String Theory Lecture Notes

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1 Introduction

If you'd like to read about string theory, check out

- String Theory, Vol 1, Polchinksi, CUP
- Superstrings, Vol 1, Green et al, CUP
- A String Theory Primer, Schomerus, CUP
- David Tong's Notes, at arXiv: 09080333
- 'Why String Theory' Conlon, CRC

What is string theory? We don't really know what string theory is. The question itself requires a little more fleshing out to make sense of it. String theory is a work in progress. A final version of string theory would be a theory which we understand physically and mathematically. It's a work in progress - the final form will be quite different from what we have now.

What do we know? Well, in some sense, string theory is an attempt to answer the question of how we quantise the gravitational field. A theory of **quantum gravity**. There are however, a number of obstacles. In particular, naive quantisation of the Einstein Hilbert action presents a number of problems.

There are deep conceptual problems associated with this.

- There's a question about the nature of time in quantum gravity. If we think about time in quantum mechanics, time is treated as a fixed clock in which the Hamiltonian governs the evolution. In GR, space and time are on the same footing. Thus, the descriptions are on a different footing in QM versus GR. This is not neseccarily a technical problem, just something to think about.
- How do we quantise things without a pre-existing causal structure? What do we mean by this? We we quantise things in QFT, it's important to know whether two operators are timelike or spacelike separated. We have a notion that for all operators which are spacelike separated, commute. (One should not be able to influence the other). If we try to quantise GR, the metric is the object which we would like to quantise. But, this determines the casual structure. So, it's not immediately obvious what the algebra of operators should look like.
- GR has a very big symmetry - diffeomorphism symmetry (symmetry under coordinate reparametrisations). This is diffeomorphism invariance. This is something we'll discuss a bit later on. This is considered to be a gauge symmetry.
- One thing we'll discuss is that there are no local diffeomorphism invariant observables in GR. It's not even clear what the observables should be.

More importantly, there are technical obstacles. The previous issues are hard, but can we make some assumptions which help us make progress? We can look at perturbation theory; we can take our metric an expand it around some classical solution

$$g_{\mu\nu}(X) = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

for our purposes, this classical solution will be Minkowski space-time. This means we can use the causal structure of the background metric (Minkowski) to learn about the causal structure of the perturbation, and quantise. This immediately neutralises the first two conceptual problems. We can call the fluctuations $h_{\mu\nu}$ as gravitons. So, if we take a background static spacetime, add a field, then quantise.

There are some unsatisfactory things about this. When we split the metric into two, we hide a lot of the deep structure that we want. Nonetheless, we can take our Einstein-Hilbert action and expand it out

$$S\left[g\right] = \frac{1}{K_0} \int d^D X \sqrt{-g} R(g)$$

Choose a gauge and expand out

$$S\left[h\right] = \frac{1}{K_0} \int d^D X \left(h_{\mu\nu} \Box h^{\mu\nu} \right)$$

Since the Ricci scalar contains inverses, this expansion goes on forever. This is called a non-polynomial action. The first quadratic term gives the propagator, and the higher order term gives us our interactions.

The propagator is represented by a wiggly line. The interaction term gives us vertices. (Three or four wiggly lines coming together) These lead to Feynman rules.

However, when we compute loops, we get divergences. But, in physical QFT, we can absorb these divergences into coupling constants. These can be dealt with using standard techniques. From advanced quantum field theory, this technique is called renormalisation.

Thus, the difficulties run deeper than conceptual ones. We simply don't know how to calculate. So, string theory provides a way to do quantum perturbation theory of the gravitational 'field' We put field in quotation marks because it's not really a field which we'll be dealing with.

String theory answers some of the questions, in the sense that it gives us a framework to ask meaningful questions in quantum gravity. But not all of these questions are questions that we can answer.

The viewpoint that we're going to take for this course will be from perturbation theory. But we'll always try to understand what this tells us about the non-perturbative physics.

1.1 Worldsheets and Embeddings

Let's try to put together some sort of language to get started. From any popular science book, you may find that particles are described as vibrating strings. The starting point is to consider a worldsheet Σ which is a 2-dimensional surface swept out by a string. This is analogous to a worldline swept out by a particle.

(Insert diagram of line parametrised by τ and pointlike object, and diagram of cylinder object with surface called Σ, with two axes called τ and σ. This diagram is in 3d Minkwoski space). We put coordinates (σ, τ) on Σ, at least locally. And, we can define an embedding of the worldsheet Σ in the background spacetime, $\mathcal{M}$ (Minkowski space), by the functions $X^\mu(\sigma, \tau)$, where the X^μ

are coordinates on $\mathcal{M}$. So if you like, we have that

$$X : \Sigma \to \mathcal{M}$$

Why is this an embedding? A choice σ and τ on Σ gives us a location in Minkwoski space. There are rules (which we shall investigate), for gluing such worldsheets together in a way which is consistent with the symmetries of the theory.

So, we can not only describe the embeddings of a single string propagating through space-time, but multiple strings coming together.

(Draw a diagram of two tubes merging into one tube, then splitting back again into two tubes - this looks like a three point vertex)

We shall see that such diagrams like the above are in one-to-one correspondence with correlation functions in some quantum theory. It is natural to interpret such diagrams as Feynman diagrams in a perturbative expansion of some theory about a given vacuum.

So what we have is a way of understanding writing down Feynman rules, and performing successively better approximations to an exact result in field theory which we don't have.

So what we have here will turn out to be Feynman rules for a theory which we don't yet have. Where do these Feynman rules come from? Quantising is tremendously restrictive.

2 The Classical Particle and String

In non-relativistic Q, we treat time (t) as a parameter and position $\hat{X}^i$ as an operator. Obviously, this kind of restiriction shouldnt survive very long in a relativistic string theory. So, there are choices to be made here. One of those choices is second quantisation. Second quantisation is when both X^i and t are parameters. Then, we quantise fields, for example $\phi(x, t)$ which are the fundamental objects of interest in our theory. We of course require that the fields transform in an appropriate way under field transformations. This is what we do in QFT. Most of what we know for example, in the standard model, omes from this approach.

However, there is another way. This is **first quantisation**. We elevate t to be an operator, and have something else in the background. This is a natural framework for describing the relativistic embedding for worldline, worldsheet or worldvolume in a spacetime.

And here, $X^\mu = \left(X^i, t\right)$ is an operator, which is the fundamental object which we quantise (our basic variable), and we have some other natural parameter entering the theory. We'll look at concrete examples of what that parameter is through this section.

This other possibility, where we think of our fundamental degrees of freedom as some object embedded into spacetime, is the path we'll take in string theory. This is because this approach has been very successful.

There is string field theory however, which takes the first approach, but this leads to most of the results of first quantised string theory.

2.1 Worldlines and Particles

Suppose we want to take this approach with a particle. We consider an embedding of a worldline $\mathcal{L}$ into spacetime $\mathcal{M}$. We assume zero curvature. The basic field is the embedding $X^\mu : \mathcal{L} \to \mathcal{M}$ and an action might be

$$S[X] = -m \int_{x_0}^{x_1} ds = -m \int_{\tau_1}^{\tau_2} d\tau \sqrt{-\eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu}$$

(Insert diagram of line connecting nodes x_1^μ and x_2^μ) where τ (a parameter) is the proper time and

$$X^\mu(\tau_2) = x_2^\mu, \quad X^\mu(\tau_1) = x_1^\mu$$

are endpoints of the worldline. It makes sense that our action should be proportional to the length of the worldline. So, a reasonable guess for our action. We're taking our space-time metric as $(-,+,+,+)$. This seems like a reasonable starting point.

The constant m has dimensions of mass, so a good guess is that this parameter is interpreted as the mass. We can do some things with this action. We can first compute the conjugate momentum to $X^\mu(\tau)$, which is

$$P_\mu(\tau) = -m \frac{\dot{X}^\mu}{\sqrt{-\dot{X}^2}}, \quad \left(\dot{X}^2 = \eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu\right)$$

This satisfies $P^2 + m^2 = 0$. This is what we call an 'on shell' condition. There are two symmetries associated with this action.

- We have a rigid symmetry, where

$$X^\mu(\tau) \to \Lambda^\mu{}_\nu X^\nu(\tau) + a^\mu$$

  where $\Lambda^\mu{}_\nu$ is a Lorentz transformation matrix and a^μ is a constant displacement.

- Also, this action has re parametrisation invariance. In other words, in the physical variable of the x 's, τ is just a parameter which measures the distance along the line. So, we can replace it. If we take

$$\tau \to \tau + \xi(\tau)$$

  The embedding X^μ changes as

$$X^\mu(\tau) \to X^\mu(\tau + \xi) = X^\mu(\tau) + \xi \dot{X}^\mu(\tau) + \dots$$

  To first order, we have that $\delta X^\mu(\tau) \xi \dot{X}^\mu(\tau)$.

There's a rewriting of this action which makes life a little bit easier. Specifically, the action above is hard to interpret in the massless case. We can rewrite the action as

$$X[X, e] = \frac{1}{2} \int d\tau \left(e^{-1} \eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu - em^2\right)$$

There are no square roots, and we can take the massless limit. We will show that this new action is equivalent to the one we wrote down earlier. $e(\tau)$ is some new field on the worldline.

If you like, you might want to think of e as some one-dimensional metric which sets the scale of distances on the line.

The equations of motion for $x^\mu(\tau)$ and $e(\tau)$ are as follows

$$\frac{d}{d\tau}\left(e^{-1}\dot{X}\right) = 0$$

We notice that interestingly, e does not appear with a time derivative. So its equation of motion is purely algebraic. If you like, you can think of e as being a lagrange multiplier for every single point τ on the worldline. The e equation of motion is

$$\dot{X}^2 + e^2 m^2 = 0$$

$e(\tau)$ enters algebraically and it can be thought of as a constraint! The momentum conjugate to X^μ is

$$P_\mu = e^{-1}\dot{X}^\mu$$

If we combine this with the algebraic constraint for e, we can combine this with the mass shell condition to get $P^2 + m^2 = 0$. So interestingly, this auxillary field e imposes a constraint but is equivalent to the space-time energy momentum condition.

We can write $e^{-1} = \frac{m}{|\dot{X}|}$, plug this into the action to find $S[X,e]$ subject to the equations of motion for $e(\tau)$ gives precisely the action

$$S[X] = -m\int_C \sqrt{-\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu}$$

What guarantees that e^{-1} exists? Well, a priori nothing. But, we can motivate e coming from the interpretation as being a metric, which is invertible.

With a bit more work, we can argue that the m goes to 0 limit gives us the description for null light. This action is overall a lot nicer.

The action $S[X,e]$ has the symmetries

- Poincare invariance, where e is invariant.
- This also has re-parametrisation invariance, but since e depends on τ, it also has to transform. Infinitesimally,

  $$\delta X^\mu = \xi\dot{X}^\mu, \quad \delta e = \frac{d}{d\tau}(\xi e)$$

  provided these variations vanish on the endpoints. e is not a scalar function on the worldline, but this is the natural way to choose how it transforms so that the action is invariant.

we have a couple of comments. The first thing we could to is add curvature to our spacetime. We could generalise $\eta_{\mu\nu} \to g_{\mu\nu}(X(\tau))$. Then, this becomes a highly non-linear model.

2.2 Classical Strings

2.2.1 Nambu-Goto Action

The Nambu-Goto action is the analog of the action above but for a string. (Diagram of cylinder with open ends with axes σ, τ). The fundamental degree of freedom is

$$X : \Sigma \to \mathcal{M}$$

In this context, we refer to the object which the sheet is embedded into as the target space. Often, the thing we're embedding into may not be space-time for historical reasons. The Nambu-Goto action is the proposed generalisation so that for $X^\mu(\sigma, \tau)$, we have

$$S[X] = -\frac{1}{2\pi\alpha'} \int d\sigma d\tau \sqrt{-\det\left(\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu\right)}$$

This is an action which is proportional to the area spread out by the wordsheet. α' is a historically labelled constant with dimensions of area as measured in spacetime. One often speaks of the string length $l_s = 2\pi\sqrt{\alpha'}$. We introduce the string tension $T = \frac{1}{2\pi\alpha'}$, where we assume throughout that $\hbar = 1 = c$. These are some characteristic scales in the theory.

The usual sort of issues from Nambu-Goto are similar to the issues we faced from the original worldline action. A much better starting point for us is the Polyakov action. We place this game of removing the square root at the price of introducing an extra non-dynamical field.

$$S[X, h] = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{-h} h^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu$$

Our entire lecture course will start from the quantisation of this. h_{ab} is a metric on Σ and is non-dynamical - there are no terms involving derivatives h - it is merely a constraint field like e was. We will find however that h plays an important role.

If we remember h as a metric, this is a two dimensional quantum Klein-Gordon field which is massless and in 2 dimensions. if we treat this as a two dimensional quantum field theory, there are other terms which we may want to add.

Let's have a look at some equations of motion

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2.3 Classical Hamiltonian Dynamics of the String

Today, we're going to be interested in the quantisation of our closed bosonic string. To a first approximation, the canonical quantisation theory is quite straight forward.

We're going to continue to work in what we're going to call conformal gauge. This is when we take the metric on the worldsheet to take the form

$$h_{ab} = e^\Phi \begin{pmatrix} -1^0 & \\ 0 & 1 \end{pmatrix}$$

So we have a sort of natural notion of time. We can then define the canonical momentum field conjugate to X^μ in the usual way. This is just

$$P_\mu(\sigma, \tau) = \frac{\delta S[X]}{\delta X^\mu(\sigma, \tau)} = \frac{1}{2\pi\alpha'}\dot{X}_\mu$$

We can also do the usual stuff and write down the Hamiltonian density. Given the Lagrangian density $\mathcal{L}$, the Hamiltonian density is

$$\mathcal{H} = P_\mu \dot{X}^\mu - \mathcal{L} = \frac{1}{4\pi\alpha'}\left(\dot{X}^2 + X'^2\right)$$

Recall that the dot derivative is the derivative with respect to τ, and the prime is the derivative with respect to σ. It is always useful when looking at Hamiltonian dynamics to define the Poisson brackets. We introduce the bracket as $\{,\}_{PB}$. In particle theory, where our coordinates $x^\mu(\tau)$ and momenta $p_\mu(\tau)$ are on fundamental variables, it is useful to define the following.

$$\{f, g,\}_{PB} = \frac{\partial f}{\partial X^\mu}\frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu}\frac{\partial g}{\partial x^\mu}$$

So, for example, we have that $\{x^\mu, p_\nu\} = \delta^\mu{}_\nu$. The Hamiltonian $\mathcal{H}$ plays the role as a generator of time translations. For example, if we have some function of x, p which is

$$\frac{df}{d\tau} = \frac{\partial f}{\partial \tau} + \{f, H\}$$

which can be shown from the chain rule. Here, we're doing something slightly more general because x, p depends two parameters, not just the τ. So, if you like, this is a field theory generalisation. Our field theoretic generalisation requires that

$$\left\{X^\mu(\sigma, \tau), P_\nu(\sigma', \tau)\right\} = \delta^\mu{}_\nu \delta(\sigma - \sigma')$$

This is a precurcsor for equal time commutation relations. This is a nice construction! If we recall the form of $X^\mu(\sigma, \tau)$ to be written in terms of fourier modes α_n^μ and $\overline{\sigma}_n^\mu$. We have the mode expansion

$$X^\mu(\sigma, \tau) = x^\mu + \sqrt{\alpha'}p^\mu\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^\mu e^{-in(\tau-\sigma)} + \overline{\alpha}_n^\mu e^{-in(\tau+\sigma)}\right), \quad \sigma \sim \sigma + 2\pi$$

So imposing the natural commutation relations on X^μ and P^μ, gives us natural Poisson bracket relations for α, α'. This requires

$$\left\{\sigma_\mu^\nu, \sigma_n^\nu\right\}_{PB} = -im\eta^{\mu\nu}\delta_{m+n,0}, \quad \{\alpha_m^\mu, \overline{\alpha}_n^\nu\}_{PB} = 0, \quad \{\overline{\alpha}_m^\mu, \overline{\alpha}_n^\nu\}_{PB} = -im\eta^{\mu\nu}\delta_{m+n,0}$$

The left hand side vanishes unless $n = -m$. Why are we considering closed strings? Apparently, they make life easier later, and closed strings give rise to gravity. Let's see if this is plausible. The relation above is for equal times, and the X, P commutation relation is valid for equal times, so let's check this holds for $\tau = 0$. We'll see later on that this choice of τ is in actual fact not a special case. Just for simplicity however, we'll do it this way.

Our string looks like, at $\tau = 0$,

$$X^\mu(\sigma) = x^\mu + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^\mu e^{in\sigma} + \overline{\sigma}_n^\mu e^{-in\sigma}\right)$$

Our conjugate momenta is then

$$P_\mu(\sigma) = \frac{p^\mu}{2\pi} + \frac{1}{2\pi}\sqrt{\frac{1}{2\alpha'}}\sum_{n\neq 0}\left(\alpha_n^\mu e^{in\sigma} + \overline{\alpha}_n^\mu e^{-in\sigma}\right)$$

Computing the commutation relation from this, we have that

2.4 The Stress Tensor and Wit Algebra

Introduce the worldsheet lightcone coordinates

$$\sigma^\pm = \tau \pm \sigma$$

which are a natural choice of coordinates in this gauge to help us understand the structure of this space a little but more.

In these coordinates, and this gauge, the worldsheet metric now looks like

$$h = e^\phi \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

and $\partial_\pm = \frac{\partial}{\partial \sigma^\pm}$. The action and equations of motion becomes

$$S = -\frac{1}{2\pi\alpha'}\int_\Sigma d\sigma^+ d\sigma^- \partial_+ X \cdot \partial_- X, \quad \partial_+\partial_- X^\mu = 0$$

The stress tensor T_{ab} becomes in these coordinates

$$T_{++} = -\frac{1}{\alpha'}\partial_+ X \cdot \partial_+ X, \quad T_{--} = -\frac{1}{\alpha'}\partial_- X \cdot \partial_- X, \quad T_{+-} = T_{-+} = 0$$

So we have two fields on the worldsheet, the Xs, and the metric which encodes the vanishing stress tensor constraint.

The constraint is $T_{\pm\pm} = 0$. Let's pause a little and think about what these constraints might look like in terms of these modes. It;s going to be very useful to introduce the Fourier modes of $T_{\pm\pm}$. We define at $\tau = 0$, the charges

$$L_n = -\frac{1}{2\pi}\int_0^{2\pi} d\sigma T_{--}(\sigma)\, e^{-in\sigma}, \quad \overline{L}_n = -\frac{1}{2\pi}\int_0^{2\pi} d\sigma T_{++}(\sigma)\, e^{in\sigma}$$

We're just starting out to take $\tau = 0$, but we'll see later that this doesn't matter. What do these modes look like in terms of α ? We'll choose to explore one of them in detail, with the awareness that the other moving mode will have similar properties.

If we differentiate X^μ as

$$\partial_- X^\mu(\sigma,\tau)\sqrt{\frac{\alpha'}{2}}\sum_n \alpha_n e^{-in\sigma^-}, \quad \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$$

We find that

$$\begin{aligned} L_n &= \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma \partial_- X^\mu(\sigma) \cdot \partial_- X_\mu(\sigma) \\ &= \frac{1}{4\pi} \sum_{m,p} \alpha_m \cdot \alpha_p \int_0^{2\pi} d\sigma e^{-i(m+p-n)\sigma} \\ &= \frac{1}{4\pi} \sum_{m,p} \alpha_m \cdot \alpha_p 2\pi \delta_{p,n-m} \end{aligned}$$

Similarly, this holds for $\overline{L}_n$. This gives us the relation that

$$L_n = \frac{1}{2} \sum_m \alpha_{n-m} \cdot \alpha_m, \quad \overline{L}_n = \frac{1}{2} \sum_m \overline{\alpha}_{n-m} \cdot \overline{\alpha}_m$$

This constraint can be written as $L_n = 0 = \overline{L}_n$. Using the algebra for the α_n^μ ($\overline{\alpha}_n^\mu$), we can compute the algebra for $L_n \left(\overline{L}_n\right)$. It's not to hard to show that the algebra of these objects obey the algebra

$$\{L_m, L_n\} = -i(m-n) L_{m+n}, \quad \left\{L_m, \overline{L}_n\right\} = 0, \quad \left\{\overline{L}_m, \overline{L}_n\right\} = -i(m-n) \overline{L}_{m+n}$$

This is often called the Witt algebra. We shall see, when we promote this to quantum operators, we see something very similar, but with an important difference. We will see that if we set $L_n = 0 = \overline{L}_n$, at a given τ , then the evolution of the system preserves $L_n = 0 = \overline{L}_n$. So what we find, underlying the constraints of this theory, that there is this underlying infinite dimensional symmetry.

2.5 A First Look at the Quantum Theory

Given the work we've done on the classical theory, quantising this to build a theory on Hilbert space will be a straightforward extension.

Recall from earlier, that we decided to work with the Plyakov action. We transformed our metric locally due to Weyl rescaling, which gives us a two dimensional massless Klein-Gordon theory. But, as a result, we get constraints from the stress-energy tensor.

As we did in quantum field theory, there are two ways we can proceed in imposing the constraints.

- We can constrain states then quantise, which is an approach that has been reasonably successful (this is called light-cone quantisation). However, we will not be following this path of quantisation.
- The second approach is to quantise the unconstrained theory, then impose constraints as a physical condition on our Hilbert space. For specifically, recall that the only algebraic constraint we got from earlier was our condition on our stress energy tensor, $T_{ab} = 0$. Thus, this is the only thing we will impose on our Hilbert space.

2.5.1 Canonical Quantisation

We quantise by replacing out Poisson brackets with commutators. So, functions on phase space for example, are replaced by operators. We do the transformation

$$\{,\}_{PB} \to -i\,[,]$$

So, the structure of quantum mechanics is similar to Hamiltonian dynamics. These give rise to equal time comuttation relations:

$$\left[X^{\mu}(\sigma), X^{\nu}(\sigma')\right] = 0, \quad \left[P_{\mu}(\sigma), P_{\nu}(\sigma')\right] = 0, \quad \left[P_{\mu}(\sigma), X^{\nu}(\sigma')\right] = -i\delta_{\mu}{}^{\nu}\delta(\sigma - \sigma')$$

Now, recall the mode expansions we get from expanding our position operator X^{μ},

$$X^{\mu}(\sigma,\tau) = x^{\mu} + \alpha' p^{\mu}\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_{\nu}^{\mu}e^{-in(\tau-\sigma)} + \overline{\alpha}_{n}^{\mu}e^{-in(\tau+\sigma)}\right)$$

$$P^{\mu}(\sigma,\tau) = \frac{p^{\mu}}{2\pi} + \frac{1}{2\pi}\sqrt{\frac{1}{2\alpha'}}\sum_{n\neq 0}\left(\alpha_{n}^{\mu}e^{-in(\tau+\sigma)} + \overline{\alpha}_{n}^{\mu}e^{-in(\sigma-\tau)}\right)$$

One can show that the commutation relations for the mode operators α and $\overline{\alpha}$ are consistent which changing our Poisson brackets to commutators. This gives our relations as

$$[\alpha_m^{\mu}, \alpha_n^{\nu}] = m\delta_{m+n,0}\eta^{\mu\nu}, \quad [\alpha_m^{\mu}, \overline{\alpha}_n^{\nu}] = 0, \quad [\overline{\alpha}_m^{\mu}, \overline{\alpha}_n^{\nu}] = m\delta_{m+n,0}\eta^{\mu\nu}$$

For each dimension μ, these are an infinite number of ladder operators, where the annihilation operators are given by α_n for $n > 0$. However, by looking at the fact that X^{μ} and P^{μ} should be real and by comparing coefficients, we find the relations

$$(\alpha_n^{\mu})^{\dagger} = \alpha_{-n}^{\mu}$$

These are creation and annihilation ladder operators. They are creating and annihilating different modes on the string. It use necessary to introduce a vacuum state on Σ, $|0\rangle$ such that

$$\alpha_n^{\mu}|0\rangle = 0, \quad n \geq 0$$

It is important to note that this is not a vacuum in spacetime, it is a vacuum of vibrational modes. We recall the Fourier modes of T_{ab} are L_n and $\overline{L}_n$. Just as before, we can write those modes in terms of α and $\overline{\alpha}$.

$$L_m = \frac{1}{2}\sum_{n}\alpha_{m-n}\cdot\alpha_n$$

We call L_m Virasoro operators. This expression is ambiguous for $m = 0$, going from the classical to the quantum theory, operator ordering starts to matter. This is because α_n and α_{-n} do not commute for $n \neq 0$, and hence we don't know how to order these in our expression for L_0. In this case, we shall take

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n>0}\alpha_{-n}\cdot\alpha_n$$

We can adopt the usual notion of normal ordering. There are other normal ordering prescriptions we could use.

2.5.2 Physics State Conditions

Instead of thinking about the stress tensor and X, P, it's more useful to talk about the Virasoro operators L_i to impose $T_{ab} = 0$ on the Hilbert space of our theory.

Let's first define some notation to make our lives easier. Define

$$N = \sum_{n>0} \alpha_{-n} \cdot \alpha_n, \quad \overline{N} = \sum_{n>0} \overline{\alpha_{-n}} \cdot \overline{\alpha}_n$$

These operators have the interpretation of being 'weighted number operators' on physical states. Given this, we can write for example, that

$$L_0 = \frac{\alpha'}{4} p^2 + N, \quad \overline{L}_0 = \frac{\alpha'}{4} p^2 + \overline{N}, \quad \left(\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu \right)$$

We require that $T_{ab} \left|\phi\right\rangle = 0$ for $\left|\phi\right\rangle$ to be a physical state. This means that

$$L_n \left|\phi\right\rangle = 0, \quad \text{for } n > 0$$

under Hermitian conjugation, this also implies that $\left\langle\phi\right| L_{-n} = 0$. We shall also require

$$L_0 \left|\phi\right\rangle = a \left|\phi\right\rangle, \quad \overline{L}_0 \left|\phi\right\rangle = a \left|\phi\right\rangle$$

allowing for the fact that L_0 may be true up to some constant, which is related to the normal ordering. The constant a reflects the ambiguity in defining L_0 in the quantum theory. We'll come back as to what values a should take depending on our perspective. For now, we should take $a = 1$. This justification is a posteori.

For convenience, we also define the new operators

$$L_0^\pm = L_0 \pm \overline{L}_0$$

Hence, in terms of our newly defined operators, our physical conditions are

$$\left(L_0^+ - 2\right) \left|\phi\right\rangle = 0, \quad L_0^- \left|\phi\right\rangle = 0, \quad L_n \left|\phi\right\rangle = 0 = \overline{L}_n \left|\phi\right\rangle \text{ for } n > 0$$

The L_n are difficult to interpret physically, but are related to polarisation conditions. The first two conditions are related to rotational invariance. With these physical constraint conditions, we can start to look at the spectrum of the system.

2.6 The Spectrum

2.6.1 The Tachyon

Now it's time to start constructing some actual states in our Hilbert space. We can construct a space-time momentum eigenstate as

$$\left|k\right\rangle = e^{ik\cdot x} \left|0\right\rangle$$

For now, we just work in the context where x is just a position variable and not an operator, like in the expression $e^{ik\cdot X} \left|0\right\rangle$. However, we will see what this looks like at a later stage. From quantum mechanics, we know how p_μ acts in the position basis. In terms of a position basis in the target space, the momentum operator is $-i\frac{\partial}{\partial x^u} = p_\mu$, so $p_\mu \left|k\right\rangle = k_\mu \left|k\right\rangle$. In addition, $L_n \left|k\right\rangle = 0 = \overline{L}_n \left|k\right\rangle$ straightforwardly, although this has yet to be shown in the lectures and I'm not sure why this is true. We can write L_0^- as

$$L_0^- = N - \overline{N}$$

So the vanishing condition $L_0^- \left|\phi\right\rangle = 0$ suggests a symmetry of right movers versus left movers. This is sometimes called level matching. It is the weighted count difference from either side. If we apply N or $\overline{N}$ to any of the states of the form $\left|k\right\rangle$, we will always find that since N is the sum of $\alpha_n \cdot \alpha_{-n}$, we can always commute the annihilation operator forward. $N = \overline{N} = 0$. We check that $\left(L_0^+ - 2\right)\left|k\right\rangle = 0$. Thus, we find that

$$\begin{aligned}\left(L_0^+ - 2\right)\left|k\right\rangle &= \left(\frac{\alpha'}{2} + N + \overline{N} - 2\right)\left|k\right\rangle \\ &= \left(\frac{\alpha'}{2}k^2 - 2\right)\left|k\right\rangle = 0\end{aligned}$$

This gives us the condition on k^2, as $k^2 - \frac{4}{\alpha'} = 0$. If we compare this with the energy-momentum condition $k^2 + M^2 = 0$, with gives

$$M^2 = -\frac{4}{\alpha'}$$

The state $\left|k\right\rangle$ has a spacetime interpretation as a tachyon. This problem is not going to go away. We do have the tachyon, and its cured by promoting this to the supersymmetric string, and adding supersymmetry.

We'll now look at the next excited state, which are the massless states. Consider states of the form

$$\left|\epsilon\right\rangle = \epsilon_{\mu\nu}\alpha_{-1}^{\mu}\overline{\alpha}_{-1}^{\nu}\left|k\right\rangle$$

NOw we want to look at the condition which allows us to view this as a physical state. Clearly, $N = \overline{N} = 1$. We can look at the energy momentum condition. The condition

$$\left(L_0^+ - 2\right)\left|\epsilon\right\rangle = 0$$

implies $\frac{\alpha'}{2}k^2 = 0$, so we require that $k^2 = 0$ (null). Consider the next condition, which gives $L_1\left|\epsilon\right\rangle = 0$. This condition gives

$$\frac{1}{2}\left(\sum_n \alpha_{1-n}\cdot\alpha_n\right)\epsilon_{\mu\nu}\alpha_{-1}^{\mu}\overline{\alpha}_{-1}^{\nu}\left|k\right\rangle = \epsilon_{\mu\nu}\overline{\alpha}_{-1}^{\nu}\alpha_0\cdot\alpha_1\alpha_{-1}^{\mu}\left|k\right\rangle$$

recall that $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$. We require then

$$k_\rho\alpha_1^\rho\alpha_{-1}^\mu\left|k\right\rangle = 0$$

we can use our commutator to show that the above is equal to

$$\epsilon_{\mu\nu}k_\rho\left(\left[\alpha_1^\rho, \alpha_{-1}^\mu\right] + \alpha_{-1}^\mu\alpha_1^\rho\right)\left|k\right\rangle = \epsilon_{\mu\nu}\eta^{\mu\rho}k_\rho\left|k\right\rangle = \epsilon_{\rho\nu}k^\rho\left|k\right\rangle = 0$$

This condition $L_1 |\epsilon\rangle = 0$ requires us to impose $\epsilon_{\nu\mu} k^\mu = 0$. In other words, we can think of this condition as the fact that there are no longitudinal polarisations. Similarly, $\overline{L}_1 |\epsilon\rangle = 0$ requires $\epsilon_{\mu\nu} k^\nu = 0$. So, we have three physical state conditions given to us. There are no further conditions on this tensor.

We then have the conditions on $|\epsilon\rangle$. Our first condition is that it is null and massless, so $k^2 = 0$, and no longitudinal polarisations $\epsilon_{\mu\nu} k^\mu = 0$ and $\epsilon_{\mu\nu} k^\nu = 0$, thinking of ϵ as a polarisation tensor.

We can decompose $\epsilon_{\mu\nu}$ into symmetric $h_{\mu\nu}$, anti-symmetric ($b_{\mu\nu}$), and trace ϕ parts. We first extract the trace bit, which we call the Dilaton.

$$|\phi\rangle = \phi \alpha^\mu_{-1} \overline{\alpha}_{-1\mu} |k\rangle$$

We also have two other particles from this

$$\begin{aligned} |h\rangle &= h_{\mu\nu} \alpha^\mu_{-1} \overline{\alpha}^\nu_{-1} |k\rangle \quad \text{Graviton}, \quad h_{\mu\nu} = h_{\nu\mu} \\ |b\rangle &= b_{\mu\nu} \alpha^\mu_{-1} \overline{\alpha}^\nu_{-1} |k\rangle, \quad \text{B-Field}, \quad b_{\mu\nu} = -b_{\nu\mu} \end{aligned}$$

Now let's look at massive states. We can now look at states with $N = \overline{N} = 2$. We have

$$A_{\mu\nu} \alpha^\mu_{-2} \overline{\alpha}^\nu_{-2} |k\rangle + A_{\mu\nu\lambda} \alpha^\mu_{-2} \overline{\alpha}^\nu_{-1} \overline{\alpha}^\lambda_{-1} |k\rangle + \tilde{A}_{\mu\nu\lambda} \overline{\alpha}^\mu_{-2} \alpha^\nu_{-1} \alpha^\lambda_{-1} |k\rangle + A_{\mu\nu\lambda\rho} \alpha^\mu_{-1} \alpha^\nu_{-1} \overline{\alpha}^\lambda_{-1} \overline{\alpha}^\rho_{-1} |k\rangle$$

and N and $\overline{N}$ count number of quanta going around the string. There's no profit in us solving the mass shell conditions. The mass of such states is $m^2 = \frac{4}{\alpha'}$. So the string is describing not only the tahcyon and massless fields, but an infinite amount of massive fields.

For the most part, string theorists have concerned themselves with the massless spectrum.

2.7 The Big(ish) Picture

We started with our Polyakov action which describes the embedding of our string in our manifold.

$$S = -\frac{1}{4\pi\alpha'} \int_\Sigma d^\sigma \eta_{\mu\nu} \partial_a X^\mu \partial^a X^\nu$$

We could deform this theory and add a perturbation to the metric. This can be achieved by adding a small plane wave deformation to the spacetime metric. For example, we might take $\eta_{\mu\nu}$ and replace it by adding some plane wave.

$$\eta_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu} e^{ik\cdot x}$$

The action changes by

$$\Delta S = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma h_{\mu\nu} \partial_a X^\mu \partial^a X^\nu e^{ikx}$$

For every deformation of the theory, there is an associated operator

$$\mathcal{O} = h_{\mu\nu} \partial_a X^\mu \partial^a X^\nu e^{ikx}$$

This is clearly associated with a deformation of the spacetime metric.

In string theory, (2-dimensional CFTs), to each operator $\mathcal{O}$ that corresponds to a physical deformation of the theory, (something that seems reasonable is what we mean by a physical deformation), there is a state in the Hilbert space.

In this case,

$$\lim_{\tau \to -\infty} \mathcal{O} |0\rangle = |h\rangle$$

We will do this more carefully later on. This is called the state-operator correspondence. One last comment on this.

What if we choose to start with a general metric? For example, we could've started off with

$$S = -\frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma g_{\mu\nu}(x) \partial_a X^\mu \partial^a X^\nu$$

Thhis is highly non-linear. How would we deal with this? We could try expand this in terms of a power series and then try to deal with the theory perturbatively. This is now an interacting theory, and this is hard to do.

Moreoever, the condition of Weyl invariance in the quantum theory constrains what $g_{\mu\nu}(X)$ we can have. Weyl invariance is $h_{ab} \to e^{\omega(\omega,\tau)} h_{ab}$. One finds that $G_{\mu\nu}$ has to satisfy

$$R_{\mu\nu}(g) + \mathcal{O}(\alpha') = 0$$

Which is the Einstein tensor up to string corrections. More generally, if we have other background fields, we find Weyl invariance requires the full Einstein equations to be satisfied to leading order in α'.

2.8 Spurious states and gauge invariance

Before, when we mentioned L_0 to have an ordering ambiguity, we put our normal ordering constant as $a = 1$. We took $a = 1$ in the conditions $(L_0 - a)|\phi\rangle = \left(\overline{L}_0 - a\right)|\phi\rangle = 0$. Why did we do this? This is so that we can interpret states easily. Consider the state

$$|\chi\rangle = \sqrt{\frac{2}{\alpha'}} \left(\lambda_\mu \alpha^\mu_{-1} \overline{L}_1 + \tilde{\lambda}_\mu \overline{\alpha}^\mu_{-1} L_1\right) |k\rangle$$

Clearly, $|\chi\rangle$ is orthogonal to all physical states. If $|\phi\rangle \in \mathcal{H}$, then $\langle\phi|\chi\rangle = 0$ because $L_1|\phi\rangle = \overline{L}_1|\phi\rangle = 0$. What conditions do $\lambda_\mu, \tilde{\lambda}_\mu, k$ have to satisfy for $|\chi\rangle$ to be physical.

Is is useful to write $|\chi\rangle$ as

$$|\chi\rangle = \left(\lambda_\mu k_\nu + \tilde{\lambda}_\nu k_\mu\right) \alpha^\mu_{-1} \overline{\alpha}^\nu_{-1} |k\rangle$$

Keeping a arbitrary, we find that

$$\left(L_0^+ - 2a\right)|\chi\rangle = 0 \implies$$

k $^2 = \frac{4(a-1)}{\alpha'}$

We also have that, by symmetry,

$$L_1 |\chi\rangle = 0 \text{ if } (\lambda \cdot k) k_\mu + \tilde{\lambda}_\mu k^2 = 0$$
$$\overline{L}_1 |\chi\rangle = 0 \text{ if } \left(\tilde{\lambda} \cdot k\right) k_\mu + \lambda_\mu k^2 = 0$$

Finally, we have that

$$\langle \chi | \chi \rangle = \lambda^2 k^2 + 2 (\lambda \cdot k) \left(\tilde{\lambda} \cdot k\right) + \tilde{\lambda}^2 k^2$$

If $a = 1$, $k^2 = 0$, $\lambda \cdot k, \tilde{\lambda} \cdot k = 0$, so $\langle \chi | \chi \rangle = 0$.

Worldline actions

- Our action is
$$S = -m \int_{s_1}^{s_2} ds = -m \int_{\tau_1}^{\tau_2} d\tau \sqrt{-\eta^{\mu\nu} \dot{x}^\mu \dot{x}^\nu}$$
- We have conjugate momenta with on-shell mass condition
$$P^\mu = -\frac{m \dot{x}^\mu}{\sqrt{-\dot{x}^2}}, \quad P^2 + m^2 = 0$$
- It makes more sense to work with Einbeins, since we can work in the $m \to 0$ limit
$$S = \frac{1}{2} \int d\tau \, \left(e^{-1} \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - e m^2\right)$$
- Two equations of motion come from the Einbein action
$$\frac{d}{d\tau} \left(e^{-1} \dot{x}^\mu\right) = 0, \quad \dot{x}^2 + e^2 m^2 = 0$$
- This has symmetries
$$\delta x^\mu = \xi \dot{x}^\mu, \quad \delta e = \frac{d}{d\tau} (\xi \dot{e})$$
- In the massless limit, if we replace our Minkowski metric with a general metric, we recover the geodesic equations
$$S = \frac{1}{2} \int d\tau e^{-1} g_{\mu\nu} \dot{x}^\mu \dot{x}^\mu, \quad \ddot{x}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0$$

Strings

- A string is two dimensional, embedded with parameters σ, τ
$$X^\mu (\sigma, \tau) = X^\mu (\sigma + 2\pi n, \tau), \quad n \in \mathbb{Z}$$
- Our associated action is the Nambu-goto action
$$S[X] = -\frac{1}{2\pi\alpha'} \int d\sigma d\tau \sqrt{-\det \left(\eta_{\mu\nu} \partial_a X^\mu \partial_b X^\beta\right)}$$

- We add an extra degree of freedom h_{ab} to introduce the Polyakov action

$$S[X,h] = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}h^{ab}\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$$

Quantising the Bosonic String

- Expand out position and conjugate momenta in terms of Fourier modes
- We use our gauge invariance to set

$$h_{ab} = e^{\phi}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

  The -1 in the diagonal gives us a rough notion of time.
- Put this in our Polyakov action to get the conjugate momenta, which is

$$P_\mu = \frac{\delta S}{\delta \dot{X}^\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu$$

- Impose equal time commutation relations which are equivalent to commutation relations of the Fourier components. We use Poisson brackets for this

$$\{X^\mu, P_\nu\} = \delta^\mu{}_\nu = \delta\left(\sigma - \sigma'\right) \iff \{\alpha_n, \alpha_m\} = -im\delta_{n+m,0}, \quad \{\overline{\alpha}_n, \overline{\alpha}_m\} = -im\delta_{n+m,0}$$

- We can change coordinates to

$$\sigma_\pm = \tau \pm \sigma$$

Example Sheet 1

2.9 Question 1

Here we are showing equivalence of the Nambu-Goto action and the Polyakov action.