# Notes on Quantum Field Theory II

Afiq Hatta

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# 1 Formulating the Path Integral

In this section, we'll be moving on from our standard procedure of quantising a given Hamiltonian in quantum mechanics. We'll be introducing the concept of a path integral. The path integral is a 'functional integral' where we integrate over all possible paths with a Gaussian probability factor.

## 1.1 Classical and Quantum Mechanics

In classical mechanics, we use the Lagrangian as a conduit to encode the information about our physical system. The Lagrangian is given by a function of position and velocity, with

$$\mathcal{L} = \mathcal{L}\left(q_a, \dot{q}_a\right)$$

where $a = 1, \ldots, N$ is an index for each particle in our system. We can convert this to the Hamiltonian formalism where we put position and momentum on the same pedestal and define our conjugate momenta

$$p_a = \frac{\partial \mathcal{L}}{\partial \dot{q}^a}$$

We then work in terms of the Hamiltonian which is the Legendre transformation of the Lagrangian, where we eliminate $\dot{q}^a$ everywhere in the Lagrangian in favour of $p^a$ as follows

$$H(q_a, p_a) = \sum_a \dot{q}_a p^a - \mathcal{L}\left(q_a, \dot{q}_a\right)$$

The quantum mechanical analog of this is the same. However, $p_i$ and $q_i$ are **promoted** to what we call operators, and obey commutation relations which as we know, eventually lead to discrete energy levels in the Hamiltonian. In quantum mechanics, we write the position and momentum operators as $\vec{q}^i$ and $\vec{p}^i$ for position and momentum respectively. In the Heisenberg picture of quantum mechanics, operators (and not states), depend on time. So, we impose the commutation relations for some fixed coordinate time $t \in \mathbb{R}$, where

$$\left[\vec{q}^i, \vec{p}_j\right] = i\delta^i{}_j$$

In classical field theory, we promote operators to fields instead. If $\phi(\vec{x}, t)$ represents a classical scalar field at some point in time $t$, then the field as well as it's conjugate momentum $\pi\left(\vec{x}, t\right)$ obey the commutation relations

$$[\phi(\vec{x}, t), \pi(\vec{y}, t)] = i\delta^3\left(\vec{x} - \vec{y}\right)$$

There is however a caveat in performing these approaches to quantisation. The theory is not manifestly Lorentz invariant. This is because when we imposed the equal time commutation relations above, we had to pick a preferred coordinate time $t$.

## 1.2 Formulating the Path Integral

We use the Hamiltonian as a starting point.

$$H(q,p) = \frac{p^2}{2m} + V(q)$$

The Schrödinger equation for a state $|\psi(t)\rangle$ which is time dependent is given by

$$i\frac{d}{dt}|\psi(t)\rangle = \hat{\mathbf{H}}|\psi(t)\rangle$$

Now, this is is a first order differential equation and can be solved provided that we have the right initial conditions. For now, let's just write down the solution in a 'formal' sense, where we 'exponentiate' the Hamiltonian whilst being vague about what this actually means. We write the solution as

$$|\psi(t)\rangle = \exp\left(-i\hat{\mathbf{H}}t\right)|\psi(0)\rangle$$

To do calculations however, we need construct an appropriate basis of states. For this section, we'll use the position basis $|q,t\rangle$, for $q,t \in \mathbb{R}$. These states are are defined to be the eigenstates of the position operator $\hat{\mathbf{q}}(t)$, so that

$$\hat{\mathbf{q}}(t)|q,t\rangle = q|q,t\rangle$$

We'll impose the condition that these states are normalised so that for a fixed time, we have

$$\langle q,t|q',t\rangle = \delta\left(q-q'\right)$$

We impose the analogous conditions as well for momentum eigenstates. For now though, we'll work in the Schrodinger picture so that $\hat{\mathbf{q}}$ is fixed and hence we have that the eigenstates $|q\rangle$ are time-independent. Since these states form a basis, we have that they obey the completeness relation

$$1 = \int d^3q\ |q\rangle\langle q|$$

We also label the time-independent momentum eigenstates as $|p\rangle$, and impose the completeness relation

$$1 = \frac{d^3p}{(2\pi)^3}|p\rangle\langle p|$$

Note the factor of $2\pi$ that we divide by. Other literature doesn't include this. With this set of basis states, we can now write the abstract state $|\psi(t)\rangle$ in terms of the position basis, where we denote

$$\psi(q,t) = \langle q|\psi(t)\rangle = \langle q|\exp\left(-i\hat{\mathbf{H}}t\right)|\psi(0)\rangle$$

We will put this into an integral form for reasons we will discuss later. To put any equation in integral form, the rule of thumb is to employ the completeness relations for either the position or momentum basis. We get that

$$\begin{aligned}
\langle q| \exp\left(-i\hat{\mathbf{H}}t\right)|\psi(0)\rangle &= \int d^3q' \ \langle q| \exp\left(-i\hat{\mathbf{H}}t\right)\left|q'\right\rangle \left\langle q'\middle|\psi(0)\right\rangle \\
&= \int d^3q' \ \langle q| \exp\left(-i\hat{\mathbf{H}}t\right)\left|q'\right\rangle \psi(q',0) \\
&= \int d^3q' \, K\left(q,q';t\right)\psi(q',0) \\
&= \int d^3q' K(q,q';t)\psi(q',0)
\end{aligned}$$

Here we've defined $K(q,q';t) = \langle q| \exp\left(-i\hat{\mathbf{H}}t\right)|q'\rangle$. Now to make progress, we need to find a meaningful expression for what $K\left(q,q;T\right)$ actually is. First, 'split up' our $\exp\left(-i\hat{\mathbf{H}}T\right)$ term into smaller pieces - that is, partition $T$ as

$$\exp\left(-i\hat{\mathbf{H}}T\right) = \exp\left(-i\hat{\mathbf{H}}\left(t_{n+1}-t_n\right)\right)\exp\left(-i\hat{\mathbf{H}}\left(t_n - t_{n-1}\right)\right)\dots\exp\left(-i\hat{\mathbf{H}}\left(t_1-t_0\right)\right) \tag{1}$$

here, we set by definition that $t_{n+1} = T > t_n > t_{n-1} > \cdots > t_1 > t_0 = 0$. For example, setting $n = 1$ and inserting one integral as part of the completeness relation, we get that

$$\begin{aligned}
K\left(q,q';T\right) &= \langle q| \exp\left(-i\hat{\mathbf{H}}\left(t_2-t_1\right)\right)\exp\left(-i\hat{\mathbf{H}}\left(t_1-t_0\right)\right)\left|q'\right\rangle \\
&= \int dq_1 \ \langle q| \exp\left(-i\hat{\mathbf{H}}\left(t_2-t_1\right)\right)|q_1\rangle\langle q_1| \exp\left(-i\hat{\mathbf{H}}\left(t_1-t_0\right)\right)\left|q'\right\rangle
\end{aligned}$$

where we've set $t_2 = T$. We can generalise this to the case where we have $n$ time slices. We have that

$$K\left(q,q',r\right) = \int \prod_{i=1}^{n}\left(dq_r \ \langle q_{r+1}| \exp\left(-i\hat{\mathbf{H}}\left(t_{r+1}-t_r\right)\right)|q_r\rangle\right)\langle q_1| \exp\left(-i\hat{\mathbf{H}}\left(t_1-t_0\right)\right)\left|q'\right\rangle \tag{2}$$