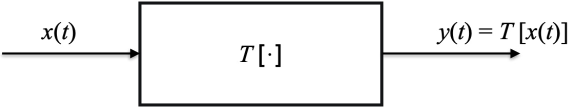
PHY408

Lecture 12: Review

April 5, 2023

Linear systems

A system maps an input signal $x(t)$ onto an output signal $y(t)$.



- Homogeneity:

$$T[ax(t)] = aT[x(t)] = ay(t) \tag{1}$$

- Additivity:

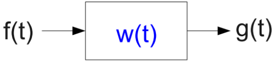
$$\begin{aligned} T[x_1(t) + x_2(t)] &= T[x_1(t)] + T[x_2(t)] \quad (2) \\ &= y_1(t) + y_2(t) \quad (3) \end{aligned}$$

- Time-invariant systems:

$$T[x(t-s)] = y(t-s) \tag{4}$$

- Linear time-invariant systems (LTI): Can be represented by the response of the system to a unit pulse. This response is called the impulse response function of the system.

Convolution



Convolution of $w(t)$ and $f(t)$ is defined as

$$g(t) = \int_{-\infty}^{\infty} w(\tau) f(t-\tau) d\tau = f(t) * w(t) \tag{5}$$

1. Commutative: $g(t) = f(t) * w(t) = w(t) * f(t)$
2. $w(t)$ also called system function: given any input signal $f(t)$, the output $g(t)$ is uniquely determined by $f(t) * w(t)$

Convolution theorem

Fourier transform of a convolution is simply the product of the Fourier transforms of the two functions being convolved.

$$g(t) = \int_{-\infty}^{\infty} w(\tau) f(t-\tau) d\tau \tag{6}$$

using IFT

$$\begin{aligned} w(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} W(\omega') e^{i\omega'\tau} d\omega' \\ f(t-\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega(t-\tau)} d\omega \end{aligned} \tag{7}$$

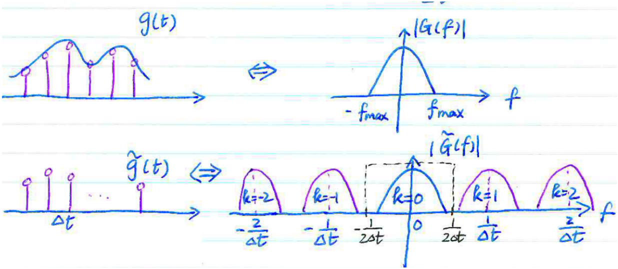
$$g(t) = \frac{1}{2\pi} \int W(\omega) F(\omega) e^{i\omega t} d\omega \tag{8}$$

According to the interchangeability of $t \leftrightarrow \omega$,

$$g(t) = w(t) f(t) \Leftrightarrow G(\omega) = \frac{1}{2\pi} W(\omega) * F(\omega) \tag{9}$$

Discrete Fourier transform

$$G_k = \Delta t \sum_{j=0}^{N-1} g_j e^{-i2\pi kj/N}, \qquad g_j = \frac{1}{N\Delta t} \sum_{k=0}^{N-1} G_k e^{i2\pi kj/N}. \tag{10}$$



- Periodic, with periodicity determined by $1/\Delta t$.
- For a signal between $[0, T]$, the frequency resolution is $\delta f = \frac{1}{T}$.
- $f_{max} < \frac{1}{2\Delta t} = \frac{1}{2} f_s$.

Fast Fourier Transform (FFT)

For an $N = 2$ sequence $x = \{x_0, x_1\}$

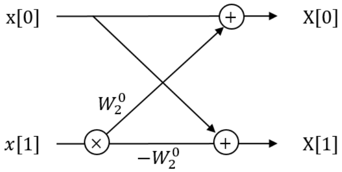
$$X_k = \sum_{n=0}^{N-1} x_n e^{-i2\pi kn/N} = \sum_{n=0}^{N-1} x_n e^{-i\pi kn} \tag{11}$$

$$= x_0 + x_1 e^{-i\pi k} \tag{12}$$

So the FT is given by

$$X_0 = x_0 + x_1 \tag{13}$$

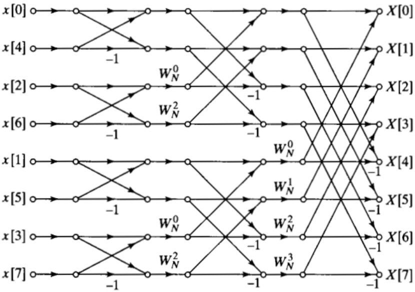
$$X_1 = x_0 - x_1 \tag{14}$$



Here $W_2^n = e^{-i2\pi n/2} = (-1)^n$.

N=8 FFT

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{bmatrix} \xrightarrow{\text{bit-reversal}} \begin{bmatrix} 000 \\ 100 \\ 010 \\ 110 \\ 001 \\ 101 \\ 011 \\ 111 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_4 \\ x_2 \\ x_6 \\ x_1 \\ x_5 \\ x_3 \\ x_7 \end{bmatrix}$$



Z-transform

- For a time series $f_0, f_1, \cdots$, the z-transform is defined as a power series of z with f_k as the coefficients,

$$F(z) = f_0 + f_1 z + f_2 z^2 + \cdots = \sum_{k=0}^{\infty} f_k z^k \tag{15}$$

- If $g_k = f_k * w_k$, then $G(z) = F(z)W(z)$.

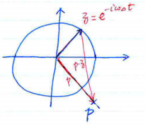
  Often, the system function $W(z)$ can be written as $\frac{Q(z)}{P(z)}$. How should $W(z)$ be applied to an input signal $F(z)$?
- two-step operation: $G(z) = F(z)W(z) = [F(z)Q(z)]\frac{1}{P(z)}$
- $D(z) = [F(z)Q(z)]$ is a simple convolution of f_n and q_n.
- $G(z) = \frac{D(z)}{P(z)}$ involves deconvolution and inverse filtering.

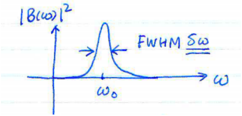
Z-transform: Filter design

For desired characteristics of a filter in the freq domain, choose a proper function form and solve for the related parameters

- For a narrow band filter at ω_0, we use the form

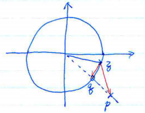
$$B(z) = \frac{p}{p - z}, \quad \text{where } p = (1+\epsilon)e^{-i\omega_0} \tag{16}$$

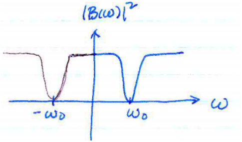




- For a notch filter, we use

$$B(z) = \frac{z - q}{z - p}, \quad \text{where } p = (1+\epsilon)e^{-i\omega_0} \text{ and } q = e^{-i\omega_0} \tag{17}$$





Cross-correlation and Power Spectra

Cross-correlation of two signals can be used to measure the similarity of two signals under time translation. It is given by

$$C_{fg}(\tau) = \int_{-\infty}^{\infty} f^*(t) g(t+\tau) dt. \tag{18}$$

1. The Fourier transform of C_{fg} is

$$C(\omega) = F^*(\omega) G(\omega). \tag{19}$$

2. If $f(t) = g(t)$, then the auto-correlation provides information on how similar the signal is to itself under time translation

$$a(\tau) = \int_{-\infty}^{\infty} f^*(t) f(t+\tau) dt. \tag{20}$$

Power spectrum of a stationary process

- Sometimes for a periodic signal or ergodic signal, $A(\omega) = \lim_{T\to\infty} \frac{1}{T}|F(\omega)|^2$, is the power spectrum, where

$$A(\tau) = \frac{1}{T}\int_0^T f^*(t)f(t+\tau)dt. \tag{21}$$

- We define Power spectrum of a stationary process as the averaged power spectrum of realizations

$$P(\omega) = \lim_{T\to\infty} \frac{1}{T} < |F_T(\omega)|^2 > \tag{22}$$

Importance of window function

1. What is the effect of estimating the autocorrelation function of $x(t)$ over the finite interval $0 \leq t \leq T$?

$$a(\tau) = \frac{1}{T} \int_0^T x(t)x(t+\tau)\, dt \Leftrightarrow \hat{P}(\omega) \tag{23}$$

2. Introduces leakage in the frequency components.
3. This can be corrected by applying window functions to $a(\tau)$ to reduce the effect of inaccuracy.
4. What is the effect of windowing?

$$\hat{P}_w(\omega) = \int_\theta W(\omega - \theta)P(\theta)d\theta \tag{24}$$

- wider center bands, lower frequency resolution
- subdued side bands, less spectral leakage, less variance.

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THANK YOU