

PHY408

Lecture 9: Z-Transform, Digital Filters

March 15, 2023

Z transform

For a time series f_0, f_1, \dots , define its z-transform as

$$F(z) = f_0 + f_1 z + f_2 z^2 + \dots = \sum_{k=0}^{\infty} f_k z^k \quad (1)$$

- ① express f_k as coefficients of a polynomial (or power series) of z , an analytic function of complex variable z , well-behaved within a region of convergence (ROC); a convenient theoretical tool
- ② why z-transform? If $f_k \rightarrow F(z)$ and $g_m \rightarrow G(z)$, then

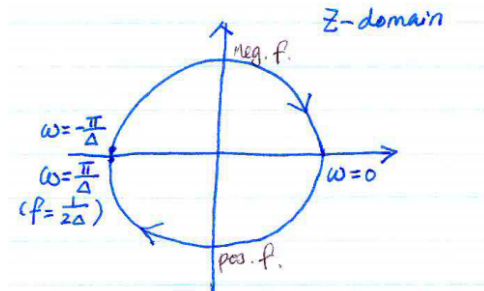
$$\begin{aligned} H(z) &= F(z)G(z) = \sum_k \sum_m f_k g_m z^{k+m} = \sum_n \sum_k f_k g_{n-k} z^n \\ &= \sum_n h_n z^n \leftarrow h_n = \sum_k f_k g_{n-k} \end{aligned} \quad (2)$$

This is the **convolution theorem** for the z-transform.

- ③ $F(z) \Rightarrow f_n$: polynomial series expansion and reading off the coefficients in front of z^n .
- ④ Note, some books follow the definition of $F(z) = \sum_{k=0}^{\infty} f_k z^{-k}$.

Relationship between z-transform and FT

Let $z = e^{-i\omega\Delta}$, i.e. z clock-wisely goes along the unit circle as $\omega \in [-\frac{\pi}{\Delta}, \frac{\pi}{\Delta}]$ or freq $f \in [-f_N, f_N]$, where $f_N = \frac{1}{2\Delta}$ is the Nyquist frequency.



- 1 $F(z = e^{-i\omega\Delta}) \Rightarrow F(\omega) = \sum_k f_k e^{-i\omega k\Delta}$ is the FT of time series f_k
- 2 implicit periodicity $e^{i\phi} = e^{i(\phi+2\pi)}$; one cycle $f \in [-f_N, f_N]$.

Properties of Z transform

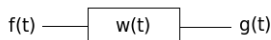
Properties of z transform

- linearity: $ax_n + by_n \rightarrow aX(z) + bY(z)$
- time shifting: $w_n \rightarrow W(z)$, then $w_{n-k} \rightarrow W(z)z^k$.
- complex conjugation: $x_n \rightarrow X(z)$, then $x_n^* \rightarrow X^*(z)$, i.e., only conjugate the coefficients of $X(z)$, not z itself ($a + bz$ becomes $a^* + b^*z$).
- time reversal: $x_n \rightarrow X(z)$, then $x_{-n} \rightarrow X(1/z)$

Often, system function $W(z)$ can be written as $\frac{Q(z)}{P(z)}$, where $P(z)$ and $Q(z)$ are polynomials of z . How should $W(z)$ be applied to an input signal $F(z)$?

- two-step operation: $G(z) = F(z)W(z) = [F(z)Q(z)]\frac{1}{P(z)}$
- $D(z) = [F(z)Q(z)]$ is a simple convolution of f_n and q_n .
- $G(z) = \frac{D(z)}{P(z)}$ involves deconvolution and inverse filtering, or $D(z) = P(z)G(z)$, i.e. what is the 'input' $G(z)$ given 'output' $D(z)$ and known system function $P(z)$.

Deconvolution by Polynomial Division



If $f_k * w_k = g_k$, then $G(z) = W(z)F(z)$. Given output $G(z)$, input

$$F(z) = \frac{G(z)}{W(z)} \quad (3)$$

which can be realized by polynomial long-division

① Example: $w = [1, 1]$, $g = [1, 4, 6, 4, 1]$, therefore

$$F(z) = \frac{1 + 4z + 6z^2 + 4z^3 + z^4}{1 + z} = 1 + 3z + 3z^2 + z^3 \quad (4)$$

② Limited to very simple time series of finite-length

Deconvolution by Inverse Filtering

If $y(t) * a(t) = x(t)$, $Y(z)A(z) = X(z)$, then given observation of output signal $x(t)$, how do we retrieve the input signal $y(t)$ if $a(t)$ is known as $[a_0, a_1, \dots]$? Define $B(z) = 1/A(z)$,

$$Y(z) = \frac{X(z)}{A(z)} = \frac{X(z)}{a_0 + a_1 z + \dots} = X(z)B(z) \quad (5)$$

Zeros of $A(z)$ are the poles of $B(z)$.

❶ Example: $A(z) = 1 - \frac{1}{2}z$ and $B(z) = \frac{1}{1 - \frac{z}{2}}$ (with pole at 2), then

$$B(z) = \frac{1}{A(z)} = \frac{1}{1 - \frac{z}{2}} = \left(1 - \frac{z}{2}\right)^{-1} = 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \frac{z^4}{16} + \dots \quad (6)$$

and $b = [1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots]$ converges (ratio test) and is causal!

- ❷ pole for $B(z)$ at $z = 2$; ROC for z : $|z| < 2$, a circular region inside the pole that includes the unit circle.
- ❸ convergence means terms at large n can be ignored, i.e. filter b_n can be truncated.
- ❹ **realizable time series** b_n : **stable** (absolutely summable) and **causal**

Inverse Filtering

For a general

$$A(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_N z^N = a_N (z - r_1)(z - r_2) \cdots (z - r_N) \quad (7)$$

i.e. the input signal goes through N filters in series, the inverse filter

$$B(z) = \frac{1}{a_N} \frac{1}{z - r_1} \frac{1}{z - r_2} \cdots \frac{1}{z - r_N} \quad (8)$$

- ❶ $B(z)$ realizable: $|r_i| > 1$, for $i = 1, \dots, N$, i.e. all poles are **outside the unit circle**.
- ❷ Implementation, $x(t)$ goes through each individual $B_i(z)$, however $B_i(z)$ may correspond to a long time series, making the entire filter time-consuming.
- ❸ If $A(z)$ is relatively short: feedback filtering

Inverse Filtering

Suppose $y(t) * a(t) \Rightarrow x(t)$, and now from the output $x(t)$ how can we retrieve the input time series $y(t)$, given $A(z) = a_0 + a_1z + \dots + a_{M-1}z^{M-1}$?

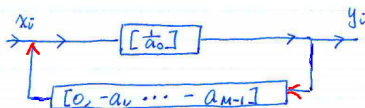
1 From convolution

$$x_i = \sum_{k=0}^{M-1} a_k y_{i-k} = a_0 y_i + \sum_{k=1}^{M-1} a_k y_{i-k} \quad (9)$$

Now we can retrieve y_i given x_i and $y_j, j = 0, \dots, i-1$, provided $a_0 \neq 0$:

$$y_i = \frac{1}{a_0} \left[x_i - \sum_{k=1}^{M-1} a_k y_{i-k} \right] \quad (10)$$

2 iterative computation: obtain the current value of y_i based on current value of x_i and previous values of y , involving only extra M computation per point, i.e., the shorter the a series, the more efficient the filter.



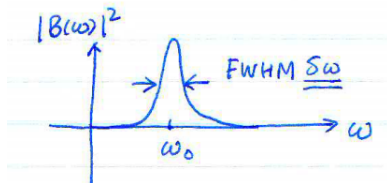
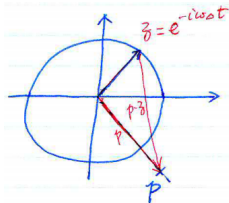
Filter Design

For given characteristics of a filter in the freq domain, choose a proper function form and solve for related parameters

- 1 real filters are often required \Rightarrow real-even and imaginary odd spectrum.
- 2 For example, a narrow band filter at ω_0 , we use the function form

$$B(z) = \frac{p}{p - z} \quad (11)$$

where $p = (1 + \epsilon)e^{-i\omega_0}$ (outside unit circle!), and assuming $\Delta t = 1$



- 3 The full-width-half-maximum (i.e. the sharpness of the narrow-band filter) is determined by parameter ϵ

Example: Narrow-band filter

Its frequency spectrum:

$$|B(z = e^{-i\omega})|^2 = \frac{(1 + \epsilon)^2}{\epsilon^2 + 4(1 + \epsilon) \sin^2 \frac{\omega - \omega_0}{2}} \quad (12)$$

- ① usually $\epsilon \ll 1$,
- ② at $\omega = \omega_0$, $|B(\omega)|^2$ takes the maximum value $\frac{(1+\epsilon)^2}{\epsilon^2}$
- ③ if at $\omega = \omega_0 \pm \delta\omega/2$, $|B(\omega)|^2 = \frac{1}{2}|B(\omega_0)|^2$ ($\delta\omega$ is so-called Full Width Half Maximum(FWHM)), then

$$\epsilon^2 = 4(1 + \epsilon) \sin^2(\delta\omega/4) \sim (1 + \epsilon)\delta^2\omega/4 \quad (13)$$

and $\epsilon \sim \frac{1}{2}\delta\omega$, relating function parameters to filter characteristics.

Narrow-band filter

In order to generate a real time filter, we need $B(-\omega) = B^*(\omega)$. Given $B(\omega) = \frac{p}{p - e^{-i\omega}}$ for $\omega > 0$, we have $B(-\omega) = B^*(\omega) = \frac{p^*}{p^* - e^{i\omega}}$, i.e.

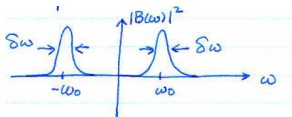
$$B(\omega) = \frac{p^*}{p^* - e^{-i\omega}}, \quad \omega < 0 \quad (14)$$

or $B(z) = \frac{p^*}{p^* - z}$ for a series corresponding to neg freq. spectrum, i.e., we need to add poles and zeros that are **complex conjugate** of the poles and zeros for original $B(z)$.

Multiply together the positive and negative freq. spectrum:

$$B(z) = \frac{p}{p - z} \frac{p^*}{p^* - z} = \frac{1}{1 - \frac{(p+p^*)z}{|p|^2} + \frac{z^2}{|p|^2}} \quad (15)$$

which can be implemented as an **inverse filter** of $a = \left[1, -\frac{p+p^*}{|p|^2}, \frac{1}{|p|^2}\right]$, a 3-pt **real** time series.

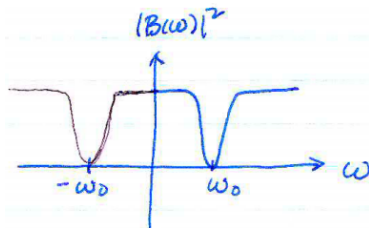
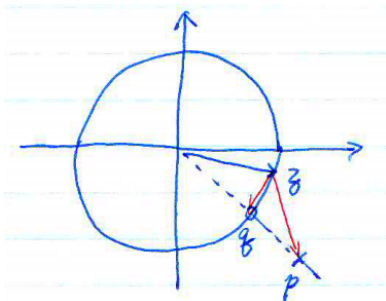


More Examples: Notch filter

- 1 bandpass filter: composed of a series of narrow band filters to cover the desired frequency range
- 2 notch filter:

$$B(z) = \frac{z - q}{z - p}, \quad \text{where } p = (1 + \epsilon)e^{-i\omega_0} \text{ and } q = e^{-i\omega_0} \quad (16)$$

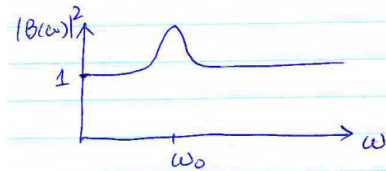
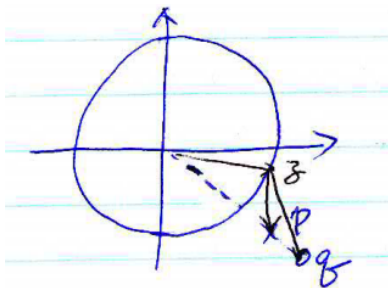
i.e. pole p outside the unit circle and zero q on the unit circle



More Examples

1 Pedestal filter

$$B(z) = \frac{z - q}{z - p}, \quad \text{where } p = (1 + \epsilon)e^{-i\omega_0} \text{ and } q = (1 + 2\epsilon)e^{-i\omega_0} \quad (17)$$



Filter design

- 1 If we require our filter to be **zero-phase**, i.e. $B(\omega)$ to be real, then b_n has to be real-even and imaginary-odd, which means b_n is **non-causal**
- 2 On the other hand, if b_n is **causal**, then $B(\omega)$ can not be a pure real function, therefore, **phase shift** is introduced to the input signal after filtering.
- 3 Sometimes filter characteristics are given in terms of $|B(\omega)|^2$ (square amplitude response). Through bilinear transformation: $i\omega\Delta t \rightarrow 2\frac{1-z}{1+z}$, it may be converted to

$$|B(i\omega)|^2 = B(i\omega)B^*(-i\omega) \rightarrow B(z)B^*(1/z) = S(z) = \frac{z - q}{z - p} \frac{\frac{1}{z} - q^*}{\frac{1}{z} - p^*} \quad (18)$$

whose poles and zeros are **conjugate reciprocal pairs** (different from the real filter requirement!). Then there exists several $B(z)$ that may have the same amplitude spectrum.