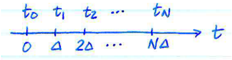
$$g(t) = \int_{-\infty}^{\infty} w(\tau) f(t-\tau) d\tau = f(t) * w(t) \tag{2}$$

1. Commutative: $g(t) = f(t) * w(t) = w(t) * f(t)$
2. $w(t)$ also called system function: given any input signal $f(t)$, the output $g(t)$ is uniquely determined by $f(t) * w(t)$
3. causal: $w(t) = 0$, $t < 0$. $g(t)$ only depends on past values of $f(t)$

Discretization

Sample continuous time series $f(t)$ and $w(t)$ both at equal time interval Δt (sometimes also denoted by Δ):

$$f_k = f(t_k) = f(k\Delta), \quad w_k = w(t_k) = w(k\Delta) \tag{3}$$

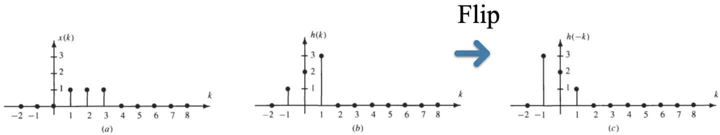


Then the discrete convolution is given by:

$$g_n = \left[\sum_{k=-\infty}^{\infty} f_{n-k} w_k \right] \Delta \tag{4}$$

1. Python g=numpy.convolve(f,w) only gives $\sum f_{n-k} w_k$; remember to multiply Δ!
2. In practical implementation, $f(t)$ and $w(t)$ are of finite duration.

Discrete Conv.: $y = x * h$; $y_n = \left(\sum_{k=-\infty}^{\infty} x_k h_{n-k}\right) \Delta$



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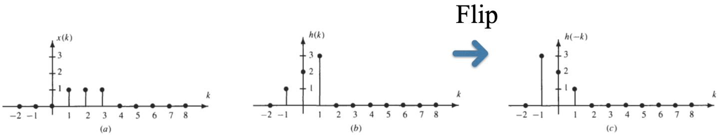
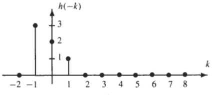


$n = 0$



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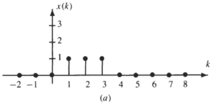
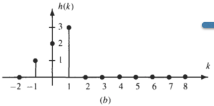
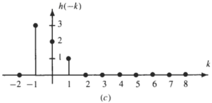
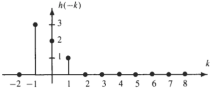




Flip

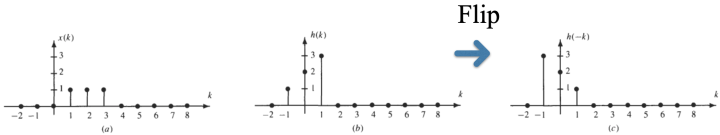


$n = 0$

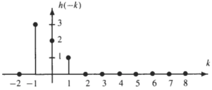


$y_0 = 1$

Discrete Conv.: $y = x * h$; $y_n = \left(\sum_{k=-\infty}^{\infty} x_k h_{n-k}\right) \Delta$

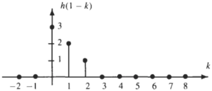


$n = 0$

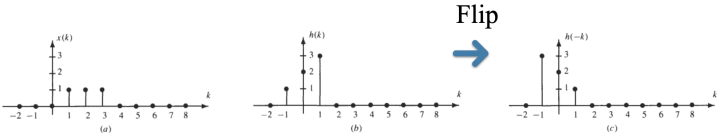


$y_0 = 1$

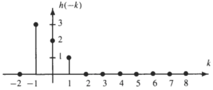
$n = 1$



Discrete Conv.: $y = x * h$; $y_n = \left(\sum_{k=-\infty}^{\infty} x_k h_{n-k}\right) \Delta$

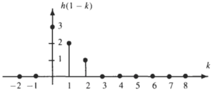


$n = 0$



$y_0 = 1$

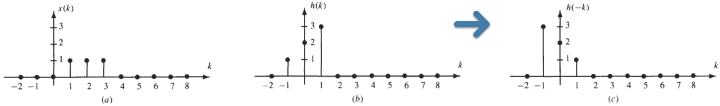
$n = 1$



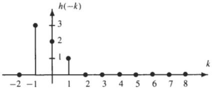
$y_1 = 3$

Discrete Conv.: $y = x * h$; $y_n = \left(\sum_{k=-\infty}^{\infty} x_k h_{n-k}\right) \Delta$

Flip

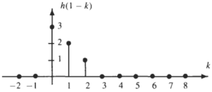


$n = 0$



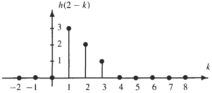
$y_0 = 1$

$n = 1$



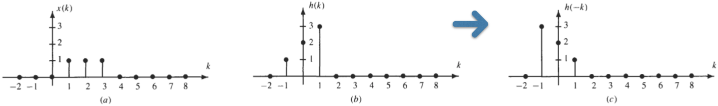
$y_1 = 3$

$n = 2$

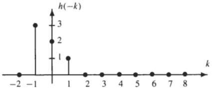


Discrete Conv.: $y = x * h$; $y_n = \left(\sum_{k=-\infty}^{\infty} x_k h_{n-k}\right)\Delta$

Flip

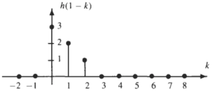


$n = 0$



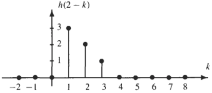
$y_0 = 1$

$n = 1$



$y_1 = 3$

$n = 2$



$y_2 = 6$

Discrete Conv.: $y = x * h$; $y_n = \left(\sum_{k=-\infty}^{\infty} x_k h_{n-k}\right) \Delta$

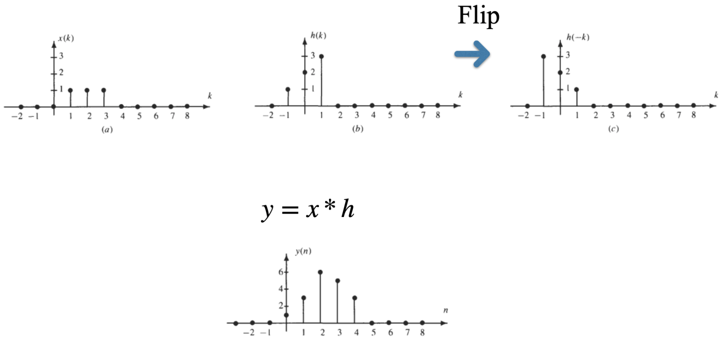


Figure: x_k, $k = 1, 2, 3$ and h_k, $k = -1, 0, 1$, then $y_k = 0, \cdots 4$ of length 5.

Convolution

$$g(t) = \int_{-\infty}^{\infty} w(\tau) f(t-\tau)\, d\tau \tag{5}$$

Suppose both $w(t)$ and $f(t)$ are of finite duration,

$$w(t) \neq 0,\ t_1 \leq t \leq t_2, \quad f(t) \neq 0,\ t_3 \leq t \leq t_4 \tag{6}$$

1. It follows that $g(t) \neq 0,\ t_1 + t_3 \leq t \leq t_2 + t_4$
2. for a particular t, the sum only needs to be performed for

$$max(t_1, t - t_4) \leq \tau \leq min(t_2, t - t_3) \tag{7}$$

3. Question: how do you implement this in discrete form?

Discrete convolution

$$g_n = \left[\sum_{k=-\infty}^{\infty} f_{n-k} w_k \right] \Delta \tag{8}$$

Suppose $f_i \neq 0, 0 \leq i \leq N_f - 1$ has total N_f points, and $w_k \neq 0, 0 \leq k \leq N_w - 1$ has total N_w points, then for a particular index n:

$$\begin{aligned} & 0 \leq n - k \leq N_f - 1 \Rightarrow n - N_f + 1 \leq k \leq n \\ & 0 \leq k \leq N_w - 1 \end{aligned} \tag{9}$$

gives $max(0, n - N_f + 1) \leq k \leq min(n, N_w - 1)$. But if

$$max(0, n - N_f + 1) > min(n, N_w - 1) \quad \text{i.e.,} \quad n < 0, n > N_f + N_w - 2, \tag{10}$$

then no non-zero terms in the summation, $g_n = 0$, ($n < 0$ or $n > N_f + N_w - 2$). In other words

$$g_n \neq 0, \quad 0 \leq n \leq N_f + N_w - 2 \tag{11}$$

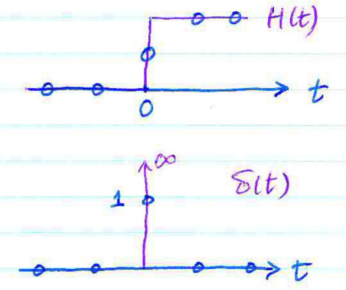
The convolved time series is of length $N_g = N_f + N_w - 1$.

Discretize step function $H(t)$ and Delta function $\delta(t)$

1. Heaviside function (step function) $H(t)$

$$H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{12}$$

   step response: $g(t) = H(t) * w(t)$
2. $\delta(t)$ function
3. Discretization:



$$\delta_0 = 1, \quad H_0 = 0.5 \tag{13}$$

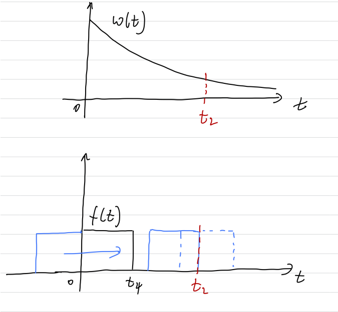
A special case

$$g(t) = \int_{-\infty}^{\infty} w(\tau) f(t-\tau)\, d\tau \tag{14}$$

In many applications, both $w(t)$ and $f(t)$ are causal, and $f(t)$ is of finite duration ($0 \leq t \leq t_4$). However, $w(t)$ may be of infinite duration with decreasing values at large time. For discrete implementation, $w(t)$ has to be truncated ($0 \leq t \leq t_2$), and according to the previous slide, the convolved time series $g(t)$ is non-zero between 0 and $t_2 + t_4$.

1. What is the effect of truncating $w(t)$? Are all the values of $g(t)$ between 0 and $t_2 + t_4$ accurate?

Convolution with a boxcar



Given

1. $f(t) \neq 0, \ 0 \leq t \leq t_4$
2. $w(t) \neq 0, \ 0 \leq t < \infty$ but truncated between $[0, t_2]$

only $g(t) \neq 0, \ 0 \leq t \leq t_2$ may be accurately computed by convolution.