Part III Statistical Field Theory

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Contents

1 Useful resources

- David Tong's notes on statistical field theory

- MIT opencourseware's lecture notes on statistical mechanics.

2 Introduction

Statistical field theory studies systems with large degrees of freedom, their different 'phases' and the transition between these phases. Think 'more is different'. When we have a large number of degrees of freedom, we can see discontinuous phenomena emerge.

3 Why Statistical Field Theory?

3.1 Phase transitions and critical exponents

This set of notes will revolve around explaining the phenomena of phase transitions. In physics so far, we've described systems in terms of smooth and differentiable functions. For example, smooth paths given by solving the Euler-Lagrange equations for N particles. However, in reality, many physical systems exhibit discontinuous behaviour. The most obvious of these would be a phase transition from liquid to gas, say. Or, for example, thje change in magnetisation for a system from an ordered state to a disordered state. One may ask, how can systems comprised of smooth dynamics exhibit discontinuous behaviour?
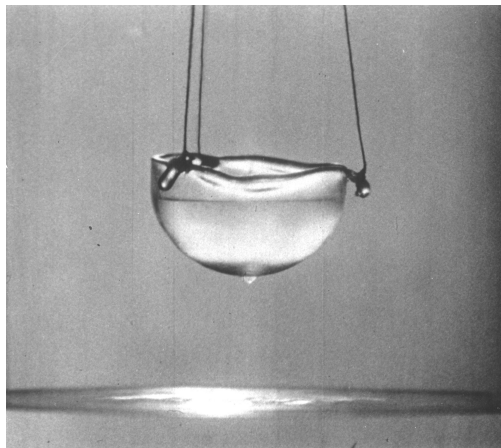
Figure 1: Helium in a 'superfluid' phase

These discontinuities appear as a result of taking an $N \to \infty$ limit for the number of particles, and our previous functions which are smooth begin to get 'squashed' into a discontinuous shape. We'll study this behaviour throughout these set of notes. The discontinuities of phase transitions occur at special values of descriptors of our physical system (for example temperature, magnetisation, pressure), which we call **critical points**. How a system behaves near critical points give

rise to critical exponents. For example, the mean-field theory prediction of how magnetisation varies as T reaches it's critical temperature is

$$m \sim (T - T_c)^\alpha, \quad \alpha = \frac{1}{2}$$

where α is a critical exponent in this case. Or, somewhat conversely, we have another critical exponent derived from how magnetisation changes due to a change the applied magnetic field B, which again mean-field theory predicts to have the relationship

$$m \sim B^{\frac{1}{3}}$$

and in this case $\frac{1}{3}$ is another example of a critical exponent appearing in nature. Other quantities we might want to examine the behaviour of as we approach critical points in different variables are

- heat capacity per unit volume, often denoted c, changing with as $T \to T_c$

- magnetic susceptibility, which is how the magentisation of a system changes in response to an ambient magentic field:
$$\chi = \frac{\partial m}{\partial B}$$
and how this quantity also varies as temperature approaches the critical temp $T \to T_c$

- how density of particles, $\nu_{\text{gas/liquid}}$ changes when either pressure or temperature approaches a critical point

- compressibility and more!

3.2 Universality

A striking experimental observation that we notice with regards to critical exponents is that they seem to match for different systems despite wildly different structures. An example is the critical exponent for the magnetic susceptibility of a material χ as $T \to T_c$ and the compressiblity of a gas also as $T \to T_c$. I'll include more examples later.

3.3 Explaining universality with the renormalization group flow

The key to understanding why universality occurs is by looking at how physics changes depending on what scale we're looking at it with. This is the essensce of the renormalisation group. We'd expect a system to exhibit different physical characteristics depending if we were looking at it from the perspective of a towering giant, or the perspective of a miniscule ant.

We provide a toy example here. Suppose that we have a grid of 16 equally spaced atoms on a grid shown in the figure below. Now, take your favourite physical model of some phenomoena. For example, we could take the Ising model which treats the atoms as either spin up or spin down, giving us the Hamiltonian as the primary descriptor of the system:

$$H(T, J) = -J \sum_{\langle ij \rangle} s_i s_j - B \sum_i s_i$$
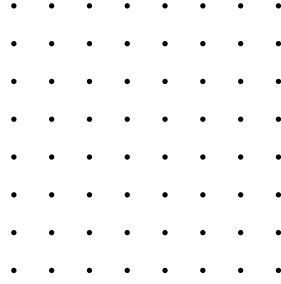
Figure 2: Just atoms neatly arranged in a lattice

where we've summed over neighbouring states, and have fixed temperature T as well as a coupling constant J, in our model. The s_i denote our spin state at each location in this lattice.

Now, having to look at every atom in this lattice is a bit difficult when fjust because of the sheer number of them. Something we could do is to group atoms which are close together, and assign something like an average composite spin at each site. One obvious way to do this is to group them up in groups of say 4 in squares.
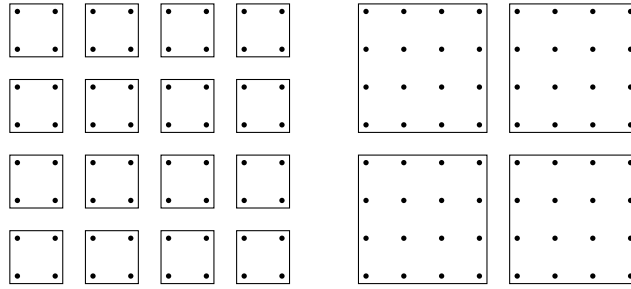
Figure 3: We can group up atoms to deal with less variables

This is shown in the first diagram. We've grouped atoms into sets of four, then we assign a single amalgamated value (like a spin average) to each group. Now, if this system exhibits **self similarity**, this means that in the new model, our structure of the Hamiltonian is the same, but our new parameters need to be changed to account for the grouping. So, our system is rewritten as

$$H(J', B') = -J' \sum_{\langle ij \rangle} \bar{s}_i \bar{s}_j - B' \sum_i \bar{s}_i$$

So, by 'zooming' out of our system, we've mapped our pair of variables for from $(J, B) \rightarrow (J', B')$. This is the study of the renormalistion group. It explains universality because it predicts that different physical descriptors are just 'zoomed out' manifestations of other descriptors.

Now, we can repeat this process even more and zoom out further, which will change the system even more.

3.3.1 Fixed points

Now, one may ask, what if we 'zoom out' according to procedures like the above, and observe the same behaviour. This is what is known as a fixed point. For example, if we had a magnetic

system which is based on T, the temperature, and a coupling constant J, then if we take $T \to \infty, J = 0$, then we'll always see the same behaviour of disorder no matter how much we zoom out. We'll touch on this in the later sections.

4 The Ising Model and Critical Exponents

4.1 The problems with just mean-field theory

4.2 Setting up the Ising model

Our simplest physical model to begin with is an array of particles situated on a lattice, each with an individual spin, $s_i = \pm 1$. From this, the Ising model gives us an expression for the energy of the system:

$$E = -B \sum_i s_i - J \sum_{<ij>} s_i s_j.$$

B is a magnetic field which is applied. We want to minimise the energy of this expression, and this leads to different behaviours depending on J

- $J > 0$ means that our spins align (ferromagnet)

- $J < 0$ Misaligned (anti-ferromagnet)

The sums are over all possible configurations of spins in the lattice. The first term arises from the magnetisation of the material, and the second term is an energy penalty applied due to the interaction of neighbouring atoms having the same or different spins. Keeping track of spins motivates us to define an idea of some average value of spin over the whole system, which ranges from -1 to 1.

$$m = \frac{1}{N} \sum_i s_i.$$

With this expression, we can calculate the probability of having a certain configuration of spins.

$$\mathbb{P}(\text{configuration is } \{s_i\}) = \frac{e^{-\beta E[s_i]}}{Z}$$

where Z is our partition function

$$Z = \sum_{\{s_i\}} e^{-\beta E[s_i]}.$$

Our expression for the partition function Z is quite important. It encodes on its own quite a lot of information we have about the physical system. Suppose we wish to compute a value for our average magnetisation $\langle m \rangle$. Well, this is the weighted sum of the magnetisation of all possible contributions to spin, which is

$$\langle m \rangle = \sum_{s_i} \frac{e^{-\beta E[s_i]}}{Z} \times \frac{1}{N} \sum_{s_i \in \{s_i\}} s_i.$$

7

Of course, given are previous expression for E we can condense this down to the compact expression

$$\langle m \rangle = \frac{1}{\beta N} \frac{\partial \log Z}{\partial B}.$$

We can also calculate the thermodynamic free energy

$$F_{thermo}(T, B) = \langle E \rangle - TS = -TlogZ$$

4.3   Reformulating the above in terms of free energy

We can repose the model above in a slightly different way. Instead of formulating the sums as sums over all possible spin configurations, could we possibly reformulate this as a function it terms of m? We can rethink the sum as summing over all possible spin states **given** a particular magnetisation, and then summing over all possible magnetisations.

$$Z = \sum_m \sum_{s_i|m} e^{-\beta E[s_i]}.$$

where we rewrite the final product in the sum as

$$Z = \sum_m e^{-\beta F(m)}, \quad \sum_{s_i|m} e^{-\beta E[s_i]} = e^{-\beta F(m)} = e^{-\beta N f(m)}.$$

where we define $F(m) = Nf(m)$. $F(m)$ is our effective free energy (which may also be temperature dependent). Since our increments for m are very small, we can rewrite this as an integral

$$Z = \frac{N}{2} \int_{-1}^{1} dm \left(e^{-\beta N f(m)} \right).$$

We've added a $\frac{N}{2}$ here since we're summing in steps of $2/N$, and we need this factor to cancel out these steps. This integral is nicely approximated by our method of steepest descent;

$$Z \approx e^{-\beta N f(m_{min})}.$$

We've dropped the factor of $\frac{N}{2}$ since scaling our partition function doesn't affect our physics. Thus, our thermodynamic free energy given by the eqution above means that

$$F_{thermo} \simeq F(m_{min})$$

4.3.1   Calculating our free energy via the mean field approximation.

We now employ an approximation to make our lives a whole lot easier. This is called the mean field approximation. For a given spin configuraiton $\{s_i\}$, we can replace

$$s_i \rightarrow \langle s \rangle = m, \quad m = \sum_i s_i,$$

so we can write our energy approximately as

$$E = -BNm - \frac{1}{2}JNqm^2 \implies \frac{E}{N} = -Bm - \frac{1}{2}Jqm^2$$

We want to figure out an expression for $f(m)$. We would like to find out

$$\sum_{s_i|m} e^{-\beta E[s_i]} = e^{-\beta N f(m)}.$$

The next step is to count the number of spin representations we can have given a certain magnetisation m. If we denote $N_\uparrow$ and $N_\downarrow$ as the number of spin up and spin down atoms in our lattice, then we have

$$N = N_\uparrow + N_\downarrow,$$
$$m = \frac{N_\uparrow - N_\downarrow}{N} = \frac{2N_\uparrow - N}{N}$$

We would like to figure out a function to express the number of spin configurations in a system given the magnetisation. We reduce the problem in terms of finding an expression of the number of configurations given a $N_\uparrow$ instead. Since $N_\uparrow$ is a determined function of m (by solving the system of equations above), we find that the number of states is

$$\Omega(N_\uparrow) = \frac{N!}{N_\uparrow!N_\downarrow!} = \frac{N!}{N_\uparrow!(N - N_\uparrow)!}$$

Stirling's formula allows us to approximate logarithms of $N!$ as

$$\log N! = N \log N - N$$

One can show that, with Stirling's approximation, that

$$\frac{\log \Omega}{N} = \log 2 - \frac{1}{2}(1 + m)\log(1 + m) - \frac{1}{2}(1 - m)\log(1 - m)$$

Our aim to to calculate our effective free energy,

$$e^{-N\beta f(m)} = \sum_{\{s_i\}|m} e^{-\beta E[s_i]}$$

But this expression, with the mean field approximation, is the given energy at magnetisation m, counted over the number of states with that magnetisation. Hence,

$$e^{-N\beta f(m)} = \Omega(m)e^{-\beta E(m)} \simeq e^{\frac{N \log \Omega(m)}{N}} e^{-\beta(-Bm - \frac{1}{2}Jqm^2)}$$

Comparing the coefficients on the exponentials, we have that our free energy is then given by

$$f(m) = -Bm - \frac{1}{2}Jqm^2 - T\left(\log 2 - \frac{1}{2}(1 + m)\log(1 + m) - \frac{1}{2}(1 - m)\log(1 - m)\right)$$

Our system lies in the state where **free energy is minimised** with respect to magnetisation. Hence, our equilibrium state lies where $\frac{\partial f}{\partial m} = 0$. This condition gives rise to something that we

call a self consistency condition, which is an equation that m satisfies on both sides. In this case it is given by

$$\beta(B + Jqm) = \frac{1}{2}\log\frac{(1 + m)}{(1 - m)}, \quad \implies m = \tanh(\beta B + \beta Jqm)$$

One can see graphically that at low temperatures, we see an emergence of two solutions $m_{\pm} \neq 0$, meaning that at low temperatures are system does have a non-trivial magnetisation. At high temperatures, this have a solution at $m = 0$, and our state is disordered since we have no average magnetisation.

4.4 Using the Landau approach

We can study the behavior of $f(m)$ more closely by Taylor expanding. From here on out, we'll think of m in a more general sense, as something that we'll call an 'order parameter', and see how f changes.

Using the expansions

$$\log(1 + m) = m - \frac{m^2}{2} + \frac{m^3}{3} - \frac{m^4}{4} + \dots$$

One can easily verify that up to $O(m^4)$, we have

$$(1 + m)\log(1 + m) + (1 - m)\log(1 - m) \simeq m^2 + \frac{m^4}{6}$$

Thus, we have an expansion for our expression

$$f(m) = -T\log 2 - Bm + \frac{1}{2}(T - Jq)m^2 + \frac{T}{12}m^4$$

The case when $B = 0$

We'll now take a look at the equilibrium states of, where $\partial f/\partial m = 0$. In the case of $B = 0$, we have a quartic that looks qualitatively different depending on whether $T > Jq$ or whether $T < Jq$. This is shown in the graphs In the case where $T > T_c$, our minimum point is at $m = 0$ as shown in the graph, which means that we have no average magnetisation. For this reason, we call this a disordered state. When $T < T_c$, we have that the system that settles at either
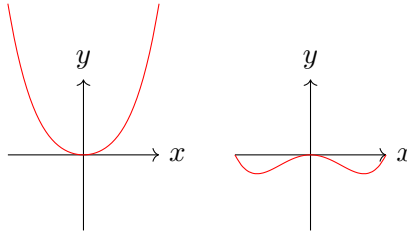
Figure 4: Here we've shown plots where $T > T_c$ on the left versus when $T < T_c$

minimum point. We can calculate this minimum point explicitly by differentiating by solving for $f'(m)$ to get that our minimum points are

$$m_0 = \pm\sqrt{\frac{3(T_c - T)}{T}}$$

This is our first example of a phase transition; when a physical quantity (in this case, average magnetisation), depends discontinuously on some parameter. We define an nth order phase transition when the nth derivative of the equilibrium value is discontinous when we differentiate it by some parameter. In other words, when the function $f'(\mu) = f(\mu, m_0)$ is discontinuous when we differentiate it n times with respect to μ, then do we have a phase transition. In this example, if we substitute our values of m_0 back into f, then

$$f(m_0, T) = \begin{cases} 0 & T > T_c \\ -\frac{3}{4}\frac{(T-T_c)^2}{T} & T < T_c \end{cases}$$

At $T = T_c$, we have that $f(m_0, T)$ and $\frac{df}{dT}$ are continuous, but we have a discontinuity at $\frac{d^2 f}{dT^2}$. Thus, our phase transition is a second order phase transition. We can also see this discontinuity exhibited when we examine heat capacity.

Our heat capacity is defined as our change in average energy as temperature changes

$$\frac{d\langle E\rangle}{dT} = c$$

But we can calulate average energy in terms of our partition function Z, which is given by

$$\langle E\rangle = \frac{-\partial \log Z}{\partial B}$$

We use the chain rule to express derivatives in T as derivatives in β

$$\frac{d}{dT} = \frac{d\beta}{dT}\frac{\partial}{\partial\beta} = -\beta^2\frac{\partial}{\partial\beta}$$

Hence our heat capacity is given by the formula

$$c = \beta^2\frac{\partial^2\langle \log Z\rangle}{\partial\beta^2}$$

Now, with our saddle point approximation, we know that

$$\log Z = -\beta N f(m_{min}) \implies \log Z = -\beta N f(m_{min}) = \begin{cases} 0 & T > T_c \\ \beta N\frac{3}{4}\frac{(T-T_c)^2}{T^2} & T < T_c \end{cases}$$

In the case when $T < T_c$, we have that setting $T = 1/\beta$ gives us that

$$\log Z = \frac{3}{4}N(1 - 2T_c\beta + T_c^2\beta^2) \implies c = \frac{3}{2}NT_c^2\beta^2$$

Hence, as $T \to T_c$, we have a discontinuity in our heat capacity

$$C = \begin{cases} 0 & T \to T_c^- \\ \frac{3}{2}N & T \to T_c^+ \end{cases}$$

Since this involves a discontinuity in a second derivative of $f(m_{min})$ (in this case β), this is a second order phase transition.

11

The case when $B \neq 0$

Now, in the case where $B \neq 0$ and we have our free energy given by

$$f(m) = -Bm + \frac{1}{2}(T - T_c)m^2 + \frac{T}{12}m^4 + \dots$$

In the case where $T < T_c$, we already have two minimum points, but as the graphs show we have a global minimum at a specific point depending on the sign of B. This is shown in the graphs below, where when $B > 0$ our equilibrium magnetisation lies at $m_0 > 0$, and when $B < 0$ our equilibrium magnetisation lies at at $m_0 < 0$.
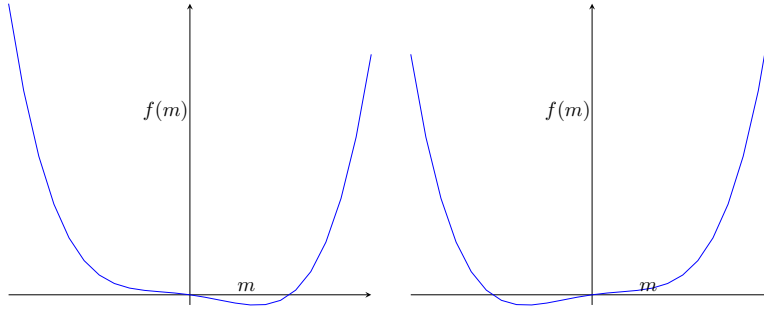
Figure 5: Our global minimum depends on the sign of B this time

4.5   A Brief look at Universality

Right now we'll explore a very strange phenomena in physics - the concept of universality which we mentioned in the introduction. Let's take a look at a different example of a phase transition - liquid and gas phase transitions given governed by the Van der Waals equation. We have the phase transition diagram as follows.

From the Van der Waals equation, we also can look at limiting behaviour as $T \to T_c$, and we can derive critical exponents

$$v_{gas} - v_{liquid} \sim (T - T_c)^\beta$$
$$v_{gas} - v_{liquid} \sim (P - P_c)^{\frac{1}{\delta}}$$

We can define compressiblity,

$$\kappa = -\frac{1}{v}\left.\frac{\partial v}{\partial p}\right|_p$$

We can show that

$$\kappa \sim \frac{1}{(T - T_c)^\gamma}, \quad \gamma = 1$$

One can notice that these are the same exponents one can find from the Ising model, which is an interesting phenomenon! We say that in this case, the theory magnetisation and liquid-gas phase transitions are in the same universality class. To gain a rough intuition of why this might be

true, we can also model gases as a lattice, where the energy cost of a particle being in a position costs an energy potential μ, and the energy cost from a Van der Waals force of neighbouring particles is given by a coupling constant J. In this case, our energy is given by

$$E = -4J \sum_{\langle ij \rangle} n_i n_i - \mu \sum_i n_i$$

The variable

$$n_i = \begin{cases} 0 & \text{if the lattice space isn't occupied} \\ 1 & \text{if the lattice space is occupied} \end{cases}$$

can be mapped to spins with the map $s_i = 2n_i - 1$, so that when $n_i = 1, s_i = 1$ and when $n_i = 0, s_i = -1$.

4.6   The Landau-Ginzburg approach

Now that we've covered using m with the Ising model as an order parameter in Landau theory, we can generalise this a bit further. How do we modify this argument to move away from lattices and into a continuous space with some dimension say $\mathbb{R}^d$? If we could achieve this, then we would be able to make sense of expressions such as $m(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n$.

$$m \to m(\mathbf{x})$$

To get started, we use something called a coarse-graining approach. Imagine we divide up the space $\mathbb{R}^d$ into a tiny gridframe work, and then assign some value $m_{\mathbf{i}}$ to each point in this tiny grid. We can then define

$$m(\mathbf{x}) = \frac{1}{N} \sum_{\mathbf{i}} m_{\mathbf{i}}$$

where we specifically average over the nearest neighbours, close to $\mathbf{x}$. Thus, our magnetisation is from $m \in (-1, 1)$. To this, we can come up with a heuristic for a partition function. We can sum up our field configurations to give our partition functions

$$Z = \sum_{m(\mathbf{x})} \sum_{\{s_i\}|m(\mathbf{x})} e^{-\beta[s_i]} = \sum_{m(\mathbf{x})} e^{-\beta F(m(\mathbf{x}))}$$

The difference with this and what we did before is that we've added a local dependence on m, called this a function, and then summed over all field configurations which give this particular configuration. For this approximation to be valid, we require that these tiny grids which divide up our space are large enough to cover a large amount of atoms, but small enough so that we still have a notion of continuity when we move from grid to grid.

We're ready to promote this concept tot he path integral formulation of the partition function, by replacing the sum with an integral.

$$\mathcal{Z} = \int \mathcal{D}m(\mathbf{x}) e^{-\beta F(m)}$$

In line with our usual expressions for probabilities with partition functions, we can then calculate the probability of obtaining a certain field configuration as

$$\mathcal{P}(m(\mathbf{x})) = \frac{e^{-\beta F(m)}}{\mathcal{Z}}$$

The measure

$$\int \mathcal{D}m$$

is meant to denote an integral over all possible field configurations $m(\mathbf{x})$, is called doing a functional integral. Intuitively, this notation is justified because the sum we're doing can be rewritten as follows, where in the second sum we're summing over all points instead, and then configurations

$$\sum_{m(\mathbf{x})} \simeq \sum_{\mathbf{x}_i} \sum_m m_i(\mathbf{x}_i) = \int dm_1(\mathbf{x}_1) dm_2(\mathbf{x}_2) \dots dm_m(\mathbf{x}_m)$$

We can extend this even more to include m being an n-dimensional vector instead of the scalar we've presented it to be here. In this case, it would our partition function would be of the form

$$\mathcal{Z} = \int \mathcal{D}\mathbf{m}(\mathbf{x}) e^{-\beta F(\mathbf{m})}$$

Okay great. Now, we're ready to try an 'construct' a theory based on some restrictions we have gained from our intuition about physical systems. by constructing a theory, we mean that we would like to construct a sensible $F(m(\mathbf{x}))$, which is an integral of something. We'll list out some guiding principles in this section.

Locality

In our theory, we would only like to consider short range interactions between points in space. This means that essentially, magnetisations at different points in space shouldn't correlate with each other. This is the principle of **locality**. It means that essentially, our free energy can integrate over one variable at each specific point. This means that an expression of the form

$$F(m(\mathbf{x})) = \int d^d x \, f(m(\mathbf{x}))$$

are allowed, but a function that integrates over different points simultaneously like

$$F(m(\mathbf{x})) = \int d^d x d^d y \, f(m(\mathbf{x}), m(\mathbf{y}))$$

is not allowed. Now, we did say that we can allow *short range* interactions. How are these incorporated in the theory? Well, we can include gradients of m, such as terms like ∇m, to encode nearest neighbour interactions. This is because philosophically, taking a gradient is like looking at the difference in magnetisation at close points on a lattice which has small spacing;

$$\frac{\partial m}{\partial \mathbf{x}} = \lim_{a \to 0} \frac{m(\mathbf{x} + a\hat{\mathbf{x}}) - m(\mathbf{x})}{a}$$

Translational and Rotational Invariance

Physical systems should be agnostic of whether we move them around in a translation sense, or if we rotate them about. Physical quantities should remain the same; systems shouldn't have a preferred vector or orientation. With this in mind, a condition we impose is that our form of $F(\mathbf{m})$ should be **invariant** under transformations of the type

$$\mathbf{m} \to \mathbf{m}' = R\mathbf{m}$$

where R is some rotation or reflection in the group $O(n)$. This is because the nature of our critical phenomena should be invariant of the orientation it's in. We can't allow terms like $\mathbf{n} \cdot \nabla \mathbf{m}$, for a fixed $\mathbf{n}$, for example.

Since we have that $R^T R = 1, \forall R \in O(n)$, it's likely that we're permitted terms of the form

$$\mathbf{m} \cdot \mathbf{m}, \quad \nabla \mathbf{m} \cdot \nabla \mathbf{m}, \quad m^4 \ldots$$

$\mathbb{Z}_2$ invariance

In the Ising model, we have that intuitively, our physics doesn't change if we swapped ↑→↓ and ↓→↑. This means that $\mathbb{Z}_2$ symmetry is manifest in our physical systems, which implies that terms in F should be invariant under the transformation

$$m(\mathbf{x}) \rightarrow -m(\mathbf{x})$$

Analyticity

In physics, we rarely have to deal with functions which are not well behaved everywhere. A function is **analytic** if it has a well behave Taylor expansion in our region of interest. This means that nice polynomial functions of our order parameter such as

$$m(\mathbf{x})^2, \ (\nabla m(\mathbf{x}))^2, \ldots$$

are okay. However, a term like

$$\log(m(\mathbf{x}))$$

is not since the Taylor expansion is not well defined at $m = 0$.

The final form of a plausible free energy

This motivates us to write an expression for our free energy as

$$F(m) = \int d^d x \, \alpha_2 m^2 + \gamma (\nabla m)^2 + \alpha_4 m^4$$

where we're working to just quadratic order. We have a μ^2 coefficient beside m^2 since, comparing with our mean field approximation we worked through earlier, we need this to me positive for the quadratic cutoff to remain valid. I'll go into more detail about this later. If our coefficient was negative then we'd have an unstable point for our magnetisation.

So far, we have a free energy which is a function of a function. Can we think of an 'extremal' function which minimises this quantity? Well, we can employ a similar approach we used in deriving the Euler-Lagrange equations, by perturbing the configuration $m(\mathbf{x}) \rightarrow m(\mathbf{x}). + \delta m$. Doing this with our expression for free energy above, we get that

$$\delta F = \int d^d x \, 2\alpha_2 m(\delta m) + 2\gamma \nabla m \cdot (\delta \nabla m) + 4\alpha_4 m^3 (\delta m)$$

$$= \int d^d x \, \delta m \left(\alpha_2 m + \alpha_4 m^2 - \gamma \nabla^2 m \right)$$

Thus our functional derivative of the integrand is of the form

$$\frac{\delta f}{\delta m} = \alpha_2 m + \alpha_4 m^3 - \gamma \nabla^2 m$$

Thus, we have that if we want to find the field configuration $m(\mathbf{x})$ which minimises the free energy, we need to solve the equation

$$\gamma \nabla^2 m = \alpha_2 m + \alpha_4 m^3$$

One can easily check if we assume that m is constant, we 3 possible solutions depending on the sign of α_2.

$$\begin{cases} m = 0 & \alpha_2 > 0 \\ m = \pm\sqrt{\frac{-\alpha_2}{\alpha_4}} & \alpha_2 < 0 \end{cases}$$

4.7 Domain Walls

However, we recognise that this obeys the same pattern as the mean field approximation! This is reassuring. However, we can go into a bit more detail, and explore field configurations in one dimension which smoothly move between $\sqrt{\frac{-\alpha_2}{\alpha_4}}$ and $-\sqrt{\frac{-\alpha_2}{\alpha_4}}$ (we can't have say, a step function because that means our gradient would be infinite). One can verify that a solution with this asymptotic property of converging to these stable state is given by

$$m = m_0 \tanh\left(\frac{x - x_0}{w}\right), \quad W = \sqrt{\frac{2\gamma}{\alpha_2}}$$

This is called a domain wall. We need to verify that this is a solution of the equation of motion. We find that

$$\frac{dm}{dx} = \frac{m^0}{W}\operatorname{sech}^2\left(\frac{x - X}{W}\right), \quad \frac{d^2m}{dx^2} = -\frac{2m_0}{W^2}\tanh\left(\frac{x - X}{W}\right) + \frac{2m_0}{W^2}tanh^2\left(\frac{x - X}{W}\right)$$

However, we can just wrangle this into the form

$$\gamma\frac{d^2m}{dx^2} = -\frac{2\gamma}{W^2}m + \frac{2\gamma}{W^2 m_0^2}m^3$$

If we substitute in our asymptotic values $\pm m_0$, as well as our characteristic width W, then we recover that indeed

$$\gamma\frac{dm}{dx^2} = \alpha_2 m + \alpha_4 m^3$$

Now, we saw that the function m has **characteristic length** W. Beyond this length we basically consider the function m to be constant. Let's try plug this function back into our expression for our free energy $F(m(\mathbf{x}))$. Let's consider how the derivative terms scale first. Since the function is constant outside of our characteristic length, we need only to integrate in this range. Our free energy derivative term is

$$F_{deriv}(m) = \int d^d x \frac{1}{2}\gamma(\nabla m)^2 \sim L^{d-1}\int dx\gamma\left(\frac{dm}{dx}\right)^2$$

Let's take a look at what we've done here. We're approximating the 3d integral into an 'radial' component, and we've factored out a characteristic length scale of L^{d-1}, much like when we do a spherical intgral in d dimensions, we take out a factor of S_{d-1}, the $d-1$ dimensional hypersphere, and leave in the radial integral. If we take just the constant term of $\frac{dm}{dx}$, then this integral becomes

$$L^{d-1}\gamma\int dx\frac{m_0^2}{W^2} \sim L^{d-1}\gamma\frac{m_0^2}{W} \simeq L^{d-1}\sqrt{\frac{-\gamma\alpha_2^3}{\alpha_4^3}}$$

17

We can also check that this is the characteristic length scale of other terms in the free energy.

We can similarly do the polynomial functions in the free energy. We have, for the term with the α_2 coefficent, that

$$F(m(\mathbf{v})) = \int d^d x \frac{1}{2}\alpha_2(T)m^2 \sim L^{d-1} \int dx \frac{1}{2}\alpha_2 m^2$$

But again, we repeat the previous argument, and integrate just over the characteristic length scale W. Once again, we just take the constant terms, so that this is approximated by

$$L^{d-1}W\alpha_2 m_0^2 = L^{d-1}\alpha_2 W \sqrt{\frac{-\alpha_2}{\alpha_4}}$$
$$= L^{d-1}W \frac{-\alpha_2^2}{\alpha_4}\sqrt{\frac{-2\gamma}{\alpha_4}}$$
$$\sim \sqrt{\frac{-\gamma\alpha_2^3}{\alpha_4}}$$

Similarly, we have that our quartic term scales in this way. Hence, this free energy of a domain wall is well defined. The free energy of a domain wall F_{DW}, compared with the free energy of a ground state scales with a power law for characteristic length,

$$F_{DW} \sim L^{d-1}\sqrt{\frac{-\gamma\alpha_2^3}{\alpha_4}}$$

There's a special value of $d = 1$ since our domain wall has no length scale. This is what we call a critical dimension. Now, we can look at the probability of a domain wall occuring. As usual, the probability to have a certain field configuration is given by

$$e^{-\beta F(m)}$$

Hence, the probabilty of us having a domain wall at a given point x is

$$\mathcal{P}(\text{ domain wall at } x = X) = \frac{e^{-\beta F_{wall}}}{\mathcal{Z}}$$

Now, the first thing to notice is that when $d = 1$, our exponential factor is constant since we have no length dependence. Overall, this constant is really small. Now, let's imagine we're in a section of $\mathbb{R}$ between $\frac{L}{2} < x < \frac{L}{2}$. Then, we can calculate the probabability of us having a domain wall in this box. This means we'd like to calculate

$$\int_{L/2}^{L/2} dx \, \mathcal{P}(\text{ domain wall at } X = x)$$

Naively, this looks like it'll just be L times the probability of a domain wall at a given point. However, we need to remember that domain walls have a characteristic length scale W. This means we need to multiply by a factor of $\frac{1}{W}$ in the measure. So, we have that

$$\mathcal{P}(\text{there's a domain wall between } \frac{L}{2} < x < \frac{L}{2}) = \frac{L}{W}\frac{e^{-\beta F_{wall}}}{\mathcal{Z}}$$

18

What's the effect of having a domain wall in the range? Well, suppose that, in this one dimensional line, that we have a fixed magnetisation at the boundary, so suppose that $m(-\frac{L}{2}) = -m_0$. Then, the effect of a domain wall would be to flip this to $m(\frac{L}{2}) = m_0$ at the other end of the line.

Now we can ask, if we have the condition that say $m(L/2) = m_0$, then what's the total probability of the magnetisation staying the same, or switching, by the time we get to the other side? The simplest way to find out what's happening is to consider the probability that we have N domain walls appearing in the system. Let's write down the expression first, then explain why this is. Our expression

$$\mathcal{P}(\text{ n domain walls}) = \frac{e^{-n\beta F_{wall}}}{\mathcal{Z}W^n} \int_{L/2}^{L/2} dx_1 \int_{x_1}^{L/2} dx_2 \cdots \int_{x_{n-1}}^{L/2} dx_n$$

To see why this is, observe that, without loss of generality, we can place a domain wall anywhere in the range, as the first domain wall to be placed. This is represented in the first integral. After that, we can place a wall any point after that wall, assuming its placed at the point x_1. We repeat, to place a wall after that, and so on, until we've placed n walls. Our probability density given by the exponential $e^{-\beta F_{wall}}$ is constant, so we pull n of these factors out.

An elegant argument to evaluate this integral is to observe that this term is symmetric amongst permutation of the variables $x_1, x_2, \ldots x_n$. So, we're essentially multiplying the first integral n times, but to take into account the permutation symmetry, we need to divide by $n!$. Thus

$$\mathcal{P}(\text{n domain walls}) = \frac{1}{n!\mathcal{Z}} \left(\frac{Le^{-\beta F_{wall}}}{W} \right)^n$$

Now that we've done this, calculating the probability of whether magnetisation will flip by the time it reaches the other side is simply a matter of looking at whether we have an odd or even number of domain walls. If we have an even number, our magnetisation flips an even number of times and thus is not changed. Hence,

$$\mathcal{P}(m_0 \to m_0) = \sum_{n \text{ even}} \mathcal{P}(\text{n walls}) \frac{1}{\mathcal{Z}} \cosh \left(\frac{Le^{-\beta F_{wall}}}{W} \right)$$

Similarly, our probability of switching sign is

$$\mathcal{P}(m_0 \to -m_0) = \frac{1}{\mathcal{Z}} \sinh \left(\frac{Le^{-\beta F_{wall}}}{W} \right)$$

What's the point of all this? Well, since our dimension was at $d = 1$, our probability diverges and $L \to \infty$. Thus, the probabilities $\mathcal{P}(m_0 \to m_0)$ and $\mathcal{P}(m_0 \to -m_0)$ are indistiuishable. We have a disordered state in one dimension, and hence mean field theory here breaks down. However, when $d = 2$, we have that $F \sim L$, so as we take the limit as $L \to \infty$, the terms don't diverge and we still retain an memory of the boundary conditions initially at $x = -\frac{L}{2}$.

5 Evaluating Path Integrals and Gaussian Path Integrals

We want to compute the path integral

$$\mathcal{Z} = \int \mathcal{D}m(\mathbf{x}) e^{-\beta F(m(\mathbf{x}))}$$

This object forms flutuations around the saddle point. This object is dominated around the saddle point (a weaklu coupled theory). Hence, we can work perturbatively. However, if the object was strongly coupled, the fluctuations matter just as much. If we set our magentic field to zero, we know our free energy takes the form

$$F(\phi(\mathbf{x})) = \int d^d x \left[\frac{1}{2}\alpha_2(T)\phi^2 + \frac{1}{4}\alpha_4(T)\phi^4 + \frac{1}{2}\gamma(T)(\nabla\phi)^2 + \dots \right]$$

We can do a gross simplification here where we take $\alpha_4 \to 0$. To justify the following approximations, we need to keep track of what our **mean field** terms are. In the case where

- $\alpha_2 > 0$, to leading order, our mean field is

$$\langle \phi \rangle = 0$$

  Since we're assuming that ϕ is close to zero, then the quartic terms disappear. Hence, we're just left with the quadratic and derivative terms.

$$F(\phi(\mathbf{v})) = \int d^d x \frac{1}{2}\alpha_2\phi^2 + \frac{1}{2}\gamma(\nabla\phi)^2$$

- In the case where $\alpha_2(T) < 0$, we redefine $hat\phi(\mathbf{v}) = \phi(\mathbf{v}) - \langle \phi \rangle$

There's a very good reason why we need to make a redefinition for the case where $\alpha_2 < 0$. This is because, when we solve for the equations of motion, just as in the mean field case, we recover two constant solutions

$$\langle \phi \rangle = \pm m_0 = \pm\sqrt{\frac{-\alpha_2}{\alpha_4}}$$

We get that $\langle \phi \rangle$ is not close to zero, but these two equilibrium terms instead. In addition, it's obvious that if we just set the quartic terms to zero with $\alpha_2 < 0$, then the whole qualitative shape of our potential would be different: we would have an upside down smiley face. However, we can make a different field which is close to zero. We define a translated field

$$\widetilde{\phi}(\mathbf{x}) = \phi(\mathbf{x}) - \langle \phi \rangle$$

These are small fluctuations around the mean field average. If we substitute these expressions into the free energy, we have

$$F(\widetilde{\phi}) = \int d^d x \frac{1}{2}\alpha_2(T)(\widetilde{\phi} + \langle \phi \rangle)^2 + \frac{1}{4}(\widetilde{\phi} + \phi)^4 + \frac{1}{2}\gamma(\nabla\widetilde{\phi})^2$$

Now, up to quadratic order, the constant terms from $\langle \phi \rangle$ come out to form the integral $F(\langle \phi \rangle)$. The non constant terms give a linear and quadratic integral as follows

$$F = F(\langle \phi \rangle) + \int d^d x \alpha_2(T) \langle \phi \rangle \widetilde{\phi} + \alpha_4 \widetilde{\phi} \langle \phi \rangle^3 + \int d^d x \frac{1}{2}\alpha_2(T)\widetilde{\phi}^2 + \frac{3}{2}\langle \phi \rangle^2 \alpha_4 + \frac{\gamma}{2}(\nabla\widetilde{\phi})^2$$

In the integral with terms linear in $\widetilde{\phi}$, the coefficient is

$$\alpha_2 \langle \phi \rangle + \alpha_4 \langle \phi \rangle^3 = 0, \quad \langle \phi \rangle = \sqrt{\frac{-\alpha_2}{\alpha_4}}$$

Thus, the linear term vanishes, and we're left with a quadratic and derivative term just like in the case with $\alpha_2 < 0$. We can write the quadratic ϕ coefficient as

$$\int d^d x \frac{1}{2} \alpha_2'(T) \widetilde{\phi}^2$$

where we have that

$$\alpha_2' = \alpha_2 + \frac{3}{2} \langle \phi \rangle^2 \alpha_4 = \alpha_2 - 3 \frac{\alpha_2}{\alpha_4} \alpha_4 = -2\alpha_2$$

Hence, in terms of our new translated field, we can write the free energy as

$$F(\widetilde{\phi}) = F(\langle \phi \rangle) + \int d^d x \frac{1}{2} \alpha_2' \widetilde{\phi} + \frac{1}{2} \gamma (\nabla \phi)^2 \dots$$

We move into Fourier space with

$$\phi_{\mathbf{k}} = \int d^d x e^{-i\mathbf{k} \cdot \mathbf{x}} \phi(\mathbf{x})$$

$$\phi_{\mathbf{k}}^* = \phi_{-\mathbf{k}}$$

However, recall that we've coarse grained, which means we have to put out some Fourier modes, by an ultraviolet cutoff. Inserting this expression into our free energy, we have that

$$F[\phi_{\mathbf{k}}] = \frac{1}{2} \int d^d \mathbf{k}_1 \int d^d \mathbf{k}_2$$

Given our form of free energy $F(m)$, our usual way of computing our partition function would be

$$Z = \sum_{\text{configurations}} e^{-\beta F(m)}$$

However, with coarse graining introduced, we now integrate over all possible configurations of $m = m(\mathbf{x})$. To do this, we introduce a measure called a path integral, and thus our partition functions is given by

$$Z = \int \mathcal{D}m \, e^{-\beta F(m)}$$

Where the measure $\int \mathcal{D}m$ is telling us to integrate over all possible field configurations.

Question is, how do we even begin to evaluate this kind of object? The key is to rewrite this measure as a product of integrals of Fourier modes, that is, we diagonalise m in Fourier space and rewrite

$$\int \mathcal{D}m = \prod_{\mathbf{k}} \mathcal{K} \int dm_{\mathbf{k}} dm_{\mathbf{k}}^*$$

21

In what follows, we'll be using both continuous and discrete decomppositions of m into Fourier modes. We decompose $m(\mathbf{x})$ into contnuous Fourier modes as

$$m(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^d} e^{-i\mathbf{x}\cdot\mathbf{k}} m_\mathbf{k}$$

and decompose m into discrete Fourier modes in some volume V as

$$m(\mathbf{x}) = \frac{1}{V} \sum_\mathbf{k} e^{-i\mathbf{x}\cdot\mathbf{k}} m_\mathbf{k}$$

At this point, as a result of our coarse graining proceduce, since we're counting averages over a lattice, we need to ensure that our Fourier modes aren't too high. So, we impose that

$$\phi_\mathbf{k} = 0, \quad \frac{\pi}{a} < \|k\|$$

Where a is the characteristic length of our coarse graining procedure.

If we rewrite our free energy $F(m)$ in terms of our Fourier modes, we have

$$F(m) = \int \frac{dm_{\mathbf{k_1}} dm_{\mathbf{k_2}} d^d x}{(2\pi)^d} \left(\mu^2 m_{\mathbf{k_1}} m_{\mathbf{k_2}} e^{-i\mathbf{k_1}\cdot\mathbf{x}} e^{-i\mathbf{k_2}\cdot\mathbf{x}} - \mathbf{k_1}\cdot\mathbf{k_2} e^{-i(\mathbf{k_1}+\mathbf{k_2}\cdot\mathbf{x})} m_{\mathbf{k_1}} m_{\mathbf{k_2}} \right)$$

At this point, it's useful to remember the identity

$$(2\pi)^d \delta(\mathbf{k}) = \int d^d x e^{-i\mathbf{x}\cdot\mathbf{k}}$$

so our final expression for our free energy is

$$F(m) = \int \frac{d\mathbf{k}}{(2\pi)^d} \left(\mu^2 |\mathbf{k}|^2 + \alpha \right) m_\mathbf{k} m_{-\mathbf{k}}$$

Substituting this transformed expression for our free energy, and using our measure which is the product of Fourier modes, our partition function

$$Z = \left(\prod_\mathbf{k} \mathcal{K} \int dm_\| dm_{-\|} \right) \exp \left(-\beta \int \frac{d\mathbf{k}}{(2\pi)^d} \left(\mu^2 |\mathbf{k}|^2 + \alpha \right) m_\mathbf{k} m_{-\mathbf{k}} \right)$$

Now the tricky part is to somehow get rid of that integral over $d\mathbf{k}$ in the exponent. To do this, we discretize the integral over the possible modes $\mathbf{k}$ in a finite volume and then take this volume to infinity.

$$\int d\frac{\mathbf{k}}{(2\pi)^d} f(\mathbf{k}) = \lim_{V\to\infty} \frac{1}{V} \sum_\mathbf{k} f(\mathbf{k})$$

where in this case in the finite volume, our modes $\mathbf{k}$ are appropriately quantized. This makes our integral above easier to hande because now we can write it as

$$\left(\prod_\mathbf{k} \mathcal{K} \int dm_\mathbf{k} dm_{-\mathbf{k}} \right) \exp \left(-\frac{\beta}{V} \sum_\mathbf{k} (\mu^2 |\mathbf{k}|^2 + \alpha) m_\mathbf{k} m_{-\mathbf{k}} \right)$$

but we can factor out our above expression over the product so that

$$Z = \prod_{\mathbf{k}} \mathcal{K} \int dm_{\mathbf{k}} dm_{-\mathbf{k}} \exp\left(-\frac{1}{TV}(\mu^2 |\mathbf{k}|^2 + \alpha) m_{\mathbf{k}} m_{-\mathbf{k}}\right)$$

(There's a factor of 2 missing here which I need to correct). How do we go about evaluating this object? Well, recall that in a fourier transform, our negative fourier mode is indeed just the complex conjugate of the corresponding positive mode:

$$dm_{-\mathbf{k}} = dm_{\mathbf{k}}^*$$

So to do the above, we need to think carefully about how we integrate expressions that look like

$$\int dx dx^* \exp\left(-\frac{|x|^2}{2\alpha}\right)$$

At first glance, one might think to treat the x, x^* variables here separately and integrate twice over, but since x clearly depends on x^*, we have that the expression above is just the same as integrating

$$\int dx \exp\left(-\frac{|x|^2}{2\alpha}\right) = \sqrt{2\pi\alpha}$$

This is our usual expression for a Gaussian-like integral. Our expression for our partition function is therefore

$$Z = \prod_{\mathbf{k}} \sqrt{\frac{2\pi V T \mathcal{K}^2}{(\mu^2 |\mathbf{k}|^2 + \alpha)}}$$

where, in calculating this, we've pulled the constant $\mathcal{K}$ inside the square root to make our calculations with it easier later on. Now that we have our partition function, we're in good shape to start calculating quantities of interest. The most obvious one we can compute is our thermodynamic free energy, $F_{thermo} = -T \log Z$. What we'll do now is naively plug our expression for Z in, but then make an approximation to get it into a nice integral form. We'll work instead with the thermodynamic free energy per unit volume.

This is given by

$$\frac{F_{thermo}}{V} = -\frac{T}{V} \log Z = -\frac{T}{2V} \sum_{\mathbf{k}} \log\left(\frac{2\pi V T \mathcal{K}^2}{(\mu^2 |\mathbf{k}|^2 + \alpha)}\right)$$

But, we can do even better than this. Previously, we swapped out our integral for a sum, but now we can do the opposite thing and swap out our sum $\frac{1}{V} \sum_{\mathbf{k}}$ with $\int \frac{d^d \mathbf{k}}{(2\pi)^d}$. This gives the condensed expression that

$$\frac{F_t hermo}{V} = -\frac{T}{2} \int \frac{d^d k}{(2\pi)^d} \log\left(\frac{2\pi V T \mathcal{K}^2}{(\mu^2 |\mathbf{k}|^2 + \alpha)}\right)$$

In the $\mathbb{C}$ case, we have

$$\mathcal{Z}_{\mathbb{C}} = \prod_{\mathbf{k}}^{\infty} \left\{ \int d\phi_{\mathbf{k}}^r d\phi_{\mathbf{k}}^I e^{-\frac{\beta}{2V}(\gamma k^2 + \mu^2)(\phi_{\mathbf{k}}^{r2} + \phi_{\mathbf{k}}^{I2}}\right\}$$

23

Our Gaussian integral identity implies that

$$\mathcal{Z}_{\mathbb{C}} = \prod \mathcal{N} \frac{2\pi TV}{\gamma k^2 + \mu^2}$$

So, in the real case we have that for the real field, related in the real case. Our free energy was $\mathcal{Z} = e^{-\beta F_{thermo}}$. This means that in the discrete case we have that

$$\begin{aligned}
\frac{F_{thermo}}{V} &= -\frac{T}{V} \log(\mathcal{Z}) \\
&= -\frac{T}{2V} \sum_{\mathbf{k}} \log\left(\frac{2\pi TVN^2}{\gamma k^2 + \mu^2}\right) \\
&\sim -\frac{T}{2} \int d^d k \frac{1}{(2\pi)^d} \log\left(\frac{2\pi TVN^2}{\gamma k^2 + \mu^2}\right)
\end{aligned}$$

For Heat capacity, we define

$$\begin{aligned}
c &= \frac{C}{V} \\
&= \frac{\beta^2}{V} \frac{\partial^2}{\partial \beta^2} \log(\mathcal{Z}) \\
&= \frac{1}{2}\left(T^2 \frac{\partial^2}{\partial T^2} + 2T \frac{\partial}{\partial T}\right) \int \frac{d^d k}{(2\pi)^2} \log\left(\frac{2\pi TVN^2}{\gamma k^2 + \mu^2}\right)
\end{aligned}$$

When we do this integral we get that

$$c = \frac{1}{2} \int_0^\lambda \frac{d^d k}{(2\pi)^2} \left[1 - \frac{2T}{\gamma k^2 + \mu^2} + \frac{T^2}{(\gamma k^2 + \mu^2)^2}\right]$$

The last two terms depend on dimension d. Here, we've added a cutoff from our lattice spacing approximation. Our middle term is

$$\int_0^\lambda dk \frac{k^{d-1}}{\gamma k^2 + \mu^2} = \begin{cases} \lambda^{d-1} & d > 2 \\ \log(\lambda) & d = 2 \\ \frac{1}{\mu} & d = 1 \end{cases}$$

Our last term gives

$$\int_0^\lambda dk \frac{k^{d-1}}{(\gamma k^2 + \mu^2)^2} = \begin{cases} \lambda^{d-1} & d > 4 \\ \log(\lambda) & d = 4 \\ \mu^{d-4} & d < 4 \end{cases}$$

For the case that $d < 4$

$$c \sim |T - T_c|^{-\alpha}, \quad \alpha = 2 - \frac{d}{2}$$

5.1 Correlation functions

In mean field theory, we had no local specific information.

$$\langle \phi(\mathbf{x}) \rangle = \begin{cases} 0 & T > T_c \\ \pm m_0 & T < T_c \end{cases}$$

Our next simplest way to think about correlations is

$$\langle \phi(\mathbf{x})\phi(\mathbf{y}) \rangle$$

this tells us how fluctuations propagate throughout time by encoding spatial variations at multiple points. A naive way to think about this is how waves propagate in the material. Moreover, we can think about functions deviating from the mean as the **connected correlation function**. This is defined as

$$\langle \phi(\mathbf{x})\phi(\mathbf{y}) \rangle_c = \langle \phi(\mathbf{x})\phi(\mathbf{y}) \rangle - \langle \phi \rangle^2$$

This is also known as the cumulant in statistics. We can compute these correlation functions explicitly.

If we define

$$F[\phi(\mathbf{x})] = \int d^d x \left[\frac{\gamma}{2}(\nabla\phi)^2 + \frac{\mu^2}{2}\phi^2 + B(x)\phi \right]$$

with the partition function

$$Z[B(x)] = \int \mathcal{D}\phi \, e^{-\beta F[\phi(\mathbf{x}), B(\mathbf{x})]}$$

We can take a functional derivative to get

$$\frac{1}{\beta} \frac{\delta \log \mathcal{Z}}{\delta B(\mathbf{x})} = \frac{1}{\beta \mathcal{Z}} \frac{\delta \mathcal{Z}}{\delta B(\mathbf{x})}$$

However, this term, the benefit of doing this is that differentiating by $B(\mathbf{v})$ pulls out a factor of ϕ in the integral. Hence, the quantity we've calculated is

$$-\frac{1}{\mathcal{Z}} \int \mathcal{D}\phi(\mathbf{x}) e^{-\beta F[\phi, B]} = -\langle \phi(\mathbf{x}) \rangle |_B$$

So, we've found a neat way to encode the average function in our partition function. This is exactly the same as encoding our average magnetisation in our partition function using mean field theory earlier. We can get even more information from this. We calculate an object called the **two-point** function. In analogy to moment generating functions in statistics, differentiating this a second time enables us to get the connected correlation function we listed earlier. If we act on what we have with $\frac{\partial}{\beta \partial}$, the product rule gives us

$$\frac{1}{\beta^2} \frac{\delta^2 \log \mathcal{Z}}{\delta B(\mathbf{x})\delta B(\mathbf{y})} = \frac{1}{\beta^2 \mathcal{Z}} \frac{\delta^2 \mathcal{Z}}{\delta B(\mathbf{x})\delta B(\mathbf{y})} - \frac{1}{\beta^2 \mathcal{Z}^2} \frac{\delta \mathcal{Z}}{\delta B(\mathbf{x})} \frac{\delta \mathcal{Z}}{\delta B(\mathbf{y})}$$

This was obtained by applying the product rule on $\frac{1}{\mathcal{Z}}$ as well as the derivative terms. The upshot is that the first term pulls down two factors, $\phi(\mathbf{x})$ and $\phi(\mathbf{y})$ from our free energy. Hence, our first term is

$$\frac{1}{Z} \int \mathcal{D}\phi \, \phi(\mathbf{x})\phi(\mathbf{y}) e^{-\beta F} = \langle \phi(\mathbf{x})\phi(\mathbf{y}) \rangle |_B$$

Further more, observe that the second term is just $\langle \phi \rangle^2$. Thus, our final correlation function can be written in the compact form, writing our subscript B to denote evaluation at the field configuration B,

$$\langle \phi(\mathbf{x})\phi(\mathbf{y}) \rangle_c = \frac{1}{\beta^2} \frac{\delta^2 \log \mathcal{Z}}{\delta B(\mathbf{x})\delta B(\mathbf{y})} = \langle \phi(\mathbf{x})\phi(\mathbf{y}) \rangle_B - \langle \phi(\mathbf{x}) \rangle_B \langle \phi(\mathbf{y}) \rangle_B$$

Now, above the critical temperature $T > T_c$, we have that our average magnetisation is given by $\langle \phi(\mathbf{x}) \rangle = 0$ when $B = 0$. Thus, we can condense what we have even further by

$$\frac{1}{\beta^2} \frac{\delta^2 \log \mathcal{Z}}{\delta B(\mathbf{x})\delta B(\mathbf{y})} \mid_{B=0} \langle \phi(\mathbf{x})\phi(\mathbf{y}) \rangle_{B=0}$$

5.2 Computing Gaussian Path Integrals

5.2.1 Our correlation function is just a Green's function

The point of this whole section is to drive the point home that diagonalising in Fourier space makes things easier!

Our total free energy, including the magnetic field, was

$$\int d^d x \, \frac{\gamma}{2}(\nabla \phi)^2 + \frac{1}{2}\mu^2 \phi^2 + B(\mathbf{x})\phi$$

Now, we already know that the Fourier expansion of the first two terms gives us an intergal in Fourier space which is equal to

$$\int \frac{d^d k}{(2\pi)^d} \left[\frac{1}{2}\left(\gamma k^2 + \mu^2\right)\phi_{\mathbf{k}}\phi_{-\mathbf{k}} \right]$$

Now, to incorporate our magentic field, we just expand out ϕ in Fourier modes and sub in our definition for what a Fourier mode is for B. So,

$$F_{mag} = \int d^d x \, \frac{d^d k}{(2\pi)^d} e^{-i\mathbf{k}\cdot\mathbf{x}}\phi_{\mathbf{k}}B(x)$$

Defining

$$B_{\mathbf{k}} = \int d^d x \, e^{i\mathbf{k}\cdot\mathbf{x}}B(\mathbf{x}) \implies F_{mag} = \int \frac{d^d k}{(2\pi)^d} B_{-\mathbf{k}}\phi_{\mathbf{k}}$$

Hence, our whole expression for our free energy in terms of Fourier modes is

$$F = \int \frac{d^d k}{(2\pi)^d} \left[\frac{1}{2}(\gamma k^2 + \mu^2)\phi_{\mathbf{k}}\phi_{-\mathbf{k}} + B_{-\mathbf{k}}\phi_{\mathbf{k}} \right]$$

To evaluate this, we can define a translated integral as

$$\tilde{\phi}_{\mathbf{k}} = \phi_{\mathbf{k}} + \frac{B_{\mathbf{k}}}{\gamma k^2 + \mu^2}$$

With this transformation, we can write

$$F[\tilde{\phi}_{\mathbf{k}}] = \int \frac{d^d k}{(2\pi)^d} \left[\frac{1}{2}(\gamma k^2 + \mu^2)|\tilde{\phi}_{\mathbf{k}}|^2 - \frac{1}{2}\frac{|B_{\mathbf{k}}|^2}{\gamma k^2 + \mu^2} \right]$$

Since we have that our partition function is given in terms of Fourier modes

$$\mathcal{Z} = \prod_{\mathbf{k}} d\phi_{\mathbf{k}} d\phi_{-\mathbf{k}} e^{-\beta F_{thermo}}$$

Our Jacobian doesn't change because we're just changing by a constant.

$$\mathcal{Z}[B_k] = e^{-\beta F_{thermo}} e^{\frac{\beta}{2} \int \frac{d^d k}{(2\pi)^d} \frac{|B_{\mathbf{k}}|^2}{\gamma k^2 + \mu^2}}$$

However, we can do an inverse Fourier transform

$$\mathcal{Z}[B(\mathbf{x})] = e^{-\beta F_{thermo}} e^{\frac{\beta}{2} \int d^d x d^d y B(\mathbf{x}) G(\mathbf{x}-\mathbf{y}) B(\mathbf{y})}$$

We define our Green's function as

$$G(\mathbf{x} - \mathbf{y}) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{\gamma k^2 + \mu^2}$$

Differentiating twice, we get that

$$\langle \phi(\mathbf{x})\phi(\mathbf{y}) \rangle_{B=0} = \frac{1}{\beta} G(\mathbf{x} - \mathbf{y})$$

This is because

$$\frac{1}{\beta} \frac{\delta \log \mathcal{Z}}{\delta B(\mathbf{x})} = \frac{1}{B\mathcal{Z}} e^{-\beta F_{thermo}} \frac{\beta}{2} \left(\frac{\delta}{\delta B(\mathbf{x})} \int d^d x d^d y\, B(\mathbf{y}) G(\mathbf{x} - \mathbf{y}) B(\mathbf{x}) \right) \exp(K(B(\mathbf{x})))$$

Where we have defined for brevity that

$$K(B(\mathbf{x})) = \frac{\beta}{2} \int d^d x d^d y\, B(\mathbf{x}) G(\mathbf{x} - \mathbf{y}) B(\mathbf{y})$$

5.2.2   An aside on functional integrals

Our functional derivative with respect to $B(\mathbf{x})$ kind of looks like if we were differentiating $B^2 G$. Let's recall how to do a functional integral, from the persepctive of deriving the Euler-Lagrange equations from a quadratic free integral. Suppose we had an expression

$$F(B(\mathbf{x})) = \int d^d z f(B(\mathbf{z}))$$

Then, our functional integral in the function $g(B(\mathbf{x}))$ that we get when we vary F with respect to $B \to B + \delta B$, such that

$$\delta F = \int d^d z \delta B(\mathbf{z})(g(B(\mathbf{z})))$$

We did this before in deriving our equations of motion associated with our quadratic terms of our free energy. Varying with respect to ϕ,

$$
\begin{aligned}
\frac{\delta F}{\delta \phi} &= \int d^d x \, \gamma (\nabla \phi) \delta(\nabla \phi) + \mu^2 \phi \delta \phi \\
&= \int d^d x \, \gamma (\nabla \phi) \nabla (\delta \phi) + \mu^2 \phi \delta \phi \\
&= \int d^d x - \gamma \delta \phi \nabla^2 \phi + \mu^2 \phi \delta \theta \\
&= \int d^d x \, \delta \phi (-\gamma \nabla^2 \phi + \mu^2 \phi)
\end{aligned}
$$

Thus, in this case we have that our functional derivative is given by

$$
\frac{\delta F}{\delta \phi} = -\gamma \nabla^2 \phi + \mu^2 \phi
$$

As we shown in the previous section, solving this equation for $\frac{\delta F}{\delta \phi} = 0$, gives us the most likely field configuration. Thus, applying this to our previous problem an setting

$$
F = \int d^d x \int d^d y \, B(\mathbf{x}) G(\mathbf{x} - \mathbf{y}) B(\mathbf{y}) \implies \delta F = 2 \int d^d x \, \delta B(\mathbf{x}) \int d^d y \, G(\mathbf{x} - \mathbf{y}) B(\mathbf{y})
$$

So, taking our functional derivative has left us with the term, relablleling variables, that

$$
\frac{\delta F}{\delta B(\mathbf{x})} = 2 \int d^d z \, G(\mathbf{x} - \mathbf{z}) B(\mathbf{z})
$$

Hence, our expression for our first derivative of the log of the partition function simplifies to

$$
\frac{1}{\beta} \frac{\delta \log \mathcal{Z}}{\delta B(\mathbf{x})} = \int d^d z \, G(\mathbf{z} - \mathbf{x}) B(\mathbf{z})
$$

If we differentiate this object using a functional derivative with respect to B again, we get that

$$
\frac{1}{\beta^2} \frac{\delta^2 \log \mathcal{Z}}{\delta B(\mathbf{x}) \delta B(\mathbf{y})} = \frac{1}{\beta} G(\mathbf{x} - \mathbf{y}) = \langle \phi(\mathbf{x}) \phi(\mathbf{y}) \rangle
$$

So, our correlation is a Green's function with a factor of $\frac{1}{\beta}$. This is reminiscnent of quantum field theory, where our Feynman propagator was also essentially a Green's function. So, to calculate this correlation function, all we have to do is calculate this Green's function explicity.

5.2.3 Evaluating our Green's function

In the section that follows, we'll use symmetry arguments as well as a saddle point approximation to calculate the Green's function

$$
G(\mathbf{x}) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\gamma k^2 + \mu^2}
$$

By rotational symmetry we can evaluate this Green's function as a function of r, the magnitude of our vector

$$G(r) = \frac{1}{\gamma} \int \frac{d^d k}{(2\pi)^d} e^{-i\mathbf{k}\cdot x} \frac{1}{k^2 + \frac{1}{\zeta^2}}$$

We've defined our correlation length $\zeta^2 = \frac{\gamma}{\mu^2}$, which is a convenient scaling parameter, which we'll analyse the properties of later. We now use a fairly obvious identity to help us evaluate this intergal; we use the simple fact that

$$\frac{1}{k^2 + \frac{1}{\zeta^2}} = \int_0^\infty dt e^{-t(k^2 + \frac{1}{\zeta^2})}$$

The upshot of writing things this way allows us to nest the integrals within one another. Hence, we have that

$$G(r) = \frac{1}{\gamma} \int \frac{d^d k}{(2\pi)^d} \int_0^\infty e^{-t(k^2 + \frac{1}{\zeta^2}) - i\mathbf{k}\cdot\mathbf{x}}$$

$$= \frac{1}{\gamma} \int \frac{d^d k}{(2\pi)^d} \int_0^\infty dt \, dt e^{-t(k + \frac{ix}{2t})^2} e^{-\frac{r^2}{4t} - \frac{t}{\zeta^2}}$$

$$= \frac{1}{\gamma} \frac{1}{(4\pi)^{d/2}} \int_0^\infty dt \, t^{-\frac{d}{2}} e^{-\frac{r^2}{4t} - \frac{t}{\zeta^2}}$$

Here we've used the fact that

$$\frac{d^d k}{(2\pi)^d} \to \frac{d\Omega_d}{(2\pi)^d} k^{d-1} dk, \quad \int \frac{d\Omega_d}{(2\pi)^d} = 2\pi^{\frac{d}{2}} \Gamma^{-1}(\frac{d}{2})$$

We can now employ the saddle point approximation

$$G(r) \simeq \int_0^\infty dt e^{-S(t)}, \quad S(t) = \frac{r^2}{4t} + \frac{t}{\zeta^2} + \frac{d}{2} \log(t)$$

Now, expanding the exponential about the minimum point,

$$S(t) = S(t^*) + \frac{S''(t^*)t^2}{2} + \dots, \quad t^* = \frac{\zeta^2}{2}(-\frac{d}{2} + \sqrt{\frac{d^2}{4} + \frac{r^2}{\zeta}}$$

So, we have that

$$G(r) \simeq \sqrt{\frac{\pi}{2S''(t^*)}} e^{-S(t^*)}$$

We now examine the limits in two regimes.

$$r \gg \zeta, t^* \simeq r\frac{\zeta}{2} \implies G(r) \sim \frac{1}{\zeta^{\frac{d-3}{2}} r^{\frac{d-1}{2}}} \frac{e^{-\frac{r}{2}}}{}$$

On the other hand, in the other regime

$$r \ll \zeta, t^* \simeq \frac{r^2}{4d} \implies G(r) \sim \frac{1}{r^{d-2}}$$

Recall, $G(r) = \langle \phi(\mathbf{y})\phi(\mathbf{x}) \rangle$

29

6   Green's functions

Consider a multi-dimensional Gaussian integral:

$$I = \int_{-\infty}^{\infty} d^n y \, e^{-\frac{1}{2} y^T \cdot G^{-1} \cdot y}$$

Notice that we can diagonalise this object in the eigenbasis to give that

$$I = det^{\frac{1}{2}} (2\pi G)$$

Similarly, we can add a 'source' term to the integral and shift the variables to get

$$I_B = \int_{-\infty}^{\infty} d^n y \, e^{-\frac{1}{2} y \cdot G^{-1} \cdot y + \mathbf{B} \cdot y} = det^{\frac{1}{2}} (2\pi G) e^{\frac{1}{2} \mathbf{B}^T \cdot G \cdot \mathbf{B}}$$

From this result our expression for free energy

$$F[\phi(\mathbf{x})] = \int d^d x \left[\frac{1}{2} \gamma (\nabla \phi)^2 + \frac{1}{2} \mu^2 \phi^2 + \mathbf{B} \phi \right]$$

can be rewritten by parts with a dummy integral so that

$$F[\phi(x)] = \int d^d x \, d^d y \, \frac{1}{2} \phi(\mathbf{x}) G^{-1}(\mathbf{x}, \mathbf{y}) \phi(\mathbf{y}) + \int d^d x \, \mathbf{B}(\mathbf{x}) \phi(\mathbf{x})$$

where we in this expression

$$G^{-1}(\mathbf{x}, \mathbf{y}) = \delta^d(\mathbf{x} - \mathbf{y})(-\gamma \nabla_y^2 + \mu^2)$$

We also have

$$(-\gamma \nabla_x^2 + \mu^2) G(\mathbf{x}, \mathbf{y}) = \delta^d(\mathbf{x} - \mathbf{y}), \quad G(\mathbf{x}, \mathbf{y}) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-i \mathbf{k} \cdot (\mathbf{x} - \mathbf{y})}}{\gamma k^2 + \mu^2}$$

Be careful with inverting things! We need to integrate over them. Hence our correlation function satisfies a differential equation

$$(-\gamma \nabla_x^2 + \mu^2) \langle \phi(\mathbf{x}) \phi(0) \rangle = \frac{1}{\beta} \delta^d(\mathbf{x})$$

Away from the origin, this differential equation tells us that away from the origin our fluctuations satisfy the Euler Lagrange equations. An aside on using Green's functions. If we have

$$\mathcal{O}(x) F(x) = H(x)$$

If we can solve $\mathcal{O}(x) G(x, y) = \delta(x - y)$, then setting

$$F(x) = \int dy H(y) G(x - y) \implies \mathcal{O}(x) F(x) = H(x)$$

6.1 Connection to Susceptibility

Let's apply these ideas to what we know about magnetic susceptiblity. Our equation that we've always used is

$$\chi = \frac{\partial m}{\partial B}$$

Now, we can promote this to a localised version by setting

$$\chi(\mathbf{x}, \mathbf{y}) = \frac{\partial \langle \phi(\mathbf{x}) \rangle}{\partial B(\mathbf{y})} = \beta(\langle \phi(\mathbf{x})\phi(\mathbf{y}) \rangle \sim \int \mathcal{D}\phi \frac{\partial \phi}{\partial B} e^{-\beta F[\phi]}$$

From this local definition, we can define a global magnetic susceptibilty as

$$\chi = \int d^d x \chi(\mathbf{x}, 0) = \beta \int d^d x \, \langle \phi(\mathbf{x})\phi(\mathbf{y}) \rangle$$

Recall our correlation function

$$\langle \phi(\mathbf{x})\phi(\mathbf{y}) \rangle = \begin{cases} \frac{1}{r^{d-2}} & r \ll \zeta \\ e^{-\frac{r}{2}}/r^{\frac{d-1}{2}} & r \gg \zeta \end{cases}$$

This implies that our critical exponents satisfy $\mu^2 \sim |T - T_c|$, and $\zeta \sim \frac{1}{|T-T_c|^{\frac{1}{2}}}$. Thus at low temperatures, our correlation length diverges, so our system because highly correlated.

6.2 Critical exponents

We say from our earlier argument that $\zeta \sim \frac{1}{|T-T_c|^\nu}, \nu = \frac{1}{2}$ And, our correlation goes like

$$\langle \phi(\mathbf{x})\phi(\mathbf{y}) \rangle \sim \frac{1}{r^{d-2+\eta}} \quad \eta = 0$$

So far, this agrees with our predictions for mean-field theory, but we still have some errors.

6.2.1 Upper critical dimension

In mean field theory, $\langle \phi(\mathbf{x}) \rangle = \pm m_0$. But going into our path integral formulation, we ask how big are the fluctuations relative to m_0. If our fluctuations are very big, we shouldn't expect our calculations to match our predictions from mean field theory. We look at our ratio

$$R = \frac{\int_0^\zeta d^d x \, \langle \phi(\mathbf{x})\phi(\mathbf{y}) \rangle}{\int_0^\zeta d^d x \, m_0^2} \sim \frac{1}{m_0^2 \zeta^d} \int_0^\zeta dr \frac{r^{d-1}}{r^{d-2}} \sim \frac{\zeta^{2-d}}{m_0^2}$$

Now substituting in our critical exponents,

$$m_0 \sim |T - T_c|^{\frac{1}{2}}, \zeta \sim |T - T_c|^{-\frac{1}{2}} \implies R = |T - T_c|^{\frac{d-4}{2}}$$

6.2.2 Translating to Quantum Field Theory

We have many links between statistical field theory and quantum field theory.

$$\text{SFT: } \mathcal{Z} = \int \mathcal{D}\phi\, e^{-\beta \int d^d x\, F[\phi]}$$

Here we have d spatial dimensions. In quantum field theory, we have that

$$\mathcal{Z} = \int \mathcal{D}e^{\frac{i}{\hbar} \int d^d x \mathcal{L}(\phi)}$$

Here we have d spactime dimensions, with an $\frac{i}{\hbar}$ factor. We do a wick rotation $\tau = it$, so that our differential operator

$$\partial_t^2 + \nabla^2 \to \partial_\tau^2 - \nabla^2$$

Similarly, our path integrals transform as

$$\mathcal{Z} = \int \mathcal{D}\phi\, e^{\frac{i}{\hbar} S[\phi]} \to \mathcal{Z} = \int \mathcal{D}\phi\, e^{-\frac{1}{\hbar} S_E[\phi]}$$

So our quantum fluctuations are identified with thermal fluctuations. $QFT_{d-1} \sim SFT_d$.

7 The Renormalisation Group

The idea of the renormalisation group is an idea that was pioneered by Kadanoff, Fisher and Wilson which formalises how physics changes depending on our scale. It's an important idea which helps explain universality

7.1 Basic Introduction

Consider the free energy

$$F[\phi] = \int d^d x \left[\frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} \mu^2 \phi^2 + g\phi^4 + \dots \right]$$

Recall that when we're writing this theory, we only used the few basic assumptions such as $\mathbb{Z}_2$ symmetry, analyticity, and so on. So in essence, we could have included terms like $g_{126} \phi^{126}$, $(\nabla^2 \phi)^2$ and so on. These terms have coupling constants in front of them, such as $\mu^2 = T - T_X$, which we call Wilson coefficients. We can also rescale our coupling coefficients so that we absorb β in our free energy, so we can write

$$\mathcal{Z} = \int \mathcal{D}\phi \, e^{-F[\phi]}$$

The final piece of the theory is that we also specificed some cutoff Λ such that

$$\phi_{\mathbf{k}} = 0, \quad |\mathbf{k}| \geq \Lambda, \Lambda \sim \frac{\pi}{a}$$

The crux of the renormalisation group is what happens when we ask what happens when we change Λ? Since what we wrote down the most general theory to begin with, when we change Λ we expect the same form of theory but with different coupling constants. When we change the cutoff we call this a 'flow'.

7.2 Flowing to the 'IR' (long distances)

Imagine we only care about predictions at on scales $L \gg a$. Then, we can write a theory with a new cutoff given by

$$\Lambda' = \frac{\Lambda}{\epsilon}, \quad \epsilon > 1$$

This means that all fourier modes $\frac{1}{L} \leq |\mathbf{k}| \leq \Lambda$ We can expand Fourier modes as

$$\phi_{\mathbf{k}} = \phi_{\mathbf{k}}^- + \phi_{\mathbf{k}}^+$$

The $\phi_{\mathbf{k}}^-$ are low frequency modes with

$$\phi_{\mathbf{k}}^- = \begin{cases} \phi_{\mathbf{k}} & |\mathbf{k}| \leq \Lambda' \\ 0 & \text{otherwise} \end{cases}$$

And our $\phi_{\mathbf{k}}^+$ are our high frequency terms with

$$\phi_{\mathbf{k}}^+ = \begin{cases} \phi_{\mathbf{k}} & \Lambda' < \|\mathbf{k}\| < \Lambda \\ 0 & \|k\| \geq \Lambda \end{cases}$$

We can write out our free energy in terms of this fourier mode split, so that

$$F[\phi_{\mathbf{k}}] = F_0[\phi_{\mathbf{k}}^-] + F_0[\phi_{\mathbf{k}}^+] + F_I[\phi_{\mathbf{k}}^+, \phi_{\mathbf{k}}^-]$$

The final term in this expression can be interpreted as an interaction between long and short length scales in the Fourier modes. We can then write out our partition function as

$$\mathcal{Z} = \int \prod_{k < \Lambda} d\phi_{\mathbf{k}} e^{-F}$$

However, the upshot of this is that now we can decompose F into the high and low frequency modes which we were derived earlier. We have that

$$\mathcal{Z} = \int \prod_{k < \Lambda'} d\phi_{\mathbf{k}}^- e^{-F_0[\phi_{\mathbf{k}}^-]} \int \prod_{\Lambda' < k < \Lambda} d\phi_{\mathbf{k}}^+ e^{-F_0[\phi_{\mathbf{k}}^+]} e^{-F_I[\phi_{\mathbf{k}}^+, \phi_{\mathbf{k}}^-]}$$

But the upshot of writing things out like this is that we can then write out an 'effective' action which only concerns integrals from our product of low frequency Fourier modes. In other words, we have that

$$\mathcal{Z} = \int d\phi_{\mathbf{k}}^- e^{-F'[\phi_{\mathbf{k}}^-]}, \quad e^{-F'[\phi_{\mathbf{k}}^-]} = e^{-F_0[\phi_{\mathbf{k}}^-]} \int \prod_{\Lambda' < k < \Lambda} d\phi_{\mathbf{k}}^+ e^{-F_0[\phi_{\mathbf{k}}^+]} e^{-F_I[\phi_{\mathbf{k}}^+, \phi_{\mathbf{k}}^-]}$$

We define F' as the Wilsonian effective action. In other words, F' is now a shifted version of our free theory, giving that

$$F'[\phi] = \int d^d x \left[\frac{1}{2} \gamma' (\nabla \phi)^2 + \frac{1}{2} \mu'^2 \phi^2 + g' \phi^4 \right]$$

We cannot direction compare $F'[\phi]$ with $F[\phi]$ because they have **different cutoff** regimes in our new theory. Nonetheless, there's a trick we can do to put these theories on the same footing. To do this, we use a technique called momentum rescaling, by rescaling momenta such that

$$\mathbf{k}' = \zeta \mathbf{k}, \quad \mathbf{x}' = \frac{\mathbf{x}}{\zeta}, \quad \zeta = \frac{\Lambda}{\Lambda'}$$

When we do this, we have that Λ' is rescaled to Λ. Finally, for convention's sake, we can do a field rescaling to absorb γ' into the definition of $(\nabla \phi)^2$.

Now, our new action F_ζ can be written in terms of this new rescaling parameter. We hence write

$$F_\zeta[\phi_{\mathbf{k}}^-] = \int d^d x \left[\frac{1}{2} (\nabla \phi')^2 + \frac{1}{2} \mu^2(\zeta) \phi'^2 + g(\zeta) \phi'^4 \right]$$

We have a diagram with one axis being g_k and on another axis we have g_i, with flows across them which represents some coupling constants. We flow in the direction of increasing ζ.
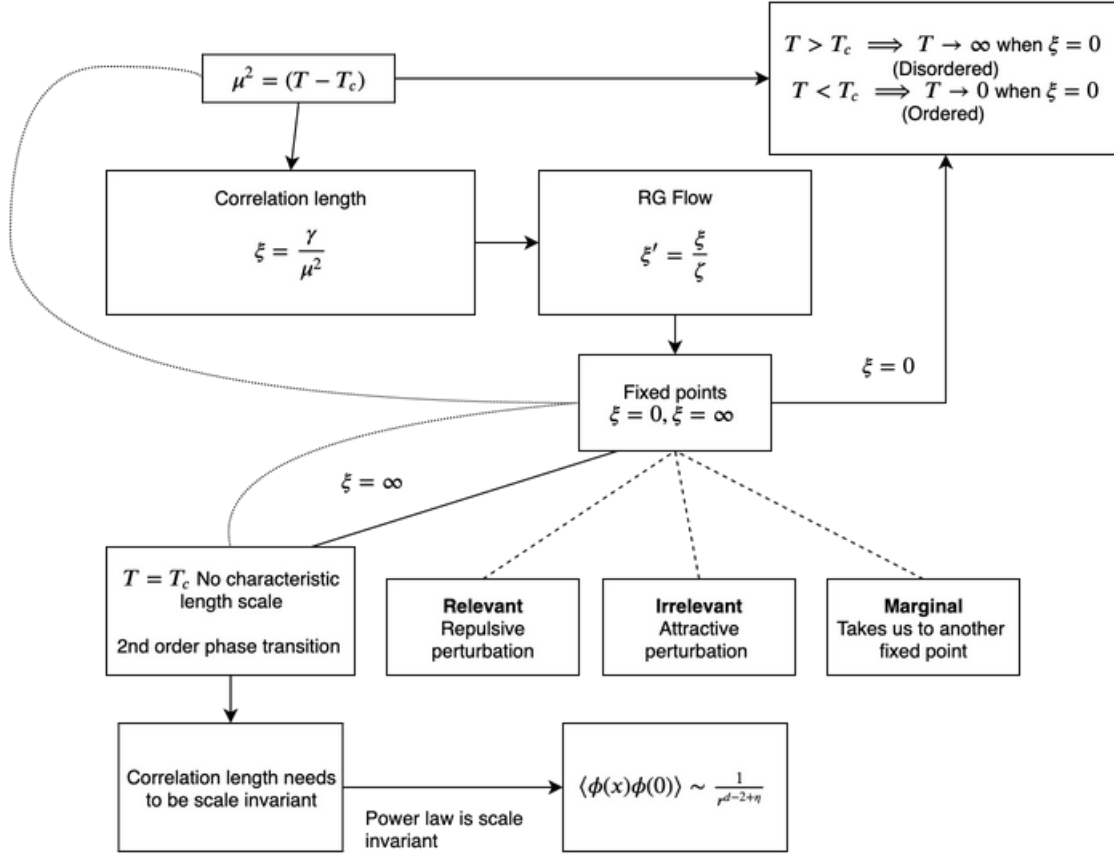
Figure 6: A schematic on how fixed points in RG flow mean different behaviours for temperatures

7.3   A Second Look at Universality

We can understand the phenomena of universality much better now using the language of renormalisation. Let's begin by starting at one point in theory space, for example in the above case where $\zeta = 1$. We then start flowing to larger and larger ζ. Where can we land? We can't have cycles in theory space! We may be able to have shapes in theory space. Check out the C-theorem for more information about this.

1. We could have that our couplings like $\mu^2(\zeta)$. or $g(\zeta)$', could flow in some direction off to infinity.

2. More interestingly, we could have the coupling constants converging towards a fixed point. At a fixed point, the theory doesn't change with scale which implies that our correlation length ξ needs to be $\xi = 0$ or $\xi = \infty$, since ξ depends on scale.

7.3.1   The Ising model

Let's think about the case when $T > T_c$, with $T \to \infty$. We expect that at this temperature, spins change so rapidly that we don't expect the things to be correlated. Hence, we have that $\xi \to 0$.

On the other hand, when we're in the cold regime with $T < T_c$, and that $T \to 0$, we also have that nothing happens so our correlation length $\xi \to 0$.

Our most interesting case is when $T = T_c$, In this case, we have that $\xi \to \infty$, and we have fluctuations on every possible length scale. Colloquially, this looks like a fractal.

7.4 Classifying fixed points

Suppose we have a set of coupling constants in theory space. We can classify deformations in theory space based on how the flow changes towards fixed points.
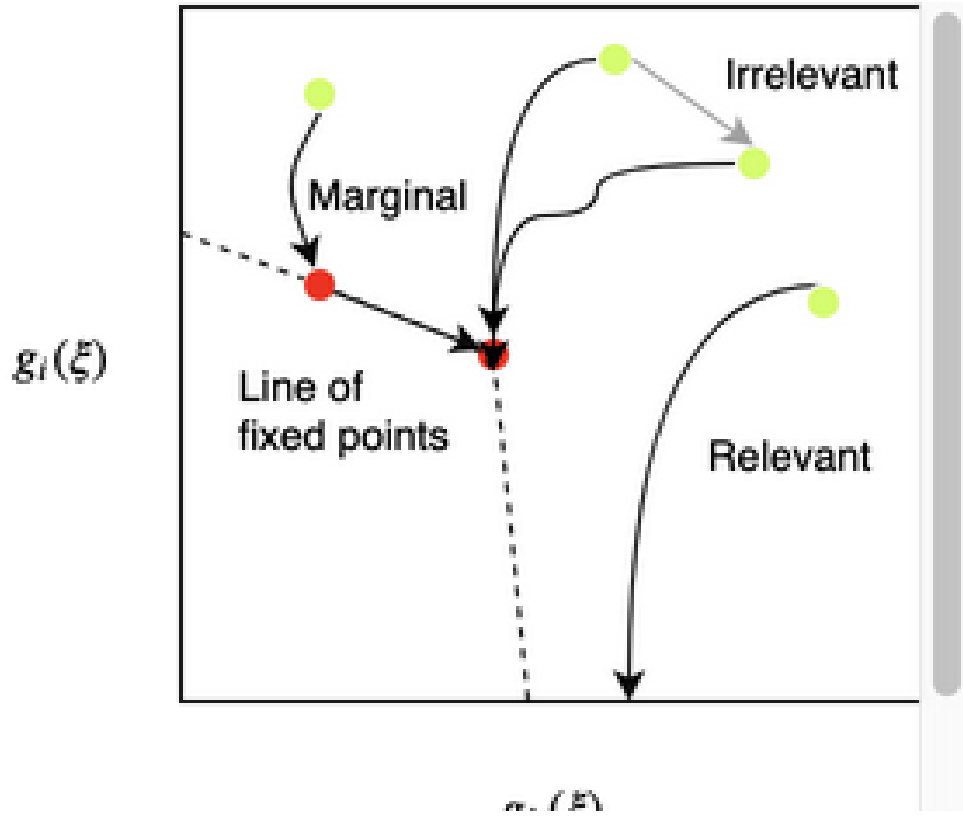

Figure 7: Classifying deformations in theory space

7.5 Scaling

Since all scales are washed away in the fixed point, we have that our correlation, which is a function of length, can only depend on this length scale. So, we must have that our correlation function must take the form of a power law.

$$\langle \phi(\mathbf{x})\phi(0) \rangle = \frac{1}{r^{d-2+\eta}}$$

In units of inverse length, we have that $[x] = -1$, $\left[\frac{\partial}{\partial x}\right] = 1$. Since our free energy needs to be dimensionless, ie that $[F[\phi]] = 0$, we have that our dimensions of our scalar field satisfies

$$[\phi(\mathbf{x})] = \frac{d-2}{2} \implies \eta = 0$$

Assuming that under the scaling $\mathbf{x} \to \frac{\mathbf{x}}{\zeta}$, that our scalar field $\phi(\mathbf{x}) \to \zeta^{\Delta_\phi} \phi(\mathbf{x})$, substituting this into our power law relation gives that

$$\Delta_\phi = \frac{d-2+\eta}{2}$$

where we have that the $\frac{d-2}{2}$ is referred to as our engineering dimension, where are our extraneous factor of $\frac{\eta}{2}$ is referred to as our anomolous dimension. There's a difference here. However, we can reconcile this by setting

$$\langle \phi(\mathbf{x})\phi(0) \rangle = \frac{a^\mu}{r^{d-2+\eta}}$$

7.6   Relation to critical exponents

Recall that our correlation length was derived as a function of our critical temperature

$$\xi \sim t^{-^i v}$$

However, since ξ is a length scale, this means it has scaling coefficient $\Delta_\xi = -1$. Assuming that our temperature scales like

$$t \to \xi^{\Delta_t} t$$

we have that $\Delta_t = \frac{1}{^i v}$. Now consider the free energy without a magnetic field. At leading order, we have that

$$F_{\text{thermo}}(t) = \int d^d x f(t) \implies f(t) \sim t^{d\nu}$$

From this we can read off variables of interest. We have that

$$c = \frac{\partial^2 f}{\partial t^2} \sim t^{d\nu-2} \sim t^{-\alpha}$$

So in summary, we have the following critical exponents,

$$c = \frac{\partial^2 f}{\partial t^2} \sim t^{-\alpha} \implies \alpha = 2 - d\nu$$

$$\phi \sim t^\beta \implies \beta = \left(\frac{d-2+\eta}{2}\right)\nu$$

$$\chi \sim t^{-\gamma} \implies \gamma = \nu(2-\eta)$$

$$\phi \sim B^{\frac{1}{\delta}} \implies \delta = \frac{d+2-\eta}{d-2+\eta}$$

This is great because we've reduced the number of parameters which we need to measure! We have four observables but only need to determine η and ν. Hence, our theory is useful!

Table 1: caption

MF	$\frac{4-d}{2}$	1	3	0	$\frac{1}{2}$	
$d = 2$	0	$\frac{1}{8}$	$\frac{7}{4}$	15	$\frac{1}{4}$	1
$d = 3$	0.11	0.33	1.24	4.79	0.04	0.63

7.7 Relevant, Irrelevant or Marginal?

Consider our free energy given by our usual expression

$$F[\phi] \sim \int d^d x \left\{ + \cdots + g_\mathcal{O} \mathcal{O}(\mathbf{x}) + \ldots \right\}$$

We could have a special case where our operator $\mathcal{O}$ scales **nicely** with our renormalisation group flow. This is not usually, the case and its hard to find operators which behave in this way, but bear with me for a moment and let's assume that this is the case.

$$\mathcal{O}(\mathbf{x}) \to \mathcal{O}'(\mathbf{x}') = \zeta^{\Delta_\mathcal{O}} \mathcal{O}(\mathbf{x})$$

So, we can think of $\mathcal{O}$ as an eigen-operator of our RG flow process. Now, in the above expression we have coupled $g_\mathcal{O}$ as the coupling constant. Due to the dimensionless constraint of our free energy, we have that

$$\Delta g_\mathcal{O} = d - \Delta_\mathcal{O}$$

This implies that $g_\mathcal{O}$ goes off to infinity, zero, or stays where it is depending on the sign on $\Delta g'$.

1. $\Delta_\mathcal{O} < d \implies$ Relevant

2. $\Delta_\mathcal{O} > d \implies$ Irrelevant

3. $\Delta_\mathcal{O} = d \implies$ Marginal

7.8 The Gaussian fixed point

We're now in the position to do some actual calculations using the renormalisation group flow. Our simplest case is, of course, the quadratic free energy. Let's consider what happens when we have our usual free energy to quadratic order. The first thing we do is to expand ϕ in terms of Fourier modes, remembering to impose the ultraviolet cutoff Λ.

$$F_0[\phi] = \int d^d x \left[\frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} \mu_0^2 \phi^2 \right] = \int_0^\Lambda \frac{d^d k}{(2\pi)^d} \frac{1}{2} \left(k^2 + \mu_0^2 \right) \phi_\mathbf{k} \phi_{-\mathbf{k}}$$

The useful thing about having a quadratic free energy like this is that this separates into low and high Fourier modes with

$$F_0[\phi] = F_0[\phi^-] + F_0[\phi^+]$$

You can prove this yourself by simply decomposing

$$\phi_\mathbf{k} = \phi_\mathbf{k}^+ + \phi_\mathbf{k}^-$$

When we integrate over, the mixed terms go to zero since there is no portion of the integral where both the high frequency and low frequency modes are non-zero. This means that when we write out our effective free energy, we get that

$$e^{-F'[\phi^-]} = \left[\int \mathcal{D}\phi^+ e^{-F_0[\phi^+]} \right] e^{-F_0[\phi^-]}$$

$$= \mathcal{N} e^{-F_0[\phi^-]}$$

We don't care about the constant $\mathcal{N}$ since it doesn't change the physics. The upshot of this is that now we have an expression for our effective free energy in terms of low frequency modes $\phi_{\mathbf{k}}^-$.

$$F'[\phi^-] = \int^{\frac{\Lambda}{\zeta}} \frac{d^d k}{(2\pi)^d} \frac{1}{2} (k^2 + \mu_0^2) \phi_{\mathbf{k}}^- \phi_{-\mathbf{k}}^-$$

Now that we've performed the cut-off procedure, we need to wrangle the upper bound of the integral into its original form. This is done by rescaling our Fourier mode variable through

$$k' = k\zeta, \phi'_{\mathbf{k}'} = \zeta^{-w} \phi_{\mathbf{k}}^- \implies F'[\phi'] = \int^{\Lambda} \frac{d^d k'}{(2\pi)^d} \zeta^{-d} \frac{1}{2} (\zeta^{-2}(k')^2 + \mu_0^2) \zeta^{2w} \phi'_{\mathbf{k}'} \phi'_{-\mathbf{k}'}$$

Now, the aim of the game is to 'canonically normalise' our first term $(k')^2$ since its associated with our kinetic gradient in our free energy (and hence should not be changed under RG group flow. This is achieved by solving for our scaling parameter w and setting it to $w = \frac{d+2}{2}$. This gives our final free energy as

$$F_0[\phi'] = \int \frac{d^d k}{(2\pi)^d} \frac{1}{2} (k^2 + \zeta^2 \mu_0^2) \phi_{\mathbf{k}} \phi_{-\mathbf{k}}$$

Now, we're in business because we can just read off the coupling constant and we find that $\mu^@$ oru coupling parameter scales like

$$\mu^2(\zeta) = \zeta^2 \mu_0^2$$

Once we have this, we can find our fixed points which satisfy the relation $\frac{d\mu^2}{d\zeta} = 0$. Hence, we have that

$$\mu^2 = \infty, \quad \mu^2 = 0$$

are our fixed points! Our point $\mu^2 = 0$ is called our Gaussian fixed point. Right now, these points are uninteresting, but let's go along with it. We add an infinite series of coupling constants, much like a Taylor expansion. Then, we'll look at the behaviour of these coupling constants near the point $\mu_0^2 = 0$.

7.8.1 Expanding around the Gaussian fixed point

Moving on from our expression above, let's generalise this further by expanding in higher orders of ϕ. We expand our free energy in a power series as

$$F[\phi] = \int d^d x \left[\frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} \mu_0^2 \phi^2 + \sum_{n=4} g_{0,n} \phi^n \right]$$

Having these extra terms mean that when we decompose our Fourier modes into low frequency and high frequency modes, we have that the coupling constants shift when we integrate them out. We'll offer a more detailed explanation in specifically calculating these shifts, but just for now, lets assume that we have that

$$g_{0,n} \to g_n' = g_{0,n} + \delta g_n$$

We're in good shape to now perform RG flow on this free energy. To do RG flow, this time in position space, we use the scaling arguments we used earlier to determine critical exponents. Since we're working in negative scaling dimension, we set that

$$\mathbf{x} \to \mathbf{x}' = \mathbf{x}/\zeta, \quad \phi(\mathbf{x}) \to \phi'(\mathbf{x}') = \zeta^{\Delta_\phi} \phi(\mathbf{x})$$

Again, as before, we have that Δ_ϕ is a scaling dimension to determine. Inverting these expressions and substituting them into the free energy, we have that in terms of rescaled parameters,

$$F[\phi] = \int d^d x \, \zeta^d \frac{1}{2} \left(\zeta^{-2\Delta_\phi - 2} \nabla \phi \cdot \nabla \phi + \frac{1}{2} \mu_0^2 \zeta^{-2\Delta_\phi} \phi^2 + \sum_{n=4}^{\infty} \zeta^{-n\Delta_\phi} \phi^n \right)$$

One should be careful when doing these kinds of calculations. First, it's important to remember to appropriately scale the Jacobian factor (if not, our dependence on dimension which we have so pressingly made a point of wouldn't be there in the first place !). Also, when rescaling the gradient terms, we have to be careful to also scale the derivatives with the RG flow as well.

Asserting that the gradient term should be invariant under our renormalisation gives us the relation that

$$\Delta_\phi = \frac{d+2}{2}$$

This is great because since we've determined what Δ_ϕ is, we can then substitute this into our interaction terms. For our coupling constants, we then see that our flow induces a scaling on g_n via

$$g_n(\zeta) = \zeta^{d - n\Delta_\phi} g_{0,n}$$

The upshot of dong this process is that now we can see under what conditions g_n flow towards, or diverge from, fixed points. Take the case when $n = 4$. Our coupling constant then evolves like

$$g_4(\zeta) = \zeta^{d-4} g_{0,4}$$

This means that we can classify our deformations around fixed points depending on our dimension. So, we have that

1. $d > 4, \phi^4$ irrelevant

2. $d < 4, \phi^4$ Relevant

3. $d = 4, \phi^4$ Marginal

7.8.2 A remark on 'tuning' to remain at a fixed point

Let's just think about the slice of theory space which includes only μ_0^2 and g_4. We've shown that in any case, when we act on out theory space with our renormalisation procedure we have that

μ_0^2 goes away from our fixed point. This means to stay at this point, we have to 'fine tune' μ_0^2 to stay there. However, depending on our critical dimension $d = 4$, we can either fine tune g_4 if our deformation is relevant (d < 4) or not care about it when it's irrelevant (d > 4) with the confidence that it'll return back to the fixed point.

In the figure below we have a diagram to show this behaviour.
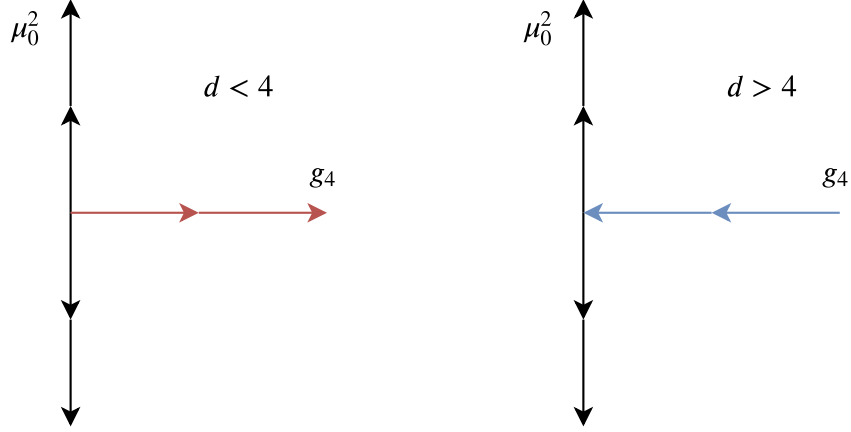
Figure 8:

7.8.3 $\mathbb{Z}_2$ Symmetry breaking

Typically, we have that RG flow doesn't break our symmetries. This symmetry is respected by momentum, so in momentum space symmetry is also preserved. This means that every term in our decomposition

$$F_0[\phi] = F_0[\phi^-] + F_0[\phi^+] + F_I[\phi^-, \phi^+]$$

At low temperature, our background magnetic field, becomes important. We need to have an RG way of visualising what's going on.

7.9 Renormalisation with Interacting Fields

7.9.1 A quick review of finding the effective free energy

In the case of a 'free theory', our energy is written in terms of quadratic terms in ϕ. Free energy cases are important, because when we split Fourier modes, we saw earlier that we could split up the free energy into a high frequency part and a low frequency part.

$$F_0[\phi] = F_0\left[\phi^+\right] + F_0\left[\phi^-\right]$$

However, much like we do in quantum field theory, these cases, while easy, are not very interesting. In this article, we'll explore how to use Wick's theorem to look at the behaviour of energies that look like

$$F[\phi] = \int d^d x \frac{1}{2}\left(\nabla\phi\right)^2 + \frac{1}{2}\mu_0^2\phi^2 + g\phi^4$$

We've added a ϕ^4 term on the end here. This is completely similar to ϕ^4 theory in quantum field theory. Now, we ask what happens when we split this up into higher and low Fourier modes. In real space, we expect that our free energy splits into free terms and an interaction term.

$$F[\phi] = F_0[\phi^+] + F_0[\phi^-] + F_I[\phi^+, \phi^-]$$

The presence of the interaction term mean that there will be some mixing of high and low frequency modes. Note the odd notation here. Since we have that $\phi = \phi(\phi^+, \phi^-)$, we write

$$F[\phi] = F_0[\phi^+] + F_0[\phi^-] + F_I(\phi), \quad F_I(\phi) = \int d^d x \, g\phi^4$$

So far we've looked at how things look in ϕ space, but let's have a quick look as well about how things evolve from the perspective of Fourier space variables $\phi_{\mathbf{k}}$. Starting from our partition function

$$\mathcal{Z} = \int \mathcal{D}\phi e^{-F[\phi]}$$

We rewrite our measure in terms of a product of Fourier coefficients. In this product, we separate high and low frequency terms, and 'integrate' out our high frequency terms. This then prompts us to write our partition function as

$$\mathcal{Z} = \int \prod_{0<k<\Lambda'} d\phi_{\mathbf{k}}^- d\phi_{-\mathbf{k}}^+ e^{-F_0[\phi_{\mathbf{k}}^-]} \int \prod_{\Lambda'<k<\Lambda} dd\phi_{\mathbf{k}}^+ \phi_{-\mathbf{k}}^+ e^{-F_0[\phi_{\mathbf{k}}^+]} e^{-F_I[\phi_{\mathbf{k}}^+, \phi_{\mathbf{k}}^-]}$$

This means we define an 'effective' free energy by writing our whole sum as just an integral over the lower Fourier modes. We rewrite the above term as

$$\mathcal{Z} = \int \prod_{0<k<\Lambda} d\phi_{\mathbf{k}}^- d\phi_{-\mathbf{k}}^- e^{-F'[\phi_{\mathbf{k}}]}$$

Comparing coefficients, our new effective free energy from the above expression is given by

$$e^{-F'[\phi_{\mathbf{k}}^-]} = e^{-F_0\left[\phi_{\mathbf{k}}^-\right]} \int \prod_{\Lambda'<k\Lambda} d\phi_{\mathbf{k}}^+ d_{\phi-\mathbf{k}}^+ e^{-F_I[\phi_{\mathbf{k}}^+, \phi_{\mathbf{k}}^-]}$$

We can write this as more familiarly as a functional integral over just the positive Fourier modes, so that

$$e^{-F'[\phi^-_{-\mathbf{k}}]} = e^{-F_0[\phi^-_{\mathbf{k}}]} \int \mathcal{D}\phi^+_{\mathbf{k}} e^{-F_0[\phi^+_{\mathbf{k}}]} e^{iF_I}$$

Let's stare hard at the nature of the term on the right hand side. We're integrating F_I over high frequency Fourier modes but with a pre-faction of e^{-F_0}. This precisely is an expression for the expectation of F_I but over high frequency Fourier modes instead. So, we write that

$$e^{-F'[\phi^-_{-\mathbf{k}}]} = e^{-F_0[\phi^-_{\mathbf{k}}]} \left\langle e^{-F_I[\phi^+_{\mathbf{k}}, \phi^-_{\mathbf{k}}]} \right\rangle_+$$

7.9.2 Logarithmic expansion of the Interaction Term

Now, a quantity immediately of interest is what the average value $\langle F_I \rangle$ is, and how it can give us an expression for our effective free energy F'. Now, there's not much we can do about finding this average value other than to attempt to Taylor expand the equation above upon taking logs.

$$F'[\phi_{\mathbf{k}}-] = F_0[\phi^-_{\mathbf{k}}] - \log \left\langle e^{-F_I} \right\rangle$$

employing the Taylor expansion for log, something special happens because we get an expression for the cumulant of F_I;

$$F' = F_0 + \langle F_I \rangle + \frac{1}{2}\left(\langle F_I \rangle^2 - \langle F_I^2 \rangle \right)$$

7.9.3 Computing moments of the Interaction Free energy

Now, the name of the game is to actually compute quantities like $\langle F_I \rangle$ Let's do this explicitly for an interaction term which has its analogue in ϕ^4 theory in quantum field theory. We set

$$F_I = g \int d^d x \, \phi^4(x)$$

where g is our small perturbation order term. The first thing to do is to expand out F_I in terms of Fourier modes. This gives a product of a bunch of Fourier modes which we need to sort through.

$$F_I[\phi] = \int d^d x \int \prod_{i=1}^{4} \frac{d^d k_i}{(2\pi)^d} e^{-i\mathbf{k}_i \cdot \mathbf{x}} \phi_{\mathbf{k}_i}$$

$$= \int \prod_{i=1}^{4} \frac{d^d k_i}{(2\pi)^d} \phi_{\mathbf{k}_i} (2\pi)^d \delta(\sum_i \mathbf{k}_i)$$

In this step, we mutliplied out the Fourier modes and then moved the $\int d^d x$ inside to get the delta function. This delta function has the physical interpretation of imposing momentum conservation. Now, let's do what we did with RG and split our Fourier modes $\phi_{\mathbf{k}_i}$ into high frequency and low frequency parts. Our product looks like

$$\prod_{k=1}^{4} \phi_{\mathbf{k}_i} = (\phi^+_{\mathbf{k}_1} + \phi^-_{\mathbf{k}_1}) \dots (\phi^+_{\mathbf{k}_4} + \phi^-_{\mathbf{k}_4})$$

Now, let's think about what kind of terms we get from this product. We count these by considering how many low frequency versus high frequency modes there are in the expansion.

1. We have terms whose all four factors are low frequency modes. There is only one term of this type, and it's $\phi_{\mathbf{k}_1}^- \phi_{\mathbf{k}_2}^- \phi_{\mathbf{k}_3}^- \phi_{\mathbf{k}_4}^-$. This means that when we compute our average value $\langle F_I \rangle_+$, since this an average over ϕ^+, this term just factors out of our intergal. We then observe that this is just, in real space, our original interaction term but this time with just low frequency modes. Thus, we have a $\int d^4x \, \phi^-$ contribution.

2. If we have just one factor of $\phi_{\mathbf{k}}^+$ with the rest of the terms being $\phi_{\mathbf{k}}^-$ (we have 4 of these in total), then, since we are averaging over an odd number of $\phi_{\mathbf{k}}^+$, this vanishes since our Gaussian integral is an even function. Terms like this look like $\phi_{\mathbf{k}_1}^- \phi_{\mathbf{k}_2}^- \phi_{\mathbf{k}_3}^- \phi_{\mathbf{k}_4}^+$. When we take the average, we're including a factor of

$$\langle \phi_{\mathbf{k}}^+ \rangle_+ = \int \mathcal{D}\phi_k^+ e^{-F_0[\phi_k^+]} \phi_{\mathbf{k}}^+$$

This goes to zero since we have an odd term and an even term integrated together.

3. Our only interesting term are terms which look like $\phi_{\mathbf{k}_1}^- \phi_{\mathbf{k}_2}^- \phi_{\mathbf{k}_3}^+ \phi_{\mathbf{k}_4}^+$. This contribution is non-trivial and we'll do some work to compute this explicitly in the following section. There are 6 of these terms in the expansion.

4. By the same logic with just one $\phi_{\mathbf{k}}^+$ terms that contain things like $\phi_{\mathbf{k}_1}^+ \phi_{\mathbf{k}_2}^+ \phi_{\mathbf{k}_3}^+ \phi_{\mathbf{k}_4}^-$ vanish since we have an odd number of terms

5. Terms with just $\phi_{\mathbf{k}}^+$ are integrated out and contribute a constant. Let's go through why this is the case, slowly. Recall that our quantity of interest to compute as

$$F'[\phi^-] = F_0[\phi^-] - \log \langle e^{-F_I} \rangle_+ = F_0[\phi^-] + \langle F_I \rangle_+$$

Now, since this term **doesn't** have any dependence on ϕ_k^-, we don't care about it because our effective free energy is only a function of ϕ_k^-. Hence, this term is only an extraneous constant addition to our effective free energy, and hence doesn't contribute.

Thus we focus on the only interesting term, with two $\phi_{\mathbf{k}}^+$ factors. This term looks like

$$\langle \phi_{\mathbf{k}}^+ \phi_{\mathbf{k}'}^+ \rangle_+$$

How do we calculate this? Well, forgetting about factors of β, we showed earlier that the correlation function in **position space** looks like

$$\langle \phi(\mathbf{x})\phi(\mathbf{y}) \rangle = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k^2 + \mu_0^2}$$

Now, one can convince themselves quite easily that when we Fourier expand out, then wrangle out $\mathbf{x}$ and $\mathbf{y}$ by integrating over them, to find that

$$\langle \phi_{\mathbf{k}}^+ \phi_{\mathbf{k}'}^+ \rangle = \frac{(2\pi)^d \delta^d(\mathbf{k} + \mathbf{k}')}{k^2 + \mu_0^2}$$

Hence, all together, the only term which contributes to the correction $\langle F_I \rangle_+$ is

$$
\begin{aligned}
\langle F_I \rangle_+ &= 6 \int \prod_{i=1}^{4} \frac{d^d k_i}{(2\pi)^d} \left\langle \phi^+_{\mathbf{k}_1} \phi^+_{\mathbf{k}_2} \right\rangle_+ \phi^-_{\mathbf{k}_3} \phi^-_{\mathbf{k}_4} (2\pi)^d \delta^d \left(\sum_i \mathbf{k}_i \right) \\
&= 6 \int \prod_{i=1}^{4} \frac{d^d k_i}{(2\pi)^d} \frac{(2\pi)^{2d} \delta^d(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \delta^d(\mathbf{k}_1 + \mathbf{k}_2) \phi_{\mathbf{k}_4} \phi_{\mathbf{k}_4}}{k_1^2 + \mu_0^2} \\
&= 6 \int d^d k_1 d^d k_2 d^d k_3 \frac{1}{(2\pi)^{2d}} \frac{\delta(\mathbf{k}_3 + \mathbf{k}_4) \phi_{\mathbf{k}_3} \phi_{\mathbf{k}_4}}{k_1^2 + \mu_0^2} \\
&= 6 \int \frac{d^d k_3}{(2\pi)^d} \phi^-_{\mathbf{k}_3} \phi^-_{-\mathbf{k}_3} \int \frac{d^d k_1}{(2\pi)^d} \frac{1}{k_1^2 + \mu_0^2} \\
&= 6 \int_0^{\Lambda/\zeta} \frac{d^d k}{(2\pi)^d} \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + \mu_0^2}
\end{aligned}
$$

In the last step, we've taken into account the ranges of integration for the high and low frequency modes. Now, we can look at the impact this on our effective free energy.

This means in our effective free energy,

$$
\begin{aligned}
F'[\phi] &= F_0[\phi_-] + \langle F_I \rangle_+ \\
&= \int_0^{\Lambda/\zeta} \frac{d^d k}{(2\pi)^d} \frac{1}{2} \left(k^2 + \mu_0^2 \right) \phi_{\mathbf{k}} \phi_{-\mathbf{k}} + 6 \int_0^{\Lambda/\zeta} \frac{d^d k}{(2\pi)^d} \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + \mu_0^2}
\end{aligned}
$$

This is equivalent however to remapping μ_0^2 to itself plus some correction term that we get from the interacting theory which incorporates high energies.

$$
\mu_0^2 \rightarrow \mu_0^2 + 12 \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + \mu_0^2}
$$

What's the point in doing all this? Well, we saying that when we turn on an interaction term in ϕ^4, we're inducing a change in how the ϕ^2 coupling term changes. In particular, we induce a RG flow. The μ^2 coefficient flows as

$$
\mu^2(\zeta) \rightarrow \zeta^2 \left(\mu_0^2 + 12 \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + \mu_0^2} \right)
$$

7.9.4 Second order corrections

7.10 Applying Wick's thoerem for the Free Energy

(Change terms here since they vanish) Wick's theorem gives us a way to expand out our four point correlation function into the symmetric sum of choices of two fields. The first term non-vanishing term in the symmetric sum is

$$
\int \prod_{j=1}^{4} d^d q_j \left\langle \phi^+_{\mathbf{q}_1} \phi^+_{\mathbf{q}_2} \right\rangle_+ + \left\langle \phi^+_{\mathbf{q}_3} \phi^+_{\mathbf{q}_4} \right\rangle_+ \delta^d(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{q}_1 + \mathbf{q}_2) \delta^d(\mathbf{k}_3 + \mathbf{k}_4 + \mathbf{q}_3 + \mathbf{q}_4)
$$

45

However, the first two terms combined with the delta functions yield a Green's function so that the above is

$$\sim \int d^d q_2 d^d q_4 \, G_0(q_2)G_0(q_4)\delta^d(\mathbf{k}_1 + \mathbf{k}_2)\delta^d(\mathbf{k}_3 + \mathbf{k}_4)$$

Now, integrating over the delta function this gives us

$$\sim \int d^d q G_0(q) G_0(\|\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{q}\|\delta^d(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{q})$$

This finally gives us a term for the average value $\langle F_I^2 \rangle$. Substituting this into our expression,

$$\langle F_I^2 \rangle_+ \sim \binom{4}{2}^2 g_0^2 \int 0^{\Lambda/\zeta} \prod_{i=1}^4 \left[\frac{d^d k_i}{(2\pi)^d} \phi_{\mathbf{k}_i}^- \right] f(\mathbf{k}_1 + \mathbf{k}_2)(2\pi)^d \delta^d(\sum_i \mathbf{k}_i)$$

By symmetry, this function f must be a function of the modulus of our vectors so that

$$f(\mathbf{k}) = \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \left[\frac{1}{q^2 + \mu_0^2} \frac{1}{(\mathbf{k} + \mathbf{q})^2 + \mu_0^2} \right]$$

Now, we Taylor expand f to first order in $O(\|\mathbf{k}\|)$, the upshot of which makes us recover a constant term.

$$f(\mathbf{k}) \simeq \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{((2\pi)^d} \frac{1}{(q^2 + \mu_0^2)^2} \left[1 + O(\|k^2\|] \right]$$

This implies that our g_0 coupling constant goes like

$$g_0 = g_0' = g_0 - 36 g_0^2 \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + \mu_0^2)^2}$$

7.11 Feynman Diagrams
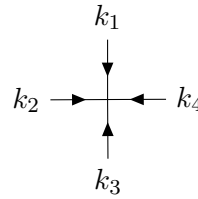
Previously we were calculating an expansion

$$\log \left\langle e^{-F_I[\phi_k^-, \phi_k^+]} \right\rangle_+$$

This yields a sum of terms which look like $g_0^p (\phi^-)^a (\phi^+)^b$. We would like an efficient way of computing this object once it's averaged in the Gaussian ensemble. In an almost perfect analogy with quantum field theory, we construct a diagrammatic way to calculate these expansions. These diagrams are called Feynman diagrams. We'll outline the rules to construct them below

1. Each $\phi_{\mathbf{k}}^-$ is represented by an external solid line. Contributions from these external lines can be written as just a factor of ϕ^- integrated over position space.

2. Each $\phi_{\mathbf{k}}^+$ is represented by an internal dotted line. These lines each contribute to adding a propagator $\langle \phi_{\mathbf{k}}^+ \phi_{\mathbf{k}}^+ \rangle$ in the integral expression.

3. Dotted lines are always connected at both ends. The fact that we used Wick's theorem to give us all possible pairwise contractions is the reason for the requirement that all the dotted lines need to be paired up an connected in all possible different ways.
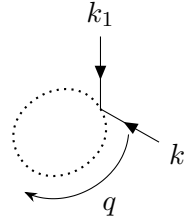
4. Each vertex joins 4 lines with g_0 . When we have a vertex, we need to impose momentum conservation b y

5. Each line has some momentum $\mathbf{k}$, conserved at vertices, which comes from the delta function we get by integrating with respect to the spatial variables first.

6. Momenta which are not determined explicitly (for example internal loops), are integrated over with a factor $\int d^d q/(2\pi)^d$.

7. Divide by symmetry factors.

Now, once we've drawn these diagrams, we are in shape to write down the integrals that they correspond to. At lowest order we have something called a tree diagram where 4 momenta go into a single vertex. These are represented by $g_0 \int d^4 x \, (\phi^-)^4$.


$$= g \int d^d x \, (\phi^-)^4$$

We then have something called a loop diagram, which contributes an overall constant since we integrate over high modes. The interesting connection is when we have a loop attached to two external vertices. This gives us an extra term

$$= 6 g_0 \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{q^2 + \mu_0^2} \int d^d x (\phi^-)^2$$

This term corresponds to the contribution which we looked at earlier; that of $\phi^-_{\mathbf{k}_1} \phi^-_{\mathbf{k}_2} \phi^+_{\mathbf{k}_3} \phi^+_{\mathbf{k}_4}$ This diagram is qualitatively new because it contains a loop which of undetermined momentum which we need to integrate over, as opposed to the *tree diagrams* which have no internal loops. You can convince yourself that these are the only possible diagram one can get at first order. Now, if we move up to second order, we need to consider diagrams which contain two vertices, but whose ends are connected. We shall consider a **connected** diagram first, whose contribution looks like
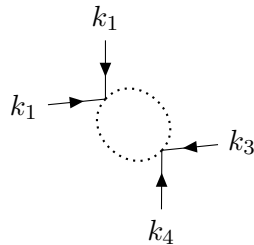

$$(1)$$

Now, this diagram is from second order and corresponds to the term given by

$$36g_0^2 \int_0^{\Lambda/\zeta} \prod_{i=1}^{4} \left[\frac{d^d k_i}{(2\pi)^d} \phi_{\mathbf{k}_i}^{-} \right] f(\mathbf{k}_1 + \mathbf{k}_2)(2\pi)^d \delta^d(\sum_i \mathbf{k}_i)$$

where

$$f(\mathbf{k}_1 + \mathbf{f}_2) = \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + \mu_0^2} \frac{1}{(\mathbf{k}_1 + \mathbf{k}_2 + q)^2}$$

Let's break this down. Our factor of f comes from the fact that, after we impose momentum conservation, we still have an undetermined momentum q to integrate over. Within this integral, we have the two internal line propagators which contribute $\frac{1}{q^2 + \mu_0^2}$ and $\frac{1}{(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{q})^2 + \mu_0^2}$. Now, this is the only diagram which contributes to this order in the interaction. One may ask, what happens to the terms like
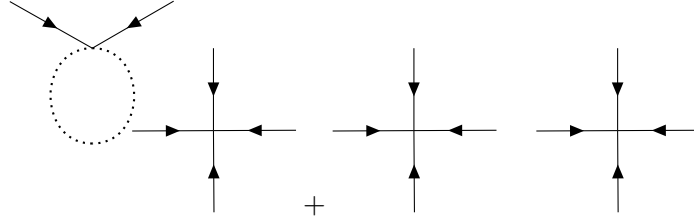
The key thing to observe here is that these diagrams are disconnected, and in fact get cancelled out by terms in the expansion of $\langle F_I \rangle^2$. Multiples of diagrams get cancelled out by cumulant terms! So, in our expansion for F_I we need only consider the sum of connected diagrams. Finally, we don't consider diagrams which look like

$$= 0$$
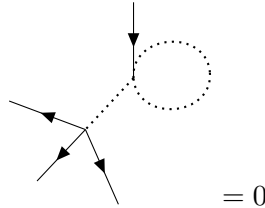
This is because we need to impose momentum conservation at each vertex, but as you can see here there's a lone external propagator meeting a bunch of internal ones, so momentums cannot match here. This means that diagrams of this form go to zero.

However, there are other interesting cases to second order that we need to think about. For example, consider the diagram

$$= \int \frac{d^d k}{(2\pi)^d} \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q_1 d^d q_2}{(2\pi)^{2d}} \frac{1}{q_1^2 + \mu_0^2} \frac{1}{q_2^2 + \mu_0^2}$$

48

The above diagram can be written in simplified fashion as

$$\cdots = \int \frac{d^d k}{(2\pi)^d} \frac{1}{2} \phi_{\mathbf{k}} \phi_{-\mathbf{k}} C(\Lambda)$$

The key point that we made here is that our function which comes from the interaction terms doesn't depend on the momentum $\mathbf{k}$ in this case, but we've still encoded the ultraviolet cutoff Λ. If we factor in this contribution to our free energy, then we've induced another contribution in how μ^2 changes:

$$\mu_0^2 \to \mu_0^2 + 12 g_0 \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + \mu_0^2} + g_0^2 C(\Lambda)$$

We can also denote general functions which **do** depend on $\mathbf{k}$. For example, consider the diagram
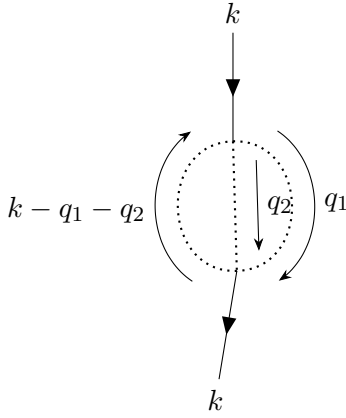
$$= \int \frac{d^d k}{(2\pi)^d} \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \int \frac{d^d q_1 d^d q_2}{(2\pi)^{2d}} \frac{1}{q_1^2 + \mu_0^2} \frac{1}{q_2^2 + \mu_0^2} \frac{1}{(k - q_1 - q_2)^2 + \mu_0^2}$$

This is qualitatively different from the previous case because now we can write our second order contribution as

$$\cdots = g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{2} A(k, \Lambda) \phi_{\mathbf{k}}^{-} \phi_{-\mathbf{k}}^{-}$$

Where $A(k, \Lambda)$ actually now depends on k. The upshot of this is that, when we return back into **real space**, we have that since we expand $A(k) = A(0) + \frac{1}{2} k^2 A''(0, \Lambda)$ (ignoring odd powers of k since they ultimately vanish), we get that now our **gradient term** picks up an extra contribution now, so we have a flow on this term which we can figure out by inverse Fourier transforming.

$$g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{2} A(k, \Lambda) \phi_{\mathbf{k}} \phi_{-\mathbf{k}} = g_0^2 \frac{1}{4} \int \frac{d^d k}{(2\pi)^d} k^2 \phi_{-\mathbf{k}} \phi_{\mathbf{k}}$$

$$= g_0^2 \int \frac{d^d k}{(2\pi)^d} d^d x_1 d^d x_2 \phi(x_1) \phi(x_2) e^{i\mathbf{k}\cdot\mathbf{x}_1} e^{-i\mathbf{k}\cdot\mathbf{x}_1} k^2 A''(0, \Lambda)$$

$$= -\frac{g_0^2 A''(\Lambda, 0)}{4} \int d^d x_1 d^d x_2 \phi(x_1) \phi(x_2) \nabla_{\mathbf{x}_2}^2 \int \frac{d^d k}{(2\pi)^d} e^{i\mathbf{k}\cdot(\mathbf{x}_1 - \mathbf{x}_2)}$$

$$= -g_0^2 \frac{A''(0, \Lambda)}{4} \int d^d x_1 d^d x_2 \phi(x_1) \phi(x_2) \nabla_{\mathbf{x}_2} \nabla_{\mathbf{x}_1} \delta(\mathbf{x}_1 - \mathbf{x}_2)$$

$$= -g_0^2 \int \frac{A''(0, \Lambda)}{4} \int d^d x_1 d^d x_2 \nabla_{\mathbf{x}_1} \phi(\mathbf{x}_1) \nabla_{\mathbf{x}_2} \phi(\mathbf{x}_2) \delta(\mathbf{x}_1 - \mathbf{x}_2)$$

Applying this delta function, this means that we get an overall correction in the free energy that

$$F'[\phi] = \int d^d x \, \frac{1}{2} \gamma' \nabla \phi \cdot \nabla \phi$$

Now, we can write things as general functions of $\mathbf{k}$ and Λ when dealing with more complicated terms. This means that a diagram of the below form gives us a term like

$$g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{2} A(\Lambda, k) \phi_{\mathbf{k}}^- \phi_{-\mathbf{k}}^-$$

We can now Taylor expand out our general function $A(k, \Lambda)$ as

$$A(k, \Lambda) \simeq A(0) + \frac{1}{2} k^2 A''(0) + \dots$$

Now, the first term here gives us a correction to μ^2, and the second term gives us a correction to $(\nabla \phi)^2$. So, in our full effective field theory we have that

$$F'[\phi] = \int d^d x \, \frac{1}{2} \gamma' (\nabla \phi)^2, \quad \gamma' = 1 - 2g_0^2 A''(0, \Lambda)$$

Hence, our rescaled momenta is given by

$$\phi_{\mathbf{k}}' = \frac{\zeta^{-\frac{d+2}{2}}}{1 - g_0^2 A''(0, \Lambda)} \phi_{\frac{\mathbf{k}}{\zeta}}^-$$

7.12 β - functions

Instead of parametrising our RG flow the way we've done it as Λ / ζ, since $\zeta \geq 1$, it's more natural to write $\zeta = e^s$ for some continuous parameter $s \geq 0$. Our new cutoff in RG flow is given by the relation

$$\Lambda' = \frac{\Lambda}{\zeta} = \Lambda e^{-s}$$

We will now introduce a piece of machinery to help us analyse RG flow a bit more. We define our beta functions to be forcing terms to the differential equation

$$\frac{dg_n}{ds} = \beta_n(g_n), \quad \begin{cases} \beta_n > 0 & \text{in the IR} \\ \beta_n < 0 & \text{otherwise} \end{cases}$$

The solutions to these equations are of course just how our coupling constants evolve with respect to changes in s. Recall, that our leading order contributions for our coupling constants are given by

$$\mu^2(\zeta) = \zeta^2 (\mu_0^2 + a g_0), \quad g(\zeta) = \zeta^{4-d} (g_0 - b g_0^2)$$

We derived this from earlier when we looked at our first and second order changes in our free energy. Recall, our constants are given by

$$a = 12 \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + \mu_0^2}, \quad \text{and}, \quad b = 36 \int_{\Lambda/\zeta}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + \mu_0^2)^2}$$

We differentiate these things with respect to ζ. To do this, we assume we only have a very small perturbation from $\Lambda \to \Lambda e^{-s}$, and we approximate the function as a constant locally. We also approximate $e^{-s} = 1 - s + \ldots$

$$\int_{\Lambda e^{-s}}^{\Lambda} dq\, f(q) \simeq \left[\Lambda - \Lambda e^{-s}\right] f(\Lambda) \simeq \Lambda f(\Lambda)s \implies \frac{d}{ds} \int_{\Lambda e^{-s}}^{\Lambda} dq\, f(q) = \Lambda f(\Lambda)$$

We do a change of coordinates in terms of spherical polars in the integrands of a and b. Then, we use the identity above to write out the beta functions. In four dimensions, we integrate this thing in terms of spherical polars so that we have that our coupling constants are solved by

$$\frac{d\mu^2}{ds} = 2\mu^2 + \frac{3g}{2\pi^2} \frac{\Lambda^4}{\Lambda^2 + \mu^2} \quad \text{and} \quad \frac{dg}{ds} = \frac{-9}{2\pi^2} \frac{\Lambda^4}{(\Lambda^2 + \mu^2)^2} g^2$$

This gives us differential equations that we can solve. To the first order, we've written $\mu^2(\zeta) \sim \mu_0^2$, and that $g \sim g_0$ and we've reflected this in the differential equation (where intially we had μ_0^2.

7.13 ϵ Expansions

We learned that our ϕ^4 interaction is irrelevant for $d \geq 4$ and relavant otherwise. Now, lets employ a strange trick and perturb our dimensions, for example by setting

$$d = 4 - \epsilon$$

Returning to our previous beta functions which we calculated in four dimensions, generalising these to beta functions in d dimensions gives us new differential equations

$$\frac{d\mu^2}{ds} = 2\mu^2 + \frac{12\Omega_{d-1}}{(2\pi)^d} \frac{\Lambda^4}{\Lambda^2 + \mu^2} \tilde{g} + \ldots$$

$$\frac{d\tilde{g}}{ds} = \epsilon \tilde{g} - \frac{36\Omega_{d-1}}{(2\pi)^d} \frac{\Lambda^4}{(\Lambda^2 + \mu^2)^2} \tilde{g}^2 + \ldots$$

We use the approximation $\Omega_3 = 2\pi^2 + O(\epsilon)$ to give approximate equations

$$\frac{d\mu^2}{ds} \simeq 2\mu^2 + \frac{3}{2\pi^2} \frac{\Lambda^4}{\Lambda^2 + \mu^2} \tilde{g}$$

$$\frac{\tilde{g}}{ds} \simeq \epsilon g - \frac{9}{2\pi^2} \frac{\Lambda^4}{(\Lambda^2 + \mu^2)^2} \tilde{g}^2$$

From these things, we have the Gaussian fixed point we derived before, but we also have a new fixed point given by

$$\mu_*^2 = -\frac{3}{4\pi^2} \frac{\Lambda^4}{\Lambda^2 + \mu_*^2} \tilde{g}_*, \quad \tilde{g}_* = \frac{2\pi^2}{9} \frac{(\Lambda^2 + \mu_*^2)^2}{\epsilon}$$

To leading order in ϵ, our solutions for these fixed points are

$$\mu_*^2 = -\frac{1}{6} \Lambda^2 \epsilon, \quad \tilde{g}_* = \frac{2\pi^2}{9} \epsilon$$

7.13.1 Wilson-Fisher Fixed Point

We now work perturbatively to understand flows near this fixed point. We write out

$$\mu^2 = \mu_*^2, \quad \tilde{g} = \tilde{g}_* + \delta\tilde{g}$$

We use the matrix method to expand this out as

$$\frac{d}{ds}\begin{pmatrix} \delta\mu^2 \\ d\tilde{g} \end{pmatrix} = \begin{pmatrix} 2 - \frac{\epsilon}{3} & \frac{3}{2\pi^2}\Lambda^2(1 + \frac{\epsilon}{6}) \\ 0 & -\epsilon \end{pmatrix}\begin{pmatrix} \delta\mu^2 \\ \delta\tilde{g} \end{pmatrix}$$

This gives us that our scaling dimension coefficients are, to first order in ϵ, that

$$\Delta_t = 2 - \frac{\epsilon}{3} + \mathcal{O}(\epsilon^2), \quad \Delta_g = -\epsilon + \mathcal{O}(\epsilon^2)$$

7.13.2 Revisiting critical exponents

Expanding out our critical exponents, we have that

$$t \to \zeta^{\Delta_t} t = e^{s\Delta_t} t$$

$$\xi \sim t^{-\nu}\nu = \frac{1}{\Delta_t} = \frac{1}{2} + \frac{\epsilon}{12}$$

From our old scaling relations, we have that

$$c \sim t^{-\frac{\epsilon}{6}}, \quad \Delta_\phi \sim \frac{d-2}{2} = 1 - \frac{\epsilon}{2}, \quad \beta = \frac{1}{2} - \frac{\epsilon}{6}, \quad \gamma = 1 + \frac{\epsilon}{6}, \quad \delta = 3 + \epsilon$$

We've made a small perturbation, set this perturbation to be one, themn recovered our exponents. Now, in the case when $d = 2$, there's no price to be paid for adding ϕ since ϕ has zero dimension. Our free energy looks like

$$F[\phi] = \int d^d x \left[\frac{1}{2}(\nabla\phi)^2 + g_{2(n+1)}\phi^{2(n+1)}\right]$$

This gives us something called conformal symmetry. This means that our theory is scale invariant, under transformations

$$\mathbf{x} \to \lambda\mathbf{x}$$

We can enhance this type of symmetry from

$$\mathbf{X} \to \tilde{\mathbf{X}}(\mathbf{x}), \quad \frac{\partial\tilde{x}^i}{\partial x_k}\frac{\partial\tilde{x}_j}{\partial x_l}\delta_{ij} = \phi(\mathbf{x})\delta_{kl}$$

This motivates the idea of conformal transformations

$$\tilde{x}^i = \frac{x^i - (\mathbf{x}\cdot\mathbf{x})a^i}{1 - 2(\mathbf{x}\cdot\mathbf{a}) + (\mathbf{a}\cdot\mathbf{a})(\mathbf{x}\cdot\mathbf{x})}$$

8 Continuous Symmetries

What we learned thus far is that phase transitions are characterised by symmetries. Up until now for example, in the Ising model, we had $\mathbb{Z}_2$ symmetry. In this section, we'll start to characterise symmetries by their type. We label these symmetries as G and H.

1. G : a symmetry of the free energy

2. H : a symmetry of the ground state.

Consider the Ising model. When we are above the critical temperature, we're at the zero state, so we have that

$$\begin{cases} G & = \mathbb{Z}_2 \\ H & = \mathbb{Z}_2 \end{cases} T > T_c$$

On the other hand, when we have $T < T_c$, we have that

$$\begin{cases} G = \mathbb{Z}_2 \\ H = 0 \end{cases} T < T_c$$

Symmetries are useful because we can then write down the order parameter and then write down the free energy. The nice thing about symmetries is that they're very general - we don;t have to worry about coupling constants. Many systems are characterised by spacetime symmetries, for example crystals. However, the most interesting class of symmetries are internal symmetries, which are symmetries of the field themselves.

8.1 $O(n)$ modes

A classic example are the $O(n)$ modes. If we have n scalar fields, represented by a vector

$$\phi(\mathbf{x}) = \begin{pmatrix} \phi_1(\mathbf{x}) \\ \vdots \\ \phi_n(\mathbf{x}) \end{pmatrix}$$

We have a symmetry when we map

$$\phi_a(\mathbf{x}) \to R_a{}^b \phi_b(\mathbf{x}), \quad R^T R = 1$$

Our free energy then looks like

$$F[\phi(\mathbf{x})] = \int d^d x \left[\frac{\gamma}{2} (\nabla \phi_j) \cdot (\nabla \phi_j) + \frac{\mu^2}{2} \phi \cdot \phi + g(\phi \cdot \phi)^2 + \dots \right]$$

The first model which we will unpack is the XY model, shorthand for the $O(2)$ model.

8.1.1 XY - Models

We can package up the two fields in the XY model as $\psi(\mathbf{x}) = \phi_1(\mathbf{x}) + i\phi_(\mathbf{x})$, using $U(1) \sim SO(2)$. Our free energy which is then generated by this symmetry is given by

$$F[\phi(\mathbf{x})] = \int d^d x \left[(\nabla \psi^*) \cdot (\nabla \psi) + \frac{\mu^2}{2} |\psi|^2 + g|\psi|^2 \right]$$

There are a lot of physical applications of this model. Objects in this universality class include magnets and superfluids!

We can play the same game and look at $O(3)$ models, which is called the Heisenberg model. What's more interesting really than just writing these down is to also think about symmetries and critical points.

8.2 Goldstone modes

The statement is that for every spontaneously broken symmetry, we get an exactly massless scalar. This shows up everywhere! For example, pions in QCD and phonons in condensed matter theory. This is really important because massless particles are important. For discrete symmetries, for example $\mathbb{Z}_2$, we have only a finite number of vacua. For example, in the Ising model, we have that

$$\langle \phi_0 \rangle = \pm m_0$$

However, if we have a continuous symmetry, the story is different. For example, in our XY model, we only constrain our magnitude.

$$\langle \|\phi\| \rangle = M_0 = \sqrt{\frac{-\mu^2}{4g}}$$

Our ground states are some n- sphere. If we generalise this to the $O(n)$ model, our generalised vacua is S^{n-1}. How much energy does it cost to 'move around'? Our polynomial terms don't depend on phase. Well, our gradient terms do contribute, bu t over long enough distances these don't contribute anything.

So, at low energies we expect to have our Goldstone bosons dominating the low energy physics.

8.2.1 Symmetry and Goldstone Bosons

The number of Goldstone bosons will be equal to the dimension of the symmetries of the free energy minus the symmetries of the ground state

$$O(N) : \langle \phi \rangle = \begin{pmatrix} M \\ 0 \\ :0 \end{pmatrix}$$

We have quotiented our our manifold by

$$S^{N-1} = O(N)/O(N-1)$$

Let's apply this to our XY model. If we write out our symmetry

$$\psi(\mathbf{x}) = \left(M_0 + \tilde{M}(\mathbf{x})\right) e^{i\theta(\mathbf{x})}$$

Our free energy shifts like

$$F[M,\theta] = \int d^d x \left\{ \frac{\gamma}{2}(\nabla \tilde{M})^2 + |\mu^2|\tilde{M}^2 + g\tilde{M}^4 + \cdots + \frac{\gamma}{2}M_0^2(\nabla\theta)^2 + \gamma M_0 \tilde{M}(\nabla\theta)^2 + \ldots \right\}$$

We don't require any tuning here. Our Goldstone modes exhibit what we call shift symmetry.

8.2.2 Critical Exponents

Let's look at the particularly interesting case of $d = 3$. Our critical exponents look like This

Table 2: caption

exponent	μ	ν
MF	0	$\frac{1}{2}$
ising	0.0363	0.6300
N = 2	a	b

means our heat capacity has a cusp, which has been observed for superfluids. We have that our tiny different in the critical exponent is actually very important. This is known as the lambda transition.

$$c \sim |T - T_c|^{-\alpha}, \quad \alpha = 2 - 3\nu, \quad \alpha_{O(2)} = -0.16$$

8.2.3 RG in the $O(n)$ model

If we return back to our naive dimensional analysis

$$[\phi] = \frac{d-2}{2}, \quad [\mu_0^2] = 2, \quad [g_0] = 4 - d$$

Now however, when we do our interaction term, we need to take into account that we are already implicitly pairing things together.

$$F_I[\phi] \sim d^d x\, g\phi^4 = \int d^d x\, g(\phi \cdot \phi)(\phi \cdot \phi)$$

(Insert diagram here) So this term contributes a term in the free energy which is $\sim g_0 \delta_{ab}\delta_{cd}$.

To order g_0, we also get new contributions that look like

$$\sim \phi_a^- \phi_a^- \left\langle \phi_b^+ \phi_B^+ \right\rangle, \quad \sim ..$$

8.2.4 Epsilon expansion

When we go to the epsilon expansion

$$\frac{d\mu^2}{ds} = 2\mu^2 + \frac{N+2}{2\pi^2}\frac{\Lambda^4}{(\Lambda^2 + \mu^2)}\tilde{g}^2, \quad \frac{\tilde{g}}{ds} = \epsilon g - \frac{N+8}{2\pi^2}\frac{\Lambda^4}{\Lambda^2 + \mu^2}\tilde{g}^2, \quad \tilde{g} = \Lambda^{-\epsilon}$$

This changes our Wilson Fisher fixed point. We see that we have different factors of N here. As N grows, the effective contribution in the loops grows. From here, we can calculate the critical exponents gained from these equations.

$$\alpha = \frac{4-N}{2(N+8)}\epsilon$$

$$\beta = \frac{1}{2} - \frac{3}{2(N+8)}\epsilon$$

$$\gamma = 1 + \frac{N+2}{2(N+8)}\epsilon$$

$$\delta = 3 + \epsilon$$

These are interesting, if we look at β as $N \to \infty$ we loose dependence on ϵ.

8.3 Goldstone Bosons in $d = 2$

Before, when we did symmetry breaking, we got an associated Goldstone boson. When we go into our ordered phase, a direction was picked out in the $O(n)$ vector. This gives rise to a submanifold in our vacuum (an $n-1$ dimensional sphere). These bosons are massless - costs very little energy. Hence, we expect them to be relevant as, and so appear at all distance scales. They show up everywhere. At $d = 2$ something special happens. In our previous derivation, we didn't have much talk of dimension itself. However, it turns out that at $d = 2$, independent of the symmetry breaking patterns, the underlying physics seems to care about the number of dimensions. For the Ising model, we had $\mathbb{Z}_2$ discrete symmetries, not continuous. But, we showed that at $d = 1$, due to domain walls, we had no ordered phased since they dominated the physics (so $d = 1$ was the critical dimension). However, in $d = 2$, the vacua are not contnuously connected, so we had to use a domain wall. Domain walls cost us free energy. Fluctuations along the manifold in n - 1 mexican hat might break the ordered phase.

In the XY model, when we start, our expectation value of Goldstone modes $\langle \mathcal{O}(\mathbf{x}) \rangle = 0$. We can calculate our fluctuation specifically about the vacuum at $\theta = 0$ as

$$\left\langle [(\mathbf{x}) - \mathcal{O}(0)]^2 \right\rangle = 2\left\langle \mathcal{O}^2(\mathbf{x}) \right\rangle - 2\left\langle \mathcal{O}(\mathbf{x})\mathcal{O}(0) \right\rangle$$

We can calculate this two point function explicitly as

$$\langle \mathcal{O}(\mathbf{x})\mathcal{O}(0) \rangle = \frac{1}{\gamma M_0^2} \int_0^\Lambda \frac{d^d k}{(2\pi)^d} \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{k^2} \sim \begin{cases} \Lambda^{d-2} - r^{2-d} & d > 2 \\ \log(\Lambda r) & d = 2 \\ r - \Lambda^{-1} & d = 1 \end{cases}$$

Something weird is going on. If we look at the fluctuations, we see that as soon as we hit $d = 2$ the fluctuations start to become arbitrarily large at large distances! This is the manifestation

of something known as the Mermin-Wagner theorem, which states that a continuous symmetry cannot be spontaneously broken in $d = 2$. These Goldstone modes start to contribute some fishy business - they give arbitrarily large fluctuations.

8.3.1 σ models

In the 60's and 70's people were trying to understand pions in the context of QCD. People started noticing that these are very similar to Goldstone bosons - they're now called pseudo-Goldstone bosons. In the ordered phase, we have that

$$\|\phi\|\rangle \neq 0, \implies \text{Vacuum Manifold} \sim S^{N-1}$$

We have another type of mode, which are called longitudinal modes. These are gapped since the second derivative is non-zero.

We have that $\phi \cdot \phi = M_0^2$. We can rescale ϕ such that $\mathbf{n} \cdot \mathbf{n} = 1$. In terms of the free energy, we have that

$$F[n] = \int d^d x \frac{1}{2e^2} (\nabla n) \cdot (\nabla n), \quad e^2 = \frac{1}{\gamma M_0^2}$$

There's no potential term since in the vacua its set to zero since $V(\phi) = \frac{\lambda}{2}(\phi \cdot \phi - M_0^2)$. What does the path integral look like? The path integral is now

$$\mathcal{Z} = \int \mathcal{D}\mathbf{n} \delta(\mathbf{n} \cdot \mathbf{n} - 1) \exp\left(-\frac{1}{2e^2} \int d^d x \, (\nabla \mathbf{n}) \cdot (\nabla \mathbf{n})\right)$$

From dimensional analysis, we have that $\left[e^2\right] = 2 - d$. Now, if e is large, the exponential term is surpressed, so we have less fluctuations. We write

$$\mathbf{n}(\mathbf{x}) = (\pi(\mathbf{x}), \sigma(\mathbf{x}))$$

where we have that $\phi(\mathbf{x})$ is $n - 1$ dimensional. Our constraint gives us that

$$\sigma^2(\mathbf{x}) = 1 - \pi(\mathbf{x}) \cdot \pi(\mathbf{x})$$

Plugging this expression in, we get that the free energy as a function of π and σ takes the form

$$F[\phi(\mathbf{x})] = \int d^d x \, \frac{1}{2e^2} \left[(\nabla \mathbf{n})^2 + (\nabla \sigma)^2\right]$$

Substituting our constraint, give us

$$\cdots = \int d^d x \, \frac{1}{2e^2} \left[(\nabla \pi)^2 + \frac{(\phi \cdot \nabla \pi)^2}{1 - \pi \cdot \pi}\right]$$

When we expand, each interaction comes with an e term which tells us that e actually acts like a coupling constant.

8.3.2 Background Field Method

We want to understand the fate of these Goldstone bosons. The thing to do is to perform standard RG calculations. We could expand the second term, but this is hard. Polyakov however, devised a clever method to do this. What we do is set

$$n^a(\mathbf{x}) \to \tilde{n}^a(\mathbf{x})\text{long wavelength}, \quad \tilde{\mathbf{n}} \cdot \tilde{\mathbf{n}} = 1$$

We introduce the "frame fields" which is a basis of $N-1$ unit vectors which we call

$$e_\alpha^a(\mathbf{x}), \quad a = 1, \ldots N, \alpha = 1, \ldots N-1$$

We also impose the constraint that these are orthogonal to $\tilde{n}^a$, such that $\tilde{n}^a e_\alpha^a = 0, \forall a$. Also, we have that $e_\alpha^a e_\beta^a = \delta_{\alpha\beta}$.

Now we can introduce short wavelength modes which we call $\chi_\alpha(x)$, and hence write

$$n^a(\mathbf{x}) = \tilde{n}^a(1 - \chi^2(x))^{\frac{1}{2}} + \int_{n=1}^{N-1} \chi_\alpha(x)e_\alpha^a(x) \implies n^2 = 1$$

8.3.3 Path Integral

We have that

$$\mathcal{Z} = \int \mathcal{D}\tilde{n}\delta(\tilde{n}^2 - 1)\exp\left(-\frac{1}{2e^2}\int d^d x\,(\nabla\tilde{n})^2\right)\int \mathcal{D}\chi \exp\left(-\frac{1}{2e^2}\int d^d x\,(\nabla\chi)^2\right)e^{-F_I[\tilde{n},\chi]}$$

Our interaction term is given by

$$F_I\left[\tilde{n}^a, \chi_\alpha\right] = \frac{1}{2e^2}\int d^d x\,\left[-\chi^2(\nabla\tilde{n})^2 + \chi_\alpha\chi_\beta\nabla e_\alpha^a\nabla e_\beta^a + 2\nabla\tilde{n}^a\nabla(\chi_\alpha e_\alpha^a)\right]$$

We want to focus on $\left\langle e^{-F_I}\right\rangle \simeq 1 - \langle F_I\rangle + \ldots$. Taking the average, our first term is

$$\langle F_I\rangle = \frac{1}{2e^2}\int d^d x\,\left(-\delta_{\alpha\beta}(\nabla\tilde{n}^a)^2 + \nabla e_\alpha^a\nabla e_\beta^\alpha\right)\langle\chi_\alpha\chi_\beta\rangle$$

We can evaluate the two point function in the integral

$$\langle\chi_\alpha\chi_\beta\rangle = e^2\delta_{\alpha\beta}I_d$$

where

$$I_d = \frac{\Omega_{d-1}}{(2\pi)^d}\Lambda^{d-2}\begin{cases}\zeta - 1 & d = 1 \\ \log(\zeta) & d = 2 \\ 1 - 1\zeta^{2-d} & d \geq 3\end{cases}$$

Using the relation

$$\tilde{n}^a\tilde{n}^b + e_\alpha^a e_\alpha^b = \delta^{ab} \implies \nabla e_\alpha^a\nabla e_\alpha^a = \nabla\tilde{n}^a\nabla\tilde{n}^a + \ldots$$

Finally we have that

$$\langle F_I\rangle = (2 - N)I_d\int d^d x\frac{1}{2}(\nabla\tilde{n})^2$$

This gives us a correction

$$\frac{1}{(e')^2} = \frac{1}{e_0^2} + (2 - N)I_d$$

when $N = 2$ no change. We hence have that

$$\frac{1}{e^2(\zeta)} = \zeta^{d-2}\left[\frac{1}{e_0^2} + (2 - N)I_d\right]$$

As we go to IR, interacting thepry of Goldstones become more strongly coupled. This is why we don't get Goldstones in $d = 2$. if we did, we would get arbitrarily large fluctuations which destroy ordered states.

8.4   Asymptotic freedom and $d = 2 + \epsilon$

Differentiating with respect to s, we can calculate the beta function for this coupling as

$$B(e) = \frac{de}{ds} = (N - 2)\frac{e^3}{4\pi}, \text{ for } N \geq 3, B(e) > 0$$

Thus, our theory is weakly coupled in the ultraviolet limit. This phenomena is called asymptotic freedom. For $N = 2$, this vanishes. This reflects the fact that when we work in the $O(2)$ model, we have that the Goldstone bosons are strongly interacting. We see that for $d = 2$ this is relevant. There are many interesting facts that we can extract from this property. The Mermin-Wagner theorem states that in 2 dimensions, we don't expect to have an ordered phase due to strong interaction. In $d = 2$, the weakly coupled phase is unstable, since it becomes strongly coupled. Now, what about the case when $d = 2 + \epsilon$? Then, in this case, we can write our beta function as

$$\frac{de}{ds} = -\frac{\epsilon}{2}e + (N - 2)\frac{e^3}{4\pi}\Lambda^3$$

Compare this to the technique we applied earlier in doing the $d = 40\epsilon$ technique. There's a suggestion here that de/ds can vanish for some given value of e^*. There is indeed a fixed point, and this exists at

$$e_*^2 = \frac{2\pi\epsilon}{N - 2}\Lambda^{-\epsilon}$$

For $\epsilon \ll 1$, this occurs at small coupling, hence exists in the UV limit. We know that if we head to the UV from the IR, it will eventually hit this fixed point. Hence, this is a Wilson Fisher fixed point.

Just as before, we dubiously set $\epsilon = 1$.

8.4.1   Critical Exponents

It is tempting to identify that $\frac{1}{e^2} \sim \frac{1}{T}$, since this is the coefficient of the quadratic term. The fixed point e_*^2 is then identified as some critical temperature T_*. Linearising around the fixed point, we get that the beta function for the perturbative corrections satisfy

$$\frac{d(\delta e^2)}{ds} = \epsilon\delta e^2 \implies \text{ this gives us the scaling dimension } \Delta_t$$

Recall that $\xi \sim t^{-\nu}$, this implies that ... (on correlation length here) Recall that $[F] = 0$, so $\eta = \frac{\epsilon}{N-2}$ Now, plugging in $\epsilon = 1$, we get that our critical exponents satisfy

Table 3: caption

Exponent	η	ν
MF	0	$\frac{1}{2}$
$d = 4 - \epsilon$	0	0.61
$d = 2 + \epsilon$	0	0
Actual	0.0386	0.702

8.4.2 The Kosterlitz-Thouless Transition for $N = 2, d = 2$

We saw, that for $O(N \geq 3)$, Goldstone modes are interacting and gapped in $d = 2$. This is great, it's confirming the Mermin Wagner theorem that we have no spontaneous symmetry break. However, for the XY model with $N = 2$, things are more interesting. Consider the two-point function of the complex field. At high T, then the 2 point function looks like

$$\left\langle \psi^+(\mathbf{x})\psi(\mathbf{0}) \right\rangle = \frac{e^{-\frac{r}{\xi}}}{\sqrt{r}}, \quad \xi \sim \frac{1}{\mu^2}$$

At low temperature however, if we work in the ordered phase, writing $\psi \sim Me^{i\theta}$, where θ is the Goldstone boson. Our long distance physics is dominated by θ. To leading order, we have that $F[\theta] = \frac{1}{2e^2} \int d^2(\nabla\theta)^2$. At low temperature, we have that $e^2 \ll 1$. We can calculate the correlation function explicitly here

$$\langle \theta(\mathbf{x})\theta(\mathbf{0}) \rangle = -\frac{e^2}{2\pi} \log(\Lambda r)$$

Recall that we can approximate our integral as $\sim \int \frac{d^d k}{(2\omega)^d} \frac{e^{-ik \cdot x}}{k^2}$. But, our previous correlation function that we wanted to calculate is given by

$$\left\langle \psi^\dagger \psi(0) \right\rangle \sim \left\langle e^{-i\theta(x)} e^{i\theta(0)} \right\rangle = \left\langle e^{-i(\theta(\mathbf{x}) - \theta(0))} \right\rangle = e^{-\left\langle \frac{(\theta(x) - \theta(0))^2}{2} \right\rangle} = \frac{1}{r^\eta}$$

where $\eta = \frac{e^2}{2\pi}$. So, at high temperature T, we have exponential behaviour. At a low temperatures, we get a power low relation over a range of temperatures, since $e^2 \sim T$. This tells us that it seems like a phase transition occurs.

8.4.3 Vortices

Our field $\theta(x)$ is periodic. We can have field configurations which wind! This means that when we integrate around some closed curve.

$$\oint_C (\nabla\theta) \cdot d\mathbf{x} = 2\pi n$$

These have different names depending on our value of n. For $n = 1$, this is called a vortex. In $n = -1$, this is called an anti-vortex. These things are called topological defects. (Insert pictures of vortices here) The vortex field is given by

$$\nabla\theta(x) = \frac{n}{r^2} \begin{pmatrix} y \\ -x \end{pmatrix}$$

60

Free energy scales like $\sim n^2$, so $n = 1$ vortices are preferred.

$$F_{\text{vortex}} = \frac{1}{2e^2} \int d^2 x \, (\nabla \theta)^2 = \frac{\pi n^2}{e^2} \log \left(\frac{l}{a} \right) + F_{\text{core}}$$

L is the IR cutoff, and a is the UV cutoff. If we insert this into the path integral, we get that

$$\mathcal{P}(\text{vortex}) = \left(\frac{L}{a} \right)^2 \frac{e^{-F_{\text{vortex}}}}{\mathcal{Z}} = \frac{e^{-F_{\text{core}}}}{\mathcal{Z}} \left(\frac{L}{a} \right)^{2 - \frac{\pi}{e^2}}$$

For $e^2 > e^2_{KT} = \frac{\pi}{2}$, vortices are unsuppressed. e^2 is like T, so phase transitions at T_{KT}, and power law behaviour is destroyed. This is a topological effect - we need to see the global picture. The Kosterlitz-Thouless transition is responsible for

- Superfluid films

- Melting of 2D lattices

This is a topological phase transition because it's driven by topological defects.

9 Example Sheet 1

9.1 Question 1

In this question, we're examining the partition function of a one dimensional, periodic N atom lattice in the Ising model. Summing over all possible states $s_i = \pm 1$, with the boundary condition that $s_0 = s_N$, it's easy to see that the partition function takes the form

$$Z = \sum_{s_1 = \pm 1} \cdots \sum_{s_N = \pm 1} \prod_i \exp\left(\beta J s_i s_{i+1} + \beta B(s_i + s_{i+1})\right).$$

Note that we've transferred the inner sum for our expression of E into the outer product of exponentials to make this object easier to deal with.

Given that the transfer matrix is

$$T = \begin{pmatrix} e^{\beta J - \beta B} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J + \beta B}. \end{pmatrix}$$

we can index this matrix in a slightly different way which reflects spins. We index the elements in the array with ± 1 instead of perhaps what we're used to with $\{0, 1\}$. Using the indices s_i, s_{i+1}, we can write that

$$T_{s_i, s_{i+1}} = \exp\left(\beta s_i s_{i+1} + \frac{\beta B}{2}(s_i + s_{i+1})\right)$$

If we're trying to find out what T^N is then in standard index notation,

$$\left(T^N\right)_{s_1, s_{N+1}} = T_{s_1, s_2} T_{s_2, s_3} \dots T_{s_N, s_{N+1}}$$

where we sum implicitly over pairs of identical indices. Due to our periodicity condition, we have that $s_1 = s_{N+1}$, and since we're taking the trace we have

$$Tr(T^N) = \left(T^N\right)_{s_1, s_1} = \sum_{s_1 = \pm 1} T_{s_1, s_2} T_{s_2, s_3} \dots T_{s_N, s_1}$$

where we've kept the sum over s_1 explicit here. Making the sum over the rest of the terms explicit, and substituting in our expressions, for $T_{s_i, s_{i+1}}$, gives the required result.

Our characteristic equation for T is

$$0 = (e^{\beta J - \beta B} - \lambda)(e^{\beta J + \beta B} - \lambda) - e^{-2\beta J}$$

Expanding out, we have the equation

$$0 = \lambda^2 - \lambda e^{\beta J}(e^{\beta B} + e^{-\beta B}) + e^{2\beta J} - e^{-2\beta J}$$

Using cosh and sinh expresions, and the quadratic formula, this is

$$0 = \lambda^2 - 2\lambda \cosh(\beta B) + 2\sinh(\beta B) \implies \lambda_\pm = e^{\beta J} \cosh(\beta B) \pm \sqrt{e^{2BJ} \cosh^2(\beta B) - 2\sinh(\beta B)}$$

Now note that, if λ is an eigenvalue of A, then λ^n is an eigenvalue of A^n, since repeated application of A on an eigenvector pulls out n factors of λ.

Since the matrix T^n is two dimensional, it has at most 2 eigenvectors. We know that, for the previous reason, that λ_+^n and λ_-^n are eigenvalues of T^n, so they're all of the eigenvalues. Since the trace of a matrix is the sum of its eigenvalues,

$$Tr(T^N) = \lambda_+^N + \lambda_-^N = \lambda_+^N(1 + (\frac{\lambda_-}{\lambda_+})^N) \to \lambda_+^N, \quad N \to \infty$$

The term in the bracket goes to 1 since in this case $|\lambda_-|$ is less than $|\lambda_+|$. Asymptotically as $N \to \infty$, we have that

$$m = \frac{1}{N\beta}\frac{\partial \log Z}{\partial B} \sim \frac{1}{N\beta}N\frac{\partial \log \lambda_+}{\partial B}$$

But we have that

$$\frac{\partial \lambda_+}{\partial B} = e^{BJ}\beta \sinh(\beta B) + \frac{\beta e^{2\beta J} \cosh(\beta B)\sinh(\beta B)}{\sqrt{e^{2BJ}\cosh^2(\beta B) - 2\sinh(\beta B)}}$$

But this is just 0 at $B = 0$ since we can factor out $\sinh(\beta B)$. Hence, our magnetisation at equilibrium is 0 regardless of temperature. So our free energy is fixed at $f(0)$, which is constant and therefore continuous. So, we have no phase transitions based on temperature.

9.2 Question 2

Our energy expression for the Ising model is

$$E = -B \sum_i s_i - J \sum_{\langle ij \rangle} s_i s_j$$

where in the last term we sum across neighbouring pairs of atoms on the lattice. Writing out

$$s_i = (s_i - m) + s_i$$

where we now assume $(s_i - m)$ is small, we have that

$$
\begin{aligned}
s_i s_j &= (s_i - m + m)(s_j - m + m) \\
&= (s_i - m)(s_j - m) + m^2 + m(s_j - m + s_i - m) \\
&\approx m(s_i + s_j) - m^2.
\end{aligned}
$$

Thus we have

$$\sum_{\langle ij \rangle} s_i s_j = \sum_{\langle ij \rangle} \left(m(s_i + s_j) - m^2 \right) = qm \sum_i s_i - \frac{m^2 qN}{2}$$

Hence we have that

$$E = -(B + Jqm) \sum_i s_i + \frac{JqmN^2}{2}$$

Since we've gotten rid of our interaction term and can factor this expression out in s_i, all we have to do is calculate Z for a single particle only, then take this to the power N. This is because the partition function is factorable. For a single particle, we sum over both spin configurations

$$Z_i = e^{-\beta Jqm^2/2} \left(e^{-\beta B_{\text{eff}}} + e^{\beta B_{\text{eff}}} \right) = 2e^{-\beta Jqm^2/2} \cosh(\beta B_{\text{eff}}), \quad B_{\text{eff}} = B + Jqm$$

For the whole system we get that

$$Z = e^{-\beta JNqm^2/2} 2^N \cosh^N(\beta B_{\text{eff}})$$

Using our expression that the partition function is well approximated by

$$Z \approx e^{-\beta N f(m_{\min})}$$

we equate this to our expression above and find that our free energy as a function of magnetisation is

$$f(m) = \frac{1}{2} Jqm^2 + \frac{1}{\beta} \log 2 \cosh(\beta Jqm + \beta B)$$

Minimising this function gives us the self-consistency equation

$$\frac{\partial f}{\partial m} = 0 \implies m = \tanh(\beta(Jqm + B)).$$

9.3 Question 3

Setting $m = \frac{1}{N} \sum_i s_i$, our expression for E is

$$E = -B \sum_i s_i - \frac{J}{2N} \left(\sum_i s_i \right) \left(\sum_j s_j \right) = -BNm - \frac{JqNm^2}{2}$$

If the extra divisor of N wasn't there, we'd have a factor of N^2 in our second term, which means that our expression for our free energy $f(m)$ would depend on N.

As before, we apply mean field theory and remove small terms quadratic in $s_i - m$ to find that

$$E = -B \sum_i s_i + \frac{JNm^2}{2} - Jm \sum_i s_i = -(B + Jm) \sum_i s_i + \frac{JNm^2}{2}.$$

so our partition function is reduced to the partition function for the neighbour Ising model with $q = 1$:

$$Z = e^{-\beta JNm^2/2} 2^N \cosh^N(\beta(Jm + B)).$$

We verify the identity in the question by completing the square in the exponent

$$\sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} e^{-N\beta J x^2/2 + \alpha \beta J x} dx = \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{N\beta J}{2}((x - \frac{\alpha}{N})^2 - \frac{\alpha^2}{N^2})} dx$$

$$= \sqrt{\frac{N\beta J}{2\pi}} e^{\frac{\alpha^2 \beta J}{2N}} \int_{-\infty}^{\infty} e^{-\frac{N\beta J}{2}(x - \frac{\alpha}{N})^2}$$

$$= e^{\frac{\alpha^2 \beta J}{2N}}$$

Our energy is of the form

$$E = -B \sum_i s_i - \frac{J}{2N} \left(\sum_i s_i \right) \left(\sum_j s_j \right)$$

but to use mean field theory, we set

$$\sum_i s_i = Nm \implies E = -BNm - \frac{JNm^2}{2}$$

so we have that for a given magnetisation m,

$$e^{-\beta E(m)} = \exp \left(\beta BNm + \frac{\beta JNm^2}{2} \right)$$

From the previous identity, we have that setting $\alpha = 0$ gives us the identity

$$1 = \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} dx \exp \left(-N\beta J \frac{x^2}{2} \right)$$

We do a clever translation by the constant m so that

$$1 = \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} dx \, e^{-N\beta J(x-m)^2/2}$$

and once we translate this, we can multiply this expression by $e^{-\beta E(m)}$

$$e^{-\beta E(m)} = \int_{-\infty}^{\infty} dx \, e^{-N\beta J(x-m)^2/2} e^{\beta BNm} e^{\beta JNm^2/2}.$$

which simplifies to

$$e^{-\beta E(m)} = \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} dx \, e^{-\frac{N\beta Jx^2}{2}} e^{-\beta Nm(Jx+b)}$$

We rewrite this in terms of our decoupled spin sums to get

$$e^{-\beta E} = \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} e^{-N\beta Jx^2/2} e^{-\beta\left(\sum_i s_i\right)(Jx+B)}$$

Our partition function Z comes from all possible configurations of our spins.

$$Z = \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} dx \, e^{-N\beta Jx^2/2} \sum_{\{s_i\}} e^{-\beta \sum_i s_i(Jx+B)}$$

We can use the fact that we have a separable form here to factor out the sum over all possible spin configurations.

$$\sum_{\{s_i\}} e^{-\beta \sum_i s_i(Jx+B)} = \prod_i \sum_{s_i=\pm 1} e^{-\beta \sum_i s_i(Jx+B)} = (2\cosh(Jx + B))^N$$

Substituting this into out integral gives us the result that we want. Items:

- Tried expanding the exponential representation of cosh binomially.

- Messing around with substitutions?

9.4 Question 4

Sketching out our functions

We're interested in the behaviour of the function

$$f(m) = \alpha_2(T)m^2 + \alpha_4 m^4 + \alpha_6 m^6, \quad \alpha_4 < 0$$

We first differentiate and see where roots exist.

$$f'(m) = 2m(\alpha_2 + 2\alpha_4 m^2 + 3\alpha_6 m^4)$$

This implies that $m = 0$ is always a stationary point. We focus our attention to solving for roots of the quadratic factor. Without caring abotu signs for now, solving the equation as a quadratic in m^2 then square rooting gives

$$m = \pm\sqrt{-\frac{\alpha_4}{3\alpha_6} \pm \frac{1}{3\alpha_6}\sqrt{\alpha_4^2 - 3\alpha_6\alpha_2}}$$

Now, the number of real solutions depends on signs. This equation has no real solutions when $\alpha_2 > \frac{\alpha_4}{3\alpha_6}$. Thus, $m = 0$ is our only stationary point. Our function is
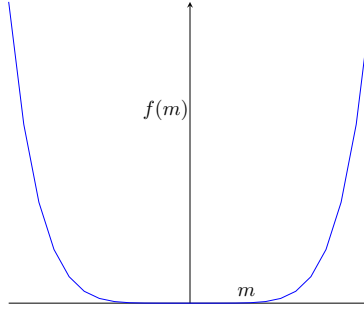
After this, we have only two real solutions in the case when $\alpha_2 < 0$, is this implies that only $\frac{-\alpha_4}{3\alpha_6} + \frac{1}{3\alpha_6}\sqrt{\alpha_4^2 - 3\alpha_6\alpha_2}$ is positive in the square root. This means our only stationary points are at 0 and two other places. we know it's this shape since around zero, m^2 term dominates and we should have a negative quadratic.
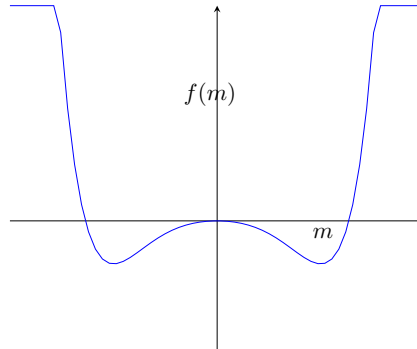

Figure 9: This our metastable state disappears when $\alpha_2(T)$ is below 0!

In our final case we have that for $0 < \alpha_2 < \frac{\alpha_4^2}{3\alpha_6}$ we have five stationary points including zero. The only possible configurations for stationary points is that $m_0 = 0, m_\pm = \pm\sqrt{-\frac{\alpha_4}{3\alpha_6} + \frac{1}{3\alpha_6}\sqrt{\alpha_4^2 - 3\alpha_6\alpha_2}}$ must be minima, and the other points are maxima.

Our condition for $m_\pm$ to be **global** minima (not metastable states) is that we must have that $f(m_\pm)$ must be smaller than $f(m_0) = 0$. Algebra shows that

$$m_\pm^2 = -\frac{\alpha_4}{\alpha_6} + \frac{1}{3\alpha_6}\sqrt{\alpha_4^2 - 3\alpha_6\alpha_2}$$

$$m_\pm^4 = \frac{1}{9\alpha_6^2}(2\alpha_4^2 - 3\alpha_6\alpha_2 - 2\alpha_4\sqrt{\alpha_4^2 - 3\alpha_6\alpha_2}$$

So substituting this into $f(m)$ gives

$$f(m_\pm) = m_\pm^2(\alpha_2 + \alpha_4\left(-\frac{\alpha_4}{3\alpha_6} + \frac{1}{3\alpha_6}\sqrt{\alpha_4^2 - 3\alpha_6\alpha_2}\right)$$

$$+ \frac{\alpha_6}{9\alpha_6^2}\left(2\alpha_4^2 - 3\alpha_6\alpha_2 - 2\alpha_4\sqrt{\alpha_4^2 - 3\alpha_6\alpha_2}\right))$$

$$= m_\pm^2(\frac{2\alpha_2}{3} - \frac{\alpha_4^2}{9\alpha_6} + \frac{\alpha_4}{9\alpha_6}\sqrt{\alpha_4 - 3\alpha_6\alpha_2})$$

Hence, we require that the term in the brackets is negative. If we multiply by $9\alpha_6 > 0$, our condition becomes that

$$6\alpha_2\alpha_6 - \alpha_4^2 + \alpha_4\sqrt{\alpha_4^2 - 3\alpha_6\alpha_2} < 0$$

$$6\alpha_2\alpha_6 - \alpha_4^2 < -\alpha_4\sqrt{\alpha_4^2 - 3\alpha_2\alpha_6}$$

Squaring both sides, cancelling an α_4^4 in the sum on each side, and then cancelling α_2 gives us the condition that

$$36\alpha_2\alpha_6^2 - 12\alpha_4^2\alpha_6 < -3\alpha_4^2\alpha6 \implies \alpha_2 < \frac{\alpha_4^2}{4\alpha_6}$$

So, we get that our magnetisation jumps by $m_\pm$ only when these points become a global minimum, at $\alpha_2 = \frac{\alpha_4^2}{4\alpha_6}$

Because of this jump, we have a phase transition.

Showing a first order phase transition

For ease of notation let's denote the global minimum of f as m_0 in either case. When $\alpha_2 > \alpha_c = \frac{\alpha_4^2}{4\alpha_6}$, we have $f(m_0) = 0$. Let's write our the energy in terms of m_0 and differentiate with respect to the parameter α_2, denoted with an apostrophe. We have

$$f(m_0) = \alpha_2(T)m_0^2 + \alpha_4 m_0^4 + \alpha_6 + m_0^6$$

$$f'(m_0) = 2\alpha_2 m_0 m_0' + m_0^2 + 4\alpha_4 m_0^3 m_0' + 6\alpha_6 m_0^5 m_0'$$

$$= m_0^2 + 2m_0 m_0'(\alpha_2 + 2\alpha_4 m_0^2 + 3\alpha_6 m_0^4)$$
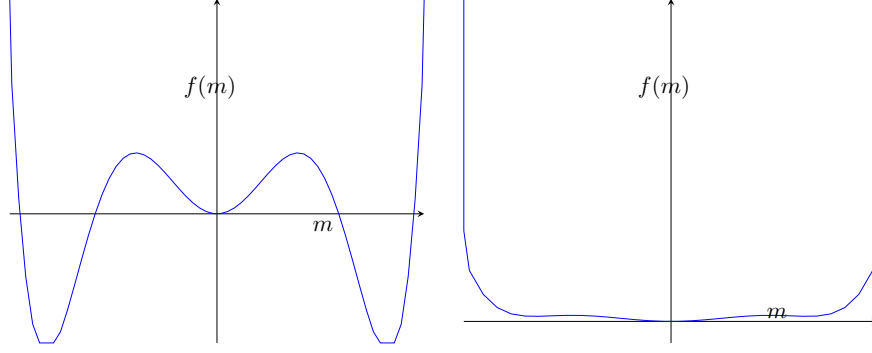
$$= m_0^2$$

Figure 10: Left: when $\frac{\alpha_4^2}{4\alpha_6} > \alpha_2 > 0$; we have a metastable state at $m = 0$. On the right is what happens otherwise.

We took the last term to zero since m_0 solved this equation earlier. Now, if we take the limit as $\alpha_2 \to \frac{\alpha_4^2}{4\alpha_6}$, it's easy to verify that

$$m_0 = -\frac{\alpha_4}{3\alpha_6} + \frac{1}{3\alpha_6}\sqrt{\alpha_4^2 - 3\alpha_2\alpha_6} \to -\frac{1}{6}\frac{\alpha_4}{\alpha_6}$$

This is non zero. So, we have a discontinuity as we vary $f(m_0)$ with respect to this parameter.

Critical exponents

At $\alpha_4 = 0$, we have that

$$m \sim (-\alpha_2)^{\frac{1}{4}}$$

The question is, how do we get a critical exponent from this with the condition that $\alpha(T_c) = 0$? Critical exponents occur when $T \to T_c$, which means that $(T - T_c)^2$ is small. We can Taylor expand α_2 around the critical temperature as

$$\alpha_2(T) = \alpha_2(T_c) + (T - T_c)\alpha_2'(T_c) + \frac{(T - T_c)^2}{2}\alpha_2''(T_c) + \cdots \sim (T - T_c) \text{ at leading order}$$

Substituting this in the expression above, we then must have that

$$m \sim (T_c - T)^{\frac{1}{4}}$$

which implies that our critical exponent $\beta = \frac{1}{4}$ Now, for heat capacity, assuming that $\alpha_2 = K(T - T_c)$ to leading order, we have that our free energy can be written as

$$f(m) = \frac{K^{\frac{3}{2}}}{(3\alpha_6)^{\frac{1}{2}}}(T_c - T)^{\frac{3}{2}} + \frac{1}{3\alpha_6}(T_c - T)^3$$

Now we can calculate our critical exponent for heat capacity! Heat capacity per unit volume is defined as

$$c = \frac{1}{N}\beta^2\frac{\partial}{\partial\beta^2}\log Z = \left(T^2\frac{\partial^2}{\partial T^2} + 2T\frac{\partial}{\partial T}\right)\frac{1}{T}f(m)$$

However, recall that since we're eventually taking a limit as $T \to T_c$, we only need to find the leading order contribution of $(T_c - T)^k$ from our expression above. This is akin to just differentiating the $(T_c - T)terms$, and not caring about anything else. So, differentiating the first term twice in our expression for $f(m)$ gives a leading order contribution of $(T_c - T)^{-\frac{1}{2}}$, so $\alpha = \frac{1}{2}$.

To calculate δ, observe that at the critical point with $\alpha_2 = 0$, our free energy is given by

$$f(m) = -Bm + \alpha_6 m^6$$

Solving for $f'(m) = 0 \implies \delta = 5$.

To calculate our magnetisation constant, we define our new free energy as

$$g(m) = -Bm + \alpha_2 m^2 + \alpha_6 m^6$$

Consider the case when $\alpha_2 < 0$. We want to find our new minimum point such that this function is minimised. We work perturbatively. The minimum point $m_0 \neq 0$ for our free energy without magnetic field satisfies

$$f(m) = \alpha_2 m^2 + \alpha_6 m^6 \implies 0 = f'(m_0) = 2m_0(\alpha_2 + 3\alpha_6 m_0^4)$$

We have that

$$g'(m) = -B + 2m(\alpha_2 + 3\alpha_6 m^4)$$

We assume that our minimum point for small B takes the form

$$m' = m_0 + \epsilon$$

So, to first order in ϵ, we have

$$\begin{aligned}
0 = g'(m') &= -B + (m_0 + \epsilon)(\alpha_2 + 3\alpha_6(m_0 + \epsilon)^4) \\
&= -B + 2(m_0 + \epsilon)(\alpha_2 + 3\alpha_6 m_0^4 + 12\alpha_6\epsilon m_0^4) \\
&= -B + 2(m_0 + \epsilon)(0 + 12\alpha_6\epsilon m_0^3) \\
&= -B + 24m_0^4\epsilon\alpha_6
\end{aligned}$$

So, we have that

$$\epsilon = \frac{B}{24m_0^4\alpha_6} = \frac{B}{24\alpha_2\alpha_6} \sim \frac{B}{24(T_c - T)\alpha_6}$$

Hence, our equilibrium magnetisation is

$$m' = m_0 + \frac{B}{24(T_c - T)\alpha_6} \implies \xi = \frac{\partial m'}{\partial B} \sim \frac{1}{T_c - T}$$

This was for the case $\alpha_2 < 0$, where $m_0 \neq 0$. Otherwise, when $\alpha_2 > 0$, $m \sim 0$ and thus we can truncate g

$$g(m) = -Bm + \alpha_2 m^2 \implies m' = \frac{B}{2\alpha_2} \implies \xi \sim \frac{1}{T - T_c}$$

where m' is simply found by differentiating g and setting to 0. Thus, on either side of the phase,

$$\xi \sim \frac{1}{|T - T_c|} \implies \gamma = 1$$

9.5 Question 5

A rough physical model

In this question, we're exploring the phase transitions one has with an Ising model which takes our whole range of unit vectors as spins. We define an average magnetisation with the vector **m** with magnitude m, which obeys

$$\mathbf{m} = \frac{1}{N} \sum_i \mathbf{s}_i$$

Substituting this into our energy, much like we did for the Ising model, we have that

$$E = -Jq \sum_{\langle ij \rangle} \mathbf{s}_i \cdot \mathbf{s}_j + g \sum_i \left((s_z)_i^2 - \frac{1}{2}((s_x)_i^2 + (s_y)_i^2) \right)$$

$$\sim -JqN^2 m^2 + N^2 g \left(m_z^2 - \frac{1}{2}(m_y^2 + m_x^2) \right)$$

Heuristically, and without much thought, we can identify the form of this model to the free energy that we get with the Ising model. Our best guess is that

$$f(m, g, T) = \alpha_2(-2g, T)m_z^2 + \alpha_2(g, T)(m_x^2 + m_y^2) + \alpha_4 m^4 + \dots$$

We've put in a factor of -2 in the first coefficient to account for the different proportions that each component has. Now, at first glance we have that (assuming the temperature is not too high such that magnetisations become disordered);

- For $g > 0$ it is energetically favourable for magnetisations to lie in the (x, y) plane due to the negative $(m_y^2 + m_x^2)$ term, and hence our set of states which minimise free energy is a contour in the plane is the limit as $g \to \infty$. This is completely analogous to the XY ordered case that we encountered in the previous question, where we can identify m_x, m_y with a complex order parameter

$$\psi = m_x + im_y$$

  In the previous question, we showed that in the XY model, heat capacity is discontinuous as a result of differentiation twice with respect to β. This means that we have a second order phase transition when we increase the temperature.

- For $g < 0$ it is energetically favourable for our spins to lie in the z direction, and, ignoring magnetisations m_x, m_y, we have that our energy is of the form

  $E \sim N^2(g - Jq)m_z^2$, akin to the Ising model but without a magnetic field

  Thus, we have an Ising type order here, and thus we again have a second order phase transition when we increase the temperature.

We can explicitly calculate that our transition is a first order phase transition when we move from $g < 0$ to $g > 0$. In the regime with $g > 0$, ignoring m_z, we have that

$$f(m, g, T) = \alpha_2(g, T)(m_x^2 + m_y^2) + \alpha_4(m_x^2 + m_y^2)^2 \implies m_{min} = \sqrt{\frac{-\alpha_2(g, T)}{2\alpha_4}}$$

Thus, our free energy with $g > 0$ is given by

$$f_{min} = \begin{cases} -\frac{\alpha_2^2(g,T)}{4\alpha_4} & g > 0 \\ -\frac{\alpha_2^2(-2g,T)}{4\alpha_4} & g < 0 \end{cases}$$

To get the order of our phase transition, we differentiate the free energy with respect to the parameter g, so that

$$\frac{df}{dg} = \begin{cases} -2\frac{\alpha_2(g,T)\alpha_2'(g,T)}{4\alpha_4} & g > 0 \\ 4\frac{\alpha_2(g,T)\alpha_2'(-2g,T)}{4\alpha_4} & g < 0 \end{cases}$$

Now, as $g \to 0$, this is discontinuous in general since we have a factor of 2 multiplier in the $g < 0$ regime.

9.6   Question 5

A rough physical model

In this question, we're exploring the phase transitions one has with an Ising model which takes our whole range of unit vectors as spins. We define an average magnetisation with the vector **m** with magnitude m, which obeys

$$\mathbf{m} = \frac{1}{N} \sum_i \mathbf{s}_i$$

Substituting this into our energy, much like we did for the Ising model, we have that

$$E = -Jq \sum_{\langle ij \rangle} \mathbf{s}_i \cdot \mathbf{s}_j + g \sum_i \left((s_z)_i^2 - \frac{1}{2}((s_x)_i^2 + (s_y)_i^2) \right)$$

$$\sim -JqN^2 m^2 + N^2 g \left(m_z^2 - \frac{1}{2}(m_y^2 + m_x^2) \right)$$

Heuristically, and without much thought, we can identify the form of this model to the free energy that we get with the Ising model. Our best guess is that

$$f(m, g, T) = \alpha_2(-2g, T)m_z^2 + \alpha_2(g, T)(m_x^2 + m_y^2) + \alpha_4 m^4 + \dots$$

We've put in a factor of -2 in the first coefficient to account for the different proportions that each component has. Now, at first glance we have that (assuming the temperature is not too high such that magnetisations become disordered);

- For $g > 0$ it is energetically favourable for magnetisations to lie in the (x, y) plane due to the negative $(m_y^2 + m_x^2)$ term, and hence our set of states which minimise free energy is a contour in the plane is the limit as $g \to \infty$. This is completely analogous to the XY ordered case that we encountered in the previous question, where we can identify m_x, m_y with a complex order parameter

$$\psi = m_x + i m_y$$

  In the previous question, we showed that in the XY model, heat capacity is discontinuous as a result of differentiation twice with respect to β. This means that we have a second order phase transition when we increase the temperature.

- For $g < 0$ it is energetically favourable for our spins to lie in the z direction, and, ignoring magnetisations m_x, m_y, we have that our energy is of the form

  $$E \sim N^2(g - Jq)m_z^2, \text{ akin to the Ising model but without a magnetic field}$$

  Thus, we have an Ising type order here, and thus we again have a second order phase transition when we increase the temperature.

We can explicitly calculate that our transition is a first order phase transition when we move from $g < 0$ to $g > 0$. In the regime with $g > 0$, ignoring m_z, we have that

$$f(m, g, T) = \alpha_2(g, T)(m_x^2 + m_y^2) + \alpha_4(m_x^2 + m_y^2)^2 \implies m_{min} = \sqrt{\frac{-\alpha_2(g, T)}{2\alpha_4}}$$

Thus, our free energy with $g > 0$ is given by

$$f_{min} = \begin{cases} -\frac{\alpha_2^2(g,T)}{4\alpha_4} & g > 0 \\ -\frac{\alpha_2^2(-2g,T)}{4\alpha_4} & g < 0 \end{cases}$$

To get the order of our phase transition, we differentiate the free energy with respect to the parameter g, so that

$$\frac{df}{dg} = \begin{cases} -2\frac{\alpha_2(g,T)\alpha_2'(g,T)}{4\alpha_4} & g > 0 \\ 4\frac{\alpha_2(g,T)\alpha_2'(-2g,T)}{4\alpha_4} & g < 0 \end{cases}$$

Now, as $g \to 0$, this is discontinuous in general since we have a factor of 2 multiplier in the $g < 0$ regime.

9.7 Question 7

We substitute our expression of $\psi = A_k e^{-ikx}$ into our expression for free energy. This gives us the free energy integral

$$F = \int d^3x \ \alpha_2(T)|A_k|^2 + \alpha_4|A_k|^4 - k^2\gamma|A_k|^2 + \kappa k^4|A_k|^2$$

Now, assuming our mode satisfies $\frac{dA_k}{dk} = 0$, differentiating the integrand with respect to k gives us a minimum value of k at

$$k_0^2 = \frac{\gamma}{2\kappa}$$

Substituting this back into our integral gives us a thermodynamic free energy of

$$F = \int dx \left(\alpha_2(T) - \frac{\gamma^2}{4\kappa}\right)|A_k|^2 + \alpha_4|A_k|^4$$

Now, we can view F as a function of the complex parameter $|A_k|$. We find that if $\alpha_2 > \frac{\gamma^2}{4\kappa}$, then our to minimise our free energy we have to force $|A_k| = 0$ Hence $F = 0$ and our field configuration $\psi = 0$. However, in the case where $\alpha_2 < \frac{\gamma^2}{4\kappa}$, we have that our parameter $|A_k|$ is minimised at

$$|A_k| = \pm\sqrt{\frac{\frac{\gamma^2}{4\kappa} - \alpha_2(T)}{2\alpha_4}}, \quad \psi \neq 0$$

But this means precisely that $\psi \sim e^{ikx}$ is spatially modulated.

9.8 Techniques in Statistical Field Theory

9.8.1 Thinking about scaling

Always explore scaling dimensions! Remember to think carefully about how the Jacobian in the measure scales.

9.8.2 Terms in our free energy

We should always think about terms in our free energy up to scaling.

9.9 Wick's Theorem

To prove this, just run through the definitions. Use symmetry arguments to wash away terms from Wick's theorem. Remember to take into account symmetry factors.