Notes on Quantum Field Theory II

Afiq Hatta

January 20, 2020

Contents

1 Path Integrals

In this section, we'll introduce the path integral in QM, look at some methods with integrals, and then explore Feynman rules. Throughout these notes, we'll leave $\hbar$, but feel free to set this to 1 throughout the course. Let's introduce the path integral from the standpoint of quantum mechanics. The goal here is to take Schrödinger's equation and reformulate it into a "path integral", which is roughly speaking, a weighted integral summing over all probable paths. Let's consider a Hamiltonian in just one dimension, which as usual we can decompose into a kinetic and potential term

$$\hat{H} = H\left(\hat{x}, \hat{p}\right) = \frac{\hat{p}^2}{2m} + V\left(\hat{x}\right), \quad \text{with } [\hat{x}, \hat{p}] = i\hbar$$

Schrödinger's equation says thst if we have a state, its time evolution is governed the equation below, which we write as its formal solution by

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \overline{H} |\psi(t)\rangle$$

Our formal solution is given by mutliplying by the time evolution operator

$$|\psi(t)\rangle = e^{-iH\frac{t}{\hbar}} |\psi(0)\rangle$$

In the Schrodinger picture, we have that

- States evolve in time
- Operators and their eigenstates are constant in time (fixed).

Wavefunctions in position space are denoted

$$\psi(x,t) = \langle x|\psi(t)\rangle$$

This gives Schrodingers equation as

$$\langle \hat{x}| \overline{H} |\psi(t)\rangle = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x,t)$$

How do we convert this differential equation into an integral equation? We have that

$$\begin{aligned}
\psi(x,t) &= \langle x| e^{-iH\frac{t}{\hbar}} |\psi(0)\rangle \\
&= \int_{-\infty}^{\infty} \langle x| e^{\frac{-iHt}{\hbar}} |x_0\rangle \langle x_0|\psi(0)\rangle \\
&= \int_{-\infty}^{\infty} dx_0 K(x, x_0; t)\psi(x_0, 0)
\end{aligned}$$

We can introduce an integral quite straightforwardly by introducing a projection operator onto initial positions. We insert a complete set of states

$$I = \int dx_0 |x_0\rangle \langle x_0|$$

We call $K(x, x_0; t)$ the Kernel. Repeat this procedure for n intermediate times and positions. Let $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$, and we factor

$$e^{-\frac{iHT}{\hbar}} = e^{-\frac{i}{\hbar}\overline{H}(t_{n+1}-t_n)} \dots e^{-\frac{i}{\hbar}\overline{H}(t_1-t_0)}$$

Then,

$$K(x, x_0, T) = \int_{-\infty}^{\infty} \left[\prod_{r=1}^{n} dx_r \langle x_{r+1}| e^{-\frac{i\overline{H}}{\hbar}(t_{r+1}-t_r)} |x_r\rangle \right] \langle x_1| e^{-\frac{i\overline{H}}{\hbar}(t_1-t_0)} |x_0\rangle$$

Integrals are over all possible position eigenstates at times $t_r, r = 1, \dots n$. Consider a free theory, with $V(\overline{x}) = 0$. Let's define a corresponding free kernel

$$K_0\left(x, x'; t\right) = \langle x| \exp\left(\frac{i\hat{p}^2}{2m}t\right) |x'\rangle$$

Insert, on the right side, the completeness relation for the identity.

$$I = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} |p\rangle \langle p| \, , \langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}}$$

This gives

$$K_0\left(x, x'; t\right) = \int_{\infty}^{\infty} \frac{dp}{2\pi\hbar} e^{-ip^2t/2m\hbar} e^{ip(x-x')/\hbar} = e^{\frac{im(x-x')^2}{2\hbar t}} \sqrt{\frac{m}{2\pi i\hbar t}}$$

Note,

$$\lim_{t\to 0} K_0\left(x, x'; t\right) = \delta(x - x')$$

which is as expected from $\langle x|x'\rangle = \delta(x - x')$. From the Baker-Campbell-Haudorf formula, we have that

$$e^{\epsilon\hat{A}} e^{\epsilon\hat{B}} = \exp\left(\epsilon\overline{A} + \epsilon\overline{B} + \frac{\epsilon^2}{2}\left[\overline{A}, \hat{B}\right] + \dots\right) \neq e^{\epsilon(\hat{A}+\hat{B})}$$

For small ϵ, we have that

$$e^{\epsilon(\hat{A}+\hat{B})} = e^{\epsilon\hat{A}} e^{\epsilon\hat{B}} \left(1 + O(\epsilon^2)\right)$$

Now let $\epsilon = \frac{1}{n}$, raise the above to the n power, so we have the result that

$$e^{\hat{A}+\hat{B}} = \lim_{n\to\infty} \left(e^{\hat{A}/n} e^{\hat{B}/n}\right)^n$$

Take $t_{r+1} - t_r = \delta t$, with $\delta t \ll T$. Also take n large such that $n\delta t = T$. Then we can write that

$$e^{\frac{-i\hat{H}\delta t}{\hbar}} = \exp\left(\frac{-i\hat{p}^2\delta t}{2m\hbar}\right) \exp\left(-\frac{iV(\hat{x})\delta t}{\hbar}\right) \left[1 + O(\delta t)^2\right]$$

Writing out the above, this gives us

$$\langle x_{r+1}| \exp\left(-\frac{i\hat{H}\delta t}{\hbar}\right) |x_r\rangle = e^{-iV(\hat{x})\delta t/\hbar} K_0(x_{r+1}, x_r; \delta t) = \sqrt{\frac{m}{2\pi i\hbar\delta t}} \exp\left[\frac{im}{2\hbar}\left(x_{r+1} - x_r/\delta t\right)^2 \delta t - \frac{i}{\hbar}V(x_r)\delta t\right]$$

with $T = n\delta t$. This gives our final expression as

$$K\left(x, x_0; T\right) = \int \left[\prod_{r=1}^{r} dx_r\right] \left(\frac{m}{2\pi i\hbar\delta t}\right)^{n+1/2} \exp\left[i\sum_{r=0}^{n}\left[\frac{m}{2}\left(\frac{x_{r+1} - x_r}{\delta t}\right)^2 - \frac{1}{\hbar}V(x_r)\right]\delta t\right]$$

In the limit $n \to \infty$, $\delta t \to 0$, with $n\delta t = T$, the exponent becomes

$$\frac{1}{\hbar}\int_0^T \left[\frac{1}{2}m\dot{X}^2 - V(x)\right] = \int_0^T dtL\left(x, \dot{x}\right)$$

where L is our classical Lagrangian. The classical action $S = \int dtL(x, \dot{x})$. The path integral

$$K\left(x, x_0; t\right) = \langle x| \, e^{-\frac{i\hat{H}t}{\hbar}} \, |x_0\rangle \int \mathcal{D}x e^{\frac{i}{\hbar}S}$$

The functional integral $\mathcal{D}x = \lim_{\delta t \to 0, n\delta T \text{ fixed}} (...) \prod_{r=1}^n (...dx_r)$. we wont need to care about normalisation factors.

Summary

1.0.1 Path Integral Derivations

- You get path integrals from repeatedly inserting the completeness relation

$$I = \int dx_0 \, |x_0\rangle \langle x_0|$$

- The kernel is

$$K\left(x, x_0, t\right) = \langle x| \, e^{-\frac{i\hat{H}t}{\hbar}} \, |x_0\rangle$$

- Our action is defined as

$$S = \int_0^T dtL\left(x, \dot{x}\right)$$

- Our measure is the two-way limit

$$\mathcal{D}x = \lim_{\delta t \to 0, n\delta \text{ fixed}} \sqrt{\frac{m}{2\pi i\hbar\delta t}} \prod_{r=1}^{n} \left(\sqrt{\frac{m}{2\pi i\hbar\delta t}} dx_r\right)$$

1.0.2 Feynman Diagrams

- For each graph with n vertices, we add a combinatoric factor of

$$\frac{|D_n|}{|G_n|} = \sum \frac{1}{|\text{Aut } \Gamma|}$$

2 Regularisation and Renormalisation

3 Gauge Theories

4 Formulating the Path Integral

In this section, we'll be moving on from our standard procedure of quantising a given Hamiltonian in quantum mechanics. We'll be introducing the concept of a path integral. The path integral is a 'functional integral' where we integrate over all possible paths with a Gaussian probability factor.

4.1 Classical and Quantum Mechanics

In classical mechanics, we use the Lagrangian as a conduit to encode the information about our physical system. The Lagrangian is given by a function of position and velocity, with

$$\mathcal{L} = \mathcal{L}\left(q_a, \dot{q}_a\right)$$

where $a = 1, \ldots, N$ is an index for each particle in our system. We can convert this to the Hamiltonian formalism where we put position and momentum on the same pedestal and define our conjugate momenta

$$p_a = \frac{\partial \mathcal{L}}{\partial \dot{q}^a}$$

We then work in terms of the Hamiltonian which is the Legendre transformation of the Lagrangian, where we eliminate $\dot{q}^a$ everywhere in the Lagrangian in favour of p^a as follows

$$H(q_a, p_a) = \sum_a \dot{q}_a p^a - \mathcal{L}\left(q_a, \dot{q}_a\right)$$

The quantum mechanical analog of this is the same. However, p_i and q_i are **promoted** to what we call operators, and obey commutation relations which as we know, eventually lead to discrete energy levels in the Hamiltonian. In quantum mechanics, we write the position and momentum operators as $\vec{q}^i$ and $\vec{p}^i$ for position and momentum respectively. In the Heisenberg picture of quantum mechanics, operators (and not states), depend on time. So, we impose the commutation relations for some fixed coordinate time $t \in \mathbb{R}$, where

$$\left[\vec{q}^i, \vec{p}_j\right] = i\delta^i{}_j$$

In classical field theory, we promote operators to fields instead. If $\phi(\vec{x}, t)$ represents a classical scalar field at some point in time t, then the field as well as it's conjugate momentum $\pi\left(\vec{x}, t\right)$ obey the commutation relations

$$[\phi(\vec{x}, t), \pi(\vec{y}, t)] = i\delta^3\left(\vec{x} - \vec{y}\right)$$

There is however a caveat in performing these approaches to quantisation. The theory is not manifestly Lorentz invariant. This is because when we imposed the equal time commutation relations above, we had to pick a preferred coordinate time t.

4.2 Formulating the Path Integral

We use the Hamiltonian as a starting point.

$$H(q,p) = \frac{p^2}{2m} + V(q)$$

The Schrödinger equation for a state $|\psi(t)\rangle$ which is time dependent is given by

$$i\frac{d}{dt}|\psi(t)\rangle = \hat{\mathbf{H}}|\psi(t)\rangle$$

Now, this is is a first order differential equation and can be solved provided that we have the right initial conditions. For now, let's just write down the solution in a 'formal' sense, where we 'exponentiate' the Hamiltonian whilst being vague about what this actually means. We write the solution as

$$|\psi(t)\rangle = \exp\left(-i\hat{\mathbf{H}}t\right)|\psi(0)\rangle$$

To do calculations however, we need construct an appropriate basis of states. For this section, we'll use the position basis $|q,t\rangle$, for $q,t \in \mathbb{R}$. These states are are defined to be the eigenstates of the position operator $\hat{\mathbf{q}}(t)$, so that

$$\hat{\mathbf{q}}(t)|q,t\rangle = q|q,t\rangle$$

We'll impose the condition that these states are normalised so that for a fixed time, we have

$$\langle q,t|q',t\rangle = \delta\left(q-q'\right)$$

We impose the analogous conditions as well for momentum eigenstates. For now though, we'll work in the Schrodinger picture so that $\hat{\mathbf{q}}$ is fixed and hence we have that the eigenstates $|q\rangle$ are time-independent. Since these states form a basis, we have that they obey the completeness relation

$$1 = \int d^3q\ |q\rangle\langle q|$$

We also label the time-independent momentum eigenstates as $|p\rangle$, and impose the completeness relation

$$1 = \frac{d^3p}{(2\pi)^3}|p\rangle\langle p|$$

Note the factor of 2π that we divide by. Other literature doesn't include this. With this set of basis states, we can now write the abstract state $|\psi(t)\rangle$ in terms of the position basis, where we denote

$$\psi(q,t) = \langle q|\psi(t)\rangle = \langle q|\exp\left(-i\hat{\mathbf{H}}t\right)|\psi(0)\rangle$$

We will put this into an integral form for reasons we will discuss later. To put any equation in integral form, the rule of thumb is to employ the completeness relations for either the position or momentum basis. We get that

$$\begin{aligned}
\langle q| \exp\left(-i\hat{\mathbf{H}}t\right)|\psi(0)\rangle &= \int d^3q' \ \langle q| \exp\left(-i\hat{\mathbf{H}}t\right)\left|q'\right\rangle\left\langle q'\middle|\psi(0)\right\rangle \\
&= \int d^3q' \ \langle q| \exp\left(-i\hat{\mathbf{H}}t\right)\left|q'\right\rangle \psi(q',0) \\
&= \int d^3q' \, K\left(q,q';t\right)\psi(q',0) \\
&= \int d^3q' K(q,q';t)\psi(q',0)
\end{aligned}$$

Here we've defined $K(q,q';t) = \langle q| \exp\left(-i\hat{\mathbf{H}}t\right)|q'\rangle$. Now to make progress, we need to find a meaningful expression for what $K\left(q,q;T\right)$ actually is. First, 'split up' our $\exp\left(-i\hat{\mathbf{H}}T\right)$ term into smaller pieces - that is, partition T as

$$\exp\left(-i\hat{\mathbf{H}}T\right) = \exp\left(-i\hat{\mathbf{H}}\left(t_{n+1}-t_n\right)\right)\exp\left(-i\hat{\mathbf{H}}\left(t_n-t_{n-1}\right)\right)\dots\exp\left(-i\hat{\mathbf{H}}\left(t_1-t_0\right)\right) \tag{1}$$

here, we set by definition that $t_{n+1} = T > t_n > t_{n-1} > \cdots > t_1 > t_0 = 0$. For example, setting $n = 1$ and inserting one integral as part of the completeness relation, we get that

$$\begin{aligned}
K\left(q,q';T\right) &= \langle q| \exp\left(-i\hat{\mathbf{H}}\left(t_2-t_1\right)\right)\exp\left(-i\hat{\mathbf{H}}\left(t_1-t_0\right)\right)\left|q'\right\rangle \\
&= \int dq_1 \ \langle q| \exp\left(-i\hat{\mathbf{H}}\left(t_2-t_1\right)\right)|q_1\rangle \langle q_1| \exp\left(-i\hat{\mathbf{H}}\left(t_1-t_0\right)\right)\left|q'\right\rangle
\end{aligned}$$

where we've set $t_2 = T$. We can generalise this to the case where we have n time slices. We have that

$$K\left(q,q',r\right) = \int \prod_{i=1}^{n}\left(dq_r \ \langle q_{r+1}| \exp\left(-i\hat{\mathbf{H}}\left(t_{r+1}-t_r\right)\right)|q_r\rangle\right)\langle q_1| \exp\left(-i\hat{\mathbf{H}}\left(t_1-t_0\right)\right)\left|q'\right\rangle \tag{2}$$

Example Sheet 1

Question 1 (2018)

We expand the exponential involving λ as

$$\begin{aligned}\mathcal{Z}(\lambda) &= \frac{1}{\sqrt{2\pi}} \int dx e^{-\frac{1}{2}x^2} \sum_{l=0}^{n} \left(-\lambda \frac{x^4}{4!}\right)^l \frac{1}{l!} \\ &= \sum_{l=0}^{n} \frac{1}{\sqrt{2\pi}} \left(-\frac{\lambda}{4!}\right)^l \frac{1}{l!} \int_{\mathbb{R}} dx \, e^{-\frac{1}{2}x^2} x^{4l}\end{aligned}$$

Now, we evaluate the integral using a trick. We arbitrarily set

$$I(\alpha) = \int dx \, e^{\frac{1}{2}\alpha x^2}$$

Differentiating this integral with respect to α, we have that

$$\frac{d^{2l} I}{d\alpha^{2l}} = \int_{\mathbb{R}} dx \left(\frac{1}{2}\right)^{2l} x^{4l} e^{-\frac{\alpha}{2}x^2} = \sqrt{2\pi} \left(\frac{1}{2}\right)^{2l} 1\,(3) \ldots (4l-1)$$

Cancelling out factors and using the standard formula for odd factorials, we get that

$$\int dx \, x^{4l} e^{-\frac{\alpha}{2}x^2} = \sqrt{2\pi} \frac{(4l)!}{4^l\,(2l)!}$$

Substituting this in means that we get our expression for our partition function as

$$\mathcal{Z}_n(\lambda) = \sum_{l=0}^{n} \left(-\frac{\lambda}{4!}\right)^l \frac{(4l)!}{4^l\,(2l)!}$$

Our contributing Feynman diagrams at $l \ll 3$ are shown in the figure. At $l = 1$, $a_l = \frac{1}{8}$, which is in agreement with a figure of 8 diagram. At $l = 2$, $a_l = \frac{35}{384}$, which agrees with the sum of the automorphism factors at 2 loops.

At $l = 3$, $a_l = \frac{385}{3072}$.

We need to sum multiple diagrams, which are connected with n loops to get terms in the expansion. There are two ways to get terms in the expansion. One is to sum all possible diagrams, the other is to sum connected diagrams with a certain number of loops!

What are the possible 3 loop diagrams? What are the automorphism factors?

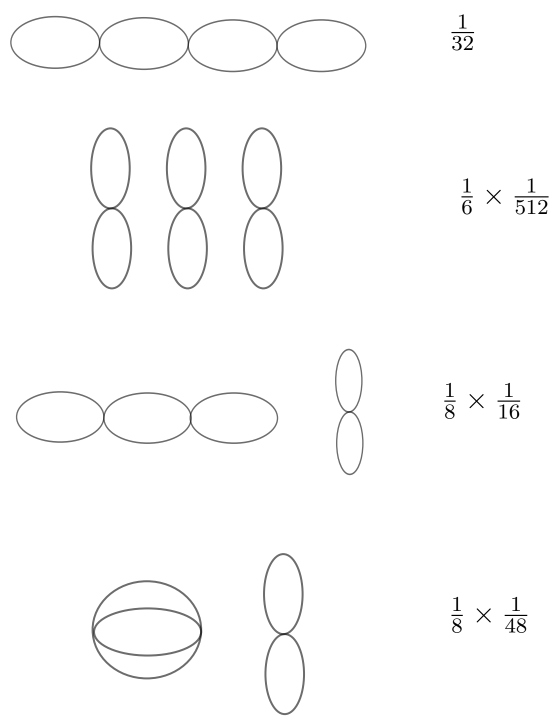

Figure 1: Feynman diagrams and their automorphism factors