An Introduction to Quantum Games: History and Mathematical Significance

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Road-map

- Parallel developments: a history of quantum and game theory, Von Neumann's role, and the point of collision with quantum information!
- The math of the matter.
- The Spin Flip game: classical to quantum.
- The Prisoner's dilemma: classical to quantum.
- (Time permitting) Number guessing games, Schor's factoring, and the RSA crypto game.

Last 15 minutes

Questions with free discussion.

History and development of quantum physics

The old and new paradigms...

Figure 1: Someone ought to recreate this iconic photo at the 1927 Solvay conference

Meanwhile: N-person games, and the linear programming problem

Cooperation may be favourable in N-person games!

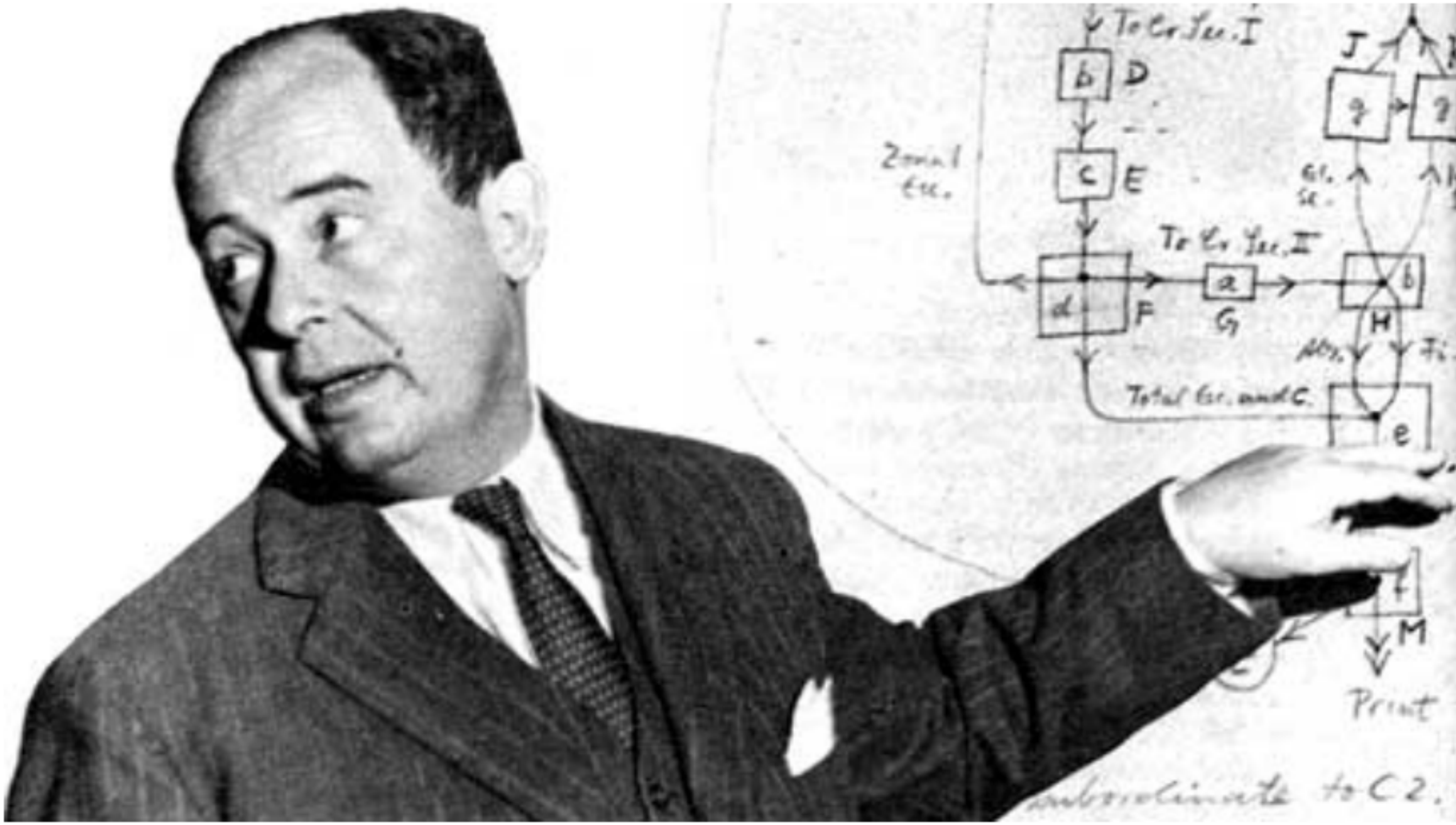

Figure 2: John Von-Neumann, considered one of the last of the great mathematicians.

Developments in Quantum Computation

- **The EPR channel:** "spooky action at a distance".
- **Feynman, von Neumann, and Ulam's cellular automata:** complex physical phenomena from simple computational rules - a first mix of physics and computation.
- **Deutsch:** quantum superposition parellelising the performance of classical computers.
- **Shor:** factoring numbers in polynomial time.
- **Meyer:** first talks on quantum games at the Microsoft Corporation.

Mathematical ingredients - basic quantum physics

- Think of this as a n-particle system in 2-d Hilbert space, each representing a Qubit in a n-bit string.
- Define up and down basis states in a 2-d Hilbert space

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Define operators to transform this basis. We'll use the Pauli spin transition matrices; we'll only need the σ_x matrix for the upcoming example

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- Now, take n tensor products to get the **n-bit register**, an element of which could be represented as $|ud \ldots udd\rangle$ for example.

Basic notions in Game theory

- A set $\Gamma = \{\text{Players, Actions, Payoffs, Outcomes}\}$.
- Induces a **strategy**, a map from a player's information set, which determines their moves.
- Strategies are often non-deterministic, so we speak in terms of the **expected payoff**!
- In classical game theory we use clear cut moves, but in quantum theory, we explore playing games with **states** that player's can **prepare and manipulate** with Hermitian operators.

The flip game

Toy game to move away from the classical picture! Introduced by Meyer:

1. Alice prepares an up state u.
2. Bob applies either $\mathbf{I}$ or σ_x.
3. Alice applies either $\mathbf{I}$ or σ_x.
4. Bob applies either $\mathbf{I}$ or σ_x.

Measure the final state. If d, Alice gets $+1$ ringgit. Otherwise, Bob gets $+1$ ringgit (represented as -1). Sequences and payoffs:

$Alice \backslash Bob$	$\mathbf{1},\mathbf{1}$	$\mathbf{1},\sigma_x$	$\sigma_x,\mathbf{1}$	σ_x,σ_x
$\mathbf{1}$	u,u,u	d,d,d	d,u,u	u,d,d
σ_x	d,d,u	u,u,d	u,d,u	d,u,d

Figure 3: Sequence of states, right to left

Payoffs and a naive strategy

Naive strategies:

- Bob plays either move with probability $\frac{1}{2}$.
- Alice plays either move with probability $\frac{1}{2}$.

$Alice \backslash Bob$	$\mathbf{1},\mathbf{1}$	$\mathbf{1},\sigma_x$	$\sigma_x,\mathbf{1}$	σ_x,σ_x
$\mathbf{1}$	-1	$+1$	$+1$	-1
σ_x	$+1$	-1	-1	$+1$

Figure 4: Payoffs

By symmetry, each have a 0 expected payout. This can be generalised to an n by m payout matrix with mixed strategy, solved with the minimax theorem. Interesting point: full information transparency on both sides gives Bob an advantage (why)?

Switching it up (no pun intended)

How do superposition and Hadamard operators change the game? Let's let **Alice** cheat by allowing her to change the initial prepared state without Bob's knowledge.

- Prepare d instead of u. No change in payoff by symmetry.
- Prepare $\frac{1}{\sqrt{2}}(u + d)$ instead of u. No change in payoff since $\mathbf{I}$ and σ_x leave state unchanged, so collapse at measurement has expected value zero.

Let's let Bob cheat!

Give him access to a wider choice of operators: the Hadamard gate as a superposition of the Pauli matrices. On the first turn (Bob's) he applies

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \implies Hu = \frac{1}{\sqrt{2}}(u + d)$$

so Alice's turn leaves this unchanged. Let Bob apply the Hadamard gate once again; but $HH(u) = u$ (check this yourself!). So he always wins.

Nash Equilibrium, Optimality, and Entanglement

Consider the classic *prisoner's dilemma* game. We have two types of optimal point

- Nash equilibrium / Dominant strategy intersection (1, 1)
- Pareto optimal point (3, 3)

They're different; but can introducing QM change this?

	Bob C	Bob D
Alice C	(3,3)	(0,5)
Alice D	(5,0)	(1,1)

Figure 5: An iconic payoff matrix

Structure of the game

- Start with initial state $U|CC\rangle$, right and left states in the tensor product representing Alice and Bob's state.
- Apply unitary matrices on each restricted qubit and then measure:

$$U^\dagger(U_A \otimes U_B)U|CC\rangle$$

, yields an element in 4 dimensional vector space.
- Introduce **entanglement** via U so that we can no longer write this state as a tensor product in $H_2 \otimes H_2$. For example $|CC\rangle + |DD\rangle$ is entangled. From now on let's write $|C\rangle, |D\rangle$ as $|0\rangle, |1\rangle$. In this example, set

$$U = \frac{1}{\sqrt{2}}(\mathbf{1}^{\otimes 2} + i\sigma_x^{\otimes 2}) \implies U^\dagger = \frac{1}{\sqrt{2}}(\mathbf{1}^{\otimes 2} - i\sigma_x^{\otimes 2})$$

If we represent C and D choices as $\mathbf{I}$ and σ_x we get a completely isomorphic game; each set of choices yields 4 states corresponding to the classical case with absolute certainty (just contract with the obvious basis).

Diverge from the classical picture by introducing the Hadamard gate

Let's introduce the Hadamard gate as a choice for both players. So, we can have the superposed outcome;

$$U^{\dagger}(H \otimes \mathbf{1})U\,|00\rangle = \frac{1}{\sqrt{2}}(|10\rangle - i\,|11\rangle)$$

, which by contraction yields an expected value of 3 for Alice, 1/2 for Bob.

	Bob $\mathbf{1}$	Bob σ_x	Bob H
Alice $\mathbf{1}$	(3,3)	(0,5)	$(\frac{1}{2},3)$
Alice σ_x	(5,0)	(1,1)	$(\frac{1}{2},3)$
Alice H	$(3,\frac{1}{2})$	$(3,\frac{1}{2})$	$(2\frac{1}{4},2\frac{1}{4})$

Figure 6: An modified payoff matrix

Introduce a pareto optimal point which is also a Nash equilibrium!

Introduce σ_z as a move!

	Bob $\mathbf{1}$	Bob σ_x	Bob H	Bob σ_z
Alice $\mathbf{1}$	(3,3)	(0,5)	$(\frac{1}{2},3)$	(1,1)
Alice σ_x	(5,0)	(1,1)	$(\frac{1}{2},3)$	(0,5)
Alice H	$(3,\frac{1}{2})$	$(3,\frac{1}{2})$	$(2\frac{1}{4},2\frac{1}{4})$	$(1\frac{1}{2},4)$
Alice σ_z	(1,1)	(5,0)	$(4,1\frac{1}{2})$	(3,3)

Figure 7: An modified payoff matrix