Notes on Black Holes

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Conventions and Housekeeping

We will set $G = c = 1$, and ignore the cosmological constant Λ. For indices, greek letters μ, ν refer to a specific basis. a, b are abstract indices which refer to any basis, from Roger Penrose. For example, we have

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma} \right), \quad R = g^{ab} R_{ab}$$

Christoffel components are basis dependent so they are written with greek indices.

As far as Black holes are concerned, the cosmological constant is incredibly small. Hence, we will ignore it from now. Check out Wald's book on General Relativity. It's great!

1 Spherical Stars

1.1 Cold Stars

If you think about a star like our sun, it's a ball of hot gas with nuclear reactions at its centre. Gravitational force makes the star contract, but nuclear reactions at the centre exert outward pressure to resist the contraction. If we wait long enough, th e star will exhaust the fuel it has and hence the star will contract. If we're interested in the final state of this star, we need to look at some source of pressure which is non-thermal in nature (so valid when the star is cold) to counter act gravity.

There is another natural source of pressure to resist gravity. This is the Pauli principle - where a gas of Fermions resists compression. This is called degeneracy pressure. This is a purely quantum mechanical phenomenon.

For example, a white dwarf is a star in which gravity is balanced by electron degeneracy pressure. A white dwarf is a very dense kind of star. If we had a white dwarf with the same mass of our sum $M = M$. the radius would be $R \sim \frac{R}{100}$. A white dwarf is how the sun will end its life.

Can all stars end their life this way? No. This is because there's a maximum mass for a white dwarf. This is called the Chandrasekhar limit, where

$$M_{WD} \leq 1.4M$$

What happens when we have a star more massive than this? When the star continues to get more and more dense, then inverse beta decay occurs where protons turn into neutrons. Neutrons are fermions and they also have degeneracy pressure. Thus, there's a second class of stars called Neutron stars where gravity is balanced by neutron degeneracy pressure. These stars are tiny.

If we took a neutron star with $M \sim M$, $R \sim 10km$. Compare this to $R \simeq 7 \times 10^5 km$. They are very dense! The gravitational field is very strong. If we have a Newtonian gravitational potential at the surface

$$|\phi| \sim 0.1$$

General relativity because important when $|\phi|$ is order 1. Hence, GR is important here. We can derive a maximum possible mass for neutrons stars as well. We'll derive this bound which is independent of our knowledge of dense matter.

We need to make some simplifying assumptions. We assume spherical symmetry.

1.2 Spherical Symmetry

Recall the unit round metric on a two dimensional sphere S^2,

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

The isometries are diffeomorphisms which preseve the metric. They form a group. The isometry group of a two sphere with that metric is $SO(3)$. Essentially, we define a space time as spherically symmetric if it has this as its isometry group.

Definition. Spherically Symmetric Spacetime A spacetime is spherically symmetric if its isometry group contains an $SO(3)$ subgroup, whose orbits are 2-spheres. In other words, if we pick a point and act on it with all $SO(3)$ elements, it will fill out a sphere.

Definition. Area Radius function In a spherically symmetric spacetime (M, g), the area radius function is

$$r : M \to \mathbb{R}, r(p) = \sqrt{\frac{A(p)}{4\pi}}$$

where $A(p)$ is the area of the S^2 orbit through the point p. So we take a point p, since its spherically symmetric, we can make a sphere out of it, and we take the area. This is linked to our understanding of the r coordinate, but this definition doesn't require a preferred origin. In other words, the S^2 has induced metric $r(p)^2 d\Omega^2$.

1.3 Time independence

Definition. Stationarity (M, g) is stationary if there exists a Killing vector field k^a (KVF) which is everywhere timelike. We're saying that $g_{ab}k^a k^b < 0$. We can introduce adapted coordinates. Let's pick some hypersurface Σ which is transverse to a vector field k. We can then pick coordinates x^i on Σ, where $i = 1, 2, 3$. This gives us coordinates on our surface. We assign coordinates $\left(t, x^i\right)$ to point parameter distance t along the integral curve of k^a through point on Σ with coords x^i. This implies

$$k = \frac{\partial}{\partial t}$$

This implies the metric is independent of t, since k^a is a Killing vector field. Therefore, the metric looks like

$$ds^2 = g_{00}\left(x^k\right)dt^2 + 2g_{0i}dtdx^i + g_{ij}\left(x^k\right)dx^i dx^j$$

This means that $g_{00} < 0$. Thus, given a stationary spacetime, we can construct coordinates which the metric is time independent. Conversely, any metric of this form is stationary.

There's a more refined version of time independence which we can use. Imagine we have a hypersurface Σ where $f = 0$, $f : M \to \mathbb{R}$ which is smooth, and $df \neq 0$ on Σ. Then, df is normal to Σ. If we let t^a tangent to Σ, then we have that

$$df(t) = t(f) = t^\mu \partial_\mu f = 0 \text{ since } f \text{ is constant}$$

A normal to a surface is not unique. Normals are not unique. What's the most general form of a covector field which is normal to Σ. If n_a also normal to Σ, them

$$n = gdf + fn'$$

where g is smooth and not equal to 0 on Σ, and n' is a smooth 1-form. Let's look at the exterior derivative of this vector field. Using the rules for the exterior derivative,

$$dn = dg \wedge df + df \wedge n' + fdn'$$

Let's evaluate this on Σ. We have that

$$dn|_\Sigma = \left(dg - n'\right) \wedge df$$

Hence, $n \wedge dn - 0$ on Σ. This is because $n \propto df$ on Σ. The wedge product vanishes.

Theorem. Frobenius If $n \neq 0$ is a one form such that $n \wedge dn = 0$ everywhere, then there exist functions g, f such that $n = gdf$. So n is normal to surfaces of constant f, so n is 'hypersurface orthogonal'. n is orthogonal to all surfaces of constant f.

Definition. Static spacetimes A spacetime (M, g) is static if there exists a hypersurface orthogonal, timelike KVF. In particular, static implies stationary. Why is this useful? Returning to the adapted coordinates, how does hypersurfaece orgonality help? We choose Σ orthogonal to k^a when defining adapted coordinates $\left(t, x^i\right)$. But, Σ is $t = 0$, therefore the normal to Σ is dt. So

$$k_\mu \mid_{t=0} \propto (1, 0, 0, 0)$$

In particular, we have that

$$k_i \mid_{t=0} = 0$$

but, $k_i = g_{0i}\left(x^k\right)$. Hence, $g_{0i} = 0$. Thus, if we write down the metric, we have that

$$ds^2 = g_{00}\left(x^k\right)dt^2 + g_{ij}\left(x^k\right)dx^i dx^j, \quad g_{00} < 0$$

Thus we have a discrete time reversal isometry $\left(t, x^i\right) \to \left(-t, x^i\right)$. Thus,

$$\text{static} \iff \text{time independent and invariant under time reversal}$$

For example, for a rotating star, the metric may be time independent but not static since they're is no time reversal symmetry since it changes the sense of rotation.

1.4 Static, Spherically Symmetric Spacetimes

The isometry group is $\mathbb{R} \times SO(3)$. We can show that this implies the metric must be static. If it wasn't static, it would be rotating, but that breaks spherical symmetry. On Σ choose coordinates $x^i = (r, \theta, \phi)$, where r is our area radius function. If we do this, then the metric is

$$ds^2 \mid_\Sigma = e^{2\psi(r)} dr^2 + r^2 d\Omega^2$$

where due to spherical symmetry, everything depends on r. If we had $drd\phi$ or $drd\theta$ terms, this would break spherical symmetry. If we then go on to write down the full spacetime metric, we get that

$$ds^2 = -e^{2\Phi(r)} ds^2 + e^{2\Psi(r)} dr^2 + r^2 d\Omega^2$$

At the moment, there's no origin - r is not the distance from the origin! Because of spherical symmetry and stacity, the functions only depend on r.

We model the matter in the star as a perfect fluid. This is a pretty good approximation, where the stress tensor looks like

$$T_{\alpha\beta} = (\rho + p)\, u_\alpha u_\beta + p g_{\alpha\beta}$$

where u_α is the velocity of the fluid, and ρ, p are the energy desnity and pressure in the fluid's rest frame. We have that $g_{ab} u^a u^b = -1$. Time independence means that

$$u^a = e^{-\Phi} \left(\frac{\partial}{\partial t} \right)^a$$

So the fluid velocity is fixed by symmetry assumptions. Symmetry implies that $\rho = \rho(r)$ and $p = p(r)$. Our final observation is that outside the star, there is no fluid. This means that $p, \rho = 0$ for $r > R$, where R is the radius of the star.

1.5 Tolman-Oppenheimer-Volkoff Equations

The equation of motion of a perfect fluid is the conservation law for the energy momentum tensor, which is guaranteed to hold from the Einstein equations (from the Bianchi identity). The Einstein tensor inherits the symmetries from the metric. There are essentially only three independent components, so only three independent equations to solve. We define a new function $m(r)$ by

$$e^{2\Psi} = \left(1 - \frac{2m}{r} \right)^{-1}$$

In particular, the LHS is positive, which means that $m(r) < \frac{r}{2}$. We can then look at the tt component of the Einstein equation, which gives

$$\frac{dm}{dr} = 4\pi r^2 \rho \quad \text{TOV 1}$$

we also get

$$\frac{d\Phi}{dr} = \frac{m + 4\pi r^3 \rho}{r(r - 2m)} \quad \text{TOV 2}$$
$$\frac{dp}{dr} = -(\rho + p) \frac{m + 4\pi r^3 p}{r\,(r - 2m)} \quad \text{TOV 3}$$

The last equation comes from imposing $\nabla_\mu T^{\mu r} = 0$. We have 3 equations in 4 unknowns $(m, \Phi, p\rho)$. But, we get extra information from thermodynamics. In the case of a cold star, we have $T = 0$, but $T = T(p, \rho)$, which means that this implier $p = p(\rho)$, which we call a 'barotropic' equation of state. We assume that $\rho, p > 0$ and also that $\frac{dp}{d\rho} > 0$. If this was not true, then we would have an unstable fluid. For example, if we had some region where the energy density went up, the the pressure would go down, which means that stuff flows into the region, and the density of that region rises up even more.

1.6 Outside the Star: Schwarzchild Solution

In the case where $r > R$, $p = \rho = 0$. This implies by TOV 1 that $m(r) = M$ which is a constant. Then TOV 2 implies that

$$\Phi(r) = \frac{1}{2}\log\left(1 - \frac{2M}{r}\right) + \Phi_0$$

where Φ_0 is a constant. This constant is not physical. Since $\Phi(r) \to \Phi_0$ as $r \to \infty$, then

$$g_{tt} \to -e^{-2\Phi_0} \text{ as } r \to \infty$$

We can eliminate Φ_0 by a coordinate transform $t' = e^{\Phi_0}t$. Putting this all together, we have that

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2 dd\Omega^2 \text{ Schw solution}$$

What is M - a free parameter. We discovered at large r this solution is Minkwoski space. If we add a small correction, we can think of M as the mass of the star. We will see later how we properly define mass in general relativity. In particular, $M > 0$. One thing we notice with this solution is that something funny happens at $r = 2M$. The metric $g_{\mu\nu}$ in these coordinates is singular. This is called the Schwarzchild radius. This means that the star must have a radius bigger than $2M$, so $R > 2M$. For example, for the sun, $2M \simeq 3km$, where as $R \simeq 7 \times 10^5 km$.

1.7 Inside the Star - the Interior Solution

TOV 1 implies that, by integrating,

$$m(r) = 4\pi \int_0^r \rho(r')r'^2 dr' + m_*$$

Let Σ_1 be a $t = const$ surface. The pullback

$$ds^2 \mid_{\Sigma_1} = e^{2\Psi(r)}dr^2 + r^2 d\Omega$$

This is smooth at $r = 0$. This means that close enough to this point, the metric is locally flat at $r = 0$. So if we consider a sphere of radius r, if that value of r is small enough, it should be Euclidean space. A point on S^2 of small radius r must be distance r from $r = 0$. This is saying that

$$r \simeq= \int_0^r e^{\Psi(r')}dr' \simeq e^{\Psi(0)}r$$

for small r. So, smoothness at the origin tells us that $\Psi(0) = 0$. If we go back to the definition of m, this means that $m(0) = 0$. This means that from the above, $m_* = 0$.

By continuity, we have that $m(R) = M$. This means that $M = 4\pi \int_0^R \rho(r) r^2 dr$. This is exactly the same equation we'd get from Newtonian physics. This is just a coincidence. Let's just think about GR here. We should be integrating things with respect to the appropriate volume form. The volume form on Σ_1 is

$$e^{\Phi(r)} r^2 \sin\theta dr \wedge d\theta \wedge d\psi$$

The energy of matter on Σ_1 is the integral of energy density with respect to the volume form. This is

$$E = 4\pi \int_0^R e^{\psi(r)} \rho(r) r^2 dr$$

The fact that $m > 0$ implies that $e^\Psi > 1$. That therefore means that $E > M =$ total energy of star. So, the energy of the matter of the star is made from is larger than that of M. The total energy includes gravitational binding energy. $E - M$ is gravitational binding energy.

Coming back to the bound $R > 2M$, if we reinsert units, the bound is

$$\frac{GM}{c^2 R} < \frac{1}{2}$$

To get to Newtonian theory, we take the speed of light $c \to \infty$. This equation becomes trivial and there is no Newtonian analogue of this bound. This is an intrinsically GR effect.

Let's continue solving the equations. If we look at TOV 3, the right hand side is negative, so

$$\frac{dp}{dr} < 0$$

But since $\frac{dp}{dr} > 0$, we have that $\frac{d\rho}{dr} < 0$. In example sheet 1, we are asked to show

$$\frac{m(r)}{r} < \frac{2}{9}\left[1 - 6\pi r^2 p(r) + \left(1 + 6\pi r^2 p(r)\right)^{\frac{1}{2}}\right]$$

If we set $r = R$ at the surface of the star, the pressure $p = 0$, we get that

$$R > \frac{9}{4} M$$

So we've improved the previous bound. This is called the Buchdahl inequality. This also holds for a hot star. Can we improve on this bound? No, since we can get close as we like to this for a star of constant density.

Going back to solving our equations, we can solve TOV 1 and TOV 3 as they are coupled ordinary differential equations using the fact that $p = p(\rho)$ is known from the equation of state, given $m(0) = 0$, and a specified parameter $\rho(0) = \rho_c$.

From TOV 3, p decreases as r increases, so we define the radius of the star where $p(R) = 0$. This is how we define R. This means that $R = R(\rho_c)$. This implies that $M = M(\rho_c)$. The last thing we need to fix is Φ by solving TOV 2 in $r < R$, with the initial condition $\Phi(R) = \frac{1}{2}\log\left(1 - \frac{2M}{R}\right)$. So we've shown that for a given equation of state, cold stars form a 1 parameter family labelled by ρ_c.

1.8 Maximum Mass of a cold star

If we plot the mass against ρ_c (insert diagram), we have that the maximum mass M_{max} depends on the equation of state. Experimentally, we only know the equation of state up to nuclear density ρ_0. We will show that the maximum mass $M_{\text{max}} \leq 5M$, whatever happens for $\rho > \rho_0$.

We have that ρ decreases with r. We define the core to be the region where $\rho > \rho_0$, or $r < r_0$. We define the envelope as where $\rho < \rho_0$, where $r_0 < r < R$.

Our core mass is $m_0 = m(r_0)$. Our equation for mass previously implies that $m_0 \geq \frac{4}{3}\pi r_0^3 \rho_0$.

The other thing we have is that if we set $r = r_0$, then we have

$$\frac{m_0}{r_0} = \frac{2}{9}\left[1 - 6\pi r_0^2 p_0 + \left(1 + 6\pi r_0^2 p_0\right)^{\frac{1}{2}}\right]$$

where $p_0 = p(r_0)$ is known from the equation of state.

The RHS of the equation above decreases with p_0. So, at $p_0 = 0$, we get the Buchdahl bound

$$m_0 < \frac{4}{9} r_0$$

so the core on its own satisfies the Buchdahl bound. We have two inequalities which we can use to constrain the parameters.

(Insert plot of m_0 against r_0).

This shows that the mass of the core satisfies

$$m_0 < \sqrt{\frac{16}{243\pi\rho_0}}\rho_0 \sim \text{ nuclear density } \leq 5M$$

for any (m_0, r_0) in allowed region, we can solve TOV 1 and TOV 3 in the envelope region with $\rho = \rho_0, m = m_0$ at $r = r_0$. This fixes M in terms of (m_0, r_0). We find numerically that M is maximised when m_0 is maximised. Furthermore, at that point, it turns out the envelope is very small. So, the upper bound of the total

$$M_{\text{max}} \leq 5$$

2 The Schwarzchild Solution

If we're given a Schwarzchild solution with no matter present, what does this solution describe? In this section, we'll explore this question and also look at the nature of singularities in this spacetime.

2.1 Birkhoff's Theorem

In Schw coordinates (t, r, θ, ϕ) the Schwarzchild solution is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega$$

the only free parameter here is M. The location $r = 2M$ is the Schwarzchild radius, and we want to understand what happens here. To derive this metric, what we assumed before was static and spherically symmetric.

Theorem. Birkhoff's Theorem Birkhoff's theorem says that any spherically symmetric solution of the vacuum Einstein equations is isometric to the Schwarzchild solution. In other words, you don't need the static condition on

So

$$\text{Spherically symmetric} + \text{Vacuum} \to \text{static}$$

This is interesting. If we have any star, even if it's time independent, if it's spherically symmetric then the metric outside the star will have a time independent solution. Conversely, we can't have spherically symmetric gravitational waves.

2.2 Gravitational Redshift

(Diagram of Alice and Bob light rays, two vertical lines with coordinate time interval Δt). So assume we have A, B at fixed (r, θ, ϕ). Suppose sends A sends 2 photons to B separated by Δt. By time translation symmetry, we have that the receiving of these photons by Bob also has coordinate time difference Δt. Alice measures proper time. We find that

$$\Delta \tau_A = \sqrt{1 - \frac{2M}{r_A}} \Delta t$$

If $r_B > r_A$, the proper time received by B is

$$\Delta \tau_B = \sqrt{1 - \frac{2M}{r_B}} \Delta t$$

Taking the ratio, we have that

$$\frac{\Delta \tau_A}{\Delta \tau_B} = \frac{\sqrt{1 - 2M/r_B}}{\sqrt{1 - 2M/r_A}} > 1$$

We can also look at this from the perspective of successive wave crests. The time interval in natural units between two successive wave crests is

$$\Delta \tau = \lambda, \quad \lambda_B > \lambda_A \text{ redshift}$$

The light has to crawl out of a potential well. A useful case to consider is when Bob is very far from the Schwarzchild radius. If Bob has radius $r_B \gg 2M$, then the numerator is just 1. We then have the redshift z as

$$1 + z = \frac{\lambda_B}{\lambda_A} = \frac{1}{1 - 2M/r_A}$$

which diverges as $r_A \to 2M$. If we had a spherical star with A on the surface, so that $r_A = R > \frac{9}{4}M$, due to the Buchdahl bound this implies that $z < 2$ for light on the surface of a spherical star.

Let's explore the geometry of the solutions.

2.3 Geodesics

Let's assume we have affinely parametrised geodesics where $x^{\mu}(\tau)$, and we define $u^{\mu} = \frac{dx^{\mu}}{d\tau}$. If we let $k = \frac{\partial}{\partial t}$, this is one killing vector field, and we have another one $m = \frac{\partial}{\partial d\phi}$.

These give us conserved quantities

$$E = -k \cdot u = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}, \quad h = m \cdot u = r^2 \sin^2\theta \frac{d\phi}{d\tau}$$

These are constant along an affinely parametrised geodesic. On a timelike geodesic, we normalise so that τ is propertime. At large r, the Schw spacetime becomes Minkowski spaecetime, so we get special relativity. Using special relativity to understand what these constants are, we have that E represents the energy per unit rest mass, and h represents the angular momentum per unit rest mass.

In the null case, these two quantities are not physical, and rescaling τ allows use to rescale E and h . However, h/E is invariant. This is the impact parameter $b = |\frac{h}{E}|$

What does this mean? If we consider a light ray far away approaching a star, we have that the impact parameter is the closest distance of the ray to the object. From the Euler Lagrange equations, we have that

$$r^2 \frac{d}{dr}\left(r^2 \frac{d\theta}{dr}\right) - h^2 \frac{\cos\theta}{\sin^2\theta} = 0$$

Rotate S^2 such that $\theta(0) = \sin\frac{\pi}{2}$. We can also rotate our coordinates again such that $\dot{\theta}(0) = 0$. In other words, our geodesic lies in, and moves tangentially, to the equatorial plane $\left(\theta = \frac{\pi}{2}\right)$. By uniqueness of solutions, this must be the unique solution to the geodesic equation above. So, the geodesic stays in the equatorial plane.

As an exercise, the normalisation of our parameter means that

$$g_{\mu\nu}u^{\mu}u^{\nu} = -\sigma, \quad \sigma = \begin{cases} 1 & \text{timelike} \\ 0 & \text{null} \\ -1 & \text{spacelike} \end{cases}$$

we can show that we get an equation for r which is

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V(r) = \frac{1}{2}E^2, \quad V(r) = \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(\sigma + \frac{h^2}{r^2}\right)$$

This looks like the equation moving in a classical potential. By looking at qualitative properties of the potential, we can figure out qualitative properties of the potential.

2.4 Eddington Finkelstein Coordinates

Let's start in the region $r > 2M$. The simplest geodesics are the radial null geodesics, where θ, ϕ are constant. This implies that $h = 0$. We can rescale τ to set $E = 1$. This gives us the equation

$$\frac{dt}{d\tau} = \left(1 - \frac{2M}{2}\right)^{-1}, \quad \frac{dr}{dt} = \pm 1$$

We have $\frac{dr}{dt} = 1$ corresponding to outgoing coordinates, and $\frac{dr}{dt} = -1$ corresponding to ingoing.

Ingoing reaches $r = 2M$ in finite time. Dividing the equations above

$$\frac{dt}{dr} = \pm \left(1 - \frac{2M}{r}\right)^{-1} \implies t \to \mp\infty \text{ as } r \to 2M$$

However, if we let $dr_* = \frac{dr}{1-\frac{2M}{r}}$, then we get that

$$r_* = r + 2M \log |\frac{r}{2M} - 1|$$

(Draw a diagram of this function).

Why have we defined this coordinate as such? Well, if we divide through, we have that $\frac{dt}{dr_*} = \pm 1$. This means that $t \mp r_* = C$. Let's define $v = t + r_*$. This is a constant on ingoing radial null geodesics. Let's now introduce new coordinates (v, r, θ, ϕ), which are called ingoing EF coordinates. If we invert this transformation, we get

$$t = v - r_* \implies dt = dv - \frac{dr}{1 - \frac{2M}{r}}$$

Thus, our new metric looks like

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2$$

The metric looks like

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2M}{r}\right) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

This is smooth $\forall r > 0$! We get that $\det g_{\mu\nu} = -r^4 \sin^2\theta$. Thus, this is non-degenerate $\forall r > 0$. Thus, $g_{\mu\nu}$ is Lorentzian $\forall r > 0$.

We can extend this spacetime through $r = 2M$ to a new region $0 < r < 2M$. (Look at analyticity). For $r < 2M$, we now define r_* by going back to the modulus sign, and t, by $t = v - r_*$. We can transform the metric to coordinates (t, r, θ, ϕ). Our exercise is to show that it is the $r < 2M$ Schwarzchild solution.

An ingoing radial null geodesic has $\frac{dr}{d\tau} = -1$ and $v = \text{const}$. This reaches $r = 0$ at finite τ. What happens here? Our curvature diverges. The simplest scalar which we can build from the Riemann tensor is

$$R_{abcd}R^{abcd} \propto \frac{M^2}{r^6} \to \infty, \quad \text{as } r \to 0$$

This scalar diverges in all charts. We can't smoothly extend like we did with the change of coordinates. The metric cannot be smoothly extended through $r = 0$. At $r = 0$, we get what is called a curvature singularity. We get ∞ tidal forces. Strictly speaking, $r = 0$ is not defined to be part of the spacetime since g_{ab} is not smooth there. All we should be talking about is the limiting behaviour as we approach $r = 0$.

Another important thing is to look at the Killing vector field $k = \frac{\partial}{\partial t}$ with $(r > 2M)$. Converting this to EF coordinates. This becomes

$$k = \frac{\partial x^\mu}{\partial t} \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial v}$$

We can use this to define k for $r \leq 2M$. If we look at the norm of the Killing field in new coordinates,

$$k^2 = g_{vv} = -\left(1 - \frac{2M}{r}\right)$$

If we cross the region $r = 2M$, k is null. And, inside the surface, it's spacelike. Since the definiton of being static requires the Killing vector field to be timelike, this means that the interior is not static. Only the $r > 2M$ region is static.

2.5 The Black Hole Region

In this section, we attempt to make rigorous the definition of a 'black hole region'.

Definition. Causal vector A vector is causal if it is either timelike or null. A curve is causal if the tangent vector is everywhere causal.

At any given point in spacetime, the set of causal vectors allows us to define two lightcones, the future lightcone and past lightcone.

Definition. Time Orientable Spacetime A spacetime is time-orientable if it admits a time orientation, which is some causal vector field T^a. We say that a causal vector X^a is future-directed if, at that given point, it lies in the same lightcone as T^a. We call it past directed otherwise.

Note that any other time orientation on the spacetime U^a necessarily lies everywhere in the past or future lightcone of T^a. This means that any given spacetime has two inequivalent time-orientations.

For ingoing Eddington-Finkelstein coordinates, we have that $k = \partial/\partial v$ can't be a valid time orientation since it's spacelike for the region $r < 2M$. But, $\pm\partial/\partial r$ is a globally null vector field.

We would however, like to choose the correct sign based on making it have a consistent orientation with the Killing vector field $\partial/\partial t = \partial/\partial v$.

We find that $k \cdot (-\partial/\partial r) = -g_{vr} = -1$. Thus, the vectors lie in the same light-cone, so $-\partial/\partial r$ is the time-orientation we'll use.

(Insert missing lecture notes here) We can see that the energy of a circular orbit is

$$E = \frac{r - 2M}{r^{\frac{1}{2}} (r - 3M)^{\frac{1}{2}}}$$

If we have $r \gg 2M$, we get that

$$E \simeq 1 - \frac{M}{2r}$$

This implies that the energy is given by

$$\text{energy } = m - \frac{mM}{2r}$$

where m is our initial rest mass, and the second term is our gravitational binding energy. If we're talking about a solar mass black hole, with $M \leq 100M$, these are the black holes formed by gravitational collapse of a star.

How do we see this? The first way that black holes were observed is by looking at binary systems. (Diagram of star and black holes) When matter from the star is stripped off due to tidal forces, the matter surrounds the black hole in an accretion disk. Let's try to understand the properties of one of these disks with a crude model.

Let's approximate the disc as particles following circular orbits. Each particle in the disc has some energy. There's friction amongst the particles. Thus, energy is lost due to friction, and hence the radius decreases, until it reaches the ISCO. The ISCO gives the disc an inner edge of $r = 6M$. When it reaches this and loses more energy, it just falls into the black hole. The energy of the ISCO is $E = \sqrt{8/9}$. The remaining energy fraction is $1 - \sqrt{8/9}$, which is the energy of the rest mass lost to friction. This is carried away from the disc in the form of electromagnetic radiation. This is about 6, which is a huge amount of energy converted. This accretion discs around black holes are very luminous. The first detections are from X-rays in the 1980s.

Now there's another way to detect black holes, which are gravitational waves.

2.6 White Holes

Taking a step back, if we define $r > 2M$ the coordinate $u := t - r_*$, constant along outgoing radial null geodesics, we have the set of coordinates $(u, r\theta, \phi)$, which are outgoing EF coordinates. $ds^2 = -(1 - 2M/r) - 2dudr + r^2 d\Omega^2$. We can smoothly extend this through $r = 2M$ to $r \leq 2M$, with a curvature singularity at $r = 0$. This is not the same as previous $r = 2M$ region. If we look at the outgoing radial null geodesics, , $u = \text{const}$, and $\frac{dr}{d\tau} = 1$. Here, r can only increase in $r \leq 2M$, so it can't be the same region as we saw before since it has different properties.

As an exercise, we can show $k = \partial/\partial u$. In outgoing EF coordinates, we can show $\partial/\partial r$, is the time orientation equivalent to k in $r > 2M$. The $r < 2M$ is a white hole. This is a region which cannot receive a signal from infinity. To understand what a white hole is, it is just the time reverse of a black hole. So, $u = -v$ is an isometry which maps the white hole to the black hole, and reverses the time orientation.

White holes are generally regarded as unphysical, because there is no mechanism for forming them. To form a white hole, we'd have to start with a singularity. White holes are also unstable due to time reversal since black holes are stable.

2.7 The Kruskal Extension

Again, we start off with $r > 2M$, the Kruskal-Szekeres coordinates are

$$U = -e^{-u/4M}, U < 0, \quad V = e^{v/4M}, V > 0$$

This gives (U, V, θ, ϕ). Taking the product, we get that

$$UV = -e^{r_*/2M} = -e^{r/2M}\left(r/2M - 1\right)$$

The right hand side is monotonic, which determines $r\left(U, V\right)$ uniquely. We $V/U = e^{t/2M}$, fixes $t\left(U, V\right)$. Transforming coordinates, we have $ds^2 = -\frac{32M^3}{r(U,V)} e^{-r(U,V)/2M} dU dV r\left(U, V\right)^2 d\Omega^2$. To extend to a larger range of U and V, we can use the equation above to define $r\left(U, V\right)$ for $U \geq 0$, or $V \leq 0$. Metric can then be analytically extended with $\det g_{\mu\nu} \neq 0$, to new regions $U > 0$ or $V < 0$.

At $r = 2M$, $UV = 0$. This corresponds to $U = 0$ or $V = 0$. At $r = 0$, we require $UV = 1$, which are equations of hyperbola.

The shaded regions are not apart of the diagram. Radial null geodesics are constant U or V, depending on ingoing or outgoing. If we pick $r > 2M$, and constant r, this has hyperbolae on either side. We have four regions. Region I has $r > 2M$, which is Schwarzchild. We have region II, which is the black hole region, region III, the white hole region, and region IV, which is new.

Region IV with $R > 2M$, is Isometric to I. We should have $(U, V) \to (-U, -V)$. The interior of a black hole is quite similar to the collapse of a universe. This is somewhat akin to the big crunch. You shouldn't think of a singularity as a place in space. It's more like a region in time.

If we have a star which undergoes collapse, the space time outside the star corresponds to this diagram. (Draw diagram of surface of the star)

Let's now think about the symmetries of this. As an exercise, we can show that

$$k = \frac{1}{4M}\left(V\frac{\partial}{\partial V} - U\frac{\partial}{\partial U}\right), \quad k^2 = -\left(1 - 2M/r\right)$$

Thus, k is timelike in I and IV, null at $U = 0$ and $V = 0$, and spacelike in II or III. Drawing the integral curves of k, we have the diagram below.

Note that the lines $\{U = 0\}, \{V = 0\}$, are mapped to themselves, and so are fixed by k. $k = 0$ is on a bifurcation 2 -sphere $U = V = 0$ which is also fixed by k.

2.8 Einstein Rosen Bridges

For t constant in region I, this happens if and only if V/U is constant. This manifold extends in to IV. If we let $r = \rho + M + \frac{M^2}{4\rho}$, the graph looks like below.

For a given value of r, there are two possible values of ρ. We choose $\rho > M/2$ in region I, and $0 < \rho < M/2$ in region IV.

It is an exercise to show that in isotropic coordinates, we have that (t, ρ, θ, ϕ) is

$$ds^2 = -\frac{\left(1 - \frac{M}{2\rho}\right)^2}{\left(1 + \frac{M}{2\rho}\right)^2} dt^2 + \left(1 + \frac{M}{2\rho}\right)^4 \left(d\rho^2 + \rho^2 d\Omega^2\right)$$

We said earlier that region I and IV are isometric. Thus, we should have some isometry which relates regions I and IV. This is

$$\rho \to \frac{M^2}{4\rho}$$

There is still a singularity in these coordinates at the Schwarzchild radius. If we set t to be a constant, then the induced metric is

$$ds^2 = \left(1 + \frac{M}{2\rho}\right)^4 \left(d\rho^2 + \rho^2 d\Omega^2\right)$$

This is smooth for $\rho > 0$. We can embed this in four dimensional Euclidean space. If we surpress the theta direction, we get that we have two asymptotically flat hypersurfaces connected by a throat region.

We're suppressing the theta directions so it looks like cirles where we have the throat. It is flat as $r \to 0$ since the space is isometric to $r \to \infty$. This is called an Einstein-Rosen bridge.

2.9 Extendibility

Definition. Extendibility We define (M, g) to be extendible if it is isometric to a proper subset of another spacetime (M', g'). This is called an extension of (M, g). For example, take (M, g) to be the $r > 2M$ section of the Schwarzchild manifold. Then, we can take (M', g') to be the extension of (M, g).

If we can make the spacetime bigger, then we do. But, the Kurskal spacetime for example is inextendible. The Kruskal spacetime is an example of a maximal analytic extension of (M, g).

2.10 Singularities

In general, we say that the metric $g_{\mu\nu}$ is singular if it is not smooth or $\det g_{\mu\nu} = 0$ somewhere. A coordinate singularity is a singularity which we can eliminate via a change of coordinates. For example, $r = 2M$ in Schwarzchild coordinates. These are unphysical.

We also have a scalar curvature singularity, where a scalar build from ${R^a}_{bcd}$ diverges. For example, $r = 0$ in Schwarchild.

We can also have examples of singularities in which the curvature does not blow up. These are called non-curvature singularities.

For example, take $M = \mathbb{R}^2$, with polar coordinates (r, ϕ), with $\phi \sim \phi + 2\pi$. Take $g = dr^2 + \lambda^2 r^2 d\phi^2$. For $\lambda > 0$, the determinant $\det g_{\mu\nu} = 0$ at $r = 0$. At $\lambda = 1$, we have that this is Cartesian coordinates, which implies $r = 0$ is a coordinate singularity.

For $\lambda \neq 1$, set $\phi' = \lambda\phi$, which implies $g = dr^2 + r^2 d\phi^2$. This is locally isometric to Euclidean space, so we have that ${R^a}_{bcd} = 0$. This isometry is only local because we've forgotten the period of ϕ. Since $\phi' \sim \phi' + 2\pi\lambda$, this is not globally isometric. At $r = 0$, let's consider the circle $r = \epsilon$. The circumference divided by the radius is

$$\frac{2\pi\lambda\epsilon}{\epsilon} = 2\pi\lambda \neq 2\pi, \quad \text{as } \epsilon \to 0$$

Taking the limit as $\epsilon \to 0$, This is not locally flat at $r = 0$, so the metric can't be smoothly extended to $r = 0$. This is an example of a conic singularity. This is not a curvature singularity since curvature is zero.

Singularities are not points in the manifold since they're not included by construction.

Definition. Future endpoints We say that $p \in \mathcal{M}$ is a future endpoint of a future directed causual curve $\gamma : (a, b) \to \mathcal{M}$, if, for any neighbourhood $\mathcal{O}$ of p, there exists some t_0 such that $\gamma(t) \in \mathcal{O}$ for all $t > t_0$. We say that γ is future inextendible if it has no future endpoint.

For example, take our spacetime (M, g) to be Minkwoski, and the curve $\gamma : (-\infty, 0) \to M$, $\gamma(t) = (t, 0, 0, 0)$. Then, $(0, 0, 0, 0)$ is a future endpoint. However, if we define $(M, g) =$ Mink $\{(0, 0, 0, 0)\}$, then the curve is future inextendible.

Definition. Completeness A geodesic is complete if an affine parameter extends to $\pm\infty$.. We say that (M, g) is geodesically complete if all inextensible causal geodesics are complete.

For example, in Miknowski space, and the spacetime of a static, spherical star, these are geodesically complete.

In the Kruskal spacetime, this is geodesically incomplete since some geodesics reach $r = 0$ in finite affine parameter.

An extendible spacetime is trivially geodesically incomplete. We say that a spacetime is singular if and only if (M, g) is inextendible and geodesically incomplete. For example, the Kruskal spacetime.

3 The Initial Value Problem

3.1 Predictability

Definition. Partial Cauchy Surface Suppose (M, g) is a time orientable spacetime. A partial Cauchy surface Σ is a hypersurface such that no two points are connected by a causal curve in M. So, this could look like a $t = \text{const}$ curve in Minkwoski spacetime, since all points are spacelike separated. The future domain of dependence of Σ is defined as

$$D^+(\Sigma) = \{p \in M : \text{every past-inextendible causal curve through } p \text{ which intersects } \Sigma\}$$

The past domain of dependence $D^-(\Sigma)$ is defined similarly. The entire domain of dependence is the union of the future and past domains of dependence

$$D(\Sigma) = D^+(\Sigma) \cup D^-(\Sigma)$$

A causal geodesic in $D(\Sigma)$ must intersect Σ, which is determined uniquely by tangent vectors on Σ. A causal geodesic is a timelike or null curve, and is either future or past inextendible. It is in the domain of dependence since geodesics are determined by the tangent vector at $p \in \Sigma$, where curve intersects.

The solutions to hyperbolic partial differential equations in $D(\Sigma)$ can be uniquely determined from initial data defined on Σ.

By hyperbolic partial differential equations, we mean equations in the tensor field $T^{(i)ab\ldots}{}_{cd\ldots}$, with equations of motion, $i = 1, \ldots N$ equations of motion

$$g^{ef}\nabla_e\nabla_f T^{iab\ldots}{}_{cd\ldots} = \ldots$$

where the right hand side depends on g and its derivatives, and depends linearly on T and it's first derivatives. These equations are satisfied for example, by Maxwell's equations in the Lorentz gauge.

Example 1. Minkowski Spacetime with the positive x axis as the partial Cauchy surface

For example, let's have (M, g) taken to be 2 dimensional Minkwoski space, and Σ to be the positive x axis. Then, we can draw the domain of dependence as shown in the figure. We draw the domain of dependence by taking a point, and seeing if there exists a timelike curve which doesn't intersect the Cauchy surface. If there is, it's not in the domain of dependence.

If we have a look at the wave equation

$$\nabla^a\nabla_a\psi = -\partial_t^2\psi + \partial_x^2\psi = 0$$

the solution in $D(\Sigma)$ us uniquely determined by the data $(\psi, \partial_t\psi)$ on Σ.

Generally, if $D(\Sigma) \neq M$, then physics in $M/D(\Sigma)$ is not determined by data on Σ.

Definition. (Globally Hyperbolic) (M, g) is globally hyperbolic if there is a Cauchy surface. A Cauchy surface is a partial Cauchy surface such that $D(\Sigma) = M$. The Cauchy Horizon is a boundary of $D(\Sigma)$ in M. A space is globally hyperbolic if there is no Cauchy hoirzon for Σ.

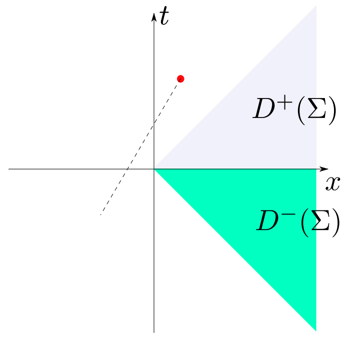

Figure 1: The domain of dependence for the positive x-axis in Minkowski spacetime

Some examples of globally hyperbolic spacetimes is the Minkowski metric ($t = const$) are Cauchy surfaces. Kruskal spacetime is also a globally hyperbolic spacetime.

(Insert diagram here)

We can also look at spherical gravitational collapse, which also is a globally hyperbolic spacetime.

The Minkowski metric with the origin removed is not since there is a future inextendible curve.

Theorem. Let (M, g) be globally hyperbolic. Then, (i), there exists a global time function $t : M \to \mathbb{R}$ such that $-(dt)^a$ is future directed and timelike. Secondly, $t =$ const surfaces are Cauchy, and all have the same topology Σ. Finally, M has topology $\mathbb{R} \times \Sigma$.

As an exercise, show that $U + V$ is a global time function for Kruskal. Note that the surface $U + V = 0$ is an Einstein-Rosen bridge. Topologically, $\Sigma \simeq \mathbb{R} \times S^2$. In this case, $M \simeq \mathbb{R}^2 \times S^2$.

We can also introduce coordinates on the spacetime. Let x^i be coords on $t = 0$ surface, Σ. Let T^a be a timelike vector field with $p \in \mathcal{M}$. The integral curve of T^a through p intersects Σ at a unique point. Let $x^i(p)$ be the coordinates of this. This defined three maps $x^i : M \to \mathbb{R}$. Use (t, x^i) as coordinates on M. So, we have a global set of coordinates. If we write down the most general metric in these coordinates,

$$ds^2 = -N^2 dt^2 + h_{ij} \left(dx^i + N^i dt\right) \left(dx^j + N^j dt\right)$$

$N(x, t)$ is called the lapse function, $N^i(t, x)$ is called the shift vector, and $h_{ij}(t, x)$ is a metric on surfaces of constant t.

3.2 Extrinsic Curvature

Definition. We say the hypersurface Σ is spacelike if the normal n_i everywhere is timelike. If X^a is tangent, then $n_a X^a = 0$ implies that X^a is spacelike.

Assume $n_a n^a = -1$. We now define the quantity $h^a{}_b = \delta^a{}_b + n^a n_b$. This implies that $h_{ab} = g_{ab} + n_a n_b$.

Thus, if X^a, Y^a are tangent vectors, then $h_{ab}X^aY^b = g_{ab}X^aY^b$. h_{ab} is the induced metric of Σ, which is the pullback of g_{ab}. Then $h^a{}_b n^b = 0$. This means that $h^a{}_c h^c{}_b = h^a{}_b$. $h^a{}_b$ is a projection onto Σ. We can decompose the vector as follows

$$X^a = \delta^a{}_b X^b = h^a{}_b X^b - n^a n_b X^b = X^a_{\parallel} + X^a_{\perp}$$

We have that N_a is perpendicular to Σ at p. Parallel transport N_a along $C : X^b \nabla_b N_a = 0$. Does N_a remain $\perp$ to Σ? Suppose that Y^a is tangent to Σ. Then

$$X\left(N \cdot Y\right) = X^b \nabla_b \left(Y^a N_a\right) = N_a X^b \nabla_b Y^a$$

If $N \cdot Y = 0$, then $\left(\nabla_X Y\right)_{\perp} = 0$

Definition. Extend n_a to neighbourhood Σ, $n_a n^a = -1$. The Extrinsic curvature tensor K_{ab} is defined at $p \in \Sigma$ by $K\left(X, Y\right) = -n_a \left(\nabla_{X_{\parallel}} Y_{\parallel}\right)^a$.

Theorem. We have that, independent of the extension of n_a, we have that

$$K_{ab} = h_a{}^c h_b{}^d \nabla_c n_d$$

Proof. We have that

$$\begin{aligned} -n_d X^c_{\parallel} \nabla_c Y^d_{\parallel} &= -X^c_{\parallel} \nabla_c \left(n_d Y^d_{\parallel}\right) + X^c_{\parallel} Y^c_{\parallel} \nabla_c n_d \\ &= \left(n_a{}^c h_b{}^d \nabla_c n_d\right) X^a Y^b \end{aligned}$$

Note that $n^b \nabla_c n_b = \frac{1}{2} \nabla_c \left(n_b n^b\right) = 0$. This implies that $K_{ab} = h_a{}^c \nabla_c n_b$. □

Theorem. Our extrinsic curvature tensor is symmetric, with $K_{ab} = K_{ba}$. Let Σ be a surface where f is constant, with $df \mid_{\Sigma} \neq 0$. Therefore, $n_a \mid_{\Sigma} = g(df)_a$ for some g, fixed by $n_a n^a = -1$. We can use this to extend n_a off Σ.

$$\nabla_c n_d = g \nabla_c \nabla_d f + \nabla_c g \nabla_d f$$

this implies $K_{ab} = g h_a{}^c h_b{}^d \nabla c \nabla_d f$, which is symmetric. A lemma that we have is $K_{ab} = \frac{1}{2} \mathcal{L}_n h_{ab}$.

3.3 Gauss-Codecci equations

The tensor at $p \in \Sigma$ is invariant under projection $h^a{}_b$ if

$$T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s} = h^{a_1}{}_{c_1} \ldots h^{a_r}{}_{c_r} h^{d_1}{}_{b_1} \ldots h^{d_s}{}_{b_s} T^{c_1 \ldots c_r}{}_{d_1 \ldots d_s}$$

can be identified with tensors on Σ. Covariant derivatives on Σ are defined as

$$D_a T^{b_1 \ldots b_r}{}_{c_1 \ldots c_s} = h_a{}^d h^{b_1}{}_{c_1} \ldots h^{b_r}{}_{e_r} h^{f_1}{}_{c_1} \ldots h^{f_s}{}_{c_s} \nabla_d T^{e_1 \ldots e_r}{}_{f_1 \ldots f_s}$$

In actual fact, D_a is the Levi-Civita connection of $\mathcal{L}_{ab}$. The Riemann tensor of D_a is $R'^a{}_{bcd}$. We have that

$$R'^a{}_{bcd} = h^a{}_e h^f{}_b h^g{}_c h^h{}_d R^e{}_{fgh} - 2K^a_{[c} K_{d]} b$$

It is a lemma that the Ricci scalar of D_a is $R' = R + 2R_{ab} n^a n^b - K^2 + K^{ab} K_{ab}$. We have that $K = K^a{}_a = g^{ab} K_{ab} = h^{ab} K_{ab}$.

Theorem.

$$D_a K_{bc} - D_b K_{ac} = h_a{}^d h_b{}^e h_c{}^f n^g R_{defg}$$

This is Codavvi's equation.

Theorem.

$$D_a K^a{}_b - D_b K = h^c{}_b R_{dc} n^d$$

3.4 The Constraint Equations

If we set $G_{ab} = 8\pi T_{ab}$, then

$$G_{ab} n^a n^b = R_{ab} n^a n^b + \frac{1}{2} R = \frac{1}{2}\left(R' - K^{ab} K_{ab} + K^2\right)$$

Thus,

$$R' - K^{ab} K_{ab} + K^2 = 16\pi\rho$$

where $T_{ab} n^a n^b$ is energy density seen by the observer with velocity n^a. This is a Hamiltonian constraint.

$$8\pi h_a{}^b T_{bc} n^c = h_a{}^b G_{bc} n^c = h_a{}^b R_{bc} n^c$$

The last proposition above tells us that

$$D_b K^b{}_a - D_a K = 8\pi h_a{}^b T_{bc} n^c$$

The last quantity is minus the momentum density seen by the observer. This involved the spatial derivative of K.

Summary

Spherical symmetry

- A spacetime is spherically symmetric if the isometry group has an $SO(3)$ subgroup.
- The orbit of $p \in M$ is a S^2 sphere.
- We define the area radius function

  $$r(p) = \sqrt{\frac{A(p)}{4\pi}}$$

  This has the interpretation of radius.

Properties of Spacetimes

To help us, we can construct hyper-surfaces as follows

- We can define a hypersurface Σ to be the surface where $f(x) = 0$
- df is normal to this surface as $df(t) = 0$ for any tangent vector.

- Normals n to the surface can be written as

$$n = gdf + fn' \implies n \wedge dn \mid_{\Sigma} = 0$$

- Frobenius says that if $n \wedge dn = 0$ everywhere, then there exist f, g such that $n = gdf$ so n is normal to surfaces of constant f.

We can classify different spacetimes as below.

- A spacetime is symmetric in a variable s if s is a coordinate but the metric doesn't depend on s.
- A spacetime is stationary if there exist coordinates x^α such that x^0 is timelike at infinity, and our metric doesn't depend on x^0 (equivalent to saying that there's a Killing vector which is timelike at infinity).
- A spacetime is static if there are no cross terms in the metric like g_{0i}.
- Construct stationary spacetimes by defining a Killing vector, then construct a hypersurface Σ nowhere tangent to that vector. Assign spatial coordinates x^i for positions in Σ. Then, construct the coordinate t by moving a distance t in the parameter orthogonal to the hypersurface. Then, the killing vector is

$$k = \frac{\partial}{\partial t}$$

- Our final metric, including spherical symmetry is

$$-e^{2\Psi(r)}dt^2 + e^{2\Phi(r)}dr^2 + r^2 d\Omega^2$$

3.5 The Schwarzchild Solution

3.5.1 Conserved quantities and Geodesics

- The Schwarzchild solution is

$$ds^2 = -\left(1 - 2M/r\right)dt^2 + \left(1 - 2M/r\right)^{-1} dr^2 + dr^2 + r^2 d\Omega^2$$

- **Birkhoff's Theorem.** A spacetime which is spherically symmetric, and a solution of the vacuum Einstein equations is isometric to the Schwarzchild solution. The Schwarzchild solution is static. So spherical symmetry and vacuum matter implies staticity!
- Redshift is the difference between two proper times

$$\frac{\Delta \tau_A}{\Delta \tau_B} = \sqrt{\frac{1 - 2M/r_B}{1 - 2M/r_A}}$$

- Conserved quantities along a geodesic $x^\mu(\tau)$ and a Killing vector field are given by $k \cdot u$, where $u^\mu = \frac{dx^\mu}{d\tau}$. These conserved quantities are E and h. With null geodesics, we are free to rescale E by reparametrisation.
- Geodesic equations from here allow rotations such that $\theta = \pi/2$.

3.5.2 Eddington-Finkelstein Coordinates

- The black hole is the region where a signal can't be sent to infinity - for any future directed causal curve, if $r(\lambda_0) \leq 2M$, then $r(\lambda) \leq 2M$ for all $\lambda \geq \lambda_0$.
- We define white holes with the new coordinate $u = t - r_*$, which is constant on outgoing radial, null geodesics.
- The metric in these new coordinates is

$$ds^2 = -(1 - 2M/r)\, du^2 - 2dudr + dr^2 + r^2 d\Omega^2$$

- Setting $v = -u$ recovers ingoing Eddington-Finkelstein coordinates. The physical interpretation of doing this is time-reversal.

3.5.3 The Kruskal Extension

- We define Kruskal coordinates as

$$U = -e^{-u/4M}, \quad V = e^{v/4M}$$

- Metric looks like

$$ds^2 = -32M^3 \frac{e^{-r/2M}}{r} dUdV + r(U,V)^2\, d\Omega^2$$

- A Kruskal diagram is a diagram showing lines of constant r and constant t on a 45 degree rotated (U, V) axis - there are four regions here.

$$\frac{a}{b} \vec{v} \vec{v} \alpha \sigma \delta ` f \frac{a}{b} \frac{a}{b} a_1 a_2 a_3 a_4$$

) (

4 The formation of black holes

In this section, we'll cover the formation of black holes. To start this discussion, we'll need to discuss the idea of symmetry on manifolds and in metrics. The word 'black hole' should be a big enough hint that the kinds of metrics we'll be considering exhibit spherical symmetry. But, since general relativity is done in the frame-work of both **space** and **time**, we need to make clear what 'spherical symmetry' actually means.

First, lets discuss how to obtain the metric for a standard 2-sphere. Working in signature $(-, +, +, +)$, we have that the metric on the 2 -sphere in Cartesian coordinates is given by the following, with the constraint

$$ds^2 = dx^2 + dy^2 + dz^2, \quad x^2 + y^2 + z^2 = 1$$

If we reparametrize the coordinates as follows, using our standard spherical coordinates with $r = 1$, we embed the 2 -sphere in $\mathbb{R}^3$

$$\begin{aligned} x &= \cos\phi \sin\theta \\ y &= \sin\phi \sin\theta \\ z &= \cos\theta \end{aligned}$$

Applying a coordinate transformation for the one forms dx, dy, dz, we get that

$$\begin{aligned} dx &= \cos\phi \cos\theta d\theta - \sin\phi \sin\theta d\phi \\ dy &= \sin\phi \cos\theta d\theta + \cos\phi \sin\theta d\phi \\ dz &= -\sin d\theta \end{aligned}$$

Substituting into the above, we can read off that the components of the metric are given by

$$ds^2 = d\theta^2 + \sin^2\phi d\phi^2$$

This metric comes from considering the 2-sphere manifold. Now, we know that the symmetry group of a 2-sphere is $O(3)$ if we consider reflections, and just $SO(3)$ if we consider only rotations. Hence, we say that the metric admits an isometry group of $SO(3)$.

Definition. (Isometries on a metric) An **isometry** is a transformation on a metric space which leaves distances between points invariant. The image one has in their mind immediately might be a rotation or a reflection on a two dimensional plane.

5 Useful Identities in General Relativity for Black Holes

In this section, we'll cover some useful identities which may prove useful for doing general relativity. The first one we'll prove is an equation that's useful for proving the divergence theorem for curved space.

Theorem. Divergence of a vector field in terms of $\sqrt{-g}$

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} V^\mu \right) \tag{1}$$

Proof. We first apply the product rule, then make use of a smart rearrangement using logs. We first have that

$$\begin{aligned} \nabla_a V^a &= \partial_\mu V^\mu + \frac{1}{\sqrt{-g}} V^\mu \partial_\mu \left(\sqrt{-g} \right) \\ &= \partial_\mu V^\mu + V^\mu \partial_\mu \left(\log \sqrt{-g} \right) \\ &= \partial_\mu V^\mu + \frac{1}{2} V^\mu \partial_\mu \left(\log\left(-\det g \right) \right) \end{aligned}$$

Where in the last line we wrote out the determinant explicitly. Now the trick here is to use the identity which relates the logarithm of the determinant to the trace of the formal logarithm of a matrix.

Observe that if A is a matrix, which we assume to be positive definite (since our metric is), then it is diagonalisable. Since it's diagonalisable, we can write it in the appropriate basis such that $\exp A = \operatorname{diag}\left(e^{\lambda_1}, \dots, e^{\lambda_n}\right)$ where $\lambda_i \in \mathbb{R}$ are the eigenvalues of the matrix A. This means that the eigenvalues of $\exp A$ are e^{λ_i} for all i. So, since the determinant of the matrix is equal to the product of its eigenvalues, we have that

$$\det\left(\exp A\right) = \exp\left(\operatorname{tr} A\right)$$

If we set $B = \exp A$, then this identity reduces to

$$\det B = \exp\left(\operatorname{tr}\log B\right)$$

Taking the logarithm of both sides once more, we have that finally

$$\log\det B = \operatorname{tr}\log B$$

We can now resume to the question at hand. We rewrite the above using this identity so that

$$\begin{aligned}
\nabla_a V^a &= \partial_\mu V^\mu + \frac{1}{2} V^\mu \partial_\mu \left(\operatorname{tr}\log\left(-g\right)\right) \\
&= \partial_\mu V^\mu + \frac{1}{2} V^\mu \operatorname{tr}\partial_\mu \log\left(-g\right) \\
&= \partial_\mu V^\mu + \frac{1}{2} V^\mu \operatorname{tr}\left(g^{-1}\partial_\mu g\right)
\end{aligned}$$

Where in this case we've put g to denote schematically the matrix $g_{\mu\nu}$ and **not** the determinant. Be careful to observe that the minus signs when differentiating the logarithm cancel out. One can easily verify that $\frac{1}{2} g^{\alpha\beta}\partial_\mu g_{\alpha\beta} = \Gamma^\nu_{\mu\nu}$. Thus, this completes the proof. □

Summary

Miscellaneous Identities

Lie Derivatives

- The Lie derivative for a general tensor is

$$\mathcal{L}_X T^{\mu_1 \dots}{}_{\nu_1 \dots} = X^\sigma \partial_\sigma T^{\mu_1 \dots}{}_{\nu_1 \dots} - \left(\partial_\lambda X^{\mu_1}\right) T^{\lambda \dots}{}_{\nu_1 \dots} - \dots + \left(\partial_{\nu_1} X^\lambda\right) T^{\mu_1 \dots}{}_{\lambda \dots}$$

Differential forms

- A p -form is a totally anti-symmetric rank $(0, p)$ tensor.
- Express p-forms in terms of wedge products

$$X = \frac{1}{p!} X_{\mu_1 \dots \mu_p} dx^{\mu_1}$$

- The exterior derivative on a $p-$form is defined as

$$\left(dX\right)_{\mu_1 \dots \mu_p \mu_{p+1}} = \left(p+1\right)\partial_{[\mu_1} X_{\mu_2 \dots \mu_{p+1}]}$$

- The exterior derivative acts using the standard Leibniz rule.

Killing equation

-

Differential forms

Example Sheet 1

Question 0

We have two causal vectors X, Y at p. We want to show that $X \cdot Y \leq 0$ if and only if they lie in the same lightcone. This is when we set

$$g_{\mu\nu}(p) = \eta_{\mu\nu} = \operatorname{diag}(-1, 1, \dots 1)$$

In the first case, where X is timelike, we can always do a local Lorentz transformation so that the time coordinate is

$$X^\mu = \left(X^0, 0, \dots 0\right)$$

Since X, Y are causal, the time component X^0 and Y^0 are non zero. This means that $X \cdot Y = -X^0 Y^0 \leq 0$. Thus, $X^0 Y^0 \leq 0$. So, they necessarily lie in the same light-cone.

In the case where X is null, we can write

$$X^\mu = \left(X^0, X^0, 0, \dots 0\right)$$

This means that

$$X \cdot Y = -X^0 Y^0 + X^1 Y^1 = X^0 \left(Y^1 - Y^0\right) \leq 0$$

If

Question 2

We want to show Cartan's magic formula.

$$\mathcal{L}_X Y = \iota_X dY + d\left(\iota_X Y\right)$$

For a p-form Y the Lie derivative is

$$\mathcal{L}_X Y = X^\alpha \partial_\alpha Y_{\mu_1 \mu_1 \dots \mu_p} + \left(\partial_{\mu_1} X^\alpha\right) Y_{\alpha \mu_2 \dots \mu_p} + \dots + \left(\partial_{\mu_p} X^\alpha\right) Y_{\mu_1 \dots \alpha}$$

The basic thing to show here is that

$$d\left(\iota_X Y\right)_{\mu_1 \dots \mu_p} = \sum_{i=1,\dots n} \partial_{\mu_i}\left(X^\alpha Y_{\mu_1 \dots \alpha \dots \mu_p}\right)$$

In addition, we have that

$$\iota_X dY = X^\alpha \partial_\alpha Y_{\mu_1 \dots \mu_p} - X^\alpha \partial_{\mu_1} Y_{\alpha \dots \mu_p} - \dots - X^\alpha \partial_{\mu_p} Y_{\mu_1 \dots \alpha}$$

We can use small cases to work out the right signs here.

5.1 Question 3

The conformal Killing equation reads

$$\nabla_a k_b + \nabla_b k_a = \phi g_{ab}$$

Consider the quantity $X_a V^a$, where V^a is the tangent vector field associated with a null geodesic.

$$\begin{aligned} \frac{d}{d\tau}\left(X_a V^a\right) &= V^b \nabla_b \left(X_a V^a\right) \\ &= V^b V^a \nabla_b X_a \\ &= V^b V^a \phi g_{ab} \\ &= 0 \end{aligned}$$

Going into the second line, we use the geodesic equation after the product rule. Going into the third line, we used the conformal Killing equation after symmetrisation. Finally, we use the fact that it's a null geodesic to take it to zero.

Using the Leibniz property of the Lie derivative, if we have $\Omega^2 g_{ab}$,

$$\mathcal{L})_k \left(\Omega^2 g_{ab}\right) = \Omega^2 \phi g_{ab} + 2 g_{ab} \Omega k^\mu \partial_\mu \Omega$$

But this is just a scalar multiple of g_{ab}, so we're good.

5.2 Question 4

I can't find any other convincing argument that $\left(\frac{m}{r^3}\right)' \leq 0$ other than the fact that at large r, we have that $\left(\frac{m}{r^3}\right)' \leq 0$.

Using the fact that $\frac{m(r_1)}{r_1} \geq m\left(r\right) \frac{r_1^3}{r^3}$, we bound the integral from below using this inequality.

$$\begin{aligned} \int_0^r dr_1 \, r_1 \left(1 - \frac{2m\left(r_1\right)}{r_1}\right)^{-\frac{1}{2}} &\geq \int_0^r r_1 \left(1 - \frac{2m\left(r\right)) r_1^2}{r^3}\right) \\ &= \frac{r^3}{2m\left(r\right)} \left[1 - \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}}\right] \end{aligned}$$

5.3 Question 5

With constant density, we have that $m = \frac{4}{3}\pi r^3 \rho$. Substituting this in our differential equation for p, gives us

$$\frac{dp}{dr} = -\left(p + \rho\right)\left(\rho + 3p\right) \frac{4\pi r^3}{3} \frac{1}{r\left(r - 2m\right)}$$

Integrating this between the centre and the surface of the star, and rearranging, we arrive at the equation

$$p_c = \rho \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} - 1$$

$\overline{1-3\left(1-\frac{2M}{R}\right)^{\frac{1}{2}}}$ This value diverges at $M/R = \frac{4}{9}$, which means that the inequality is saturated.

If the inequality was not saturated, then p_c would not have diverged there, which suggests 'room to grow'.

5.4 Question 7

Starting from the expression for a particle moving in the Schwarzchild potential, we have our equation as

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1 - \frac{2M}{r}\right)\frac{b^2}{r^2} = \frac{1}{2}$$

Here, we've rescaled $E = 1$ since we're dealing with null geodesics. At the closest point to the origin, where $\dot{r} = 0$, we have that

$$b = \sqrt{\frac{r^3}{r - 2M}}$$

If we seek a stationary point for b, this is at $r_* = 3M$ for $r_* > 2M$. This corresponds to $b_{\max} = 3\sqrt{3}M$.

b is the distance

Hence, if you fire a photon with a parameter smaller than this radius, it will be 'captured'. So, we can view this as equivalent to an absorbing disk of area $\pi 27M^2$.

5.5 Question 9

The region II in a Kruskal diagram is enclosed by $r = 0$ on the top, and the positive axes for U and V. In this region, $0 < r < 2M$.

For any time-like geodesic, we have the equation of motion as

$$\dot{r}^2 = E^2 - 1 + \frac{2M}{r}$$

Now, we can bound E^2 below as $E^2 \leq 0$. Thus, we have that

$$\dot{r} = \sqrt{\frac{(E^2 - 1)\, r + 2M}{r}} \geq \sqrt{\frac{-r + 2M}{r}}$$

Hence, integrating over this inequality, we have

$$0 \leq \int_0^r dr \sqrt{\frac{r}{2M - r}} = \int_0^{\tau_{\text{total}}} \leq \int_0^{2M} 2 \sqrt{\frac{r}{2M - r}} = M\pi$$

Thus, we get a bound on the total proper time. The right-hand expression can actually be evaluated using the substitution $r = 2M\sin^2\theta$.

A curve which yields $E^2 = 0$ has $\frac{dr}{d\tau}\,|_{\tau=0} = 0$ and $r_0 = 2M$.

5.6 Question 10

Setting $R = R(\rho)$ and $z = z(\rho)$, changing variables in our metric gives

$$ds^2 = d\rho^2 \left(\left(\frac{dR}{d\rho} \right)^2 + \left(\frac{dz}{d\rho} \right)^2 \right) + R(\rho)^2 \, d\Omega^2$$

Comparing coefficients, we have that

$$\frac{R^2}{(R')^2 + (z')^2} = \rho^2, \quad (R')^2 + (z')^2 = \left(1 + \frac{M}{2\rho} \right)^4$$

This implies that

$$R = \rho \left(1 + \frac{M}{2\rho} \right)^2, \quad (R')^2 = 1 - \frac{M^2}{2\rho^2} + \frac{M^4}{16\rho^4}$$

Substituting this into the above, we get that

$$(z')^2 = \frac{M}{2\rho} \left(2 + \frac{M}{\rho} \right)^2, \quad \implies z' = \sqrt{\frac{M}{2\rho}} (2 + \frac{M}{\rho})$$

We just integrate this expression to find z as a function of ρ.