Notes on Black Holes

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Conventions and Housekeeping

We will set $G = c = 1$, and ignore the cosmological constant Λ. For indices, greek letters μ, ν refer to a specific basis. a, b are abstract indices which refer to any basis, from Roger Penrose. For example, we have

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma} \right), \quad R = g^{ab} R_{ab}$$

Christoffel components are basis dependent so they are written with greek indices.

As far as Black holes are concerned, the cosmological constant is incredibly small. Hence, we will ignore it from now. Check out Wald's book on General Relativity. It's great!

1 Spherical Stars

1.1 Cold Stars

If you think about a star like our sun, it's a ball of hot gas with nuclear reactions at its centre. Gravitational force makes the star contract, but nuclear reactions at the centre exert outward pressure to resist the contraction. If we wait long enough, th e star will exhaust the fuel it has and hence the star will contract. If we're interested in the final state of this star, we need to look at some source of pressure which is non-thermal in nature (so valid when the star is cold) to counter act gravity.

There is another natural source of pressure to resist gravity. This is the Pauli principle - where a gas of Fermions resists compression. This is called degeneracy pressure. This is a purely quantum mechanical phenomenon.

For example, a white dwarf is a star in which gravity is balanced by electron degeneracy pressure. A white dwarf is a very dense kind of star. If we had a white dwarf with the same mass of our sum $M = M$. the radius would be $R \sim \frac{R}{100}$. A white dwarf is how the sun will end its life.

Can all stars end their life this way? No. This is because there's a maximum mass for a white dwarf. This is called the Chandrasekhar limit, where

$$M_{WD} \leq 1.4M$$

What happens when we have a star more massive than this? When the star continues to get more and more dense, then inverse beta decay occurs where protons turn into neutrons. Neutrons are fermions and they also have degeneracy pressure. Thus, there's a second class of stars called Neutron stars where gravity is balanced by neutron degeneracy pressure. These stars are tiny.

If we took a neutron star with $M \sim M$, $R \sim 10km$. Compare this to $R \simeq 7 \times 10^5 km$. They are very dense! The gravitational field is very strong. If we have a Newtonian gravitational potential at the surface

$$|\phi| \sim 0.1$$

General relativity because important when $|\phi|$ is order 1. Hence, GR is important here. We can derive a maximum possible mass for neutrons stars as well. We'll derive this bound which is independent of our knowledge of dense matter.

We need to make some simplifying assumptions. We assume spherical symmetry.

1.2 Spherical Symmetry

Recall the unit round metric on a two dimensional sphere S^2,

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

The isometries are diffeomorphisms which preseve the metric. They form a group. The isometry group of a two sphere with that metric is $SO(3)$. Essentially, we define a space time as spherically symmetric if it has this as its isometry group.

Definition. Spherically Symmetric Spacetime A spacetime is spherically symmetric if its isometry group contains an $SO(3)$ subgroup, whose orbits are 2-spheres. In other words, if we pick a point and act on it with all $SO(3)$ elements, it will fill out a sphere.

Definition. Area Radius function In a spherically symmetric spacetime (M, g), the area radius function is

$$r : M \to \mathbb{R}, r(p) = \sqrt{\frac{A(p)}{4\pi}}$$

where $A(p)$ is the area of the S^2 orbit through the point p. So we take a point p, since its spherically symmetric, we can make a sphere out of it, and we take the area. This is linked to our understanding of the r coordinate, but this definition doesn't require a preferred origin. In other words, the S^2 has induced metric $r(p)^2 d\Omega^2$.

1.3 Time independence

Definition. Stationarity (M, g) is stationary if there exists a Killing vector field k^a (KVF) which is everywhere timelike. We're saying that $g_{ab}k^a k^b < 0$. We can introduce adapted coordinates. Let's pick some hypersurface Σ which is transverse to a vector field k. We can then pick coordinates x^i on Σ, where $i = 1, 2, 3$. This gives us coordinates on our surface. We assign coordinates $\left(t, x^i\right)$ to point parameter distance t along the integral curve of k^a through point on Σ with coords x^i. This implies

$$k = \frac{\partial}{\partial t}$$

This implies the metric is independent of t, since k^a is a Killing vector field. Therefore, the metric looks like

$$ds^2 = g_{00}\left(x^k\right) dt^2 + 2g_{0i} dt dx^i + g_{ij}\left(x^k\right) dx^i dx^j$$

This means that $g_{00} < 0$. Thus, given a stationary spacetime, we can construct coordinates which the metric is time independent. Conversely, any metric of this form is stationary.

There's a more refined version of time independence which we can use. Imagine we have a hypersurface Σ where $f = 0$, $f : M \to \mathbb{R}$ which is smooth, and $df \neq 0$ on Σ. Then, df is normal to Σ. If we let t^a tangent to Σ, then we have that

$$df(t) = t(f) = t^\mu \partial_\mu f = 0 \text{ since } f \text{ is constant}$$

A normal to a surface is not unique. Normals are not unique. What's the most general form of a covector field which is normal to Σ. If n_a also normal to Σ, them

$$n = g df + f n'$$

where g is smooth and not equal to 0 on Σ, and n' is a smooth 1-form. Let's look at the exterior derivative of this vector field. Using the rules for the exterior derivative,

$$dn = dg \wedge df + df \wedge n' + f dn'$$

Let's evaluate this on Σ. We have that

$$dn|_\Sigma = \left(dg - n'\right) \wedge df$$

Hence, $n \wedge dn - 0$ on Σ. This is because $n \propto df$ on Σ. The wedge product vanishes.

Theorem. Frobenius If $n \neq 0$ is a one form such that $n \wedge dn = 0$ everywhere, then there exist functions g, f such that $n = g df$. So n is normal to surfaces of constant f, so n is 'hypersurface orthogonal'. n is orthogonal to all surfaces of constant f.

Definition. Static spacetimes A spacetime (M, g) is static if there exists a hypersurface orthogonal, timelike KVF. In particular, static implies stationary. Why is this useful? Returning to the adapted coordinates, how does hypersurfaece orgonality help? We choose Σ orthogonal to k^a when defining adapted coordinates $\left(t, x^i\right)$. But, Σ is $t = 0$, therefore the normal to Σ is dt. So

$$k_\mu \mid_{t=0} \propto (1, 0, 0, 0)$$

In particular, we have that

$$k_i \mid_{t=0} = 0$$

but, $k_i = g_{0i}\left(x^k\right)$. Hence, $g_{0i} = 0$. Thus, if we write down the metric, we have that

$$ds^2 = g_{00}\left(x^k\right)dt^2 + g_{ij}\left(x^k\right)dx^i dx^j, \quad g_{00} < 0$$

Thus we have a discrete time reversal isometry $\left(t, x^i\right) \to \left(-t, x^i\right)$. Thus,

$$\text{static} \iff \text{time independent and invariant under time reversal}$$

For example, for a rotating star, the metric may be time independent but not static since they're is no time reversal symmetry since it changes the sense of rotation.

1.4 Static, Spherically Symmetric Spacetimes

The isometry group is $\mathbb{R} \times SO(3)$. We can show that this implies the metric must be static. If it wasn't static, it would be rotating, but that breaks spherical symmetry. On Σ choose coordinates $x^i = (r, \theta, \phi)$, where r is our area radius function. If we do this, then the metric is

$$ds^2 \mid_\Sigma = e^{2\psi(r)}dr^2 + r^2 d\Omega^2$$

where due to spherical symmetry, everything depends on r. If we had $drd\phi$ or $drd\theta$ terms, this would break spherical symmetry. If we then go on to write down the full spacetime metric, we get that

$$ds^2 = -e^{2\Phi(r)}ds^2 + e^{2\Psi(r)}dr^2 + r^2 d\Omega^2$$

At the moment, there's no origin.

Summary

Spherical symmetry

- A spacetime is spherically symmetric if the isometry group has an $SO(3)$ subgroup.
- The orbit of $p \in M$ is a S^2 sphere.
- We define the area radius function

  $$r(p) = \sqrt{\frac{A(p)}{4\pi}}$$

  This has the interpretation of radius.

Properties of Spacetimes

To help us, we can construct hyper-surfaces as follows

- We can define a hypersurface Σ to be the surface where $f(x) = 0$
- df is normal to this surface as $df(t) = 0$ for any tangent vector.
- Normals n to the surface can be written as

$$n = gdf + fn' \implies n \wedge dn \mid_{\Sigma} = 0$$

- Frobenius says that if $n \wedge dn = 0$ everywhere, then there exist f, g such that $n = gdf$ so n is normal to surfaces of constant f.

We can classify different spacetimes as below.

- A spacetime is symmetric in a variable s if s is a coordinate but the metric doesn't depend on s.
- A spacetime is stationary if there exist coordinates x^α such that x^0 is timelike at infinity, and our metric doesn't depend on x^0 (equivalent to saying that there's a Killing vector which is timelike at infinity).
- A spacetime is static if there are no cross terms in the metric like g_{0i}.
- Construct stationary spacetimes by defining a Killing vector, then construct a hypersurface Σ nowhere tangent to that vector. Assign spatial coordinates x^i for positions in Σ. Then, construct the coordinate t by moving a distance t in the parameter orthogonal to the hypersurface. Then, the killing vector is

$$k = \frac{\partial}{\partial t}$$

- Our final metric, including spherical symmetry is

$$-e^{2\Psi(r)} dt^2 + e^{2\Phi(r)} dr^2 + r^2 d\Omega^2$$

2 The formation of black holes

In this section, we'll cover the formation of black holes. To start this discussion, we'll need to discuss the idea of symmetry on manifolds and in metrics. The word 'black hole' should be a big enough hint that the kinds of metrics we'll be considering exhibit spherical symmetry. But, since general relativity is done in the frame-work of both **space** and **time**, we need to make clear what 'spherical symmetry' actually means.

First, lets discuss how to obtain the metric for a standard 2-sphere. Working in signature $(-, +, +, +)$, we have that the metric on the 2 -sphere in Cartesian coordinates is given by the following, with the constraint

$$ds^2 = dx^2 + dy^2 + dz^2, \quad x^2 + y^2 + z^2 = 1$$

If we reparametrize the coordinates as follows, using our standard spherical coordinates with $r = 1$, we embed the 2 -sphere in $\mathbb{R}^3$

$$\begin{aligned} x &= \cos\phi \sin\theta \\ y &= \sin\phi \sin\theta \\ z &= \cos\theta \end{aligned}$$

Applying a coordinate transformation for the one forms dx, dy, dz, we get that

$$\begin{aligned} dx &= \cos\phi \cos\theta d\theta - \sin\phi \sin\theta d\phi \\ dy &= \sin\phi \cos\theta d\theta + \cos\phi \sin\theta d\phi \\ dz &= -\sin d\theta \end{aligned}$$

Substituting into the above, we can read off that the components of the metric are given by

$$ds^2 = d\theta^2 + \sin^2\phi d\phi^2$$

This metric comes from considering the 2-sphere manifold. Now, we know that the symmetry group of a 2-sphere is $O(3)$ if we consider reflections, and just $SO(3)$ if we consider only rotations. Hence, we say that the metric admits an isometry group of $SO(3)$.

Definition. (Isometries on a metric) An **isometry** is a transformation on a metric space which leaves distances between points invariant. The image one has in their mind immediately might be a rotation or a reflection on a two dimensional plane.

3 Useful Identities in General Relativity for Black Holes

In this section, we'll cover some useful identities which may prove useful for doing general relativity. The first one we'll prove is an equation that's useful for proving the divergence theorem for curved space.

Theorem. Divergence of a vector field in terms of $\sqrt{-g}$

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} V^\mu \right) \tag{1}$$

Proof. We first apply the product rule, then make use of a smart rearrangement using logs. We first have that

$$\begin{aligned} \nabla_a V^a &= \partial_\mu V^\mu + \frac{1}{\sqrt{-g}} V^\mu \partial_\mu \left(\sqrt{-g} \right) \\ &= \partial_\mu V^\mu + V^\mu \partial_\mu \left(\log \sqrt{-g} \right) \\ &= \partial_\mu V^\mu + \frac{1}{2} V^\mu \partial_\mu \left(\log\left(-\det g \right) \right) \end{aligned}$$

Where in the last line we wrote out the determinant explicitly. Now the trick here is to use the identity which relates the logarithm of the determinant to the trace of the formal logarithm of a matrix.

Observe that if A is a matrix, which we assume to be positive definite (since our metric is), then it is diagonalisable. Since it's diagonalisable, we can write it in the appropriate basis such that $\exp A = \operatorname{diag}\left(e^{\lambda_1}, \ldots, e^{\lambda_n}\right)$ where $\lambda_i \in \mathbb{R}$ are the eigenvalues of the matrix A. This means that the eigenvalues of $\exp A$ are e^{λ_i} for all i. So, since the determinant of the matrix is equal to the product of its eigenvalues, we have that

$$\det(\exp A) = \exp(\operatorname{tr} A)$$

If we set $B = \exp A$, then this identity reduces to

$$\det B = \exp(\operatorname{tr} \log B)$$

Taking the logarithm of both sides once more, we have that finally

$$\log \det B = \operatorname{tr} \log B$$

We can now resume to the question at hand. We rewrite the above using this identity so that

$$\begin{aligned}
\nabla_a V^a &= \partial_\mu V^\mu + \frac{1}{2} V^\mu \partial_\mu \left(\operatorname{tr} \log(-g)\right) \\
&= \partial_\mu V^\mu + \frac{1}{2} V^\mu \operatorname{tr} \partial_\mu \log(-g) \\
&= \partial_\mu V^\mu + \frac{1}{2} V^\mu \operatorname{tr}\left(g^{-1} \partial_\mu g\right)
\end{aligned}$$

Where in this case we've put g to denote schematically the matrix $g_{\mu\nu}$ and **not** the determinant. Be careful to observe that the minus signs when differentiating the logarithm cancel out. One can easily verify that $\frac{1}{2} g^{\alpha\beta} \partial_\mu g_{\alpha\beta} = \Gamma^\nu_{\mu\nu}$. Thus, this completes the proof. □

Summary

Miscellaneous Identities

Lie Derivatives

- The Lie derivative for a general tensor is

$$\mathcal{L}_X T^{\mu_1 \cdots}{}_{\nu_1 \ldots} = X^\sigma \partial_\sigma T^{\mu_1 \cdots}{}_{\nu_1 \ldots} - \left(\partial_\lambda X^{\mu_1}\right) T^{\lambda \ldots}{}_{\nu_1 \ldots} - \cdots + \left(\partial_{\nu_1} X^\lambda\right) T^{\mu_1 \cdots}{}_{\lambda \ldots}$$

Differential forms

- A p -form is a totally anti-symmetric rank $(0, p)$ tensor.
- Express p-forms in terms of wedge products

$$X = \frac{1}{p!} X_{\mu_1 \ldots \mu_p} dx^{\mu_1}$$

- The exterior derivative on a $p-$form is defined as

$$(dX)_{\mu_1 \ldots \mu_p \mu_{p+1}} = (p+1)\, \partial_{[\mu_1} X_{\mu_2 \ldots \mu_{p+1}]}$$

- The exterior derivative acts using the standard Leibniz rule.

Killing equation

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Differential forms

Example Sheet 1

Question 2

We want to show Cartan's magic formula.

$$\mathcal{L}_X Y = \iota_X dY + d\left(\iota_X Y\right)$$

For a p-form Y the Lie derivative is

$$\mathcal{L}_X Y = X^\alpha \partial_\alpha Y_{\mu_1 \mu_1 \ldots \mu_p} + \left(\partial_{\mu_1} X^\alpha\right) Y_{\alpha \mu_2 \ldots \mu_p} + \cdots + \left(\partial_{\mu_p} X^\alpha\right) Y_{\mu_1 \ldots \alpha}$$

The basic thing to show here is that

$$d\left(\iota_X Y\right)_{\mu_1 \ldots \mu_p} = \sum_{i=1,\ldots n} \partial_{\mu_i} \left(X^\alpha Y_{\mu_1 \ldots \alpha \ldots \mu_p}\right)$$

In addition, we have that

$$\iota_X dY = X^\alpha \partial_\alpha Y_{\mu_1 \ldots \mu_p} - X^\alpha \partial_{\mu_1} Y_{\alpha \ldots \mu_p} - \cdots - X^\alpha \partial_{\mu_p} Y_{\mu_1 \ldots \alpha}$$

We can use small cases to work out the right signs here.