Notes on Black Holes

Afiq Hatta

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1 The formation of black holes

In this section, we'll cover the formation of black holes. To start this discussion, we'll need to discuss the idea of symmetry on manifolds and in metrics. The word 'black hole' should be a big enough hint that the kinds of metrics we'll be considering exhibit spherical symmetry. But, since general relativity is done in the frame-work of both **space** and **time**, we need to make clear what 'spherical symmetry' actually means.

First, lets discuss how to obtain the metric for a standard 2-sphere. Working in signature $(-,+,+,+)$, we have that the metric on the 2 -sphere in Cartesian coordinates is given by the following, with the constraint

$$ds^2 = dx^2 + dy^2 + dz^2, \quad x^2 + y^2 + z^2 = 1$$

If we reparametrize the coordinates as follows, using our standard spherical coordinates with $r = 1$, we embed the 2 -sphere in $\mathbb{R}^3$

$$\begin{aligned} x &= \cos\phi \sin\theta \\ y &= \sin\phi \sin\theta \\ z &= \cos\theta \end{aligned}$$

Applying a coordinate transformation for the one forms dx, dy, dz, we get that

$$\begin{aligned} dx &= \cos\phi \cos\theta d\theta - \sin\phi \sin\theta d\phi \\ dy &= \sin\phi \cos\theta d\theta + \cos\phi \sin\theta d\phi \\ dz &= -\sin d\theta \end{aligned}$$

Substituting into the above, we can read off that the components of the metric are given by

$$ds^2 = d\theta^2 + \sin^2\phi d\phi^2$$

This metric comes from considering the 2-sphere manifold. Now, we know that the symmetry group of a 2-sphere is $O(3)$ if we consider reflections, and just $SO(3)$ if we consider only rotations. Hence, we say that the metric admits an isometry group of $SO(3)$.

Definition. (Isometries on a metric) An **isometry** is a transformation on a metric space which leaves distances between points invariant. The image one has in their mind immediately might be a rotation or a reflection on a two dimensional plane.

2 Useful Identities in General Relativity for Black Holes

In this section, we'll cover some useful identities which may prove useful for doing general relativity. The first one we'll prove is an equation that's useful for proving the divergence theorem for curved space.

Theorem. Divergence of a vector field in terms of $\sqrt{-g}$

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} V^\mu \right) \tag{1}$$

Proof. We first apply the product rule, then make use of a smart rearrangement using logs. We first have that

$$
\begin{aligned}
\nabla_a V^a &= \partial_\mu V^\mu + \frac{1}{\sqrt{-g}} V^\mu \partial_\mu \left(\sqrt{-g}\right) \\
&= \partial_\mu V^\mu + V^\mu \partial_\mu \left(\log \sqrt{-g}\right) \\
&= \partial_\mu V^\mu + \frac{1}{2} V^\mu \partial_\mu \left(\log\left(-\det g\right)\right)
\end{aligned}
$$

Where in the last line we wrote out the determinant explicitly. Now the trick here is to use the identity which relates the logarithm of the determinant to the trace of the formal logarithm of a matrix.

Observe that if A is a matrix, which we assume to be positive definite (since our metric is), then it is diagonalisable. Since it's diagonalisable, we can write it in the appropriate basis such that $\exp A = \operatorname{diag}\left(e^{\lambda_1}, \ldots, e^{\lambda_n}\right)$ where $\lambda_i \in \mathbb{R}$ are the eigenvalues of the matrix A. This means that the eigenvalues of $\exp A$ are e^{λ_i} for all i. So, since the determinant of the matrix is equal to the product of its eigenvalues, we have that

$$
\det\left(\exp A\right) = \exp\left(\operatorname{tr} A\right)
$$

If we set $B = \exp A$, then this identity reduces to

$$
\det B = \exp\left(\operatorname{tr} \log B\right)
$$

Taking the logarithm of both sides once more, we have that finally

$$
\log \det B = \operatorname{tr} \log B
$$

We can now resume to the question at hand. We rewrite the above using this identity so that

$$
\begin{aligned}
\nabla_a V^a &= \partial_\mu V^\mu + \frac{1}{2} V^\mu \partial_\mu \left(\operatorname{tr} \log\left(-g\right)\right) \\
&= \partial_\mu V^\mu + \frac{1}{2} V^\mu \operatorname{tr} \partial_\mu \log\left(-g\right) \\
&= \partial_\mu V^\mu + \frac{1}{2} V^\mu \operatorname{tr}\left(g^{-1} \partial_\mu g\right)
\end{aligned}
$$

Where in this case we've put g to denote schematically the matrix $g_{\mu\nu}$ and **not** the determinant. Be careful to observe that the minus signs when differentiating the logarithm cancel out. One can easily verify that $\frac{1}{2} g^{\alpha\beta} \partial_\mu g_{\alpha\beta} = \Gamma^\nu_{\mu\nu}$. Thus, this completes the proof. $\square$