Standard Model Lecture Notes

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1 Introduction

1.1 Introduction

What even is the standard model? You may have heard of this before but we'll try to start from scratch in this course.

Definition. Standard Model The Standard model is a theoretical physics construction (what we call a theory, or a model), which describes all known elementary particles and their interactions, based on relativistic quantum field theory (QFT). This is a major thing - one of the biggest achievements in the history of science so far.

What are the 'ingredients' of the Standard Model? The SM is mostly based on the principle of symmetries, which is the key word here. The most important symmetry to consider here are spacetime symmetries, and we'll list these as well as some other important ones below.

- Spacetime: 3 + 1 dimensional Minkwoski space. Symmetry: The Poincare group.
- We can list the particles of the SM.
  - spin $s = 0$: Higgs. A big part of this course will be to describe the statistics of the Higgs boson.
  - spin $s = \frac{1}{2}$: 3 families of quarks and leptons.
- How do these particles interact? The interactions are
  - $s = 1$: 3 gauge interactions
  - $s = 2$: 1 gravity

  The above gauge interactions are based on Gauge symmetry. This is a local symmetry. In QFT for example, we saw an example of a $U(1)$ gauge symmetry. For the standard model this symmetry is $SU(3)_C \times SU(2)_L \times U(1)_\gamma$. The c stands for colour, which describes the strong interaction. The L stands for left, which describes the elctroweak force, and the γ stands for hypercharge. This symmetry is spontenously broken to get

  $$SU(3)_c \times SU(2)_L \times U(1)_\gamma \to SU(3)_c \times U(1)_{EM}$$

  The Higgs particle, can take a value different from zero at the minimum. The symmetry of the vacuum is then no longer the same as the symmetry of the theory. This is called symmetry breaking.

  We have the following particle representations

  - Quarks and leptons have the representation

    $$3\left[\left(3, 2; \frac{1}{6}\right) + \left(\overline{3}, 1; -\frac{2}{3}\right) + \left(\overline{3}, 1; -\frac{1}{3}\right) + \left(1, 2; -\frac{1}{2}\right) + (1, 1; -1) + (1, 1; 0)\right]$$

    In this order, we have $Q_L, U_R, d_R, L_L, e_R, \nu_R$ The factor of 3 is non-trivial (this is called flavour). So, we have three flavours of quarks, three flavours of leptons, etc. This applies for $s = \frac{1}{2;}$

  - The Higgs representation is $\left(1, 2; -\frac{1}{2}\right)$, which applies for $s = 1$.
  - Our gauge is represented by

$$(8, 1; 0) + (1, 3; 0) + (1, 1; 0)$$

    From left to right, these are gluons, $W^{\pm}, \mathbb{Z}$ bosons, and γ, which is for $s = 1$.
  - Gravity is $s = 2$.

We have some comments

- Interactions given by QFT.
- Our main tool us symmetry!
- Total symmetry: spacetime and internal symmetries (gauge).
- There are also accidental (global) symmetries like lepton and baryon number.
- We also have symmetries which are approximate symmetries like flavour.
- Look at the sum of the hypercharges $\sum \gamma = \sum \gamma^3 = 0$. We also have that $3 = \overline{3}$, and 2 even. (Multiply everything in the vector to get hypercharge).
- Gluons are confined. The same way that quarks are confined. This is an open question to prove that quarks are confined.
- There is rich structure (3 phases, Coulomb charge, Higgs and confinement).

There are other symmetries like baryon number and lepton number, which are accidental and not fundamental - they just happen to be there.

1.2 Motivation

Why should we learn about the SM?

- It is fundamental!
- It is based on elegant principles of symmetry.
- It is true! The SM has been tested in many different ways, with some outstanding prediction. For example ($\left(\mathbb{Z}^0, W^{\pm}\right)$, the Higgs, and so on. There are also precision tests; but there is an anomalous magnetic dipole moment electron.

$$a = \frac{g-2}{2}$$

  We also have measured the fine structure constant

$$\alpha^{-1} =$$

  $\overline{h}\frac{c}{e^2}$

- The standard model is incomplete!

Just to recall; $SU_L(2)$ means that it's the left handed fields which transform.

1.3 The History of the Standard Model

We should know the history of theories to gain a better understanding of why they work. Before the 20th Century, there were only two interactions known, which were gravity and electromagnetism. These interactions were studied famously by Newton and Maxwell from Cambridge. At this point, the discreteness of matter wasn't discovered, and it was not clear that matter was made of atoms.

In 1896, radioactivity was studied by the Curies, Becquerel and Rutherford. α, β and γ rays were discovered. α rays are nuclei of helium, β rays are just electrons, and γ is light. Each of these are indicators of other interactions. For example, α and β decay comes from the strong and weak interactions respectively. In 1897, J. Thompson was playing with some cathode rays, and from computing charge and mass, discovered the electron. He computed the ratio of charge to mass. Years later, his son proved that electrons behaved as waves using diffraction experiments.

From 1900 - 1930, quantum mechanics developed. In 1905, special relativity was discovered by Einstein. These are two basic theories which hints at the existence of the photon. In the same decade, Rutherford and Geiger carried out the gold scattering experiment which gave rise to the model of the atom.

Once we had this, in the 1910s, Aston in 1919 found the 'whole numbers rule', where the nuclei of atoms had whole multiples, which implied the existence of the proton. In that decade, it was also an important experimental development that cosmic rays were discovered through cloud chambers. Cloud chambers are chambers of vapour where you can see the tracks of particles. Also, Einstein discovered general relativity.

In the 1920s, Bose and Fermi came up with quantum statistics (bosons behave differently from fermions). During this time, we also saw the start of QFT, from Jordan, Heisenberg and Dirac. The Dirac equation described the electron but also predicted the existence of the positron (he initially thought it was the proton). In 1931, Dirac came out with this prediction (e^+). On a side note, he also mentioned this in the introduction of his paper on monopoles.

In 1932, Anderson discovered the positron experimentally. Anderson was in Caltech, and in London Blackett also discovered the positron. But, Anderson got there first. This was found by finding particles that reflect in a different direction in a magnetic field.

Chadwick in 1932 discovered the neutron. Pauli predicted the neutrino in 1930. In 1934, Fermi came up with the theory of weak interactions. This was found in β decay,

$$\eta \to p + e^- + \overline{\nu}$$

(Diagram of n splitting to $e^-, p, \overline{\nu}$). There is a coupling here called the Fermi coupling. We still use this theory today at low energies.

In 1934, Yukawa had a theory of strong interactions - which is the force which keeps nucleons together in the atom. He proposed some particles called pions - which are scalar mediators of strong interactions. He did the calculation and computed the strength of the interaction,

$$V(r) \sim \frac{e^{-mr}}{r}$$

This force goes very sharply to zero, since the strong interactions are very short range. He estimated m, the mass of the pion, was $100MeV$. With this mass, the strong interaction would be very important at very short distances.

People started searching for pions, and in 1936, Anderson discovered what we now call the muon, with a mass similar to the pion ($m \sim 100MeV$). But, then people discovered the muon was interacting via the weak interaction. The pion was discovered later, which decay into muons.

In 1932, Heisenberg and in 1936, Condor et al, introduced a nice idea called isospin. People started seeing too many particles and Heisenberg realised that the neutron and proton were very similar. So, they thought that in the same way that electrons have two spins, there should be another internal symmetry that relates the newton and the proton.

In the 1940s, nothing happened in the early 1940s, but 1947 was a wonderful year for physics. This is when people started thinking again about understanding the world and the Lamb shift was discovered. This is what happens when we look at the two energy levels of hydrongen, which have slight difference of energies which could not be explained using quantum mechanics. Lamb presented at a conference, QED was discovered by Scwhinger, Feynman, Dyson and Tomonaga. QED started, renormalisation was a success, and this was a huge triumph for QFT. After this, people started believing in the theory. Three of the above won a Nobel prize for this. At this time, pions π were discovered.

In the 1950s, particle accelerators and bubble chambers were improved. This allowed scientists to produce energies $E \geq MeV$. People always said that the 1950s were a decade of wealth, and this also applies to the discovery of particles. Dozens or particles were discovered, mostly from the strong interaction, and these were called hadrons. They came with different names (kaons, hyperons...) but in the end they classified t into two classes, mesons and baryons.

Mesons are bosons and baryons were fermions. to put some order on this, new ideas were introduced. ONe of these ideas was strangeness, which came from Gell-Mann, Nishijima and Pais. This was kind of like a new charge, which also acts in pairs in decays. Another funny thing happened, which was Parity violation discovered by Lee and Yang. This was in 1956. They went to talk to Wu and she did an experiment with cobalt to produce beta decay, and changed the orientation. She discovered this in 1957.

Parity is not a fundamental symmetry of nature.

After this, there was the discovery of the (anti) neutrino by Cowan. At the same time the V- A property of weak interactions (vector and axial vector) proposed by Marshak and Sudarshan In the same year Pontecorvo proposed neutrino oscillations. Then, Yang and Mills in 1954 came out with an interesting idea which would be a very important component. They realised that something that Schwinger proposed about symmetry. They thought about symmetries which are non-Abelian. In QED, there is $U(1)$ symmetry mediated by the photon. In Yang-Mills, non-Abelian symmetry is proposed with also mediating particles which have not been discovered, so Pauli said it was nonsense.

In the 1950s so many particles were discovered so no one new which particles were fundamental and which were not. However, in 1961 the Eightfold way was discovered by Gell-Mann and Neeman. This was a great development. Neeman was less famous and did a PhD at Imperial after working at the Israeli embassy. They had the idea of adding some structure to the particles, by organising particles as multiplets of representations of groups.

They looked at $SU(3)_{\text{ Flavour }}$. The neutron and proton were two states of one single multiplet, which was part of the 8 dimensional representation of $SU(3)$. $SU(2)$ has rank 1 so it's a one dimensional lattice, and $SU(3)$ has rank 2 so it's a two dimensional lattice. This is represented in the diagram below, of Isospin verus Hypercharge.

There is also a 10 dimensional representation of $SU(3)$, and the Ω^- particle was not yet discovered, but from the lattice picture, Gell-Mann predicted its existence and then found shortly after.

Then, in 1964, again, Gell-Mann and Zweig both in Caltech and CERN came out with a very nice idea. We all know that from $SU(3)$ has fundamental representations, and from those we get the other representations. The fundamental representation of $SU(3)$ is three dimensional, and this motivated the idea of quarks. The quarks are up, down and strange:

$$u, d, s$$

However, people didn't take this seriously at first. But, how did people figure this out?

$$3 \otimes 3 \rightarrow \text{mesons}, \quad s = 0, \quad 3 \times \overline{3} = 8 + 1$$

$$3 \otimes 3 \otimes 3 \rightarrow \text{baryons}, \quad s = \frac{1}{2}, \quad 3 \times 3 \times 3 = 10 + 8 + 8 + 1$$

There was a problem with this. In the 10 representation, we have fermions so we couldn't have 3 quarks in one state. However, in 1964 Greenberg and in 1965 Nambu and Han introduced the idea of colour so that we could resolve this issue.

In 1967, Deep inelastic scattering gave the first piece of evidence that there was something inside protons and neutrons. There was evidence of substructure in protons and neutrons. Then people started to take the idea of quarks seriously.

Back in 1961, the idea of symmetry breaking by Nambu, Goldstone, Salam and Weinberg was put out. Yang originally proposed the idea of non-Abelian symmetries. Nambu got the idea of symmetry breaking from superconductivity. Goldstone realised that when you break a symmetry into a smaller symmetry group there should be one particle that should be massless.

This was the idea of Goldstone bosons which are massless.

In 1964, the Higgs mechanism was discovered. This was due to Higgs, Brout, Englert, Kibble. If there is a broken symmetry locally then the gauge field is massive and and Goldstone boson is eaten to give a massive particle. This solves two problems at once. We can have non-Abelian gauge symmetries and brokem gauge symmetries.

Everyone wanted to do this to learn about the strong interactions. In 1967 and 1968, Weinberg and Salam did this for the weak interaction and this gave rise to electroweak unification.

$$SU(2)_L \times U(1)_\gamma \rightarrow U(1)_{\text{ EM}}$$

Glashow identified in 1962 $SU(2) \times SU(1)$.

In 1964, CP violation by Cronin and Fitsch was shown experimentally. This is when a particle changes to and anti-particle.

In the 1970s, the Glashow-Iliopoulos-Maiani (GIM) mechanism was discovered to explain no flavour changing NCs, which implies the existence of a new quark - the charm quark (C).

In hindsight, it was totally obvious that was had to have a fourth quark. In 1969, Jackiw, Bell and Adler discovered anomalies, which suggested that the strange quark needed a partner.

In 1974, J/ψ particles were discoverd so this implied the existence of the charm quark. In 1973, weak neutral currents were discovered. This was one of the thinks put forward by the Weinberg-Salam model.

Then, in 1975 and 1979, jets (quarks, gluons). Take a pion give it more and more energy, to get the quarks out of it. But, this stretches out the pion into a string and creates quark and anti-quark pairs which is evidence for their existence. For example,

$$e^+e^- \to \vec{q}\vec{q} \to 2 - \text{jets}$$

We also have

$$\vec{q}g\vec{q} \to 3 - \text{ jets}$$

This was evidence for 3 colours.

In 1973, asymptotic freedom was discovered by Gross and Politzer. The strength of the coupling of the strong interactions (Insert graphs here).

In the 1970s, the τ lepton was discovered. In the 1980s, the $\mathcal{Z}^0, \mathcal{W}^+$ bosons were discovered. In the 1990s, the top quark was discovered. Now we have the three families

$$\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix}$$

which are 3 families. In 2000s, the tau neutrino was discovered. And, the Higgs boson was discovered. Why was it that the 8 and 10 dimensional representations correspond to particles and other representations don't?

2 Spacetime Symmetries

In the standard model, we can decompose symmetries into a tensor product of spacetime and internal symmetries.

$$\text{symmetries} = \text{spacetime} \otimes \text{internal}$$

In 1967, Coleman and Mandula showed that you cannot mix the two. Internal symmetries are how fields which describe particles transform. As for spacetime symmetries, we know that special relativity provides the right description locally, so we know that the spacetime symmetries are the Poincare group.

In this chapter, we'll be looking at spacetime symmetries.

2.1 Poincare Symmetries and Spinors

A general transformation in the Poincare group is of the form

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \quad \mu = 0,1,2,3$$

where $\Lambda \in O(3,1,)$, the Lorentz group, and a^μ is a translation in ^{4}. The whole transformation group is referred to as the semi-direct product $O(3,1) \wedge^4$, since the two component groups don't commute.

This set of transformations are motivated by the fact that they leave invariant

$$ds^2 = dx^\mu \eta_{\mu\nu} dx^\nu, \quad \eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$$

The Lorentz boost is defined such that for $\Lambda \in O(3,1)$,

$$\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\rho\sigma} \quad \Lambda^T \eta \Lambda = \eta$$

This implies that, taking the determinant of both sides of the equation implies that $\det \Lambda = \pm 1$. Taking the 00 component of the equation implies that

$$\left(\Lambda^0{}_0\right)^2 - \left(\Lambda^1{}_0\right)^2 - \left(\Lambda^2{}_0\right)^2 - \left(\Lambda^3{}_0\right)^2 = 1$$

This implies that $|\Lambda^0{}_0| \geq 1$, so we have that $\Lambda^0{}_0 \leq 1$ or $\Lambda^0{}_0 \geq 1$. With the choice of determinant and this choice, we have that $O(3,1,)$ has four disconnected components.

We denote $SO(3,1)^\uparrow$ as the proper orthochronous Lorentz group as the choice of $\det \Lambda = 1, \Lambda^0{}_0 \geq 1$ This keeps the orientation and keeps the object in its respective lightcone.

Any other element of $O(3,1)$ can be obtained by combining elements of $SO(3,1)^\uparrow$ with

$$\{I, \Lambda_P, \Lambda_T, \Lambda_{PT}\}$$

which corresponds to identity, parity, time-reversal, and the combination of the two.

$$\begin{aligned} \Lambda_P &= \mathrm{diag}(+1,-1,-1,-1) \quad \text{Parity} \\ \Lambda_T &= \mathrm{diag}(-1,+1,+1,+1) \quad \text{Time reversal} \end{aligned}$$

Λ_{PT} is the product of the above. These four elements form the Klein group. From now on, to save ink, we'll just call $SO(3,1)^\uparrow \to SO(3,1)$. From now on, we're also working with the component which is connected to the identity. From this, we can construct the corresponding Lie algebra, which is called the Poincare Algebra. We do the standard thing and do an infinitesimal transformation. We write

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \quad a^\mu = \epsilon^\mu, \quad \omega^\mu{}_\nu, \epsilon^\mu \ll 1$$

When we do the identity transformation infinitesimally, we have that

$$\Lambda^T \eta \Lambda = \eta \implies \left(\delta^\mu{}_\rho + \omega^\mu{}_\rho\right) \eta_{\mu\nu} \left(\delta^\nu{}_\sigma + \omega^\nu{}_\sigma\right) = \eta_{\rho\sigma}$$

This implies that $\omega_{\rho\sigma} = -\omega_{\sigma\rho}$. This implies that there are only 6 parameters for the Lorentz transformations. Also, we have an extra 4 dimensions which come from translations from ϵ^μ. Therefore, in total, we have 10 dimensions in the Poincare group.

The Poincare group is a group with ten dimensions and ten non trivial generators. The group is a manifold. We can take the coset of the Poincare group, quotient it out by the Lorentz group, then we get Minkowski spacetime.

We want to study the algebra of representations of the Poincare group on Hilbert space. We're working with a state $|\psi\rangle$, and then mapping it with a unitary transformation

$$|\psi\rangle \to U\left(\Lambda, a\right)|\psi\rangle\,, \quad U^{-1} = U^\dagger$$

Near the identity, we have

$$U\left(1+\epsilon, \epsilon\right) = I + \frac{-i}{2}\omega_{\mu\nu}\mathcal{M}^{\mu\nu} + ii\epsilon_\mu P^\mu + \dots$$

$\mathcal{M}^{\mu\nu}$ are generators of $SO(3,1)$, and P^μ generates translations. Thus, $\mathcal{M}^{\mu\nu} = -\mathcal{M}^{\nu\mu}$.

In terms of the algebra

- Translators commuting implies that

$$[P_\mu, P_\nu] = 0$$

- What is $[P^\sigma, M^{\mu\nu}]$? For this we will use the following. P^μ has an index, so it's a vector. But, it's also an operator acting on Hilbert space. So since it's a vector, we know how it transforms around boosts, namely,

$$P^\sigma = \Lambda^\sigma{}_\rho P^\rho \simeq \left(\delta^\sigma{}_\rho + \omega^\sigma{}_\rho\right)P^\rho = P^\sigma + \frac{1}{2}\left(\omega_{\alpha\rho} + \omega_{\rho\alpha}\right)\eta^{\sigma\alpha}P^\rho = P^\sigma + \frac{1}{2}\omega_{\alpha\rho}\left(\eta^{\sigma\alpha}P^\rho - \eta^{\sigma\rho}P^\alpha\right)$$

  We're just playing with the indices here. That comes from the fact that P transforms as a vector. But, we also require that P transforms as an operator in Hilbert space. This means that

$$P^\sigma \to U^{\dagger q}P^\sigma U = \left(1 + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right)P^\sigma\left(1 - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right)$$

  Comparing this transformation from operator with the transformation as a vector, we get that

$$[P^\sigma, M^{\mu\nu}] = -i\left(P^\mu\eta^{\nu\sigma} - P^\nu\eta^{\mu\sigma}\right)$$

  It is left the reader to prove why the commutator of the translation generators vanishes.

- Similarly, we have that

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(M^{\mu\sigma}\eta^{\nu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\rho} - M^{\nu\sigma}\eta^{\mu\rho}\right)$$

  This is how we define the generators of the transformation.

- A four dimensional matrix representation of $M^{\mu\nu}$ is

$$\left(M^{\rho\sigma}\right)^\mu{}_\nu = -i\left(\eta^{\mu\sigma}\delta^\rho{}_\nu - \eta^{\rho\mu}\delta^\sigma{}_\mu\right)$$

We have some comments about what there transformations really are.

- $P^0 = \mathcal{H}$ implies that $\left[P^0, P^\mu\right] = 0$, which is energy and momentum conservation.
- $\left[P^0, M^{ij}\right] = 0$ implies angular momentum conservation. M^{ij} are the generators of rotations.
- $\left[P^0, M^{0i}\right] \neq 0$ This means that energy is not conserved under Lorentz boosts! implies no conservation laws here.

The algebra of $SO(3,1)$ is determined by the algebra of $SU(2) \otimes SU(2)$, not equal to. We will see why this is the case. Define $J_i = \frac{1}{2}\epsilon_{ijk}M^{jk}$. We also define $M_{0i} = K_i$, Hermitian. We can see from this that these satisfy the algebra

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k, \quad [K_i, K_j] = -i\epsilon_{ijk}J_K$$

We define

$$A_i = \frac{1}{2}\left(J_i + iK_i\right), \quad B_i = \frac{1}{2}\left(J_i - iK_i\right)$$

Note that these operators are not Hermitian. These satisfy the algebra

$$[A_i, A_j] = i\epsilon_{ijk}A_k, \quad [B_i, B_j] = i\epsilon_{ijk}B_k, \quad [A_i, B_j] = 0$$

The commutation relations from A_i and B_i follow the $SU(2)$ algebra. Since we know how to handle the algebra of $SU(2)$, we can write down some representations.

For representations of $SU(2) \otimes SU(2)$, we have states labelled by $j = 0, \frac{1}{2}, \ldots$. Then, the A_i algebra has states labelled by $A = 0, \frac{1}{2}, \ldots$, and the B_i algebra has states labelled by $B = 0, \frac{1}{2} \ldots$. We then label representations of $SO(3,1)$ as (A, B).

Note that under parity, we map

$$P : J_i \to J_i; \quad K_i \to -K_i$$

this can be seen because J has two spatial indices, where as K_i only relies on one. Thus parity swaps A_i and B_i. Thus,

$$\begin{aligned} P : & A_i \leftrightarrow B_i \\ (A, B) & \leftrightarrow (B, A) \\ \text{left} & \leftrightarrow \text{right} \\ \text{right} & \leftrightarrow \text{left} \end{aligned}$$

We have another comment on this as well.

We have $SO(3,1) \simeq SL(2, \mathbb{C})$, homomorphic. This is important, so we'll prove this. We have the four vector

$$X = X_\mu e^\mu = (X_0, X_1, X_2, X_3)$$

Under a Lorentz transformation, $X \to \Lambda X$, with $|X|^2 = X_0^2 - X_1^2 - X_2^2 - X_3^2$ invariant. Now we do something a bit different. Instead of using e_i as a basis, we now change basis. Take the basis of 2×2 matrices.

$$\mathcal{B} = \{I, \sigma_x, \sigma_y, \sigma_z\}$$

which we denote with an index as σ^μ. We write $\tilde{X}$ as a linear combination of these matrices, and this is

$$\tilde{X} = X_\mu \sigma^\mu = \begin{pmatrix} X_0 + X_3 & X_1 - iX_2 \\ X_1 + iX + 2 & X_0 - X_3 \end{pmatrix}$$

Furthermore, an action of $SL\left(2, \mathbb{C}\right)$ on this matrix will do the following. Define the action of $SL(2, \mathbb{C})$ acting on $\tilde{X}$ as

$$\tilde{X} \to N\tilde{X}N^\dagger \quad N \in SL(2, \mathbb{C})$$

Since the determinant $\det N = 1$, this leaves $X_0^2 - X_1^2 - X_2^2 - X_3^2$ invariant.

This is a map from $SL(2, \mathbb{C}) \to SO(3, 1)$. Hence, this these groups are homomorphic. The map is 2 to 1 since $N = \pm 1$ is mapped to $\Lambda = 1$. $SL(2, \mathbb{C})$ is nice to work with since as a manifold, it is simply connected unlike $SO(3, 1)$. This means that we can work with the geometry of the whole space. $SL(2, \mathbb{C})$, is the covering group of $SO(3, 1)$.

We will prove that $SL(2, \mathbb{C})$ is simply connected, which then implies that $SO(3, 1)$ is doubly connected.

we use the fact that we can use polar decomposition to decompose N.

$$N = e^h U, \quad h\text{hermitian}, \quad U\text{unitary}$$

Since $\det N = 1$, this implies that $\operatorname{tr} h = 1$ and $\det U = 1$. This is because $\det e^h = e^{\operatorname{tr} h}$. The most general h we can write down is

$$h = \begin{pmatrix} a & b + ic \\ b - ic & -a \end{pmatrix} \quad U = \begin{pmatrix} x + iy & z + iw \\ -z + iw & x - iy \end{pmatrix}$$

This imposes the condition that $a, bc \in \mathbb{R}$. We also have that $x^2 + y^2 + z^2 + w^2 = 1$, which is S^3. This means that the $SL(2, \mathbb{C})$ manifold is $\mathbb{R}^3 \times S^3$. Then the $SO(3, 1)$ manifold is $\mathbb{R}^3 \times S^3/\mathbb{Z}_2$.

Let's look at The representations of $SL(2, \mathbb{C})$. The fundamental representation of $\psi_\alpha, \alpha = 1, 2$ is given by

$$\psi_\alpha = N_\alpha{}^\beta, \quad \alpha, \beta = 1, 2$$

We call ψ_α left-handed Weyl spinors. We also have the conjugate representation with with the right handed Weyl spinors $\overline{\chi}_{\dot{\alpha}}$.

$$\overline{\xi}'_{\dot{\alpha}} = N^*{}_{\dot{\alpha}}{}^{\dot{\beta}} \overline{\chi}_{\dot{\beta}}$$

We have the contra variant representations.

$$\psi'^\alpha = \psi^\beta \left(N^{-1}\right)_\beta{}^\alpha ; \overline{\chi}'^{\dot{\alpha}} = \overline{\chi}^{\dot{\beta}} \left(N^{*-1}\right)$$

We use invariant tensors

- $SO(3, 1)$, which has invariant tensor $\eta^{\mu\nu} = \left(\eta_{\mu\nu}\right)^{-1}$. Invariance means we rase and lower indices.
- For $SL(2, \mathbb{C})$, we have that

  $$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{\alpha\beta} = -\epsilon_{\dot{\alpha}\dot{\beta}}$$

  which is the invariant tensor.

- We have ${\epsilon'}^{\alpha\beta} = \epsilon^{\rho\sigma} {N_\rho}^\alpha {N_\sigma}^\beta = \epsilon^{\alpha\beta} \det N = \epsilon^{\alpha\beta}$.

Recall that Dirac spinors are given by $\psi_D = \begin{pmatrix} \phi_\alpha \\ \overline{\chi}^{\dot{\alpha}} \end{pmatrix}$ the top spinor is represented as $\left(\frac{1}{2}, 0\right)$, and the bottom spinor is represented as $\left(0, \frac{1}{2}\right)$.

2.2 Unitary Representations of the Poincare Group

2.2.1 Warming up with representations of $SO(3)$

So far, using special relativity and quantum mechanics, we've achieved alot. To proceed further, we need to build representations of the Poincare group. A lot of this work was done by Wigner in 1939.

To warm up, we'll refresh our memory of how to build finite irreducible representations of the rotation group in three dimensions. The rotation group in 3D is $SO(3) \simeq SU(2)$. The Lie algebra has the generators J_i with the algebra $[J_i, J_k] = i\epsilon_{ijk} J_k$. In terms of matrices, these things cannot be simultaneously diagonalised since they do not commute with one another.

However, there is an operator called the Casimir operator. This operator is something we've seen before; it's the total angular momentum operator

$$J^2 = J_1^2 + J_2^2 + J_2$$

which has the property that $\left[J^2, J_i\right] = 0$, so it commutes with everything in our algebra. This means, we can label representations with the eigenvalues of this operator. J^2 labels a particular representation or 'multiplet'. We set

$$J^2 \left|j\right\rangle = j\left(j+1\right)\left|j\right\rangle, \quad j = 0, \frac{1}{2}, 1, \dots$$

So, we want to find representation which are labelled by this. For compact group, the number of Kashmir operators is equal to the rank of the group. The rank of $SU(2)$ is one, so this is the only Casimir operator we have.

Now, within one single representation, we can have several states which are part of the representation. How do we differentiate between states in one representation? To do this, we pick one of the J_i's, then simultaneously diagonalise with the Casimir operator.

For instance, take J_3, with eigenvalues j_3. So now we have the states

$$J_3 \left|j; j_3\right\rangle = j_3 \left|j; j_3\right\rangle, \quad j_3 = -j, -j+1, \dots, j$$

2.2.2 Building representations of the Poincare group

We will follow exactly the same steps, but now for the Poincare group. For the Poincare group, we have the generators P^μ and $M^{\mu\nu}$. The next step, is to identify the Casimirs.

Our two operators are

$$C_1 = P^\mu P_\mu, \quad C_2 = W^\mu W_\mu$$

where
$$W_\mu := \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}, \quad \text{the Pauli-Ljubanski vector}$$
These Casimir operators have the property that the commute with the generators. One can easily check that
$$[C_{1,2}, P^\mu] = [C_{1,2}, M^{\mu\nu}] = 0$$
Since we have two Casimir operators, we require two labels for a given representation. Note that we have not fully justified that these indeed are the only two Casimir operators, but this is the case. Just for completeness, we'll write down the algebras for P and W. Using the fact that $[W_\mu, P_\nu] = 0$, we have that
$$[W_\mu, M_{\rho\sigma}] = i\left(\eta_{\mu\rho}W_\sigma - \eta_{\mu\sigma}W_\rho\right), \quad [W_\mu, W_\nu] = -i\epsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma$$
Be careful, however. Note that this is not strictly an algebra since the last commutation involves a product of operators, not a linear combination.

So, we can label representations by eigenvalues of C_1 and C_2. Now we have to pick some set of operators in the algebra which mutually commute. We can pick the momenta for this, since we found earlier that the set $\{P^\mu\}$ commute.

Hence, within this representation, we've picked a subset of the generators that can be simultaneously diagonalised, which is $\{C_1, C_2\} \cup \{P^\mu\}$ For example, P^μ with eigenvalue p^μ. Now, $C_1 = P^\mu P_\mu$ has eigenvalues $p^\mu p_\mu > 0$, or equal to zero, or less than zero. In the case that $p^\mu p_\mu > 0$, we use a standard technique to Lorentz transform to a frame where $p^\mu = (m, 0, 0, 0)$. Rotations M^{ij} with leave p^μ invariant, and hence J_i keep p^μ invariant. This is called the Little group $SO(3)$.

We can also thus simultaneously diagonalise by J^i, which gives us a total of four parameters to label our states with. We call this the multiplet labelled by
$$|C_1, C_2; p^\mu, j_3\rangle$$
Now, we wish to find the associated eigenvalues for the C_1 and C_2 operators. We have that
$$\begin{aligned} C_1 &= P^\mu P_\mu = m^2 \\ C_2 &= W^\mu W_\mu = ?, \quad W_\mu = (0, -mJ_i) \implies C_2 = m^2 J^2 \end{aligned}$$
C_1 is fairly obvious, but W_μ less so. If we work in the rest frame, we necessarily require that the ν index in the definition is zero, and this leaves the spatial generator part of $M^{\mu\nu}$. This implies that the multiplet, in terms of eigenvalues, is labelled by
$$|m, j; p^\mu, j_3\rangle$$
This is a particle for a massive one particle state!

The next case we could have is taking, for $P^\mu P_\mu = 0$, that
$$P^\mu = (E, 0, 0, E)$$
In this case, $C_1 = 0$. Computing the vector of (W_0, W_1, W_2, W_3), is done similarly as before, by looking over the components. We have that
$$(W_0, W_1, W_2, W_3) = E\left(J_3, -J_1 + K_2, -J_2 - K_1, -J_3\right)$$

this gives us the commutation relations

$$[W_1, W_2] = 0, \quad [W_3, W_1] = -iEW_2, \quad [W_3, W_2] = iEW_1$$

This algebra gives us a combination of rotations and translations in 2 dimensions, which is the 2D Euclidean group. This group has infinite dimensional representations, and thus we haven't seen these particles before. These are infinite dimensional representations which cause a problem!

For now, we remedy this problem by setting some things to zero. To remedy this, set $W_1 = W_2 = 0$, which is still consistent with the algebra above. It's then easy to see that W_3 generates $SO(2)$ on its own, which are rotations around x_3. Setting W_1 and W_2 to zero means that our new W_μ vector is as follows

$$W_\mu = EJ_3\,(1, 0, 0, -1) \propto p_\mu$$

Crucially, we have that the vector $W_\mu = \lambda p_\mu$, where λ is called the Lorentz scalar. Hence, $C_1 = p^\mu p_\mu = 0$, and $C_2 = W^\mu W_\mu = 0$. Thus, we have an irrep represented by

$$|0, 0, p^\mu, \lambda\rangle := |p^\mu, \lambda\rangle$$

the eigenvalues of J_3 is called helicity. $\lambda = 0, \pm\frac{1}{2}, \pm 1, \ldots$, so in this case

$$e^{2\pi i\lambda}\,|p^\mu, \lambda\rangle = \pm\,|p^\mu, \lambda\rangle$$

$\lambda = 0$ is the Higgs, $\frac{1}{2} = \lambda$ are quarks and leptons, $\pm 1 = \lambda$ are gauge fields, ± 2 are gravitons. We haven't seen $\lambda = \pm\frac{3}{2}$, which are called gravitini. Beyond $\lambda = \pm 2$.

When $p^\mu = (0, 0, 0, 0)$, this is called the vacuum. When $p^\mu p_\mu < 0$, these are called Tachyons.

2.3 Discrete Spacetime Symmetries

3 Gauge Symmetries

4 Symmetry Breaking

5 Electroweak Unification

6 QCD

7 Phenomenology of the SM

8 EFT's and open questions

Example Sheet 1

Question 2

If our spinor transforms under parity as $\psi \to P\psi P^{-1} = \gamma^0 \psi(x_P)$, then it is easy to see that

$$\overline{\psi} \to \overline{\psi}(x_P)\gamma^0$$

Thus, if we transform our Lagrangian with $g' = 0$ by inserting parity operators, we get that

$$\begin{aligned} P\mathcal{L}_I P^{-1} &= g\left(P\overline{\psi}P^{-1}\right)\left(P\psi P^{-1}\right)\left(P\phi P^{-1}\right) \\ &= g\overline{\psi}(x_P)\gamma^0\gamma^0\psi(x_P)P\phi P^{-1} \end{aligned}$$

If we wanted to enforce invariance, we require that $P\phi(x)P^{-1} = \phi(x_P)$. This is because now,

$$\int d^x \mathcal{L}(x) \to \int d^4x_P\, P\mathcal{L}(x)P = \int d^4x_P\, \mathcal{L}(x_P)$$

Now we look at the case with $g = 0$. Similarly, we use the fact that γ^5 anti-commutes with γ^0 to show that for

$$\mathcal{L}_I = ig'\overline{\psi}(x)\gamma^5\psi(x)\phi(x)$$

we require $O\phi(x)P^{-1} = -\phi(x_P)$, for our Lagrangian to be invariant.

For parity to be conserved, we need that $[P, H] = 0$, or equivalently that $[P, \mathcal{L}] = 0$. If g and g' are both non-zero, then parity is not even well defined.

Note: commuting with the parity operator is equivalent to $P\mathcal{L}P^{-1} = \mathcal{L}$. The axial vector transforms as follows.

$$\begin{aligned} j^\mu &\to P\overline{\psi}P^{-1}\gamma^\mu\gamma^5 P\psi P^{-1} \\ &= \overline{\psi}(x_P)\gamma^0\gamma^\mu\gamma^5\gamma^0\phi(x_P) \\ &= \overline{\psi}(x_P)\gamma^0\gamma^\mu\gamma^0\gamma^0\gamma^5\gamma^0\phi(x_P) \\ &= -\overline{\psi}(x_P)\gamma^0\gamma^\mu\gamma^0\gamma^5\phi(x_P) \\ &= \begin{cases} -\overline{\psi}(x_P)\gamma^\mu\gamma^5\phi(x_P) & \mu = 0 \\ \overline{\psi}(x_P)\gamma^i\gamma^5\psi(x_P) & \mu = 1, 2, 3 \end{cases} \end{aligned}$$

Question 3

Our strategy is to show that $B^{-1}\psi^*(x)$ is a solution to the time-reversed Dirac equation satisfied by $\psi(x_T)$:

$$(i\gamma^\mu\partial_{\mu T} - m)\psi(x_T) = 0$$

This can be shown to be true by just using the substitution $y = x_T$, which reduces it to the Dirac equation.

We start from the Dirac equation then apply operations.

$$\begin{aligned}
(i\gamma^\mu \partial_\mu - m)\,\psi_r &= 0 \\
(-i\gamma^{\mu *}\partial_\mu - m)\,\psi_r^* &= 0 \\
B^{-1}(-i\gamma^{\mu *}\partial_\mu - m)\,\psi_r^* &= 0 \\
\left(-iB^{-1}\gamma^{\mu *}\partial_\mu - B^{-1}m\right)\psi_r^* &= 0 \\
\left(-i\left(\gamma^0, -\gamma\right)\partial_\mu - m\right)B^{-1}\psi_r^* &= 0 \\
(i\gamma^\mu \partial_{\mu T} - m)\,B^{-1}\psi_r^* &= 0
\end{aligned}$$

Therefore, by uniqueness of solutions, we have shown that up to scaling, $B^{-1}\psi_r^* = \psi_{r'}(x_T)$.

Dirac spinors transform under time reversal as $T\psi(x)T^{-1} = \eta_T B\psi(x_T)$. Therefore, under the whole transformation, we have that

$$\begin{aligned}
T\hat{\psi}(x)T^{-1} &= \sum_r T a_r \psi_r(x)T^{-1} \\
&= \sum_r T a_r T^{-1} T\psi_r(x)T^{-1} \\
&= \sum_r a_{r'}\eta_T B\psi_r(x_T) \\
&= \sum_r a_{r'}\eta_T BB^{-1}\psi_{r'}^*(x) \\
&= \eta_T \hat{\psi}^*(x), \qquad \text{assuming } a_r \in \mathbb{R}
\end{aligned}$$

What is the phase factor for this question? Does it factor out of the sum?

Question 4

Note that $\hat{C}$ is unitary, and therefore linear. Thus, to find out how $\hat{C}\overline{\psi}X\psi\hat{C}^{-1}$ transforms, we insert $\hat{C}^{-1}\hat{C}$ between $\overline{\psi}$ and X, and then commute it past X. So,

$$\begin{aligned}
\hat{C}\overline{\psi}X\psi\hat{C}^{-1} &= \hat{C}\overline{\psi}\hat{C}^{-1}X\hat{C}\psi\hat{C}^{-1} \\
&= \hat{C}\overline{\psi}\hat{C}^{-1}XC\overline{\psi}^T
\end{aligned}$$

Using the unitary property of $\hat{C}$, and the fact that $\gamma^0 C\gamma^0 = -C$, we can show that

$$\hat{C}\overline{\psi}\hat{C}^{-1} = -\psi^T C^{-1}$$

So, we have that

$$\hat{C}\overline{\psi}X\psi\hat{C}^{-1} = -\psi^T C^{-1}XC\overline{\psi}^{-1}$$

But since this is just a number, we can take the transpose of this object. Because ψ and $\overline{\psi}$ anti-commute, this picks up a minus sign. So, the above is equal to $\overline{\psi}C^T X^T\left(C^{-1}\right)^T\psi$ Now, use the fact that $C^T = -C$ and this recovers the result.

The case for $\hat{T}\overline{\psi}X\psi\hat{T}^{-1}$ is entirely similar, except we need to use anti-linearity to commute the $\hat{T}$ past X.

Substituting the appropriate matrices into this expression, we find that

$$\begin{aligned}
\overline{\psi}\left(x\right)\psi\left(x\right) &\rightarrow_{\hat{C}}\overline{\psi}\left(x\right)\psi(x) \\
\overline{\psi}\left(x\right)\psi\left(x\right) &\rightarrow_{\hat{T}} \overline{\psi}\left(x_T\right)\psi\left(x_T\right) \\
\overline{\psi}i\gamma^5\psi &\rightarrow_{\hat{C}}\overline{\psi}i\gamma^5\psi \\
\overline{\psi}i\gamma^5\psi &\rightarrow_{\hat{T}} -\overline{\psi}\left(x_T\right)i\gamma^5\overline{\psi}\left(x_T\right) \\
\overline{\psi}\gamma^\mu\gamma^5\psi &\rightarrow_{\hat{C}}\overline{\psi}\gamma^\mu\gamma^5\psi
\end{aligned}$$

Finally we have that for our time transformed quantity

$$\overline{\psi}\left(x\right)\gamma^\mu\gamma^5\psi\left(x\right) \rightarrow_{\hat{T}} \begin{cases}\overline{\psi}\left(x_T\right)\gamma^0\gamma^5\psi\left(x_T\right) \\ -\overline{\psi}\left(x_T\right)\gamma^i\gamma^5\psi\left(x_T\right)\end{cases}$$