PHY408

Lecture 7: Fast Fourier Transforms

March 1, 2023

Final report: Due April 26th (no extensions)

1. Maximum of 8 pages single spaced, with a 12 pt font size.
2. You can use extra pages for large figures; additional math references can also be on additional pages.
3. You can use any format for the references, and there is no lower or upper limit for references.
4. 5% for overall formatting of the report.
5. 20% for introduction and laying out the question you want to address clearly.
6. 50% for analysis and results.
7. 25% for final discussion of the results, discuss caveats, how this may be applicable more broadly, etc.
8. See the syllabus for links to possible sources of data.
9. We will use the University's plagiarism detection tool to provide to a similarity assessment of the final reports.

Fourier Transforms

$$\begin{aligned} G(f) &= \int_{-\infty}^{\infty} g(t) e^{-i2\pi ft}\, dt \\ g(t) &= \int_{-\infty}^{\infty} G(f) e^{i2\pi ft}\, df \end{aligned} \tag{1}$$

DFT:

$$\begin{aligned} G_k &= \Delta t \sum_j g_j e^{-i2\pi kj/N}, \\ g_j &= \frac{1}{N\Delta t} \sum_{k=0}^{N-1} G_k e^{i2\pi kj/N} \end{aligned} \tag{2}$$

Fast Fourier Transform (FFT)

Suppose N is even, $N = 2M$, $g_0, g_1, \cdots, g_{N-1} =$, take the odd and even points out as two separate time series a_k and b_k, $0 \le k \le M-1$:

$$a_0, b_0, a_1, b_1, \cdots a_{M-1}, b_{M-1} \tag{3}$$

Assume we already know the DFT of a_k and b_k, and let $v = e^{-i\pi/M}$

$$A_k = \sum_{j=0}^{M-1} a_j e^{-i2\pi kj/M} = \sum_j a_j v^{2kj} \quad B_k = \sum_{j=0}^{M-1} b_j e^{-i2\pi kj/M} = \sum_j b_j v^{2kj} \tag{4}$$

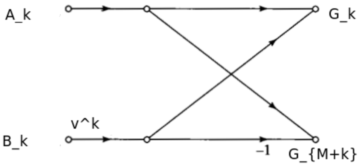
while

$$\begin{aligned} G_k &= \sum_{n=0}^{N-1} g_n e^{-i2\pi kn/N} = \sum_n g_n v^{nk} = \sum_{n=0,2}^{2M-2} g_n v^{nk} + \sum_{n=1,3}^{2M-1} g_n v^{nk} \\ &= \sum_{j=0}^{M-1} a_j v^{2jk} + \sum_{j=0}^{M-1} b_j v^{2jk+k} = A_k + B_k v^k \end{aligned} \tag{5}$$

computation cost: N operations from $\{A_k, B_k\}$ to G_k.

$v = e^{-i\pi/M}$ and $e^{-i\pi} = -1$, therefore $v^{M+k} = -v^k$, and given $A_{M+k} = A_k$ and $B_{M+k} = B_k$,

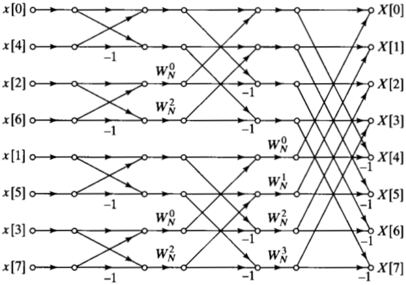
$$G_k = A_k + B_k v^k, \quad G_{M+k} = A_k - B_k v^k \tag{6}$$



FFT

computationally most efficient when $N = 2^p$

1. Divide x_k into even and odd series a_k and b_k, each of length $N/2$
2. Continue dividing a_k and b_k into even-and-odd series $p-1$ times until each sub-series has length of 1.
3. DFT of 1-point series is itself.
4. combine above DFT results p times to obtain X_k.



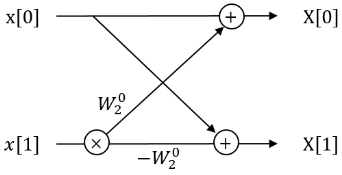
total number of computation is Np as compared to N^2 for DFT.

For an $N = 2$ sequence $x = \{x_0, x_1\}$

$$\begin{aligned} X_k &= \sum_{n=0}^{N-1} x_n e^{-i2\pi kn/N} = \sum_{n=0}^{N-1} x_n e^{-i\pi kn} && (7) \\ &= x_0 + x_1 e^{-i\pi k} && (8) \end{aligned}$$

So the FT is given by

$$\begin{aligned} X_0 &= x_0 + x_1 && (9) \\ X_1 &= x_0 - x_1 && (10) \end{aligned}$$



Here $W_2^n = e^{-i2\pi n/2} = (-1)^n$.

FFT

For an $N = 4$ sequence $x = \{x_0, x_1, x_2, x_3\}$

$$X_k = \sum_{n=0}^{N-1} x_n e^{(-i\pi/2)nk} \tag{11}$$

$$X_k = x_0 + x_1(-i)^k + x_2(-i)^{2k} + x_3(-i)^{3k} \tag{12}$$

since $e^{-i\pi/2} = -i$.

FFT

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since $e^{-i\pi/2} = -i$.
Thus the FT is given by

$$X_0 = x_0 + x_1 + x_2 + x_3 = (x_0 + x_2) + (x_1 + x_3) \tag{13}$$

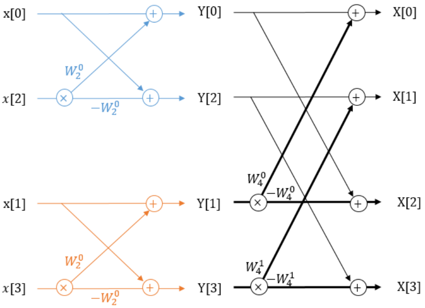
$$X_1 = x_0 - ix_1 - x_2 + ix_3 = (x_0 - x_2) - i(x_1 - x_3) \tag{14}$$

$$X_2 = x_0 - x_1 + x_2 - x_3 = (x_0 + x_2) - (x_1 + x_3) \tag{15}$$

$$X_3 = x_0 + ix_1 - x_2 - ix_3 = (x_0 - x_2) + i(x_1 - x_3) \tag{16}$$

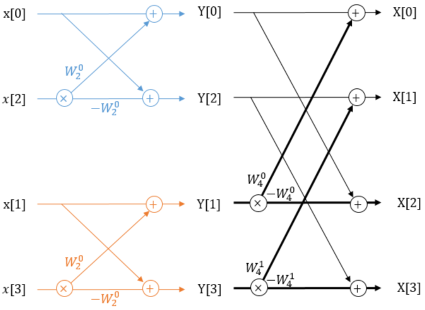
$N = 4$ FFT

$$
\begin{aligned}
X_0 &= x_0 + x_1 + x_2 + x_3 = (x_0 + x_2) + (x_1 + x_3) && (17)\\
X_1 &= x_0 - ix_1 - x_2 + ix_3 = (x_0 - x_2) - i(x_1 - x_3) && (18)\\
X_2 &= x_0 - x_1 + x_2 - x_3 = (x_0 + x_2) - (x_1 + x_3) && (19)\\
X_3 &= x_0 + ix_1 - x_2 - ix_3 = (x_0 - x_2) + i(x_1 - x_3) && (20)
\end{aligned}
$$



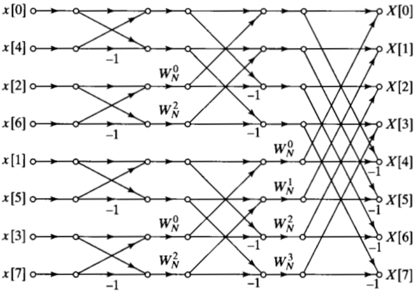
N=4 FFT

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 00 \\ 01 \\ 10 \\ 11 \end{bmatrix} \xrightarrow{\text{bit-reversal}} \begin{bmatrix} 00 \\ 10 \\ 01 \\ 11 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_2 \\ x_1 \\ x_3 \end{bmatrix}$$



N=8 FFT

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{bmatrix} \xrightarrow{\text{bit-reversal}} \begin{bmatrix} 000 \\ 100 \\ 010 \\ 110 \\ 001 \\ 101 \\ 011 \\ 111 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_4 \\ x_2 \\ x_6 \\ x_1 \\ x_5 \\ x_3 \\ x_7 \end{bmatrix}$$



Fast Fourier Transform (FFT)

1. How significant an improvement is this? For example, For $N = 1024$, $Np = 10 * 1024 \sim 10^4$ vs. $N^2 \sim 10^6$, a factor of 100 speed-up!
2. what if $N \neq 2^p$?

Fast Fourier Transform (FFT)

1. How significant an improvement is this? For example, For $N = 1024$, $Np = 10 * 1024 \sim 10^4$ vs. $N^2 \sim 10^6$, a factor of 100 speed-up!
2. what if $N \neq 2^p$? pad series to a length of nearest 2^p.

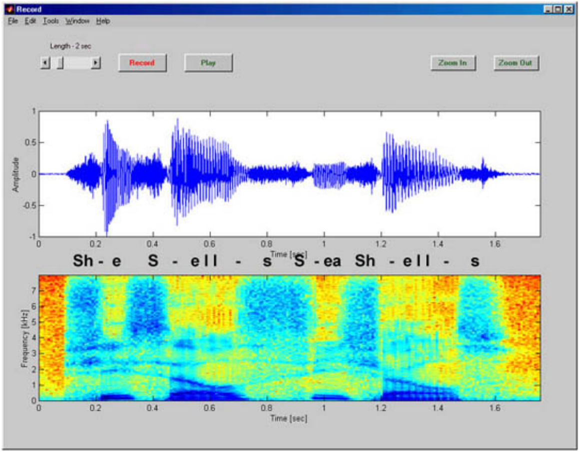
Short-time Fourier transform (time-dependent FT)

For a time-dependent (or non-stationary) signal $x(t)$, its STFT is given by

$$X(\omega, t) = \int_{-\infty}^{\infty} x(\tau)w(\tau - t)e^{-i\omega\tau}d\tau \tag{21}$$

1. time and frequency localized $w(\tau - t)$
2. trade-off in frequency and time domain resolution
3. useful for non-stationary signal
4. also known as spectrogram

Spectrogram of Voice



- voiced/unvoiced (vowels/consonants) sounds
- formants: acoustic resonance for vowels of human vocal tract.
- https://musiclab.chromeexperiments.com/Spectrogram/
- https://auditoryneuroscience.com/acoustics/spectrogram

Spectrogram of Music / Music of Spectrogram

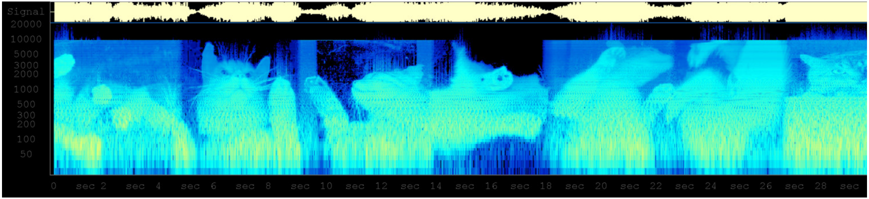


Figure: Hidden spectrogram image in track 14: ”Look” of *Songs About My Cats (2001)* by Venetian Snares.

Spectrogram of Music / Music of Spectrogram

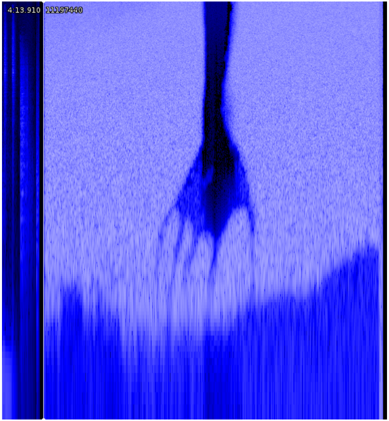


Figure: Hidden spectrogram in the leaked version of *My Violent Heart* by Nine Inch Nails, from the *Year Zero album*. (Source: Omegatron)