Notes on Quantum Field Theory II

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Contents

1 Path Integrals

In this section, we'll introduce the path integral in QM, look at some methods with integrals, and then explore Feynman rules. Throughout these notes, we'll leave $\hbar$, but feel free to set this to 1 throughout the course. Let's introduce the path integral from the standpoint of quantum mechanics. The goal here is to take Schrödinger's equation and reformulate it into a "path integral", which is roughly speaking, a weighted integral summing over all probable paths. Let's consider a Hamiltonian in just one dimension, which as usual we can decompose into a kinetic and potential term

$$\hat{H} = H(\hat{x}, \hat{p}) = \frac{\hat{p}^2}{2m} + V(\hat{x}), \quad \text{with } [\hat{x}, \hat{p}] = i\hbar$$

Schrödinger's equation says thst if we have a state, its time evolution is governed the equation below, which we write as its formal solution by

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \overline{H} |\psi(t)\rangle$$

Our formal solution is given by mutliplying by the time evolution operator

$$|\psi(t)\rangle = e^{-iH\frac{t}{\hbar}} |\psi(0)\rangle$$

In the Schrodinger picture, we have that

- States evolve in time
- Operators and their eigenstates are constant in time (fixed).

Wavefunctions in position space are denoted

$$\psi(x,t) = \langle x|\psi(t)\rangle$$

This gives Schrodingers equation as

$$\langle \hat{x}| \overline{H} |\psi(t)\rangle = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x,t)$$

How do we convert this differential equation into an integral equation? We have that

$$\begin{aligned}
\psi(x,t) &= \langle x| e^{-iH\frac{t}{\hbar}} |\psi(0)\rangle \\
&= \int_{-\infty}^{\infty} \langle x| e^{\frac{-iHt}{\hbar}} |x_0\rangle \langle x_0|\psi(0)\rangle \\
&= \int_{-\infty}^{\infty} dx_0 K(x, x_0; t)\psi(x_0, 0)
\end{aligned}$$

We can introduce an integral quite straightforwardly by introducing a projection operator onto initial positions. We insert a complete set of states

$$I = \int dx_0 \, |x_0\rangle \langle x_0|$$

We call $K(x, x_0; t)$ the Kernel. Repeat this procedure for n intermediate times and positions. Let $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$, and we factor

$$e^{-\frac{iHT}{\hbar}} = e^{-\frac{i}{\hbar}\overline{H}(t_{n+1}-t_n)} \ldots e^{-\frac{i}{\hbar}\overline{H}(t_1-t_0)}$$

Then,

$$K(x, x_0, T) = \int_{-\infty}^{\infty} \left[\prod_{r=1}^{n} dx_r \left\langle x_{r+1} \right| e^{-\frac{i\overline{H}}{\hbar}(t_{r+1}-t_r)} \left| x_r \right\rangle \right] \left\langle x_1 \right| e^{-\frac{i\overline{H}}{\hbar}(t_1-t_0)} \left| x_0 \right\rangle$$

Integrals are over all possible position eigenstates at times $t_r, r = 1, \ldots n$. Consider a free theory, with $V(\overline{x}) = 0$. Let's define a corresponding free kernel

$$K_0\left(x, x'; t\right) = \langle x| \exp\left(\frac{i\hat{p}^2}{2m}t\right) \left|x'\right\rangle$$

Insert, on the right side, the completeness relation for the identity.

$$I = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} \left|p\right\rangle \left\langle p\right|, \langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}}$$

This gives

$$K_0\left(x, x'; t\right) = \int_{\infty}^{\infty} \frac{dp}{2\pi\hbar} e^{-ip^2 t/2m\hbar} e^{ip(x-x')/\hbar} = e^{\frac{im(x-x')^2}{2\hbar t}} \sqrt{\frac{m}{2\pi i\hbar t}}$$

Note,

$$\lim_{t\to 0} K_0\left(x, x'; t\right) = \delta(x - x')$$

which is as expected from $\langle x|x'\rangle = \delta(x - x')$. From the Baker-Campbell-Haudorf formula, we have that

$$e^{\epsilon\hat{A}} e^{\epsilon\hat{B}} = \exp\left(\epsilon\overline{A} + \epsilon\overline{B} + \frac{\epsilon^2}{2}\left[\overline{A}, \hat{B}\right] + \ldots\right) \neq e^{\epsilon\left(\hat{A}+\hat{B}\right)}$$

For small ϵ, we have that

$$e^{\epsilon(\hat{A}+\hat{B})} = e^{\epsilon\hat{A}} e^{\epsilon\hat{B}} \left(1 + O(\epsilon^2)\right)$$

Now let $\epsilon = \frac{1}{n}$, raise the above to the n power, so we have the result that

$$e^{\hat{A}+\hat{B}} = \lim_{n\to\infty} \left(e^{\hat{A}/n} e^{\hat{B}/n}\right)^n$$

Take $t_{r+1} - t_r = \delta t$, with $\delta t \ll T$. Also take n large such that $n\delta t = T$. Then we can write that

$$e^{\frac{-i\hat{H}\delta t}{\hbar}} = \exp\left(\frac{-i\hat{p}^2 \delta t}{2m\hbar}\right) \exp\left(-\frac{iV(\hat{x})\delta t}{\hbar}\right) \left[1 + O(\delta t)^2\right]$$

Writing out the above, this gives us

$$\langle x_{r+1}| \exp\left(-\frac{i\hat{H}\delta t}{\hbar}\right) |x_r\rangle = e^{-iV(\hat{x})\delta t/\hbar} K_0(x_{r+1}, x_r; \delta t) = \sqrt{\frac{m}{2\pi i\hbar\delta t}} \exp\left[\frac{im}{2\hbar}\left(x_{r+1} - x_r/\delta t\right)^2 \delta t - \frac{i}{\hbar}V(x_r)\delta t\right]$$

with $T = n\delta t$. This gives our final expression as

$$K\left(x, x_0; T\right) = \int \left[\prod_{r=1}^{r} dx_r\right] \left(\frac{m}{2\pi i\hbar\delta t}\right)^{n+1/2} \exp\left[i \sum_{r=0}^{n} \left[\frac{m}{2}\left(\frac{x_{r+1} - x_r}{\delta t}\right)^2 - \frac{1}{\hbar}V(x_r)\right] \delta t\right]$$

In the limit $n \to \infty$, $\delta t \to 0$, with $n\delta t = T$, the exponent becomes

$$\frac{1}{\hbar}\int_0^T \left[\frac{1}{2}m\dot{X}^2 - V(x)\right] = \int_0^T dtL\left(x, \dot{x}\right)$$

where L is our classical Lagrangian. The classical action $S = \int dtL(x, \dot{x})$. The path integral

$$K\left(x, x_0; t\right) = \langle x| \, e^{-\frac{i\hat{H}t}{\hbar}} \, |x_0\rangle = \int \mathcal{D}x e^{\frac{i}{\hbar}S}$$

The functional integral $\mathcal{D}x = \lim_{\delta t \to 0, n\delta T \text{ fixed}} (...) \prod_{r=1}^{n} (...dx_r)$. we wont need to care about normalisation factors.

This has the interpretation of the particle having associated probabilities of all possible paths, and then summing these. We will also talk about analytic continuation which allows us to turn the imgainary phase into a real exponential. We analytically continue to imaginary time. Let $\tau = it$, then in terms of this imaginary time, we have

$$\langle x| \, e^{\hat{\mathbf{H}}\frac{\tau}{\hbar}} \, |x_0\rangle = \int \mathcal{D}x e^{-\frac{S}{\hbar}}$$

The $\hbar = 0$ argument is more clear. Here we see the connection to statistical physics, where $e^{-\frac{S}{\hbar}}$ plays the role of the Boltzmann factor $e^{-\beta H}$. In this case, integrals are more clearly convergent in this framework.

Quantum mechanics is just quantum field theory in $0+1$ dimensions, where $\hat{\mathbf{x}}(t)$ is a field and t is a variable. To develop a field theory, we need to be consistent with Lorentz invariance, and therefore, t and x must be on the same footing. In QFT, we solve that problem by demoting x to be just another label, a variable. For example, $\phi(x, t)$. String theory offers another choice, where we demote things in a different way.

What we next want to do is devewlop the methods we want to use in analysing theories. We will work perturbatively and use Feynman diagrams. In order to simplify things a bit, we will first work in zero dimensions. We want to look at the integrals themselves and not worry about position or momentum.

2 Integrals and their Diagrams

The first thing that we'll look at are correlation functions. In quantum mechanics, time is our only variable and we look at the evolution of a wavefunction. When we demote x, we now want to look at the behaviour in spacetime, and see how a field in one place affects the field in another place. We will see how they are connected to correlation functions.

For simplicity, consider a zero dimensional field ϕ which is just a real valued variable $\mathbb{R}$. What we want to do is look at the partition function as if we are in imaginary time. Let

$$\mathcal{Z} = \int_{\mathbb{R}} d\phi e^{-S(\phi)/\hbar}$$

We will add some assumptions for this. We assume that $S(\phi)$ is a polynomial, which is even, and we want it to be well behaved as $S(\phi) \to \infty$, as $\phi \to \pm\infty$. What we are concerned with are our expectation values, where

$$\langle f \rangle = \frac{1}{\mathcal{Z}} \int d\phi f(\phi) e^{-S/\hbar}$$

Again, we assume that f is well behaved and does not grow too fast as $\phi \to \infty$. Usually, f is a polynomial in ϕ. So we've set the generic notation. Let's start with the simplest case which we call the free theory.

2.1 Free Theory

For the time being, let's for the time being think about having N scalar fields (variables) instead of just one. We label these $\phi_a, a = 1, \ldots N$. The action will be denoted as

$$S_0(\phi) = \frac{1}{2} M_{ab} \phi_a \phi_b = \frac{1}{2} \phi^T M \phi$$

we want M to be symmetric, $N \times N$ and positive definite, so that $\det M > 0$. So, as we go to a large number of dimensions, this is the kind of term which has both the kinetic term as well as the mass term. We can currently think of these labels as just being flavour labels. We'll generalise this when we go to higher dimensions. Here, we can just diagonalise. Looking at the partition function,

$$M = P \Lambda P^T$$

where Λ is diagonal and P is orthogonal. Lets also do a field redefinition where $\chi = P^T \phi$. Then, we get that the free partition function

$$\mathcal{Z}_0 = \int d^N \phi \, \exp\left(-\frac{1}{\hbar} \phi^T M \phi \right) = \int d^N \chi \exp\left(-\frac{1}{\hbar} \chi^T \Lambda \chi \right)$$

We can write this as the product of independent integrals.

$$\cdots = \int_{c=1}^{N} \int d\chi_c e^{-\lambda_c \chi^2 / 2\hbar} = \sqrt{(2\pi\hbar)^N / \det M}$$

This is a very useful result. When we have to introduce anti-commuting numbers, we will see something similar.

Let's introduce another concept which is useful, another trick from statistical physics. We want to get correlations out of partition functions. The way to do that is to introduce external sources, which we call J, an N component external force. In this case, we map

$$S_0(\phi) \to S_0(\phi) + J^T \phi$$

Now, we extend the definition of the partition function which we call the generating function.

$$Z_0(J) = \int d^4 \phi \exp\left[-\frac{1}{2\hbar} \phi^T M \phi - \frac{1}{\hbar} J^T \phi \right]$$

we now have to complete the square, and set $\tilde{\phi} = \phi + M^{-1}J$. This allows us to rewrite the generating function as

$$Z_0\left(J\right) = \mathcal{Z}_0\left(0\right)\exp\left(\frac{1}{2\hbar}J^T M^{-1} J\right)$$

We see here that this is our free theory multiplied by the sources coupled to the matrix that appears in the action. This is something that we call a 'generating function', and it will allow us to calculate correlation functions from differentiating with respect to J.

$$\langle\phi_a\phi_b\rangle = \frac{1}{\mathcal{Z}_0\left(0\right)}\int d^N\phi\,\phi_a\phi_b\exp\left(-\frac{1}{2\hbar}\phi^T M\phi - \frac{1}{\hbar}J^T\phi\right)|_{J=0}$$

we can get the ϕ in the integrand by differentiating the exponential with respect to J. So,

$$\begin{aligned}\langle\phi_a\phi_b\rangle &= \frac{1}{\mathcal{Z}_0}\int d^N\phi\left(-\hbar\frac{\partial}{\partial J_a}\right)\left(-\hbar\frac{\partial}{\partial J_b}\right)\exp\left(\dots\right)|_{J=0} \\ &= \frac{1}{\mathcal{Z}_0\left(0\right)}\left(-\hbar\frac{\partial}{\partial J_a}\right)\left(-\hbar\frac{\partial}{\partial J_b}\right)\mathcal{Z}_0\left(J\right)|_{J=0} \\ &= \hbar\left(M^{-1}\right)_{ab} \\ &= \text{(diagram of two nodes connected by a line, called free propagator)}\end{aligned}$$

This is a pairing of the two fields which are given by the indices. We can extend this to see how this works more generally. Let's invent some notation which allows us to be a little more general. Let $l(\phi)$ be a linear combination of $\phi_a, a = 1, \dots, N$. All these expectation values are linear so we can do this. So we write

$$l\left(\phi\right) = \int_{a=1}^{N} l_a\phi_a, \quad l_a \in \mathbb{R}$$

Then, the steps above are equivalent to swapping

$$l\left(\phi\right)\text{ for } l\left(-\hbar\frac{\partial}{\partial J}\right) = -\hbar\sum_{a=1}^{N} l_a\frac{\partial}{\partial J_a}$$

Our correlation function is thus

$$\left\langle l^{(1)}\left(\phi\right)\dots l^{(p)}\left(\phi\right)\right\rangle = \frac{1}{\mathcal{Z}_0}\int d^N\phi\prod_{i=1}^{p} l^{(i)}\left(\phi\right)e^{-\frac{1}{2\hbar}\phi^T M\phi - \frac{1}{\hbar}J^T\phi}|_{J==0}$$

Moving the functions of ϕ to functions of derivatives, we find that this is equal to

$$\left\langle\lambda^{(1)}\left(\phi\right)\dots l^{(p)}\left(\phi\right)\right\rangle = (-\hbar)^p\prod_{i=1}^{p} l^{(i)}\left(\frac{\partial}{\partial J}\right)\exp\left(\frac{1}{2\hbar}J^T M^{-1}J\right)|_{J=0}$$

Now, if p is odd the answer is zero, then the integrand is odd in some ϕ_a and the integral over $\phi_a \in (-\infty, \infty)$ vanishes. For $p = 2k$, the terms which are non-zero has $J \to 0$: half the derivatives to bring down components of $M^{-1}J$ and half to remove J dependence from the prefactor. This establishes that we get exactly k factors of M^{-1}. Let's look at the four point function

$$\langle\phi_b\phi_c\phi_d\phi_f\rangle = \hbar^2\left[\left(M^{-1}\right)_{bc}\left(M^{-1}\right)_{df} + \left(M^{-1}\right)_{bd}\left(M^{-1}\right)_{cf} + \left(M^{-1}\right)_{bf}\left(M^{-1}\right)_{cd}\right]$$

In terms of Feynman diagrams, we've just got various propagators here. In terms of connecting the ϕs, we have different components. We can represent this as connecting different lines. (Insert diagrams of lines here). The number of terms is the number of ways of forming pairs, which os

$$\frac{(2k)!}{2^k k!} = \text{ number ways of permutating points /(permute inside pair } \times \text{ permute pairs)}$$

If we have a complex matrix, ϕ_a complex and M hermitian, then $\langle \phi_a \phi_b^* \rangle = \hbar (M-1)^{ab}$ is represented by a line with an arrow from a to b.

2.2 Interacting Theory

We want to go beyond the free theory and add higher power terms of ϕ. The way to do this is to expand about $\hbar = 0$, which is the classical result. We will be expanding about the minimum of the action. On the other hand, we will not be as satisfactory as one may imagine, because the expansion may not even be convergent.

Lets look at integrals like

$$\int d\phi f(\phi) e^{-S/\hbar}$$

which do not have a Taylor expansion about $\hbar = 0$. The proof is by Dyson. If we did have a Taylor expansion about $\hbar = 0$ which existed for $\hbar > 0$, then in the complex $\hbar$ plane, there must be some finite radius of convergence. If there's some finite radius of convergence in the complex $\hbar$ plane. then the expansion needs to exist for some negative real values of $\hbar$. But this is manifestly not true.

For $S(\phi)$ to have a minimum, the integral is divergent if $\mathrm{Re}(\hbar) < 0$. Therefore, the radius of convergence cannot be greater than zero. So we're not going to have convergent expansions here. (Insert diagram of complex $\hbar$ plane with small circle at origin).

So, our $\hbar$ - expansion is at best asymptotic.

$$I(\hbar) \sim \sum_{n=0}^{\infty} c_n \hbar^n$$

where $\sim$ means asymptotic to. This means that

$$\lim_{\hbar \to 0} \frac{1}{\hbar^N} |I(\hbar) - \sum_{n=0}^{N} c_n \hbar^n| = 0$$

where the limit is taken with N fixed. As we take $\hbar$ to zero, the series is arbitrarily close to zero.

It is important to note that the series misses out a transcendental terms like $e^{-\frac{1}{\hbar^2}} \sim 0$. But, $e^{-\frac{1}{\hbar^2}}$ for finite $\hbar$. These are what we call 'non-perturbative contributions'. Let's get back to the theory we're interested in. Take our action to be

$$S(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4$$

where we set $S_0(\phi)$ to be the first term, and $S_1(\phi)$ to be the second term. Here, we also assume that $m^2 > 0$ and $\lambda > 0$.

We can expand about the minimum of $S(\phi)$, where $\phi = 0$. We expand about the saddle point, so that

$$\begin{aligned} \mathcal{Z} &= \int d\phi e^{-S/\hbar} \\ &= \int d\phi e^{-S_0/\hbar} \sum_{v=0}^{\infty} \frac{1}{V!} \left(-\frac{\lambda}{4!\hbar} \right)^v \phi^{4v} \end{aligned}$$

what we'll do is that we'll truncate the series so that we miss out transcendental terms. In order to make progress, we need to truncate the series and swap summation and integration. This misses out transcendental terms like what we had before.

In the end what we have, is a series that

$$\mathcal{Z} \sim \frac{\sqrt{2\hbar}}{m} \sum_{v=0}^{N} \frac{1}{v!} \left(-\frac{\hbar\lambda}{4!m^4} \right)^v 2^{2v} \int_0^\infty dx\, e^{-x} x^{2v+\frac{1}{2}-1}$$

where $x = \frac{1}{2\hbar} m^2 \phi^2$. The integrand is just a gamma function, where

$$\int_0^\infty dx\, e^{-x} x^{2v+\frac{1}{2}-1} = \Gamma\left(2v + \frac{1}{2}\right) = \frac{(4v)!\sqrt{\pi}}{4^{2v}(2v)!}$$

In closed form, we have

$$Z \sim \frac{\sqrt{2\pi\hbar}}{m} \sum_{v=0}^{N} \left(-\frac{\hbar\lambda}{m} \right)^v \frac{1}{(4!)^v v!} \frac{(4v)!}{}$$

From String's approximation, we have that $v! \sim e^{v \log v}$, then the factors which are multiplied together are approximately $v!$. We have that factorial growth : asymptotic series. The first term in the product comes from the Taylor expansion of $e^{-S_1/\hbar}$. The second term in the product comes from pairing the $4v$ fields of the v copies of ϕ^4.

We will now follow the diagrammatic method. If we write the action including a source term

$$\begin{aligned} \mathcal{Z}(J) &= \int d\phi \exp\left\{ -\frac{1}{\hbar} \left(S_0(\phi) + S_1(\phi) + J\phi \right) \right\} \\ &= \exp\left[-\frac{1}{\hbar} S_1 \left(-\hbar \frac{\partial}{\partial J} \right) \right] \int d\phi \exp\left\{ -\frac{1}{\hbar} \left(S_0 + J\phi \right) \right\} \\ &\propto \exp\left[-\frac{\lambda}{4!\hbar} \left(\hbar \frac{\partial}{\partial J} \right)^4 \right] \exp\left(\frac{1}{2\hbar} J^T M^{-1} J \right) \\ &\sim \sum_{v=0}^{N} \frac{1}{v!} \left[-\frac{\lambda}{4!\hbar} \left(\hbar \frac{\partial}{\partial J} \right)^4 \right]^V \sum_{p=0} \frac{1}{p!} \left(\frac{1}{2\hbar} J m^{-2} J \right)^P \end{aligned}$$

This has the associated diagrams below. We should check $\mathcal{Z}(0)$. For a term to be non-zero when $J = 0$, we require that the number of derivative is equal to the number of propagators. This means we require

$$E = 2P - 4V = 0$$

where E is the number of sources left undifferentiated. Our first non-trivial terms include $(V,P) = (1,2) = (2,4)$. For $\mathcal{Z}(0)$, our first terms are ...

We can count the number of times each diagram appears. Consider the figure 8 graph, the 'pre-diagram', it has one vertex with four free vertices, and $p = 2$ propagators. There are four factorial ways of matching derivatives to sources. $A = 4!$. The denominator od the $\mathcal{Z}(J)$ expansion is just, reading off, is

$$F = (V!)\,(4!)^v\,(p!)\,2^p = 4! \cdot 2 \cdot 2$$

Thus, the figure of 8 comes with a pre factor of $\frac{A}{F} = \frac{1}{8}$ multiplied by $-\frac{\hbar\lambda}{m^4}$. More generally, F accounts for permutations of

- all vertices $v!$
- each vertex legs $4!$
- All propagators $p!$
- both ends of each propagator 2

Symmetry of particular graph is important. For example, take the figure of eight diagram. Take the pairing $(1a, 2a', 3b, 4b')$. Consider swapping $a \iff a'$ and $1 \iff 2$, gives exactly the same graph. So

$$\frac{A}{F} = \frac{1}{\mathcal{S}}, \quad \mathcal{S} \text{ is the symmetry factor}$$

$\mathcal{S}$ is the number of ways of redrawing unlabelled graph, leaving it unchanged. For example, for the figure of eight graph, we can swap the direction fo the upper and lower loops, and swap the upper and lower loops.

Looking at the basket ball diagram, we have 4! for each of the four lines attaching the two vertices, and swapping the vertices. (Insert pre-diagram drawing here)

$$\frac{Z(0)}{Z_0(0)} = 1 - \frac{\hbar\lambda}{8m^4} + \frac{\hbar^2\lambda^2}{m^8}\left(\frac{1}{48} + \frac{1}{16} + \frac{1}{128}\right)$$

From last time we have that

$$\mathcal{Z}(J) \sim \sum_{v=0}^{N} \frac{1}{v!}\left[-\frac{\lambda}{4!\hbar}\left(\hbar\frac{\partial}{\partial J}\right)^4\right]^v \sum_{p=0} \frac{1}{p!}\left[\frac{1}{2\hbar}\frac{J^2}{m^2}\right]^p$$

If we focus on the case with $E = 2$, we have that

$$Z(J) \subset \text{all diagrams with two external points}$$

We can factor out the vacuum bubble diagrams so that

$$Z(J) = [\text{No vacuum bubbles}]\,[Z(0) \text{ vacuum bubbles }]$$

Our expectation values are hence

$$\begin{aligned} \langle \phi^2 \rangle &= \frac{(-\hbar)^2}{\mathcal{Z}(0)}\left(\frac{\partial}{\partial J}\right)^2 \mathcal{Z}(J)\,|_{J=0} \\ &= [\text{ connected diagrams}] \end{aligned}$$

In terms of our symmetry factors, from $Z(J)$ the $E = 2$, $V = 0$, $P = 1$ term gives a contribution of

$$= \frac{1}{2\hbar}\frac{J^2}{m^2}, \quad F = 2, A = 1, \frac{A}{F} = \frac{1}{2} = \frac{1}{\mathcal{S}}$$

So the first order contribution is $\left\langle \phi^2 \right\rangle = \frac{\hbar}{m^2} =$ line diagram., $\left\langle \phi^{2n} \right\rangle$ proceeds similarly, but note there are disconnected diagrams. (Insert diagram here).

2.3 Effective actions

We will now define **effective actions**, which are ways of simplifying calculations by manipulating our partition function, and then cutting out different kinds of graphs we can have. The first kind of action we'll explore is the Wilsonian effective action, which allows us to consider connected diagrams only.

Definition. Wilsonian Effective Action The Wilsonian effective action is defined as

$$Z(J) = e^{-\mathcal{W}(J)/\hbar}, \quad \mathcal{W}(J) = -\hbar \log Z(J)$$

This action is useful since we can calculate it by summing just over connected diagrams, which we will prove. For example, we know that in the absence of a source, we have

$$\begin{aligned} \frac{\mathcal{Z}(0)}{\mathcal{Z}_0(0)} &= \sum \text{ all diagrams with no external points (vacuum diagrams)} \\ &= \exp\left(\sum \text{ connected vacuum diagrams }\right) \end{aligned}$$

But the above fact tells us that this is indeed the exponential of connected diagrams. Since $\frac{Z}{Z_0} = e^{-(\mathcal{W}-\mathcal{W}_0)\ \hbar}$, we then have that

$$-\mathcal{W} + \mathcal{W}_0 = \hbar \sum \text{connected diagrams}$$

In this case, we have that $\mathcal{W}_0$ is a constant and we can ignore it.

Theorem. The Wilsonian effective action is proportional to sum of connected diagrams.

Proof. First observe that we can break down any diagram into it's connected components. We write a given diagram D as

$$D = \frac{1}{S_D}\prod_I (C_I)^{n_I}$$

where C_I are connected diagrams indexed by the set I, whose own symmetry factors are already absorbed into the definition of C_I. We denote n_I as the number of C_I diagrams in D. S_D is the symmetry factor of the diagram, and since we can swap each of the n_I C_I diagrams,

$$S_D = \prod_I (n_I)!$$

Now we it's a matter of just rearranging the sums to get what we want

$$\begin{aligned}
\frac{\mathcal{Z}}{\mathcal{Z}_0} &= \sum_{n_I} D \\
&= \sum_{\{n_I\}} \prod_I \frac{1}{n_I!} (C_I)^{n_I} \\
&= \prod_I \sum_{n_I} \frac{1}{n_I!} (C_I)^{n_I} \\
&= \exp\left(\sum_I c_I \right) \\
&= \exp(\text{ sum of unique connected diagrams }) \\
&= e^{-(\mathcal{W}-\mathcal{W}_0)/\hbar}
\end{aligned}$$

so that we have $\mathcal{W} = \mathcal{W}_0 - \hbar \sum_I c_I$. $\mathcal{W}(J)$ is the generating functional for connected correlation functions.

$\square$

This means that we have

$$\begin{aligned}
-\frac{1}{\hbar} W(J) &= \log Z(J) \\
-\frac{1}{\hbar} \frac{\partial^2}{\partial J^2} W \mid_{J=0} &= \frac{1}{\mathcal{Z}(0)} \frac{\partial^2 \mathcal{Z}}{\partial J^2} \mid_{J=0} - \frac{1}{(\mathcal{Z}(0))^2} \left(\frac{\partial \mathcal{Z}}{\partial J} \right)^2 \mid_{J=0} \\
&= \frac{1}{\hbar^2} \left[\left\langle \phi^2 \right\rangle - \left\langle \phi \right\rangle^2 \right] \\
&= \frac{1}{\hbar^2} \left\langle \phi^2 \right\rangle_{\text{connected}}
\end{aligned}$$

Less trivially, we may encounter theories where the expecation of ϕ is non zero, but we're not discussing that. Less trivially,

$$-\frac{1}{\hbar} \frac{\partial^4 \mathcal{W}}{\partial J^4} \mid_{J=0} = \frac{1}{\mathcal{Z}(0)} \frac{\partial^4 \mathcal{Z}}{\partial J^4} \mid_{J=0} - \left(\frac{1}{\mathcal{Z}(0)} \frac{\partial^2 \mathcal{Z}}{\partial J^2} \right) \mid_{J=0}$$

This implies that

$$\left\langle \phi^4 \right\rangle_{\text{connected}} = \left\langle \phi^4 \right\rangle - \left\langle \phi^2 \right\rangle^2$$

Let's consider an action with two real fields.

$$S(\phi, \chi) = \frac{m^2}{2} \phi^2 + \frac{M^2}{2} \chi^2 + \frac{\lambda}{4} \phi^2 \chi^2$$

Note that we don't have a factorial here. With two fields, we now have two sets of Feynman rules. We can look at the Wilson effective action by looking at the connected vacuum diagrams.

$$-\mathcal{W}/\hbar = \text{diagram of connected vacuum diagrams}$$

Counting the symmetry factors of this diagram are

$$-\frac{\mathcal{W}}{\hbar}=-\frac{\hbar\lambda}{4m^2M^2}+\frac{\hbar^2\lambda^2}{m^4M^4}\left[\frac{1}{16}+\frac{1}{16}+\frac{1}{8}\right]$$

Also, from Feynman diagrams, the

$$\left\langle\phi^2\right\rangle=(\text{connected diagrams with external lines})$$

Again, counting the symmetry factors, we get that

$$\left\langle\phi^2\right\rangle=\frac{\hbar}{m^2}-\frac{\hbar^2\lambda}{2m^4M^2}+\frac{\hbar^3\lambda^2}{m^6M^4}\left[\frac{1}{4}+\frac{1}{2}+\frac{1}{4}\right]$$

say we don't care about χ explicitly, maybe because we don't know that much about χ, we may want to integrate it out. This may be because $M\gg m$, never produced on experimental scales. We define $\mathcal{W}(\phi)$, to give

$$e^{-\mathcal{W}(\phi)/\hbar}=\int d\chi e^{-S(\phi,\chi)/\hbar}$$

Thus, $\phi^2\chi^2$ is treated as a source term with $J=\phi^2$ in earlier notation. We're using our low energy particles, bashing them together, and using them as a source.

We want to look at correlation functions only involving ψ fields.

$$\langle f(\phi)\rangle=\frac{1}{2}\int d\phi d\chi f(\phi)\, e^{-S(\phi,\chi)/\hbar}=\frac{1}{\mathcal{Z}}\int d\phi f(\phi)\, e^{-W(\phi)/\hbar}$$

In this simple example, we have that

$$\int d\chi e^{-S(\phi,\chi)/\hbar}=e^{-m^2\phi^2/2\hbar}\sqrt{\frac{2\pi\hbar}{M^2+(\lambda\phi^2)/2}}$$

This implies that, solving for $W(\phi)$, we have

$$W(\phi)=\frac{1}{2}m^2\phi^2+\frac{\hbar}{2}\log\left(1+\frac{\lambda}{2M^2}\phi^2\right)+\frac{\hbar}{2}\log\frac{M^2}{2\pi\hbar}$$

The final term is a constant. One way to think about this is in the context of the cosmological constant problem. This cancels out in expectation values in QFTs. Let's expand the logarithm. We've been explicit in the inclusion of $\hbar$, so expansions in the logarithm term are quantum effects. This gives us

$$W(\phi)=\left(m^{\frac{2}{2}}+\frac{\hbar\lambda}{4M^2}\right)\phi^2-\frac{\hbar\lambda^2}{16M^4}\phi^4+\frac{\hbar\lambda^3}{48M^6}\phi^6+\ldots$$

One can think of these as an effective mass term. So, we have that

$$W(\phi)=\frac{m_{\text{eff}}^2}{2}\phi^2+\frac{\lambda^4}{4!}\phi^4+\frac{\lambda^6}{6!}\phi^6+\ldots\frac{\lambda_{2k}}{(2k)!}\phi^{2k}+\ldots$$

where we have defined $m^2_{\text{eff}}=m^2+\frac{\hbar\lambda}{2M}$, and we define

$$\lambda_{2k}=(-1)^{k+1}\hbar\frac{(2k)!}{2^{k+1}k}\frac{\lambda^k}{M^{2k}}$$

In dim > 0 , we usually need to calculate $W(\phi)$ perturbatively. From the action $S(\phi, \chi)$, and the path integral over χ, we have the Feynman rules.

The dotted lines come from the ϕ field. Putting the integrals together, we have that this makes it equal to

$$W(\phi) = S(\phi) + \frac{1}{2}\frac{h\lambda}{2M^2}\phi^2 - \frac{1}{4}\frac{\hbar\lambda^2}{4M^4}\phi^4 + \frac{1}{3!}\frac{\hbar\lambda^3}{8M^6}\phi^6 + \dots$$

Using the effective action, we can also calculate the correlation function

$$\begin{aligned}\langle\phi^2\rangle &= \frac{1}{\mathcal{Z}}\int d\phi\phi^2 e^{-W(\phi)/\hbar} \\ &= \frac{\hbar}{m_{\text{eff}}^2} - \frac{\lambda_4\hbar^2}{2m_{\text{eff}}^6}\end{aligned}$$

2.4 Quantum Effective Action

We represent the quantum effective action by Γ. We want to define the average field in the presence of an external source.

$$\Phi := \frac{\partial W}{\partial J} = \langle\phi\rangle_J = -\frac{\hbar}{Z(J)}\frac{\partial}{\partial J}\int d\phi e^{-(S+J\phi)/\hbar}, S(\phi) \text{ same as before}$$

We define the Legendre transformation from $W(J) \to \Gamma(\Phi)$. This is

$$\Gamma(\Phi) = W(J) - \Phi J$$

Note that

$$\frac{\partial\Gamma}{\partial\Phi} = \frac{\partial W}{\partial\Phi} - J - \Phi\frac{\partial J}{\partial\Phi} = \frac{\partial W}{\partial J}\frac{\partial J}{\partial\Phi} - J - \Phi\frac{\partial J}{\partial\Phi}$$

This means that

$$\frac{\partial J}{\partial\Phi} = -J$$

so J is the minimum of the quantum effective action. If $J = 0$, then

$$\left.\frac{\partial\Gamma}{\partial\Phi}\right|_{J=0} = 0$$

So, $J \to 0$ corresponds to an extremum of $\Gamma(\Phi)$.

In higher dimensions, one performs a derivative expansion

$$\Gamma(\Phi) = \int d^dx\left[-V(\Phi) - \frac{1}{2}\partial^\mu\Phi\partial_\mu\Phi + \dots\right]$$

where $V(\Phi)$ is the effective potential. There is some analogy here with statistical mechanics. h is a magnetic field.

$$e^{-\beta F(h)} = \int \mathcal{D}s\exp(-\beta\mathcal{H})$$

The magentization $M = -\frac{\partial F}{\partial h}$. The Gibbs free energy is

$$G(M) = F(h) + Mh$$

as $h \to 0$, M of the system is the minimum of G.

We do a perturbative calculation of ,$\Gamma(\Phi)$ We write

$$e^{-W_\Gamma(J)/g} = \int d\Phi e^{-(\Gamma(\Phi)+J\Phi)/g}$$

We define a new Planck fictitious constant, and include the source term g. We know that $W_\Gamma(J)$ is the sum of connected vacuum diagrams.

$$W_\Gamma(J) = \int_{l=0}^{\infty} g^l W_\Gamma^{(l)}(J)$$

We know that $W_\Gamma^{(0)}$ is composed of tree diagrams. In the $g \to 0$ limit, we have that $W_\Gamma(J) = W_\Gamma^{(0)}(J)$. Also as $g \to 0$, integral over Φ is dominated by the minimum of the exponent, in other words the Φ such that

$$\frac{\partial \Gamma}{\partial \Phi} = -J$$

But then

$$W_\Gamma^{(0)}(J) = \Gamma(\Phi) + J\Phi = W(J)$$

where the $W(J)$ is from earlier with action $S(\phi) + J\phi$. The moral of the story is that the sum of the connected diagrams in theory with action $S(\phi) + J\phi$ which is $W(J)$, can be constructed from the sum of tree diagrams with action $\Gamma(\Phi) + J\Phi$.

We make the definition that an internal line in a graph is a bridge if cutting it would make a graph disconnected. A connected graph is called a one particle irreducible (1PI) if it has no bridges.

The irreducible parts are loops. Buried within it, Γ sums up the loop diagrams (1PI).

2.5 Fermions and Grassman Variables

In zero dimensions, we don't have a concept of spin, since we don't even have a way to orientate things correctly. Thus, the best we can do is to construct a set of N variables, fermion fields, which anti-commute.

We call these fields $\theta^a, a = 1, \dots N$. This has the characteristic property that

$$\theta_a \theta_b = -\theta_b \theta_a, \quad a = 1, \dots N$$

This in particular implies that for a given field θ^a, we have

$$\theta^a \theta^a = 0$$

They also have the property that they commute with scalar fields, so we have that

$$\phi_a \psi_b = \psi_b \phi_a, \quad a = 1, \dots N$$

With this data, we can define functions of these variables as a finite expansion.

$$F(\theta) = f + \rho_a \theta^a + \frac{1}{2!} g_{ab} \theta^a \theta^b + \frac{1}{n!} \dots h_{ab \dots n} \theta^a \theta^b \dots \theta^n$$

In these expansions, we have that each of the tensors are totally anti-symmetric. Suppose that we only had one field in the case that $a = 1$ only. Then, the most general function we could write with this is

$$F(\theta) = f + \rho\theta$$

since any terms of higher order go to zero by antisymmetry.

We define differentiation and integration on these variables as follows. For differentiation, we have that

$$\frac{\partial}{\partial\theta^a}\theta^b + \theta^b\frac{\partial}{\partial\theta^a} = \delta^b{}_a$$

We also have the integration rules.

The integrals should be invariant under translation, so

$$\int d\theta\,(\theta + \eta) = \int d\theta\theta$$

If we're dealing with a single variable, we can then integrate by parts, since clearly

$$\int d\theta\frac{\partial}{\partial\theta}F(\theta) = 0$$

We can extend this to integration rules for n variables. Namely, we only have a non-vanishing integral when our integrand is a product of one power of each Grassman number.

$$\int d^n\theta^1\ldots\theta^n = 1$$

By antisymmetry, we have that for a given ordering of the Grassman variables, we have that

$$\int d^n\theta\,\theta^{a_1}\ldots\theta^{a_n} = \epsilon^{a_1\ldots a_n}$$

We can compute the Jacobian of this measure as follows. Suppose that we relate a set of new Grassman variables $\theta'^a = N^a{}_b\theta^b$, where $N \in GL(2,\mathbb{C})$. This means that we have, integrating over the variables,

$$\begin{aligned}\int d^n\theta\,\theta'^{a_1}\theta'^{a_2}\ldots\theta'^{a_n} &= N^{a_1}{}_{b_1}N^{a_2}{}_{b_2}\ldots N^{a_n}{}_{b_n}\int d^n\theta^{b_1}\ldots\theta^{b_n} \\ &= N^{a_1}{}_{b_1}\ldots N^{a_n}{}_{b_n}\,\epsilon^{b_1\ldots b_n} \\ &= \det N\epsilon^{a_1\ldots a_n} \\ &= \det N\int d^n\theta'\theta'^{a_1}\ldots\theta'^{a_n}\end{aligned}$$

This implies that $d^n\theta = \det N d^n\theta'$.

2.5.1 Fermionic Free Field Theory

If we want to build a bosonic theory out of fermions, we need to include an even number of fermions. In full generality, this means our action has to take the form

$$S(\theta) = \frac{1}{2} A_{ab} \theta^a \theta^b$$

We have that this is

$$\begin{aligned}
\mathcal{Z}_0 &= \int d^{2m}\theta e^{-S(\theta)} \\
&= \int d^{2m}\theta e^{-\frac{1}{2}A(\theta,\theta)} \\
&= \int d^{2m}\theta \sum_{n=0}^{2m} \frac{(-1)^n}{(2\hbar)^n n!} \left(A_{ab}\theta^a\theta^b\right)^n
\end{aligned}$$

Notice however, that when we perform Berezin integration, only terms which have a single power of each variable don't vanish. Hence, the only term which doesn't vanish is when $n = 2m$,

$$\begin{aligned}
\mathcal{Z}_0 &= \int d^2 m\theta \frac{(-1)^n}{(2\hbar)^n n!} A_{a_1 a_2} \ldots A_{a_{2m-1} a_{2m}} \theta^{a_1}\theta^{a_2} \ldots \theta^{a_{2m}} \\
&= \frac{(-1)^n}{(2\hbar)^n n!} A_{a_1 a_2} \ldots A_{a_{2m-1} a_{2m}} \epsilon^{a_1 \ldots a_{2m}}
\end{aligned}$$

3 LSZ Reduction Formula

We're going to do an illustrative example here, so we can get some intuition going on in terms of a free theory. The main result we'll be exploring today is scattering amplitudes in terms of correlation functions. For example, let's look at $2 \to 2$ scattering of scalar particles. Recall from the previous set of notes from quantum field theory, that our scattering amplitude can be written in the form $\langle f| S |i\rangle$. Now, it is in our interest to try and find out what this is in terms of our correlation functions of our fields $\phi(x)$. Recall that a correlation function looks like $\langle 0| \mathcal{T}\phi(x_1)\phi(x_2) |0\rangle$. This is what our LSZ reduction formula is.

Our motivation for proceeding is as follows. Since initial and final states $|i\rangle, |f\rangle$ are written in terms of creation operators, we will need to invert these to get expressions in ϕ.

We write our free scalar field which can be built out of plane waves.

$$\phi(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3 2E}\left[a(\mathbf{k})e^{-ik\cdot x} + a^\dagger(\mathbf{k})e^{ik\cdot x}\right]$$

where we have $k\cdot x = Et - \mathbf{k}\cdot\mathbf{x}$. We have relativistic normalisation for $a(\mathbf{k})$. Now, it's convenient to invert this expression via the inverse Fourier transform of the field and it's derivative

$$\int d^3x e^{ik\cdot x}\phi(\mathbf{x}), \quad \int d^3x e^{ik\cdot x}\partial_0\phi(x)$$

One can easily verify that the identities below hold, using the standard identities.

$$a(\mathbf{k}) = \int d^3x e^{ik\cdot x}\left[i\partial_0\phi(x) + E\phi(x)\right]$$
$$a^\dagger(\mathbf{k}) = \int d^3x e^{-ik\cdot x}\left[-i\partial_0\phi(x) + E\phi(x)\right]$$

We set our initial and final states for the free theory, to be one-particle momentum states, created by applying a creation operator to the particle vacuum.

$$|k\rangle = a^\dagger(k)|\Omega\rangle$$

where $|\Omega\rangle$ is the true vacuum, which in a weakly interacting theory is not too different from the true free vacuum. This is a key assumption which we have to make. We have that $|\Omega\rangle$ satisfies $a(k)|\Omega\rangle = 0$, for all k, and $\langle\Omega|\Omega\rangle = 1$. We have the norm

$$\langle\mathbf{k}|\mathbf{k}\rangle = (2\pi)^3(2E)\delta^3\left(\mathbf{k}-\mathbf{k}'\right), \quad E = \sqrt{\mathbf{k}^2+m^2}$$

The initial and final states we're interested in will be time moving Gaussian wavepackets which we construct from the creation operators. We introduce a Gaussian wavepacket

$$a_1^\dagger := \int d^3k f_1(\mathbf{k})a^\dagger(\mathbf{k}), \quad f_1(\mathbf{k}) \propto \exp\left[-\frac{(\mathbf{k}_1-\mathbf{k}_2)^2}{4\sigma^2}\right]$$

similarly, we define a different moving Gaussian wavepacket for $a_2^\dagger$ Now, we want to see what happens when these wavepackets collide with each other.

We can evolve these Gaussians into the distant past and future, where the overlap in coordinate space is negligible. Assume this works when including interactions.

We are going to evolve including the full Hamiltonian, so, there will be a complication that there is some time dependence. $a^\dagger(\mathbf{k})$ becomes time dependent, so we will get that $a_1^\dagger(t)$ and $a_2^\dagger(t)$ depend on time.

Assume that as $t \to \pm\infty$, $a_1^\dagger$ and $a_2^\dagger$ coincide with their free theory expansions, and that we can Fourier transform without worrying about this complication too much.

Define the initial and final states as two Gaussian wavepackets moving.

$$
\begin{aligned}
|i\rangle &= \lim_{t\to-\infty} a_1^\dagger(t)\, a_2^\dagger(t)\, |\Omega\rangle \\
|f\rangle &= \lim_{t\to\infty} a_{1'}^\dagger(t)\, a_{2'}^\dagger(t)\, |\Omega\rangle
\end{aligned}
$$

We assume that $\langle i|i\rangle = \langle f|f\rangle = 1$, and that $\mathbf{k}_1 \neq \mathbf{k}_2$. We want $\langle f|i\rangle$, the scattering amplitude. To do this, we need to use a trick. Note for example, that

$$
\begin{aligned}
a_1^\dagger(\infty) - a_1^\dagger(-\infty) &= \int_{-\infty}^{\infty} dt \partial_0 a_1^\dagger(t) \\
&= \int d^3k_1 f_1(k) \int d^4x \partial_0 \left[e^{-ik\cdot x} \left(-i\partial_0\phi + E\phi \right) \right] \\
&= -i \int d^3k_1 f_1(k) \int d^4x e^{-ik\cdot x} \left(\partial_0^2 + E^2 \right) \phi \\
&= \int \cdots \int d^4x e^{-ik\cdot x} \left(\partial_0^2 - \nabla^{\leftarrow 2} + m^2 \right) \phi^2 \\
&= -i \int d^3k f_1(x) \int d^4x e^{-ik\cdot x} \left(\partial^2 + m^2 \right) \phi
\end{aligned}
$$

This is great, because in the last line, we simply have the Klein-Gordon operator. Note in free theory, we have that our fields solve the Klein-Gordon equation, so in this case we have that the difference is zero.

$$
a_1^\dagger(\infty) - a_1^\dagger(-\infty) = 0
$$

We are however looking at the weakly interacting case, so the integrand doesn't necessarily evaluate to zero. Now we can start to calculate the scattering amplitude.

$$
\langle f|i\rangle = \langle\Omega|\, \mathcal{T} a_{1'}(\infty)\, a_{2'}(\infty)\, a_1^\dagger(-\infty)\, a_2^\dagger(-\infty)\, |\Omega\rangle
$$

use $a_j^\dagger(-\infty) = a_j^\dagger(\infty) + i\int d^3k f_j(k) \int d^4x e^{-ik\cdot x} \left(\partial^2 + m^2\right)\phi$. Similarly, we have that

$$
a_j(\infty) = a_j(-\infty) + i \int \ldots e^{ik\cdot x} \ldots
$$

Then, the only non-zero term is

$$
\begin{aligned}
\langle f|i\rangle = (i)^4 \int & d^4x_1 d^4x_2 d^4x_1' d^4x_2' e^{-k_1\cdot x} e^{-ik_2\cdot x_2} e^{ik_1'\cdot x_1'} e^{ik_2'\cdot x_2'} \\
& \times \left(\partial_1^2 + m^2\right)\left(\partial_2^2 + m^2\right)\left(\partial_{1'}^2 + m^2\right)\left(\partial_{2'}^2 + m^2\right) \\
& \times \langle\Omega|\, \mathcal{T}\phi(x_1)\,\phi(x_2)\,\phi\left(x_1'\right)\phi\left(x_2'\right) |\Omega\rangle
\end{aligned}
$$

having taken $\sigma \to 0$ such that $f(\mathbf{k}_j) \to \delta^3(\mathbf{k} - \mathbf{k}_j)$. Let's examine assumptions here. The general deviation requires only weaker assumptions.

- We need a unique Ω, such that the first excited state is a single particle.
- We want $\phi |\Omega\rangle$ to be a single particle state. In other words, we want

$$\langle \Omega | \phi | \Omega \rangle = 0$$

  If not, and $\langle \Omega | \phi | \Omega \rangle = v \neq 0$, then let $\tilde{\phi} = \phi - v$.

- We want ϕ normalised such that

$$\langle k | \phi(x) | 0 \rangle = e^{ik \cdot x}$$

  as in the free case. Usually interactions require us to rescale $\phi \to Z_\phi^{\frac{1}{2}} \phi$. We see the need to renormalise, for example,

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4$$

  From here, the coefficients may spoil the LSZ formula.

$$\mathcal{L} = \frac{Z_\phi}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} Z_m m^2 \phi^2 - \frac{\lambda}{4!} Z_\lambda \phi^4$$

Summary

3.0.1 Path Integral Derivations

- You get path integrals from repeatedly inserting the completeness relation

$$I = \int dx_0 \, |x_0\rangle \langle x_0|$$

- The kernel is

$$K(x, x_0, t) = \langle x | e^{-\frac{i\hat{H}t}{\hbar}} | x_0 \rangle$$

- Our action is defined as

$$S = \int_0^T dt L(x, \dot{x})$$

- Our measure is the two-way limit

$$\mathcal{D}x = \lim_{\delta t \to 0, n\delta \text{ fixed}} \sqrt{\frac{m}{2\pi i \hbar \delta t}} \prod_{r=1}^{n} \left(\sqrt{\frac{m}{2\pi i \hbar \delta t}} dx_r \right)$$

3.0.2 Free Partition Functions

- The free theory is defined as
$$S_0(\phi) = \frac{1}{2} M_{ab}\phi_a\phi_b$$
- The free partition function
$$\mathcal{Z}_0 = \int d^N \phi e^{-S(\phi)/\hbar}$$
- With a source term, $S_0 + J\phi$, this free partition function as a function of J is
$$\mathcal{Z}(J) = Z(0)\exp\left(\frac{1}{2\hbar} J^T M J\right)$$

3.0.3 Feynman Diagrams

- For each graph with n vertices, we add a combinatoric factor of
$$\frac{|D_n|}{|G_n|} = \sum \frac{1}{|\text{Aut } \Gamma|}$$
- There are two ways to generate diagrams. See which combinations of exponents reduce the source terms to zero, then construct the possible diagrams.

3.0.4 Effective Actions

- The Wilsonian effective action is the logarithm of the partition function
$$\mathcal{W} = -\hbar \log \mathcal{Z}$$
- The connected correlation function for n variables is the n the derivative of the Wilsonian effective action.
- With two real fields, derive the Feynman rules
$$S(\phi, \chi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{2} M^2 \xi^2 + \frac{\lambda}{4} \chi^2 \phi^2$$
The Wilsonian effective action
$$-\mathcal{W}/\hbar = \text{connected vacuum bubbles}$$
- Integrate out high energy fields with
$$e^{-\mathcal{W}(\phi)/\hbar} = \int d\chi\, e^{-S(\phi,\chi)\ \hbar}$$

3.1 Scalar Field Theory

In this section, we'll first make our integrals more convergent by Wick rotating from Minkowski space (with metric $(+,-,-,-)$) to Euclidean space, with metric $(+,+,+,+)$. We know the standard form of a Lagrangian of a scalar field. Recall, doing a Wick rotation is when we simultaneous change our time coordinate and set $t \to t' = it$and switch our metric from Minkowski to Euclidean.

Recall from the previous course in quantum field theory that our expression for our free propagator here is

$$\Delta_0(k) = \frac{i}{(k_0)^2 - |\mathbf{k}|^2 - m^2 + i\epsilon}$$

We will derive the expression for this in an alternate way shortly below. The reason why we do the Wick rotation is because the free propagator above has the addition of $+i\epsilon$ since our poles land on the real line. Thus, we need the Feynman prescription to shift the poles. This is annoying. We Wick rotate in Euclidean space to get a different propagator with imaginary poles, so we don't have any issues. Thus, if we have the Lagrangian in Minkowski space,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi), \quad V(\phi) = \frac{1}{2}m^2\phi^2 + \sum_{n>2} V^{(n)}\phi^n$$

and our associated partition function is $\mathcal{Z} = \int \mathcal{D}\phi \exp\left(i \int dt \mathcal{L}\right)$, then doing a change of variables and setting $it = \tau$ gives

$$\begin{aligned} i\int dt \mathcal{L} &= i \int dt \ (\partial_t \phi)^2 - (\nabla\phi)^2 - V(\phi) \\ &= \int d\tau \ -(\partial_\tau\phi)^2 - (\nabla\phi)^2 - V(\phi) \\ &= \int d\tau - \delta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) \\ &= -\int d\tau \, \mathcal{L}' \end{aligned}$$

where we have now redefined L' to to use the Euclidean metric, along with the shown change of sign, so that

$$L' = \delta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)$$

So, up to relabelling and switching signs, we have that a Wick rotation gives us $\mathcal{Z} = \int \mathcal{D}\phi \exp\left(-\int dt\mathcal{L}\right)$. Now we are in a good place, since our propagator can be written, as we shall see, as

$$\Delta_0(k) = \frac{1}{k^2 + m^2} = \frac{1}{(k_0)^2 + |\mathbf{k}|^2 + m^2}$$

Now our poles lie on the imaginary axis. This means that we don't have the issue of evaluating the contour integral.

We show how to calculate the free propagator as in chapter 2, where we looked at the vacuum expectation $\langle \phi_a \phi_b \rangle$, which turned out to be $\hbar\left(M^{-1}\right)_{ab}$. This was obtained by first finding the partition function with a source term, differentiating twice, where the source term is zero.

Following the same process but with infinite degrees of freedom, we consider the partition function but with a source term

$$\mathcal{Z}_0(J) = \int \mathcal{D}\phi \exp\left(-S_0(\phi, J)\right)$$

in the free case, where

$$S_0(\phi, J) = \int d^4x \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 + J\phi$$

where we're working with the Euclidean metric as we've already Wick rotated. Now, it's easier to work with this in Fourier space. Fourier transforming gives us

$$\begin{aligned}
S_0\left(\tilde{\phi}, \tilde{J}\right) &= \int d^4k\, \frac{1}{2}k^2\tilde{\phi}(k)\,\tilde{\phi}(-k) + \frac{1}{2}m^2\tilde{\phi}(k)\,\tilde{\phi}(-k) + \tilde{J}(k)\,\tilde{\phi}(-k) \\
&= \frac{1}{2}\int d^4k\left(k^2+m^2\right)\tilde{\phi}(k)\,\tilde{\phi}(-k) + \tilde{J}(k)\,\tilde{\phi}(-k) + \tilde{J}(-k)\,\tilde{\phi}(k) \\
&= \frac{1}{2}\int d^4k\,\left(k^2+m^2\right)\left(\tilde{\phi}(k) + \frac{\tilde{J}(k)}{k^2+m^2}\right)\left(\tilde{\phi}(-k) + \frac{\tilde{J}(-k)}{(k^2+m^2)}\right) - \frac{\tilde{J}(k)\,\tilde{J}(-k)}{k^2+m^2} \\
&= \frac{1}{2}\int d^4k\left(k^2+m^2\right)\tilde{\chi}(k)\,\tilde{\chi}(-k) - \frac{\tilde{J}(k)\,\tilde{J}(-k)}{k^2+m^2}
\end{aligned}$$

In the second step, we symmetrised and switched integration variables so that we could complete the square above. Now, we stick this into our path integral!

$$\mathcal{Z}_0(J) = \int \mathcal{D}\phi \exp\left(-\frac{1}{2}\int d^4k\left(k^2+m^2\right)\tilde{\chi}(k)\,\tilde{\chi}(-k)\right)\exp\left(\frac{1}{2}\frac{\tilde{J}(k)\,\tilde{J}(-k)}{k^2+m^2}\right) = \exp\left(\frac{1}{2}\frac{\tilde{J}(k)\,\tilde{J}(-k)}{k^2+m^2}\right)Z_0(0)$$

where we realise that the factor which is actually integrated over is just the partition function without a source term included. We then assume a normalisation factor where $\mathcal{Z}_0(0) = 1$. Hence, for a source term in our free theory, we have that in a Fourier expansion, our free theory partition function with a source is given by

$$Z_0\left[\tilde{j}\right] = \exp\left[\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{\tilde{j}(-k)\,\tilde{j}(k)}{k^2+m^2}\right]$$

Our Fourier space propagator is given by the second functional derivative of this function, much like our derivation before in zero dimensional quantum field theory.

$$\begin{aligned}
\Delta(q) &= \frac{\delta^2 Z_0\left[\tilde{j}\right]}{\delta\tilde{j}(q)\,\delta\tilde{j}(-q)}\,\Big|_{\tilde{j}=0} \\
&= \frac{1}{q^2+m^2}
\end{aligned}$$

As convention due to the fact that we have momentum conservation, we've stripped off the $(2\pi)^4\,\delta(0)$ which comes automatically from momentum conservation.

The position space propagator is achieved simply by Fourier transforming this back into position space. Thus, our position space propagator is

$$\Delta\left(x-x'\right)=\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot(x-x')}}{k^2+m^2}$$

Fourier transforming our partition function with source back into position space, we can write it nicely as

$$Z_0\left[j\right]=\exp\left(\frac{1}{2}\int dx_1dx_2j\left(x_1\right)\Delta\left(x_1-x_2\right)j\left(x_2\right)\right)$$

This has the nice interpretation of averaging across the propagator of all points in space, and then summing this to get the partition function.

3.1.1 Incorporating Interactions

We now incorporate interaction terms into this picture. Our Lagrangian splits up into $\mathcal{L}=\mathcal{L}_0+\mathcal{L}_{\text{int}}$

3.2 Vertex Functions

Recall that we have our Wilsonian effective action $W\left[J\right]$ which corresponds to the sum of connected diagrams, and our quantum effective action $\Gamma\left[\Phi\right]$, which corresponds to the sum of our 1PI diagrams (diagrams with no bridges). We generalise our definition of the quantum effective action to the case of scalar fields, by integrating in the Legendre transform

$$\Gamma\left[\Phi\right]=W\left[J\right]-\int d^4xJ\left(x\right)\Phi\left(x\right)$$

. As a result, doing some functional integration, this implies that

$$\frac{\delta W\left[J\right]}{\delta J\left(x\right)}=\Phi\left(x\right),\quad\frac{\delta\Gamma\left[\Phi\right]}{\delta\Phi\left(x\right)}=-J\left(x\right)$$

As we shown for three points in the example sheet, our connected n-point functions in full generality is

$$G^{(n)}\left(x_1,\ldots,x_n\right)=(-1)^{n+1}\prod_{i=1}^{n}\frac{\delta}{\delta J\left(x_i\right)}W\left[J\right]=\left\langle\phi\left(x_1\right)\ldots\phi\left(x_n\right)\right\rangle^{\text{ conn}}$$

Recall that what we mean by connected is as follows. Suppose we compute $\langle\phi_a\phi_b\rangle_J$ from differentiating our partition function twice with respect to J for example. Then, in this case, we might get some disconnected diagrams, encoded in $\langle\phi_a\rangle$. In this case, we would define

$$\langle\phi_a\phi_b\rangle^{\text{conn}}=\langle\phi_a\phi_b\rangle-\langle\phi_a\rangle_J\langle\phi_b\rangle_J$$

We also define the n-point vertex functions

$$\Gamma^{(n)}\left(x_1,\ldots x_n\right)=(-1)^n\prod_{i=1}^{n}\frac{\delta}{\delta\Phi\left(x_i\right)}\Gamma\left[\Phi\right]$$

In this case, we get the expectation value of 1PI diagrams. It's worth noting that the set of all connected diagrams are diagrams whose vertices are 1PI diagrams. There's also a duality between n-point vertex functions and connected n-point functions. For example, in the case where $n = 2$, we have that

$$G^{(2)}(x, y) = -\frac{\delta^2 W}{\delta J(x)\,\delta J(y)} = -\frac{\delta \Phi(y)}{\delta J(x)}$$

$$\Gamma^{(2)}(x, y) = \frac{\delta^2 \Gamma}{\delta \Phi(x)\,\delta \Phi(y)} = -\frac{\delta J(x)}{\delta \Phi(y)}$$

Note that the expressions are inverses of each other, so that

$$\int d^4 z G^{(2)}(x, z)\,\Gamma^{(2)}(z, y) = \delta^4(x - y)$$

This can be seen from just substituting the functional derivatives above and taking out the Dirac delta functions. From example sheet 1, we have that

$$G^{(3)}(x_1, x_2, x_3) = \int d^4 z_1 d^4 z_2 d^4 z_3 G^{(1)}(x_1, z_1)\,G^{(2)}(x_2, z_2)\,G^{(3)}(x_3, z_3)\left(-\frac{\delta^3 \Gamma}{\delta \Phi(z_1)\,\delta \Phi(z_2)\,\delta \Phi(z_3)}\right)$$

where the last term is defined as $\Gamma^{(3)}(z_1, z_2, z_3)$. Diagrammatically, we decompose the three point function into IPI and two point functions. The above expression can be inverted to give

$$\Gamma^{(3)}(y_1, y_2, y_3) = \int d^4 x_1 d^4 x_2 d^4 x_3 \Gamma^{(2)}(x_1, y_1)\,\Gamma^{(2)}(x_2, y_2)\,\Gamma^{(3)}(x_3, y_3)\,G^{(3)}(x_1, x_2, x_3)$$

which we can compare to the LSZ reduction formula. These arguments carry forward to general n and in momentum space. This is discussed in Ryder, section 7.3.

4 Regularisation and Renormalisation

Consider ϕ^4 theory in Euclidean space time.

$$S[\phi] = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!}\phi^4$$

Let's look at the full propagator $\tilde{G}^2(p) = \int d^4 x e^{-ip\cdot x} \langle \phi(x)\,\phi(0)\rangle^{\text{conn}}$. Even without perturbation theory, we can still write this as a geometric series. This is given by

$$\begin{aligned}\tilde{G}^{(n)}(p) &= \frac{1}{p^2 + m^2} + \frac{1}{p^2 + m^2}\Pi\left(p^2\right)\frac{1}{p^2 + m^2} \\ &= \frac{1}{p^2 + m^2 - \pi\left(p^2\right)}\end{aligned}$$

where $\Pi\left(p^2\right) = \tilde{\Gamma}^{(2)}(p)$. Our contribution from the first loop integral with amputations is

$$= \frac{-\lambda}{2}\int \frac{d^4 k}{(2\pi)^4}\frac{1}{k^2 + m^2}$$

When doing these things, since it depends just on radial coordinates, we have that $d^4k = k^{4-1}dkd\Omega$. In general we have that $d^dk = S_d|k|^{d-1}d|k|$, with $S_d = \frac{2\lambda^{d/2}}{\Gamma(\frac{d}{2})}$. This integral diverges. To examine this integral and it's divergence, we set

$$\begin{aligned} I &= -\frac{\lambda}{2}\int^{\Lambda} \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2} \\ &= -\frac{\lambda S_4}{4\,(2\pi)^4}\int^{\Lambda/m^2} \frac{udu}{1+u}, \quad u = \frac{k^2}{m^2} \\ &= -\frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2\log\left(1+\frac{\Lambda^2}{m^2}\right)\right] \end{aligned}$$

this is divergent as $\Lambda \to \infty$, which is what we call UV divergent. Let's look at the four point function at one loop. We sum over these diagrams, which give a contribution of

$$I = \int^{\Lambda} \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\sum_{P\in\{p_1+p_2,p_1+p_3,p_1+p_4\}}\frac{1}{(P+k)^2+m^2} = \tilde{\Gamma}^{(4)}\left(p_1,p_2,p_3,p_4\right)$$

as $k \to \infty$, we have out leading contribution as $\frac{d^4k}{k^4}$, expect $\log(\Lambda/m)$. We evaluate integral with zero external momenta. This gives

$$\begin{aligned} \tilde{\Gamma}^{(4)}\left(0,0,0,0\right) &= \frac{3\lambda^2}{2}\int^{\Lambda}\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+m^2)^2} \\ &= \frac{3\lambda^2}{16\pi^2}\int^{\Lambda}\frac{k^3dk}{(k^2+m^2)^2} \\ &= \frac{3\lambda^2}{32\pi^2}\left[\log\left(1+\frac{\Lambda^2}{m^2}\right)+\frac{\Lambda^2}{m^2+\Lambda^2}\right] \end{aligned}$$

These divergences must be dealt with. On general grounds, we expect the full propagator to have the form

$$\begin{aligned} \tilde{G}^2\left(p\right) &= \sum_n \frac{|\left\langle\Omega\right|\tilde{\phi}\left(0\right)\left|1\right\rangle|^2}{p^2+m_n^2} \\ &= \frac{|\left\langle\Omega\right|\tilde{\phi}\left(0\right)\left|1\right\rangle|^2}{p^2+m_{\text{phys}}^2} + \ldots \end{aligned}$$

The sum is over a continuum of physical states. The first term is an excited state, and crucially, we have that m_{phys} is some measured mass from experiment. We will now go through the process of renormalisation to understand how to make these divergences finite. From the LSZ formula, we required the normalisation $\left\langle k\right|\phi\left(x\right)\left|\Omega\right\rangle = e^{ik\cdot x}$. This is consistent with the condition that we keep $\left\langle\Omega\right|\tilde{\phi}\left(0\right)\left|1\right\rangle = 1$. You can check this independently (admittedly I don't know how to do this with Fourier transforms yet).

Due to the contribution from loop diagrams, we have unfortunately that $\left\langle\Omega\right|\tilde{\phi}\left(0\right)\left|1\right\rangle \neq 1$, and that the mass of our Lagrangian doesn't match our physical mass that we have measured, so $m \neq m_{\text{phys}}$ How do we remedy this? We use a renormalisation scheme.

Take our original Lagrangian; the very first Lagrangian which we have written down. We differentiate this from a 'new' Lagrangian by writing down a 0 subscript to the original Lagrangian fields and couplings. So, our original Lagrangian, our 'bare' Lagrangian as it is sometimes called, is denoted as

$$\mathcal{L}_0 = \frac{1}{2}\left(\partial\phi_0\right)^2 + \frac{1}{2}m_0^2\phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4$$

From this original Lagrangian, we apply a rescaling so that it fits with our LSZ normalisation convention. In general, rescale $\phi_0 = Z_\phi^{\frac{1}{2}}\phi$, with Z_ϕ determined by requiring proper normalisation for LSZ $\langle\Omega|\tilde{\phi}(0)|1\rangle = 1$. This is okay since the normalisation factor is just one condition, so we can do this. We write these normalisation factors explicit in $\mathcal{L}_0$, so that

$$\mathcal{L}_0 = \frac{Z_\phi}{2}\left(\partial\phi\right)^2 + \frac{Z_\phi}{2}m_0^2\phi^2 + \frac{Z_\phi^2\lambda_0}{4!}\phi^4$$

Now, we separate out 2 sets of terms into a renormalised part of our Lagrangian and counter terms.

$$\mathcal{L}_0 = \mathcal{L}_{\text{ren}} + \mathcal{L}_{\text{ct}}$$

Inventing new notation for new couplings, we want to write our new Lagrangian as

$$\mathcal{L}_0 = \left[\frac{1}{2}\left(\partial\phi\right)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right] + \left[\frac{\delta Z_\phi}{2}\left(\partial\phi\right)^2 + \frac{\delta m^2}{2}\phi^2 + \frac{\delta\lambda}{4!}\phi^4\right]$$

Now, we equate coefficients

$$\delta Z_\phi = Z_\phi - 1, \quad \delta m^2 = Z_\phi m_0^2 - m^2, \quad \delta\lambda = Z_\phi^2\lambda_0 - \lambda$$

Now we need to calculate some Feynman rules. What we can do is to construct Feynman rules for the renormalised Lagrangian as well as the counter term Lagrangian separately.

For $\mathcal{L}_{\text{ren}}$, it's the same as $\mathcal{L}_0$, with m^2 and λ the renormalised values. For $\mathcal{L}_{\text{ct}}$, we need new ones. For example, we have derivative terms and quadratic terms to worry about which wouldn't have been in the original Lagrangian. The new coupling terms have their own Feynman rules and are denoted with the new vertices as shown in figure 1.

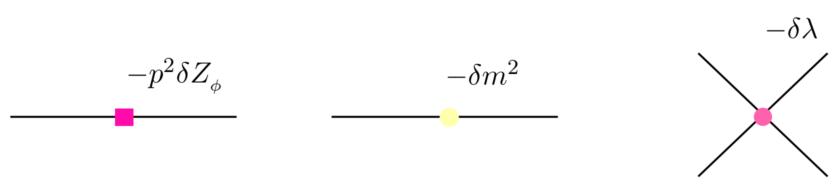


Figure 1: Vertices from our renormalised theory

The terms here are order $\hbar$ to counter quantum effects. Generally, δZ_ϕ, δm^2 and $\delta\lambda$ are order $O(\hbar)$ at most. Therefore, tree diagrams containing $\mathcal{L}_{CT}$ vertices are the same order as l opp diagrams from $\mathcal{L}_{\text{ren}}$.

4.0.1 Renormalising with our first loop contribution

Recall that our contribution from our 1-loop diagram is denoted $\Pi_1\left(p^2\right)$, and originally, it diverges. However, we define a new renormalised one-loop contribution by setting

$$\Pi_{1,\text{ren}}\left(p^2\right) = \Pi_1\left(p^2\right) + \Pi_{1,\text{ct}}\left(p^2\right)$$

Once again, our intention is to cancel out the one-loop divergence with our counter terms.

We had from earlier the two point vertex expression, which we obtained by summing over a geometric series of 1PI diagrams.

$$\tilde{\Gamma}^{(2)}(p)=\left[\tilde{G}^{(2)}(p)\right]^{-1}=p^2+m^2-\Pi\left(p^2\right)$$

From $\mathcal{L}_{\text{ren}}$, we still compute $\Pi_1\left(p^2\right)$ in the exact same way as before, but this time our values of m^2 and λ are the different, rescaled and renormalised quantities of our original Lagrangian. $\mathcal{L}_0=\mathcal{L}_{ren}+\mathcal{L}_{ct}$. From $\mathcal{L}_{\text{ct}}$, we have The finite result for $\Pi_{1,\text{ren}}=\Pi_1\left(p^2\right)+\Pi_{1,\text{ct}}$, is obtained by choosing $\delta Z_\phi=0$, and

$$\delta m^2=-\frac{\lambda}{32\pi^2}\left[\Lambda^2-m^2\log\left(1+\frac{\Lambda^2}{m^2}\right)\right]$$

With this choice, $\Pi_{1,\text{ren}}=0$. We will explain this a bit more. We previously suppressed a factor of $\hbar$ since we're working in natural units, but we have that when we expand this object out again, we have that

$$\mathcal{L}\rightarrow\mathcal{L}+\hbar\mathcal{L}_{CT}$$

so we add a loop contribution from the first part, which is order $\hbar$ and the vertices, which are also order $\hbar$. We get that

$$\Pi_{1,\text{ren}}=\Pi_1\left(p^2\right)+\Pi_{1,\text{CT}}=-\frac{\lambda}{32\pi^2}\left[\Lambda^2-m^2\log\left(1+\frac{\Lambda^2}{m^2}\right)\right]-p^2\delta Z-\delta m^2$$

The first term is just our loop contribution which we computed before, and the second terms come from our counter term vertex contributions. We can't include $\delta\Lambda$ yet because this term is of order $\hbar^2$. Correspondingly, we require at two loops the following counter term contributions as well. The freedom to choose where to put finite points is called the renormalisation scheme. The scheme above is called the on shell scheme, because of the fact that we require

- Require

  $$\Pi_{\text{ren}}\left(-m_{\text{phys}}^2\right)=m^2-m_{\text{phys}}^2$$

  which is usually 0

- We also require this to have a finite derivative

  $$\frac{\partial\Pi_{\text{ren}}}{\partial p^2}\Big|_{p^2=-m_{\text{phys}}^2}=0$$

Then, we have that

$$\tilde{G}^{(2)}(p)=\frac{1}{p^2+m^2-\Pi_{\text{ren}}\left(p^2\right)}=\frac{1}{p^2+m_{\text{phys}}^2}$$

Here, we have a pole at $p^2=-m_{\text{phys}}^2$, and the residue is 1 from LSZ. The m^2 in the renormalised Lagrangian should be equal to the observed mass. Next, we choose $\delta\lambda$ to cancel divergences in $\Lambda^{(4)}(0,0,0,0)$. We set $\Phi_{1,\text{ct}}^{(4)}=-\delta\lambda$, choose $\delta\lambda=\frac{3\lambda^2}{32\lambda^2}\log\left(\frac{\Lambda^2}{m^2}-1\right)$ This gives

$$\lambda_{eff}:=\tilde{\Gamma}_{\text{ren}}^{(4)}(0,0,0,0)=\lambda+\tilde{\Gamma}_1^{(4)}(0,0,0,0)+\tilde{\Gamma}_{1,\text{ct}}^{(4)}\quad=\lambda-\frac{3\lambda^2}{32\pi^2}\left[\log\left(1+\frac{m^2}{\Lambda^2}+\frac{m^2}{m^2+\Lambda^2}\right)\right]$$

This is finite as $\Lambda^2\rightarrow\infty$. In fact, the finite piece chosen so that $\lambda_{\text{eff}}\rightarrow\lambda$.

4.1 Dimensional Regularisation

Applying a momentum cutoff as a strategy to regute the 1-loop integrals doesn't work if we are dealing with non-Abelian gauge theories. What we need to do instead is work by varying the dimension. In the context of perturbation theory, divergences can be regulated by working in $d = 4 - \epsilon$ Usually, this is $0 < \epsilon \ll 1$. If we start with

$$S = \int d^d \left[\frac{1}{2} \left(\partial \phi^2 \right) + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right]$$

Dimensional analysis can be used here. Given that $[S] = 0, [\partial] = [m] = [X] = 1$. So,

$$\left[\phi^2 m^2 \right] = 2 [m] + 2 [\phi] = d, \to [\phi] = \frac{d}{2} - 1$$

Also, $\left[\lambda \phi^4 \right]$, thus $[\lambda] = 4 - d = \epsilon$. Introducing arbitrary scale u, we have $[u] = 1$. We write $\lambda = \mu^\epsilon g(u)$. such that g is dimensionless coupling in terms of mass dimension. We interpret μ as a a renormalisation scale, but in this case it is not to be taken to ∞.

4.2 Mathematical Identities which are useful

Before we go into the mechanics of regularisation, we will prove some useful identities which will help us calculate things. These identities are important, so remember them!

Theorem. The surface area of an n-dimensional sphere We evaluate the surface area of an n-dimensional sphere in terms of gamma functions. To do this, we start from integrating over d gaussian integrals then do a change of variables. Furthermore, once we figure out an expression for d when d is real, we can analytically continue this for when $d \in \mathbb{C}$, which we will need to do when we are doing the epsilon expansion.

$$\begin{aligned}
\left(\sqrt{\pi} \right)^d &= \left(\int_{-\infty}^{\infty} e^{-x^2} \right)^d \\
&= \int_{\mathbb{R}^d} dx_1 \dots dx_d e^{-x_1^2 + \dots + x_d^2} \\
&= S_d \int dr \, r^{d-1} e^{-r^2} \\
&= \frac{S_d}{2} \int du \, u^{\frac{d}{2} - 1} e^{-u} \quad r^2 = u \\
&= \frac{S_d}{2} \Gamma \left(\frac{d}{2} \right)
\end{aligned}$$

This means that

$$S_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$$

which is an expression which we can analytically continue. Be careful - sometimes the definitions of S_d can differ!

Our gamma function $\Gamma(\alpha)$ is initially defined only in the domain where $\alpha \in \mathbb{R}_+$. Hwoever, we can use analytic continuation by appealing to the fact that $\alpha\Gamma(\alpha) = \Gamma(\alpha+1)$, and extended the definition of the gamma function to $\alpha \in \mathbb{R}$ and include negative numbers. It's worth remembering that

$$\Gamma(1) = 1, \quad \Gamma\left(\frac{1}{2}\right) = \frac{1}{2}, \quad \Gamma(\alpha) = \int_0^\infty dx x^{\alpha-1} e^{-x}$$

4.2.1 Deriving a small expansion of the Gamma function

We will, without proof, use the fact that the log Taylor expansion of the gamma function is

$$\log\Gamma(\alpha+1) = -\gamma\alpha - \sum_{k=2}^{\infty} (-1)^k \frac{1}{k}\zeta(k)$$

The first constant γ has a special name, it is called the Euler-Mascheroni constant, with $\gamma = \gamma_E \sim 0.58$, and we have that $\zeta(k)$ is the Riemann-Zeta function

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$$

We can use the expansion above to show derive a small α expansion for $\Gamma(\alpha)$. Taking the first term of the expansion, we have

$$\begin{aligned}
\log\Gamma(\alpha+1) &= -\gamma\alpha + \dots \\
\alpha\Gamma(\alpha) &= e^{-\gamma\alpha} \\
\alpha\Gamma(\alpha) &= 1 - \gamma\alpha + O\left(\alpha^2\right) \\
\Gamma(\alpha) &= \frac{1}{\alpha} - \gamma + O(\alpha)
\end{aligned}$$

We also define a new function $B(s,t)$ which is called the Euler beta function, where

$$B(s,t) = \int_0^1 du\, u^{s-1}(1-u)^{t-1} = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}$$

4.2.2 Regulating our 1-loop integral

We replace our constant λ with $\lambda = g(\mu)\mu^\epsilon$, and then, use our new found knowledge to write our divergent one loop integral as the following.

$$\begin{aligned}
\Pi_1 &= -\frac{\lambda}{2}\int \frac{d^d k}{(2\pi)^d}\frac{1}{k^2+m^2} \\
&= -\frac{g(\mu)\mu^\epsilon}{2(2\pi)^d} S_d \int_0^\infty \frac{k^{d-1}}{k^2+m^2} dk
\end{aligned}$$

Let's focus on the integrand. Writing the integrand as an integral over dk^2, and then using a $u = m^2/\left(k^2+m^2\right)$ substitution, we can show that

$$\mu^\epsilon \int_0^\infty \frac{k^{d-1}}{k^2+m^2} dk = \frac{m^2}{2}\left(\frac{\mu}{m}\right)^\epsilon \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)}{\Gamma(1)}$$

Now, using our value for the surface area S_d, we substitute in the value $(\pi)^{d/2} = \frac{S_d}{2}\Gamma\left(\frac{d}{2}\right)$, which gives us

$$\Pi_1 = -\frac{gm^2}{2\,(4\pi)^{\frac{d}{2}}}\left(\frac{\mu}{m}\right)^{\epsilon}\Gamma\left(1-\frac{d}{2}\right)$$

Now, the idea here is to set $d = 4-\epsilon$, and expand out the gamma function perturbatively. First we use the fact that in our scheme, we're setting $d = 4-\epsilon$. Then, we use our logarithm expansion to show that

$$\Gamma\left(1-\frac{d}{2}\right) = \Gamma\left(\frac{\epsilon}{2}-1\right) = -\frac{1}{1-\frac{\epsilon}{2}}\Gamma\left(\frac{\epsilon}{2}\right) = -\frac{2}{\epsilon}+\gamma-1+O(\epsilon)$$

Furthermore, we expand out our exponent $\left(\frac{4\pi\mu^2}{m^2}\right)^{\frac{\epsilon}{2}} = 1+\frac{\epsilon}{2}\log\left(\frac{4\pi\kappa^2}{m^2}\right)+O\left(\epsilon^2\right)$. Substituting this in all together, we have that

$$\Pi_1\left(p^2\right) = -\frac{gm^2}{32\pi^2}\left[\frac{2}{\epsilon}-\gamma+1+\log\left(\frac{4\pi\mu^2}{m^2}\right)+O\left(\epsilon^2\right)\right]$$

So effectively, what we've done here is isolate the divergence and factored it out it in the form of $\frac{2}{\epsilon}$.

This means that we now ought to add counter-terms into the mix, which come from $\frac{\delta m^2}{2}\phi^2$, with $\mathcal{L}_0 = \mathcal{L}_{\text{ren}}+\mathcal{L}_{\text{CT}}$. We can choose one of two different schemes to subtract our factor of $\frac{gm^2}{16\pi^2\epsilon}$. Note the funny signs here. Recall, when we're taking the contribution of the counter-terms we add by $-\delta m^2$.

- We can employ the minimal subtraction scheme where we set $\delta m^2 = -\frac{gm^2}{16\pi^2\epsilon}$, which is basically the simplest thing we can do. We call this the **MS** scheme, for short.
- We can subtract more constants from the mix in the modified minimal subtraction scheme, denoted $\overline{\text{MS}}$, where in this case we set

$$\delta m^2 = -\frac{gm^2}{32\pi^2}\left(\frac{2}{\epsilon}-\gamma+\log 4\pi\right)$$

Note that with these schemes, m doesn't necessarily denote the physical mass of the system. In the case of the modified minimal subtraction scheme, we have that

$$\Pi_1^{\overline{\text{MS}}} = \frac{gm^2}{32\pi^2}\left(\log\left(\frac{\mu^2}{m^2}\right)-1\right)$$

At this point, we have several ways to subtract this off so that we get a finite answer. We can repeat this same story exactly with the four-point vertex function. Just to the same integral but this time with the four point function and approximate it using the same Taylor expansions as above. We find that, at 1-loop, we have a counter term that needs to be added which looks like

- In the case of the minimal subtraction scheme, our counter term removes simply just the divergent part, so we have that

$$\mu^{\epsilon}\delta g = \frac{3g^2\mu^{\epsilon}}{32\pi^2}\left(\frac{2}{\epsilon}\right)$$

  where we recall that $\lambda_0 = (g+\delta g)\,\mu^{\epsilon}$.

- In the case of the modified minimal subtraction scheme, we subtract a bit more from the mix, and include the Euler-Mascheroni constant.

$$\mu^{\epsilon}\delta g = \frac{3g^2\mu^{\epsilon}}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log 4\pi\right)$$

Now one may ask the obvious question. We've put in this dimensionless constant μ, but our physics shouldn't rely on this. Thus, we need to fix how μ evolves. To learn how to do this, we take a look at our original form for the Lagrangian. This technique is called subtracting to infinity and is an old-fashioned approach to eliminating μ dependence. We had

$$\begin{aligned}\mathcal{L}_0 &= \frac{1}{2}\left(\phi_0\right)^2 + \frac{1}{2}m_0^2 + \frac{\lambda_0}{4!}\phi_0^4 \\ \mathcal{L}_{\text{ren}} + \mathcal{L}_{\text{CT}} &= \frac{1+\delta\mathcal{Z}_{\phi}}{2}\left(\partial\phi\right)^2 + \frac{m^2+\delta m^2}{2}\phi^2 + \frac{g+\delta g}{4!}\mu^{\epsilon}\phi^4\end{aligned}$$

Now, the point is that our original parameters λ_0, m_0^2 and so on were μ independent. We use this as a condition to see how g depends on μ and how to coupling runs. For this purpose, we define the **beta** function.

Definition. Beta function. Our beta function of some coupling constant, say g, denoted as $\beta\left(g\right)$, is defined as

$$\beta\left(g\right) = \mu\frac{d}{d\mu}g = \frac{dg}{d\left(\log\mu\right)}$$

With this definition, we can solve for the β function of g knowing full well that $\lambda_0 = \left(g+\delta g\right)u^{\epsilon}$ shouldn't depend on μ. In our next calculation, we crucially work to lowest order in ϵ. Writing $u^{\epsilon} = e^{\epsilon\log\mu}$ to simplify the calculations, we have that in the MS scheme,

$$\begin{aligned}0 &= \frac{d}{d\log\mu}\lambda_0 \\ &= \frac{d}{d\log\mu}\left[\left(\gamma+\delta g\right)\mu^{\epsilon}\right] \\ &= \epsilon g\left(1+\frac{3g}{16\pi^2\epsilon}\right) + \beta\left(g\right)\left(1+\frac{3g}{8\pi\epsilon^2}\right)\end{aligned}$$

This means we can write our β function explicitly as

$$\begin{aligned}\beta\left(g\right) &= -\left(\frac{3g^2}{16\pi^2}+\epsilon g\right)\left(1+\frac{3g}{8\pi\epsilon^2}\right)^{-1} \\ &= \frac{3g^2}{16\pi^2} - \epsilon g + O\left(\frac{g^2}{\epsilon^2}\right) \\ &= \mu\frac{dg}{d\mu} + \text{two-loop order calculations}, \quad \beta\left(g\right) > 0\end{aligned}$$

Let's now try to solve this differential equation we have to first order. To first order in ϵ, from the above derivation, we have that

$$\beta\left(g\right) = \mu\frac{dg}{d\mu} = \frac{3g^2}{16\pi^2} + \ldots$$

since the first thing came from the 1-loop diagram we have that the first term is of order $\hbar$, and the rest of the terms are of order $\hbar^2$. This has the associated differential equation

$$\frac{dg}{g^2} = \frac{3}{16\pi^2}\frac{d\mu}{\mu}$$

When we integrate $\mu \to \mu'$, we have that the solution is

$$\frac{1}{g(\mu')} = \frac{1}{g(\mu)} - \frac{3}{16\pi^2}\log\frac{\mu'}{\mu}$$

Rearranging, we get that

$$g(\mu') = \frac{g(\mu)}{1 - \frac{3g}{16\pi^2}\log\left(\frac{\mu'}{\mu}\right)} \simeq g(\mu) + \frac{3\left(g(\mu)\right)^2}{16\pi^2}\log\frac{\mu'}{\mu}$$

We can use this to show that the running of our coupling g increases with our dimensionless parameter μ. For $\mu' > \mu$, the added second term is positive, and hence we have that $g(\mu') > g(\mu)$. The coupling runs to larger values as μ increases. This is a clever way to show this without having to differentiate our function in the first place. We also have that, looking at the denominator of our fraction, that our value of $g(\mu')$ diverges when μ approaches μ^* such that

$$\frac{6g}{16\pi^2}\log\frac{\mu^*}{\mu} = 1$$

We call this dimensional parameter Λ_{ϕ^4}, so that if $\mu' \to \Lambda_{\phi^4}$, where Λ_{ϕ^4} is defined via

$$\frac{3g}{16\pi^2}\log\frac{\Lambda_{\phi^4}}{\mu} = 1, \quad \text{1-loop}$$

then we have that $g(\mu') \to \infty$. This Λ_{ϕ^4} can be used as a scheme-dependent reference mass scale.

$$g(\mu) = \frac{16\pi}{3}\frac{1}{\log\frac{\Lambda_{\phi^4}}{\mu}}$$

The appearance of Λ_ϕ scale is called dimensional transmutation, since this parameter has a mass scale, but comes out from the parameter μ which was dimensionless by construction. In this spirit, we only trust perturbation theory when $\mu \ll \Lambda_{\phi^4}$, although I'm not quite sure how to interpret our cutoff Λ_{ϕ^4}.

4.3 The second approach to Renormalisation

We will now explore a different way to explore how the coupling $g(\mu)$ runs, using the Callan-Symanzik equations. We have our quantum effective action $\Gamma(\phi)$ and the vertex functions $\Gamma^{(n)}$. These quantities go into the LSZ formula, which means that they should be physical quantities. should be physical. These go into LSZ formula, from which we obtain physical predictions. This is a big hint, because as a result, our quantities Γ should be independent of μ, which means we should get zero if we differentiate the quantities by μ. Recall that, when we had to rescale our fields ϕ was properly normalised, we set $\phi_0 = Z_\phi^{\frac{1}{2}}\phi$.

$$\Gamma_0^{(n)}(x_1,\ldots x_n) = (-1)^n \frac{\delta^{(n)}\Gamma[\phi_0]}{\delta\phi_0(x_1)\ldots\delta\phi_0(x_n)} = (-1)^n Z_\phi^{-n/2}\frac{\delta^{(n)}\Gamma[\phi]}{\delta\phi(x_1)\ldots\delta\phi(x_n)}$$

Here, we used the chain rule to take out the rescaling when moving from ϕ_0 to ϕ. We define the anomalous dimension of ϕ, so that

$$\gamma_\phi = -\frac{\mu}{2}\frac{d}{d\mu}\log Z_\phi$$

We then have

$$\mu\frac{d}{d\mu}Z_\phi^{-n/2} = -\frac{n}{2}Z_\phi^{-n/2}\mu\frac{d}{d\mu}\log Z_\phi = (n\gamma_\phi)Z_\phi^{-n/2}$$

The steps here can be shown straightforwardly by just writing $Z_\phi^{-n/2} = e^{-\frac{n}{2}\log Z_\phi}$. We require that terms in Γ should be independent of scale. So, differentiating this object with respect to μ should equal zero. We have that Γ has both explicit dependence on μ, as well as implicit dependence via our couplings $g(\mu)$. This means that we require

$$\mu\frac{d}{d\mu}\Gamma^{(n)} = 0 = \left(\mu\frac{\partial}{\partial\mu} + \mu\frac{dm^2}{d\mu}\frac{\partial}{\partial m^2} + \beta(g)\frac{\partial}{\partial g} + n\gamma_\phi\right)\Gamma_{\text{ren}}^{(n)}(x_1\ldots x_n)$$

These are called the Callan-Symanzik equations. We can substitute different values of n in this case, to how our beta functions evolve. We will repeat this procedure in the case of $n=2$ and $n=4$ and check we get the same answers, and also see how this relates to the anomalous dimension. The last three terms are of order $\hbar$ at 1-loop. In ϕ^4 theory, we have that $Z_\phi = 1$ to one loop order, so we don't have to consider this in the calculations below? Recall that our value for the one loop integral, in the modified minimal subtraction scheme is

$$\Pi_1^{\overline{\text{MS}}} = \frac{gm^2}{32\pi^2}\left(\log\left(\mu^2/m^2\right) - 1\right)$$

This gives us an expression for our vertex function, which we previously derived using the geometric series argument behaves like

$$\tilde{\Gamma}^{(2)}\left(p^2=0\right) = p^2 + m^2 - \frac{gm^2}{32\pi^2}\left(\log\frac{\mu^2}{m^2} - 1\right) = 0 + \ldots$$

We then have that

$$0 = \mu\frac{d}{d\mu}\tilde{\Gamma}^{(2)}(0) = \mu\frac{dm^2}{d\mu} - \frac{gm^2}{16\pi^2} + O\left(\hbar^2\right) \implies \mu\frac{dm^2}{\mu} = \frac{gm^2}{16\pi^2}$$

Similarly, for our fourth order term, we have that

$$\tilde{\Gamma}^{(4)}(0,0,0,0) = -g\mu^\epsilon + \frac{3g^2\mu^\epsilon}{32\pi^2}\log\frac{\mu^2}{m^2} +$$
$$0 = \mu\frac{d}{d\mu}\tilde{\Gamma}^{(4)} = -\beta(g) + \frac{3g^2\mu^\epsilon}{16\pi^2} + \ldots$$

This implies that $\beta(g) = \frac{3g^2}{16\pi^2}$.

5 Renormalisation Group

Quantum field theory is not defined by the Lagrangian alone. To do this, we have a renormalisation scheme - for example the minimal subtraction scheme or the modified minimal subtraction scheme. However, having a renormalisation scheme doesn't fix parameters on its own, and we need to fix parameters with renormalisation conditions (for example, doing things on shell).

The idea is that QFT is not well defined without a regulator. This regulator is introduced, and we see how things change as we change this.

We explore the concept of universality. This is when we impose an arbitrary cutoff in UV, and then at low energies we see universal IR physics emerging from theories with different regularisation and renormalisation schemes (and scale). The renormalisation group is the study of how microscopic details change along lines of constant IR physics. For a real scalar field, momentum cutoff Λ_0, in d dimensions $d \in \mathbb{Z}_+$, we write the action

$$S_{\Lambda_0}[\phi] = \int d^d x \left[\frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \sum_{i \text{ terms}} \frac{1}{\Lambda_0^{d_i - d}} g_{i0} \mathcal{O}_i(x)\right]$$

where $\mathcal{O}_i[\phi(x)]$ local operators with mass dimension $d_i > 0$, and they can be made up of fields and their derivatives, for example $\mathcal{O}_i = (\partial\phi)^{r_i} \phi^{s_i}$. Right now, we're ignoring the mass coupling. The partition function is defined as

$$\mathcal{Z}_{\Lambda_0}(g_{i\mathcal{O}}) = \int^{\Lambda_0} \mathcal{D}\phi e^{-S_{\Lambda_0}[\phi]}$$

The integral is over fields such that $|\phi| \leq \Lambda_0$. What do we mean in this context that the path integral has an upper cutoff Λ_0? We mean that the Fourier mode expansion has momentum modes bounded by this limit. In other words, the ϕ in $\mathcal{Z}_{\Lambda_0}$ is such that

$$\phi(x) = \int_{|p| \leq \Lambda_0} \frac{d^d p}{(2\pi)^d} e^{ip\cdot x} \tilde{\phi}(p)$$

5.1 Effective actions

We will now study effective actions, where we will provide a low momentum cutoff and integrate over our high momentum modes. When we say momentum modes, we mean the values of p in our Fourier space decomposition. This is what we mean by 'physics in the IR'.

$$S_{\Lambda_0}[\phi] = \int d^d x \left[\frac{1}{2}(\partial\phi)^2 + \sum_i \frac{g_{i0}}{\Lambda^{d_i - d}} \mathcal{O}_i(x)\right]$$

We write our field ϕ and decompose it into two functions. One function retains just the low momentum modes, and is the one to keep, whilst the other function integrates over the high momentum modes.

$$\begin{aligned}
\phi(x) &= \phi^-(x) + \phi^+(x) \\
&= \int_{|p|<\Lambda} \frac{d^d p}{(2\pi)^d} e^{ip\cdot x} \tilde{\phi}(p) + \int_{\Lambda < p < \Lambda_0} \frac{d^d p}{(2\pi)^d} \tilde{\phi}(p) e^{ip\cdot x}
\end{aligned}$$

In this step, we integrate out the ϕ^+ to get the Wilsonian effective action, like a W. In our path integral notation, this looks like a path integral from Λ to Λ_0. Remembering to keep the exponent in the first step, we thus have that

$$e^{S_{\text{eff}}[\phi_-]} = \int_{\Lambda}^{\Lambda_0} \mathcal{D}\phi_+ e^{-S[\phi_+ + \phi_-]}$$
$$S_{\text{eff}}[\phi_-] = -\log \int_{\Lambda}^{\Lambda_0} \mathcal{D}\phi_+ e^{-S[\phi_+ + \phi_-]}$$

From this the RG equations will tell us how S_Λ^{eff} and S_{Λ_0} are related. We write

$$S_{\Lambda_0}\left[\phi^- + \phi^+\right] = S^0\left[\phi^-\right] + S^0\left[\phi^+\right] + S_{\Lambda_0}^{\text{int}}\left[\phi^-, \phi^+\right]$$

with free $S^0[\phi] = \int d^dx \frac{1}{2}\left[(\partial\phi)^2 + m^2\phi^2\right]$ There is no quadratic term $\phi^-\phi^+$. In Fourier space, this gives us $\tilde{\phi}^-(k)\,\tilde{\phi}^+(k')\,\delta^{(d)}(k+k')$. Vanish because of disjoint support. A term like the following is non-zero

$$\tilde{\phi}^-(k)\,\tilde{\phi}^-\left(k'\right)\tilde{\phi}^+\left(k''\right)\delta\left(k+k'+k''\right)$$

We use these to construct effective interactions.

$$S_\Lambda^{\text{int}}[\phi] = -\log \int \mathcal{D}\phi^+ e^{-S^0[\phi^+] - S_{\Lambda_0}^{\text{int}}[\phi^-, \phi^+]}$$

5.2 Running couplings

The physics being independent of Λ, Λ_0 implies that our partition functions should look the same regardless of momentum cutoff. In other words, we should have the condition that

$$\zeta_\Lambda\left(g_i(\Lambda)\right) = \zeta_{\Lambda_0}\left(g_{i0}; \Lambda_0\right)$$

The right hand side independent of Λ implies that the left hand side is independent of Λ. This means that the couplings $g_i(\Lambda)$ must run to compensate.

$$\Lambda \frac{d\zeta_\Lambda(g)}{d\Lambda} = \left(\Lambda \frac{\partial}{\partial\Lambda}\,|_{g_i} + \Lambda \frac{dg_i}{d\Lambda}\frac{\partial}{\partial g_i}\right)\zeta_\Lambda(g) = 0$$

This is the Callan-Symanzik equation or the RG equation. We have that S_Λ^{eff} has the same form as S_{Λ_0}.

$$S_\Lambda^{\text{eff}}[\phi] = \int d^dx \left[\frac{Z_\Lambda}{2}(\partial\phi)^2 + \sum_i \frac{Z_\Lambda^{n_i/2}}{\Lambda^{d_i - d}} g_i(\Lambda)\,\mathcal{O}_i(x)\right]$$

here, n_i is the number of ϕ fields in $\mathcal{O}_i(x)$. Integrating out ϕ^+ modes may force $Z_\Lambda \neq 1$. So, we renormalise the fields so that $\phi^r = Z_\Lambda^{1/2}\phi$. Any remaining Λ dependence must be described by $g_i(\Lambda)$. In terms of beta functions

$$\beta_i^{\text{cl}} = (d_i - d)g_i, \quad \beta_i^{\text{qu}} = \Lambda \frac{dg_i}{d\Lambda}, \beta = \beta^{cl} + \beta^{qu}$$

5.3 Vertex functions

We'll now look at the procedure in how we rescale vertex functions. When we renormalise, in general we have that $\delta Z \neq 0$, which means that Z changes non-trivially. This information is encoded in the anomoulous dimension. Recall the definition of the anomalous dimension, which is a derivative of our partition function multiplied by a cutoff

$$\gamma_\phi = -\frac{\Lambda}{2}\frac{d}{d\Lambda}\log Z_\Lambda$$

To study what goes on in terms of how couplings run, we once again appeal to functions which are phsyical, and therefore don't depend on our chosen momentum cutoff Λ. In this case, we'll look at n-point vertex functions. We will iterate over a 'mode thinning' procedure, where we repeatedly adjust the threshold of our momentum cutoff. Let $0 < s < 1$.

$$Z_{s\Lambda}^{-n/2}\Gamma_{s\Lambda}^{(n)}\left(x_1, \ldots, x_n; g_i\left(s\Lambda\right)\right) = Z_\Lambda^{-n/2}\Gamma_\Lambda^{(n)}\left(x_1, \ldots, x_n; g\left(\Lambda\right)\right)$$

Infinitesimally, if we Taylor expand out the left hand side by doing a perturbation $s = 1 - \delta s$, then expanding out both Z and well as $\Gamma^{(n)}$ gives us the relation

$$0 = \Lambda\frac{d}{d\Lambda}\Gamma_\Lambda^{(n)}\left(x_1, , x_n; g\left(\Lambda\right)\right) = \left(\Lambda\frac{\partial}{\partial\Lambda} + \beta_i\frac{\partial}{\partial g_i} + n\gamma_\phi\right)\Gamma_\Lambda^{(n)}\left(x_1, \ldots x_n; g\left(\Lambda\right)\right)$$

For example, let $s\Lambda = \Lambda'$ for fixed Λ. Differentiate this with respect to s.

$$s\frac{d}{ds}Z_s^{-n/2} = n\gamma_\phi, \quad \text{using } s\frac{d}{ds} = \Lambda'\frac{d}{d\Lambda'}$$

then relabel Λ' as Λ. The whole RG transformation is

- Integrate out momentum modes $(s\Lambda, \Lambda]$.
- Rescale coordinates $x' = sx$.

Under rescaling, our kinetic term must be made properly normalised.

$$\phi^r\left(sx\right) = s^{1-\frac{d}{2}}\phi^r\left(x\right)$$

Then the rest of the action is invariant is we also rescale $\Lambda \to \Lambda/s$.

$$\begin{aligned}\Gamma_\Lambda^{(n)}\left(x_1, \ldots, x_n; g(\Lambda)\right) &= \left(\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{n/2}\Gamma_{s\Lambda}^{(n)}\left(x_1, \ldots, x_n; g(s\Lambda)\right) \\ &= \left(s^{2-d}\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{n/2}\Gamma_\Lambda^{(n)}\left(sx_1, \ldots, sx_n; g\left(s\Lambda\right)\right)\end{aligned}$$

Going into the second step, we've rescaled coordinates, cutoff and fields. But, numerical values of $Z_{s\Lambda}$ and $g\left(s\Lambda\right)$ don't get rescaled. Now, we reconsider the points which we look at. Instead of x_i in argument, look at x_i/s.

$$\Gamma_\Lambda^{(n)}\left(\frac{x_1}{s}, \ldots, \frac{x_n}{s}; g((\Lambda)\right) = \left(s^{2-d}\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{n/2}\Gamma_\Lambda^{(n)}\left(x_1, \ldots, x_n, g\left(s\Lambda\right)\right)$$

As s get s smaller, on the left hand side, we have that $|x_i - x_j|$ gets bigger. In the right hand side, the coupling runs to the infrared. For small $\delta\sigma = 1 - s$. we have that

$$\left(s^{2-d}\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{1/2} = 1 + \left(\frac{d-2}{2} + \gamma_\phi\right)\delta s$$

The fields behave as if their mass dimension were

$$\frac{d-2}{2} + \gamma_\phi = \Delta_\phi$$

where the first term is our engineering dimension, and our second term is the anomalous dimension. These combined give our scaling dimension. The anomalous dimension comes from a correction from our rescaling. In the running of the n point functions, they don 't run exactly as they should.

5.4 Renormalisation group flow

We look at coupling constant space, which is a high dimensional space parametrised with values of g_i. Renormalisation group flows are lines in coupling constant space corresponding to how the set of couplings $k\{g_i\}$ change as we integrate modes. This is governed by $\{\beta_i(\{g_i\})\}$ which is the set of beta functions.

Theories lying along the same flow line describe the same infrared physics. We're thinking of the physics now in terms of differential equations of the β functions. Thus, we're interested in fixed points and such.

5.4.1 Fixed Points of RG equations (β functions)

Fixed points are where β functions vanish. Denote the points with an asterisk, so the set of fixed point $\left\{g_j^*\right\}$, such that

$$\beta_i \mid_{\beta_j} = 0$$

Recall that

$$\beta_i(\{g_i\}) = (d_i - d) + \Lambda\frac{dg_i}{d\Lambda}(\{g_i\})$$

which is the sum of the classical part and the quantum part. For example, Gaussian fixed point $g)j^* = 0, \forall j$ in the free massless theory. We also have non-trivial fixed points, which require cancellation of β^{cl} abd β^{qu}.

5.4.2 Scale invariance at fixed points

g_i^* is independent of scale implies that other dimensionless functions of g_i are scale invariant, for example, $\gamma_\phi(g_i^*) = \gamma_\phi^*$. Now, if the β functions vanish, the Callan-Symanzik equations become

$$\Lambda\frac{\partial}{\partial\Lambda}\Gamma_\Lambda^{(2)}(x,y) = -2\gamma_\phi^*\Gamma^{(2)}(x,y)$$

If we impose translational and rotational invariance, then $\Gamma^{(2)}(x,y) = \Gamma^{(2)}(|x-y|)$. We know the mass dimensions. Thus, we have that like $\langle \phi(x)\phi(y)\rangle$, the engineering dimension of $\Gamma^{(2)} = \Lambda^{d-2}$. Accounting for anomalous dimension, we have

$$\Gamma^{(2)}_{\Lambda}(x,y;g_i^*) = \frac{\Lambda^{d-2}}{\Lambda^{2\Delta_\phi}} \frac{c(g_i^*)}{|x-y|^{2\Delta_\phi}}$$

where $\Delta_\phi = \frac{1}{2}(d-2) + \gamma_\phi^*$ is the scaling dimension of ϕ. This power law behaviour of two point functions is characteristic of scale invariant theories. We can contrast this to theories with a characteristic scale $M = \frac{1}{\xi}$ which is inverse correlation length. In a theory with scale $M \sim \frac{1}{\xi}$, then we have that $\Gamma^{(2)}(x,y) \sim \frac{e^{-M|x-y|}}{|x-y|^{2\Delta_\phi}}$ Near a fixed point, we can linearise the RG equations. If we let $\delta g_i = g_i - g_i^*$, then we can write the RG equations as

$$\Lambda \frac{dg_i}{d\Lambda} |_{g_i^*+\delta g_i} = \beta_{ij}\delta g_j + O\left(\delta g^2\right)$$

So now what we want to do is to look at eigenvalues and eigenvectors of β to see how these things behave. Let's call the eigenvector σ_i, and the eigenvalue $\Delta_i - d$, where Δ_i is the scaling dimension associated with σ_i. Note that the σ_i generally represent some combination of directions in coupling space, which is a linear combination of operators O_i in $S[\phi]$. Our linearised RG flow equations are now

$$\Lambda \frac{d\sigma_i}{d\Lambda} = (\Delta_i - d_i)\sigma_i$$

This means that

$$\sigma_i(\Lambda) = \left(\frac{\Lambda}{\Lambda_0}^{\Delta_i - d}\right)\sigma_i(\Lambda_0)$$

with initial scale $\Lambda_0 > \Lambda$. We have a few cases to consider. $\Delta_i > d$ implies $\sigma_i(\Lambda) < \sigma_i(\Lambda_0)$. These flow back to the fixed point as Λ decreases. These are called irrelevant directions. Conversely, we have $\Lambda_i < d$ which implies $\sigma_i(\Lambda) > \sigma_i(\Lambda_0)$. These flow away from the fixed point. These are called relevant interactions. When $\Delta_i = d$, this is called a marginal interaction. We have a diagram here in infinite dimensional coupling space. The critical surface we have infinite dimensions. The codimension is finite, which is the number of relevant operators. The trajectory leaving the fixed point is the 'renormalised trajectory'. It takes an infinite amount of time to leave the fixed point.

5.5 Quantum Electrodynamics

Quantum electrodynamics studies interactions between electrons, which is mediated by photons. We will revise some concepts from QED from QFT. Recall that the action is constructed from our electromagnetic field tensor as well as a Dirac Lagrangian which dictates what the electrons do. In this case, we put our new found knowledge of Grassman variables to good use here. Our full action is

$$S\left[\phi,\overline{\phi},A\right] = \int d^4x \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\left(\not{D} + m\right)\psi$$

in this expression, $\not{D} = \gamma^\mu \left(\partial_\mu + ieA_\mu\right)$ is required so that we have gauge invariance when under the $U\left(1\right)$ transformation given by

$$\begin{aligned}\psi &\to e^{ie\alpha(x)}\psi \\ \overline{\psi} &\to e^{-ie\alpha(x)}\overline{\psi} \\ A_\mu &\to A_\mu - \partial_\mu \alpha\end{aligned}$$

We need the covariant derivative in the Lagrangian term since our $U\left(1\right)$ field $\alpha\left(x\right)$ varies in space, to cancel out some extra terms when we do a derivative. For now, we will work in Euclidean spacetime, where our matrices γ^μ satisfy the Clifford algebra (which now involves the Dirac delta function)

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$$

This means of course that we have to adapt our representation slightly.

5.5.1 The Propagator in the free Case

In the free Fermion action, there is an absence of the field A_μ and the field tensor $F_{\mu\nu}$. This means that our action is reduced to

$$S\left[\psi, \overline{\psi}\right] = \int d^4x \overline{\psi}\left(\gamma^\mu \partial_\mu + m\right)\psi$$

If we Fourier transform, we get that that our action in terms of Fourier modes is

$$S_0\left[\psi, \overline{\psi}\right] = \int \frac{d^4p}{\left(2\pi\right)^4} \tilde{\overline{\psi}}\left(-p\right)\left(i\not{p} + m\right)\tilde{\psi}$$

Now looking at the source electromagnetic fields with source term, we have that our equation of motion is

$$\partial_\nu F^{\mu\nu} = j^\mu$$

This motivates our action to be of the form

$$Z_0\left[J\right] = \int \mathcal{D}A \exp\left(-\int d^4x \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + j^\mu A_\mu\right)$$

We want to check that this action is gauge invariant under the transformation $A_\mu \to A_\mu - \partial_\mu \alpha$ We have that our $F_{\mu\nu}F^{\mu\nu}$ part is invariant under this gauge transformation, but we need to check that the $j_\mu A^\mu$ integral of our action is also invariant. Under this transformation, we have that

$$\int d^4x j^\mu \left(A'_\mu - A_\mu\right) = \int d^4x j^\mu \partial_\mu \alpha = \int \alpha \partial_\mu j^\mu = 0$$

since we have that $\partial_\mu j^\mu = 0$ by the antisymmetry of $F_{\mu\nu}$. Now, when we consider the full action S we have that our action is, upon Fourier transforming

$$S_g\left[\tilde{A}, \tilde{j}\right] = \frac{1}{2}\int \frac{d^4k}{\left(2\pi\right)^4}\left[\tilde{A}_\mu\left(-k\right)\left(k^2\delta^{\mu\nu} - k^\mu k^\nu\right)\tilde{A}_\nu\left(k\right) + \tilde{j}^\mu\left(-k\right)\tilde{A}_\mu\left(k\right) + \tilde{j}^\mu\left(k\right)\tilde{A}_\mu\left(-k\right)\right]$$

Now, something to note is that we encounter problems when we consider fields which are gauge equivalent to $A_\mu\left(x\right) = 0$, and correspondingly when $\tilde{A}\left(k\right) = 0$. This is due to the fact that the partition function will start to diverge. Let's see why. Under a $U(1)$ gauge transformation, we can transform our field to pick up a gauge term $A_\mu = \partial_\mu \alpha$. The corresponding Fourier transform is $\tilde{A}_\mu(k) = k_\mu \tilde{\alpha}(k)$. We want to show that the gauge fields cause the action to vanish. We can write the part of the action

$$\tilde{A}_\mu(-k)\left(k^2\delta^{\mu\nu} - k^\mu k^\nu\right)\tilde{A}_\nu(k) = \tilde{A}_\mu(-k)\, k^2 P^{\mu\nu}\tilde{A}_\nu(k)$$

One can easily confirm that P is indeed actually a projection operator, in other words the condition

$$P^{\mu\nu}P_\nu{}^\rho = P^{\mu\rho}$$

A nice side comment about this is that we know that P has eigenvalues which are either 0 or 1. Observe further that the trace of P is 3, which means that we have one eigenvector with eigenvalue 0 and three other vectors which have eigenvalue 1. One can also check that $P^{\mu\nu}k_\nu = 0$. This means that we have the above term in the action vanishing. In addition, the fact that our source is conserved $\partial_\mu j^\mu = 0$ implies that the Fourier transform $k_\mu j^\mu = 0$. This in turn means that our $\tilde{j}\tilde{A}$ terms in our action vanish. The point of this exercise is that since we have an infinite amount of fields which are gauge equivalent to $A_\mu = 0$, then $\int \mathcal{D}A = \infty$.

We choose to integrate over the subspace where $k_\mu \tilde{A}^\mu(k) = 0$, or in position space, where $\partial^\mu A_\mu(x) = 0$. In this subspace, we have that our projection operator acts as the identity on all $\tilde{A}(k)$. This means that we have

$$P^{\mu\nu}(k)\, A_\nu(k) = A^\mu$$

Since the inverse of $P^{\mu\nu}$ is $\frac{1}{k^2}P^{\mu\nu}$, we have that, repeating the same procedure of completing the square, that our partition function with source is given by

$$Z_0\left[\tilde{J}\right] = \exp\left[\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\tilde{j}_\mu(-k)\,\frac{P^{\mu\nu}}{k^2}\tilde{j}_\nu(k)\right]$$

Now, we can simplify this expression even more. Since our expression for $P^{\mu\nu}$ involves $k^\mu k^\nu$, when we contract this with $\tilde{j}$, since $\partial_\mu j^\mu = 0$, these terms cancel out. Thus we're left with a partition function which is akin to working in the Feynman gauge.

$$Z_0\left[\tilde{j}\right] = \exp\left[\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\tilde{j}_\mu(-k)\,\frac{\delta^{\mu\nu}}{k^2}\tilde{j}_\nu(k)\right]$$

Our original action involved the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$. The interaction term in all this which couples the field A_μ (electromagnetism), and our fermion field ψ, is precisely the A_μ term which comes from D_μ. This term, making the indices explicit is the term

$$ieA_\mu(x)\,\overline{\psi}^\alpha(x)\,(\gamma^\mu)_{\alpha\beta}\,\psi^\beta(x)$$

Now, to add in an interaction term, we follow the same procedure as we did previously. Our free partition function with source term has action

$$S_0 = \int d^4x \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\partial\!\!\!/\psi + m\overline{\psi}\psi + \overline{\psi}\overline{\eta} + \eta\psi + A_\mu j^\mu$$

We differentiate with respect to these source terms to pull out the required terms. This means our full partition function is

$$Z\left[\eta, \overline{\eta}, j\right] \propto \exp\left[-ie\,(\gamma^\mu)^{\mu\nu}\int d^4x\left(-\frac{\delta}{\delta J(x)}\right)(-\right.$$

5.6 Vacuum Polarisation

We'll now look at quantum corrections to the photon propagator, which is the set of all graphs with two external photon legs. Using the standard argument with a geometric series, we have that the full propagator gives the contribution

$$I = \frac{1}{1 - \Pi(p)}$$

Now, at one loop order, the set of all 1PI graphs with amputations on both external legs is given by

$$\Pi^{\mu\nu} = 1 \text{ loop contribution} + 2 \text{ loops contribution}$$

The 1 loop contribution is the contribution we computed earlier but without our external photon legs contributing a factor of A_μ. To compute this in a meaningful fashion, we resort to our previous technique of dimensional regularisation. We have two vertices, each contributing a factor of $(ie)^2$. We introduce the dimensional regularisation by setting $d = 4 - \epsilon$, and also by setting $e^2 = \mu^\epsilon g^2(\mu)$. Each vertex contributes a gamma matrix indexed by $ie\gamma^\mu$ and $ie\gamma^\nu$. As before, our fermionic propagator contributes factors of $\frac{1}{i\not{p}+m}$ and $\frac{1}{i(\not{q}-\not{p})+m}$. If we combine this all together and take the trace of this object, we get that our 1-loop contribution is

$$\Pi^{\mu\nu}_{\text{1-loop}}\left(q^2\right) = -\mu^\epsilon (ig)^2 \int \frac{d^d p}{(2\pi)^d} \frac{\operatorname{tr}\left[\left(-i\not{p}+m\right)\gamma^\mu\left(-i\left(\not{p}-\not{q}\right)+m\right)\gamma^\nu\right]}{(p^2+m^2)\left((p-q)^2+m^2\right)}$$

Now here, we use Feynman's trick to split a product into integral form;

$$\frac{1}{AB} = \int_0^1 dx \int_0^1 dy \frac{\delta(x+y-1)}{(Ay+Bx)^2}$$

If we look just at the denominator here, we get that our integral is given by

$$\int_0^1 \frac{dx}{\left[(p-qx)^2 + m^2 + q^2 x(1-x)\right]^2}$$

Now, we can translate this integral and redefine $p' = p - qx$, so that our total one loop contribution is

$$\Pi^{\mu\nu}_{\text{1-loop}}(q) = \mu^\epsilon g^2 \int \frac{d^d p}{(2\pi)^d} \int_0^1 dx \frac{\operatorname{tr}\left\{\left[-i\left(\not{p}+\not{q}x\right)+m\right]\gamma^\mu\left[-i\left(\not{p}-\not{q}(1-x)\right)+m\right]\right\}}{(p^2+\Delta)^2}$$

where our adjusted mass is $\Delta = m^2 + q^2 x(1-x)$.

To proceed with this calculation, we need to following trace indices

$$\begin{aligned} \operatorname{tr}\left(\gamma^\mu\gamma^\nu\right) &= 4\delta^{\mu\nu} \\ \operatorname{tr}\left(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma\right) &= 4\left(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\nu}\delta^{\rho\sigma} + \delta^{\mu\sigma}\delta^{\nu\rho}\right) \end{aligned}$$

Hence, the numerator becomes

$$\operatorname{tr}\{\} = 4\left\{-(p+xq)^\nu \{p-q(1-x)\}^\nu + (p+qx)\cdot[p-q(1-x)]\,\delta^{\mu\nu} - (p+qx)^\nu[p-q(1-x)]^\nu + m^2\delta^{\mu\nu}\right\}$$

We can see a lot of terms will vanish since the integrand is odd in terms of changes in indices. As $d \to 4$, integrals over odd powers of p^ν vanish, we can drop these terms. Similarly, only the diagonal parts of $p^\nu p^\mu$ will integrate to something non-zero, since this is an even power. Thus, we are able to replace the following expressions

$$\begin{aligned} p^\mu p^\nu &\to \frac{1}{d}\delta^{\mu\nu}p^2 \\ p^\mu p^\rho p^\nu p^\sigma &\to \frac{\left(p^2\right)^2}{d\left(d+2\right)}\left(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\nu}\delta^{\rho\sigma} + \delta^{\mu\sigma}\delta^{\nu\rho}\right) \end{aligned}$$

This means that we're left with an integrand which depends only on p^2. This means we can change the measure as follows. We change variables

$$\frac{d^dp}{(2\pi)^d} \to S_d\frac{dp^{d-1}}{(2\pi)^d} = \frac{\left(p^2\right)^{\frac{d}{2}-1}dp^2}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}$$

Putting things together, we then find that

$$\begin{aligned} \Pi_1^{\mu\nu}(q) = 4\mu^\epsilon\frac{g^2}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\int_0^1 dx\int_0^\infty dp^2\left(p^2\right)^{\frac{d}{2}-1}\frac{1}{\left(p^2+\Delta\right)^2} \\ \times\left[p^2\left(1-\frac{2}{d}\right)\delta^{\mu\nu} + \left(2q^\mu q^\nu - q^2\delta^{\mu\nu}\right)x\left(1-x\right) + m^2\delta^{\mu\nu}\right] \end{aligned}$$

These are Euler-beta functions. If we let $u = \frac{\Delta}{p^2+\Delta}$, then we get that these objects are

$$\begin{aligned} \int dp^2\frac{\left(p^2\right)^{\frac{d}{2}-1}}{\left(p^2_\Delta\right)^2} &= \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}\frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(2\right)} \\ \int_0^\infty dp^2\frac{\left(p^2\right)^{\frac{d}{2}}}{\left(p^2+\Delta\right)^2} &= \left(\frac{1}{\Delta}\right)^{1-\frac{d}{2}}\frac{\Gamma\left(1+\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)}{\Gamma\left(2\right)} \end{aligned}$$

This calculation gives the result

$$\Pi_1^{\mu\nu}(q) = \frac{4q^2\mu^\epsilon}{(4\pi)^{\frac{d}{2}}}\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 dx\frac{1}{\Delta^{\frac{\epsilon}{2}}}\left[\delta^{\mu\nu}\left[m^2 - x\left(1-x\right)q^2\right] - \delta^{\mu\nu}\left[m^2 + x\left(1-x\right)q^2\right] + 2x\left(1-x\right)q^\mu q^\nu\right]$$

This simplifies to

$$\begin{aligned} ... &= \frac{8g^2\mu^\epsilon}{(4\pi)^{\frac{d}{2}}}\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 dx\frac{1}{\Delta^{\frac{\epsilon}{2}}}\left(-q^2\delta^{\mu\nu} + q^\mu q^\nu\right)x\left(1-x\right) \\ &= \left(q^2\delta^{\mu\nu} - q^\mu q^\nu\right)\Pi_1\left(q^2\right) \end{aligned}$$

where

$$\Pi_1(q) = -\frac{8g^2\Gamma\left(\frac{\epsilon}{2}\right)}{(4\pi)^{\frac{d}{2}}}\int_0^1 dx x\left(1-x\right)\left(\frac{\mu^2}{\Delta}\right)^{\frac{\epsilon}{2}}$$

Note that $q_\mu\Pi_1^{\mu\nu} = 0$. In the $d \to 4$ limit, we have that

$$\Pi_1\left(q^2\right) = \frac{-g^2}{2\pi^2}\int_0^1 dx x\left(1-x\right)\left[\frac{2}{\epsilon} - \gamma + \log\frac{4\pi\mu^2}{\Delta}\right] + \mathcal{O}\left(e^2\right)$$

Now, we renormalise this object so that

$$\begin{aligned}\mathcal{L}_0 &= \mathcal{L} + \mathcal{L}_{CT} \\ S_0 &= S + S_{CT} \\ e_0 &= Z_e e = (1 + \delta Z_e)\, e \\ m_0 &= Z_m m = (1 + \delta Z_m)\, m \\ \psi_0 &= \sqrt{Z_2}\psi \\ A_0 &= \sqrt{Z_3} A\end{aligned}$$

We write the RHS as

$$S + S_{CT} = \int d^4x \left[\frac{1}{4} Z_3 F_{\mu\nu}F^{\mu\nu} + Z_2 \overline{\psi}\gamma\psi + Z_m Z_2 m\overline{\psi}\psi + ieZ_1 \overline{\psi}\not{A}\psi\right]$$

where $Z_1 = Z_e Z_2 \sqrt{Z_3}$. Let $Z_K = 1 + \delta Z_k$ for $k = e, m, 1, 2, 3$, and note that

$$\delta Z_e = \delta Z_1 - \delta Z_2 - \frac{1}{2}\delta Z_3 + \text{small}$$

and we will show that gauge invariance implies that $\delta Z_e = -\frac{1}{2}\delta Z_3$ since $Z_1 = Z_2$

Below we show a counter-term diagram

$$S^{CT} \subset \int d^4x \frac{\delta Z_3}{4} F^2$$

The new 2-point interaction gives us a contribution

$$= -\left(k^2\delta_{\mu\nu} - (1-\epsilon)\, k^\mu k^\nu\right)\delta Z_3$$

with δZ_3 chosen so that $\Pi_1^{\text{ren}}\left(q^2\right)$ is finite. We then have

$$\delta Z_3 = \frac{-g^2(\mu)}{12\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log 4\pi\right)$$

in the minimal subtraction scheme. Then, we have that $\left(\Pi_1^{\mu\nu}(q)\right)^{\text{ren}}$ calculated in renormalised perturbation theory yields

5.7 QED β functions

We can infer the QED beta functions from what we did just now.

$$e_0 = Z_e e = Z_1^{-1} Z_2^{-1} Z_3^{-\frac{1}{2}} e$$

we will show why $Z_1 Z_2^{-1} = 1$. We have that

6 Gauge Theories

We showed in the previous section that QED has a beta function which looks like

$$\beta(g) = \frac{g^3}{12\pi^2}$$

which is positive for positive g. We will show in this section that in the non-Abelian gauge theory case, we can have signs that flip.

6.1 Lie groups - Facts and Conventions

We will assume that all the Lie groups which we are dealing with are connected, such that all group elements are connected to the identity. For any $u \in G$, we can write the element as the exponential $u = \exp(i\theta^a T^a) 1$, where θ^a are numbers and a runs over the index of our Lie group.

The T^a are the generators of our Lie algebra, and under the Lie bracket we have that

$$\left[T^a, T^b\right] = if^{abc}T^c$$

where f^{abc} are our structure constants. We can always choose a basis such that f^{abc} is anti-symmetric in all its indices and we will do so. The bracket is also endowed with the Jacobi identity

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$$

If we set $A = T^a, B = T^b, C = T^c$ and factor out T, we get the relation

$$f^{abd}f^{dce} + f^{bcd}f^{dae} + f^{cad}f^{dbe} = 0$$

We can normalise our structure constants according to the convention that

$$f^{acd}f^{bcd} = N\delta^{ab}$$

We have several classes of Lie groups - unitary, orthogonal, symplectic. In this section, we will focus on unitary groups, which have the property that $U \in G$ means that $U^\dagger U = 1$. Being in the special unitary group means that we have $\det(U) = 1$. $G = SU(N)$ has $N^2 - 1$ generators (in the fundamental representation where $SU(N)$ is represented by $N \times N$ matrices, we have $N^2 - 1$ generators).

6.2 Representations

There are several ways to represent a Lie group or Lie algebra. Firstly, we have in our convention that the Lie algebra of $SU(N)$, in the fundamental representation, consists of $N \times N$ traceless, Hermitian matrices. In this space, infinitesimally we have that a $SU(N)$ transformation is given by

$$\phi_i \to \phi_i + i\alpha^a \left(T^a_{fund}\right)_{ij}\phi_j, \quad \alpha^a \in \mathbb{R}, \quad a = 1, \dots N^2 - 1, \quad i, j = 1, \dots N$$

Additionally, we can make use of the anti-fundamental representation defined as

$$\left(T^a_{\text{a-fund}}\right)^a = -\left(T^a_{\text{fund}}\right)^*$$

Acting on the anti-fundamental representation space, we have that since T is hermitian

$$\phi_i^* \to \phi_i^* + i\alpha^a \left(T^a_{\text{a-fund}}\right)_{ij} \phi_j^* = \phi_i^* - i\alpha^a \phi_j^* \left(T^a_{\text{fund}}\right)_{ji}$$

To save ink, from now on we'll be dropping the 'fund' subscript. In the adjoint representation, we have that the vector space on which we act on is precisely the set of generation matrices. To this end, our adjoint representation of an element T, denoted $\left(T^a_{\text{adj}}\right)_{ij} = -if^{aij}$ is itself the structure constant. Gauge fields transform in the adjoint representation.

6.3 Classifying representations

Each representation R has an associated index which we can construct, which we shall call $T(R)$, defined by the inner product

$$\text{tr}\left(T^a_R T^b_R\right) = T(R)\,\delta^{ab}$$

where we do the sum over the indices of our representation. In the fundamental representation, this turns out to be $\frac{1}{2}$, where

$$T^a_{ij} T^b_{ji} = \frac{1}{2}\delta^{ab}$$

We can check this condition for $SU(2)$ and $SU(3)$ using our standard generators for the Lie algebra. In our adjoint representation

$$f^{acd} f^{bcd} = N\delta^{ab}, \quad \implies T(\text{adj}) = N$$

Here, N comes from $SU(N)$. From this we can construct a quadratic Casimir given by $C_2(R)$ with

$$T^a_R T^a_R = C_2(R)\,1$$

If we set $a = b$ and sum over the representation indices, we get that

$$\begin{aligned} T(R)\,d(G) &= C_2(R)\,d(R) \\ C_2(\text{fund }) = C_F &= \frac{N^2-1}{2N} \\ C_2(\text{adj}) = C_a &= N \end{aligned}$$

The dimension of the representation space, V in which representations of group elements act on is the dimension of the representation. For instance, we have that the dimension of the fundamental representation of $SU(N)$ is N since our matrices act on vectors, and the dimension of our adjoint representation is $N^2 - 1$ since this is the number of generators themselves.

6.4 Gauge invariance and Wilson Lines

Wilson lines are a natural construction to construct gauge invariant fields. They allow us to meaningfully compare to different points in spacetime, since fields should be invariant under a phase transformation $\phi(x) \to e^{i\alpha(x)}\phi(x)$, but this phase is not generally the same at each point. Thus, gauge fields are in themselves a **connection**, as in general relativity, since they allow us to meaningfully compare fields at different points. This is elaborated more in Schwartz. Let's revisit QED, where fermions have a $U(1)$ gauge symmetry. Under this gauge transformation, we have that

$$\phi(x) \to e^{i\alpha(x)}\phi(x), \quad \overline{\phi}(x) \to \overline{\phi}(x) e^{-i\alpha(x)}$$

Under this transformation however, since $\alpha(x)$ depends on x, our kinetic term $\overline{\phi}\partial\!\!\!/\phi$ is not invariant. Consider a derivative in the direction

$$\phi(x+an) - \phi(x) = e^{i\alpha(x+an)}\phi(x+an) - e^{i\alpha(x)}\phi(x)$$

$$n^\mu \partial_\mu \phi = \lim_{a\to 0} \frac{1}{a}\left[\phi(x+an) - \phi(x)\right]$$

The gauge covariant derivative is defined to be the object which just picks up a phase under this transformation, so

$$D_\mu \phi(x) = e^{i\alpha(x)} D_\mu \phi(x)$$

Definition. Wilson Lines We define a Wilson line (or parallel transporter) as a function (denoted as a line from x to y), such that its gauge transformation takes it $e^{i\alpha(y)}W(y,x)e^{-i\alpha(x)}$. The convention here is that $W(x,x) = 1$. This then implies that $W(y,x)$ is a pure phase, so we can write $W(y,x) = e^{i\phi(y,x)}$. We will assume the convention that $W(x,y) = (W(y,x))^*$.

With Wilson lines, we can also define the covariant derivative in a given direction as follows. We set

$$n^\mu D_\mu \phi = \lim_{a\to 0} \frac{1}{a}\left[\phi(x+an) - W(x+an,x)\phi(x)\right]$$

One can check that this is gauge invariant by just transforming the expression in the brackets under $U(1)$, and we find that the whole term just adjusts by the phase $e^{i\alpha(x+an)}$. Since our n was arbitrary, we thus have that $\overline{\psi}D\!\!\!\!/\,\psi$ is gauge invariant. Now, from this Wilson line, we can define the gauge field A_μ as the infinitesimal parameter in the exponential. We write

$$W(x+an,x) = \exp\left[iean^\mu A_\mu\left(x + \frac{1}{2}an\right)\right]$$

If we expand our exponential in a since we're taking the limit where it's small, we get that the form our our covariant derivative is

$$D_\mu \phi(x) = \left[\partial_\mu - ieA_\mu(x)\right]\phi(x)$$

From doing the gauge transformations infinitesimally, we have the following identities

$$D_\mu \psi \to e^{i\alpha(x)} D_\mu \phi, \quad A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \phi(x)$$

Thus, since D_μ doesn't add anything extra to the transformation of a field which transforms as $\phi(x) \to e^{i\alpha(x)}\phi(x)$, we then get that the composite operator $D_\nu D_\mu$ doesn't add anything. In particular, this means that

$$[D_\mu, D_\nu]\phi \to ie^{i\alpha(x)} D_\nu (D_\mu \phi)$$

It's easy to see that

$$[D_\mu, D_\nu] = ie(\partial_\mu A_\nu - \partial_\nu A_\nu) = ieF_{\mu\nu}$$

This is precisely the electromagnetic field. Alternatively, we can build a 'plaquette', which is constructed as a square with vertices

$$\begin{aligned} y_1 &= x \\ y_2 &= x + a\hat{e}_1 \\ y_3 &= x + a(\hat{e}_1 + \hat{e}_2) \\ y_4 &= x + a\hat{e}_2 \end{aligned}$$

We can expand this about small a, and once we cancel everything we get something which looks like

$$P_{12}(x) = 1 - iea^2 F_{12}(x) + O\left(a^3\right)$$

So we get the electromagnetic tensor dropping out for free when we construct this square. On the other hand, Wilson lines need not be small and with the field A_μ being continuous, we can construct a curve C which is a Wilson line

$$W(z, y) = \exp\left[ie \int_C dx^\mu A_\mu(x)\right]$$

This is a loop if $y = z$, with $C = \partial\Sigma$ where Σ is the enclosed area. From Stokes' theorem, we get that

$$\oint_{C=\partial\Sigma} A_\mu dx^\mu = \frac{1}{2}\int_\Sigma F_{\mu\nu} d\sigma^{\mu\nu}$$

6.5 Fadeev-Popov Gauge Fixing

We'll start by explaining this concept with an analogy. Consider integrating the exponential of an action over two real variables instead of one;

$$Z = \int dxdy e^{-S(x)}$$

Now, in this case, since our action S depends on x only, we have that the variable y is redundant to any physics. However, nevertheless, integrating y from $(-\infty, \infty)$ still leads to a divergent integral. In $U(1)$, Abelian gauge theory, we have that we can just ignore this and remove y so that $Z = \int dx e^{-S(x)}$. However, this doesn't work in non-Abelian gauge theory. Instead, we require that

$$\begin{aligned} Z &= \int dxdy \delta(y) e^{-S(x)} \\ &= \int dxdy \delta(y - f(x)) e^{-S(x)} \end{aligned}$$

This fixes $y = 0$ and $y = f(x)$ respectively. Maybe we don't have y explicitly in terms of x, and in this case all we need is that $y = f(x)$ is the unique solution to the equation

$$\begin{aligned} G(x, y) &= 0 \\ f(x) &= y \end{aligned}$$

7 Formulating the Path Integral

In this section, we'll be moving on from our standard procedure of quantising a given Hamiltonian in quantum mechanics. We'll be introducing the concept of a path integral. The path integral Iis a 'functional integral' where we integrate over all possible paths with a Gaussian probability factor.

7.1 Classical and Quantum Mechanics

In classical mechanics, we use the Lagrangian as a conduit to encode the information about our physical system. The Lagrangian is given by a function of position and velocity, with

$$\mathcal{L} = \mathcal{L}(q_a, \dot{q}_a)$$

where $a = 1, \ldots, N$ is an index for each particle in our system. We can convert this to the Hamiltonian formalism where we put position and momentum on the same pedestal and define our conjugate momenta

$$p_a = \frac{\partial \mathcal{L}}{\partial \dot{q}^a}$$

We then work in terms of the Hamiltonian which is the Legendre transformation of the Lagrangian, where we eliminate $\dot{q}^a$ everywhere in the Lagrangian in favour of p^a as follows

$$H(q_a, p_a) = \sum_a \dot{q}_a p^a - \mathcal{L}(q_a, \dot{q}_a)$$

The quantum mechanical analog of this is the same. However, p_i and q_i are **promoted** to what we call operators, and obey commutation relations which as we know, eventually lead to discrete energy levels in the Hamiltonian. In quantum mechanics, we write the position and momentum operators as $\mathbf{q}^i$ and $\mathbf{p}^i$ for position and momentum respectively. In the Heisenberg picture of quantum mechanics, operators (and not states), depend on time. So, we impose the commutation relations for some fixed coordinate time $t \in \mathbb{R}$, where

$$\left[\mathbf{q}^i, \mathbf{p}_j\right] = i\delta^i{}_j$$

In classical field theory, we promote operators to fields instead. If $\phi(\mathbf{x}, t)$ represents a classical scalar field at some point in time t, then the field as well as it's conjugate momentum $\pi(\mathbf{x}, t)$ obey the commutation relations

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta^3(\mathbf{x} - \mathbf{y})$$

There is however a caveat in performing these approaches to quantisation. The theory is not manifestly Lorentz invariant. This is because when we imposed the equal time commutation relations above, we had to pick a preferred coordinate time t.

7.2 Formulating the Path Integral

We use the Hamiltonian as a starting point.

$$H(q,p) = \frac{p^2}{2m} + V(q)$$

The Schrödinger equation for a state $|\psi(t)\rangle$ which is time dependent is given by

$$i\frac{d}{dt}|\psi(t)\rangle = \hat{\mathbf{H}}|\psi(t)\rangle$$

Now, this is is a first order differential equation and can be solved provided that we have the right initial conditions. For now, let's just write down the solution in a 'formal' sense, where we 'exponentiate' the Hamiltonian whilst being vague about what this actually means. We write the solution as

$$|\psi(t)\rangle = \exp\left(-i\hat{\mathbf{H}}t\right)|\psi(0)\rangle$$

To do calculations however, we need construct an appropriate basis of states. For this section, we'll use the position basis $|q,t\rangle$, for $q,t \in \mathbb{R}$. These states are are defined to be the eigenstates of the position operator $\hat{\mathbf{q}}(t)$, so that

$$\hat{\mathbf{q}}(t)|q,t\rangle = q|q,t\rangle$$

We'll impose the condition that these states are normalised so that for a fixed time, we have

$$\left\langle q,t\middle|q',t\right\rangle = \delta\left(q-q'\right)$$

We impose the analogous conditions as well for momentum eigenstates. For now though, we'll work in the Schrodinger picture so that $\hat{\mathbf{q}}$ is fixed and hence we have that the eigenstates $|q\rangle$ are time-independent. Since these states form a basis, we have that they obey the completeness relation

$$1 = \int d^3q\ |q\rangle\langle q|$$

We also label the time-independent momentum eigenstates as $|p\rangle$, and impose the completeness relation

$$1 = \frac{d^3p}{(2\pi)^3}|p\rangle\langle p|$$

Note the factor of 2π that we divide by. Other literature doesn't include this. With this set of basis states, we can now write the abstract state $|\psi(t)\rangle$ in terms of the position basis, where we denote

$$\psi(q,t) = \langle q|\psi(t)\rangle = \langle q|\exp\left(-i\hat{\mathbf{H}}t\right)|\psi(0)\rangle$$

We will put this into an integral form for reasons we will discuss later. To put any equation in integral form, the rule of thumb is to employ the completeness relations for either the position or momentum basis. We get that

$$\begin{aligned}
\langle q| \exp\left(-i\hat{\mathbf{H}}t\right)|\psi(0)\rangle &= \int d^3q' \ \langle q| \exp\left(-i\hat{\mathbf{H}}t\right)\left|q'\right\rangle \left\langle q'\middle|\psi(0)\right\rangle \\
&= \int d^3q' \ \langle q| \exp\left(-i\hat{\mathbf{H}}t\right)\left|q'\right\rangle \psi(q',0) \\
&= \int d^3q' \, K\left(q,q';t\right)\psi(q',0) \\
&= \int d^3q' K(q,q';t)\psi(q',0)
\end{aligned}$$

Here we've defined $K(q,q';t) = \langle q| \exp\left(-i\hat{\mathbf{H}}t\right)|q'\rangle$. Now to make progress, we need to find a meaningful expression for what $K\left(q,q;T\right)$ actually is. First, 'split up' our $\exp\left(-i\hat{\mathbf{H}}T\right)$ term into smaller pieces - that is, partition T as

$$\exp\left(-i\hat{\mathbf{H}}T\right) = \exp\left(-i\hat{\mathbf{H}}\left(t_{n+1}-t_n\right)\right)\exp\left(-i\hat{\mathbf{H}}\left(t_n-t_{n-1}\right)\right)\dots\exp\left(-i\hat{\mathbf{H}}\left(t_1-t_0\right)\right) \tag{1}$$

here, we set by definition that $t_{n+1} = T > t_n > t_{n-1} > \cdots > t_1 > t_0 = 0$. For example, setting $n=1$ and inserting one integral as part of the completeness relation, we get that

$$\begin{aligned}
K\left(q,q';T\right) &= \langle q| \exp\left(-i\hat{\mathbf{H}}\left(t_2-t_1\right)\right)\exp\left(-i\hat{\mathbf{H}}\left(t_1-t_0\right)\right)\left|q'\right\rangle \\
&= \int dq_1 \ \langle q| \exp\left(-i\hat{\mathbf{H}}\left(t_2-t_1\right)\right)|q_1\rangle\langle q_1| \exp\left(-i\hat{\mathbf{H}}\left(t_1-t_0\right)\right)\left|q'\right\rangle
\end{aligned}$$

where we've set $t_2 = T$. We can generalise this to the case where we have n time slices. We have that

$$K\left(q,q',r\right) = \int \prod_{i=1}^{n}\left(dq_r \ \langle q_{r+1}| \exp\left(-i\hat{\mathbf{H}}\left(t_{r+1}-t_r\right)\right)|q_r\rangle\right)\langle q_1| \exp\left(-i\hat{\mathbf{H}}\left(t_1-t_0\right)\right)\left|q'\right\rangle \tag{2}$$

8 Review of Renormalisation

Let's review the steps we need to make a physical prediction.

Example Sheet 1

Question 1 (2020)

We use the completeness relation in the position basis.

$$\begin{aligned}\int dx' K(x,t,x',t')K\left(x',t';x_0,t_0\right) &= \int dx' \, \langle x| \, e^{-iH(t-t')} \left|x'\right\rangle \left\langle x'\right| e^{-iH(t'-t_0)} \left|x_0\right\rangle \\ &= \langle x| \, e^{-iH(t-t')} e^{-iH(t'-t_0)} \left|x_0\right\rangle \\ &= K(x,t,x_0,t_0)\end{aligned}$$

To show that $f(x) = \delta(x)$, we need to show that $f(x) = 0 \, \forall x \neq 0$, and that $\int dx \, f(x) = 1$.

Question 1 (2018)

We expand the exponential involving λ as

$$\begin{aligned}\mathcal{Z}\left(\lambda\right) &= \frac{1}{\sqrt{2\pi}} \int dx e^{-\frac{1}{2}x^2} \sum_{l=0}^{n} \left(-\lambda \frac{x^4}{4!}\right)^l \frac{1}{l!} \\ &= \sum_{l=0}^{n} \frac{1}{\sqrt{2\pi}} \left(-\frac{\lambda}{4!}\right)^l \frac{1}{l!} \int_{\mathbb{R}} dx \, e^{-\frac{1}{2}x^2} x^{4l}\end{aligned}$$

Now, we evaluate the integral using a trick. We arbitrarily set

$$I(\alpha) = \int dx \, e^{\frac{1}{2}\alpha x^2}$$

Differentiating this integral with respect to α, we have that

$$\frac{d^{2l} I}{d\alpha^{2l}} = \int_{\mathbb{R}} dx \, \left(\frac{1}{2}\right)^{2l} x^{4l} e^{-\frac{\alpha}{2}x^2} = \sqrt{2\pi} \left(\frac{1}{2}\right)^{2l} 1\,(3) \ldots (4l-1)$$

Cancelling out factors and using the standard formula for odd factorials, we get that

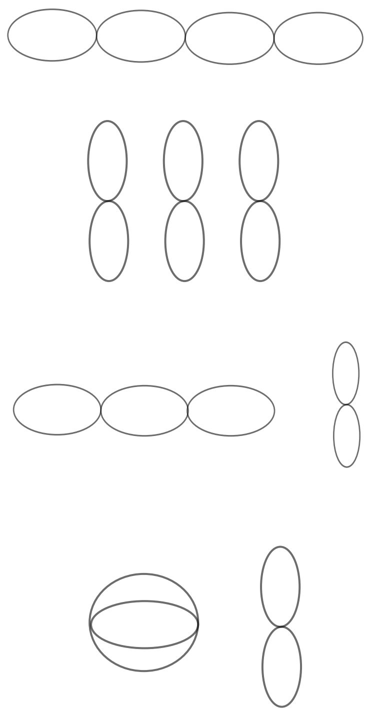
$$\int dx \, x^{4l} e^{-\frac{\alpha}{2}x^2} = \sqrt{2\pi} \frac{(4l)!}{4^l\,(2l)!}$$

Substituting this in means that we get our expression for our partition function as

$$\mathcal{Z}_n\left(\lambda\right) = \sum_{l=0}^{n} \left(-\frac{\lambda}{4!}\right)^l \frac{(4l)!}{4^l\,(2l)!}$$

Our contributing Feynman diagrams at $l \ll 3$ are shown in the figure. At $l = 1$, $a_l = \frac{1}{8}$, which is in agreement with a figure of 8 diagram. At $l = 2$, $a_l = \frac{35}{384}$, which agrees with the sum of the automorphism factors at 2 loops.

At $l = 3$, $a_l = \frac{385}{3072}$.



$\frac{1}{32}$

$\frac{1}{6} \times \frac{1}{512}$

$\frac{1}{8} \times \frac{1}{16}$

$\frac{1}{8} \times \frac{1}{48}$

Figure 2: Feynman diagrams and their automorphism factors

We need to sum multiple diagrams, which are connected with n loops to get terms in the expansion. There are two ways to get terms in the expansion. One is to sum all possible diagrams, the other is to sum connected diagrams with a certain number of loops!

What are the possible 3 loop diagrams? What are the automorphism factors? I've tried exponentiating the sum of connected vacuum bubbles - doesn't seem to add up!

9 Useful Identities

9.1 Integral Identities

- The gamma function is defined as

$$\Gamma\left(Z\right) = \int_0^\infty dx\, x^{z-1} e^{-x}$$