Notes on Quantum Field Theory II

Afiq Hatta

February 4, 2020

Contents

1 Path Integrals

In this section, we'll introduce the path integral in QM, look at some methods with integrals, and then explore Feynman rules. Throughout these notes, we'll leave $\hbar$, but feel free to set this to 1 throughout the course. Let's introduce the path integral from the standpoint of quantum mechanics. The goal here is to take Schrödinger's equation and reformulate it into a "path integral", which is roughly speaking, a weighted integral summing over all probable paths. Let's consider a Hamiltonian in just one dimension, which as usual we can decompose into a kinetic and potential term

$$\hat{H} = H\left(\hat{x}, \hat{p}\right) = \frac{\hat{p}^2}{2m} + V\left(\hat{x}\right), \quad \text{with } [\hat{x}, \hat{p}] = i\hbar$$

Schrödinger's equation says thst if we have a state, its time evolution is governed the equation below, which we write as its formal solution by

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \overline{H} |\psi(t)\rangle$$

Our formal solution is given by mutliplying by the time evolution operator

$$|\psi(t)\rangle = e^{-iH\frac{t}{\hbar}} |\psi(0)\rangle$$

In the Schrodinger picture, we have that

- States evolve in time
- Operators and their eigenstates are constant in time (fixed).

Wavefunctions in position space are denoted

$$\psi(x, t) = \langle x|\psi(t)\rangle$$

This gives Schrodingers equation as

$$\langle \hat{x}| \overline{H} |\psi(t)\rangle = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x, t)$$

How do we convert this differential equation into an integral equation? We have that

$$\begin{aligned}
\psi(x, t) &= \langle x| e^{-iH\frac{t}{\hbar}} |\psi(0)\rangle \\
&= \int_{-\infty}^{\infty} \langle x| e^{\frac{-iHt}{\hbar}} |x_0\rangle \langle x_0|\psi(0)\rangle \\
&= \int_{-\infty}^{\infty} dx_0 K(x, x_0; t)\psi(x_0, 0)
\end{aligned}$$

We can introduce an integral quite straightforwardly by introducing a projection operator onto initial positions. We insert a complete set of states

$$I = \int dx_0 |x_0\rangle \langle x_0|$$

We call $K(x, x_0; t)$ the Kernel. Repeat this procedure for n intermediate times and positions. Let $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$, and we factor

$$e^{-\frac{iHT}{\hbar}} = e^{-\frac{i}{\hbar}\overline{H}(t_{n+1}-t_n)} \dots e^{-\frac{i}{\hbar}\overline{H}(t_1-t_0)}$$

Then,

$$K(x, x_0, T) = \int_{-\infty}^{\infty} \left[\prod_{r=1}^{n} dx_r \left\langle x_{r+1} \right| e^{-\frac{i\overline{H}}{\hbar}(t_{r+1}-t_r)} \left| x_r \right\rangle \right] \left\langle x_1 \right| e^{-\frac{i\overline{H}}{\hbar}(t_1-t_0)} \left| x_0 \right\rangle$$

Integrals are over all possible position eigenstates at times $t_r, r = 1, \dots n$. Consider a free theory, with $V(\overline{x}) = 0$. Let's define a corresponding free kernel

$$K_0 \left(x, x'; t \right) = \left\langle x \right| \exp \left(\frac{i\hat{p}^2}{2m} t \right) \left| x' \right\rangle$$

Insert, on the right side, the completeness relation for the identity.

$$I = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} \left| p \right\rangle \left\langle p \right|, \left\langle x | p \right\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}}$$

This gives

$$K_0 \left(x, x'; t \right) = \int_{\infty}^{\infty} \frac{dp}{2\pi\hbar} e^{-ip^2 t/2m\hbar} e^{ip(x-x')/\hbar} = e^{\frac{im(x-x')^2}{2\hbar t}} \sqrt{\frac{m}{2\pi i\hbar t}}$$

Note,

$$\lim_{t \to 0} K_0 \left(x, x'; t \right) = \delta(x - x')$$

which is as expected from $\left\langle x | x' \right\rangle = \delta(x - x')$. From the Baker-Campbell-Haudorf formula, we have that

$$e^{\epsilon\hat{A}} e^{\epsilon\hat{B}} = \exp \left(\epsilon\overline{A} + \epsilon\overline{B} + \frac{\epsilon^2}{2} \left[\overline{A}, \hat{B} \right] + \dots \right) \neq e^{\epsilon\left(\hat{A}+\hat{B}\right)}$$

For small ϵ, we have that

$$e^{\epsilon(\hat{A}+\hat{B})} = e^{\epsilon\hat{A}} e^{\epsilon\hat{B}} \left(1 + O(\epsilon^2) \right)$$

Now let $\epsilon = \frac{1}{n}$, raise the above to the n power, so we have the result that

$$e^{\hat{A}+\hat{B}} = \lim_{n \to \infty} \left(e^{\hat{A}/n} e^{\hat{B}/n} \right)^n$$

Take $t_{r+1} - t_r = \delta t$, with $\delta t \ll T$. Also take n large such that $n\delta t = T$. Then we can write that

$$e^{\frac{-i\hat{H}\delta t}{\hbar}} = \exp \left(\frac{-i\hat{p}^2 \delta t}{2m\hbar} \right) \exp \left(-\frac{iV(\hat{x})\delta t}{\hbar} \right) \left[1 + O(\delta t)^2 \right]$$

Writing out the above, this gives us

$$\left\langle x_{r+1} \right| \exp \left(-\frac{i\hat{H}\delta t}{\hbar} \right) \left| x_r \right\rangle = e^{-iV(\hat{x})\delta t/\hbar} K_0(x_{r+1}, x_r; \delta t) = \sqrt{\frac{m}{2\pi i\hbar\delta t}} \exp \left[\frac{im}{2\hbar} \left(x_{r+1} - x_r/\delta t \right)^2 \delta t - \frac{i}{\hbar} V(x_r) \delta t \right]$$

with $T = n\delta t$. This gives our final expression as

$$K\left(x, x_0; T \right) = \int \left[\prod_{r=1}^{r} dx_r \right] \left(\frac{m}{2\pi i\hbar\delta t} \right)^{n+1/2} \exp \left[i \sum_{r=0}^{n} \left[\frac{m}{2} \left(\frac{x_{r+1} - x_r}{\delta t} \right)^2 - \frac{1}{\hbar} V(x_r) \right] \delta t \right]$$

In the limit $n \to \infty$, $\delta t \to 0$, with $n\delta t = T$, the exponent becomes

$$\frac{1}{\hbar}\int_0^T \left[\frac{1}{2}m\dot{X}^2 - V(x)\right] = \int_0^T dtL\left(x, \dot{x}\right)$$

where L is our classical Lagrangian. The classical action $S = \int dtL(x, \dot{x})$. The path integral

$$K\left(x, x_0; t\right) = \langle x| e^{-\frac{i\hat{H}t}{\hbar}} |x_0\rangle = \int \mathcal{D}x e^{\frac{i}{\hbar}S}$$

The functional integral $\mathcal{D}x = \lim_{\delta t \to 0, n\delta T \text{ fixed}} (...) \prod_{r=1}^n (...dx_r)$. we wont need to care about normalisation factors.

This has the interpretation of the particle having associated probabilities of all possible paths, and then summing these. We will also talk about analytic continuation which allows us to turn the imgainary phase into a real exponential. We analytically continue to imaginary time. Let $\tau = it$, then in terms of this imaginary time, we have

$$\langle x| e^{\hat{\mathbf{H}}\frac{\tau}{\hbar}} |x_0\rangle = \int \mathcal{D}x e^{-\frac{S}{\hbar}}$$

The $\hbar = 0$ argument is more clear. Here we see the connection to statistical physics, where $e^{-\frac{S}{\hbar}}$ plays the role of the Boltzmann factor $e^{-\beta H}$. In this case, integrals are more clearly convergent in this framework.

Quantum mechanics is just quantum field theory in $0 + 1$ dimensions, where $\hat{\mathbf{x}}\left(t\right)$ is a field and t is a variable. To develop a field theory, we need to be consistent with Lorentz invariance, and therefore, t and x must be on the same footing. In QFT, we solve that problem by demoting x to be just another label, a variable. For example, $\phi\left(x, t\right)$. String theory offers another choice, where we demote things in a different way.

What we next want to do is devewlop the methods we want to use in analysing theories. We will work perturbatively and use Feynman diagrams. In order to simplify things a bit, we will first work in zero dimensions. We want to look at the integrals themselves and not worry about position or momentum.

2 Integrals and their Diagrams

The first thing that we'll look at are correlation functions. In quantum mechanics, time is our only variable and we look at the evolution of a wavefunction. When we demote x, we now want to look at the behaviour in spacetime, and see how a field in one place affects the field in another place. We will see how they are connected to correlation functions.

For simplicity, consider a zero dimensional field ϕ which is just a real valued variable $\mathbb{R}$. What we want to do is look at the partition function as if we are in imaginary time. Let

$$\mathcal{Z} = \int d\phi e^{-S(\phi)/\hbar}$$

We will add some assumptions for this. We assume that $S(\phi)$ is a polynomial, which is even, and we want it to be well behaved as $S(\phi) \to \infty$, as $\phi \to \pm\infty$. What we are concerned with are our expectation values, where

$$\langle f \rangle = \frac{1}{\mathcal{Z}} \int d\phi f(\phi) e^{-S/\hbar}$$

Again, we assume that f is well behaved and does not grow too fast as $\phi \to \infty$. Usually, f is a polynomial in ϕ. So we've set the generic notation. Let's start with the simplest case which we call the free theory.

2.1 Free Theory

For the time being, let's for the time being think about having N scalar fields (variables) instead of just one. We label these $\phi_a, a = 1, \dots N$. The action will be denoted as

$$S_0(\phi) = \frac{1}{2} M_{ab} \phi_a \phi_b = \frac{1}{2} \phi^T M \phi$$

we want M to be symmetric, $N \times N$ and positive definite, so that $\det M > 0$. So, as we go to a large number of dimensions, this is the kind of term which has both the kinetic term as well as the mass term. We can currently think of these labels as just being flavour labels. We'll generalise this when we go to higher dimensions. Here, we can just diagonalise. Looking at the partition function,

$$M = P \Lambda P^T$$

where Λ is diagonal and P is orthogonal. Lets also do a field redefinition where $\chi = P^T \phi$. Then, we get that the free partition function

$$\mathcal{Z}_0 = \int d^N \phi \, \exp\left(-\frac{1}{\hbar} \phi^T M \phi \right) = \int d^N \chi \exp\left(-\frac{1}{\hbar} \chi^T \Lambda \chi \right)$$

We can write this as the product of independent integrals.

$$\dots = \int_{c=1}^{N} \int d\chi_c e^{-\lambda_c \chi^2 / 2\hbar} = \sqrt{(2\pi\hbar)^N / \det M}$$

This is a very useful result. When we have to introduce anti-commuting numbers, we will see something similar.

Let's introduce another concept which is useful, another trick from statistical physics. We want to get correlations out of partition functions. The way to do that is to introduce external sources, which we call J, an N component external force. In this case, we map

$$S_0(\phi) \to S_0(\phi) + J^T \phi$$

Now, we extend the definition of the partition function which we call the generating function.

$$Z_0(J) = \int d^4 \phi \exp\left[-\frac{1}{2\hbar} \phi^T M \phi - \frac{1}{\hbar} J^T \phi \right]$$

we now have to complete the square, and set $\tilde{\phi} = \phi + M^{-1}J$. This allows us to rewrite the generating function as

$$Z_0(J) = \mathcal{Z}_0(0)\exp\left(\frac{1}{2\hbar}J^T M^{-1} J\right)$$

We see here that this is our free theory multiplied by the sources coupled to the matrix that appears in the action. This is something that we call a 'generating function', and it will allow us to calculate correlation functions from differentiating with respect to J.

$$\langle \phi_a \phi_b \rangle = \frac{1}{\mathcal{Z}_0(0)} \int d^N\phi\, \phi_a\phi_b \exp\left(-\frac{1}{2\hbar}\phi^T M\phi - \frac{1}{\hbar}J^T\phi\right)|_{J=0}$$

we can get the ϕ in the integrand by differentiating the exponential with respect to J. So,

$$\begin{aligned}
\langle \phi_a \phi_b \rangle &= \frac{1}{\mathcal{Z}_0} \int d^N\phi \left(-\hbar\frac{\partial}{\partial J_a}\right)\left(-\hbar\frac{\partial}{\partial J_b}\right)\exp(\dots)|_{J=0} \\
&= \frac{1}{\mathcal{Z}_0(0)}\left(-\hbar\frac{\partial}{\partial J_a}\right)\left(-\hbar\frac{\partial}{\partial J_b}\right)\mathcal{Z}_0(J)|_{J=0} \\
&= \hbar\left(M^{-1}\right)_{ab} \\
&= \text{(diagram of two nodes connected by a line, called free propagator)}
\end{aligned}$$

This is a pairing of the two fields which are given by the indices. We can extend this to see how this works more generally. Let's invent some notation which allows us to be a little more general. Let $l(\phi)$ be a linear combination of $\phi_a, a = 1, \dots, N$. All these expectation values are linear so we can do this. So we write

$$l(\phi) = \int_{a=1}^{N} l_a\phi_a, \quad l_a \in \mathbb{R}$$

Then, the steps above are equivalent to swapping

$$l(\phi) \text{ for } l\left(-\hbar\frac{\partial}{\partial J}\right) = -\hbar\sum_{a=1}^{N} l_a\frac{\partial}{\partial J_a}$$

Our correlation function is thus

$$\left\langle l^{(1)}(\phi)\dots l^{(p)}(\phi)\right\rangle = \frac{1}{\mathcal{Z}_0}\int d^N\phi \prod_{i=1}^{p} l^{(i)}(\phi)\, e^{-\frac{1}{2\hbar}\phi^T M\phi - \frac{1}{\hbar}J^T\phi}|_{J==0}$$

Moving the functions of ϕ to functions of derivatives, we find that this is equal to

$$\left\langle \lambda^{(1)}(\phi)\dots l^{(p)}(\phi)\right\rangle = (-\hbar)^p\prod_{i=1}^{p} l^{(i)}\left(\frac{\partial}{\partial J}\right)\exp\left(\frac{1}{2\hbar}J^T M^{-1}J\right)|_{J=0}$$

Now, if p is odd the answer is zero, then the integrand is odd in some ϕ_a and the integral over $\phi_a \in (-\infty, \infty)$ vanishes. For $p = 2k$, the terms which are non-zero has $J \to 0$: half the derivatives to bring down components of $M^{-1}J$ and half to remove J dependence from the prefactor. This establishes that we get exactly k factors of M^{-1}. Let's look at the four point function

$$\langle \phi_b\phi_c\phi_d\phi_f\rangle = \hbar^2\left[\left(M^{-1}\right)_{bc}\left(M^{-1}\right)_{df} + \left(M^{-1}\right)_{bd}\left(M^{-1}\right)_{cf} + \left(M^{-1}\right)_{bf}\left(M^{-1}\right)_{cd}\right]$$

In terms of Feynman diagrams, we've just got various propagators here. In terms of connecting the ϕs, we have different components. We can represent this as connecting different lines. (Insert diagrams of lines here). The number of terms is the number of ways of forming pairs, which os

$$\frac{(2k)!}{2^k k!} = \text{ ways of permutating points } /(\text{ permute inside pair } \times \text{ permute pairs })$$

If we have a complex matrix, ϕ_a complex and M hermitian, then $\langle \phi_a \phi_b^* \rangle = \hbar M^{ab-1}$ is represented by a line with an arrow from a to b.

2.2 Interacting Theory

We want to go beyond the free theory and add higher power terms of ϕ. The way to do this is to expand about $\hbar = 0$, which is the classical result. We will be expanding about the minimum of the action. On the other hand, we will not be as satisfactory as one may imagine, because the expansion may not even be convergent.

Lets look at integrals like

$$\int d\phi f(\phi) e^{-S/\hbar}$$

which do not have a Taylor expansion about $\hbar = 0$. The proof is by Dyson. If we did have a Taylor expansion about $\hbar = 0$ which existed for $\hbar > 0$, then in the complex $\hbar$ plane, there must be some finite radius of convergence. If there's some finite radius of convergence in the complex $\hbar$ plane. then the expansion needs to exist for some negative real values of $\hbar$. But this is manifestly not true.

For $S(\phi)$ to have a minimum, the integral is divergent if $\mathrm{Re}(\hbar) < 0$. Therefore, the radius of convergence cannot be greater than zero. So we're not going to have convergent expansions here. (Insert diagram of complex $\hbar$ plane with small circle at origin).

So, our $\hbar$ - expansion is at best asymptotic.

$$I(\hbar) \sim \sum_{n=0}^{\infty} c_n \hbar^n$$

where $\sim$ means asymptotic to. This means that

$$\lim_{\hbar \to 0} \frac{1}{\hbar^N} |I(\hbar) - \sum_{n=0}^{N} c_n \hbar^n| = 0$$

where the limit is taken with N fixed. As we take $\hbar$ to zero, the series is arbitrarily close to zero.

It is important to note that the series misses out a transcendental terms like $e^{-\frac{1}{\hbar^2}} \sim 0$. But, $e^{-\frac{1}{\hbar^2}}$ for finite $\hbar$. These are what we call 'non-perturbative contributions'. Let's get back to the theory we're interested in. Take our action to be

$$S(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4$$

where we set $S_0(\phi)$ to be the first term, and $S_1(\phi)$ to be the second term. Here, we also assume that $m^2 > 0$ and $\lambda > 0$.

We can expand about the minimum of $S(\phi)$, where $\phi = 0$. We expand about the saddle point, so that

$$\begin{aligned}\mathcal{Z} &= \int d\phi e^{-S/\hbar} \\ &= \int d\phi e^{-S_0/\hbar} \sum_{v=0}^{\infty} \frac{1}{V!}\left(-\frac{\lambda}{4!\hbar}\right)^v \phi^{4v}\end{aligned}$$

what we'll do is that we'll truncate the series so that we miss out transcendental terms. In order to make progress, we need to truncate the series and swap summation and integration. This misses out transcendental terms like what we had before.

In the end what we have, is a series that

$$\mathcal{Z} \sim \frac{\sqrt{2\hbar}}{m} \sum_{v=0}^{N} \frac{1}{v!}\left(-\frac{\hbar\lambda}{4!m^4}\right)^v 2^{2v} \int_0^\infty dx\, e^{-x} x^{2v+\frac{1}{2}-1}$$

where $x = \frac{1}{2\hbar} m^2\phi^2$. The integrand is just a gamma function, where

$$\int_0^\infty dx\, e^{-x} x^{2v+\frac{1}{2}-1} = \Gamma\left(2v + \frac{1}{2}\right) = \frac{(4v)!\sqrt{\pi}}{4^{2v}(2v)!}$$

In closed form, we have

$$Z \sim \frac{\sqrt{2\pi\hbar}}{m} \sum_{v=0}^{N} \left(-\frac{\hbar\lambda}{m}\right)^v \frac{1}{(4!)^v v!} \frac{(4v)!}{}$$

From String's approximation, we have that $v! \sim e^{v\log v}$, then the factors which are multiplied together are approximately $v!$. We have that factorial growth : asymptotic series. The first term in the product comes from the Taylor expansion of $e^{-S_1/\hbar}$. The second term in the product comes from pairing the $4v$ fields of the v copies of ϕ^4.

We will now follow the diagrammatic method. If we write the action including a source term

$$\begin{aligned}\mathcal{Z}(J) &= \int d\phi \exp\left\{-\frac{1}{\hbar}\left(S_0(\phi) + S_1(\phi) + J\phi\right)\right\} \\ &= \exp\left[-\frac{1}{\hbar} S_1\left(-\hbar\frac{\partial}{\partial J}\right)\right] \int d\phi \exp\left\{-\frac{1}{\hbar}\left(S_0 + J\phi\right)\right\} \\ &\propto \exp\left[-\frac{\lambda}{4!\hbar}\left(\hbar\frac{\partial}{\partial J}\right)^4\right] \exp\left(\frac{1}{2\hbar} J^T M^{-1} J\right) \\ &\sim \sum_{v=0}^{N} \frac{1}{v!}\left[-\frac{\lambda}{4!\hbar}\left(\hbar\frac{\partial}{\partial J}\right)^4\right]^V \sum_{p=0} \frac{1}{p!}\left(\frac{1}{2\hbar} J m^{-2} J\right)^P\end{aligned}$$

This has the associated diagrams below. We should check $\mathcal{Z}(0)$. For a term to be non-zero when $J = 0$, we require that the number of derivative is equal to the number of propagators. This means we require

$$E = 2P - 4V = 0$$

where E is the number of sources left undifferentiated. Our first non-trivial terms include $(V,P) = (1,2) = (2,4)$. For $\mathcal{Z}(0)$, our first terms are ...

We can count the number of times each diagram appears. Consider the figure 8 graph, the 'pre-diagram', it has one vertex with four free vertices, and $p = 2$ propagators. There are four factorial ways of matching derivatives to sources. $A = 4!$. The denominator od the $\mathcal{Z}(J)$ expansion is just, reading off, is

$$F = (V!)\,(4!)^v\,(p!)\,2^p = 4! \cdot 2 \cdot 2$$

Thus, the figure of 8 comes with a pre factor of $\frac{A}{F} = \frac{1}{8}$ multiplied by $-\frac{\hbar\lambda}{m^4}$. More generally, F accounts for permutations of

- all vertices $v!$
- each vertex legs $4!$
- All propagators $p!$
- both ends of each propagator 2

Symmetry of particular graph is important. For example, take the figure of eight diagram. Take the pairing $(1a, 2a', 3b, 4b')$. Consider swapping $a \iff a'$ and $1 \iff 2$, gives exactly the same graph. So

$$\frac{A}{F} = \frac{1}{\mathcal{S}}, \quad \mathcal{S} \text{ is the symmetry factor}$$

$\mathcal{S}$ is the number of ways of redrawing unlabelled graph, leaving it unchanged. For example, for the figure of eight graph, we can swap the direction fo the upper and lower loops, and swap the upper and lower loops.

Looking at the basket ball diagram, we have 4! for each of the four lines attaching the two vertices, and swapping the vertices. (Insert pre-diagram drawing here)

$$\frac{Z(0)}{Z_0(0)} = 1 - \frac{\hbar\lambda}{8m^4} + \frac{\hbar^2\lambda^2}{m^8}\left(\frac{1}{48} + \frac{1}{16} + \frac{1}{128}\right)$$

From last time we have that

$$\mathcal{Z}(J) \sim \sum_{v=0}^{N} \frac{1}{v!}\left[-\frac{\lambda}{4!\hbar}\left(\hbar\frac{\partial}{\partial J}\right)^4\right]^v \sum_{p=0} \frac{1}{p!}\left[\frac{1}{2\hbar}\frac{J^2}{m^2}\right]^p$$

If we focus on the case with $E = 2$, we have that

$$Z(J) \,\text{all diagrams with two external points}$$

We can factor out the vacuum bubble diagrams so that

$$Z(J) = [\text{No vacuum bubbles}]\,[Z(0) \text{ vacuum bubbles }]$$

Our expectation values are hence

$$\begin{aligned}\langle \phi^2 \rangle &= \frac{(-\hbar)^2}{\mathcal{Z}(0)}\left(\frac{\partial}{\partial J}\right)^2 \mathcal{Z}(J)\,|_{J=0} \\ &= [\text{ connected diagrams}]\end{aligned}$$

In terms of our symmetry factors, from $Z(J)$ the $E = 2$, $V = 0$, $P = 1$ term gives a contribution of

$$= \frac{1}{2\hbar}\frac{J^2}{m^2}, \quad F = 2, A = 1, \frac{A}{F} = \frac{1}{2} = \frac{1}{\mathcal{S}}$$

So the first order contribution is $\left\langle \phi^2 \right\rangle = \frac{\hbar}{m^2}$ = line diagram., $\left\langle \phi^{2n} \right\rangle$ proceeds similarly, but note there are disconnected diagrams. (Insert diagram here).

Effective actions Define the Wilsonian effective action $\mathcal{W}(J)$ such that

$$\mathcal{Z}(J) = e^{-\mathcal{W}(J)/\hbar}$$

We want to show that

$$\mathcal{W}(0) = \sum \text{ all connected vacuum diagrams}$$

and

$$\mathcal{W}(J) = \sum \text{ all connected diagrams}$$

Any diagram D is a product of connected diagrams C_I.

$$D = \frac{1}{S_D}\prod_I (C_I)^{n_I} \text{ I : index over unique connected diagrams }, \quad n_I = \text{numbers of times } C_I \text{ appears in}$$

D Assume C_I includes appropriate symmetry factor $\frac{1}{S_I}$. where S_D is a symmetry factor associated with rearranging $C_I's$.

$$S_D = \prod_I (n_I)!$$

For example, if we take $D =$ (Insert diagram here) The since the disconnected parts commute, we have $n_1 = 3, n_2 = 1$. This means the total symmetry factor $S_D = 3!1! = 6$.

If we write $\{n_I\}$ as the set of integers specifying D, we have that

$$= e^{-(\mathcal{W}-\mathcal{W}_0)/\hbar}$$

$$\frac{\mathcal{Z}}{\mathcal{Z}_0} = \sum_{n_I} D \quad = \sum_{\{n_I\}}\prod_I \frac{1}{n_I!}(C_I)^{n_I} \ = \prod_I\sum_{n_I}\frac{1}{n_I!}(C_I)^{n_I} \quad = \exp\left(\sum_I c_I\right) = \exp\Big(\sum of unique connected diagra$$

so that we have $\mathcal{W} = \mathcal{W}_0 - \hbar\sum_I c_I$. $\mathcal{W}(J)$ is the generating functional for connected correlation functions. This means that we have

$$\begin{aligned}
-\frac{1}{\hbar}W(J) &= \log Z(J) \\
-\frac{1}{\hbar}\frac{\partial^2}{\partial J^2}W\mid_{J=0} &= \frac{1}{\mathcal{Z}(0)}\frac{\partial^2\mathcal{Z}}{\partial J^2}\mid_{J=0} - \frac{1}{(\mathcal{Z}(0))^2}\left(\frac{\partial\mathcal{Z}}{\partial J}\right)^2\mid_{J=0} \\
&= \frac{1}{\hbar^2}\left[\left\langle\phi^2\right\rangle - \left\langle\phi\right\rangle^2\right] \\
&= \frac{1}{\hbar^2}\left\langle\phi^2\right\rangle_{\text{connected}}
\end{aligned}$$

Less trivially, we may encounter theories where the expecation of ϕ is non zero, but we're not discussing that. Less trivially,

$$-\frac{1}{\hbar}\frac{\partial^4 \mathcal{W}}{\partial J^4}\mid_{J=0}=\frac{1}{\mathcal{Z}(0)}\frac{\partial^4 \mathcal{Z}}{\partial J^4}\mid_{J=0}-\left(\frac{1}{\mathcal{Z}(0)}\frac{\partial^2 \mathcal{Z}}{\partial J^2}\right)\mid_{J=0}$$

This implies that

$$\left\langle \phi^4 \right\rangle_{\text{connected}} = \left\langle \phi^4 \right\rangle - \left\langle \phi^2 \right\rangle^2$$

Let's consider an action with two real fields.

$$S(\phi,\chi)=\frac{m^2}{2}\phi^2+\frac{M^2}{2}\chi^2+\frac{\lambda}{4}\phi^2\chi^2$$

Note that we don't have a factorial here. With two fields, we now have two sets of Feynman rules. We can look at the Wilson effective action by looking at the connected vacuum diagrams.

$$-\mathcal{W}/\hbar = \text{diagram of connected vacuum diagrams}$$

Counting the symmetry factors of this diagram are

$$-\frac{\mathcal{W}}{\hbar}=-\frac{\hbar\lambda}{4m^2M^2}+\frac{\hbar^2\lambda^2}{m^4M^4}\left[\frac{1}{16}+\frac{1}{16}+\frac{1}{8}\right]$$

Also, from Feynman diagrams, the

$$\left\langle \phi^2 \right\rangle = (\text{connected diagrams with external lines})$$

Again, counting the symmetry factors, we get that

$$\left\langle \phi^2 \right\rangle = \frac{\hbar}{m^2}-\frac{\hbar^2\lambda}{2m^4M^2}+\frac{\hbar^3\lambda^2}{{}^6M^4}\left[\frac{1}{4}+\frac{1}{2}+\frac{1}{4}\right]$$

say we don't care about χ explicitly, maybe because we don't know that much about χ, we may want to integrate it out. This may be because $M \gg m$, never produced on experimental scales. We define $\mathcal{W}(\phi)$, to give

$$e^{-\mathcal{W}(\phi)/\hbar}=\int d\chi e^{-S(\phi,\chi)/\hbar}$$

Thus, $\phi^2\chi^2$ is treated as a source term with $J=\phi^2$ in earlier notation. We're using our low energy particles, bashing them together, and using them as a source.

We want to look at correlation functions only involving ψ fields.

$$\langle f(\phi)\rangle = \frac{1}{2}\int d\phi d\chi f(\phi)\, e^{-S(\phi,\chi)/\hbar}=\frac{1}{\mathcal{Z}}\int d\phi f(\phi)\, e^{-W(\phi)/\hbar}$$

In this simple example, we have that

$$\int d\chi e^{-S(\phi,\chi)/\hbar}=e^{-m^2\phi^2/2\hbar}\sqrt{\frac{2\pi\hbar}{M^2+(\lambda\phi^2)/2}}$$

This implies that, solving for $W(\phi)$, we have

$$W(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\hbar}{2}\log\left(1 + \frac{\lambda}{2M^2}\phi^2\right) + \frac{\hbar}{2}\log\frac{M^2}{2\pi\hbar}$$

The final term is a constant. One way to think about this is in the context of the cosmological constant problem. This cancels out in expectation values in QFTs. Let's expand the logarithm. We've been explicit in the inclusion of $\hbar$, so expansions in the logarithm term are quantum effects. This gives us

$$W(\phi) = \left(m^{\frac{2}{2}} + \frac{\hbar\lambda}{4M^2}\right)\phi^2 - \frac{\hbar\lambda^2}{16M^4}\phi^4 + \frac{\hbar\lambda^3}{48M^6}\phi^6 + \dots$$

One can think of these as an effective mass term. So, we have that

$$W(\phi) = \frac{m_{\text{eff}}^2}{2}\phi^2 + \frac{\lambda^4}{4!}\phi^4 + \frac{\lambda^6}{6!}\phi^6 + \dots \frac{\lambda_{2k}}{(2k)!}\phi^{2k} + \dots$$

where we have defined $m^2_{\text{eff}} = m^2 + \frac{\hbar\lambda}{2M}$, and we define

$$\lambda_{2k} = (-1)^{k+1}\,\hbar\frac{(2k)!}{2^{k+1}k}\frac{\lambda^k}{M^{2k}}$$

In dim > 0 , we usually need to calculate $W(\phi)$ perturbatively. From the action $S(\phi,\chi)$, and the path integral over χ, we have the Feynman rules.

The dotted lines come from the ϕ field. Putting the integrals together, we have that this makes it equal to

$$W(\phi) = S(\phi) + \frac{1}{2}\frac{\hbar\lambda}{2M^2}\phi^2 - \frac{1}{4}\frac{\hbar\lambda^2}{4M^4}\phi^4 + \frac{1}{3!}\frac{\hbar\lambda^3}{8M^6}\phi^6 + \dots$$

Using the effective action, we can also calculate the correlation function

$$\langle\phi^2\rangle = \frac{1}{\mathcal{Z}}\int d\phi\phi^2 e^{-W(\phi)/\hbar}$$

$$= \frac{\hbar}{m_{\text{eff}}^2} - \frac{\lambda_4\hbar^2}{2m^6_{\text{eff}}}$$

2.3 Quantum Effective Action

We represent the quantum effective action by Γ. We want to define the average field in the presence of an external source.

$$\Phi := \frac{\partial W}{\partial J} = \langle\phi\rangle_J = -\frac{\hbar}{Z(J)}\frac{\partial}{\partial J}\int d\phi e^{-(S+J\phi)/\hbar}, S(\phi) \text{ same as before}$$

We define the Legendre transformation from $W(J) \to \Gamma(\Phi)$. This is

$$\Gamma(\Phi) = W(J) - \Phi J$$

Note that

$$\frac{\partial\Gamma}{\partial\Phi} = \frac{\partial W}{\partial\Phi} - J - \Phi\frac{\partial J}{\partial\Phi} = \frac{\partial W}{\partial J}\frac{\partial J}{\partial\Phi} - J - \Phi\frac{\partial J}{\partial\Phi}$$

This means that

$$\frac{\partial J}{\partial \Phi} = -J$$

so J is the minimum of the quantum effective action. If $J = 0$, then

$$\left.\frac{\partial \Gamma}{\partial \Phi}\right|_{J=0} = 0$$

So, $J \to 0$ corresponds to an extremum of $\Gamma(\Phi)$.

In higher dimensions, one performs a derivative expansion

$$\Gamma(\Phi) = \int d^d x \left[-V(\Phi) - \frac{1}{2}\partial^\mu \Phi \partial_\mu \Phi + \ldots \right]$$

where $V(\Phi)$ is the effective potential. There is some analogy here with statistical mechanics. h is a magnetic field.

$$e^{-\beta F(h)} = \int \mathcal{D}s \exp(-\beta \mathcal{H})$$

The magentization $M = -\frac{\partial F}{\partial h}$. The Gibbs free energy is

$$G(M) = F(h) + Mh$$

as $h \to 0$, M of the system is the minimum of G.

We do a perturbative calculation of ,$\Gamma(\Phi)$ We write

$$e^{-W_\Gamma(J)/g} = \int d\Phi e^{-(\Gamma(\Phi)+J\Phi)/g}$$

We define a new Planck fictitious constant, and include the source term g. We know that $W_\Gamma(J)$ is the sum of connected vacuum diagrams.

$$W_\Gamma(J) = \int_{l=0}^{\infty} g^l W_\Gamma^{(l)}(J)$$

We know that $W_\Gamma^{(0)}$ is composed of tree diagrams. In the $g \to 0$ limit, we have that $W_\Gamma(J) = W_\Gamma^{(0)}(J)$. Also as $g \to 0$, integral over Φ is dominated by the minimum of the exponent, in other words the Φ such that

$$\frac{\partial \Gamma}{\partial \Phi} = -J$$

But then

$$W_\Gamma^{(0)}(J) = \Gamma(\Phi) + J\Phi = W(J)$$

where the $W(J)$ is from earlier with action $S(\phi) + J\phi$. The moral of the story is that the sum of the connected diagrams in theory with action $S(\phi) + J\phi$ which is $W(J)$, can be constructed from the sum of tree diagrams with action $\Gamma(\Phi) + J\Phi$.

We make the definition that an internal line in a graph is a bridge if cutting it would make a graph disconnected. A connected graph is called a one particle irreducible (1PI) if it has no bridges.

The irreducible parts are loops. Buried within it, Γ sums up the loop diagrams (1PI).

2.4 Fermions and Grassman Variables

In zero dimensions, we don't have a concept of spin, since we don't even have a way to orientate things correctly. Thus, the best we can do is to construct a set of N variables, fermion fields, which anti-commute.

We call these fields $\theta^a, a = 1, \dots N$. This has the characteristic property that

$$\theta_a \theta_b = -\theta_b \theta_a, \quad a = 1, \dots N$$

This in particular implies that for a given field θ^a, we have

$$\theta^a \theta^a = 0$$

They also have the property that they commute with scalar fields, so we have that

$$\phi_a \psi_b = \psi_b \phi_a, \quad a = 1, \dots N$$

With this data, we can define functions of these variables as a finite expansion.

$$F(\theta) = f + \rho_a \theta^a + \frac{1}{2!} g_{ab} \theta^a \theta^b + \frac{1}{n!} \dots h_{ab \dots n} \theta^a \theta^b \dots \theta^n$$

In these expansions, we have that each of the tensors are totally anti-symmetric. Suppose that we only had one field in the case that $a = 1$ only. Then, the most general function we could write with this is

$$F(\theta) = f + \rho\theta$$

since any terms of higher order go to zero by antisymmetry.

We define differentiation and integration on these variables as follows. For differentiation, we have that

$$\frac{\partial}{\partial \theta^a} \theta^b + \theta^b \frac{\partial}{\partial \theta^a} = \delta^b{}_a$$

We also have the integration rules.

The integrals should be invariant under translation, so

$$\int d\theta \, (\theta + \eta) = \int d\theta \theta$$

If we're dealing with a single variable, we can then integrate by parts, since clearly

$$\int d\theta \frac{\partial}{\partial \theta} F(\theta) = 0$$

We can extend this to integration rules for n variables. Namely, we only have a non-vanishing integral when our integrand is a product of one power of each Grassman number.

$$\int d^n \theta^1 \dots \theta^n = 1$$

By antisymmetry, we have that for a given ordering of the Grassman variables, we have that

$$\int d^n \theta \, \theta^{a_1} \dots \theta^{a_n} = \epsilon^{a_1 \dots a_n}$$

We can compute the Jacobian of this measure as follows. Suppose that we relate a set of new Grassman variables ${\theta'}^a = N^a{}_b \theta^b$, where $N \in GL(2, \mathbb{C})$. This means that we have, integrating over the variables,

$$
\begin{aligned}
\int d^n\theta \, {\theta'}^{a_1} {\theta'}^{a_2} \dots {\theta'}^{a_n} &= N^{a_1}{}_{b_1} N^{a_2}{}_{b_2} \dots N^{a_n}{}_{b_n} \int d^n \theta^{b_1} \dots \theta^{b_n} \\
&= N^{a_1}{}_{b_1} \dots N^{a_n}{}_{b_n} \, \epsilon^{b_1 \dots b_n} \\
&= \det N \epsilon^{a_1 \dots a_n} \\
&= \det N \int d^n \theta' {\theta'}^{a_1} \dots {\theta'}^{a_n}
\end{aligned}
$$

This implies that $d^n\theta = \det N d^n \theta'$.

2.4.1 Fermionic Free Field Theory

If we want to build a bosonic theory out of fermions, we need to include an even number of fermions. In full generality, this means our action has to take the form

$$
S(\theta) = \frac{1}{2} A_{ab} \theta^a \theta^b
$$

We have that this is

$$
\begin{aligned}
\mathcal{Z}_0 &= \int d^{2m} \theta e^{-S(\theta)} \\
&= \int d^{2m} \theta e^{-\frac{1}{2} A(\theta, \theta)} \\
&= \int d^{2m} \theta \sum_{n=0}^{2m} \frac{(-1)^n}{(2\hbar)^n n!} \left(A_{ab} \theta^a \theta^b \right)^n
\end{aligned}
$$

Notice however, that when we perform Berezin integration, only terms which have a single power of each variable don't vanish. Hence, the only term which doesn't vanish is when $n = 2m$,

$$
\begin{aligned}
\mathcal{Z}_0 &= \int d^2 m \theta \frac{(-1)^n}{(2\hbar)^n n!} A_{a_1 a_2} \dots A_{a_{2m-1} a_{2m}} \theta^{a_1} \theta^{a_2} \dots \theta^{a_{2m}} \\
&= \frac{(-1)^n}{(2\hbar)^n n!} A_{a_1 a_2} \dots A_{a_{2m-1} a_{2m}} \epsilon^{a_1 \dots a_{2m}}
\end{aligned}
$$

3 LSZ Reduction Formula

We're going to do an illustrative example here, so we can get some intuition going on in terms of a free theory. The main result we'll be exploring today is scattering amplitudes in terms of correlation functions. For example, let's look at $2 \to 2$ scattering of scalar particles. Recall from the previous set of notes from quantum field theory, that our scattering amplitude can be written in the form $\langle f| S |i\rangle$. Now, it is in our interest to try and find out what this is in terms of our correlation functions of our fields $\phi(x)$. Recall that a correlation function looks like $\langle 0| \mathcal{T}\phi(x_1)\phi(x_2) |0\rangle$. This is what our LSZ reduction formula is.

Our motivation for proceeding is as follows. Since initial and final states $|i\rangle, |f\rangle$ are written in terms of creation operators, we will need to invert these to get expressions in ϕ.

We write our free scalar field which can be built out of plane waves.

$$\phi(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3 2E}\left[a(\mathbf{k}) e^{-ik\cdot x} + a^\dagger(\mathbf{k}) e^{ik\cdot x}\right]$$

where we have $k\cdot x = Et - \mathbf{k}\cdot\mathbf{x}$. We have relativistic normalisation for $a(\mathbf{k})$. Now, it's convenient to invert this expression via the inverse Fourier transform of the field and it's derivative

$$\int d^3x e^{ik\cdot x}\phi(\mathbf{x}), \quad \int d^3x e^{ik\cdot x}\partial_0\phi(x)$$

One can easily verify that the identities below hold, using the standard identities.

$$a(\mathbf{k}) = \int d^3x e^{ik\cdot x}\left[i\partial_0\phi(x) + E\phi(x)\right]$$
$$a^\dagger(\mathbf{k}) = \int d^3x e^{-ik\cdot x}\left[-i\partial_0\phi(x) + E\phi(x)\right]$$

We set our initial and final states for the free theory, to be one-particle momentum states, created by applying a creation operator to the particle vacuum.

$$|k\rangle = a^\dagger(k)|\Omega\rangle$$

where $|\Omega\rangle$ is the true vacuum, which in a weakly interacting theory is not too different from the true free vacuum. This is a key assumption which we have to make. We have that $|\Omega\rangle$ satisfies $a(k)|\Omega\rangle = 0$, for all k, and $\langle\Omega|\Omega\rangle = 1$. We have the norm

$$\langle\mathbf{k}|\mathbf{k}\rangle = (2\pi)^3(2E)\delta^3\left(\mathbf{k} - \mathbf{k}'\right), \quad E = \sqrt{\mathbf{k}^2 + m^2}$$

The initial and final states we're interested in will be time moving Gaussian wavepackets which we construct from the creation operators. We introduce a Gaussian wavepacket

$$a_1^\dagger := \int d^3k f_1(\mathbf{k}) a^\dagger(\mathbf{k}), \quad f_1(\mathbf{k}) \propto \exp\left[-\frac{(\mathbf{k}_1 - \mathbf{k}_2)^2}{4\sigma^2}\right]$$

similarly, we define a different moving Gaussian wavepacket for $a_2^\dagger$ Now, we want to see what happens when these wavepackets collide with each other.

We can evolve these Gaussians into the distant past and future, where the overlap in coordinate space is negligible. Assume this works when including interactions.

We are going to evolve including the full Hamiltonian, so, there will be a complication that there is some time dependence. $a^\dagger(\mathbf{k})$ becomes time dependent, so we will get that $a_1^\dagger(t)$ and $a_2^\dagger(t)$ depend on time.

Assume that as $t \to \pm\infty$, $a_1^\dagger$ and $a_2^\dagger$ coincide with their free theory expansions, and that we can Fourier transform without worrying about this complication too much.

Define the initial and final states as two Gaussian wavepackets moving.

$$
\begin{aligned}
|i\rangle &= \lim_{t \to -\infty} a_1^\dagger(t)\, a_2^\dagger(t)\, |\Omega\rangle \\
|f\rangle &= \lim_{t \to \infty} a_{1'}^\dagger(t)\, a_{2'}^\dagger(t)\, |\Omega\rangle
\end{aligned}
$$

We assume that $\langle i|i\rangle = \langle f|f\rangle = 1$, and that $\mathbf{k}_1 \neq \mathbf{k}_2$. We want $\langle f|i\rangle$, the scattering amplitude. To do this, we need to use a trick. Note for example, that

$$
\begin{aligned}
a_1^\dagger(\infty) - a_1^\dagger(-\infty) &= \int_{-\infty}^{\infty} dt \partial_0 a_1^\dagger(t) \\
&= \int d^3k_1 f_1(k) \int d^4x \partial_0 \left[e^{-ik\cdot x} \left(-i\partial_0 \phi + E\phi\right)\right] \\
&= -i \int d^3k_1 f_1(k) \int d^4x e^{-ik\cdot x} \left(\partial_0^2 + E^2\right) \phi \\
&= \int \cdots \int d^4x e^{-ik\cdot x} \left(\partial_0^2 - \nabla^{\leftarrow 2} + m^2\right) \phi^2 \\
&= -i \int d^3k f_1(x) \int d^4x e^{-ik\cdot x} \left(\partial^2 + m^2\right) \phi
\end{aligned}
$$

This is great, because in the last line, we simply have the Klein-Gordon operator. Note in free theory, we have that our fields solve the Klein-Gordon equation, so in this case we have that the difference is zero.

$$
a_1^\dagger(\infty) - a_1^\dagger(-\infty) = 0
$$

We are however looking at the weakly interacting case, so the integrand doesn't necessarily evaluate to zero. Now we can start to calculate the scattering amplitude.

$$
\langle f|i\rangle = \langle \Omega | \mathcal{T} a_{1'}(\infty)\, a_{2'}(\infty)\, a_1^\dagger(-\infty)\, a_2^\dagger(-\infty) |\Omega\rangle
$$

use $a_j^\dagger(-\infty) = a_j^\dagger(\infty) + i \int d^3k f_j(k) \int d^4x e^{-ik\cdot x} \left(\partial^2 + m^2\right)\phi$. Similarly, we have that

$$
a_j(\infty) = a_j(-\infty) + i \int \ldots e^{ik\cdot x} \ldots
$$

Then, the only non-zero term is

$$
\begin{aligned}
\langle f|i\rangle = (i)^4 \int & d^4x_1 d^4x_2 d^4x_1' d^4x_2' e^{-k_1\cdot x} e^{-ik_2\cdot x_2} e^{ik_1'\cdot x_1'} e^{ik_2'\cdot x_2'} \\
& \times \left(\partial_1^2 + m^2\right)\left(\partial_2^2 + m^2\right)\left(\partial_{1'}^2 + m^2\right)\left(\partial_{2'}^2 + m^2\right) \\
& \times \langle \Omega | \mathcal{T} \phi(x_1)\, \phi(x_2)\, \phi\left(x_1'\right) \phi\left(x_2'\right) |\Omega\rangle
\end{aligned}
$$

having taken $\sigma \to 0$ such that $f(\mathbf{k}_j) \to \delta^3(\mathbf{k} - \mathbf{k}_j)$. Let's examine assumptions here. The general deviation requires only weaker assumptions.

- We need a unique Ω, such that the first excited state is a single particle.
- We want $\phi |\Omega\rangle$ to be a single particle state. In other words, we want

$$\langle\Omega| \phi |\Omega\rangle = 0$$

  If not, and $\langle\Omega| \phi |\Omega\rangle = v \neq 0$, then let $\tilde{\phi} = \phi - v$.

- We want ϕ normalised such that

$$\langle k| \phi(x) |0\rangle = e^{ik\cdot x}$$

  as in the free case. Usually interactions require us to rescale $\phi \to Z_\phi^{\frac{1}{2}}\phi$. We see the need to renormalise, for example,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

  From here, the coefficients may spoil the LSZ formula.

$$\mathcal{L} = \frac{Z_\phi}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}Z_m m^2\phi^2 - \frac{\lambda}{4!}Z_\lambda\phi^4$$

Summary

3.0.1 Path Integral Derivations

- You get path integrals from repeatedly inserting the completeness relation

$$I = \int dx_0 \; |x_0\rangle \langle x_0|$$

- The kernel is

$$K(x, x_0, t) = \langle x| e^{-\frac{i\hat{H}t}{\hbar}} |x_0\rangle$$

- Our action is defined as

$$S = \int_0^T dt L(x, \dot{x})$$

- Our measure is the two-way limit

$$\mathcal{D}x = \lim_{\delta t \to 0, n\delta \text{ fixed}} \sqrt{\frac{m}{2\pi i\hbar\delta t}} \prod_{r=1}^{n} \left(\sqrt{\frac{m}{2\pi i\hbar\delta t}} dx_r\right)$$

3.0.2 Free Partition Functions

- The free theory is defined as
$$S_0(\phi) = \frac{1}{2} M_{ab}\phi_a\phi_b$$
- The free partition function
$$\mathcal{Z}_0 = \int d^N\phi e^{-S(\phi)/\hbar}$$
- With a source term, $S_0 + J\phi$, this free partition function as a function of J is
$$\mathcal{Z}(J) = Z(0)\exp\left(\frac{1}{2\hbar}J^T M J\right)$$

3.0.3 Feynman Diagrams

- For each graph with n vertices, we add a combinatoric factor of
$$\frac{|D_n|}{|G_n|} = \sum \frac{1}{|\text{Aut }\Gamma|}$$
- There are two ways to generate diagrams. See which combinations of exponents reduce the source terms to zero, then construct the possible diagrams.

3.0.4 Effective Actions

- The Wilsonian effective action is the logarithm of the partition function
$$\mathcal{W} = -\hbar \log \mathcal{Z}$$
- The connected correlation function for n variables is the n the derivative of the Wilsonian effective action.
- With two real fields, derive the Feynman rules
$$S(\phi, \chi) = \frac{1}{2}m^2\phi^2 + \frac{1}{2}M^2\xi^2 + \frac{\lambda}{4}\chi^2\phi^2$$
The Wilsonian effective action
$$-\mathcal{W}/\hbar = \text{connected vacuum bubbles}$$
- Integrate out high energy fields with
$$e^{-\mathcal{W}(\phi)/\hbar} = \int d\chi\, e^{-S(\phi,\chi)\ \hbar}$$

4 Regularisation and Renormalisation

5 Gauge Theories

6 Formulating the Path Integral

In this section, we'll be moving on from our standard procedure of quantising a given Hamiltonian in quantum mechanics. We'll be introducing the concept of a path integral. The path integral is a 'functional integral' where we integrate over all possible paths with a Gaussian probability factor.

6.1 Classical and Quantum Mechanics

In classical mechanics, we use the Lagrangian as a conduit to encode the information about our physical system. The Lagrangian is given by a function of position and velocity, with

$$\mathcal{L} = \mathcal{L}\left(q_a, \dot{q}_a\right)$$

where $a = 1, \ldots, N$ is an index for each particle in our system. We can convert this to the Hamiltonian formalism where we put position and momentum on the same pedestal and define our conjugate momenta

$$p_a = \frac{\partial \mathcal{L}}{\partial \dot{q}^a}$$

We then work in terms of the Hamiltonian which is the Legendre transformation of the Lagrangian, where we eliminate $\dot{q}^a$ everywhere in the Lagrangian in favour of p^a as follows

$$H(q_a, p_a) = \sum_a \dot{q}_a p^a - \mathcal{L}\left(q_a, \dot{q}_a\right)$$

The quantum mechanical analog of this is the same. However, p_i and q_i are **promoted** to what we call operators, and obey commutation relations which as we know, eventually lead to discrete energy levels in the Hamiltonian. In quantum mechanics, we write the position and momentum operators as $\mathbf{q}^i$ and $\mathbf{p}^i$ for position and momentum respectively. In the Heisenberg picture of quantum mechanics, operators (and not states), depend on time. So, we impose the commutation relations for some fixed coordinate time $t \in \mathbb{R}$, where

$$\left[\mathbf{q}^i, \mathbf{p}_j\right] = i\delta^i{}_j$$

In classical field theory, we promote operators to fields instead. If $\phi(\mathbf{x}, t)$ represents a classical scalar field at some point in time t, then the field as well as it's conjugate momentum $\pi\left(\mathbf{x}, t\right)$ obey the commutation relations

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta^3\left(\mathbf{x} - \mathbf{y}\right)$$

There is however a caveat in performing these approaches to quantisation. The theory is not manifestly Lorentz invariant. This is because when we imposed the equal time commutation relations above, we had to pick a preferred coordinate time t.

6.2 Formulating the Path Integral

We use the Hamiltonian as a starting point.

$$H(q,p) = \frac{p^2}{2m} + V(q)$$

The Schrödinger equation for a state $|\psi(t)\rangle$ which is time dependent is given by

$$i\frac{d}{dt}|\psi(t)\rangle = \hat{\mathbf{H}}|\psi(t)\rangle$$

Now, this is is a first order differential equation and can be solved provided that we have the right initial conditions. For now, let's just write down the solution in a 'formal' sense, where we 'exponentiate' the Hamiltonian whilst being vague about what this actually means. We write the solution as

$$|\psi(t)\rangle = \exp\left(-i\hat{\mathbf{H}}t\right)|\psi(0)\rangle$$

To do calculations however, we need construct an appropriate basis of states. For this section, we'll use the position basis $|q,t\rangle$, for $q,t \in \mathbb{R}$. These states are are defined to be the eigenstates of the position operator $\hat{\mathbf{q}}(t)$, so that

$$\hat{\mathbf{q}}(t)|q,t\rangle = q|q,t\rangle$$

We'll impose the condition that these states are normalised so that for a fixed time, we have

$$\langle q,t|q',t\rangle = \delta\left(q-q'\right)$$

We impose the analogous conditions as well for momentum eigenstates. For now though, we'll work in the Schrodinger picture so that $\hat{\mathbf{q}}$ is fixed and hence we have that the eigenstates $|q\rangle$ are time-independent. Since these states form a basis, we have that they obey the completeness relation

$$1 = \int d^3q\ |q\rangle\langle q|$$

We also label the time-independent momentum eigenstates as $|p\rangle$, and impose the completeness relation

$$1 = \frac{d^3p}{(2\pi)^3}|p\rangle\langle p|$$

Note the factor of 2π that we divide by. Other literature doesn't include this. With this set of basis states, we can now write the abstract state $|\psi(t)\rangle$ in terms of the position basis, where we denote

$$\psi(q,t) = \langle q|\psi(t)\rangle = \langle q|\exp\left(-i\hat{\mathbf{H}}t\right)|\psi(0)\rangle$$

We will put this into an integral form for reasons we will discuss later. To put any equation in integral form, the rule of thumb is to employ the completeness relations for either the position or momentum basis. We get that

$$\begin{aligned}
\langle q| \exp\left(-i\hat{\mathbf{H}}t\right)|\psi(0)\rangle &= \int d^3q' \ \langle q| \exp\left(-i\hat{\mathbf{H}}t\right)\left|q'\right\rangle \left\langle q'\middle|\psi(0)\right\rangle \\
&= \int d^3q' \ \langle q| \exp\left(-i\hat{\mathbf{H}}t\right)\left|q'\right\rangle \psi(q',0) \\
&= \int d^3q' \, K\left(q,q';t\right)\psi(q',0) \\
&= \int d^3q' K(q,q';t)\psi(q',0)
\end{aligned}$$

Here we've defined $K(q,q';t) = \langle q| \exp\left(-i\hat{\mathbf{H}}t\right)|q'\rangle$. Now to make progress, we need to find a meaningful expression for what $K\left(q,q;T\right)$ actually is. First, 'split up' our $\exp\Big(-i\hat{\mathbf{H}}T\Big)$ term into smaller pieces - that is, partition T as

$$\exp\left(-i\hat{\mathbf{H}}T\right) = \exp\left(-i\hat{\mathbf{H}}\left(t_{n+1}-t_n\right)\right)\exp\left(-i\hat{\mathbf{H}}\left(t_n-t_{n-1}\right)\right)\dots\exp\left(-i\hat{\mathbf{H}}\left(t_1-t_0\right)\right) \qquad (1)$$

here, we set by definition that $t_{n+1} = T > t_n > t_{n-1} > \cdots > t_1 > t_0 = 0$. For example, setting $n = 1$ and inserting one integral as part of the completeness relation, we get that

$$\begin{aligned}
K\left(q,q';T\right) &= \langle q| \exp\left(-i\hat{\mathbf{H}}\left(t_2-t_1\right)\right)\exp\left(-i\hat{\mathbf{H}}\left(t_1-t_0\right)\right)\left|q'\right\rangle \\
&= \int dq_1 \ \langle q| \exp\left(-i\hat{\mathbf{H}}\left(t_2-t_1\right)\right)|q_1\rangle \langle q_1| \exp\left(-i\hat{\mathbf{H}}\left(t_1-t_0\right)\right)\left|q'\right\rangle
\end{aligned}$$

where we've set $t_2 = T$. We can generalise this to the case where we have n time slices. We have that

$$K\left(q,q',r\right) = \int \prod_{i=1}^{n}\left(dq_r \ \langle q_{r+1}| \exp\left(-i\hat{\mathbf{H}}\left(t_{r+1}-t_r\right)\right)|q_r\rangle\right)\langle q_1| \exp\left(-i\hat{\mathbf{H}}\left(t_1-t_0\right)\right)\left|q'\right\rangle \qquad (2)$$

Example Sheet 1

Question 1 (2020)

We use the completeness relation in the position basis.

$$\begin{aligned}\int dx' K(x,t,x',t') K\left(x',t';x_0,t_0\right) &= \int dx' \left\langle x\right| e^{-iH(t-t')} \left|x'\right\rangle \left\langle x'\right| e^{-iH(t'-t_0)} \left|x_0\right\rangle \\ &= \left\langle x\right| e^{-iH(t-t')} e^{-iH(t'-t_0)} \left|x_0\right\rangle \\ &= K(x,t,x_0,t_0)\end{aligned}$$

To show that $f(x)=\delta(x)$, we need to show that $f(x)=0\,\forall x\neq 0$, and that $\int dx\, f(x)=1$.

Question 1 (2018)

We expand the exponential involving λ as

$$\begin{aligned}\mathcal{Z}(\lambda) &= \frac{1}{\sqrt{2\pi}}\int dx e^{-\frac{1}{2}x^2}\sum_{l=0}^{n}\left(-\lambda\frac{x^4}{4!}\right)^l \frac{1}{l!} \\ &= \sum_{l=0}^{n}\frac{1}{\sqrt{2\pi}}\left(-\frac{\lambda}{4!}\right)^l \frac{1}{l!}\int_{\mathbb{R}} dx\, e^{-\frac{1}{2}x^2} x^{4l}\end{aligned}$$

Now, we evaluate the integral using a trick. We arbitrarily set

$$I(\alpha) = \int dx\, e^{\frac{1}{2}\alpha x^2}$$

Differentiating this integral with respect to α, we have that

$$\frac{d^{2l}I}{d\alpha^{2l}} = \int_{\mathbb{R}} dx\left(\frac{1}{2}\right)^{2l} x^{4l} e^{-\frac{\alpha}{2}x^2} = \sqrt{2\pi}\left(\frac{1}{2}\right)^{2l} 1\,(3)\ldots(4l-1)$$

Cancelling out factors and using the standard formula for odd factorials, we get that

$$\int dx\, x^{4l} e^{-\frac{\alpha}{2}x^2} = \sqrt{2\pi}\frac{(4l)!}{4^l\,(2l)!}$$

Substituting this in means that we get our expression for our partition function as

$$\mathcal{Z}_n(\lambda) = \sum_{l=0}^{n}\left(-\frac{\lambda}{4!}\right)^l \frac{(4l)!}{4^l\,(2l)!}$$

Our contributing Feynman diagrams at $l \ll 3$ are shown in the figure. At $l=1$, $a_l=\frac{1}{8}$, which is in agreement with a figure of 8 diagram. At $l=2$, $a_l=\frac{35}{384}$, which agrees with the sum of the automorphism factors at 2 loops.

At $l=3$, $a_l=\frac{385}{3072}$.

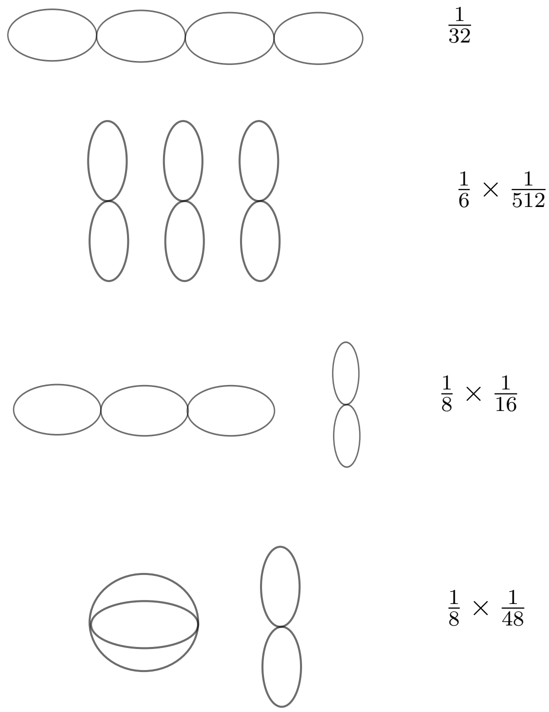

Figure 1: Feynman diagrams and their automorphism factors

We need to sum multiple diagrams, which are connected with n loops to get terms in the expansion. There are two ways to get terms in the expansion. One is to sum all possible diagrams, the other is to sum connected diagrams with a certain number of loops!

What are the possible 3 loop diagrams? What are the automorphism factors? I've tried exponentiating the sum of connected vacuum bubbles - doesn't seem to add up!

7 Useful Identities

7.1 Integral Identities

- The gamma function is defined as

$$\Gamma\left(Z\right) = \int_{0}^{\infty} dx\, x^{z-1} e^{-x}$$