String Theory Lecture Notes

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Contents

1 Introduction

If you'd like to read about string theory, check out

- String Theory, Vol 1, Polchinksi, CUP
- Superstrings, Vol 1, Green et al, CUP
- A String Theory Primer, Schomerus, CUP
- David Tong's Notes, at arXiv: 09080333
- 'Why String Theory' Conlon, CRC

What is string theory? We don't really know what string theory is. The question itself requires a little more fleshing out to make sense of it. String theory is a work in progress. A final version of string theory would be a theory which we understand physically and mathematically. It's a work in progress - the final form will be quite different from what we have now.

What do we know? Well, in some sense, string theory is an attempt to answer the question of how we quantise the gravitational field. A theory of **quantum gravity**. There are however, a number of obstacles. In particular, naive quantisation of the Einstein Hilbert action presents a number of problems.

There are deep conceptual problems associated with this.

- There's a question about the nature of time in quantum gravity. If we think about time in quantum mechanics, time is treated as a fixed clock in which the Hamiltonian governs the evolution. In GR, space and time are on the same footing. Thus, the descriptions are on a different footing in QM versus GR. This is not neseccarily a technical problem, just something to think about.
- How do we quantise things without a pre-existing causal structure? What do we mean by this? We we quantise things in QFT, it's important to know whether two operators are timelike or spacelike separated. We have a notion that for all operators which are spacelike separated, commute. (One should not be able to influence the other). If we try to quantise GR, the metric is the object which we would like to quantise. But, this determines the casual structure. So, it's not immediately obvious what the algebra of operators should look like.
- GR has a very big symmetry - diffeomorphism symmetry (symmetry under coordinate reparametrisations). This is diffeomorphism invariance. This is something we'll discuss a bit later on. This is considered to be a gauge symmetry.
- One thing we'll discuss is that there are no local diffeomorphism invariant observables in GR. It's not even clear what the observables should be.

More importantly, there are technical obstacles. The previous issues are hard, but can we make some assumptions which help us make progress? We can look at perturbation theory; we can take our metric an expand it around some classical solution

$$g_{\mu\nu}(X) = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

for our purposes, this classical solution will be Minkowski space-time. This means we can use the causal structure of the background metric (Minkowski) to learn about the causal structure of the perturbation, and quantise. This immediately neutralises the first two conceptual problems. We can call the fluctuations $h_{\mu\nu}$ as gravitons. So, if we take a background static spacetime, add a field, then quantise.

There are some unsatisfactory things about this. When we split the metric into two, we hide a lot of the deep structure that we want. Nonetheless, we can take our Einstein-Hilbert action and expand it out

$$S\left[g\right] = \frac{1}{K_0} \int d^D X \sqrt{-g} R(g)$$

Choose a gauge and expand out

$$S\left[h\right] = \frac{1}{K_0} \int d^D X \left(h_{\mu\nu} \Box h^{\mu\nu}\right)$$

Since the Ricci scalar contains inverses, this expansion goes on forever. This is called a non-polynomial action. The first quadratic term gives the propagator, and the higher order term gives us our interactions.

The propagator is represented by a wiggly line. The interaction term gives us vertices. (Three or four wiggly lines coming together) These lead to Feynman rules.

However, when we compute loops, we get divergences. But, in physical QFT, we can absorb these divergences into coupling constants. These can be dealt with using standard techniques. From advanced quantum field theory, this technique is called renormalisation.

Thus, the difficulties run deeper than conceptual ones. We simply don't know how to calculate. So, string theory provides a way to do quantum perturbation theory of the gravitational 'field' We put field in quotation marks because it's not really a field which we'll be dealing with.

String theory answers some of the questions, in the sense that it gives us a framework to ask meaningful questions in quantum gravity. But not all of these questions are questions that we can answer.

The viewpoint that we're going to take for this course will be from perturbation theory. But we'll always try to understand what this tells us about the non-perturbative physics.

1.1 Worldsheets and Embeddings

Let's try to put together some sort of language to get started. From any popular science book, you may find that particles are described as vibrating strings. The starting point is to consider a worldsheet Σ which is a 2-dimensional surface swept out by a string. This is analogous to a worldline swept out by a particle.

(Insert diagram of line parametrised by τ and pointlike object, and diagram of cylinder object with surface called Σ, with two axes called τ and σ. This diagram is in 3d Minkwoski space). We put coordinates (σ, τ) on Σ, at least locally. And, we can define an embedding of the worldsheet Σ in the background spacetime, $\mathcal{M}$ (Minkowski space), by the functions $X^\mu(\sigma, \tau)$, where the X^μ

are coordinates on $\mathcal{M}$. So if you like, we have that

$$X : \Sigma \to \mathcal{M}$$

Why is this an embedding? A choice σ and τ on Σ gives us a location in Minkwoski space. There are rules (which we shall investigate), for gluing such worldsheets together in a way which is consistent with the symmetries of the theory.

So, we can not only describe the embeddings of a single string propagating through space-time, but multiple strings coming together.

(Draw a diagram of two tubes merging into one tube, then splitting back again into two tubes - this looks like a three point vertex)

We shall see that such diagrams like the above are in one-to-one correspondence with correlation functions in some quantum theory. It is natural to interpret such diagrams as Feynman diagrams in a perturbative expansion of some theory about a given vacuum.

So what we have is a way of understanding writing down Feynman rules, and performing successively better approximations to an exact result in field theory which we don't have.

So what we have here will turn out to be Feynman rules for a theory which we don't yet have. Where do these Feynman rules come from? Quantising is tremendously restrictive.

2 The Classical Particle and String

In non-relativistic Q, we treat time (t) as a parameter and position $\hat{X}^i$ as an operator. Obviously, this kind of restiriction shouldnt survive very long in a relativistic string theory. So, there are choices to be made here. One of those choices is second quantisation. Second quantisation is when both X^i and t are parameters. Then, we quantise fields, for example $\phi(x, t)$ which are the fundamental objects of interest in our theory. We of course require that the fields transform in an appropriate way under field transformations. This is what we do in QFT. Most of what we know for example, in the standard model, omes from this approach.

However, there is another way. This is **first quantisation**. We elevate t to be an operator, and have something else in the background. This is a natural framework for describing the relativistic embedding for worldline, worldsheet or worldvolume in a spacetime.

And here, $X^\mu = \left(X^i, t\right)$ is an operator, which is the fundamental object which we quantise (our basic variable), and we have some other natural parameter entering the theory. We'll look at concrete examples of what that parameter is through this section.

This other possibility, where we think of our fundamental degrees of freedom as some object embedded into spacetime, is the path we'll take in string theory. This is because this approach has been very successful.

There is string field theory however, which takes the first approach, but this leads to most of the results of first quantised string theory.

2.1 Worldlines and Particles

Suppose we want to take this approach with a particle. We consider an embedding of a worldline $\mathcal{L}$ into spacetime $\mathcal{M}$. We assume zero curvature. The basic field is the embedding $X^\mu : \mathcal{L} \to \mathcal{M}$ and an action might be

$$S[X] = -m \int_{x_0}^{x_1} ds = -m \int_{\tau_1}^{\tau_2} d\tau \sqrt{-\eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu}$$

(Insert diagram of line connecting nodes x_1^μ and x_2^μ) where τ (a parameter) is the proper time and

$$X^\mu(\tau_2) = x_2^\mu, \quad X^\mu(\tau_1) = x_1^\mu$$

are endpoints of the worldline. It makes sense that our action should be proportional to the length of the worldline. So, a reasonable guess for our action. We're taking our space-time metric as $(-,+,+,+)$. This seems like a reasonable starting point.

The constant m has dimensions of mass, so a good guess is that this parameter is interpreted as the mass. We can do some things with this action. We can first compute the conjugate momentum to $X^\mu(\tau)$, which is

$$P_\mu(\tau) = -m \frac{\dot{X}^\mu}{\sqrt{-\dot{X}^2}}, \quad \left(\dot{X}^2 = \eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu\right)$$

This satisfies $P^2 + m^2 = 0$. This is what we call an 'on shell' condition. There are two symmetries associated with this action.

- We have a rigid symmetry, where

  $$X^\mu(\tau) \to \Lambda^\mu{}_\nu X^\nu(\tau) + a^\mu$$

  where $\Lambda^\mu{}_\nu$ is a Lorentz transformation matrix and a^μ is a constant displacement.

- Also, this action has re parametrisation invariance. In other words, in the physical variable of the x 's, τ is just a parameter which measures the distance along the line. So, we can replace it. If we take

  $$\tau \to \tau + \xi(\tau)$$

  The embedding X^μ changes as

  $$X^\mu(\tau) \to X^\mu(\tau + \xi) = X^\mu(\tau) + \xi \dot{X}^\mu(\tau) + \dots$$

  To first order, we have that $\delta X^\mu(\tau) \xi \dot{X}^\mu(\tau)$.

There's a rewriting of this action which makes life a little bit easier. Specifically, the action above is hard to interpret in the massless case. We can rewrite the action as

$$X[X, e] = \frac{1}{2} \int d\tau \left(e^{-1} \eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu - em^2\right)$$

There are no square roots, and we can take the massless limit. We will show that this new action is equivalent to the one we wrote down earlier. $e(\tau)$ is some new field on the worldline.

If you like, you might want to think of e as some one-dimensional metric which sets the scale of distances on the line.

The equations of motion for $x^\mu(\tau)$ and $e(\tau)$ are as follows

$$\frac{d}{d\tau}\left(e^{-1}\dot{X}\right) = 0$$

We notice that interestingly, e does not appear with a time derivative. So its equation of motion is purely algebraic. If you like, you can think of e as being a lagrange multiplier for every single point τ on the worldline. The e equation of motion is

$$\dot{X}^2 + e^2 m^2 = 0$$

$e(\tau)$ enters algebraically and it can be thought of as a constraint! The momentum conjugate to X^μ is

$$P_\mu = e^{-1}\dot{X}^\mu$$

If we combine this with the algebraic constraint for e, we can combine this with the mass shell condition to get $P^2 + m^2 = 0$. So interestingly, this auxillary field e imposes a constraint but is equivalent to the space-time energy momentum condition.

We can write $e^{-1} = \frac{m}{|\dot{X}|}$, plug this into the action to find $S[X, e]$ subject to the equations of motion for $e(\tau)$ gives precisely the action

$$S[X] = -m \int_C \sqrt{-\eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu}$$

What guarantees that e^{-1} exists? Well, a priori nothing. But, we can motivate e coming from the interpretation as being a metric, which is invertible.

With a bit more work, we can argue that the m goes to 0 limit gives us the description for null light. This action is overall a lot nicer.

The action $S[X, e]$ has the symmetries

- Poincare invariance, where e is invariant.
- This also has re-parametrisation invariance, but since e depends on τ, it also has to transform. Infinitesimally,

$$\delta X^\mu = \xi \dot{X}^\mu, \quad \delta e = \frac{d}{d\tau}(\xi e)$$

  provided these variations vanish on the endpoints. e is not a scalar function on the worldline, but this is the natural way to choose how it transforms so that the action is invariant.

we have a couple of comments. The first thing we could to is add curvature to our spacetime. We could generalise $\eta_{\mu\nu} \to g_{\mu\nu}(X(\tau))$. Then, this becomes a highly non-linear model.

2.2 Classical Strings

2.2.1 Nambu-Goto Action

The Nambu-Goto action is the analog of the action above but for a string. (Diagram of cylinder with open ends with axes σ, τ). The fundamental degree of freedom is

$$X : \Sigma \to \mathcal{M}$$

In this context, we refer to the object which the sheet is embedded into as the target space. Often, the thing we're embedding into may not be space-time for historical reasons. The Nambu-Goto action is the proposed generalisation so that for $X^{\mu}(\sigma, \tau)$, we have

$$S[X] = -\frac{1}{2\pi\alpha'}\int d\sigma d\tau \sqrt{-\det\left(\eta_{\mu\nu}\partial_a X^{\mu}\partial_b X^{\nu}\right)}$$

This is an action which is proportional to the area spread out by the wordsheet. α' is a historically labelled constant with dimensions of area as measured in spacetime. One often speaks of the string length $l_s = 2\pi\sqrt{\alpha'}$. We introduce the string tension $T = \frac{1}{2\pi\alpha'}$, where we assume throughout that $\hbar = 1 = c$. These are some characteristic scales in the theory.

The usual sort of issues from Nambu-Goto are similar to the issues we faced from the original worldline action. A much better starting point for us is the Polyakov action. We place this game of removing the square root at the price of introducing an extra non-dynamical field.

$$S[X, h] = -\frac{1}{4\pi\alpha'}\int_{\Sigma} d^2\sigma\sqrt{-h}h^{ab}\eta_{\mu\nu}\partial_a X^{\mu}\partial_b X^{\nu}$$

Our entire lecture course will start from the quantisation of this. h_{ab} is a metric on Σ and is non-dynamical - there are no terms involving derivatives h - it is merely a constraint field like e was. We will find however that h plays an important role.

If we remember h as a metric, this is a two dimensional quantum Klein-Gordon field which is massless and in 2 dimensions. if we treat this as a two dimensional quantum field theory, there are other terms which we may want to add.

2.3 Equations of Motion

In this section, we'll be varying the Polyakov action to find our equations of motion. The first thing we do is to vary the metric. When varying the metric, we use an important identity. Taking the Polyakov action, if we vary it with respect to h_{ab}, we get that our change is

$$\delta S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{-h}h^{ab}T_{ab}$$

$$T_{ab} = \partial_a X^{\mu}\partial_b X_{\mu} - \frac{1}{2}h_{ab}h^{cd}\partial_c X^{\mu}\partial_d X_{\mu}$$

Here, we used the important fact that

$$\delta\sqrt{-h} = -\frac{1}{2}\sqrt{-h}h^{ab}\delta h_{ab}$$

Since our action doesn't depend on derivatives of our added metric h_{ab}, we then impose that $T_{ab} = 0$, which is our stress-energy tensor condition.

There are two important remarks to make here about T_{ab}. Since we're in two dimensions, we have that $h^{ab}T_{ab} = 0$ since we can diagonalise h to be Minkowski, and then $h_{ab}h^{ab} = 2$.

We get the equation of motion from the Euler-Lagrange equations on just X^{μ}.

$$\partial_a \left(\frac{\partial L}{\partial \partial_a X_\mu} \right) = 0 \implies = \partial_a \left(\sqrt{-h} h^{ab} \partial_b X^\mu \right) = 0$$

As we will show, we can use diffeomorphisms and Weyl transformations so that we can pick h_{ab} to be the Minkwoski metric. In this case, the equation of motion above reduces to the wave equation.

2.3.1 Classical Equivalence of the Nambu-Goto action with the Polyakov action

We show that the Polyakov action is equivalent to the Nambu-Goto action. To do this, we take the equations of motion with the stress tensor and back substitute into the action. We use a trick and some notation here. We write firstly that $G_{ab} = \partial_a X \cdot \partial_b X$. Our stress tensor condition then reads

$$G_{ab} = \frac{1}{2} h_{ab} h^{cd} G_{cd}$$

Now, if we take the determinant on both sides, remembering to square the scale factor of the right hand side, we have that

$$G = \frac{1}{4}\left(h^{cd}G_{cd}\right)^2 h$$
$$\sqrt{-G} = \pm\frac{1}{2}\sqrt{-h}\left(h^{cd}G_{cd}\right)$$

We took the negative and square root going into the second line. The Polyakov action is

$$S[X,h] = -\frac{1}{4\pi\alpha'}\int d^2\sigma h^{ab}\sqrt{-h}G_{ab} = -\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-G}$$

The final expression is the Nambu-Goto action!

There's a remark to be made about which symmetries we have associated with each action. We had that the Nambu-Goto action had reparametrisation (or diffeomorphism) invariance, but the Polyakov action had Weyl invariance and reparametrisation invariance. Namely, these symmetries are represented as follows.

- Our diffeomorphism invariance comes from an infinitesimal change of coordinates

$$\sigma^a(\sigma,\tau) \to \sigma^a(\sigma,\tau) + \epsilon^a(\sigma,\tau)$$

We have that our embedding X^μ changes the normal way through a Taylor expansion, with

$$X^\mu \to X^\mu + \partial_\nu \epsilon^\nu X^\mu$$

Our metric on the other hand transforms with the Lie derivative of h_{ab} with ϵ^ν as the generating vector field. We have that

$$\delta h_{ab} = \nabla_a \epsilon_b + \nabla_b \epsilon_a$$

We can write this explicitly by expanding the covariant derivative using h_{ab} as the metric on our worldsheet.

- We also have Weyl invariance which comes from leaving the embedding itself fixed, but changing the metric h^{ab} by a Weyl transformation. So,

$$X^\mu(\sigma, \tau) \to X^\mu(\sigma, \tau), \quad h_{ab} \to e^{2\Lambda} h_{ab}$$

What's going on? Well, since we're adding a field h_{ab}, since it's a 2 by 2 symmetric matrix, we're adding 3 degrees of freedom. However, our Weyl parametrisation and diffeomorphism invariance allows us to fix these degrees of freedom back to zero, so we have no net change in the amount of degrees of freedom. Now previously, notice that we had

2.4 Extending the Polyakov action

There are a couple of things we can do to extend our Polyakov action further. We list them here.

- Recall that in our action we're always using the Minkowski metric to contract the derivatives of our embedding field X^μ. We could explore changing this to a general metric $G_{\mu\nu}(X)$. We'll look at this more in detail later.
- We could add an Einstein-Hilbert term into the mix, which is equal to what we call an Euler characteristic, which is a topologically invariant term.

$$S + \int_\Sigma d^2\sigma \sqrt{-h} R(h) = \chi$$

- We could explore adding a cosmological constant term, which looks like

$$S + \Lambda \int_\Sigma d^2\sigma \sqrt{-h}$$

- We could try include a background field (more detail on this later).

2.5 Classical solutions

From our action, we can use diffeomorphism invariance to take away two degrees of freedom from our metric field h_{ab}, and to set our field to have one scalar degree of freedom:

$$h_{ab} = e^{2\phi} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

This on it's own is enough to force our action to look like

$$S = \int_{\Sigma} d^2\sigma - \dot{X}^2 + X'^2$$

Our scalar factor of $e^{2\phi}$ cancels out in the end. In fact, we could've used our Weyl invariance to take our metric $h_{ab} = \eta_{ab}$ right off the bat. Alternatively, we can go straight to our equation of motion and then set our metric to be the flat Minkowski metric at a given point locally.

$$\partial_a \left(\sqrt{-h} h^{ab} \partial_b X^\mu \right) = \Box X^\mu = 0$$

So, we have the wave equation which govern our dynamics. This is good because we already know how to deal with the wave equation - we split it up into left and right moving modes.

2.6 Classical Hamiltonian Dynamics of the String

Today, we're going to be interested in the quantisation of our closed bosonic string. To a first approximation, the canonical quantisation theory is quite straight forward.

We're going to continue to work in what we're going to call conformal gauge. This is when we take the metric on the worldsheet to take the form

$$h_{ab} = e^{\Phi} \begin{pmatrix} -1^0 & \\ 0 & 1 \end{pmatrix}$$

So we have a sort of natural notion of time. We can then define the canonical momentum field conjugate to X^μ in the usual way. This is just

$$P_\mu (\sigma, \tau) = \frac{\delta S[X]}{\delta X^\mu (\sigma, \tau)} = \frac{1}{2\pi\alpha'} \dot{X}_\mu$$

We can also do the usual stuff and write down the Hamiltonian density. Given the Lagrangian density $\mathcal{L}$, the Hamiltonian density is

$$\mathcal{H} = P_\mu \dot{X}^\mu - \mathcal{L} = \frac{1}{4\pi\alpha'} \left(\dot{X}^2 + X'^2 \right)$$

Recall that the dot derivative is the derivative with respect to τ, and the prime is the derivative with respect to σ. It is always useful when looking at Hamiltonian dynamics to define the Poisson brackets. We introduce the bracket as $\{,\}_{PB}$. In particle theory, where our coordinates $x^\mu (\tau)$ and momenta $p_\mu (\tau)$ are on fundamental variables, it is useful to define the following.

$$\{f, g,\}_{PB} = \frac{\partial f}{\partial X^\mu} \frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu} \frac{\partial g}{\partial x^\mu}$$

So, for example, we have that $\{x^\mu, p_\nu\} = \delta^\mu{}_\nu$. The Hamiltonian $\mathcal{H}$ plays the role as a generator of time translations. For example, if we have some function of x, p which is

$$\frac{df}{d\tau} = \frac{\partial f}{\partial \tau} + \{f, H\}$$

which can be shown from the chain rule. Here, we're doing something slightly more general because x, p depends two parameters, not just the τ. So, if you like, this is a field theory generalisation. Our field theoretic generalisation requires that

$$\left\{X^\mu\left(\sigma,\tau\right), P_\nu\left(\sigma',\tau\right)\right\} = \delta^\mu{}_\nu \delta\left(\sigma-\sigma'\right)$$

This is a precurcsor for equal time commutation relations. This is a nice construction! If we recall the form of $X^\mu\left(\sigma,\tau\right)$ to be written in terms of fourier modes α_n^μ and $\overline{\sigma}_n^\mu$. We have the mode expansion

$$X^\mu\left(\sigma,\tau\right) = x^\mu + \sqrt{\alpha'}p^\mu\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^\mu e^{-in(\tau-\sigma)} + \overline{\alpha}_n^\mu e^{-in(\tau+\sigma)}\right), \quad \sigma\sim\sigma+2\pi$$

So imposing the natural commutation relations on X^μ and P^μ, gives us natural Poisson bracket relations for α, α'. This requires

$$\left\{\sigma_\mu^\nu, \sigma_n^\nu\right\}_{PB} = -im\eta^{\mu\nu}\delta_{m+n,0}, \quad \left\{\alpha_m^\mu, \overline{\alpha}_n^\nu\right\}_{PB} = 0, \quad \left\{\overline{\alpha}_m^\mu, \overline{\alpha}_n^\nu\right\}_{PB} = -im\eta^{\mu\nu}\delta_{m+n,0}$$

The left hand side vanishes unless $n=-m$. Why are we considering closed strings? Apparently, they make life easier later, and closed strings give rise to gravity. Let's see if this is plausible. The relation above is for equal times, and the X, P commutation relation is valid for equal times, so let's check this holds for $\tau=0$. We'll see later on that this choice of τ is in actual fact not a special case. Just for simplicity however, we'll do it this way.

Our string looks like, at $\tau=0$,

$$X^\mu\left(\sigma\right) = x^\mu + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^\mu e^{in\sigma} + \overline{\sigma}_n^\mu e^{-in\sigma}\right)$$

Our conjugate momenta is then

$$P_\mu\left(\sigma\right) = \frac{p^\mu}{2\pi} + \frac{1}{2\pi}\sqrt{\frac{1}{2\alpha'}}\sum_{n\neq 0}\left(\alpha_n^\mu e^{in\sigma} + \overline{\alpha}_n^\mu e^{-in\sigma}\right)$$

Computing the commutation relation from this, we have that

2.7 The Stress Tensor and Wit Algebra

Introduce the worldsheet lightcone coordinates

$$\sigma^\pm = \tau\pm\sigma$$

which are a natural choice of coordinates in this gauge to help us understand the structure of this space a little but more.

In these coordinates, and this gauge, the worldsheet metric now looks like

$$h = e^\phi\begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}$$

and $\partial_\pm = \frac{\partial}{\partial \sigma^\pm}$. The action and equations of motion becomes

$$S = -\frac{1}{2\pi\alpha'}\int_\Sigma d\sigma^+ d\sigma^- \partial_+ X \cdot \partial_- X, \quad \partial_+\partial_- X^\mu = 0$$

The stress tensor T_{ab} becomes in these coordinates

$$T_{++} = -\frac{1}{\alpha'}\partial_+ X \cdot \partial_+ X, \quad T_{--} = -\frac{1}{\alpha'}\partial_- X \cdot \partial_- X, \quad T_{+-} = T_{-+} = 0$$

So we have two fields on the worldsheet, the Xs, and the metric which encodes the vanishing stress tensor constraint.

The constraint is $T_{\pm\pm} = 0$. Let's pause a little and think about what these constraints might look like in terms of these modes. It;s going to be very useful to introduce the Fourier modes of $T_{\pm\pm}$. We define at $\tau = 0$, the charges

$$L_n = -\frac{1}{2\pi}\int_0^{2\pi} d\sigma T_{--}(\sigma) e^{-in\sigma}, \quad \overline{L}_n = -\frac{1}{2\pi}\int_0^{2\pi} d\sigma T_{++}(\sigma) e^{in\sigma}$$

We're just starting out to take $\tau = 0$, but we'll see later that this doesn't matter. What do these modes look like in terms of α ? We'll choose to explore one of them in detail, with the awareness that the other moving mode will have similar properties.

If we differentiate X^μ as

$$\partial_- X^\mu(\sigma,\tau)\sqrt{\frac{\alpha'}{2}}\sum_n \alpha_n e^{-in\sigma^-}, \quad \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$$

We find that

$$\begin{aligned} L_n &= \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma \partial_- X^\mu(\sigma) \cdot \partial_- X_\mu(\sigma) \\ &= \frac{1}{4\pi}\sum_{m,p}\alpha_m \cdot \alpha_p \int_0^{2\pi} d\sigma e^{-i(m+p-n)\sigma} \\ &= \frac{1}{4\pi}\sum_{m,p}\alpha_m \cdot \alpha_p 2\pi\delta_{p,n-m} \end{aligned}$$

Similarly, this holds for $\overline{L}_n$. This gives us the relation that

$$L_n = \frac{1}{2}\sum_m \alpha_{n-m}\cdot\alpha_m, \quad \overline{L}_n = \frac{1}{2}\sum_m \overline{\alpha}_{n-m}\cdot\overline{\alpha}_m$$

This constraint can be written as $L_n = 0 = \overline{L}_n$. Using the algebra for the α_n^μ ($\overline{\alpha}_n^\mu$), we can compute the algebra for $L_n\left(\overline{L}_n\right)$. It's not to hard to show that the algebra of these objects obey the algebra

$$\{L_m, L_n\} = -i(m-n)L_{m+n}, \quad \left\{L_m, \overline{L}_n\right\} = 0, \quad \left\{\overline{L}_m, \overline{L}_n\right\} = -i(m-n)\overline{L}_{m+n}$$

This is often called the Witt algebra. We shall see, when we promote this to quantum operators, we see something very similar, but with an important difference. We will see that if we set $L_n = 0 = \overline{L}_n$, at a given τ , then the evolution of the system preserves $L_n = 0 = \overline{L}_n$. So what we find, underlying the constraints of this theory, that there is this underlying infinite dimensional symmetry.

2.8 A First Look at the Quantum Theory

Given the work we've done on the classical theory, quantising this to build a theory on Hilbert space will be a straightforward extension.

Recall from earlier, that we decided to work with the Plyakov action. We transformed our metric locally due to Weyl rescaling, which gives us a two dimensional massless Klein-Gordon theory. But, as a result, we get constraints from the stress-energy tensor.

As we did in quantum field theory, there are two ways we can proceed in imposing the constraints.

- We can constrain states then quantise, which is an approach that has been reasonably successful (this is called light-cone quantisation). However, we will not be following this path of quantisation.
- The second approach is to quantise the unconstrained theory, then impose constraints as a physical condition on our Hilbert space. For specifically, recall that the only algebraic constraint we got from earlier was our condition on our stress energy tensor, $T_{ab} = 0$. Thus, this is the only thing we will impose on our Hilbert space.

2.8.1 Canonical Quantisation

We quantise by replacing out Poisson brackets with commutators. So, functions on phase space for example, are replaced by operators. We do the transformation

$$\{,\}_{PB} \to -i\,[,]$$

So, the structure of quantum mechanics is similar to Hamiltonian dynamics. These give rise to equal time comuttation relations:

$$\left[X^\mu(\sigma), X^\nu(\sigma')\right] = 0, \quad \left[P_\mu(\sigma), P_\nu(\sigma')\right] = 0, \quad \left[P_\mu(\sigma), X^\nu(\sigma')\right] = -i\delta_\mu{}^\nu \delta(\sigma - \sigma')$$

Now, recall the mode expansions we get from expanding our position operator X^μ,

$$X^\mu(\sigma,\tau) = x^\mu + \alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\left(\alpha^\mu_\nu e^{-in(\tau-\sigma)} + \overline{\alpha}^\mu_n e^{-in(\tau+\sigma)}\right)$$

$$P^\mu(\sigma,\tau) = \frac{p^\mu}{2\pi} + \frac{1}{2\pi}\sqrt{\frac{1}{2\alpha'}} \sum_{n\neq 0} \left(\alpha^\mu_n e^{-in(\tau+\sigma)} + \overline{\alpha}^\mu_n e^{-in(\sigma-\tau)}\right)$$

One can show that the commutation relations for the mode operators α and $\overline{\alpha}$ are consistent which changing our Poisson brackets to commutators. This gives our relations as

$$[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu}, \quad [\alpha^\mu_m, \overline{\alpha}^\nu_n] = 0, \quad [\overline{\alpha}^\mu_m, \overline{\alpha}^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu}$$

For each dimension μ, these are an infinite number of ladder operators, where the annihilation operators are given by α_n for $n > 0$. However, by looking at the fact that X^μ and P^μ should be real and by comparing coefficients, we find the relations

$$(\alpha^\mu_n)^\dagger = \alpha^\mu_{-n}$$

These are creation and annihilation ladder operators. They are creating and annihilating different modes on the string. It use necessary to introduce a vacuum state on Σ, $|0\rangle$ such that

$$\alpha_n^\mu |0\rangle = 0, \quad n \geq 0$$

It is important to note that this is not a vacuum in spacetime, it is a vacuum of vibrational modes. We recall the Fourier modes of T_{ab} are L_n and $\overline{L}_n$. Just as before, we can write those modes in terms of α and $\overline{\alpha}$.

$$L_m = \frac{1}{2}\sum_n \alpha_{m-n} \cdot \alpha_n$$

We call L_m Virasoro operators. This expression is ambiguous for $m = 0$, going from the classical to the quantum theory, operator ordering starts to matter. This is because α_n and α_{-n} do not commute for $n \neq 0$, and hence we don't know how to order these in our expression for L_0. In this case, we shall take

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n>0} \alpha_{-n} \cdot \alpha_n$$

We can adopt the usual notion of normal ordering. There are other normal ordering prescriptions we could use.

2.8.2 Physics State Conditions

Instead of thinking about the stress tensor and X, P, it's more useful to talk about the Virasoro operators L_i to impose $T_{ab} = 0$ on the Hilbert space of our theory.

Let's first define some notation to make our lives easier. Define

$$N = \sum_{n>0} \alpha_{-n} \cdot \alpha_n, \quad \overline{N} = \sum_{n>0} \overline{\alpha_{-n}} \cdot \overline{\alpha}_n$$

These operators have the interpretation of being 'weighted number operators' on physical states. Given this, we can write for example, that

$$L_0 = \frac{\alpha'}{4}p^2 + N, \quad \overline{L}_0 = \frac{\alpha'}{4}p^2 + \overline{N}, \quad \left(\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu\right)$$

We require that $T_{ab}|\phi\rangle = 0$ for $|\phi\rangle$ to be a physical state. This means that

$$L_n |\phi\rangle = 0, \quad \text{for } n > 0$$

under Hermitian conjugation, this also implies that $\langle\phi| L_{-n} = 0$. We shall also require

$$L_0 |\phi\rangle = a|\phi\rangle, \quad \overline{L}_0 |\phi\rangle = a|\phi\rangle$$

allowing for the fact that L_0 may be true up to some constant, which is related to the normal ordering. The constant a reflects the ambiguity in defining L_0 in the quantum theory. We'll come back as to what values a should take depending on our perspective. For now, we should take $a = 1$. This justification is a posteori.

For convenience, we also define the new operators

$$L_0^{\pm} = L_0 \pm \overline{L}_0$$

Hence, in terms of our newly defined operators, our physical conditions are

$$\left(L_0^+ - 2\right)|\phi\rangle = 0, \quad L_0^- |\phi\rangle = 0, \quad L_n |\phi\rangle = 0 = \overline{L}_n |\phi\rangle \text{ for } n > 0$$

The L_n are difficult to interpret physically, but are related to polarisation conditions. The first two conditions are related to rotational invariance. With these physical constraint conditions, we can start to look at the spectrum of the system.

2.9 The Spectrum

2.9.1 The Tachyon

Now it's time to start constructing some actual states in our Hilbert space. We can construct a space-time momentum eigenstate as

$$|k\rangle = e^{ik\cdot x}|0\rangle$$

For now, we just work in the context where x is just a position variable and not an operator, like in the expression $e^{ik\cdot X}|0\rangle$. However, we will see what this looks like at a later stage. From quantum mechanics, we know how p_μ acts in the position basis. In terms of a position basis in the target space, the momentum operator is $-i\frac{\partial}{\partial x^u} = p_\mu$, so $p_\mu |k\rangle = k_\mu |k\rangle$. In addition, $L_n |k\rangle = 0 = \overline{L}_n |k\rangle$ straightforwardly, although this has yet to be shown in the lectures and I'm not sure why this is true. We can write L_0^- as

$$L_0^- = N - \overline{N}$$

So the vanishing condition $L_0^- |\phi\rangle = 0$ implies that $N = \overline{N}$, and suggests a symmetry of right movers versus left movers. This is sometimes called level matching. It is the weighted count difference from either side. If we apply N or $\overline{N}$ to any of the states of the form $|k\rangle$, we will always find that since N is the sum of $\alpha_n \cdot \alpha_{-n}$, we can always commute the annihilation operator forward, so on the space of states $|k\rangle$, we have that the eigenvalues are $N = \overline{N} = 0$. We check that $\left(L_0^+ - 2\right)|k\rangle = 0$. Thus, we find that

$$\begin{aligned}
\left(L_0^+ - 2\right)|k\rangle &= \left(\frac{\alpha'}{2} + N + \overline{N} - 2\right)|k\rangle \\
&= \left(\frac{\alpha'}{2}k^2 - 2\right)|k\rangle = 0
\end{aligned}$$

This gives us the condition on k^2, as $k^2 - \frac{4}{\alpha'} = 0$. If we compare this with the energy-momentum condition $k^2 + M^2 = 0$, with gives

$$M^2 = -\frac{4}{\alpha'}$$

The state $|k\rangle$ has a spacetime interpretation as a tachyon, which is a state with negative mass, and is therefore unphysical. This problem is not going to go away. We do have the tachyon, and its cured by promoting this to the supersymmetric string, and adding supersymmetry.

2.9.2 The First Excited State

We'll now look at the next excited state, which are the massless states. Since we have that for a state to be physical, we need $N = \overline{N}$, the next best thing you can do is to excite our state with α_{-1} as well as $\overline{\alpha}_{-1}$. Consider states of the form

$$|\epsilon\rangle = \epsilon_{\mu\nu}\alpha^{\mu}_{-1}\overline{\alpha}^{\nu}_{-1}|k\rangle$$

One thing to spot here is that by construction $N = \overline{N} = 1$, which can be shown by commuting the operators. Is this state even physical? Let's look at some conditions which we need to view this as a physical state. We can look at the energy momentum condition. Let's look at the first condition we have to check:

$$\begin{aligned}\left(L_0^+ - 2\right)|\epsilon\rangle &= \left(\frac{\alpha'}{2}p^2 + N + \overline{N} - 2\right)|\epsilon\rangle \\ &= (\frac{\alpha'}{2}k^2 + 2 - 2)|\epsilon\rangle \\ &= 0\end{aligned}$$

This condition implies $\frac{\alpha'}{2}k^2 = 0$, so we require that $k^2 = 0$ (null). Since we have α_{-1} and $\overline{\alpha}_{-1}$ involved, we have another physical condition to check. Consider the next condition, which gives $L_1|\epsilon\rangle = 0$. This condition gives

$$\frac{1}{2}\left(\sum_n \alpha_{1-n}\cdot\alpha_n\right)\epsilon_{\mu\nu}\alpha^{\mu}_{-1}\overline{\alpha}^{\nu}_{-1}|k\rangle = \frac{1}{2}\epsilon_{\mu\nu}\overline{\alpha}^{\nu}_{-1}\left(\alpha_0\cdot\alpha_1\right)\alpha^{\mu}_{-1}|k\rangle$$

We arrived at the equation above by recognising that the only thing which produces a non-trivial equation is when $n = 0$. Recall that $\alpha_0^{\mu} = \sqrt{\frac{\alpha'}{2}}p^{\mu}$. We use this fact to commute p^{μ} past the modes, and then pick up a factor of k^{μ}.

We require then

$$k_\rho\alpha_1^\rho\alpha^\mu_{-1}|k\rangle = 0$$

we can use our commutator to show that the above is equal to

$$\epsilon_{\mu\nu}k_\rho\left(\left[\alpha_1^\rho,\alpha^\mu_{-1}\right] + \alpha^\mu_{-1}\alpha_1^\rho\right)|k\rangle = \epsilon_{\mu\nu}\eta^{\mu\rho}k_\rho|k\rangle = \epsilon_{\rho\nu}k^\rho|k\rangle = 0$$

This condition $L_1|\epsilon\rangle = 0$ requires us to impose $\epsilon_{\nu\mu}k^\mu = 0$. In other words, we can think of this condition as the fact that there are no longitudinal polarisations. Similarly, $\overline{L}_1|\epsilon\rangle = 0$ requires $\epsilon_{\mu\nu}k^\nu = 0$. So, we have three physical state conditions given to us. There are no further conditions on this tensor.

We then have the conditions on $|\epsilon\rangle$. Our first condition is that it is null and massless, so $k^2 = 0$, and no longitudinal polarisations $\epsilon_{\mu\nu}k^\mu = 0$ and $\epsilon_{\mu\nu}k^\nu = 0$, thinking of ϵ as a polarisation tensor.

We can decompose $\epsilon_{\mu\nu}$ into symmetric $h_{\mu\nu}$, anti-symmetric ($b_{\mu\nu}$), and trace ϕ parts. We first extract the trace bit, which we call the Dilaton.

$$|\phi\rangle = \phi\alpha^\mu_{-1}\overline{\alpha}_{-1\mu}|k\rangle$$

We also have two other particles from this

$$|h\rangle = h_{\mu\nu}\alpha^{\mu}_{-1}\overline{\alpha}^{\nu}_{-1}|k\rangle \quad \text{Graviton}, \quad h_{\mu\nu} = h_{\nu\mu}$$
$$|b\rangle = b_{\mu\nu}\alpha^{\mu}_{-1}\overline{\alpha}^{\nu}_{-1}|k\rangle, \quad \text{B-Field}, \quad b_{\mu\nu} = -b_{\nu\mu}$$

Now let's look at massive states. We can now look at states with $N = \overline{N} = 2$. We have

$$A_{\mu\nu}\alpha^{\mu}_{-2}\overline{\alpha}^{\nu}_{-2}|k\rangle + A_{\mu\nu\lambda}\alpha^{\mu}_{-2}\overline{\alpha}^{\nu}_{-1}\overline{\alpha}^{\lambda}_{-1}|k\rangle + \tilde{A}_{\mu\nu\lambda}\overline{\alpha}^{\mu}_{-2}\alpha^{\mu}_{-1}\alpha^{\lambda}_{-1}|k\rangle + A_{\mu\nu\lambda\rho}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}\overline{\alpha}^{\lambda}_{-1}\overline{\alpha}^{\rho}_{-1}|k\rangle$$

and N and $\overline{N}$ count number of quanta going around the string. There's no profit in us solving the mass shell conditions. The mass of such states is $m^2 = \frac{4}{\alpha'}$. So the string is describing not only the tahcyon and massless fields, but an infinite amount of massive fields.

For the most part, string theorists have concerned themselves with the massless spectrum.

2.10 The Big(ish) Picture

We started with our Polyakov action which describes the embedding of our string in our manifold. We then used diffeomorphism and Weyl symmetry which allowed us to choose h_{ab} to be Minkowksi.

$$S = -\frac{1}{4\pi\alpha'}\int_{\Sigma} d^2\sigma \eta_{\mu\nu}\partial_a X^{\mu}\partial^a X^{\nu}$$

We could deform this theory and add a perturbation to the metric. This can be achieved by adding a small plane wave deformation to the spacetime metric. For example, we might take $\eta_{\mu\nu}$ and replace it by adding some plane wave.

$$\eta_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu}e^{ik\cdot x}$$

The action changes by

$$\Delta S = -\frac{1}{4\pi\alpha'}\int_{\Sigma} d^2\sigma h_{\mu\nu}\partial_a X^{\mu}\partial^a X^{\nu}e^{ikx}$$

For every deformation of the theory, there is an associated operator that we get from perturbing the action. In this case, it's the

$$\mathcal{O} = h_{\mu\nu}\partial_a X^{\mu}\partial^a X^{\nu}e^{ikx}$$

This is clearly associated with a deformation of the spacetime metric.

In string theory, (2-dimensional CFTs), to each operator $\mathcal{O}$ that corresponds to a physical deformation of the theory, (something that seems reasonable is what we mean by a physical deformation) we have a Hilbert space This state in the Hilbert space is given by applying the operator onto our vacuum, then taking the limit as τ goes to $-\infty$. In this case,

$$\lim_{\tau \to -\infty} \mathcal{O}|0\rangle = |h\rangle$$

We will do this more carefully later on. This is called the state-operator correspondence. One last comment on this.

What if we choose to start with a general metric? For example, we could've started off with

$$S = -\frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma g_{\mu\nu}(x) \partial_a X^\mu \partial^a X^\nu$$

Thhis is highly non-linear. How would we deal with this? We could try expand this in terms of a power series and then try to deal with the theory perturbatively. This is now an interacting theory, and this is hard to do.

Moreoever, the condition of Weyl invariance in the quantum theory constrains what $g_{\mu\nu}(X)$ we can have. Weyl invariance is $h_{ab} \to e^{\omega(\omega,\tau)} h_{ab}$. One finds that $G_{\mu\nu}$ has to satisfy

$$R_{\mu\nu}(g) + \mathcal{O}(\alpha') = 0$$

Which is the Einstein tensor up to string corrections. More generally, if we have other background fields, we find Weyl invariance requires the full Einstein equations to be satisfied to leading order in α'.

2.11 Spurious states and gauge invariance

Before, when we mentioned L_0 to have an ordering ambiguity, we put our normal ordering constant as $a = 1$. We took $a = 1$ in the conditions $(L_0 - a)|\phi\rangle = (\overline{L}_0 - a)|\phi\rangle = 0$. Why did we do this? This is so that we can interpret states easily. Consider the state

$$|\chi\rangle = \sqrt{\frac{2}{\alpha'}} \left(\lambda_\mu \alpha^\mu_{-1} \overline{L}_1 + \tilde{\lambda}_\mu \overline{\alpha}^\mu_{-1} L_1 \right) |k\rangle$$

Clearly, $|\chi\rangle$ is orthogonal to all physical states. If $|\phi\rangle \in \mathcal{H}$, then $\langle\phi|\chi\rangle = 0$ because $L_1|\phi\rangle = \overline{L}_1|\phi\rangle = 0$. What conditions do $\lambda_\mu, \tilde{\lambda}_\mu, k$ have to satisfy for $|\chi\rangle$ to be physical.

Is is useful to write $|\chi\rangle$ as

$$|\chi\rangle = \left(\lambda_\mu k_\nu + \tilde{\lambda}_\nu k_\mu \right) \alpha^\mu_{-1} \overline{\alpha}^\nu_{-1} |k\rangle$$

Keeping a arbitrary, we find that

$$\left(L_0^+ - 2a \right) |\chi\rangle = 0 \implies k^2 = \frac{4(a-1)}{\alpha'}$$

We also have that, by symmetry,

$$\begin{aligned} L_1|\chi\rangle = 0 \text{ if } & (\lambda \cdot k) k_\mu + \tilde{\lambda}_\mu k^2 = 0 \\ \overline{L}_1|\chi\rangle = 0 \text{ if } & \left(\tilde{\lambda} \cdot k \right) k_\mu + \lambda_\mu k^2 = 0 \end{aligned}$$

Finally, we have that

$$\langle\chi|\chi\rangle = \lambda^2 k^2 + 2(\lambda \cdot k)\left(\tilde{\lambda} \cdot k \right) + \tilde{\lambda}^2 k^2$$

If $a = 1$, $k^2 = 0$, $\lambda \cdot k, \tilde{\lambda} \cdot k = 0$, so $\langle\chi|\chi\rangle = 0$.

Worldline actions

- Our action is
$$S = -m \int_{s_1}^{s_2} ds = -m \int_{\tau_1}^{\tau_2} d\tau \sqrt{-\eta^{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$$
- We have conjugate momenta with on-shell mass condition
$$P^\mu = -\frac{m\dot{x}^\mu}{\sqrt{-\dot{x}^2}}, \quad P^2 + m^2 = 0$$
- It makes more sense to work with Einbeins, since we can work in the $m \to 0$ limit
$$S = \frac{1}{2} \int d\tau \, \left(e^{-1}\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - em^2\right)$$
- Two equations of motion come from the Einbein action
$$\frac{d}{d\tau}\left(e^{-1}\dot{x}^\mu\right) = 0, \quad \dot{x}^2 + e^2 m^2 = 0$$
- This has symmetries
$$\delta x^\mu = \xi \dot{x}^\mu, \quad \delta e = \frac{d}{d\tau}\left(\xi \dot{e}\right)$$
- In the massless limit, if we replace our Minkowski metric with a general metric, we recover the geodesic equations
$$S = \frac{1}{2} \int d\tau e^{-1} g_{\mu\nu}\dot{x}^\mu \dot{x}^\mu, \quad \ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = 0$$

2.12 Recap

We have that
$$\delta h_{ab} = (P_v)_{ab} + 2\varpi h_a b + t^I P_{abI}$$
where $(P_v)_{ab}$ represents our traceless Lie derivative, the second term represents Weyl transformations, and $p_{abI} = \frac{\partial h_{ab}}{\partial m^I}$. m^I are coordinates on the moduli space of Riemann surfaces M_g, and $t^I \simeq \delta m^2$ are tangent vectors to M_g.

2.13 Moduli space of unpunctured Riemann Surfaces

Moduli spaces are a space of objects like a metric, but we quotient out some symmetry. In this case, we quotient out the space of metrics by diffeomorphisms and Weyl transformations.

Definition. Moduli Space The moduli space M_g of a genus g Riemann surface Σ_g is
$$M_g = \frac{\{\text{Metrics}\}}{\{\text{Diff} \times \text{Weyl}\}}$$

We can view this as a slice of physically inequivalent metrics on the whole space of h_{ab}.

Amazingly this is a finite dimensional object. We won't prove this, but the dimension, over the field of the reals, of M_g is

$$\dim\left(M_g\right) := s = \begin{cases} 0 \text{ if} & g = 0 \\ 2 \text{ if} & g = 1 \\ 6g - 6 \text{ if} & g \geq 2 \end{cases}$$

The first thing might be a sphere, the second thing might be a donut, and the last thing might include double donuts.

Let's look at some examples.

- For $g = 0$, all metrics on $g = 0$ surfaces may be brought to the form $e^{2\omega}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ by diffeomorphisms, so they're trivial up to Weyl transformations. One way to show this is via the Riemann-Roch theorem or the Atiyah-Singer index theorem. Alternatively, we can see this by just looking at our dimension for moduli spaces - we have that $\dim\left(M_g\right) = 0$, and roughly speaking, forgetting about diffeomorphism transformations, we have that $\{\text{metric}\} \simeq \{\text{Weyl}\}$.

  This agrees with the thinking we did last time about using reparametrization to take h_{ab} to be the identity.

  In terms of the interaction picture, we relate and are sort of thinking about these things as perhaps tree level contributions from $g = 0$, and not want to worry too much about moduli space.

- Now what about genus 1? Genus 1 objects are tori, and there's a standard way to construct these objects in the complex plane. We take the complex plane and two lattice vectors $\lambda_1, \lambda_2 \in \mathbb{C}$, and identify points which differ by these lattice vectors with one another.

  $$z \sim z + \lambda_1 m + \lambda_2 n, \quad \lambda_1, \lambda_2 \in \mathbb{Z}$$

  Choosing different λ_i makes us choose tori with different shapes. Notice that λ_i change under diffeomorphisms of the complex plane. Thus, they're not really a good set of parameters in which to choose a tori modulo Weyl transformations and diffeomorphisms. But, it can be shown that the ratio

  $$\tau = \frac{\lambda_1}{\lambda_2}$$

  is invariant under the transformation group Diff $\times$ Weyl.

  Definition. Complex Structure We call τ the complex structure, not to be confused with our time parameter.

  $$\tau = \frac{\lambda_1}{\lambda_2}$$

  It can be shown that our metric on the complex plane can be rewritten as $ds^2 = |dz^2 + \tau d\bar{z}|^2$. with Diff $\times$ Weyl transformations. In fact, we can choose as well that $\text{Im}\tau > 0$. So, we're restricting to the upper half plane. Even **if** we're just restricting to the upper half plane, there is **still** some extra redundancy to get rid of.

However, there's a lot of redundancy here. We can write the identification

$$z \sim z + n_a \lambda^a, \quad n_a = (m, n), \quad \lambda^a = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

If we act with an $SL(2)$ transformation $n_a \to U^b{}_a n_b$ and $\lambda^a \to \left(U^{-1}\right)^a{}_b \lambda^b$, this statement is preserved. Moreover, if $U \in SL(2, \mathbb{Z})$, then (n, m) remain integers, and we still have a valid identification. Thus, we still have a symmetry of the lattice.

So, the moduli space at $g = 1$ can be identified with the upper half plane modulo $SL(2, \mathbb{Z})$, which is denoted

$$\frac{\text{UHP}}{SL(2, \mathbb{Z})} = M_1$$

What does this look like?

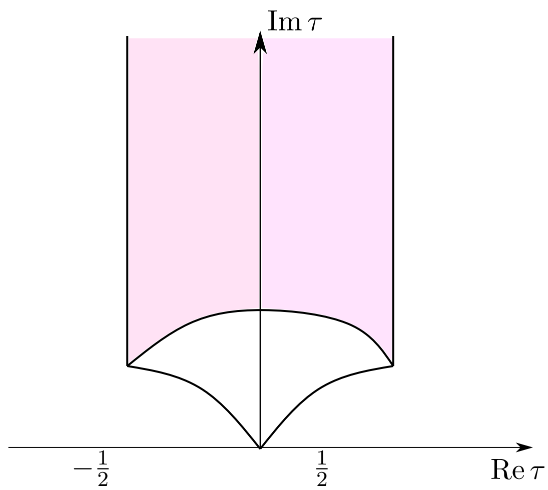

Figure 1: The space of non-physically equivalent τ

We show this in the diagram. There's no analogue for this on world lines. For higher genus, things are hard. For $g \geq 2$, it gets hard. But, there are analogues of the $SL(2, \mathbb{Z})$ (modular group) in all cases. So, we can make very strong statements from modular groups that order by order, we have UV finiteness. This is one of the first times we see that string theory is very different from field theory. Other data we can introduce, as well as counting inequivalent worldsheets and Riemann surfaces, are punctures.

Before we do that, there's a subtlety we have to address.

2.14 Conformal Killing Vectors

There is an overlap Diff$\cap$Weyl, called the conformal Killing group (CKG), where a diffeomorphism is also a Weyl transformation. So, you can think of a diffeomorphism that can be undone by a Weyl transformation. In other words, is it possible such that

$$\delta h_{ab} = \nabla_a v_b + \nabla_b v_a + 2\omega h_{ab} = 0$$

So that the diffeomorphism and Weyl transformation cancel out. We take the trace to find that

$$\omega = -\frac{1}{2}\nabla_a v^a$$

We defined

$$(P_v)_{ab} = \nabla_a v_b + \nabla_b v_a - h_{ab}\left(\nabla_c v^c\right)$$

Vectors that satisfy $(P_v)_{ab} = 0$ can be what we shall call conformal Killing vectors, which we shall abbreviate as CKV's. This is a sensible name because we want Killing vectors up to conformal transformations. So the CKG is generated by the CKVs. For a given genus, the dimension of a conformal Killing group

$$\dim_{\mathbb{R}}(\text{CKG}) := \kappa = \begin{cases} 6, & g = 0: \quad SL(2,\mathbb{C}) \\ 2, & g = 1: \quad U(1) \times U(1) \\ 0, & g \geq 2: \quad - \end{cases}$$

On the sphere, this is the Mobius group. For the torus, it's two circles. For example,

- $g = 0$ the CKG is $SL(2,\mathbb{C})$. This is the Mobius group. If $z \in \mathbb{C}$, then this is the group

  $$z \to \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1$$

  We can specify a particular element of the CKG at genus $g = 0$ by describing how three distinct points transform under a given map. We just need three points since we're dealing with complex coordinates, which takes two real numbers each.

2.15 Moduli Space of Punctured Riemann Surfaces $M_{g,n}$

Imagine we have a Riemann surface of genus g, with n punctures (or marked points). What is it's moduli space? The naive thing we might expect

$$M_{g,n} = M_g \otimes \Sigma^n$$

This is the product of the non-trivial information we get from the locations of the punctures. But, we also have to worry about the CKG. If we have a $g = 0$ surface with n punctures, we can use the conformal Killing group to fix 3 of the locations of the punctures. In other words, 3 degrees of freedom. This leaves only $n - 3$ free punctures.

Similarly, we can fix the conformal Killing group on the torus $g = 1$ by fixing the location of one puncture. So, we should really take

$$M_{g,n} = M_g \otimes \Sigma^{n-\kappa/2}$$

and

$$\dim\left(M_{g,n}\right) = 6g - 6 + 2n$$

So the space which we will be integrating over, which is the space of physically inequivalent metrics. When the above quantity is negative, or correlation function goes to zero.

3 The Faddeev-Popov Determinant

We recall how we classify small changes from our underlying metric. We will need this later. Remember, $(P_v)_{ab}$ is defined as the function whose kernel is the conformal Killing group, which is the set of diffeomorphisms which also happen to be Weyl transformations.

When we vary δh_{ab}, we need to vary h from diffeomorphisms, Weyl transformations, and changes in the moduli space.

Our task in this section will be to calculate a Jacobian factor associated with the path integral. As we have seen before, the partition function in absence of a source term is the path integral over all Riemann surfaces, and physically inequivalent metrics. This is given by the following path integral, dividing out by diffeomorphisms and Weyl transformations

$$\mathcal{Z}[0] = \frac{1}{|\text{Diff} \times \text{Weyl}|} \int \mathcal{D}X\mathcal{D}h e^{-iS[h,X]}$$

Introduce the Faddeev-Popov determinant Δ_{FP} as the factor which gives the following identity relation:

$$1 = \int_{M_g} d^s t \int_{\text{Diff}\times\text{Weyl}} \mathcal{D}\omega\mathcal{D}v\, \delta\left[h - \hat{h}\right](\sigma,\tau)\, \Delta_{FP}[h] \prod_{i,a=1,2}^{\kappa} \delta\left(v^a\left(\tilde{\sigma}_i\right)\right)$$

where the integrand is a δ-functional. In the factor $\int_{\text{Diff}\times\text{Weyl}}$, we are integrating over Weyl transformations parametrised by the function ω, and diffeomorphisms generated by the function v. The factor which is the product of $\delta\left(v\left(\tilde{\sigma}_i\right)\right)$ represents where we have punctures on our Riemann surface. We define

$$\delta\left[h - \hat{h}\right] = \prod_{a,b,\sigma,\tau} \delta\left(h_{ab}(\sigma,\tau) - \hat{h}_{ab}(\sigma,\tau)\right)$$

where $h_{ab} - \hat{h}_{ab} = \delta h_{ab} = (Pv)_{ab} + 2\overline{\omega} h_{ab} + t^I \mu_{Iab}$. In this case, a common choice for $\hat{h}_{ab}$ is η_{ab} or some other fixed metric. h_{ab} is the metric then obtained by applying a transformation. This is somewhat analogous to looking at $\int dx \delta(f(x))$, where we have to be a bit careful about the vanishing set. We need some way to fix the conformal killing group, which is $\text{Diff} \cap \text{Weyl}$. This is only worth worrying about in genus 0 and genus 1. To fix the conformal killing group, we ask that the Diffeomorphisms vanish at particular points, which is why we added a factor at the end. We now want a explicit expression for Δ_{FP}.

Since we have a delta function inside the integral, we can actually take out the Jacobian and evaluate it on the support of the delta functions.

$$1 = \Delta_{FP}\left[\hat{h}\right] \int_{M_g} d^3 t \int_{\text{Diff}\times\text{Weyl}} \mathcal{D}\omega\mathcal{D}v \delta\left[h - \hat{h}\right](\sigma,\tau) \prod_{i,a=1,2}^{\kappa} \delta\left(v^a\left(\tilde{\sigma}_i\right)\right)$$

3.1 Gauge-Fixing the Path Integral

Let us now try to gauge fix the path integral. We substitute the identity integral into the expression for our partition function. If we do this, we find that

$$\mathcal{Z}[0] = \frac{1}{|\text{Diff} \times \text{Weyl}|} \int_{\text{Diff } \times \text{Weyl}} D\omega D v \prod_{a,i} \delta\left(v^a\left(\sigma_i\right)\right) \int_{M_g} d^s t \int \mathcal{D}X \mathcal{D}h \delta\left[h - \hat{h}\right] \Delta_{FP}\left[\hat{h}\right] e^{iS[X,h]}$$

Now, rearranging this object, we have that this is equal to

$$\mathcal{Z}[0] = \frac{1}{|\text{Diff} \times \text{Weyl}|} \int \mathcal{D}X \int_{M_g} d^s t \left(\int_{\text{Diff } \times \text{ Weyl}} \mathcal{D}\omega \mathcal{D}v \prod_{a,i} \delta\left(v^a\left(\tilde{\sigma}_i\right)\right)\right) e^{iS\left[\hat{h},X\right]} \Delta_{FP}\left[\hat{h}\right]$$

Throughout this, we are considering over a particular genus. In the brackets, we have

$$\int_{\text{Diff}\times\text{Weyl}} \mathcal{D}\omega \mathcal{D}v \prod_{a,i} \delta\left(v^a(\tilde{\sigma}_i)\right) = \frac{|\text{Diff} \times \text{Weyl}|}{|\text{CKG}|}$$

Let's go over this expression. It looks like that we are just integrating over all of the diffeomorphism and Weyl transformation space, but we still have these points given by the delta function which fixes our conformal killing group, so we have to divide out by this. We find perhaps a more respectable starting point. This substituting this into the above integral gives us the somewhat more reasonable expression

$$\mathcal{Z}[0] = \frac{1}{|\text{CKG }|} \int \mathcal{D}X \int_{M_g} d^s t \Delta_{FP}\left[\hat{h}\right] e^{iS\left[\hat{h},t\right]}$$

Let's find a field theory representation for Δ_{FP}. Now, we can invert this expression, and move the Faddeev-Popov determinant over to the other side to give

$$\Delta_{FP}^{-1}\left[\hat{h}\right] = \int_{M_g} d^s t \int_{\text{Diff}\times\text{Weyl}} \mathcal{D}w \mathcal{D}v \, \delta\left[\delta h\right] \prod_{i,a} \delta\left(v^a\left(\tilde{\sigma}_i\right)\right)$$

In this case, we write, somewhat confusingly, that $\delta h = h - \hat{h}$, and we have that since the difference is generated by a diffeomorphism and a Weyl transformation, $\delta h_{ab} = (Pv)_{ab} + 2\overline{\omega} h_{ab} + \mu_{Iab} t^I$, and $\overline{\omega} = \omega +$ some other terms. We introduce what we might think of as auxiliary fields $\beta_{ab}(\sigma,\tau)$ and ξ_a^i and write these δ functionals as integrals. The idea is that we write the deltas as integrals, exactly how $\delta(x) = \int d^k e^{ik \cdot x}$.

$$\Delta_{FP}^{-1}\left[\hat{h}\right] = \int_{M_g} d^3 t \int \mathcal{D}v \mathcal{D}\omega \mathcal{D}\beta d^{2k} \xi \exp\left(i\left(\beta \mid Pv + 2\overline{\omega} h + t^I \mu_I\right) + i \sum_{i=1}^{k} \xi_a^i v^a\left(\tilde{\sigma}_i\right)\right)$$

where we define which is a result over taking the continuum product over σ, τ and absorbing this into the exponential

$$\left(\beta \mid Pv + 2\overline{\omega} h + t^I \mu_I\right) := \int_\Sigma d^2\sigma \sqrt{-h} \beta^{ab} \left((Pv)_{ab} + 2\overline{\omega} h_{ab} + t^I \mu_{Iab}\right)$$

3.1.1 An aside on Grassman Variables

As an aside, we have Δ_{FP}^{-1}, and we want Δ_{FP}. We introduce Grassman variables. These are variables θ_i that anti-commute, so we have that $\theta_1\theta_2 = -\theta_2\theta_1$. These obey lots of interesting properties, for example $\theta_i^2 = 0$. We also have the Berezin integration rules, where when we integrate over a constant, we get zero, and when we integrate over a sole θ variable, we get 1.

$$\int d\theta 1 = 0, \quad \int d\theta\theta = 1$$

We have that

$$f(x,\theta) = f(x) + \theta f'(x), \delta(\theta) = \theta, \quad \int d\theta f(\theta) = \frac{\partial f}{\partial \theta}$$

where the differentiation is with respect to θ, and when we omit θ we mean where $\theta = 0$. Let us consider a finite dimensional Gaussian integral.

$$\frac{1}{\det M} = \int_V dz d\bar{z} e^{-(\bar{z}, Mz)}$$

Writing the same type of integral with Grassmann variables, we have that

$$\det M = \int d\theta d\bar{\theta} e^{-(\bar{\theta}, M\theta)}$$

See Ryder's QFT book if curious This suggests that what we want to do is replace everything with Grassman valued fields, so that we invert this thing properly. We replace all of our fields we integrate over in Δ_{FP}^{-1} Grassman valued fields. We replace

$$v^a(\sigma,\tau) \to c^a(\sigma,\tau), \qquad \beta(\sigma,\tau) \to b_{ab}(\sigma,\tau), \quad t^I \to \zeta^I, \quad \xi_a^i \to \eta_a^i$$

and we take $\Delta_{FP}\left[\hat{h}\right]$ to be given by

$$\Delta_{FP}\left[\hat{h}\right] = \int d^s\zeta \mathcal{D}c\mathcal{D}c\mathcal{D}b d^{2k}\eta \exp\left(ib \mid Pc + \zeta^I \mu_I + i\sum_{i=1}^{\kappa} \eta_a^i c^a(\tilde{\sigma}_i)\right)$$

We can do the finite dimensional integralw. We have done the $\overline{\omega}$ integral which gives us a delta function which constrains $\beta_{ab}h^{ab} = 0$ everywhere. When we move to Grassmann valued fields, we can take b_{ab} to be tracelss also. Due to the fact that we're dealing with Grassman variables, the integral over ζ^I gives us a delta functional, and we have that

$$\prod_{I=1}^{s} \delta(b \mid \mu_I) = \prod_{I=1}^{s} (b \mid \mu_I)$$

On the other hand, we have that the integral involving η_a^i gives us

$$\prod_{a=1,2,i=1\ldots\kappa} \delta(c^a(\tilde{\sigma}_i)) = \prod_{i,a} c^a(\tilde{\sigma}_i)$$

Thus, our final result for our determinant is

$$\Delta_{FP}\left[\hat{h}\right] = \int \mathcal{D}c\mathcal{D}b e^{iS[b,c]} \prod_{I=1}^{s} (b \mid \mu_I) \prod_{i,a} c^a(\tilde{\sigma}_i)$$

In this case, our action for the b, c Grassmann fields is given by

$$S[b,c] = (b \mid Pc) = \int d^2\sigma \sqrt{-h} b^{ab} (Pc)_{ab} = 2 \int_\Sigma d^2\sigma \sqrt{-\hat{h}} b^{ab} (\nabla_a c_b)$$

We shall refer to the (Grassmann) fields b_{ab} and c^a as Faddeev-Popov ghosts. From this, we can calculate observables from our path integral.

3.2 Calculating Observables

We substitute our expression for the Faddeev-Popov determinant into our entire path integral. In the previous section, we were only working with a particular worldsheet with a genus g. Now, we aim to sum over all genera, and add an exponential factor to make sure this sum converges.

We now piece together everything we did in the last section. Our Faddeev-Popov determinant is given by

$$\Delta_{FP} = \int \mathcal{D}b\mathcal{D}c e^{iS[b,c]} \prod_I (b \mid \mu_I) \prod_a c^a \left(\tilde{\sigma}^i\right)$$

where our action here is defined as

$$S[b,c] = (b \mid Pc) = \int d^2\sigma \sqrt{-\hat{h}} b^{ab} (\nabla_a c_b)$$

Our final form for $\mathcal{Z}[0]$, summing over all possible Riemann surfaces, with some weight, is

$$\mathcal{Z}[0] = \sum_{g=0}^{\infty} e^{\lambda\chi} \frac{1}{|\mathrm{CKG}|} \int_{M_g} \int d^s t \int \mathcal{D}X\mathcal{D}b\mathcal{D}c\, e^{iS\left[X,\hat{h},b,c\right]} \prod_I (b \mid \mu_I) \prod_{i,a} c^a (\tilde{\sigma}_i)$$

Here, λ is a constant and $\chi = 2g - 2$ is the Euler characteristic of the worldsheet. Now we want to think about computing observables. We can compute correlation functions of observables, by including them in our path integral.

$$\langle \phi_1, \quad \phi_n \rangle = \mathcal{N} \int \mathcal{D}\phi\, \phi_1 \ldots \phi_n e^{iS[\phi]}$$

We're summing over connected worldsheets, and oriented worldsheets. The orientedness condition is not necessarily needed though. What might observables look like? They need to be invariant under the symmetries. So, we can build diff invariant observables by taking an operator $\mathcal{O}(\sigma, \tau)$ on the genus g worldsheet Σ_g and integrating it over Σ_g. For instance,

$$\mathcal{O} = \int_{\Sigma_g} d^2\sigma \mathcal{O}(\sigma, \tau)$$

We also need $\mathcal{O}$ to be Weyl-invariant, which means that when we change our metric on our worldsheet and integrate by a Weyl transformation, when we scale this metric by a Weyl transformation we get the same answer.

Much like how we deal with expectation values of observables in quantum mechanics by inserting them into a probability distribution, observables are the sort of thing you could think about inserting into this path integral. So, a correlation function of such observables would be

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle = \sum_{g=0}^{\infty} \frac{e^{\lambda\chi}}{|\text{CKG}|} \int \prod_{i=1}^{n} d^2\sigma_i \int_{M_g} d^s t \int \mathcal{D}X\mathcal{D}b\mathcal{D}c e^{iS[X,h]} \prod_I (b \mid \mu_I) \prod_{i,a} c^a(\tilde{\sigma}_i) \mathcal{O}_1(\sigma_1) \ldots \mathcal{O}_n(\sigma_n)$$

Notice that

$$\frac{1}{|\text{CKG}|} \int_{M_g} d^s t \int \prod_{i=1}^{n} d^2\sigma_i = \int_{\mathcal{M}_g} d^s t \int_{i=1}^{n-\kappa} d^2\sigma_I = \int_{M_{g,n}}$$

This is because we're dividing out by the space with some points fixed. So, we have

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle = \sum_{g=0}^{\infty} e^{\lambda\chi} \int_{M_{g,n}} \int \mathcal{D}X\mathcal{D}b\mathcal{D}c e^{iS[X,b,c]} \prod_I (b \mid \mu_I) \prod_{i,a} c^a(\tilde{\sigma}_i) \mathcal{O}_1 \ldots \mathcal{O}_n$$

4 Conformal Field Theory

CFT is a wonderful topic with important applications in condensed matter theory, string theory and more. If you really want to understand quantum field theory with mathematical rigour, the best place to start is with low dimensional CFT.

4.1 Conformal invariance in general dimension

We're going to highlight how interesting things become when our dimension is 2, the dimension of our worldsheet. In two dimensions, we explore transformations which preserve angles, which are called conformal transformations. This happens when our metric at some point is scaled by a particular function.

Definition. Conformal transfomation A conformal transformation is a transformation which leaves the angles between any two given vectors on our space invariant. In other words, we require that the metric (which determines the inner product) only be transformed up to a local scaling.

More precisely, imagine some coordinate $x^\mu \to x'^\mu(x)$ such that the metric on our space transforms as

$$\eta_{\rho\sigma}\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} = \Lambda(x)\,\eta_{\mu\nu}, \Lambda(x) > 0,$$

Infinitesimally, we can view this as

$$x^\mu \to x^\mu(x) = x^\mu + \epsilon v^\mu(x) + \dots$$

where ϵ is a small number, and v^μ is the what we call the generator of our transformation.

Now, in a similar fashion to coming up with a condition that a vector lies in the conformal Killing group, we will now cook up a condition to determine whether the transformation associated with v^μ is a conformal transformation. To first order in ϵ, we have that

$$\eta_{\mu\nu} \to \eta_{\mu\nu} + \epsilon\left(\partial_\mu v_\nu + \partial_\nu v_\mu\right) = \Lambda(x)\,\eta_{\mu\nu}$$

Now we parametrise our scaling parameter and let $\Lambda(x) = e^{\epsilon w(x)} \simeq 1 + \epsilon w(x)$. Substituting this on both sides, we arrive at the equation

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \omega(x)\,\eta_{\mu\nu}$$

We now solve for ω, and the obvious way to do this is to take the trace of the equation. Taking the trace of this equation, we require

$$\omega(x) = \frac{2}{d}\left(\partial_\mu v^\mu\right)$$

Vector fields that generate such conformal transformations satisfy

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \frac{2}{d}\left(\partial_\lambda v^\lambda\right)\eta_{\mu\nu}$$

In two dimensions, we have that $d = 2$, and something special happens. Let us take the coordinates to be σ, τ, and we have enough gauge freedom choose η to be the Euclidean metric $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. With this metric, our condition for a vector field which generates conformal transformations becomes, by choosing either $(\mu, \nu) = (\sigma, \tau)$ or $(\mu, \nu) = (\tau, \tau)$,

$$\frac{\partial v_\tau}{\partial \tau} = \frac{\partial v_\sigma}{\partial \sigma}, \quad \frac{\partial v_\sigma}{\partial \tau} = -\frac{\partial v_\tau}{\partial \sigma}$$

This is amazing, since you may recognise these equations as the Cauchy Riemann equations.

Let's make these conditions more compact. Let's first write $v = v_\tau + i v_\sigma$. If we introduce complex coordinates

$$w = \tau + i\sigma, \quad \overline{w} = \tau - i\sigma, \quad w \in \mathbb{C}$$

The condition for v to generate a conformal transformation is that v is holomorphic, where

$$\overline{\partial} v = 0$$

So, we have a conserved charge here, and we will show a way to generate an infinite amount of conserved charges which govern a lot of the dynamics here. A particularly useful set of coordinates for our worldsheet is

$$z = e^{\tau + i\sigma}, \quad \overline{z} = e^{\tau - i\sigma}$$

Under this map, the cylindrical worldsheet is mapped to the complex plane, if we choose a Euclidean metric. Time evolution (τ) on the cylinder becomes radial evolution on the complex plane.

4.1.1 Connection to the Witt Algebra

Let's return to the Witt algebra in the context of CFT. We saw last time that the conformal transformations are generated by holomorphic and anti-holomorphic vector fields, $v(z)$, and $\overline{v}(\overline{z})$. In other words, we have the map

$$z \to z + v(z) + \dots, \quad \overline{z} \to \overline{z} + \overline{v}(\overline{z}) + \dots$$

We can write $v(z)$ as a series expansion

$$v(z) = \sum_n v_n z^{n+1}$$

we might have some issues here if $v(z)$ has a pole. We may find the $n + 1$ factor a weird convention to put, but we'll see why this is later on. Then, we have infinitesimally, our generator gives $z \to z + \sum_n v_n z^{n+1}$. We can think of this infinitesimal transformation as being generated by the operators $l_n = -z^{n+1} \frac{\partial}{\partial z}$. In other words, we have that

$$z \to z - \sum_n v_n l_n z$$

The negative sign here right now is a stylistic convention. We have the similar analog for the conjugate $\bar{z}$. The l_n satisfy

$$[l_n, l_m] = (n - m)\, l_{n+m}$$

which is the Witt algebra. This can be checked directly by substituting in our expressions. The proof is straightforward. We're going to talk about these sorts of transformations and how things in our field theory transform with these types of transformations. The defining feature of the string theory is that stress tensor.

4.2 Conformal Fields

We'll need a bit of Jargon first. Let's have some definitions. To begin with.

Definition. Chiral Field. A chiral field is a field $\Phi(z)$ that depends on z alone. This is also called a homomorphic field.

Definition. Conformal Dimension. The conformal dimension is a number $\Delta = h + \bar{h}$ which tells us how a field transforms under scaling. Suppose we rescale $(z, \bar{z})$ by some number as $(z, \bar{z}) \to \left(\lambda z, \bar{\lambda}\bar{z}\right)$. If our field transforms as

$$\Phi(z, \bar{z}) \to \Phi'\left(z', \bar{z}'\right) = \lambda^h \bar{\lambda}^{\bar{h}} \Phi\left(\lambda z, \bar{\lambda}\bar{z}\right)$$

A chiral field has $\bar{h} = 0$

Definition. Primary Field. Under the transformation $z \to f(z)$, a primary field transforms as

$$\Phi(z, \bar{z}) \to \left(\frac{\partial f}{\partial h}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{z}} \Phi\left(f(z), \bar{f}(\bar{z})\right)$$

Here we will find that the theory factors into two sectors. But we say almost factors because we still have this level matching condition which link the anti-holomorphic and holomorphic parts. When we study the chiral theory, we might have the exact same thing in the anti-chiral sector. However, there are theories in which the left and right moving parts behave differently, and this is called heterotic string theory.

We can write the transformation functions $f(z)$ and $\bar{f}(\bar{z})$ infinitesimally. Note however that $\bar{f}$ isn't necessarily the complex conjugate, just a different function! For an infinitesimal transformation $f(z) = z + v(z) + \dots$, and the corresponding anti-holomorphic function, we have that

$$\left(\frac{\partial f}{\partial z}\right)^h \simeq (1 + \partial v)^h = 1 + h\partial v, \quad \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \simeq 1$$

Our corresponding field changes as

$$\phi(f(z)) = \phi(z + v + \dots) = \phi(z) + v\partial\phi(z) + \dots$$

and similarly, we find that for $\bar{z} \to \bar{z} + \bar{v}(\bar{z})$, and we find that

$$\delta_{v,\bar{v}}\phi(z, \bar{z}) = \left(h\partial v + \overline{h\partial}\bar{v} + v\partial + \bar{v}\bar{\partial}\right)\phi(z, \bar{z})$$

4.3 Conformal transformations and the Stress Tensor

We start with the action

$$S[X] = -\frac{1}{4\pi\alpha'}\int_{\Sigma} d^2\sigma \partial_a X^\mu \partial^a X^\nu \eta_{\mu\nu}$$

Under a conformal transformation, we have that $\delta_v X^\mu = v^a \partial_a X^\mu$. Under such a transformation, we have that the action changes by

$$\delta_v S[X] = \frac{1}{2\pi}\int_{\Sigma} d^2\sigma \left(\partial^a v^b\right) T_{ab} = -\frac{1}{2\pi}\int_{\Sigma} d^2\sigma v^a \left(\partial^b T_{ab}\right) = 0$$

The first step is shown quite straightforwardly by substituting in our form of T_{ab} where $h_{ab} = \eta_{ab}$, and then doing some integration by parts, and you should find that the factors of $\frac{1}{2}$ in our definition of T_{ab} add up to the right amount. Then, we use the symmetry of the stress tensor and then integrate by parts again to get the second identity. This implies a conserved current, which satisfies

$$\partial^a T_{ab} = 0 = \partial^b T_{ab}$$

We could define conserved charges $Q_\pm$ which in light cone coordinates $\sigma^\pm = \tau \pm \sigma$ look like

$$Q_\pm = \frac{1}{2\pi}\oint d\sigma v^\pm(\sigma) T_{\pm\pm}(\sigma)$$

Why are these charges conserved? I am not sure. Infinitesimal conformal transformations are given by the Poisson bracket of a field with $\phi_\pm$. We have that

$$\delta_v X^\mu = \{Q_+ + Q_-, X^\mu\}_{PB}$$

where our bracket here is the classical Poisson bracket. We want to understand how symmetries give rise to space time symmetries. We want to recast this into having conformal symmetries give rise to space time structure.

4.4 Complex Coordinates

To simplify things, we'll change our coordinate system We shall use $z = e^{\tau + i\sigma}$, $\overline{z} = e^{\tau - i\sigma}$ and take h_{ab} to be Euclidean on Σ in the σ, τ coordinates. First, if we write down our Polyakov action with h_{ab} chosen to be Euclidean, and then change coordinates such that

$$\partial_\tau = z\partial + \overline{z}\overline{\partial}, \quad \partial_\sigma = i\left(z\partial - \overline{z}\overline{\partial}\right)$$

In these coordinates, the stress tensor becomes:

$$T_{zz} := T = -\frac{1}{\alpha'}\partial X^\mu X^\nu \eta_{\mu\nu}, \quad T_{\overline{z}\overline{z}} = \overline{T} = -\frac{1}{\alpha'}\overline{\partial} X^\mu \overline{\partial} X^\nu \eta_{\mu\nu}$$

Recall that, regardless of coordinate choice, we have that the stress tensor is traceless, since $h_{ab}T^{ab} = 0$. In these new set of coordinates, we have that $T_{z\overline{z}} = 0$ as it is the trace of the stress tensor. We now explore how our action an equations of motion change under these change of coordinates. First, we look at how our measure in our integral for the action changes. Here we have that

$$d\tau d\sigma = -\frac{dz d\overline{z}}{2i|z|^2}$$

The path integral becomes

$$\int \mathcal{D}Xe^{iS[X]} \to \int \mathcal{D}Xe^{-S[X]}$$

our action is

$$S[X] = -\frac{1}{2\pi\alpha'} \int_{\Sigma} d^2z \partial X^{\mu} \overline{\partial} X^{\nu} \eta_{\mu\nu}$$

The equation of motion for the X^{μ} is now $\partial\overline{\partial}X^{\mu} = 0$. We have that $X^{\mu}(z, \overline{z}) = X_L^{\mu}(z) + X_R^{\mu}(\overline{z})$ The conservation equation $\partial^a T_{ab} = 0$, becomes

$$\overline{\partial}T(z) = 0, \quad \partial\overline{T}(\overline{z}) = 0$$

4.5 Ward Identities and Conformal Transformations

The point of this section is to explore the quantum version of Noether's theorem. Previously, we saw that if we transform a field to give $\delta\phi = \epsilon f(\phi, \partial\phi, \dots)$, and the action stays invariant then if ϵ is a constant, we have a **classical** symmetry of the action.

However, our generator ϵ^a might also depend on our location on the worldsheet. As we saw before, this means that our change in the action δS (which is zero by construction), is now

$$\delta S = \int_{\Sigma} d^2\sigma \left(\partial^a v^b\right) T_{ab}$$

Using integration by parts and shifting the derivative to hit the stress tensor, we get that out of this, we have a conservation law

$$\partial^a T_{ab} = 0$$

Now we will focus on the quantum analogue of this principle. We will describe a general primary field as some field which is a function of $z, \overline{z}$, sufficiently well-behaved. $\phi(z, \overline{z})$. Later we shall choose ϕ to be either X^{μ} or one of the b, c ghost fields. The first thing we do will be to consider the change in a correlation function

$$\langle \phi_1 \dots \phi_n \rangle = \int \mathcal{D}\phi e^{-S[\phi]} \phi_1 \dots \phi_n$$

resulting from an infinitesimal transformation $\phi \to \phi + \delta\phi = \phi'$. In particular we would like to see what happens when we change the field under a conformal transformation. We have

$$\left\langle \phi'(z_1) \dots \phi'(z_n) \right\rangle = \int \mathcal{D}\phi' e^{-S[\phi']} \phi_1'(z_1) \dots \phi'(z_n)$$

We shall assume that $\mathcal{D}\phi = \mathcal{D}\phi'$. When we change the field, we have that

$$\left\langle \phi'(z_1) \dots \phi'(z_n) \right\rangle = \int \mathcal{D}\phi e^{-S[\phi] - \delta S[\phi]} \left(\phi(z_1) + \delta\phi(z_1)\right) \dots \left(\phi(z_n) + \delta\phi(z_n)\right)$$

This is, to the first order in $\delta\phi$,

$$\dots = \langle \phi(z_1) \dots \phi(z_n) \rangle - \int \mathcal{D}\phi e^{-S[\phi]} \delta S[\phi] \phi(z_1) \dots \phi(z_n) + \int \mathcal{D}\phi e^{-S[\phi]} \sum_{k=1}^{n} \phi(z_1) \phi(z_{k-1}) \delta\phi(z_k) \dots \phi(z_n)$$

to leading order. Now matter what kinds of fields we use, we expect that the physics of our theory to be invariant under the change $\phi \to \phi'$. This means that we impose the condition that the correlation functions should be the same. We thus require

$$\left\langle \phi_1' \dots \phi_n' \right\rangle = \left\langle \phi_1 \dots \phi_n \right\rangle$$

This means that we're left with the identity that the correlation function, including a change in the metric, is a sum of correlation functions with some insertions.

$$\left\langle \delta S\left[\phi\right] \phi\left(x_1\right) \dots \phi\left(z_n\right) \right\rangle = \sum_{k=1}^{n} \left\langle \phi\left(z_1\right) \dots \delta\phi\left(z_k\right) \dots \phi\left(z_n\right) \right\rangle$$

For a change $\phi \to \phi + \delta\phi$, the action undergoes some $S\left[\phi\right] \to S\left[\phi\right] + \delta S\left[\phi\right]$. We then found that our associated correlation function

$$\left\langle \delta S\left[\phi\right] \phi_1 \dots \phi_n \right\rangle = \sum_{k=1}^{n} \left\langle \phi_1 \dots \delta\phi_k \dots \phi_n \right\rangle$$

where we have that $\phi_i = \phi\left(z_i, \overline{z}_i\right)$. Let's now take S to be the Polyakov action with fixed worldsheet metric, and $\delta\phi = v^a \partial_a \phi$. We can use the fact that the variation of the action under such a transformation gives

$$\delta S\left[\phi\right] = \frac{1}{2\pi} \int_{\Sigma} d^2\sigma \left(\partial_a v_b\right) T^{ab}$$

where T^{ab} is the stress tensor. We have that v^a are just vector parameters which we can pull out of the integrand and the correlation function. From the equation above, we have

$$\frac{1}{2\pi} \int_{\Sigma} d^2\sigma \left(\partial^a v^b\right) \left\langle T_{ab} \phi_1 \dots \phi_n \right\rangle = \sum_{k=1}^{n} \left\langle \phi_1 \dots \delta_k \dots \phi_n \right\rangle$$

What does our worldsheet look like? In this case, in our (σ, τ) coordinates, it looks like multiple cylinders coming in and then emerging out into different cylinders. However, we observed before that we can map cylinders in (σ, τ) coordinates to circles in $(z, \overline{z})$ coordinates. Thus, our worldsheet looks like, in the $(z, \overline{z})$ plane, as the complex plane except with discs centred on each insertion z_i. We call these discs $\mathcal{D}_i$, and we label the boundary c_i. (diagram here).

we map each of these cylinders to the complex place using $z = e^{\tau + i\sigma}$, and glue the regions together using holomorphic transition functions. If we take the boundaries c_i to points (the local τ coordinate goes to $\tau \to -\infty$), then these circles go to points. We can associate the state on the boundary c_i to a local operator $\phi\left(z, \overline{z}_i\right)$. We'll choose some v which is completely general on the bulk of the worldsheet, and vanishes on all the boundaries except 1. We choose our parameter v^a to be

- zero on all c_i, as well as in the disk $\mathcal{D}_i$, except c_ω which is the boundary associated to the operator at $z_i = \omega$. In other words, $\delta_v \phi_i = 0$ except for $\delta_v \phi\left(\omega, \overline{\omega}\right)$.
- We will choose v^a to be of the form $v^a = \left(v\left(z\right), \overline{v}\left(\overline{z}\right)\right)$.
- We could choose multiple fields to vary, but for simplicity we'll choose just one field to vary right now.

- v^a is arbitrary otherwise in Σ otherwise.

To summarise, we have that for $i \neq k$, our value of the field v^a satisfies

$$v^a(z, \bar{z})\mid_{C_i} = 0$$

On the other hand, we on the boundary of the circle which contains $z_k = \omega$, we have that our generator field v^a is arbitrary, and is defined to be split up into the holomorphic and anti-holomorphic parts

$$v^z(z, \bar{z}) = v(z), \quad v^{\bar{z}}(z, \bar{z}) = \bar{v}(\bar{z})$$

By construction, we take the boundary of our worldsheet to be these discs, specifically so that

$$\partial\Sigma = C_\omega \cup_{i\neq k} C_i$$

and we have that the value of v^a is arbitrary everywhere else on the worldsheet.

Thus, substituting this in the only value of $\delta\phi_k$ which is non-zero on this world sheet is when $z_k = \omega$, for a particular choice of disc indexed with k . We denote $\delta\phi_k = \delta\phi(\omega, \bar{\omega})$. This means that, for this choice of v^a, we have

$$\frac{1}{2\pi}\int_\Sigma d^2\sigma\,(\partial_a v_b)\left\langle T^{ab}\phi_1 \ldots \phi_n\right\rangle = \langle \phi_1 \ldots \delta\phi(\omega, \bar{\omega}) \ldots \phi_n\rangle$$

The left hand side expression is, using integration by parts to take out a boundary term, is

$$\frac{1}{2\pi}\int d^2\sigma\partial^a\left(v^b\left\langle T_{ab}(\sigma)\,\phi_1 \ldots \phi_n\right\rangle\right) - \frac{1}{2\pi}\int_\Sigma d^2\sigma v^b\partial^a\left\langle T_{ab}(\sigma)\,\phi_1 \ldots \phi_n\right\rangle$$

To evaluate the first term in the expression above, we appeal to Stokes' theorem on the complex plane. In differential geometry, Stokes theorem is the statement that, in two dimensions

$$\int_M dw = \int_{\partial M} w,\ dw = \left(\frac{\partial w_2}{\partial x^1} - \frac{\partial w_1}{\partial x^2}\right)dx^1 \wedge dx^2, \quad w = w_1 dx^1 + w_2 dx^2$$

We set, in the above, that $x^2 = z, x^2 = \bar{z}, w_1 = -j^z, w_2 = j^{\bar{z}}$. The first term is a boundary term $\partial\Sigma = \cup_i C_i$. The only boundary contribution comes from c_ω where v^a has homomorphic $v(z)$ and anti-holomorphic $\bar{v}(\bar{z})$ components and is

$$\frac{1}{2\pi i}\oint_{c_\omega} dz v(z)\left\langle T(z)\,\phi_1 \ldots \phi_n\right\rangle - \frac{1}{2\pi i}\oint_{c_\omega} d\bar{z}\bar{v}(\bar{z})\left\langle \bar{T}(\bar{z})\,\phi_1 \ldots \phi_n\right\rangle$$

where $T_{zz}(z) = T(z)$, Thus, we have that

$$\begin{aligned}\langle \phi_1 \ldots\ldots \delta_v\phi(\bar{\omega}, \omega) \ldots \phi_n\rangle &= -\frac{1}{2\pi}\int_\Sigma d^2\sigma v^b\left(\partial^a\left\langle T_{ab}\phi_1 \ldots \phi_n\right\rangle\right) \\ &\quad + \frac{1}{2\pi i}\oint_{c_\omega} dz v(z)\left\langle T(z)\phi_1 \ldots \phi_n\right\rangle - \frac{1}{2\pi i}\oint_{c_\omega} d\bar{z}\bar{v}(\bar{z})\left\langle \bar{T}(\bar{z})\,\phi_1 \ldots \phi_n\right\rangle\end{aligned}$$

Since this is valid for arbitrary v we have that this is

$$\partial^a\left\langle T_{ab}(\sigma), \phi_1 \ldots \phi_n\right\rangle = 0$$

This is the analogue of the classical Nether statement that $\partial^a T_{ab} = 0$. This then means that

$$\langle \phi_1 \ldots \ldots \delta_v \phi(\overline{\omega}, \omega) \ldots \phi_n \rangle = \frac{1}{2\pi i} \oint_{c_\omega} - \frac{1}{2\pi i} \oint_{c_\omega} d\overline{z}\overline{v}(\overline{z}) \left\langle \overline{T}(\overline{z}) \phi_1 \ldots \phi_n \right\rangle$$

Note that $\phi_i \neq \phi(\omega, \overline{\omega})$ didn't contribute to the calculation and we conclude that

$$\delta_v \phi(\omega, \overline{\omega}) = \oint_{c_\omega} \frac{dz}{2\pi i} v(z) T(z) \phi(\omega, \overline{\omega}) - \oint_{c_\omega} \frac{d\overline{z}}{2\pi i} \overline{v}(\overline{z}) \overline{T}(\overline{z}) \phi(\omega, \overline{\omega})$$

this is understood to hold up on insertion into a correlation function.

4.6 Radial Ordering

Here, we're working with the Euclidean metric, so we don't have an a-priori notion of time. There is a notion that radial separation plays an analogous role of time. We see that time ordering $\tau_1 > \tau_2$ translates into radial ordering $|z_1| > |z_2|$. We introduce the notion of radial ordering. The radial ordering of two operators is

$$\mathcal{R}(A(z) B(w)) = \begin{cases} A(z) B(w) & \text{if } |z| > |w| \\ B(w) A(z) & \text{if } |z| < |w| \end{cases}$$

Consider the expression $\oint_{C(w)} dz \mathcal{R}(A(z) B(w))$, where $C(w)$ is the contour around $z = w$. How do we make sense of this? As shown in the diagram, on the circle around w, some parts of the contour have $|z|$ larger than $|w|$, and smaller on other parts. We can make sense of this contour by looking at it as the difference of two contours, which both have a well defined ordering. On

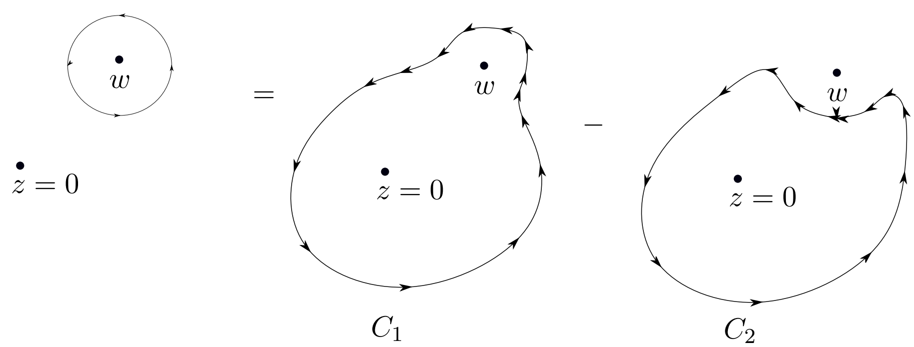

Figure 2:

C_1, we have $|z| > |w|$, and otherwise on C_2. This means that on C_1, we have that

$$\oint_{C_1} dz \mathcal{R}(A(z) B(w)) = \oint dz A(z) B(w)$$

On C_2, we have that

$$\oint_{C_2} dz \mathcal{R}(A(z) B(w)) = \oint_{C_2} dz B(w) A(z)$$

Thus, we have that, on the specific contour C_w around w, we have decomposed this into a commutator of sorts.

$$\oint_{C_w} dz \mathcal{R}\left(A\left(z\right)B\left(w\right)\right) = \oint_{C_1} dz A\left(z\right)B\left(w\right) - \oint_{C_2} B\left(w\right)A\left(z\right)$$

In particular, consider

$$O = \oint dz A\left(z\right)\ldots,\ldots B\left(w\right)$$

Consider the integral which we define as the radial ordering part.

$$\oint_{C(w)} dz \mathcal{R}\left(A\left(z\right), B\left(w\right)\right) := \left[O, B\left(w\right)\right]$$

This plays the role of a commutator in our radially ordered theory. If we consider the variation of a chiral primary field $\phi\left(w\right)$, we get a contribution from our stress energy tensor $T_{zz}\left(z\right)$ (since ϕ is chiral we have zero contribution from the anti-holomorphic component of our stress tensor $\overline{T}$). This can be encoded into a charge Q, and so the δ function can be written as such.

$$\begin{aligned}
\delta_v \phi\left(w\right) &= \oint_{C(w)} \frac{dz}{2\pi i} \mathcal{R}\left(v\left(z\right)T\left(z\right)\phi\left(w\right)\right) \\
&= \oint_{|z|>|w|} \frac{dz}{2\pi i} v\left(z\right)T\left(z\right)\phi\left(w\right) - \oint_{|z|\leq|w|} \frac{dz}{2\pi i}\phi\left(w\right)v\left(z\right)T\left(z\right) \\
&= \left[Q, \phi\left(w\right)\right]
\end{aligned}$$

where the charge Q is given by

$$Q = \int_{C(w)} \frac{dz}{2\pi i} v\left(z\right)T\left(z\right)$$

This is similar to the classical poisson bracket expression for the transformation of a field $\phi\left(w\right)$. We're extracting some information about the pole structure here, and we want to explore this.

4.7 Mode expansions and Conformal Weights

We previously considered the transformation from a cylinder to a flat plane. We want to see how the mode expansions change from the cylindrical worldsheet to the plane. On the cylinder the mode expansion for a field $\phi\left(\sigma,\tau\right)$ which only depends on $w = \tau + i\sigma$, and therefore only depends on $z = e^w$ (and is hence a chiral field) might be naturally written as

$$\phi_{\text{cyl}}\left(\omega\right) = \sum_n \phi_n e^{-n\omega}$$

and similarly for a field with $\overline{\omega}$ dependence. This would be the left-moving part. Suppose ϕ_{cyl} has conformal weight h, in other words $\phi_{\text{ cyl}}$ is a chiral primary. The definition of a chiral primary field is that it transforms like

$$\phi\left(w\right) \to \phi'\left(w\right) = \left(\frac{\partial z}{\partial w}\right)^h \phi\left(z\left(w\right)\right)$$

Now, if we switch around our z and w variables, we get that

$$\phi'(z) = \left(\frac{\partial w}{\partial z}\right)^h \phi(w)$$

If we now transform to the complex plane $z = e^w = e^{\tau + i\sigma}$, we have that

$$\phi_{\text{cyl}}\ (w) \to \phi_{\text{plane}}(z) = \left(\frac{\partial z}{\partial w}\right)^{-h} \phi_{\text{cyl}}(w) = z^{-h}\phi_{\text{cyl}}(w)$$

This is what our field on our cylinder looks like mapped to our complex plane. To save ink, we'll instead write $\phi_{\text{plane}}(z) = \phi(z)$. We have that

$$\phi(z) = \sum_n \phi_n z^{-n-h}$$

Previously, the natural thing to do was to do a Fourier expansion. Now, we have that on the complex plane with Euclidean worldsheet that this is the natural form of mode expansion, for chiral primary fields. More generally, for a primary of weight $\left(h, \overline{h}\right)$, we have that

$$\phi(z, \overline{z}) = \sum_{n,m} \phi_{n,m} z^{-n-h} \overline{z}^{-m-\overline{h}}$$

For example, as we shall see, the stress tensor has weight $\left(h, \overline{h}\right) = (2, 0)$ for $T(z)$ and $(0, 2)$ for $\overline{T}(\overline{z})$. We're shifting how we count the modes here.

$$T(z) = \sum_n L_n z^{-n-2}, \quad \overline{T}(\overline{z}) = \sum_n \overline{L}_n \overline{z}^{-n-2}$$

For reference

$$X^\mu(z, \overline{z}) = x^\mu - i\frac{\alpha'}{2} p^\mu \ln|z|^2 + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left(\alpha_n^\mu z^{-n} - \overline{\alpha}_n^\mu \overline{z}^{-n}\right)$$

and we have that our holomorphic derivative gives

$$\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^\mu z^{-n-1}, \quad \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$$

in other words, it looks like ∂X^μ looks to be a conformal field with weight $(1, 0)$. In some sense, we're working backwards here.

4.8 Radial Ordering and Normal Ordering

To a large extent we can go through the same route in QFT to relate radial ordering to normal ordering. Consider the weight $(1, 0)$ chiral field $j^\mu(z) = \sum_n \alpha_n^\mu z^{-n-1}$. We shall think of $j^\mu(z)$ as

$$j^\mu(z) = i\sqrt{\frac{\alpha}{2}} \partial X^\mu(z)$$

Since $\alpha n^{\mu} \left|0\right\rangle$ for $n \geq 0$, we can split $j^{\mu}\left(z\right)$ into creation and annihilation parts.

$$j^{\mu}\left(z\right) = j^{\mu}_{+}(z) + j^{\mu}_{-}\left(z\right)$$

where we have $j^{\mu}_{+}\left(z\right) = \sum_{n\geq 0} \alpha^{\mu}_{n} z^{-n-1}$. We define normal ordering :: in the usual way, as moving all creation operators to the left, in any string of operators. In this case, in particular, that means

$$: j^{\mu}\left(z\right) j^{\nu}\left(w\right) := j^{\mu}_{+}\left(z\right) j^{\nu}_{+}\left(w\right) + j^{\mu}_{-}\left(z\right) j^{\nu}_{+}\left(w\right) + j^{\nu}_{-}\left(w\right) j^{\mu}_{+}\left(z\right) + j^{\mu}_{-}\left(z\right) j^{\nu}_{-}\left(w\right)$$

Note that

$$: j^{\mu}\left(z\right) j^{\nu}\left(w\right) := j^{\mu}\left(z\right) j^{\nu}\left(w\right) + \left[j^{\nu}_{-}\left(w\right), j^{\mu}_{+}\left(z\right)\right]$$

Recall from the point pf Wick's theorem that we could think of this commutator as a contraction. There are many ways we could evaluate this commutator. We can use the mode expansion to evaluate the commutator. So,

$$\left[j^{\mu}_{-}\left(w\right), J^{\nu}_{+}\left(z\right)\right] = \sum_{m\geq 0, n\geq 0} \left[\alpha^{\mu}_{-m}, \alpha^{\nu}_{n}\right] w^{m-1} z^{-n-1} = -n \sum_{m,n} n\delta_{m,n}\eta^{\mu\nu} w^{m-1} z^{-n-1}$$

This gives us

$$\cdots = -\frac{\eta^{\mu\nu}}{z^2} \sum_{n>0} n \left(\frac{w}{z}\right)^{n-1} = \frac{\eta^{\mu\nu}}{\left(z-w\right)^2}$$

This will converge if $|z| > |w|$. This tells us that the relationship

$$: j^{\mu}\left(z\right) j^{\nu}\left(w\right) := j^{\mu}\left(z\right) j^{\nu}\left(w\right) + \frac{\eta^{\mu\nu}}{\left(z-w\right)^2}$$

where $|z| > |w|$. We see that this relationship, the different between the normal product and the generic product is something that is singular. This is useful when evaluating contour integrals.

Recall that $j\left(z\right)$ is a chiral primary, with conformal weight $\left(h, \overline{h}\right) = (1, 0)$. We also said that for $|z| > |w|$, we have that

$$j^{\mu}\left(z\right) j^{\nu}\left(w\right) =: j^{\mu}\left(z\right) j^{\nu}\left(w\right) : + \frac{\eta^{\mu\nu}}{\left(z-w\right)^2}$$

Similarly, we have that the converse statement reads for $|w| > |z|$,

$$j^{\nu}\left(w\right) j^{\mu}\left(z\right) =: j^{\mu}\left(z\right) j^{\nu}\left(w\right) : + \frac{\eta^{\mu\nu}}{\left(z-w\right)^2}$$

This means we have an expression for the radial ordered product. In the spirit for Wick's theorem, we write the radial ordering of the j operators as the sum of a normal ordered part plus a contraction part. We have

$$\mathcal{R}\left(j^{\mu}\left(z\right) j^{\nu}\left(w\right)\right) =: j^{\mu}\left(z\right) j^{\nu}\left(w\right) : + \overbrace{j^{\mu}\left(z\right) j^{\nu}}\left(w\right)$$

Where, we have that in this case, $\overbrace{j^{\mu}\left(z\right) j^{\nu}}\left(w\right)$. This is referred to as a contraction. Recall from earlier that we defined the derivative of the mode expansion as $\partial X^{\mu}\left(z\right) = -i\sqrt{\frac{\alpha'}{2}} j^{\mu}\left(z\right)$. This means that the corresponding contraction between these derivative fields is

$$\overbrace{\partial_z X^{\mu}\left(z\right) \partial_w} X^{\nu}\left(w\right) = \frac{-\alpha'}{2} \frac{\eta^{\mu\nu}}{\left(z-w\right)^2}$$

We get a contraction for free between the fields $X^{\mu}(z)$ and $X^{\nu}(w)$ when we integrate this object through. When we integrate with respect to w then z, we then get

$$X^{\mu}(z)\,X^{\nu}(w) = -\frac{\alpha'}{2}\eta^{\mu\nu}\log(z-w)$$

This is generalised in Wick's theorem where we take all possible contractions of the fields in normal ordering.

$$\begin{aligned}\mathcal{R}(\phi_1(z_1)\ldots\phi_n(z_n)) =&: \phi_1(z_1)\ldots\phi_n(z_n): \\ &+\sum_{i,j}:\phi_1(z_1)\ldots\phi_i(z_i)\ldots\phi_j(z_j)\,\ldots\phi_n(z_n) \\ &+\ldots\end{aligned}$$

4.9 Operator Product Expansions

We construct something called an operator expansion. This is a way to characterise the behaviour of a set of local operators at short distances. In particular, we are interested in the limit

$$\lim_{z\to w} O_i(w)\,O_j(z)\sum_k f_{ij}^k(z-w)\,O_k(w)$$

For example, we showed just now that

$$\mathcal{R}(\partial X^{\mu}(z)\,X^{\nu}(w)) =: \partial X^{\mu}(z)\,\partial X^{\nu}(w): -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-w)^2}$$

This term is regular as $z\to w$. We write that the average expected value, since our normal ordering term vanishes

$$\Big\langle\mathcal{R}(\partial X^{\mu}(z)\,\partial X^{\nu}(w)\rangle = -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-w)^2}$$

From now on, we weill implicitly assume radial ordering everywhere unless otherwise stated. We could compute the correlation function, which as in quantum field theory is just the correlation function between

$$\begin{aligned}\langle X^{\mu}(z)\,X^{\nu}(w)\rangle &= \langle: X^{\mu}(z)\,X^{\nu}(w):\rangle - \frac{\alpha'}{2}\eta^{\mu\nu}\log(z-w) \\ &= -\frac{\alpha'2}{2}\eta^{\mu\nu}\log(z-w)\end{aligned}$$

Recall from QFT last term that the two point propagator is precisely the Green's function for our Klein Gordon equation. Thus, we have that $-\frac{\alpha'}{2}\eta^{\mu\nu}\log(z-w)$ is the solution for the Green's function for $\frac{1}{2\pi\alpha'}\partial\overline{\partial}$. Similarly, we have our OPE expansions for $\overline{X}^{\mu}$ and their respective anti-holomorphic variables. This means that

$$\overline{X}^{\mu}(\overline{z})\,\overline{X}^{\nu}(\overline{w}) = -\frac{\alpha'}{2}\eta^{\mu\nu}\log(\overline{z}-\overline{w})$$

So, we still have a singular part when we have z approach w. On the other hand, when we take the radial ordering $\overline{X}^{\mu}X^{\nu}$, we have that this object is regular. We can use the knowledge of the

X^μ OPE to now define operators like the stress tensor. In the classical treatment of our theory, we had that our stress tensor in terms of holomorphic variables is given by

$$T(z) = -\frac{1}{\alpha'} \partial X^\mu(z) \partial X_\mu(z)$$

But, this doesn't apply when we are working with conformal field theory. In this case, we need to define the normal ordered part as the physical part of what we require. This means we take the sum of our radial ordered part as well as our contraction. We can take a look at why we need to do the $T(z) X^\mu(z)$ OPE, for the purpose of finding out how things transform conformally. and take the sum. Since by definition, we have that the stress tensor is $T(z)$ physical, and therefore normal ordered, we have the product

$$T(z) X^\mu(w) = -\frac{1}{\alpha'} : \partial X^\nu(z) \partial X_\nu(z) : X^\mu(w)$$

We make use of the contraction expression for $\partial^\mu(z) X^\nu(w) = -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{z-w}$. Observing that the contraction of a normal ordered product has a zero contraction, we get that when we apply Wick's theorem,

$$\begin{aligned}
T(z) X^\mu(w) &= -\frac{2}{\alpha'} : \partial X^\nu \overline{\partial X_\nu(z) : X^\mu}(w) + \ldots \\
&= -\frac{2}{\alpha'} \partial X_\nu(z) \left(-\frac{\alpha'}{2} \frac{\eta^{\mu\nu}}{z-w} \right) + \ldots \\
&= \frac{\partial X^\mu(z)}{z-w} + \ldots
\end{aligned}$$

We could now just Taylor expand $\partial X^\mu(z)$ about $z = w$, which gives us

$$\partial X^\mu(z) = \partial X^\mu(w) + (z-w) \partial^2 X^\mu(w) + \ldots$$

The second part of this equation is regular so we don't really care about this. Substituting this into the above, we find that our expression for the stress tensor combined with our position field is

$$T(z) X^\mu(w) = \frac{\partial X^\mu}{z-w}$$

We can now compute the conformal transformation associated with the generating vector v, of $X^\mu(w)$. We expect that since this is a field, that we have $X^\mu(w)$ not to transform, so that $h = 0$. Let's confirm our suspicions here. Our conformal transformation of the field is

$$\begin{aligned}
\delta_v X^\mu(w) &= \oint_{z=w} \frac{dz}{2\pi i} v(z) T(z) X^\mu(w) \\
&= \oint_{z=w} \frac{dz}{2\pi i} v(z) \frac{\partial X^\mu(w)}{(z-w)} \\
&= v(w) \partial X^\mu(w)
\end{aligned}$$

One can check that this gives the correct conformal transformation for $X^\mu(z)$. Recall that a general operator transforms under conformal transformations $f(z) = z + v(z)$ exactly and infinitesimally as follows:

$$\phi(z) = \left(\frac{df}{dz}\right)^h \phi(f(z)), \quad \delta_v \phi(z) = v\partial\phi + h(\partial v)\phi$$

Comparing this to the expression of $\delta_v \phi(v)$, this suggest that we have $h = 0$ for the conformal transformation of $X^\mu(z)$. Recall that in general, we have that a conformal transformation can be written as and this gives us

$$\delta_v \phi(z) = \int \frac{dw}{2\pi i} \mathcal{R}\left(v(w) T(w) \phi(z)\right)$$

Making use of the residue theorem, which is

$$\frac{1}{(n-1)!} \partial_z^{n-1} f(z) = \oint_{w=z} \frac{dw}{2\pi i} \frac{f(w)}{(w-z)^n}$$

This requirement fixes the $T(w)\phi(z)$ OPE to have the form

$$T(w)\phi(z) = \frac{h}{(z-w)^2}\phi(z) + \frac{1}{z-w}\partial\phi(z) + \dots$$

For formal purposes, we will now take this AS the definition for a field $\phi(z)$ to be a chiral primary of weight $(h, 0)$.

4.10 The OPE of $T(z) : e^{ik\cdot X(w)}$

Let's calculate the OPE expansion of the object $T(z) : e^{ik\cdot X(w)}$:. Throughout this calculation we'll use a few tricks, including contracting things multiple times. The first thing to do is to recall the definition of the stress tensor in holomorphic coordinates, the $T_z z(z) = T(z)$ component of the stress tensor is given by

$$T(z) = -\frac{1}{\alpha'} : \partial_\mu X_1$$

After single contractions, and Taylor expanding out $\partial^\nu(z) = \partial X^\nu(w) + (z-w) + \partial^2 X^\nu(w)$, we find that the expansion from single contractions is $\frac{1}{z-w}\partial\left(e^{ik\cdot X(w)}\right)$. The only other contraction we can have are with two sets of contractions from both of the terms in $T(z)$. Using the fact that we can have $n(n-1)$ contractions.

This term is given by

$$\begin{aligned}
T_2 &= -\frac{1}{\alpha'} \sum_{n\geq 2} k_{\mu_3} \dots k_{\mu_n} \frac{i^n}{n!} n(n-1) X^{\mu_3}(w) \dots X^{\mu_n} \left(-\frac{\alpha'}{2}\right)^2 \frac{k^2}{(z-w)^2} \\
&= -\frac{\alpha'}{4} \frac{k^2}{(z-w)^2} \sum_{n\geq 2} (k\cdot X)^{n-2} i^2 i^{n-2} \frac{n!}{n!(n-2)!} \\
&= \frac{\alpha'}{4} \frac{k^2}{(z-w)^2} : e^{ik\cdot X} :
\end{aligned}$$

Since this is all the possible contractions, we have that our OPE is

$$T(z) : e^{iK\cdot X(w)} := \left(\frac{\alpha' k^2/4}{(z-w)^2} + \frac{\partial}{(z-w)}\right) e^{ik\cdot X}$$

This gives us an indication of what the value of $h = \frac{\alpha' k^2}{4}$. It is easy to see that a similar result holds for $\overline{T}(\overline{z}) e^{ik\cdot\overline{X}(\overline{w})}$. More generally, the operator which is a function of both variables

$: \exp\left(ik_\mu X^\mu\left(w,\overline{w}\right)\right):$ has the conformal weight $\left(h,\overline{h}\right)=\left(\frac{\alpha' k^2}{4},\frac{\alpha' k^2}{4}\right)$. I'm not sure why this is true, but there's a sense this is a purely quantum mechanical result.

Now, since T is such an important object, it is of interest of us to explore how T itself transforms conformally. What is it's conformal weight, for example?

We now try to find out what the $T\left(z\right)T\left(w\right)$ operator product expansion is. This is related to something called the Virasoro algebra. We use the fact that

$$T\left(z\right)=-\frac{1}{\alpha'}:\partial X\partial X:$$

From our previous analysis we found that $\partial X^\mu\left(z\right)\partial X^\nu\left(w\right)=-\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{\left(z-w\right)^2}$. Again, we use Wick's theorem and do some contractions. In terms of single contractions, we have 4 of the same type of contraction, where we contract $\partial X_\mu\left(z\right)$ with $\partial X_\nu\left(w\right)$. In terms of double contractions, we have two contributions. This means that we ultimately have

$$T\left(z\right)T\left(w\right)=-\frac{2}{\alpha'}\frac{\eta^{\mu\nu}}{\left(z-w\right)^2}:\partial X^\mu\left(z\right)\partial X^\nu\left(w\right):+\frac{1}{2}\frac{\eta^{\mu\nu}\eta_{\mu\nu}}{\left(z-w\right)^4}$$

Ideally we want to rewrite this in terms of $T\left(w\right)$ so we can compare it to the form of the conformal transformation. To do this, we Taylor expand $\partial X^\mu\left(z\right)=\partial X^\mu\left(w\right)+\left(z-w\right)\partial^2X^\mu\left(w\right)+\dots$. This means we get a total expansion of the form

$$T\left(z\right)T\left(w\right)=\frac{D/2}{\left(z-w\right)^4}-\frac{2}{\alpha'}\frac{1}{\left(z-w\right)^2}:\partial X^\mu\partial X_\mu\left(w\right):-\frac{2}{\alpha'}\frac{1}{z-w}:\partial X_\mu\partial^2X^\mu:$$

Therefore, we have that, in terms of the expression $T\left(w\right)$, that our operator product expansion

$$T\left(z\right)T\left(w\right)=\frac{D}{2}\frac{1}{\left(z-w\right)^4}+\frac{2}{\left(z-w\right)^2}T\left(w\right)+\frac{1}{z-w}\partial T+\dots$$

The pole of order 4 only vanishes if $D=0$, which suggests that T only transforms conformally if $D=0$. This suggests that T in fact isn't the full form of the stress tensor - we need to consider ghost fields.

4.11 The Virasoro Algebra

We think a bit harder about this and introduce a mode expansion. From our reasoning above, we have that the stress tensor is a field with weight 2, so we can expand it in terms of these modes L_n.

$$T\left(z\right)=\sum_n L_n z^{-n-2}$$

This may be inverted to give

$$L_n=\oint_{z=0}\frac{dz}{2\pi i}z^{n+1}T\left(z\right)$$

and similarly for $\overline{T}\left(\overline{z}\right)$. Consider the commutator of these fields.

$$\left[L_m,L_n\right]=\oint_{z=0}\frac{dz}{2\pi i}\oint_{w=0}\frac{dw}{2\pi i}z^{m+1}w^{n+1}\left[T\left(z\right),T\left(w\right)\right]$$

We have already argued that we should think of

$$\oint_{z=0} \frac{dz}{2\pi i} z^{m+1} \left[T(z), T(w) \right] = \oint_{z=w} R\left(T(z) T(w) \right) z^{m+1}$$

and so, what we're interested in then is

$$[L_m, L_n] = \oint_{w=0} \frac{dw}{2\pi i} w^{n+1} \oint_{z=w} \frac{dz}{2\pi i} z^{m+1} \mathcal{R}\left(T(z) T(w) \right)$$

we use the TT OPE to evaluate this. So, the only thing we care about is the singular structure, since this is a countour integral and regular terms don't contribute. Hence, this is

$$= \oint_{w=0} \frac{dw}{2\pi i} w^{n+1} \oint \frac{dz}{2\pi i} z^{m+1} \left(\frac{D/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \right)$$

Noting that

$$\oint \frac{dz}{2\pi i} \frac{z^{m+1}}{(z-w)^2} = (m+1) w^m$$

$$\oint_{z=w} \frac{dz}{2\pi i} \frac{z^{m+1}}{(z-w)^4} = \frac{1}{3!} (m+1) m (m-1) w^{m-2}$$

And so, we have that the final result of the commutator is that

$$\begin{aligned} [L_m, L_n] &= \oint_{w=0} \frac{dw}{2\pi i} w^{n+1} \left(\frac{D}{12} m \left(m^2 - 1 \right) w^{m-2} + 2T(w)(m+1) w^m + \partial T(w) w^{m+1} \right) \\ &= \frac{D}{12} m \left(m^2 - 1 \right) \delta_{m+n,0} + (m-n) L_{m+n} \end{aligned}$$

The additional term is sometimes referred to as a central extension or an anomaly term. This is the Virasoro algebra. If you look at just the terms generated by $L_0, L_{\pm 1}$, there is no anomaly. Similarly, for $\left[\overline{L}_m, \overline{L}_n \right]$ this result holds. Note that the $T(z) \overline{T}(w)$ OPE is regular so $\left[L_m, \overline{L}_n \right] = 0$.

4.12 The b, c ghost system

We will now work on a way to remove the anomalous terms that we get in the Virasoro algebra. To do this, we will need to introduce ghost fields as we did looking at our FP determinant. The Faddeev Popov procedure required us to to introduce anti-commuting ghost fields: $b_{ab}(z)$ and $c^a(z)$ such that $c^a(z) c^b(w) = -c^b(w) c^a(z)$.

Our ghost action is

$$S[b, c] = \frac{1}{2\pi} \int_\Sigma d^2\sigma \sqrt{-h} h^{ac} b_{ab} \left(\nabla_c c^b \right)$$

So, the total action $S = S_p[X] + S_{gh}[b, c]$. This allows us to cook up a stress tensor which exhibits the conformal symmetry that we want. The ghost stress tensor is then, upon varying the metric h_{ab} appropriately in our action,

$$T_{ab} = -i \left(\frac{1}{2} c^c \nabla_{(a} b_{b)c} + \left(\nabla_{(a} c^c \right) b_{b)c} - h_{ab} \text{trace} \right)$$

Notice that by construction, this object is traceless, since we've removed the trace term at the end. This is a bit of a difficult expression to work with, so let's fix $h_{ab} = e^{\phi}\delta_{ab}$ by working in the conformal gauge. The total stress tensor is the contribution that we get from the field X and the stress tensor which we got from the ghost field.

$$\mathcal{T} = T_X + T_{\text{gh}}$$

Instead of working in general indices b, c lets now work with a Euclidean metric on Σ and complex coordinates as before. We cam write the two degrees of freedom in b_{ab} as $b(z)$ and $\overline{b}(\overline{z})$, and the 2 degrees of freedom in c^a as $c(z)$ and $\overline{c}(\overline{z})$. The action then becomes

$$S_{\text{gh}}[b, c] = \frac{1}{2\pi}\int_{\Sigma} d^2 z b\overline{\partial}c + \frac{1}{2\pi}\int_{\Sigma} d^2 z\bar{b}\partial\overline{c}$$

The total stress tensor

$$\mathcal{T}(z) = T_X(z) + T_{\text{gh}}\ (z)$$

where we have

$$\begin{aligned} T(z) &= -\frac{1}{\alpha'} : \partial X^{\mu}\partial X_{\mu}(z) : \\ T_{\text{gh}}(z) &=: (\partial b)\, c(z) : -2\partial\left(: bc(z) :\right) \end{aligned}$$

with similar expressions for the anti-homomorphic variables. Since we have now X dependence from the ghost fields and vice versa, we have only regular terms when we do $T_{\text{gh}} T_X$ OPE. So, we want to find the OPE of the ghost field on it's own, which is going to be spacetime independent.

4.13 The b, c OPE

The two point function (a Green's function) gives the singular part of the $b(z)\, c(w)$ OPE. The two point function is

$$\langle b(z)\, c(w)\rangle = \langle : b(z)\, c(w) : \rangle + \overset{\sqcap}{b(z)\, c}(w) = \overset{\sqcap}{b(z)\, c}(w)$$

This is also the green's function for the holomorphic part of the action. Recall that the propagator of some field configuration is the green's function of the associated equation of motion. In terms of the holomorphic part, our action is $\int d^2 z b\overline{p}c$. Our associated operator then, for the equation of motion of c, is the $\overline{\partial}$ operator. The classical green's function for $\overline{\partial}$ on the sphere is just $\frac{1}{z-w}$ is just $\frac{1}{z-w}$. This can be shown using Stokes' theorem. We want to show that

$$\overline{p}G(b, c) = 2\pi\delta(z - w)$$

so we have

$$b(z)\, c(w) = \frac{1}{z-w} + \ldots c(z)\, b(w)$$

if we swap w and z we get a change in sign, but if we commute b and see past each other we also pick up a minus sign.

4.13.1 The conformal weight of $b(z)$

The stress tensor is $T = (\partial b)\, c - 2\partial\,(bc)$, and so the OPE we want is

$$\begin{aligned}
T(z)\, b(w) &= (\partial b(z))\, \overbracket{c(z)\, b}(w) \; - 2\partial_z (b(z)\, \overbracket{c(z))b}(w) \; \dots \\
&= \frac{\partial b(z)}{z-w} - 2\partial_z \left(\frac{b(z)}{z-w} \right) + \dots \\
&= \frac{\partial b(z)}{(z-w)} - 2\frac{\partial b(z)}{z-w} + 2\frac{b(z)}{(z-w)^2} + \dots \\
&= -\frac{\partial b(z)}{z-w} + \frac{2b(z)}{(z-w)^2} + \dots \\
&= -\frac{\partial b(w)}{z-w} + 2\frac{b(w)}{(z-w)^2} + 2\frac{\partial b(w)}{(z-w)} + \dots
\end{aligned}$$

Now, putting this in the form where we can read off the conformal weight, we get the result as

$$T_{gh}(z)\, b(w) = 2\frac{b(w)}{(z-w)^2} + \frac{\partial b(w)}{z-w}$$

We conclude $b(w)$ has weight $h = 2$. We cought do $c(w)$ to find $h = -1$, with similar expressions for the anti-holomorphic part. This gives the expression

$$T_{gh}(z)\, c(w) = -\frac{1}{(z-w)^2} c(w) + \frac{1}{z-w} \partial c(w) + \dots$$

Hence we have that $c(w)$ has weight $\left(h, \overline{h}\right) = (-1, 0)$. This feels like the sort of thing we should be getting, because we have the object b_{ab} which is some modified two form, and c^a has index upstairs. So, this is roughly what we expect. We can now find the OPE between the ghost stress tensor and itself, which is $T_{gh}(z)\, T_{gh}(w)$. Life is two short to do this explicitly, and we get that

$$T_{gh}(z)\, T_{gh}(w) = -\frac{26/2}{(z-w)^4} + \frac{2}{(z-w)^2} T_{gh}(w) + \frac{1}{z-w} \partial T_{gh}(w) + \dots$$

Recall that from our fields we have that

$$T_X(z)\, T_X(w) = \frac{D/2}{(z-w)^4} + \frac{2}{(z-w)^2} T_X(w) + \frac{1}{z-w} \partial T_X(w)$$

and we have that $T_X(z)\, T_{gh}(w)$ is regular. This means that the OPE of the total stress tensor, which is the stress tensor of the whole theory $T(z) = T_X(z) + T_{gh}(z)$ has the OPE expansion

$$\mathcal{T}(z)\, \mathcal{T}(w) = \frac{(D-26)/2}{(z-w)^4} + \frac{2}{(z-w)^2} \mathcal{T}(w) + \frac{1}{z-w} \partial \mathcal{T}(w)$$

This is assuming that we can treat the ghost fields independently. If $D = 26$, the anomaly term, which is a pole of order 4, vanishes and we have a consistent quantum conformal theory (the bosonic theory). We got this result by looking at the contributions of two sectors. The solution to resolving the tachyonic problem is using supersymmetry. To make this supersymmetric we need to add some world sheet fermions, and add this on to the stress tensor. If we do this, we get $D = 10$. We'll say something briefly about reactions to this fact. One reaction is that this is nonsense. Another reaction is that this is interesting because we have a theory which spits out a dimension, unlike other theories. It could be that macroscopically, we have 4 dimensions with the other dimensions compactified. It may be that in quantum theories, that dimensions of space-time are emergent from quantum theories in some classical limit.

4.14 Mode expansions for ghosts

In this section, we'll explore how to build anti-commutation relation We have that the conventional mode expansion for the ghosts, given by the conformal weights, are

$$b(z) = \sum_n b_n z^{-n-2}, \quad c(z) = \sum_n c_n z^{-n+1}$$

thus, we have that

$$b_n = \oint \frac{dz}{2\pi i} z^{n+1} b(z), \quad c_n = \oint \frac{dz}{2\pi i} z^{n-2} c(z)$$

Then, we have that

$$b_m c_n + c_n b_m = \{b_m, c_n\}$$

They obey fermionic statistics, integer spin. The above expression is

$$\cdots = \oint_{z=0} \frac{dz}{2\pi i} z^{m+1} \oint_{z=0} \frac{dw}{2\pi i} w^{n-2} \{b(z), c(w)\} = \oint_{z=0} \frac{dz}{2\pi i} \oint_{z=w} \frac{dw}{2\pi i} z^{m+1} w^{n-2} \mathcal{R}(b(z) c(w))$$

Now, using the $b(z) c(w)$ OPE we can evaluate this anti-commutation relation as

$$\{b_m, c_n\} = \delta_{m+n,0}$$

4.15 The State-Operator Correspondence

For example, we have the states $|k\rangle = e^{ik\cdot X}|0\rangle$, and $\epsilon_{\mu\nu}\alpha_{-1}\overline{\alpha}_{-1}|0\rangle$. In 2 dimensional conformal field theory, to each physical state there is an operator in the operator algebra of the theory. For example,

$$\partial X^\mu(z) = -i\sqrt{\frac{\alpha}{2}} \sum_n \alpha_n^\mu z^{-n-1}$$

We can construct a non-physical state as follows.

$$\lim_{z\to 0} \partial X^\mu(z)|0\rangle$$

recalling that $\alpha_n^\mu |0\rangle = 0$ for $n \geq 0$. This gives

$$-i\sqrt{\frac{\alpha'}{2}} \lim_{z\to 0} \sum_n \frac{\alpha_n^\mu}{z^{n+1}}|0\rangle = -i\sqrt{\frac{\alpha'}{2}} \lim_{z\to 0} \sum_{n\geq -1} \frac{\alpha_n^\mu}{z^{n+1}}|0\rangle$$

but since $\alpha_n^\mu \left|0\right\rangle = 0$ for $n \geq 0$, this is the state

$$-i\sqrt{\frac{\alpha}{2}}\alpha_{-1}^{\mu}\left|0\right\rangle$$

There are operators that do not ... Similarly, we have that $\lim_{z,\overline{z}\to 0} \partial X^\mu \overline{\partial} X^\nu e^{ik\cdot X(z,\overline{z})}\left|0\right\rangle = \alpha_{-1}^{\mu}\overline{\alpha}_{-1}^{\nu}\left|k\right\rangle$. More generally, if we have a weight h chiral field $\phi(z)$, then $\phi(z) = \sum_m \phi_m z^{-n-h}$. Then, for

$$\lim_{z\to 0}\phi(z)\left|0\right\rangle \text{ to exist}$$

we require that $\phi_n \left|0\right\rangle = 0$ for $n > -h$. Then, we have that

$$\lim_{z\to 0}\phi(z)\left|0\right\rangle = \phi_{-h}\left|0\right\rangle$$

We will now look at the conditions for this to be a physical state.

4.16 BRST Symmetry

After Faddeev-Popov we had the action

$$S[X,b,c] = S[X] + S_{gh}[b,c]$$

and h_{ab} was fixed to $\hat{h}_{ab}$. We would like to see how the choice $\hat{h}_{ab}$ (doesn't influence the physics). Let usa introduce a Lagrange multiplier field B_{ab} and the gauge fixing term in the action

$$S_{gf}[h,B] = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{-h}B^{ab}\left(\hat{h}_{ab} - h_{ab}\right)$$

This thoery has a residual rigid symmetry. Invariance gives us clear physcial consistency conditions and operators of the theory.

Strings

- A string is two dimensional, embedded with parameters σ, τ

$$X^\mu(\sigma,\tau) = X^\mu(\sigma + 2\pi n, \tau), \quad n \in \mathbb{Z}$$

- Our associated action is the Nambu-goto action

$$S[X] = -\frac{1}{2\pi\alpha'}\int d\sigma d\tau \sqrt{-\det\left(\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\beta\right)}$$

- We add an extra degree of freedom h_{ab} to introduce the Polyakov action

$$S[X,h] = -\frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{-h} h^{ab}\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$$

Quantising the Bosonic String

- Expand out position and conjugate momenta in terms of Fourier modes
- We use our gauge invariance to set

$$h_{ab} = e^\phi \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

  The -1 in the diagonal gives us a rough notion of time.
- Put this in our Polyakov action to get the conjugate momenta, which is

$$P_\mu = \frac{\delta S}{\delta \dot{X}^\mu} = \frac{1}{2\pi\alpha'}\dot{X}^\mu$$

- Impose equal time commutation relations which are equivalent to commutation relations of the Fourier components. We use Poisson brackets for this

$$\{X^\mu, P_\nu\} = \delta^\mu{}_\nu = \delta\left(\sigma - \sigma'\right) \iff \{\alpha_n, \alpha_m\} = -im\delta_{n+m,0}, \quad \{\overline{\alpha}_n, \overline{\alpha}_m\} = -im\delta_{n+m,0}$$

- We can change coordinates to

$$\sigma_\pm = \tau \pm \sigma$$

Example Sheet 1

4.17 Question 1

Here we are showing equivalence of the Nambu-Goto action and the Polyakov action.

5 Example Sheet 2

5.1 Question 1

We are asserting that our variations in moduli space should be orthogonal to arbitrary diffeomorphism variations generated by the vector v^a. This means for arbitrary v^a, we have that

$$(\delta_v h_{ab} \mid \delta_t h_{ab}) = 0$$

This implies that, by the definition of orthogonality of tensors provided in the definition of the question,

$$\begin{aligned} 0 &= \int_\Sigma d^2\sigma \sqrt{-h} h^{ab} h^{cd} \delta_v h_{ab} \delta_t h_{cd} \\ &= \int_\Sigma d^2\sigma \sqrt{-h} h^{ab} h^{cd} \left(\nabla_a v_b + \nabla_b v_a \right) \delta_t h_{cd} \\ &= 2 \int_\Sigma d^2\sigma \left(\sqrt{-h} h^{ab} \nabla_a v_b \right) h^{cd} \delta_t h_{cd} \\ &= 0 \end{aligned}$$

Now the thing to notice here is that this holds for arbitrary v. This means that supposing we can choose v appropriately, we find $h^{cd}\delta_t h_{cd} = 0$. This is because we have an integral of the form $\int d^2\sigma f(x) g(x) = 0$ for arbitrary $f = \sqrt{-h} h^{ab} \nabla_a v_b$. So, $g(x) = 0$. This means we're left with the result that

$$h^{ab} \delta_t h_{ab} = 0$$

For the second part of the question, we know we can write a diffeomorphsim and Weyl transformation as the sum of it's traceless part and a modified Weyl part.

$$\delta_v h_{ab} + \delta_\omega h_{ab} = (Pv)_{ab} + (2\omega - \partial_c v^c) h_{ab} = (Pv)_{ab} + (2\overline{\omega}) h_{ab} = 0$$

This means that we have

$$\begin{aligned} 0 &= (\delta_t h_{ab} \mid (Pv)_{ab} + 2\overline{\omega} h_{ab}) \\ &= (\delta_t h_{ab} \mid (Pv)_{ab}) + 2 \int d^2\sigma \sqrt{-h} h^{cd} h^{ab} \overline{\omega} h_{cd} \delta_t h_{ab} \\ &= (\delta_t h_{ab} \mid (Pv)_{ab}) + 4 \int d^2\sigma \sqrt{-h} \overline{\omega} h^{ab} \delta_t h_{ab} \\ &= (\delta_t h_{ab} \mid (Pv)_{ab}) \end{aligned}$$

5.2 Question 2

To calculate the variation with respect to h in this question, we apply the usual rules of differentiation but with variations instead.

$$\delta_h \langle O_1 \dots O_n \rangle = \int \mathcal{D}X i \delta_h S e^{iS[X,h]} O_1 \dots O_n$$

However, from lectures, we know what the variation of S with respect to h is.

$$\delta_h S = -\frac{1}{4\pi} \int d^2\sigma \delta h^{ab} T_{ab}$$

This means that our corresponding variation in our expectation value of the observables is

$$\delta_h (O_1 \dots O_n) = \int \mathcal{D}X \left(\int d^2\sigma \sqrt{-h} \delta h^{ab} T_{ab} \right) e^{iS[X,h]} O_1 \dots O_n$$

However, we switch the order of integration here and absorb T_{ab} into the definition of our expectation value. Thus, we find that

$$\delta_h \langle O_1 \dots O_n \rangle = -\frac{1}{4\pi} \int d^2\sigma \sqrt{-h} \delta h^{ab} \langle T_{ab} O_1 \dots O_n \rangle$$

In the case of a Weyl transformation, we have that $\delta h_{ab} = \omega h_{ab}$. Since our expectation value should be invariant of transformations to the metric, we have that

$$\begin{aligned}
0 &= -\frac{1}{4\pi} \int d^2\sigma \sqrt{-h} \delta h^{ab} \langle T_{ab} O_1 \dots O_n \rangle \\
&= -\frac{1}{4\pi} \int d^2\sigma \sqrt{-h} \omega h^{ab} \langle T_{ab} O_1 \dots O_n \rangle \\
&= -\frac{1}{4\pi} \int d^2\sigma \sqrt{-h} \omega \langle T^a{}_a O_1 \dots O_n \rangle
\end{aligned}$$

Since we can pick ω to be arbitrary here, we must necessarily have

$$0 = \langle T^a{}_a O_1 \dots O_n \rangle$$

5.3 Question 3

In this question, it's a matter of going through the motions.

$$\begin{aligned}
\delta S &= -\frac{1}{4\pi\alpha'}\int d^2\sigma 2\delta\left(\partial_a X\right)\cdot\partial^a X \\
&= -\frac{1}{2\pi\alpha'}\int d^2\sigma\partial_a\left(v^b\partial_b X\right)\cdot\partial^a X \\
&= -\frac{1}{2\pi\alpha'}\int d^2\sigma\,\left(\partial_a v^b\right)\partial_b X\cdot\partial^{aX} + v^b\partial_a\partial_b X\cdot\partial^a X \\
&= -\frac{1}{2\pi\alpha'}\int d^2\sigma\,\left(\partial^a v^b\right)\partial_b X\partial_a X + \frac{1}{2}v^b\partial_b\left(\partial_c X\cdot\partial^c X\right) \\
&= -\frac{1}{2\pi\alpha'}\int d^2\sigma\left(\partial^a v^b\right)\partial_a X\cdot\partial_b X - \frac{1}{2}\left(\partial^a v^b\right)h_{ab}\left(\partial_c X\cdot\partial^c X\right) \\
&= \frac{1}{2\pi}\int d^2\sigma\partial^a v^b\left[\left(-\frac{1}{\alpha'}\right)\left(\partial_a X\cdot\partial_b X - \frac{1}{2}h_{ab}\partial_c\cdot\partial^c X\right)\right] \\
&= \frac{1}{2\pi}\int d^2\sigma\left(\partial^a v^b\right)T_{ab}
\end{aligned}$$