PHY408

Lecture 11: Power Spectrum Estimation

March 29, 2023

Final report: Due April 26th (no extensions)

1. Maximum of 8 pages single spaced, with a 12 pt font size.
2. You can use extra pages for large figures; additional math references can also be on additional pages.
3. You can use any format for the references, and there is no lower or upper limit for references.
4. 5% for overall formatting of the report.
5. 20% for introduction and laying out the question you want to address clearly.
6. 50% for analysis and results.
7. 25% for final discussion of the results, discuss caveats, how this may be applicable more broadly, etc.
8. See the syllabus for links to possible sources of data.
9. We will use the University's plagiarism detection tool to provide to a similarity assessment of the final reports.

Cross-correlation

Cross-correlation of two signals can be used to measure the similarity of two signals under time translation. It is given by

$$C_{fg}(\tau) = \int_{-\infty}^{\infty} f^*(t) g(t+\tau) dt. \tag{1}$$

1. shift $g(t)$ by τ with respect to $f(t)$, multiply it to $f^*(t)$ and integrate over time
2. $C_{fg}(\tau) \neq C_{gf}(\tau)$, $C_{fg}(\tau) = C_{gf}(-\tau)$
3. The Fourier transform of C_{fg} is

$$C(\omega) = F^*(\omega) G(\omega) \tag{2}$$

Auto-correlation

If $f(t) = g(t)$, then cross-correlation is simplified to the auto-correlation of $f(t)$ itself:

$$a(\tau) = \int_{-\infty}^{\infty} f^*(t) f(t+\tau) dt \tag{3}$$

1. provides information on how similar the signal is to itself under time translation.
2. $C(\omega) = F^*(\omega)F(\omega) = |F(\omega)|^2$, i.e. proportional to energy spectrum
3. sometimes for periodic signal or ergodic signal, auto-correlation is defined as:

$$a(\tau) = \frac{1}{T} \int_0^T f^*(t) f(t+\tau) dt \tag{4}$$

then $A(\omega) = \lim_{T\to\infty} \frac{1}{T} |F(\omega)|^2$, is the power spectrum.

Power spectrum of a stationary process

Recall that

$$F_T(\omega) = \int_{-T/2}^{T/2} x(t) e^{-i\omega t} dt. \tag{5}$$

Therefore, the power spectrum can be expressed as follows:

$$\begin{aligned}
P(\omega) &= \lim_{T\to\infty} \frac{1}{T} < |F_T(\omega)|^2 > \\
&= \lim_{T\to\infty} \frac{1}{T} < \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} x(t)x(t') e^{-i\omega t} e^{i\omega t'} dt' dt > \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} < x(t)x(t') > e^{-i\omega(t-t')} dt' dt \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} A(\tau) e^{-i\omega\tau} dt' d\tau \\
&= \lim_{T\to\infty} \int_{-T/2}^{T/2} A(\tau) e^{-i\omega\tau} d\tau = \int_{-\infty}^{\infty} A(\tau) e^{-i\omega\tau} \, d\tau
\end{aligned} \tag{6}$$

Power spectrum of a stationary process

- Sometimes for a periodic signal or ergodic signal, $A(\omega) = \lim_{T\to\infty} \frac{1}{T}|F(\omega)|^2$, is the power spectrum, where

$$A(\tau) = \frac{1}{T}\int_0^T f^*(t)f(t+\tau)dt \tag{7}$$

- We define Power spectrum of a stationary process as the averaged power spectrum of realizations

$$P(\omega) = \lim_{T\to\infty} \frac{1}{T} < |F_T(\omega)|^2 > \tag{8}$$

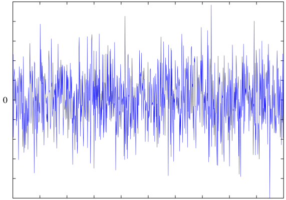
Stationary Process and LTI system

1. At different frequencies

$$
\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T} < F_T(\omega_1) F_T(\omega_2) > \\
= \quad &\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} A(\tau) e^{-i\omega_1\tau} d\tau \int_{-T/2}^{T/2} e^{-i(\omega_1-\omega_2)t'}\, dt' = 0 \quad (9)
\end{aligned}
$$

   i.e. $F(\omega_1)$ and $F(\omega_2)$ are uncorrelated if $\omega_1 \neq \omega_2$, i.e. a stochastic process consists of an infinite set of uncorrelated oscillators.

2. White noise X_i: X_i's are indep., $< X_i >= 0$ and $Var(X_i) = \sigma^2$



$a(t) = \delta(t)$ and its power spectrum $P(\omega) = 1$

Stationary Process

1. Stationary process can be considered as a convolution of a white noise with a LTI system. $P_s(\omega) = P_{LTI}(\omega)$, where $P_{LTI}(\omega)$ is the power spectrum of the deterministic system function.
2. How do we estimate the Power spectrum?

$$P(\omega) = \lim_{T \to \infty} \frac{1}{T} < |F_T(\omega)|^2 > \tag{10}$$

3. Periodogram estimates: for one realization $x_T(t) \Leftrightarrow F_T(\omega)$:

$$\hat{P}(\omega) = \frac{1}{T} |F_T(\omega)|^2 \tag{11}$$

4. Increasing T, or increasing number of points n will give better (finer) frequency resolution $1/T$. However, taking a long time series does not accomplish the averaging. Average over different realizations is required.

Periodogram

1. We know that

$$A(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)x(t+\tau)dt \Leftrightarrow P(\omega). \tag{12}$$

   However a finite sample of $x(t)$, $0 \leq t \leq T$ makes the integration less accurate as $t + \tau$ falls out of $[0, T]$, which will in turn affect the accuracy of values at every single ω of $\hat{P}(\omega)$ (fluctuations at all f). Indeed,

$$Var(\hat{P}(\omega)) = P^2(\omega). \tag{13}$$

2. How to reduce this scatter? Averaging over realizations!
3. Make multiple measurements: split data into M sub-series, and average over periodogram estimates of all, then $Var(\hat{P_M}(\omega)) = \frac{1}{M} P^2(\omega)$, however, frequency spacing becomes M times of the entire series.
4. Or sum M neighbouring frequencies of $\hat{P}(\omega)$.

Importance of window function

1. What is the effect of estimating the autocorrelation function of $x(t)$ over the finite interval $0 \leq t \leq T$?

$$a(\tau) = \frac{1}{T} \int_0^T x(t)x(t+\tau)\, dt \Leftrightarrow \hat{P}(\omega) \tag{14}$$

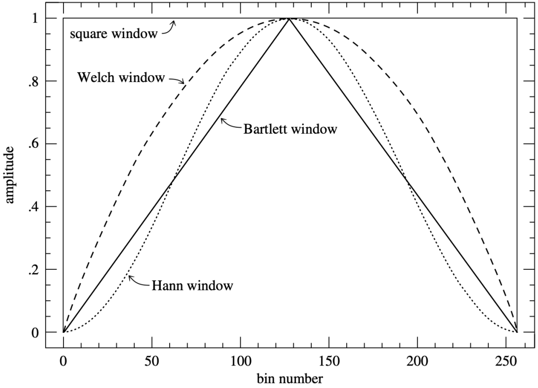
2. $a(\tau)$ values are less accurate as τ approaches T.
3. Introduces leakage in the frequency components.
4. This can be corrected by applying window functions to $a(\tau)$ to reduce the effect of inaccuracy: $w(\tau)a(\tau) \Leftrightarrow \hat{P_w}(\omega)$
5. What is the effect of windowing?

$$\hat{P}_w(\omega) = \int_\theta W(\omega - \theta)P(\theta)d\theta \tag{15}$$

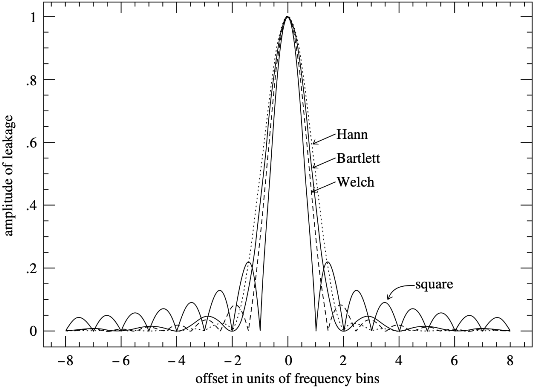
local average of the true spectrum.

Hann/Hanning/Cosine bell window

$$w(t) = \frac{1}{2}(1 - \cos\frac{2\pi t}{T}) \tag{16}$$



Response of window functions



Effect of windowing:

- wider center bands, lower frequency resolution
- subdued side bands, less spectral leakage, less variance.