Part III Quantum Field Theory

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1 Why do we care about Quantum Field Theory?

If you're reading these set of notes, you've probably done a course at university on basic quantum mechanics. However, there are several philosophical and observed issues in quantum mechanics that we've yet to address. I'll go over them here.

1.1 The Problem of Locality

1.2 Particle indistinguishability and Particle conservation

In quantum mechanics, when we talk about multiparticle systems, we usually tensor product their state representation like $|\psi\rangle \otimes |\phi\rangle$, and we usually don't care about how these things interact. In quantum mechanics, we view particle number as conserved. When we do this however, and attempt to make Schrodinger's equation relativistic, we run into problems with regards to causality violation and unbounded energy. However, in QFT we have multiparticle states, with creation and annihilation operators which remedy this situation.

1.3 Problems with QFT

In these notes, we'll have a 'free theory' where particles don't interact, and we solve for states analytically. Later, we'll also add on 'interaction terms', where particles do interact and we have to resort to perturbation techniques to study physical quantities of interest. Within these notes, we assume that these interaction terms are small. Physics where we try to understand how to deal with larger interactions is still a big area of research (see lattice QCD where we try to discretise space time and solve QFT numerically).

1.4 Scaling in QFT

Throughout QFT, we scale out the natural constants $c, \hbar$ to have $c = \hbar = 1$. Because of this, we can form relationship between units of time, length and mass. c is a unit of speed, but since this is set to 1 we have that

$$[L][T]^{-1} = 1 \implies [L] = [T]$$

So we have that in this regime, which we call 'natural units' length has units of time. In immediate consequence of this is that energy has the units of mass.

$$[E] = [M][L]^2[T]^{-2} = [M]$$

Our condition that the Planck constant goes to 1 implies that

$$[M][L]^2[T]^{-1} = 1 \implies [M] = [L]^{-1}$$

2 Classical Field Theory

2.1 Euler Lagrange equations for a point particle

We'll start off by reviewing some ideas in classical mechanics to motivate our approach to quantum field theory. In particular, the simple harmonic oscillator is a good place to start. In classical mechanics, we describe the motion of a particle $\mathbf{x}$ as a function of time, so $\mathbf{x} = \mathbf{x}(t)$. In high school, we learnt that Newton's laws dictate that in one dimension, mass times acceleration is equal to the force on the particle. So, in the case of the simple harmonic oscillator,

$$m\ddot{x} = -kx$$

, where k is our spring constant. We then learnt as undergraduates that this is a more specific case of the motion of a particle in a potential V, where in the case of a simple harmonic osccilator, $V(x) = \frac{1}{2}kx^2$. In this formalism, our equations of motion are governed by the equation

$$m\ddot{x} = -\nabla V$$

You can check that this recovers the above equation of motion.

Generalising one step further, we see that we can encode all of this infomation into a single useful quantity, the Lagrangian, which is the quantity $L = T - V$, where T is a kinetic energy term. The Lagrangian should always be a scalar function, and in the case of our harmonic oscilator in 3 dimensions it is

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 - \frac{1}{2}k\mathbf{x}^2$$

To derive the equations of motion, we want to apply the principle of least action. This principle is that the particle will move along a path which minimises the integral of the Lagrangian over time. In other words, to find out the path of the particle between to points in time, say t_1 and t_2, we would like to minimise the action.

$$S = \int_{t_1}^{t_2} L(\mathbf{x}, \dot{\mathbf{x}})dt$$

A condition that we impose here is that the endpoints should be fixed, so $\mathbf{x}(t_1) = \mathbf{x}_1$, and $\mathbf{x}(t_2) = \mathbf{x}_2$. To minimise the action, we vary the curve a tiny bit by sending $\mathbf{x} \to \mathbf{x} + \delta\mathbf{x}$. This also induces a variation in $\dot{\mathbf{x}}$. We use the chain rule with respect to the variables $\mathbf{x}$ and $\dot{\mathbf{x}}$ and to give

$$\delta S = \int_{t_1}^{t_2} \delta\mathbf{x} \cdot \frac{\partial L}{\partial \mathbf{x}} + \delta\dot{\mathbf{x}} \cdot \frac{\partial L}{\partial \dot{\mathbf{x}}}$$

But, we can perform integration by parts on the second term, by integrating the $\delta\dot{\mathbf{x}}$ and differentiating in the integrand. This gives us a surface term which we can push to zero.

$$\delta S = \left[\delta\mathbf{x} \cdot \frac{\partial L}{\partial \dot{\mathbf{x}}}\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} dt\ \delta\mathbf{x} \cdot \left(\frac{\partial L}{\partial \mathbf{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{x}}}\right)\right)$$

Now, since our variation was arbitrary, we have that the integrand needs to be zero, so we have the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{x}}}\right) = \frac{\partial L}{\partial \mathbf{x}}$$

2.2 Promotion of Point Particles to the Classical Field

Previously, we were formulating our problem in a point particle sense. We now alter the physics slightly. Instead of a point particle, suppose we assign a value to every point in spacetime, and work with that instead. This function $\phi = \phi(x)$, we call a **field**. We write $\phi(x)$ synonymously with $\phi(\mathbf{x}, t)$. In this regime, much like how we previously viewed i as a label for a spatial coordinate in x^i, we now have **infinite degrees of freedom**, where now the $\mathbf{x}$ term in $\phi(\mathbf{x}, t)$ acts as a continuous, infinite label. In essense, fields are like coordinates but with infinite degrees of freedom. In fact, we don't have to stop there.

We can attach components to this field, thus adding a few more degrees of freedom, and indexing them like $\phi_a(\mathbf{x}, t)$. So, if we have a indexing over $a = 0, 1, 2, 3$, then we've added an additional four degrees of freedom.

Our most familiar example of this are the electromagnetic fields (which yield 6 degrees of freedom in all). These are $E_i(\mathbf{x}, t)$ and $B_i(\mathbf{x}, t)$ where our index labels spatial directions. These six fields are derived from the familiar electromagnetic vector potential $A_\mu = (\phi, \mathbf{A})$, given by the following

$$
\begin{aligned}
E_i &= -\frac{\partial A_i}{\partial t} - \frac{\partial A_0}{\partial x^i} \\
B_i &= \epsilon_{ijk} \frac{\partial A_k}{\partial x_j}
\end{aligned}
$$

Since we have a field over space now, our Lagrangian itself needs to be an integral over space. So, we formulate our problem in terms of something we call the Lagrangian density.

$$L = \int d^3x \, \mathcal{L}(\phi, \partial_\mu \phi)$$

where our density now is a function of both the field itself and also the first derivative (not so different from the point particle case). And, like the point particle case, our action is once again the integral over time so that

$$S = \int dt \, L = \int dt \int d^3x \, \mathcal{L} = \int d^4x \ \mathcal{L}$$

Our Euler Lagrange equations are derived in the same way, but in this case over the 4 space-time coordinates, not just the time coordinate like we did earlier. Our change in action is

$$
\begin{aligned}
\delta S &= \int d^4x \, \delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + \delta \partial_\mu \phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \\
&= \int d^4x \, \delta\phi \frac{\partial \mathcal{L}}{\partial \phi} - \delta\phi \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}
\end{aligned}
$$

And since our variation $\delta S = 0$, we have the Euler-Lagrange equations

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

2.2.1 The Klein Gordon field

We'll now introduce one of the main equations for a free field theory, the Klein-Gordon field. The Klein-Gordon field looks roughly like a Lagrangian for a simple harmonic oscillator

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2$$

$\eta^{\mu\nu}$ denotes our Minkowski metric, which in the course of these notes we'll take to be the 'mostly negative' metric $\eta^{\mu\nu} = diag(+1, -1, -1, -1)$. In terms of time and spatial derivatives, our Lagrangian reads

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2$$

Which is reminiscent of our Lagrangian being of the form $L = T - V$, where $T = \frac{1}{2}\dot{\phi}^2$ is our kinetic term and the rest is our potential term. We can compute our equations of motion from this. We have that

$$\frac{\partial \mathcal{L}}{\partial \phi} = m^2\phi, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi$$

where we've used the chain rule for differentiation both times. Substituting this into the Euler-Lagrange equations, we have the Klein-Gordon equation which reads

$$\partial_\mu\partial^\mu\phi - m^2\phi = 0$$

If we have an ansatz for a plane wave solution, that $\phi(x) = e^{-ip\cdot x}$, then this yields the relataivistic dispersion relation that

$$p^\mu p_\mu = m^2$$

which we know is true.

2.2.2 The Lagrangian for Electromagnetism

Another important example of a Lagrangian we'll be encountering is the Lagrangian associated with electromagnetism. This is given by

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A^\mu)^2$$

where A^μ is our familiar vector potential we described earlier. Notice that this Lagrangian doesn't depend on A^μ, so if we want to figure out the equations of motion we need to calculate the term $\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}$. Using the identity that

$$\frac{\partial(\partial^\alpha A^\beta)}{\partial_\mu A_\nu} = \eta^{\alpha\mu}\eta^{\beta\nu}$$

our use of the product rule gives us that

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} &= -\partial^\mu A^\nu + \partial_\rho A^\rho \frac{\partial(\partial_\alpha A^\alpha)}{\partial(\partial_\mu A_\nu)} \\
&= -\partial^\mu A^\nu + \partial_\rho A^\rho \eta_\alpha{}^\mu \eta^{\alpha\nu} \\
&= -\partial^\mu A^\nu + \partial_\rho A^\rho \eta^{\mu\nu}
\end{aligned}$$

Our Euler Lagrange condition implies that

$$0 = \partial_\mu(-\partial^\mu A^\nu) + \partial^\nu \partial_\rho A^\rho = \partial_\mu(-\partial^\mu A^\nu + \partial^\nu A^\mu)$$

where in the second term, we relabelled the summed over indices from $\rho \to \mu$ because they're dummy indices. za

2.3 Lorentz invariance

In QFT and special relativity, theories should be invariant under Lorentz transformations. In this section, we will consider active transformations, where we're changing the direction of the field under a Lorentz boost. Under a Lorentz boost, our scalar field ϕ changes like

$$\phi(x) \to \phi'(x) = \phi(x') = \phi(\Lambda^{-1}x)$$

This is what we call an **active transformation** since our field is actually, physically shifted. We're shifting our field by a Lorentz boost in our frame, with $x \to \Lambda x$ which induces a change $\phi(x) \to \phi'(x)$, but this change is equivalent to rotation our frame of reference in the opposite direction first (I'll show a diagram below). Our Lorentz boosts are defined by the property that

$$\Lambda^\mu{}_\rho \eta^{\rho\tau} \Lambda_\tau{}^\nu = \eta^{\mu\nu}$$

Lorentz boosts can simultaneously represent rotations in 3-space, as well as boosts in a particular axis in which mixes time and space. For a rotation, our boost is represented as

$$\Lambda^\mu{}_\rho = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

and for a boost, Lorentz transformations are represented by

$$\Lambda^\mu{}_\rho = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In terms of group theory, Lorentz transformations are representations of the Lie group $SO(3,1)$ on scalar fields. Let's look at Lorentz invariant theories, which are theories where our physical action S is unchanged by Lorentz transformations. Consider the action

$$S = \int d^4x\, \partial_\mu\phi\partial^\mu\phi + U(\phi(x))$$

We've declared here that $U(\phi(x))$ is some polynomial of $\phi(x)$. How does this object change under transformations of our field $\phi(x) \to \phi'(x)$? Let's do the polynomial term first. Under the active transformation $\phi(x) \to \phi'(x) = \phi(x')$, we have that

$$U(x) = U(\phi(x)) \to U(\phi'(x)) = U(\phi(x')) = U(x')$$

This means that $U(x) \to U(x')$. Now, to do the kinetic part of the Lagrangian. To do this, we observe that our partial derivative of a field transforms as

$$\partial_\mu \phi \to \partial_\mu \phi'(x) = \partial_\mu \phi(x') = (\Lambda^{-1})^\rho{}_\mu \partial'_\rho \phi(x'), \quad x' = \Lambda^{-1} x$$

Something important to note here is that we're **not** transforming x simultaneously here as well, since this is an active transform of the actual field. We've only expressed x diffferently to express our change in our field in terms of new coordinates, but the same function. Thus, our Kinetic term transforms as

$$\begin{aligned}\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \to \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi'\partial_\nu\phi' \\ = \frac{1}{2}\eta^{\mu\nu}(\Lambda^{-1})^\sigma{}_\mu(\Lambda^{-1})^\tau{}_\nu\partial'_\rho(x')\partial'_\tau(x') \\ = \frac{1}{2}\eta^{\rho\tau}\partial'_\rho\phi(x')\partial'_\tau(x')\end{aligned}$$

where we've used the transformation property of the Minkowski metric. Thus, every term in our Lagrangian density has replaced x with a modified $x' = \Lambda^{-1}x$. We thus have that our action changes like

$$S = \int d^4x\, L(x) \to S' = \int d^4x\, L(x'), \quad x' = \Lambda^{-1}x$$

However, since the determinant (associated Jacobian) of our Lorentz boost is 1 (as Lorentz boosts are part of the special group $SO(3,1)$, we have that our measure doesn't change:

$$d^4x = det(\Lambda)d^4x' = d^4x' \implies S' = \int d^4x'\, L(x') \implies S = S'$$

In the last step we have equality since we're integrating over a dummy variable! Thus, Lagrnagians of this form are Lorentz invariant.

2.4 Symmetries and Noether's theorem

We'll now take a look at the role of symmetries in this formalism. In physics, we can have multiple types of symmetries, including Lorentz symmetries, internal symmetries, gauge symmetries and supersymmetries. In terms of Lagrangians, we'll now show that every continuous symmetry of a Lagrangian density gives rise to a conserved current j^μ, which satisfies the conservation condition

$$\partial_\mu j^\mu = 0$$

It's easy to check that, from this quantity, we can construct a conserved charge given by

$$Q = \int d^3x\, j^0$$

This is converved with respect to changes in time since

$$\dot{Q} = \int d^3x\, \frac{\partial j^0}{dt} = -\int d^3x\, \nabla \cdot \mathbf{j} \to 0$$

where the final term goes to zero, since the divergence theorem allows us to re-express this as a surface integral. We're assuming that $j^\mu(x) \to 0$ as $x^\mu \to 0$. What do we mean by a symmetry? An infinitesimal transformastion of our scalar field

$$\phi(x) \to \phi(x) + \alpha\delta\phi(x)$$

where α is small is a symmetry provided that the Lagrangian only changes by a total four derivative:

$$\mathcal{L} \to \mathcal{L} + \partial_\mu F^\mu$$

for some function F^μ. This is a symmetry because since our action integrates over our Lagrangian density, due to the divergence theorem this extra term vanishes, leaving our action S invariant. We can now derive Noether's theorem. If we Taylor expand out our Lagrangian,

$$L(x) \to L(x) + \alpha\frac{\partial L}{\partial\phi}\delta\phi + \alpha\frac{\partial L}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi)$$

We can rewrite the above term as

$$L(x) + \alpha\partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\phi)}\right)\delta\phi + \alpha\delta\phi\left(\frac{\partial L}{\partial\phi}\right) - \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\phi)}\right)$$

But, from the Euler Lagrange equations the final term vanishes, and comparing with our total derivative term we find that a conserved current is

$$j^\mu = \frac{\partial L}{\partial(\partial_\mu\phi)}\delta\phi - F^\mu$$

Let's consider a Lagrangian with a complex scalar field ψ given by

$$\psi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + \phi_2(x))$$

We treat $\psi(x)$ and $\psi(x)^*$ as separate fields, and write a complex scalar lagrangian as

$$\mathcal{L} = \partial_\mu \psi \partial^\mu \psi^* - m^2 \psi^* \psi - \frac{\lambda}{2}(\psi^* \psi)^2$$

It's clear that this Lagrangian is invariant under the phase symmetry

$$\psi \to e^{i\alpha}\psi, \quad \psi^* \to e^{-i\alpha}\psi^*$$

This induces the infinitesimal symmetry

$$\delta\psi = i\alpha\psi, \quad \delta psi^* = -i\alpha\psi^*$$

With Noether's theorem, one can verify that the conserved current is

$$j^\mu = i(\psi\partial^\mu\psi^* - \psi^*\partial^\mu\psi)$$

Physically, conserved charges could be anything from conserved electric charge, or conserved particle number such as baryon or lepton number.

2.4.1 The Energy-Momentum Tensor

In this subsection, we'll be constructing important conserved quantities from Noether's theorem. These conserved quantities will arise specifically from translational symmetry in spacetime. In classical mechanics, energy conservation arose from time translational symmetry, and momentum conservation arose from spatial translational symmetry. In classical field theory, this is no different. Our concept of conserved energy and momentum arise from translational symmetry in our Lagrangian, and we'll combine them to form a single conserved tensor. Consider a translation transformation in our spacetime coordinates

$$x^\nu \to x^\nu + \alpha\epsilon^\nu$$

We expect that our symmetry here will give rise to a higher rank tensor because we have 4 linearly independent directions arising from ϵ^ν. Taylor expanding out, we have

$$\phi_a(x) \to \phi_a(x) - \alpha\epsilon^\nu\partial_\nu\phi_a(x)$$

And this induces a similar transformation on our Lagrangian, since $\mathcal{L} = \mathcal{L}(\phi(x)) = \mathcal{L}(x)$, so we can Taylor expand our expression in exactly the same way.

$$\mathcal{L}(x) \to \mathcal{L}(x) - \alpha\epsilon^\nu\partial_\nu\mathcal{L}(x) = \mathcal{L}(x) - \alpha\epsilon^\nu\partial_\mu(\delta^\nu{}_\mu\mathcal{L})$$

In the second expression, we've written things in a slightly weird yet more suggestive manner. Our rationale is as follows. Since ϵ^ν has four degrees of freedom, we can write this out as four separate conserved currents as written in the bracket, where the distinct currents are indexed by ν. Hence, with Noether's theorem, we can construct four different Noether currents using this form of $\delta\mathcal{L}$.

$$T^\mu{}_\nu = (j^\mu)_\nu = \partial_\nu\phi_a\frac{\partial\mathcal{L}}{\partial_\mu\phi_a} - \delta^\mu{}_\nu\mathcal{L}, \quad \partial_\mu T^{\mu\nu} = 0$$

Different components of this tensor correspond to different quantities. We have

$$\text{Total Energy } E = \int d^3x T^{00}$$
$$\text{Total momentum } P^i = \int d^3x T^{i0}$$

We apply this to our Klein-Gordon field.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

This has an associated energy-momentum tensor, where raising the index on the delta function yields the Minkowski

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \eta^{\nu\mu}\mathcal{L}$$

Substituting our expression for conserved energy, and momentum

$$E = \int d^3x T^{00} = \frac{1}{2}\int d^3x\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2, \quad P^i = \int \dot{\phi}\partial^i\phi$$

If $T^{\mu\nu}$ is non-symmetric, we can massage it into a form that's symmetric which makes it easier to deal with by adding an arbitrary, antisymmetric 'gauge' term

$$T^{\mu\nu} \to T^{\mu\nu} + \partial_\rho\Gamma^{\rho\mu\nu}, \quad \Gamma^{(\rho\mu)\nu} = 0$$

This doesn't affect our conservation law, because we have

$$\partial_\mu\partial_\rho\Gamma^{\rho\mu\nu} = 0$$

2.5 Switching over to the Hamiltonian Formalism

In quantum mechanics, we're already used to using the Hamiltonian H to investigate the energy spectrum and energy eigenstates of a system. Right now, we'll build a way to switch over from our current Lagrangian understanding of a scalar field to a derivation of the associated Hamiltonian. To do this, we need to define what conjugate momenta, π is. Conjugate momenta is defined as

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}$$

where $\dot{\phi}$ is the partial derivative of our field with respect to time. Perhaps some of our intuition about this is as follows. If we were given the Lagrangian $L = \frac{1}{2}\mathbf{x}\cdot\mathbf{x} - V(\mathbf{x})$ in the discrete particle case, then our conjugate momenta is given by

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\mathbf{x}}}$$

which is in this case is $\dot{\mathbf{x}}$, the expression for our standard notion of momentum with unit mass.

In the discrete particle case, our Hamiltonian is related to our Lagrangian by a Legendre transform, where

$$H = \sum_i \pi \cdot \dot{\mathbf{x}} - L$$

Our sum over i denotes the sum of over particles in the system. However, in field theory, we make this continuous by integrating over space instead, and then writing in the Lagrangian density in the integrand. Thus, our expression for the Hamiltonian in field theory is

$$H = \int d^3x \; \pi(x)\phi(x) - \mathcal{L}$$

In the case of the Lagrangian density, we can calculate our conjugate momenta

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi), \quad \phi = \dot{\phi}$$

This gives ys an expression for our Hamiltonian density $\mathcal{H}$, which is defined as the integrand of $H = \int d^3x\mathcal{H}$.

$$\mathcal{H} = \frac{1}{2}(\pi^2 + (\nabla\phi)^2) + V(\phi)$$

Completely analogously to Hamilton's equations in classical dynamics, we have Hamilton's equations defined in terms of partial field derivatives of our Hamiltonian density.

$$\dot{\phi} = \frac{\partial\mathcal{H}}{\partial\pi}, \quad \dot{\pi} = -\frac{\partial\mathcal{H}}{\partial\phi}$$

Our Hamiltonian formalism is not obviously a Lorentz invariant theory, but it is!

2.6 Promoting discrete operators to fields

We know from standard quantum mechanics how to find the spectrum of the quantum harmonic oscillator H for unit mass, given by

$$H = \frac{1}{2}p^2 + \frac{1}{2}m\omega^2 q^2.$$

p, q represent our momentum and position operators respectively. Last year, we promoted generalised coordinates involving position and momentum to operator values obeying commutation relations. One recalls that we have our canonical commutation relations (which we can derive by examining infinitesimal transformations)

$$[p_a, p_b] = [q_a, q_b] = 0, \quad [q_a, p^b] = i{\delta_a}^b.$$

We can intuitively promote our scalar and momentum fields in the same way, where now a field is an operator valued function of space and time. For simplicity's sake, let's work in the Schrödinger picture where these operators don't depend on time but just space (and we push all our time dependence on states instead)

$$\begin{aligned}
[\phi_a(\mathbf{x}), \pi^b(\mathbf{y})] &= i\delta^3(\mathbf{x} - \mathbf{y}){\delta_a}^b \\
[\phi_a(\mathbf{x}), \phi^b(\mathbf{y})] &= [\pi_a(\mathbf{x}), \pi^b(\mathbf{y})] = 0.
\end{aligned}$$

Our final goal is to compute the spectrum from our Hamiltonian derived from quantum fields. However, this is a hard thing to do. In a free theory however, at any given point in space time, the field evolves independently on other points, and so one useful thing to do is find a basis where we can diagonalise this Hamiltonian. One thing we could do is to Fourier transform $\phi(\mathbf{x})$ into momentum space, so that (where we'll put back in our time dependence)

$$\phi(\mathbf{x}, t) = \int \frac{d^3x}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{p}, t)$$

Substituting this into the Klein-Gordon equation by differentiating from within the integrand gives us a simple harmonic oscillator with a momentum dependent frequency, we have the equation

$$\left(\frac{\partial}{\partial t^2} + (\mathbf{p}^2 + m^2) \right) \phi(\mathbf{p}, t) = 0$$

This is a simple harmonic oscillator vibrating at frequency

$$\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$$

So, we have that for a field satisfying the free theory, at every point, we have an infinite superposition of simple harmonic oscillators!

2.7 Reviewing the Harmonic Oscillator in One Dimension in Quantum Mechanics

Going back to the problem of the QM harmonic oscillator, our Lagrangian and Hamiltonian are given by

$$H = \frac{1}{2}p^2 + \frac{1}{2}m\omega^2 q^2, \quad L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2$$

Recall that we can reduce our description of the discrete Hamiltonian via creation and annihilation operators, with their important commutation relation

$$a = \left(\sqrt{\frac{\omega}{2}} q + i\sqrt{\frac{1}{2\omega}} p \right), \quad a^\dagger = \left(\sqrt{\frac{\omega}{2}} q - i\sqrt{\frac{1}{2\omega}} p \right), \quad [a, a^\dagger] = 1$$

We can invert these to give expressions for momentum and position as

$$q = \frac{1}{\sqrt{2\omega}}(a + a^\dagger) \quad , p = -\frac{i}{\sqrt{2\omega}}(a - a^\dagger)$$

The whole point of the exercise we've done above is to write our Hamiltonian in terms of $a, a^\dagger$, and hence make use of our commutation relations. We can write our Hamiltonian as follows, with the raising and lowering commutation relations

$$H = \omega \left(a^\dagger a + \frac{1}{2} \right), \quad [H, a^\dagger] = \omega a^\dagger, \quad [H, a] = -\omega a$$

This has the effect of raising and lowering energy eigenstates, which can be shown by applying the Hamiltonian to a raised or lowered state and then using our commutation relations which we derived earlier.

$$H a^\dagger |E\rangle = (E + \omega) a^\dagger |E\rangle$$
$$H a |E\rangle = (E - \omega) a |E\rangle$$

Since we can step up or step down energy in this way, we've given rise to a ladder of energy values

$$\ldots, E - 2\omega, E - \omega, E, E + \omega, E + 2\omega, \ldots$$

There's a caveat here because we can have infinite negative energy! Thus, lowering by a has to stop somewhere. This means that there's a unique state $|0\rangle$such that

$$a\,|0\rangle = 0 \implies H\,|0\rangle = \frac{\omega}{2}\,|0\rangle$$

We hence have a unique ground state with energy $\omega/2$! We call this the zero point energy. Ignoring normalisation, we can step energy up n amount of times by applying raising operators, so that

$$|n\rangle = (a^\dagger)^n\,|0\rangle\,, \implies H\,|n\rangle = (n + \frac{1}{2})\omega\,|n\rangle$$

In physics however, we usually only care about energy differences, so this non-zero zero point energy is a bit annoying. We apply a procedure called normal ordering, where we reorder terms in an expression so that lowering terms are shoved to the right and raising terms are shoved to the left. The has the notation $: A :$, so in the case of the Hamiltonian

$$: H := \frac{1}{2}(: a^\dagger a : + : aa^\dagger :)\omega = \omega a^\dagger a$$

We now have the effect that our zero point energy is set to zero, with $H\,|0\rangle = 0$. We get for free that we can solve for this wavefunction. Since our momentum operator in position space is represented as $p = -i\frac{\partial}{\partial q}$, substituting this into $a\,|0\rangle = 0$ gives us a differential equation for the state $|0\rangle$ one we contract this in the position basis.

Rearranging, our expressions for the operators x, p are

$$x = \frac{1}{\sqrt{2\omega}}\left(a + a^\dagger\right), p = -i\sqrt{\frac{\omega}{2}}\left(a - a^\dagger\right).$$

This gives rise to commutation relations, which we will show below.

2.8 Promoting fields to operators

From our definitions for position and momentum operators in the one dimensional case, we can generalise this notion to scalar and conjugate momentum fields. We do this by expanding over all modes, in what looks like a Fourier decomposition. Another motivation for the following definitions is the fact that our operators should be hermitian, which is an extra reason for why we've included terms for both operators.

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right)$$

Again using the form of momentum in 1 dimensional quantum mechanics to motivate definitions, we have a definition for conjugate momentum in terms of Fourier modes as well. This is given by

$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right)$$

This procedure is called second quantization, we've reconstructed the scalar and momentum fields completely in terms of and infinite amount of simple harmonic oscillators in momentum space. Again, just as in quantum mechanics, we should impose the commutation relations that

$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = [\pi(\mathbf{x}), \pi(\mathbf{y})] = 0, \quad [\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y})$$

However, there's an equivalence here which we'll prove, which is that if we impose that our commutation relations in terms of $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$ as

$$[a_{\mathbf{p}}, a_{\mathbf{q}}] = [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = 0, \quad [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3\delta^3(\mathbf{p} - \mathbf{q})$$

then this induces the commutation relations in terms of $\phi(\mathbf{x})$ and $\pi(\mathbf{x})$, and vice versa.

Theorem. The commutation relations above are equivalent to each other.

Proof. We'll prove first that the commutation relations on our scalar and momentum fields imply the commutation relations on the annihilation and creation operators. To do do this, we need sensible expansions of these. One can verify with our standard delta function identity that

$$\int d^3x\, \phi(\mathbf{x})e^{-i\mathbf{p}\cdot\mathbf{x}} = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}a_{\mathbf{p}} + \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}a_{-\mathbf{p}}^\dagger$$
$$\int d^3x\, \pi(\mathbf{x})e^{-i\mathbf{p}\cdot\mathbf{x}} = -i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}a_{\mathbf{p}} + i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}a_{\mathbf{p}}^\dagger$$

Solving this system gives us an expression for $a_{\mathbf{p}}$. Then, contracting the integral but with $e^{i\mathbf{p}\cdot\mathbf{x}}$ instead gives us $a_{\mathbf{p}}^\dagger$. Hence, we have the expressions:

$$a_{\mathbf{p}} = \sqrt{\frac{\omega_{\mathbf{p}}}{2}}\int d^3x\, \phi(\mathbf{x})e^{-i\mathbf{p}\cdot\mathbf{x}} + i\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\int d^3x\, \pi(\mathbf{x})e^{-i\mathbf{p}\cdot\mathbf{x}}$$
$$a_{\mathbf{p}}^\dagger = \sqrt{\frac{\omega_{\mathbf{p}}}{2}}\int d^3x\, d^3x\, \phi(\mathbf{x})e^{i\mathbf{p}\cdot\mathbf{x}} - i\sqrt{\frac{1}{2\omega_{\mathbf{p}}}}\int d^3x\, \pi(\mathbf{x})e^{i\mathbf{p}\cdot\mathbf{x}}$$

Now we integrate these over two variables to get

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = [\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\int d^3x\, \phi(\mathbf{x})e^{-i\mathbf{p}\cdot\mathbf{x}} + i\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\int d^3x\, \pi(\mathbf{x})e^{-i\mathbf{p}\cdot\mathbf{x}}$$
$$, \sqrt{\frac{\omega_{\mathbf{q}}}{2}}\int d^3y\, \phi(\mathbf{y})e^{i\mathbf{y}\cdot\mathbf{q}} - i\sqrt{\frac{1}{2\omega_{\mathbf{q}}}}\int d^3y\, \pi(\mathbf{y})e^{i\mathbf{y}\cdot\mathbf{q}}]$$

We pull the commutators by linearity into the integral. Now, to make our lives easier, since the scalar fields commute with other scalar fields, and since the momentum fields commute with other momentum fields, we have that the above expression

$$\begin{aligned}
[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] &= -\frac{i}{2}\sqrt{\frac{\omega_{\mathbf{p}}}{\omega_{\mathbf{q}}}}\int d^3x d^3y\, e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{q}\cdot\mathbf{y}}[\phi(\mathbf{x}), \pi(\mathbf{y})] + \frac{i}{2}\sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{p}}}}\int d^3x d^3y\, e^{i\mathbf{q}\cdot\mathbf{y}} e^{-i\mathbf{p}\cdot\mathbf{x}}[\pi(\mathbf{x}), \phi(\mathbf{y})] \\
&= -\frac{i}{2}\sqrt{\frac{\omega_{\mathbf{p}}}{\omega_{\mathbf{q}}}}\int d^3x d^3y\, i\delta(\mathbf{x}-\mathbf{y}) e^{-i\mathbf{p}\cdot\mathbf{x}} e^{i\mathbf{q}\cdot\mathbf{y}} + \frac{i}{2}\sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{p}}}}\int d^3x d^3y\, -i\delta(\mathbf{x}-\mathbf{y}) e^{i\mathbf{q}\cdot\mathbf{y}} e^{-i\mathbf{p}\cdot\mathbf{x}} \\
&= \frac{1}{2}\sqrt{\frac{\omega_{\mathbf{p}}}{\omega_{\mathbf{q}}}}\int d^3x\, e^{i(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}} + \frac{1}{2}\sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{p}}}}\int d^3x\, e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} \\
&= \frac{1}{2}\sqrt{\frac{\omega_{\mathbf{p}}}{\omega_{\mathbf{q}}}}\delta(\mathbf{q}-\mathbf{p})(2\pi)^3 + \frac{1}{2}\sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{p}}}}\delta(\mathbf{p}-\mathbf{q})(2\pi)^3 \\
&= \delta(\mathbf{p}-\mathbf{q})(2\pi)^3, \qquad \text{since } \omega_{\mathbf{q}} = \omega_{\mathbf{p}} \text{ when } \mathbf{p} = \mathbf{q}
\end{aligned}$$

Similarly, in the opposite direction, we have that

$$\begin{aligned}
[\phi(\mathbf{x}), \pi(\mathbf{y})] &= \int \frac{d^3p d^3q}{(2\pi)^3}\,(-i)\sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{p}}}}\left(-[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}]e^{i\mathbf{x}\cdot\mathbf{p}-i\mathbf{q}\cdot\mathbf{y}} + [a^\dagger_{\mathbf{p}}, a_{\mathbf{q}}]e^{-i\mathbf{p}\cdot\mathbf{x}+i\mathbf{q}\cdot\mathbf{y}}\right) \\
&= \int \frac{d^3p}{(2\pi)^3}\,(-\frac{i}{2})\left(-e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right) \\
&= i\delta(\mathbf{x}-\mathbf{y})
\end{aligned}$$

□

2.8.1 Calculating the Hamiltonian

Now that we've promoted our scalar and momentum fields as an infinite expansion of annihilation and creation operators, we can start computing objects of interest in terms of Fourier expansions of these operators as well. We'll start by calculating the most important quantity, our Hamiltonian. This is given by, as we calculated earlier,

$$H = \int d^3x \left(\frac{1}{2}\Pi^2 + \frac{1}{2}\left(\nabla\phi\right)^2 + \frac{1}{2}m^2\phi^2\right)$$

We go term by term, and first calculate the most tricky term

$$\begin{aligned}
\int d^3x\,(\nabla\phi)^2 &= \int \frac{d^3x d^3p d^3q}{(2\pi)^3} - \frac{1}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\mathbf{p}\cdot\mathbf{q}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} - a^\dagger_{\mathbf{p}}e^{-i\mathbf{x}\cdot\mathbf{p}}\right)\left(a_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}} - a^\dagger_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{x}}\right) \\
&= \int \frac{d^3x d^3p d^3q}{(2\pi)^3} - \frac{1}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\mathbf{p}\cdot\mathbf{q}(a_{\mathbf{p}}a_{\mathbf{q}}e^{i(\mathbf{q}+\mathbf{q})\cdot\mathbf{x}} \\
&+ a^\dagger_{\mathbf{p}}a^\dagger_{\mathbf{q}}e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} - a^\dagger_{\mathbf{p}}a_{\mathbf{q}}e^{i(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}} - \mathbf{q}^\dagger a_{\mathbf{p}}e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}})
\end{aligned}$$

But recall that we can integrate over $\mathbf{x}$ first, and make use of the identity

$$\int d^3x e^{i\mathbf{a}\cdot\mathbf{x}} = (2\pi)^3\delta(\mathbf{a})$$

This means that the above expression is equal to

$$\int d^3x\,(\nabla\phi)^2 = \int \frac{d^3p}{(2\pi)^3}(-1)\frac{\mathbf{p}\cdot\mathbf{p}}{2\omega_{\mathbf{p}}}\left(a_{\mathbf{p}}a_{-\mathbf{p}} - a^\dagger_{\mathbf{p}}a_{\mathbf{p}} - a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + a^\dagger_{\mathbf{p}}a^\dagger_{-\mathbf{p}}\right)$$
$$= \int \frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}^2}{2\omega_{\mathbf{p}}}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + a_{\mathbf{p}}a^\dagger_{\mathbf{p}}\right)$$

Going into the second line, we've vanished the first and last terms since the function depends on just the modulus of p, but taking the change of variables from $\mathbf{p} \to \mathbf{p}$ means that the expression is equal to the negative of itself. It's easy to show that

$$\int d^3\Pi^2 = 0, \quad \int d^3x\,\frac{1}{2}m^2\phi^2 = \int \frac{d^3p}{(2\pi)^3}\frac{m^2}{2\omega_{\mathbf{p}}}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + a_{\mathbf{p}}a^\dagger_{\mathbf{p}}\right)$$

Hence our final expression for our full Hamiltonian is, using the equation for our dispersion relation and commuting the annihilation and creation operators,

$$H = \int \frac{d^3p}{(2\pi)^3}\frac{\mathbf{p}^2+m^2}{2\omega_{\mathbf{p}}}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + a_{\mathbf{p}}a^\dagger_{\mathbf{p}}\right) = \int d^3p\frac{\omega_{\mathbf{p}}}{(2\pi)^3}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + \frac{1}{2}(2\pi)^3\delta(0)\right)$$

2.8.2 Infinity issues with the Hamiltonian

Let's calculate the ground state of this Hamiltonian by letting it hit zero, to get $H|0\rangle$. When we hit the annihilation operator in the first term, the term vanishes. Thus, we're only left with the second term. Since $\delta^3(0)$ is constant, we naively pull this out of the integral to get

$$H|0\rangle = 4\pi^3\delta(0)\int d^3p\,\omega_{\mathbf{p}}, \quad \omega_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2+m^2}$$

However, since we're integrating over all momentum space, we get that $\omega_{\mathbf{p}} \to \infty$, and hence the integral diverges (let's ignore the problems associated with the delta function for now)! This is called a high-frequency, or ultraviolet, divergence. However, since we're doing non-gravitational physics, all we care about are energy differences. So, to subtract of this infinity, we could've naively wrote (as similar in the case of the one dimensional harmonic oscillator), that upon reordering of our indices we have that

$$H = \int \frac{d^3p}{(2\pi)^3}\,\omega_{\mathbf{p}}a^\dagger_{\mathbf{p}}a_{\mathbf{p}} \implies H|0\rangle = 0$$

So with reordering, we have that zero point energy is just zero. One way to interpret this is that in QFT, we have 'operator ambiguity' in a sense that all operators are defined up to a reordering. Now, we're in a place to properly define normal ordering.

Definition. We denote a normal ordered string of operators, denoted

$$:\phi(\mathbf{x_1})\phi(\mathbf{x_2})\ldots\phi(\mathbf{x_n}):$$

to be the same operator but with all annihilation operators pulled to the right.

Applying normal operators has the effect of **removing unwanted infinities** in our expression. It's easy to check that

$$: H := \int \frac{d^3p}{(2\pi)^3} \, \omega_{\mathbf{p}} a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$$

Theorem. Our normal ordered Hamiltonian raises and lowers energy as in QM. With this definition, our commutation relations with raising and lowering operators are now completely analogous to the case we saw in 1 dimensional QM.

$$[H, a_{\mathbf{p}}^\dagger] = \omega_{\mathbf{p}} a_{\mathbf{p}}^\dagger, \quad [H, a_{\mathbf{p}}] = -\omega_{\mathbf{p}} a_{\mathbf{p}}$$

Proof. We substitute in our definitions to find that

$$\begin{aligned}
[H, a_{\mathbf{p}}^\dagger] &= \int \frac{d^3q}{(2\pi)^3} \omega_{\mathbf{q}} [a_{\mathbf{q}}^\dagger a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger] \\
&= \int \frac{d^3q}{(2\pi)^3} \omega_{\mathbf{q}} a_{\mathbf{q}}^\dagger [a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger] \\
&= \int \frac{d^3q}{(2\pi)^3} \omega_{\mathbf{q}} a_{\mathbf{q}}^\dagger (2\pi)^3 \delta(\mathbf{q} - \mathbf{p}) \\
&= \omega_{\mathbf{p}} a_{\mathbf{p}}^\dagger
\end{aligned}$$

The other case is completely similar! □

2.9 Creating particle states

With this framework in mind, we can create excited states with a given energy $\omega_{\mathbf{p}}$ from our vacuum by applying raising operators. Let's denote $|\mathbf{p}'\rangle = a_{\mathbf{p}'}^\dagger |0\rangle$. Then, applying our Hamiltonian, we have that

$$\begin{aligned}
H |\mathbf{p}'\rangle &= \int \frac{d^3p}{(2\pi)^3} \, \omega_{\mathbf{p}} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} a_{\mathbf{p}'}^\dagger |0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3} \, \omega_{\mathbf{p}} a_{\mathbf{p}}^\dagger [a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger] |0\rangle \\
&= \omega_{\mathbf{p}'} a_{\mathbf{p}'}^\dagger |0\rangle = \omega_{\mathbf{p}'} |\mathbf{p}'\rangle
\end{aligned}$$

We thus interpret the raised state as a particle with mass m, energy $\omega_{\mathbf{p}'} = \sqrt{\mathbf{p}'^2 + m^2}$ and momentum $\mathbf{p}'$, where the mass comes from the scalar term $\frac{1}{2}m^2\phi^2$ in the Lagrangian. From now on, we will relabel the energy $\omega_{\mathbf{p}}$ as $E_{\mathbf{p}}$. From our energy momentum tensor, we could also promote our expression for momentum as an operator, to find that

$$\mathbf{P} = -\int d^3x \, \pi(\mathbf{x}) \phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \, \mathbf{p} a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$$

One can verify that this definition makes sense by acting on our momentum eigenstate via

$$\mathbf{P} |\mathbf{p}\rangle = \mathbf{p} |\mathbf{p}\rangle$$

Similarly, constructing our angular momentum operator, J^i as a cross product of our momentum and position operators, to find that for zero momentum eigenstates, $J^i |\mathbf{p} = 0\rangle = 0$, in other words, the intrinsic angular momentum, or spin, in zero. Now, to create multi-particle states, we can apply a bunch of raising operators to the vacuum state, and interpret this as a system of multiple particles. This is denoted as

$$|\mathbf{p}_1, \dots \mathbf{p}_n\rangle = a^\dagger_{\mathbf{p_1}} \dots a^\dagger_{\mathbf{p_n}} |0\rangle$$

Because creation operators commute, we have that particles are symmetric under exchange ($|\mathbf{p}, \mathbf{q}\rangle = |\mathbf{q}, \mathbf{p}\rangle$), which means that this particle species are bosons.

Our full Hilbert space is then the space of all particles spanned by states of the form $|0\rangle, a^\dagger_{\mathbf{p}} |0\rangle, a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{q}} |0\rangle, \dots$ This is what we call Fock space. We can create another interesting operator on this space, called the number operator, defined as

$$\mathcal{N} = \int \frac{d^3p}{(2\pi)^3} \, a^\dagger_{\mathbf{p}} a_{\mathbf{p}}$$

This has the property that it counts particle states. This is because since

$$\begin{aligned}
\mathcal{N} |\mathbf{p_1} \dots \mathbf{p_n}\rangle &= \int \frac{d^3p}{(2\pi)^3} \, a^\dagger_{\mathbf{p}} a_{\mathbf{p}} a^\dagger_{\mathbf{p_1}} a^\dagger_{\mathbf{p_2}} \dots a^\dagger_{\mathbf{p_n}} |0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3} \, a^\dagger_{\mathbf{p}} ([a_{\mathbf{p}}, a^\dagger_{\mathbf{p_1}}] + a^\dagger_{\mathbf{p_1}} a_{\mathbf{p}}) a^\dagger_{\mathbf{p_2}} \dots a^\dagger_{\mathbf{p_n}} [ket0 \\
&= \int \frac{d^3p}{(2\pi)^3} \, a^\dagger_{\mathbf{p}} (2\pi)^3 \delta(\mathbf{p} - \mathbf{p_1}) a^\dagger_{\mathbf{p_2}} \dots a^\dagger_{\mathbf{p_n}} |0\rangle \\
&+ \int \frac{d^3p}{(2\pi)^3} \, a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{p_1}} a_{\mathbf{p}} a^\dagger_{\mathbf{p_2}} a^\dagger_{\mathbf{p_n}} |0\rangle \\
&= |\mathbf{p_1} \dots \mathbf{p_n}\rangle + \int \frac{d^3p}{(2\pi)^3} \, a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{p_1}} a_{\mathbf{p}} a^\dagger_{\mathbf{p_2}} \dots a^\dagger_{\mathbf{p_n}} |0\rangle \\
&= N |\mathbf{p_1} \dots \mathbf{p_n}\rangle
\end{aligned}$$

The last line is obtained by repeating the procedure until $a_{\mathbf{p}}$ hits $|0\rangle$. To show that particle number is conserved in our Fock space, we need to show that this operator commutes with the Hamiltonian. We simply write this out as

$$\begin{aligned}
[H, \mathcal{N}] &= \int \frac{d^3p d^3q}{(2\pi)^6} \omega_{\mathbf{q}} [a^\dagger_{\mathbf{q}} a_{\mathbf{q}}, a^\dagger_{\mathbf{p}} a_{\mathbf{p}}] \\
&= \int \frac{d^3p d^3q}{(2\pi)^6} \omega_{\mathbf{q}} \left(a^\dagger_{\mathbf{q}} [a_{\mathbf{q}}, a^\dagger_{\mathbf{p}} a_{\mathbf{p}}] + [a^\dagger_{\mathbf{q}}, a^\dagger_{\mathbf{p}} a_{\mathbf{p}}] a_{\mathbf{q}} \right) \\
&= \int \frac{d^3p d^3q}{(2\pi)^6} \omega_{\mathbf{q}} \left(a^\dagger_{\mathbf{q}} [a_{\mathbf{q}}, a^\dagger_{\mathbf{p}}] a_{\mathbf{p}} + a^\dagger_{\mathbf{p}} [a^\dagger_{\mathbf{q}}, a_{\mathbf{p}}] a_{\mathbf{q}} \right) \\
&= \int \frac{d^3p d^3q}{(2\pi)^6} \omega_{\mathbf{q}} (2\pi)^3 \delta(\mathbf{p} - \mathbf{q}) a^\dagger_{\mathbf{q}} a_{\mathbf{p}} - (2\pi)^3 \delta(\mathbf{p} - \mathbf{q}) a^\dagger_{\mathbf{p}} a_{\mathbf{q}} \\
&= 0
\end{aligned}$$

Hence we have that the particle operator commutes with the Hamiltonian, and thus the particle number is conserved.

Note momentum eigenstates aren't localised. We can write a localised state via

$$|x\rangle = \int \frac{d^3p}{(2\pi)^3}\, e^{-i\mathbf{p}\cdot\mathbf{x}} \,|\mathbf{p}\rangle$$

More generally, we describe a wave packet partially localised in both position and momentum space. We can write

$$|\psi\rangle = \int \frac{d^3p}{(2\pi)^3}\, e^{-i\mathbf{p}\cdot\mathbf{x}}\psi(\mathbf{p})\,|\mathbf{p}\rangle\,, \quad \psi(\mathbf{p}) \propto e^{-\mathbf{p}^2/2m^2}$$

Neither $|\mathbf{x}\rangle$ nor $|\psi\rangle$ are eigenstates of H like in usual quantum mechanics.

2.9.1 Relativistic normalisation

From now on, let's define our vacuum state to be normalised such that $\langle 0|0\rangle = 1$. Let's do a simple thing, and just contract two momentum eigenstates, and see what we get;

$$\langle \mathbf{p}|\mathbf{q}\rangle = \langle 0|\, a_{\mathbf{p}} a^\dagger_{\mathbf{q}}\,|0\rangle = \langle 0|\,[a_{\mathbf{p}} a^\dagger_{\mathbf{q}}]\,|0\rangle = (2\pi)^3\delta(\mathbf{p}-\mathbf{q})$$

But, this our delta function Lorentz invariant? If not, it would be in our interest to come up with some expression that is Lorentz invariant, since manifest Lorentz invariance is easy (since things hold in all frames). We know definitely how four momentum transforms under a Lorentz boost, this is just given by

$$p^\mu \to \Lambda^\mu{}_\nu p^\nu = p'^\mu$$

This induces a transformation on our actual **eigenstates**, where we map $|\mathbf{p}\rangle \to |\mathbf{p}'\rangle$. In our usual notion of quantum theory, we would want $|\mathbf{p}\rangle$ to be related to $|\mathbf{p}'\rangle$ by a unitary transformation, because this forces the contraction to be Lorentz invariant. So we would want

$$\begin{aligned} |\mathbf{p}\rangle \to \left|\mathbf{p}'\right\rangle &= U(\Lambda)\,|\mathbf{p}\rangle \\ (2\pi)^3\delta(\mathbf{p}-\mathbf{q}) = \langle \mathbf{p}|\mathbf{q}\rangle \to \langle \mathbf{p}|\, U^\dagger(\Lambda)U(\Lambda)\,|\mathbf{p}\rangle &= \left\langle \mathbf{p}'\middle|\mathbf{q}'\right\rangle = (2\pi)^3\delta(\mathbf{p}'-\mathbf{q}') \end{aligned}$$

But we can'y assume that this holds in our quantum field theory case - we've made no study into how states change under relativistic transformations. So, the best we can do is to start from a different perspective, and use objects we know which are Lorentz invariant to construct a Lorentz invariant dot product.

Notice that our identity transformation

$$1 = \int \frac{d^3p}{(2\pi)^3}\, |\mathbf{p}\rangle\langle \mathbf{p}|$$

is Lorentz invariant, despite the individual states $|\mathbf{p}\rangle$ not being Lorentz invariant. This means that or integration expression, or **measure**, is not Lorentz invariant. Thus, a good place to start is to find a Lorentz invariant measure.

Theorem. We have that the measure

$$\int \frac{d^3p}{2E_{\mathbf{p}}}, \quad E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$$

is an invariant measure.

Proof. We construct this item from a chain of Lorentz invariant arguments. First, we assert that the measure

$$\int d^4p$$

is Lorentz invariant. We know that this is true because when we transform 4-vectors $d^4p \to d^4p'$, the factor is changed by $det(\Lambda)$. But since Lorentz transformations lie in the special orthogonal group $SO(1,3)$ this is just 1. In addition. we know that $p^2 = p_\mu p^\mu = p_0^2 - \mathbf{p}^2 - m^2$ is a Lorentz invariant quantity, hence putting this inside a delta function and sticking it inside the integrand also gives us a Lorentz invariant quantity! So

$$\int d^4p\, \delta\left(p_0^2 - \mathbf{p}^2 - m^2\right) \quad \text{is Lorentz invariant.}$$

This all seems a bit rabbit-out-of-the-hat at the moment, but bear with us. Let's separate our the time and space terms of our integral like this

$$\int d^3p \int dp_0\, \delta\left(p_0^2 - \mathbf{p}^2 - m^2\right) = \int d^3p \int dp_0\, \delta\left(f(p_0)\right), \text{ where } \quad f(p_0) = p_0^2 - \mathbf{p}^2 - m^2$$

Now, we appeal to an identity we learned in our undergraduate years which allows us to work with delta functions of functions.

$$\delta(f(x)) = \sum_{x_i} \frac{\delta(x - x_i)}{|f'(x_i)|} \quad \text{where } x_i \text{ are roots of f .}$$

Now, our roots of f are at

$$p_0^{\pm} = \pm\sqrt{\mathbf{p}^2 + m^2}$$

And, our derivative if just $f'(p_0) = 2p_0$. Thus, our full function for our delta function is

$$\delta\left(p_0^2 - \mathbf{p}^2 - m^2\right) = \frac{\delta(p_0 - \sqrt{\mathbf{p}^2 + m^2})}{2\sqrt{\mathbf{p}^2 + m^2}} + \frac{\delta(p_0 + \sqrt{\mathbf{p}^2 + m^2})}{2\sqrt{\mathbf{p}^2 + m^2}} = \frac{\delta(p_0 - E_{\mathbf{p}})}{2E_{\mathbf{p}}} + \frac{\delta(p_0 + E_{\mathbf{p}})}{2E_{\mathbf{p}}}$$

Now, we don't have to include both of these delta functions in our integral. It's okay just to choose $p_0 > 0$, since we cannot Lorentz boost from positive time into negative time. Thus, our notion of sign choice for p_0 is also Lorentz invariant. So, selecting just the positive component gives us

$$\int d^3p \int dp_0 \delta\left(p_0^2 - \mathbf{p}^2 - m^2\right)\Big|_{p_0 > 0} = \int d^3p \int dp_0 \frac{\delta(p_0 - E_{\mathbf{p}})}{2E_{\mathbf{p}}} = \int \frac{d^3p}{2E_{\mathbf{p}}}$$

since the delta function absorbed into the $\int p_0$ just goes to one. □

With this fact, we motivate the concept of a relativistically normalised state; which we denote as

$$|p^\mu\rangle := |p\rangle = \sqrt{2E_{\mathbf{p}}}\, |\mathbf{p}\rangle$$

This is manifestly Lorentz invariant because

$$I = \int \frac{d^3p}{2E_{\mathbf{p}}} |p\rangle \langle p|$$

is Lorentz invariant, and so is the measure. Hence the integrand must be Lorentz invariant. Thus, our normalised delta function looks like

$$\langle p|q\rangle = (2\pi)^3 2\sqrt{E_{\mathbf{p}} E_{\mathbf{q}}}\delta^3(\mathbf{p} - \mathbf{q})$$

2.10 Quantising the free complex scalar field

Let's explore what we get when we play around with a complex, free-like Lagrangian. We have the Lagrangian attached with a mass term

$$\mathcal{L} = \partial^\mu \psi^* \partial_\mu \psi - \mu^2 \psi^* \psi$$

This Lagrangian is associated with the following Euler-Lagrange equations

$$\begin{aligned} \partial_\mu \partial^\mu \psi + \mu^2 \psi = 0 \\ \partial_\mu \partial^\mu \psi^* + \mu^2 \psi^* = 0 \end{aligned}$$

Now since the complex conjugate of our field ψ is not the same as ψ^*, when we promote ψ to a Schrodinger picture operator there's no notion which suggests that the operator should be Hermitian. So, we write out a normalised expansion that's similar to what we had for a scalar field but, add different operators inside. We have

$$\begin{aligned} \psi(\mathbf{x}) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (b_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + c^\dagger_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}) \\ \psi(\mathbf{x})^\dagger &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (b^\dagger_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} + c_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}}) \end{aligned}$$

In addition, we motivate the same expressions for our conjugate momenta. Our conjugate momenta classically is found by differentiating the Lagrangian:

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \dot{\psi}^*$$

With the same Heuristic we applied to find conjugate momenta for our scalar field, we tentatively write that

$$\begin{aligned} \pi &= \int \frac{d^3p}{(2\pi)^3} \, i \sqrt{\frac{E_{\mathbf{p}}}{2}} (b^\dagger_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} - c_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}) \\ \pi^\dagger &= \int \frac{d^3p}{(2\pi)^3} \, (-i) \sqrt{\frac{E_{\mathbf{p}}}{2}} (b_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} - c^\dagger_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}}) \end{aligned}$$

Here, we've defined two operators b, c, which have the interpretation of creating and annihilating a particle and antiparticle respectively. With this expansion one can check that the commutation relations

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y}), \quad [\phi(\mathbf{x}), \pi(\mathbf{y})^\dagger] = 0, \qquad \text{all other relations derived from this}$$

are equivalent to the commutation relations

$$[b_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] = (2\pi)^3 \delta(\mathbf{p} - \mathbf{q}), \quad [c_{\mathbf{p}}, c^\dagger_{\mathbf{q}}] = (2\pi)^3 \delta(\mathbf{p} - \mathbf{q}), \qquad \text{all other relations } 0$$

We interpret the operators as particle and anti particle creation. These particles have spin 0, and the same mass since we have a μ^2 coupling term in there. Since a real scalar field obeys $\psi^* = \psi$, our interpretation of a particle created by a real scalar field is that it its' own antiparticle. Using Noether's theorem, one can show that our conserved charge which is obtained by the symmetry of a phase rotation

$$\psi \to e^{i\theta}\psi$$

gives rise to the conserved charge

$$Q = i\int d^3x\, \dot{\psi}^*\psi - \psi^*\dot{\psi}$$

Using the relation we derived earlier for conjugate momentum, this can be written as

$$Q = \int d^3x\, \pi\psi - \psi^\dagger\pi^\dagger$$

Expanding this in terms of the creation and annihilation operators b, c, with normal ordering we have that a conserved charge is

$$Q = \int \frac{d^3p}{(2\pi)^3}\,(c^\dagger_{\mathbf{p}}c_{\mathbf{p}} - b^\dagger_{\mathbf{p}}b_{\mathbf{p}}) = N_c - N_b$$

This counts the number of of particles minus the number of anti-particles.

2.11 Introducing time with the Heisenberg picture

So far, in the Schrodinger picture, we aren't entirely confident that our operators are manifestly Lorentz invariant. In the Schrodinger picture, operators don't depend on time but states do since the evolve according to the equation

$$i\frac{d\left|\mathbf{p}\right\rangle}{dt} = H\left|\mathbf{p}\right\rangle = E_{\mathbf{p}}\left|\mathbf{p}\right\rangle$$

This however, is entirely equivalent to placing time dependence on operators instead of states, and force operators to evolve according to

$$O_H(t) = e^{iHt}O_s e^{-iHt}$$

where O_s is our Schrodinger picture operator which coincides with our Heisenberg picture operator at $t = 0$. At a fixed time, substituting this expressions now give us fixed time commutation relations. Recall that

$$[UAU^{-1}, UBU^{-1}] = U[A, B]U^{-1}$$

Application of this identity yields us the fixed time commutation relations

$$[\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = [\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = 0, \quad [\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta(\mathbf{x} - \mathbf{y})$$

Taylor expanding out we have

$$O_H(t + \delta t) = 1 + \delta t i[H, O_{\mathcal{H}}(t)] + \dots$$

so operators obey the Heisenberg equations of motion, that

$$i\frac{dO_H}{dt} = i[H, O_H]$$

We can use this to calculate our time derivatives of our operators, to get $\dot{\pi}$ and $\dot{\phi}$. We can then relate $\pi = \dot{\phi}$ to obtain the Klein Gordon equation. For notation purposes, we write

$$\psi(x) := \phi(\mathbf{x}, t)$$

We can check, by computing the commutator $[a_{\mathbf{p}}, e^{-iHt}]$, and $[a^\dagger_{\mathbf{p}}, e^{-iHt}]$, that

$$\begin{aligned} e^{iHt}a_{\mathbf{p}}e^{-iHt} &= e^{-iE_{\mathbf{p}}t}a_{\mathbf{p}} \\ e^{iHt}a^\dagger_{\mathbf{p}}e^{-iHt} &= e^{iE_{\mathbf{p}}t}a_{\mathbf{p}} \end{aligned}$$

Hence, we have

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}(a_{\mathbf{p}}e^{-ip\cdot x} + a^\dagger_{\mathbf{p}}e^{ip\cdot x})$$

with this we can calculate $\dot{\phi}$ amd $\dot{\pi}$. (Insert section to check that this recovers the relativistic Klein Gordon equation). We promote $\phi(\mathbf{x}), \pi(\mathbf{x})$ to adopt time dependence, so we write

$$\phi(\mathbf{x}, t) = \phi(x)$$

and similiarly with $\pi(x)$.

Our Hamiltonian which didn't depend on time was

$$H = \int \frac{d^3x}{(2\pi)^3} a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{x}\cdot\mathbf{p}}$$

but to calculate the time dependent Hamiltonian we'd need to calculate

$$e^{iHt} a_{\mathbf{p}} e^{-iHt}, \quad e^{iHt} a_{\mathbf{p}} e^{-iHt}$$

The first operator is only non-zero when we contract it on both sides for the configuration

$$\langle 0| e^{iHt} a_{\mathbf{p}} e^{-iHt} |\mathbf{p}\rangle ,$$

and this gives us

$$\langle 0| e^{iHt} a_{\mathbf{p}} e^{-iHt} |\mathbf{p}\rangle = e^{-iE_{\mathbf{p}}t}$$

thus our conjugated operator is

$$e^{iHt} a_{\mathbf{p}} e^{-iHt} = a_{\mathbf{p}} e^{-iE_{\mathbf{p}}t}$$

and similarly we have that

$$e^{iHt} a_{\mathbf{p}}^\dagger e^{-iHt} = a_{\mathbf{p}}^\dagger e^{iE_{\mathbf{p}}t}$$

Since $p = (E_{\mathbf{p}}, \mathbf{p})$, altogether this gives our expression for our scalar field

$$\phi(x) = \int \frac{d^3x}{(2\pi)^3} \frac{1}{\sqrt{E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x} \right)$$

3 Causality in QFT

We still have that potentially, a hint of Lorentz variance because the commutation relations of ϕ and π are equal time relations. If we shift to a differed time, this may not hold. Our previous fixed time commutation relations involved expressions of the form

$$[\mathcal{O}_2(\vec{x}, t), \mathcal{O}_1(\vec{y}, t)]$$

but, if we transform the coordinates with a Lorentz transformation, we would get this turning into

$$[\mathcal{O}_1(\vec{x}', t'), \mathcal{O}_2(\vec{y}', t'')]$$

where t', t'' are not necessarily the same since $\vec{x}, \vec{y}$ are different. What about arbitrary space time separations? The real requirement of causality is that all space like separated operators commute

$$[\mathcal{O}(x_1), \mathcal{O}(x_2)] = 0, \quad \forall (x-y)^2 < 0$$

The reason why commutation is synonymous with the fact that operators don't affect one another, comes from the fact that they have a shared eigenbasis. Suppose we have A, B, hermitian operators which commute. Then, we have that they share an eigenbasis which we label as $\{|i\rangle\}$, with the property that

$$A|i\rangle = \lambda_i |i\rangle, \quad \lambda_i \in \mathbb{R}$$
$$B|i\rangle = \mu_i |i\rangle, \mu_i \in \mathbb{R}$$

In particular, this means that if we expand a given quantum state in terms of the eigenbasis, we get

$$|\psi\rangle = \sum_i a_i |i\rangle \implies \langle\psi| BA |\psi\rangle = \sum_i a_i \lambda_i \mu_i$$

where in the last sum we used the orthogonality property of the eigenstates. However, notice that this is the same by symmetry of calculating $\langle\psi| AB |\psi\rangle$. Hence, applying an operator in the process of measuring the other one doesn't change our outcomes. Thus, we have no causality.

If we define $\Delta(x-y) = [\phi(x), \phi(y)]$, we'd like to check if this holds for space-time separation.

$$[\phi(x), \phi(y)] = \int \frac{d^3p\, d^3p'}{(2\pi)^6 \sqrt{4E_p E_p'}} \left\{ [a_{\vec{p}}, a^\dagger_{\vec{p}'}] e^{i(p\cdot x - p'\cdot y)} + [a^\dagger_{\vec{p}}, a_{\vec{p}'}] e^{i(p\cdot x - p'\cdot y)} \right\}$$

which is just

$$[\phi(x), \phi(y)] = \int \frac{d^3p}{2E_p (2\pi)^3} \left\{ e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right\}$$

Well, what do we know about this function? We know that this is Lorentz invariant because the measure is, and so is the integrand. If $x-y$ is space-like, it vanishes because we can take a Lorentz transformation to $y-x$ in the first term, giving 0. Another way to say this is that if we have space-like separated events, then we don't care about breaching causality in time, so we can freely transform events in this way. We can't however do this for time like events because that would mean time reversal. This integral doesn't vanish, however, for timelike separations.

If we have time like separated events, we can Lorentz transform them such that they're constant in space.

$$[\phi(\vec{x},0),\phi(\vec{x},t)] = \int \frac{d^3p}{(2\pi)^3 2E_p}\,(e^{-iE_pt} - e^{iE_pt}) \neq 0$$

We assert that this is non-zero. To show that this is non-zero, we can evaluate this expression explicitly. Let's do the first term.

$$\begin{aligned}\int \frac{d^3p}{(2\pi)^3 2E_p} e^{-iE_pt} &\sim \int dp\,\frac{p^2}{\sqrt{p^2+m^2}} e^{-i\sqrt{p^2+m^2}t} \\ &= \int_m^\infty dE\sqrt{E^2-m^2}e^{-iEt}\end{aligned}$$

In the second line, we expressed the integral in polar form using p as a radial coordinate. In the second line, we used a change of variables and set $E = \sqrt{p^2+m^2}$. To evaluate the final integral, we need to analytically continue it to $\mathbb{C}$. Since we have a square root, this induces a branch cut at $E = m$. By the residue theorem, our whole contour evaluates to zero. However, the part on the x-axis is what we'd like to calculate. To do this, we calculate the arc, as well as the contour which is show as the vertical line. Our integral along the arc is given by

$$C_R = \int dz\sqrt{z^2-m^2}e^{-izt}, z = m + Re^{i\theta}$$

If we do this integral in polars, for large R this integral goes like

$$\begin{aligned}C_R &\sim i\int d\theta e^{i\theta}R\sqrt{R^2}e^{-imt}e^{Re^{i\theta}t} \\ &\sim e^{-imt}i\int d\theta R^2 e^{iR\cos\theta t}e^{R\sin\theta t}\end{aligned}$$

Now, since we're taking the contour in $-\frac{\pi}{2} < \theta < 0$, and ignoring our oscillating $e^{iR\cos\theta t}$ term, this integral goes like

$$\sim e^{-imt}i\int d\theta e^{-\theta}R^2e^{-\epsilon Rt}$$

The negative epsilon comes from the fact that we're taking a negative contour. Due to the negative in the exponential, this term is bounded. So, $C_R \sim e^{-imt}$. A similar analysis of the downwards vertical contour shows that it grows in the same way. Hence, we have that

$$\int \frac{d^3p}{(2\pi)^2 2E_p}e^{-E_pt} \sim e^{-imt}$$

Thus, our time-like separated integral goes like

$$[\phi(\vec{x},0),\phi(\vec{x},t)] \sim e^{-imt} - e^{imt}$$

This is non-zero.

At equal times, we have

$$[\phi(\vec{x},t),\phi(\vec{y},t)] = \int \frac{d^3p}{(2\pi)^3 2E_p}\,(e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}) = 0$$

where in the second term we apply a Lorentz transformation put $x \vec{-} y \to -y \vec{-} x$. We can do a Lorentz transformation from within this integral because the whole integral was Lorentz invariant in the first place. Thus, the second term cancels, which agrees with equal time commutation relations, since it goes to zero.

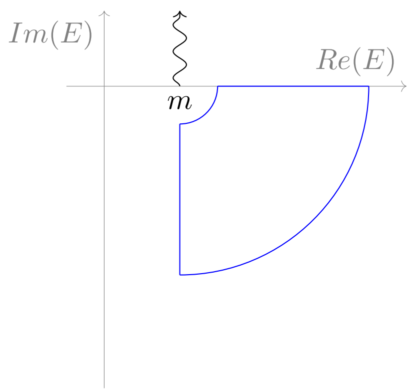


Figure 1: Our contour integral which we use to calculate our timelike integrals

3.1 Propagators

If we prepare a particle at a point y, what's the probability it ends up at x? Well, we could identify the creation of a particle at the point x, with $\phi(x)$. It's instructive to view the similarities with QM. We have that $\langle x| X |p\rangle = e^{ip\cdot x}$ in regular quantum mechanics, but we have

$$\langle 0| \phi(x) |p\rangle = e^{-ip\cdot x}$$

in QFT. Hence, we can somewhat identify $\phi(x)|0\rangle$ as the position presentation of a created particle.

Our amplitude for a particle to propagate at x then travel to y is then

$$\langle 0| \phi(x)\phi(y) |0\rangle = \int \frac{d^3p d^3q}{(2\pi)^6\sqrt{4E_pE_q}} \langle 0| [a_{\vec{p}}, a_{\vec{p}'}] |0\rangle\, e^{-i(p\cdot x i-p'\cdot y}$$

which is

$$= \int \frac{d^3p}{(2\pi)^3 2E_p} e^{-ip\cdot(x-y)} := D(x-y)$$

This quantity we call the propagator. For spacelike separations $(x-y)^2 < 0$, we can show that it decays as

$$D(x-y) \sim e^{|\vec{x}-\vec{y}|}$$

The quantum field leaks out of the light cone. But, we just saw that the space like separations commute;

$$\Delta(x-y) = [\phi(x), \phi(y)] = D(y-x) - D(x-y) = 0, \quad \forall (x-y)^2 < 0$$

There's no Lorentz invariant way of ordering the events and a particle can just as easily travel from $x \to y$ as it can from $y \to x$. In a measurement, these amplitudes cancel. For a $\mathbb{C}$ scalar field, we have that

$$[\psi(x), \psi(y)] = 0, \text{ outside the light cone}$$

The amplitude to go from $x \to y$ cancels the one for an antiparticle to go from $y \to x$. For real fields, the particle is the antiparticle.

3.2 The Feynman Propagator

Our Feynman propagator is given by

$$\Delta_F(x-y) = \langle 0|\, T\phi(x)\phi(y)\, |0\rangle = \begin{cases} \langle 0|\phi(x)\phi(y)\rangle & x^0 > y^0 \\ \langle 0|\, \phi(y)\phi(x)\, |0\rangle & x^0 < y^0 \end{cases}$$

Our claim is that we can write this propagator in the compact form

$$\Delta_F = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip\cdot(x-y)}$$

This is Lorentz invariant. Note that this is ill defined because for each p, the integral over p^0 has a pole where $(p^0)^2 = \vec{p}^2 + m^2$. We need a prescription here, which means that we **resolve the ambiguity of the integral** by choosing a specific contour. In our case, we need a contour that recovers the exact expression above. We define the integration contour to be a straight line which gues under the residue at $-E_p$ and above the residue at E_p. (draw diagram here). Notice that

$$\frac{1}{p^2 - m^2} = \frac{1}{(p^0)^2 - E_p^2} = \frac{1}{(p^0 - E_p)(p^0 + E_p)}$$

The residue of the pole at $p^0 = \pm E_p$ is $\pm\frac{1}{2E_p}$. When $x^0 > y^0$, we close the contour in the lower half plane. When $p^0 \to -i\infty$, $e^{-ip^0\cdot(x^0-y^0)} \to e^{-\infty} \to 0$. So, the lower contour doesn't contribute. Hence, we have

$$\Delta_F(x-y) = \int \frac{d^3p}{(2\pi)^3 2E_p} (-2\pi i) i e^{-iE_p(x^0-y^0) + i\vec{p}\cdot(\vec{x}-\vec{y})} = \int \frac{d^3p}{(2\pi)^3 2E_p} e^{-ip\cdot(x-y)}$$

This is $D(x-y)$, which agrees which what we want. We had the minus sign due to clockwise.

When $x^0 < y^0$, we close the contour in the upper half place to get

$$\Delta_F(x-y) = \int \frac{d^3p}{(2\pi)^4} \frac{(2\pi i)}{(-2E_p} i e^{iE_p(x^0-y^0) + i\vec{p}\cdot(\vec{x}-\vec{y})}$$

We flip the sign in the second term to get this as

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-iE_p(y^0-x^0) - i\vec{p}\cdot(\vec{y}-\vec{x})} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(y-x)} = D(y-x)$$

Instead of this laborious process of specifying the contour, we can instead do the $i\epsilon$ prescription where we set

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} i \frac{e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}$$

This propagator Δ_F is the Green's function of the Klein Gordon operator. We should get that

$$(\partial_t^2 - \nabla^2 + m^2)\Delta(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} (-p^2 - m^2) e^{-ip\cdot(x-y)}$$

which is

$$= -i \int \frac{d^4p}{(2\pi)^4} e^{-p\cdot(x-y)} = -i\delta(x-y)$$

It can be useful in some circumstances to pick other contours. For example, the retarded Green's function (diagram of camel humps) over both poles. This is when

$$\Delta_F(x-y) = \begin{cases} [\phi(x), \phi(y)] & x^0 > y^0 \\ 0 & y^0 < x^0 \end{cases}$$

This is useful if we start with an initial field configuration and look at the evolution in the presence of a source

$$\partial_\mu \partial^\mu \phi + m^2 \phi = J(x)$$

The Feynman propagator Δ_F is the most useful object in QFT.

4 Interacting fields and the approach with perturbation theory

4.1 Why do we need interacting fields and what are 'valid' conditions for perturbation expansions?

This section follows David Tong's notes, and some of Peskin and Schroesder

So far our Lagrangian $\mathcal{L}$ has only been quadratic in the field ϕ. We've referred to this as our free theory; it gives rise to the second order, linear PDE

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0$$

which we can solve and quantise exactly. However, even though our analysis of this has considered multi particle states, we have yet to consider theories in which there particles interact. For example, scattering scenarios.

Interaction terms can be written as higher order terms to our free Lagrangian;

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2 - \sum_{n=3} \frac{\lambda_n \phi^n}{n!}$$

The λ_n coefficients are called coupling constants, because they couple our free theory with interaction terms. For example, we could choose our constants to give rise to what we call ϕ^4 theory, with $\lambda_4 = \lambda$ and λ_n zero otherwise:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

However, if we have hopes of trying to solve our theory for expansions like this in a perturbative fashion, we then need to ensure that the λ_n terms are small. In other words, $\lambda_n \ll 1$. But what does this even mean? For something to be smaller than a number, it better first be dimensionless. So, let's try figure out what the dimensions of our objects in question are.

Our action is

$$S = \int d^4x \, \mathcal{L}, \quad [S] = 0$$

Our action should be dimensionless. We've denoted our **mass dimension** of a quantity Q as $[Q]$, which is why we wrote $[S] = 0$ as the above. d^4x is an integral, which means it has dimensions of 4 in length terms. But, in natural units, we have that $[M] = [L]^{-1}$. So, we have that $[d^4x] = -4$. This implies that our Lagrangian must have dimensions $[\mathcal{L}] = 4$ so compensate.

So, what is $[\phi]$? To find this out, we look at our kinetic term $\partial_\mu \phi \partial^\mu \phi$. Since we have two length differentials which contribute a -2 to the dimension, we can then infer that $[\phi] = 1$.

This means that since $[\mathcal{L}] = 4$, our dimension of $[\lambda_n]$ must satisfy $[\lambda_n] + n = 4 \implies$

$$[\lambda_n] = 4 - n$$

Let's go case by case

- If we have $n = 3$, our dimension of λ_3 is 1. This means, to force a dimensionless parameter what we actually require is that $\lambda_3/E \ll 1$, since E has the same dimensions as mass. This means that a perturbation of this form is only valid at high energies! High energy scattering is one case where we could apply this theory. This is called a **relevant** perturbation, because the perturbation is significant at low energies.
- In the case where $d = 4$, our mass dimensions $[\lambda_4] = 0$, so we require that the perturbation is small when $\lambda_4 \ll 1$. This, the boundary case, is called a **marginal** operator. We can deal with the infinities that come from this perturbation, so we call it a renormalisable theory.
- The perturbation $\lambda_n \phi^n/n!$ are called **irrelevant** operators. This is because their associated dimensionless parameter is λE^{n-4}, so the perturbation expansion is only valid at low energies. Their effect is only significant at high energies. These are non re normalisable theories.

Let's look at some of the properties of ϕ^4 theory. This has Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda \phi^4}{4!}, \lambda \ll 1$$

We can already see from first glance that our Hamiltonian is going to contain a bunch of extra terms which are inherited from the last term. This means that our number operator

$$N = \int \frac{d^3p}{(2\pi)^3} a_{\vec{p}}^\dagger a_{\vec{p}}$$

will satisfy $[H, N] \neq 0$. This already tells us that particle number will not be conserved. We can see this explicitly by expanding the last term, which will yield combinations of the form

$$\cdots \int a_{\vec{p}}^\dagger a_{\vec{p}'}^\dagger a_{\vec{p}''}^\dagger a_{\vec{p}'''}^\dagger$$

and other terms which create or destroy particles.

Another example is scalar Yukawa theory. THis is what we get when we lump together a a $\mathbb{C}$ scalar field, a real scalar field, as well as an interaction term.

$$\mathcal{L} = \partial_\mu \psi \partial^\mu \psi - \mu^2 \psi \psi^* + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 - g\psi\psi^*\phi$$

This Lagrangian has been used for example in the interaction of mesons. Now, our g term here is combined with three fields. This means that it's like a λ_3 term.

This means it's a relevant operator. Hence, working perturbatively works for low energies. In a relativistic setting, $E > m$. So, we can make perturbations small by taking $g \ll m$. We can start to make preliminary statements about this Lagrangian by observing that its invariant under the transformation

$$\psi \to e^{i\theta}\psi, \quad \psi^* \to e^{-i\theta}\psi^*, \quad \phi \to \phi$$

This means that our Noether current for this symmetry exists. This Noether current Q is our particle minus anti particle number, and we have that $[Q, H] = 0$. Thus, with this Lagrangian the number of particles minus the number of antiparticles stays constant.

We have no conservation however, for scalar particles generated by ϕ.

4.2 The interaction picture

This section combines Prof. B. Allanach's lecture course, page 53 of QFT 1 Notes, University of Heidelberg, and the interaction section of Peskin and Schroesder

The whole point of this section is to reduce solutions in our full picture, and express them perturbatively in terms of free solutions. We may wish to compute a term like

$$\langle\Omega| T\phi(x)\phi(y) |\Omega\rangle$$

where we're contracting by the ground states of the **full** theory, and using solutions of the full theory. Our aim is to do find out what this is in terms of free theory objects like $|0\rangle$!.

4.2.1 The Heisenberg and Schrodinger Picture

In quantum mechanics, our Schrodinger picture is when states evolve in time, and not operators. The obey the equation

$$i\frac{d\,|\psi\rangle_s}{dt} = H\,|\psi\rangle_s$$

Our operators, $\mathcal{O}_s$, are time independent. On the other hand, we can also choose to be in the Heisenberg picture, where operators evolve instead. We define operators in the Heisenberg picture to obey

$$\mathcal{O}_H = e^{iHt}\mathcal{O}_s e^{-iHt}, \quad |\psi\rangle_H = e^{iHt}\,|\psi\rangle_s$$

This ensures consistency; one can verify that

$$\langle\psi|_H\,\mathcal{O}_H\,|\psi\rangle_H = \langle\psi|_S\,\mathcal{O}_H\,|\psi\rangle_S$$

We see from the above that we can switch between our Schrodinger and Heisenberg pictures by multiplying by an appropriate time evolution factor. Heisenberg operators obey the Heisenberg equation of motion

$$\frac{d\mathcal{O}^H}{dt} = i[H, \mathcal{O}_H]$$

4.2.2 The Interaction picture is a hybrid!

In the **interaction picture**, however, is a hybrid of the Schrodinger and Heisenberg picture. We separate out the free and interaction parts of the Hamiltonian as

$$H = H_0 + H_{int}$$

This split is arbitrary, but we choose H_0 typically as the section which we can solve exactly.

Example 1. In ϕ^4 theory, our Lagrangian and Hamiltonian terms associated with our interaction terms are

$$\mathcal{L}_{int} = -\frac{\lambda}{4!}\psi^4, \quad H_{int} = -\int d^3x\,\mathcal{L}_{int} = \int d^3x\frac{\lambda}{4!}\phi^4$$

Our Hamiltonian associated with the free term H_0 is given by

$$H_0 = \int d^3x\,\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

Our definition of our interaction picture is similar to the Heisenberg picture, exact that we are evolving states with the **free Hamiltonian**. This is in place of the full Hamiltonian. The interaction picture for a general operator is

$$\mathcal{O}_I = e^{iH_0 t}\mathcal{O}_s e^{-iH_0 t}, \implies \phi_I(x) = e^{iH_0 t}\phi_s(\vec{x})e^{-H_0 t}$$

There's a slightly more specific way to handle this however, which makes use of a general reference time t_0. To construct the interaction picture, choose some t_0 in the Heisenberg picture, then evolve from $t_0 \to t$ (philosophically, however, our Schrodinger picture is the same as our Heisenberg picture at fixed time).

$$\phi_I(t,\vec{x}) = e^{iH_0(t-t_0)}\phi_H(t_0,\vec{x})e^{-iH_0(t-t_0)}, \quad \pi_I(t,\vec{x}) = e^{iH_0(t-t_0)}\pi_H(t_0,\vec{x})e^{-iH_0(t-t_0)}$$

4.2.3 We can Fourier expand Interaction picture states!

Now, ϕ_I obeys the same commutation relations as ϕ did in the free theory: it obeys the same dynamics in the Heisenberg picture

$$\dot{\phi}_I = i[H_0,\phi_I], \quad \dot{\pi}_I = i[H_0,\pi_I]$$

Theorem. We have that ϕ_I also obeys the Klein-Gordon equation.

$$(\partial^2 + m^2)\phi_I(x) = 0$$

Proof. Observe that for ϕ^4 theory, our interaction term Hamiltonian is a function of just ϕ. Thus, we have that our commutation relations remain exactly the same for a operators at fixed time. Thus, we have the relations for our conjugate momenta at fixed time, despite having an added Lagrangian

$$\mathcal{L}_{\text{kinetic}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \implies \pi(t_0,\vec{x}) = \dot{\phi}(t_0,\vec{x})$$

We use this fact to our advantage in finding what $\phi_I(t,\vec{x})$ is. Substituting our definitions, we compute the Heisenberg equation motion

$$\begin{aligned}
\dot{\phi}_I(t_0,\vec{x}) &= i[H_0,\phi_I(x,)] \\
&= i[H_0, e^{iH_0(t-t_0)}\phi(t_0,\vec{x})e^{-iH_0(t-t_0)}] \\
&= ie^{iH_0(t-t_0)}[H_0,\phi(t_0,\vec{x})]e^{-iH_0(t-t_0)} \\
&= ie^{iH_0(t-t_0)}(-i\pi)(t_0,x)e^{-iH_0(t-t_0)} \\
&= \pi_I(t,x)
\end{aligned}$$

In the above, we used the commutation relation that $[\phi(\vec{x}),\pi(\vec{y})] = i\delta(\vec{x}-\vec{y})$ to calculate $[H_0,\phi(t_0,\vec{x})]$. We can play exactly the same game with

$$\begin{aligned}
\dot{\pi}_I(t,x_0) &= i[H_0,\pi_I(t,\vec{x})] \\
&= ie^{iH_0(t-t_0)}[H_0,\pi(t_0,\vec{x})]e^{-iH_0(t-t_0)} \\
&= ie^{iH_0(t-t_0)}(-i\nabla^2\phi(t_0,\vec{x}) + im^2\phi(t_0,\vec{x}))ie^{-iH_0(t-t_0)} \\
&= \nabla^2\phi_I(t,\vec{x}) - m^2\phi_I(t,\vec{x})
\end{aligned}$$

Equating the above equation means that we deduce the Klein-Gordon equation for states in the interaction picture

$$\partial_\mu \partial^\mu \phi_I(x) + m^2 \phi_I(x) = 0$$

$\square$

This means we could apply the exact same procedure to quantise this thing as we did for the free case.

4.2.4 Operators defined from the Interaction picture obey the same commutation relations

Since, in the interaction picture we evolve this state by conjugating it either side with $e^{iH_0 t}$. This gives us the same result that we has for the free field. We have that

$$\phi_I(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_{\vec{p}} e^{-ip\cdot x} + a_{\vec{p}}^\dagger e^{ip\cdot x})$$

There's an important point to be made here. Philosophically, these are different annihilation and creation operators that in our free case, so**really**, we should be writing,

$$\phi_I(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_{\vec{p}}^* e^{-ip\cdot x} + a_{\vec{p}}^{*\dagger} e^{ip\cdot x})$$

In this case, we denote $a_{\vec{p}}^*, a_{\vec{p}}^{*\dagger}$ as operators which are in the **interaction** picture, not necessarily in the free picture.

Despite this however, we still have that as before, these annihilation and creation operators satisfy our commutation relation as we had in the free theory, where

$$[a_{\vec{p}}, a_{\vec{p'}}^\dagger] = (2\pi)^3 \delta(\vec{p} - \vec{p'})$$

A natural question to ask then is how these interaction picture creation and operators act commute with the free-field part of the Hamiltonian, as well as act on the ground state. If ϕ, π are operators associated with our full theory, and H_0 is our free Hamiltonian, then particular

$$\begin{aligned} H_0 = e^{iH_0(t-t_0)} H_0 e^{-iH_0(t-t_0)} &= e^{iH_0(t-t_0)} \left(\int d^3x \, \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right) e^{-iH_0(t-t_0)} \\ &= \int d^3x \, \frac{1}{2}\pi_I^2 + \frac{1}{2}(\nabla\phi_I)^2 + \frac{1}{2}m^2\phi_I^2 \end{aligned}$$

The upshot of doing this is that we can transfer our previous analysis of raising and lowering operators to our interaction picture. In particular, we have that our interaction picture annihilation and creation operators obey

$$[H_0, a_{\vec{p}}^*] = -\vec{E}_p a_{\vec{p}}^*, \quad [H_0, a_{\vec{p}}^{*\dagger}] = E_{\vec{p}} a_p^{*\dagger}$$

This means that we have a state $|0\rangle$ such that

$$H_0 \,|0\rangle = 0, \quad a_{\vec{p}}^* \,|0\rangle$$

where H_0 is the **free part of our full Lagrangian**. Again, a philosophical point to be made is that the free part of our interacting Lagrangian in of the same from as just in the case of a purely free Lagrangian. The whole point of the interacting picture was to introduce a new state which obeyed the Klein Gordon relation, hence having a Fourier mode expansion, and hence had the same commutation relations as in the free case.

In addition, $a_{\vec{p}}$ annihilates the ground state of the free theory part in the Hamiltonian, so $a_{\vec{p}} |0\rangle = 0$.

An interesting question to would be whether the physical ground state of the free part of the full Hamiltonian is the same as if we were to treat the full Hamiltonian by itself.

We have a diagram of our logical flow below

$$
\begin{tikzcd}
\text{Get a Lagrangian } \mathcal{L}_{\text{ free }} + \mathcal{L}_{\text{ interaction}} \arrow[d] & & \\
\begin{array}{c}\text{Construct } \pi \\ \text{same as free case when } \mathcal{L}_{\text{ interaction }} \text{ doesn't depend on } \partial_\mu \phi\end{array} \arrow[d] & & \\
\text{Construct } \mathcal{H}_I \text{ 'free'} \arrow[rr] \arrow[d] & & \begin{array}{c}\text{Heisen. to show KG equation} \\ \phi_I \text{ solves KG}\end{array} \arrow[d] \\
\text{derive commutation relations with } H_0 & \text{KG} \arrow[l] \arrow[r, Rightarrow] & \text{possible } a_p, a_p^\dagger \text{ expansion} \\
\begin{array}{c}\text{annihilation and ground state } |0\rangle \text{ defined} \\ H_0 |0\rangle = 0, \quad a_p |0\rangle = 0\end{array} \arrow[from=4-1, to=5-1] & &
\end{tikzcd}
$$

4.2.5 Introducing the time unitary operator

Let's reparse things for the sake of generality for a bit. There's nothing special in our choice of $t = 0$. We could have easily just fixed a time t_0 as our reference point, and in our Heisenberg operator our states evolve as

$$\phi(x,t) = e^{iH(t-t_0)}\phi(x,t_0)e^{-iH(t-t_0)}$$

In the interaction picture, we set our perturbation constant $\lambda = 0$, and thus get the result that

$$\phi_I(x,t) = e^{iH_0(t-t_0)}\phi(x,t_0)e^{-iH_0(t-t_0)}$$

Now, how do we switch the between the two pictures? Well, we invert our interaction picture then multiply like so:

$$\phi(x,t) = e^{iH(t-t_0)}e^{-iH_0(t-t_0)}\phi_I(x,t)e^{iH_0(t-t_0)}e^{-iH(t-t_0)}$$

Now, if we define $U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}$ The above equation simply reads as

$$\phi(x,t) = U(t,t_0)^{-1}\phi_I(t,x)U(t,t_0)$$

Our operator U is a unitary time evolution operators. It has the property that

$$U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3), \quad U(t,t) = 1$$

It also satisfies the equation that

$$\begin{aligned}
i\frac{\partial U(t,t_0)}{\partial t} &= e^{iH_0(t-t_0)}(H - H_0)e^{-iH(t-t_0)} \\
&= e^{iH_0(t-t_0)}H_{int}e^{-iH(t-t_0)} \\
&= e^{iH_0(t-t_0)}H_{int}e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \\
&= H_I(t)U(t,t_0)
\end{aligned}$$

In this case $H_I(t)$ is our interaction picture Hamiltonian with

$$H_I(t) = e^{iH_0(t-t_0)}H_{int}e^{-iH_0(t-t_0)}$$

If H_I were just a function, we could solve this by setting

$$U = \exp\left[-i\int_{t_0}^{t} H_I(t')dt'\right]$$

but this doesn't work if there are ordering ambiguities. This is because we have that $[H_I(t'), H_I(t'')] \neq= 0$. However, our differential equation for $U(t,t_0)$ implies that it satisfies the equation

$$U(t,t_0) = 1 - i\int_{t_0}^{t} dt'\, H_I(t')U(t',t_0)$$

We substitute this back into itself to get the infinite series expansion

$$U(t,t_0) = 1 + (-i)\int_{t_0}^{t} dt' H_I(t') + (-i)^2\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' H_I(t')H_I(t'') + \dots$$

We can already intuitively see that this satisfies the differential equation. Also, the boundary condition that $U(t_0, t_0)$ works out.

For the ranges of integration, the H_I product is automatically time ordered! We'll see why this is the case - it has to do with clever relabelling and the diagram we'll show below. Let's look specifically at the second order term

$$\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' H_I(t') H_I(t'')$$

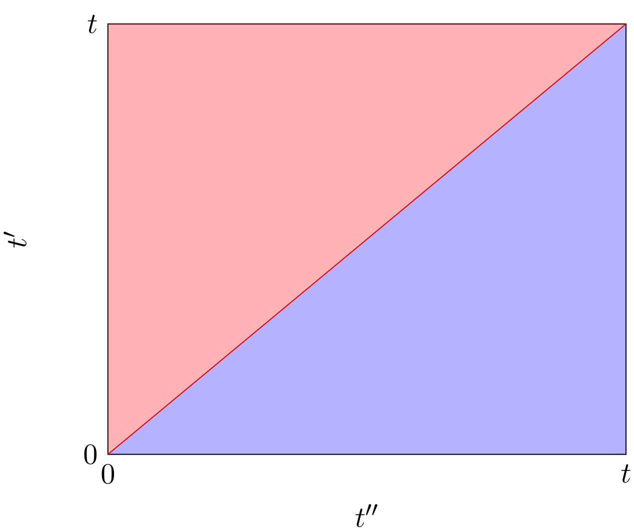


Figure 2: The integration variables add up to a square!

With this choice of dummy variable our integration variables are defined so that

$$t \geq t' \geq t'' > t_0$$

This is shown in the figure 2. So, by the definition of time ordering, in this **range**, we have that

$$\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' H_I(t') H_I(t'') = \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \mathcal{T}(H_I(t') H_I(t''))$$

However, integration variables are just dummies, so we can relabel them. If we relabel $t' \to t''$, and $t'' \to t'$, then we have that

$$\int_{t_0}^{t} dt'' \int_{t_0}^{t''} dt' H_I(t'') H_I(t') = \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' H_I(t') H_I(t'')$$

But in this case, our first expression has ranges defined by $t \geq t'' \geq t' > t_0$. This is the red portion of our square in the diagram. This is also consistent with time ordering, so we can put our time ordering symbol in the integrand so that in this range, $H_I(t'') H_I(t') = \mathcal{T}(H_I(t') H_I(t''))$. Hence, we can write our second term as

$$\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' H_I(t')H_I(t'') = \frac{1}{2}\left(\int_{t_0}^{t} dt'' \int_{t_0}^{t''} dt' \mathcal{T}(H_I(t')H_I(t'') + \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \mathcal{T}(H_I(t')H_I(t''))\right)$$

So, we have the same term in the integrand, $\mathcal{T}(H_I(t')H_I(t''))$! But, the ranges combined on the right hand side of this expression is just the full square. Hence, we can just integrate over the whole square to get that that

$$\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' H_I(t')H_I(t'') = \frac{1}{2}\int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' \mathcal{T}(H_I(t')H_I(t''))$$

In full generality, we can expand this idea to more integrals and write

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n H_I(t_1)\dots H_I(t_n) = \frac{1}{n!}\int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} \mathcal{T}\left(H_I(t_1)\dots H_I(t_n)\right)$$

Thus, we can write

$$U(t,t_0) = 1 + (-i)\int_{t_0}^{t} dt' H_I(t') + \frac{(-i)^2}{2}\int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' T\left\{H_I(t')H_I(t'')\right\}$$

This is Dyson's formula.

$$U(t,t_0) = T \text{ exp } \left\{-i\int_{t_0}^{t} dt'\, H_I(t')\right\}$$

Alternatively was have that, using the Lagrangian instead, that

$$U(t,t_0) = T\exp\left\{i\int_{t_0}^{t} d^3x dt' \mathcal{L}_I(t')\right\}$$

For scalar Yukawa theory, this would be

$$T\exp\left\{-ig\int d^4x\psi^*\psi\phi\right\}$$

This is something of a formal result. We expand this to finite order to get some results for scattering amplitudes which we will derive below in the next section.

4.3 Scattering

The time evolution used in scattering theory is called the 'S-matrix', where

$$S = \lim_{t\to\infty, t_0\to-\infty} U(t,t_0)$$

The initial state $|i\rangle$ and final state $|\phi\rangle$ are some sense 'far away' form each other and the interaction. We assume that $|i\rangle, |f\rangle$ behave like free particles; they're eigenstates of H_0 . THe amplitude is

$$\lim_{t\to\infty, t_0\to-\infty} \langle f|\, U(t,t_0)\, |i\rangle = \langle f|\, S\, |i\rangle$$

The latter expression is called the 'S-matrix' matrix element.

4.3.1 An example with Yukawa theory

Going back to scalar Yukawa theory, the interaction Hamiltonian is

$$H_I = g\psi_I^*\psi_I\phi_I$$

We model the creation of mesons with ϕ, and the creation of nucleons or anti nucleons with ψ. Concentrating on the creation and annihilation operators in the Fourier mode expansion,

$$\phi \sim a_{\vec{p}} + a_{\vec{p}}^\dagger$$

These things destroy and create mesons respectively. We have that

- $\psi \sim b_{\vec{p}} + c_{\vec{p}}^\dagger$ which destroys a nucleon and create an anti nucleon respectively.
- $\psi^* \sim b_{\vec{p}}^\dagger + c_{\vec{p}}$ which creates a nucleon and destroys an anti nucleon.

With this interaction, we have the commutation relations

$$[a_{\vec{p}}, a_{\vec{p}'}^\dagger] = [b_{\vec{p}}, b_{\vec{p}'}^\dagger] = [c_{\vec{p}}, c_{\vec{p}'}^\dagger] = (2\pi)^3\delta(\vec{p} - \vec{p}')$$

All other commutation relations are zero. To our first order our interaction, expanding out our interaction Hamiltonian in g, we might have an integral term like

$$\int dp' dp'' dp''' b_{\vec{p}}^\dagger c_{\vec{p}'}^\dagger a$$

How we we interpret this? Well, we could interpret this as a term which destroys a meson, then creates a nucleon and anti-nucleon.

If we expand to second order in g, we find more involved terms which stem from our integral. For example, we could have the term

$$\int (c^\dagger b^\dagger a)(bca^\dagger)$$

This is a two stage process. First, from our first bracket term we annihilate a nucleon and an anti-nucleon, and then create a scalar meson. Then, we annihilate that scalar meson an then create a nucleon and anti-nucleon. Thus, this is nucleon and anti-nucleon scattering, and is represented by the process

$$\psi\overline{\psi} \to \phi \to \psi\overline{\psi}$$

Thus, this term **specifically** contributes to nucleon and anti-nucleon scattering.

4.3.2 First order term in meson decay

In this part, we will be focusing on probability amplitudes that arise from certain processes. Let's focus specifically on the case of our first order interaction with a meson ϕ decaying into a nucleon and anti-nucleon

$$\phi \to \psi\overline{\psi}$$

Let's give a momentum to our meson, $\vec{p}$. Treating this as an initial state, we assume that we can use the ground state of our free theory, $|0\rangle$, as a springboard to excite states from. Thus, we write down the momentum eigenstate for our initial state as

$$|i\rangle = \sqrt{2E_p}a_{\vec{p}}^{\dagger}|0\rangle$$

If we assign our final state $|f\rangle$ to be a nucleon and anti nucleon with momenta $\vec{q}_1, \vec{q}_2$, then we have

$$|f\rangle = \sqrt{4E_{q_1}E_{q_2}}b_{\vec{q}_1}^{\dagger}c_{\vec{q}_2}^{\dagger}|0\rangle$$

Our scattering amplitude is given by

$$\langle f|S|i\rangle = \langle 0|bca^{\dagger}|0\rangle - ig\langle f|\int d^4x\, \psi_I^*(x)\psi_I(x)\phi_I(x)|0\rangle + O(g^2)$$

The zeroth order term is just zero, because $a^{\dagger}, c$ commute. Once we commute the c past $a^{\dagger}$, it hits our vacuum state and sends it to zero. The form of the first order term deserved some explanation. Since we're taking the limit as $t \to \infty$ and $t_0 \to -\infty$, the form of our first order term is

$$\langle f|\int_{-\infty}^{\infty} dt \int d^3x\psi_I^*(x)\psi_I(x)\phi_I(x)|i\rangle$$

we can compose the integrals together to get $\int d^4x$. Let's go slowly. We expand on the right hand side our $|i\rangle$ term to get that

$$\begin{aligned}\langle f|S|i\rangle &= -ig\langle f|\int d^4x\,\psi^*(x)\psi(x)\int\frac{d^3k}{(2\pi)^3\sqrt{2E_k}}\left(a_{\vec{k}}a_{\vec{p}}^{\dagger}e^{-ik\cdot x} + a_{\vec{k}}^{\dagger}a_{\vec{p}}^{\dagger}e^{-k\cdot x}\right)|0\rangle \\ &= \int d^4x\,d^3k\,\langle f|\psi^*(x)\psi(x)\frac{1}{(2\pi)^3\sqrt{2E_k}}\left(a_{\vec{k}}a_{\vec{p}}^{\dagger}e^{-ik\cdot x} + a_{\vec{k}}^{\dagger}a_{\vec{p}}^{\dagger}e^{-k\cdot x}\right)|0\rangle\end{aligned}$$

Now, since our $a_{\vec{k}}^{\dagger}$ commutes with $b, c, b^{\dagger}, c^{\dagger}$, we have that the second term trivially commutes past the ψ and ψ^* term, and it the part goes to zero. As for the first term, we commute $a_{\vec{k}}$ past and $a_{\vec{p}}^{\dagger}$, to pick up a $(2\pi)^3\delta(\vec{k}-\vec{p})$. Thus we replace

$$a_{\vec{k}}a_{\vec{p}}^{\dagger}|0\rangle = [a_{\vec{k}}, a_{\vec{p}}^{\dagger}]|0\rangle = (2\pi)^3\delta(\vec{p}-\vec{k})|0\rangle$$

This means that our term above is

$$\begin{aligned}\langle f|S|i\rangle = -ig\int\frac{d^4xd^3k_1d^3k_2}{(2\pi)^6}\sqrt{4E_{q_1}E_{q_2}}\langle 0|\frac{1}{\sqrt{4E_{k_1}E_{k_2}}}c_{\vec{q}_2}b_{\vec{q}_1}\left(b_{\vec{k}_1}^{\dagger}e^{ik_1\cdot x} + c_{\vec{k}_1}e^{-ik_1\cdot x}\right) \\ \left(b_{\vec{k}_2}e^{-ik_2\cdot x} + c_{\vec{k}_2}^{\dagger}e^{ik_2\cdot x}\right)e^{-ip\cdot x}|0\rangle\end{aligned}$$

We can create a caricature of this product by ignoring the momenta and integrals. We get a string of operators that look like

$$\sim cbb^{\dagger}b + cbb^{\dagger}c^{\dagger} + cbcb + cbcc^{\dagger}$$

Now, all strings ending with b vanish since they operate on the $|0\rangle$ term to the right.

We're left with the terms

$$\sim cb b^\dagger c^\dagger + cbcc^\dagger$$

Now, in the second term, if we commute c past $c^\dagger$, we pick up a delta function. But this leaves a term of the form δcb, and the b still annihilates. So, our only surviving term is

$$\sim cbb^\dagger c^\dagger = c_{\vec{q}_2} b_{\vec{q}_1} b^\dagger_{\vec{k}_1} c^\dagger_{\vec{k}_2}$$

Commuting b past $b^\dagger$ gives the following term in our integral!

$$(2\pi)^3 \delta(\vec{q}_1 - \vec{k}_1) c_{\vec{q}_2} c^\dagger_{\vec{k}_2} e^{ix\cdot(k_1+k_2-p)}$$

Once again, commuting c past $c^\dagger$ gives us another delta function to include. So the only term preserved in our integral is

$$(2\pi)^6 \delta(\vec{q}_1 - \vec{k}_1)\delta(\vec{q}_2 - \vec{k}_2) e^{ix\cdot(k_1+k_2-p}$$

Thus, performing our integrals over $\vec{k}_1$ and $\vec{k}_2$ gives us our only term at first order which contributes to our scattering.

$$\langle f| S |i\rangle = -ig \langle 0| \int d^4x\, e^{i(q_1+q_2-p)\cdot p} |0\rangle = -ig\delta^4(q_1 + q_2 - p)$$

But, this gives a final amplitude which is a delta function in 4 space. This means that for a non zero scattering amplitude, we require our four momentum to be conserved!

4.4 Wick's theorem

4.5 Feynman Diagrams

Feynman diagrams are a visual way to represent our complicated integrals we've defined above. We draw Feynman diagrams to represent the expansion of $\langle f| (S-1) |i\rangle$ and learn to associate functions to them (functions of the four momentum). We draw an external line for each particle in $|i\rangle$ and $|f\rangle$, assigning a four dimensional 4-momentum to each. We add an arrow for $\mathbb{C}$ fields to show flow of charge.

The first step is to set up our initial and final states. We choose an in (out) going arrow in $|i\rangle$ for a particle (antiparticle), and the opposite for final states $\langle f|$. These are 'external lines' which don't contribute an integral since we're just exciting by an operator like $b^\dagger_{\vec{p}}|0\rangle$.

We show what kind of diagrams you might have for different scattering cases in figure 3. The next step is to 'fill this in'. We have to be a bit careful here. This is is when we need to be aware of what our interaction term looks like. For the Yukawa potential, we have our interaction Hamiltonian

$$H_I = \int d^3x \psi(x)^* \psi(x)\phi(x)$$

this means that **each set of connecting lines** (what we call a vertex) in our Feynman diagram needs to look like what we have in figure 4. We then fill up this diagram with vertices. The order of our expansion in g corresponds to how many vertices we have, since each vertex corresponds to

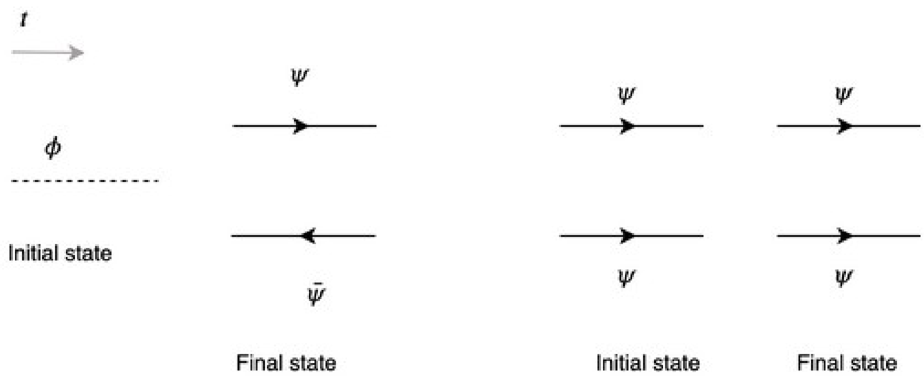


Figure 3: A template diagram. On the left we have $\phi \to \psi\overline{\psi}$ scattering and on the right we have a template for $\psi\psi \to \psi\psi$ scattering

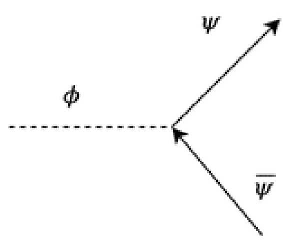


Figure 4: This is what we must have at each point

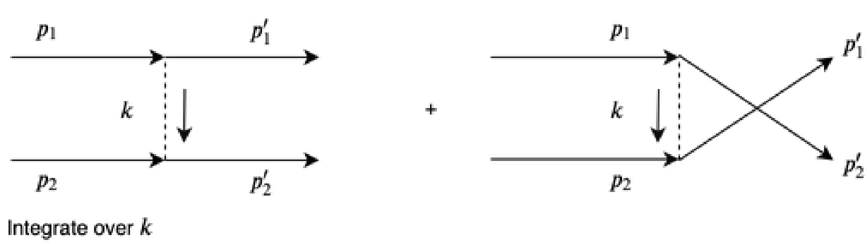


Figure 5: We have some $O(g^2)$ diagrams for $\psi\psi \to \psi\psi$ scattering

an integral $\int d^4xH_I(x)$ contributed in Dyson's formula. Once we've joined together our vertices, we then assign a momenta to each line. We have simple examples in the figure.

Our 'internal lines' which are lines with vertices on both sides are dummy momenta which we integrate over. On the other hand, we keep external lines with fixed momenta. Each diagram is in 1: 1 correspondence with terms in our expansion. To each diagram, we evaluate total amplitude using Feynman rules for this problem.

1. Associate a momentum to each internal line, and keep the momenta associated with external legs of the problem fixed.
2. Assign a factor of $(-ig)(2\pi)^4\delta(\sum k_i)$ to each vertex, where $\sum_i k_i$ is the sum of four momenta flowing **in** to the vertex. If momenta flows out, this gives a negative momentum contribution.
3. For each internal line with 4-momenta k, write factor

$$\int \frac{d^4k}{(2\pi)^4} \text{ where } D(k^2) = \begin{cases} \frac{i}{k^2-m^2-i\epsilon} & \text{for } \phi \\ \frac{i}{k^2-\mu^2+i\epsilon} & \text{for } \psi \end{cases}$$

The total amplitude is the sum of all diagrams at the given order.

4.6 Scattering revisited

Now, to order $O(g^2)$From our first diagram, we have that our contribution is given by the sum of the diagrams in figure 5. Applying our Feynman rules, we have that these diagrams give an integral of

$$\begin{aligned} = (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\epsilon} (2\pi)^8 (\delta(p_1 - p_1' - k)\delta(p_2 + k - p_2') \\ + \delta(p_1 - p_2' - k)\delta(p_2 + k - p_1')) \end{aligned}$$

Our final scattering momenta to second order is therefore

$$i(-ig)^2 \left\{ \frac{1}{(p_1 - p_1')^2 - m^2 + i\epsilon} + \frac{1}{(p_1 - p_2')^2 - m^2 + i\epsilon} \right\} (2\pi)^4 \delta(p_1 + p_2 - p_1' - p_2')$$

Note that the meson doesn't necessarily satisfy the relation that $k^2 = m^2$. If its doesn't it's called an off shell or virtual particle. One might also think that the second diagram in figure 5 is superfluous, but we need to count this as a distinct diagram since the ψ are indistinguishable under exchange, and hence obey Bose-Einstein statistics for identical particles.

We can go to even higher order to the point where we introduce loops into our diagrams.

4.7 Amplitudes

In the above, we will always have a factor to impose momentum conservation between final and initial states. Hence, this fact helps us simply things a bit. We will define the matrix element

$\mathcal{M}$ by

$$\langle f| (S-1) |i\rangle = i\mathcal{M}(2\pi)^4\delta(\sum_{\text{final state particles}} p_i - \sum_{\text{initial state particles}} p_i)$$

Our factor of i in our expression is done to match conventions with non-relativistic quantum mechanics. Our delta function expression follows from translation invariance, and is common to all S-matrix elements we compute.

We now can define a new set of Feynman rules to compute $i\mathcal{M}$, which makes things slightly easier.

1. We draw all possible diagrams with appropriate external legs, and impose 4 momentum conservation at each vertex with appropriate use of delta functions.
2. We assign a factor of $(-ig)$ at each vertex to take into account the order of our expansion.
3. We integrate over closed loops in the diagram, since they give rise to dummy variables which we can integrate over. So, for closed loops, we do the integral $\int \frac{d^4k}{(2\pi)^4}$.

We'll now do an example with meson scattering.

For example, for meson-meson scattering $\phi\phi \to \phi\phi$, we need to draw diagrams. Our lowest order diagram is a bit tricky and our requirement that our vertex has to be a meson, nucleon and anti-nucleon trio means that there'll be a loop. This is shown in figure 6.

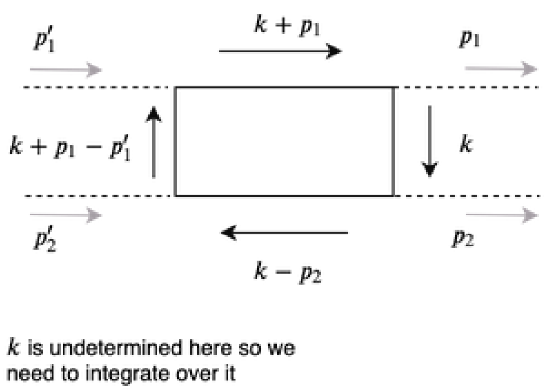


Figure 6: Our lowest order scattering term for $\phi\phi \to \phi\phi$ decay

In the loop, we build up our momentum labels by picking a side as k, then just working around whilst using conservation of momentum. This is

$$\mathcal{M} = \int \frac{d^4k}{(2\pi)^4}(-ig)^4 \frac{i^4}{(k^2-\mu^2+i\epsilon)((k+p_1)^2-\mu^2+i\epsilon)}$$
$$\times \frac{1}{((k+p_1-p_1')^2-\mu^2+i\epsilon)(k+p_1)^2-\mu^2+i\epsilon}$$

The integrand goes as $\frac{1}{k^8}$, so we're sure that this thing converges.

4.8 Computing the two point correlation function for the ground state of our perturbed Hamiltonian

Let's call $|\omega\rangle$ our ground state for our Hamiltonian. A natural question to ask would be what $\langle\omega|\,\phi(x)\phi(y)\,|\omega\rangle$ is in terms of the spectrum we already know for the free Hamiltonian H_0.

One way to do this would be to take the ground state of the free Hamiltonian which we call $|0\rangle$ and expand it in terms of the energy eignestates of the full Hamiltonian. with the full Hamiltonian, the state evolves as

$$\begin{aligned} e^{-iHt}\,|0\rangle &= e^{-iHt}\,|\omega\rangle\,\langle\omega|0\rangle + \sum_{n\geq 1} e^{-iHt}\,|n\rangle\,\langle n|0\rangle \\ &= e^{-iE_0t}\,|\omega\rangle\,\langle\omega|0\rangle + \sum_{n\geq 1} e^{-iE_nt}\,|n\rangle\,\langle n|0\rangle \end{aligned}$$

but by construction we've assigned these energy eignestates in terms of the magnitudes of their energy eigenvectors, so $E_0 < E_1 \ldots E_n < \ldots$. Thus, to make the terms in the sum disappear, we employ a clever trick. This trick is to basically make the e^{-iE_0t} term decay slower than the e^{-iE_nt} terms, by taking the limit of t to $t \to (1-i\epsilon)\infty$, where ϵ is chosen sufficiently small as to make the terms in the sum decay, but not the exponential term in front of the ground state. Hence our final result is that, our ground state for the full hamiltonian $|\omega\rangle$ can be written as

$$|\omega\rangle = lim_{t\to(1-i\epsilon)\infty} e^{-iHt}\left(e^{-iE_0t}\,\langle\omega|0\rangle\right)^{-1}$$

4.9 Wick's theorem

4.10 Applying Wick's theorem for the interaction term

Let's apply Wick's theorem to calculate the next term in the series of our quantity

$$\langle 0|\,\mathcal{T}\phi(x)\phi(y)\exp\left(-i\int dtH_I(t)\right)|0\rangle$$

. Let's resume our discussion in the case of ϕ^4 theory, where our interaction Hamiltonian is $\int d^3z\phi^4(z)$. Recall, our first term in the series expansion is just

$$\langle 0|\,\phi(x)\phi(y)\,|0\rangle = \Delta_F(x-y)$$

and our expression in the series expansion is

$$\langle 0|-i\frac{\lambda}{4!}\mathcal{T}\phi(x)\phi(y)\int dt\int d^3z\phi^4(z)\,|0\rangle$$

We can write this a little more concisely by combining writing $\int dt\int d^3z = \int d^4z$, so we're aware that it's an integral over four space time dimensions. To make it clearer to understand the nature of the contractions in this expression, we do something a bit weird and write out the terms in the ϕ^4 term as $\phi(z)\phi(z)\phi(z)\phi(z)$. We want to cookup a simplified expression for the term

$$-i\frac{\lambda}{4!}\,\langle 0|\,\mathcal{T}\phi(x)\phi(y)\int d^4z\phi(z)\phi(z)\phi(z)\phi(z)\,|0\rangle$$

Since we have a time ordering operator there, the only non zero terms which surivive once we expand this thing into a sum are the terms where every scalar ϕ in contracted with another. The important thing to remember is that this includes the $\phi(z)$ terms incorporated into the integral. There are several contraction we could do here, but the most obvious one to start with is to first contract $\phi(x)$ and $\phi(y)$ together, and then pair up the terms in the integral as follows

$$-\frac{i\lambda}{4!}\phi(x)\phi(y)\int d^4z\phi(z)\phi(z)\phi(z)\phi(z)$$

but this corresponds to, replacing the contractions with Feynman propagators,

$$-\frac{i\lambda}{4!}\Delta_F(x-y)\int d^4\Delta_F(z-z)\Delta_F(z-z)$$

Alternatively, we could have contracted the $\phi(z)$ terms differently in the integrand, and instead could have contracted the terms like so:

$$-i\frac{\lambda}{4!}\phi(x)\phi(y)\int d^4z\phi(z)\phi(z)\phi(z)\phi(z)$$

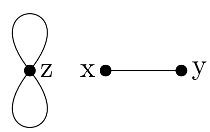
This term here still corresponds to the integral

$$-\frac{i\lambda}{4!}\Delta_F(x-y)\int d^4z\Delta_F(z-z)\Delta_F(z-z)$$

We count that there are in total 3 different sets of contractions which yield this integral. So the contribution that we have from contractions of this type is

$$-\frac{3i\lambda}{4!}\Delta_F(x-y)\int d^4z\Delta_F(z-z)\Delta_F(z-z)$$

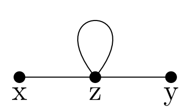
We represent this type of contraction diagramatically like before in the following diagram.



We also have another type of contraction that we can include. We contract $\phi(x)$ with one of the $\phi(z)$, and contract $\phi(y)$ with another one. This gives us 12 choices for our contraction. One of these may look like

$$-\frac{i\lambda}{4!}\phi(x)\phi(y)\int d^4z\,\phi(z)\phi(z)\phi(z)\phi(z)$$

we count 12 of these, and represent this diagramatically in the figure adjacent. Our total first



order contribution is therefore

$$-\frac{3i\lambda}{4!}D_F(x-y)\int d^4z\,D_F(z-z)D_F(z-z)-\frac{12i\lambda}{4!}\int d^4z\,D_F(x-z)D_F(y-z)D_F(z-z)$$

which exhausts all of the possible ways to contract terms in our sum above.

4.10.1 Symmetry factors

We now present a convinient way to find the coefficent of our contribution which a diagram contributes. This is called the symmetry factor of the diagram. It is as follows. Given a particular diagram,

1. Ascribe a symmetry factor of 2 for edges which loop to the same node.
2. Ascribe a symmetry factor of n for edges which can be interchanged
3. Ascribe a symmetry factor of 2 for vertices which are equivalent.

As an example, consider the diagram below.

4.11 Feynman Rules in ϕ^4 theory

When we consider interaction terms which contain the same species of particle, things get a bit more complicated, and we need to take into account different permutations we can get. When considering something like the Yukawa interaction, $H_I = g \int d^3x \psi\psi^*\phi$, our interaction term contains distinct particles. That means, for example, that our Feynman rules we derived earlier doesn't come with extraneous factors and we only have to add a prefactor $(-ig)$ at each vertex point. However, for the case of an interacting theory like

$$\mathcal{H}_I = \frac{\lambda}{4!}\phi^4$$

we have to change our thinking a bit for the sake of making life easier. On might think that, similar to the above case, that a vertex drawn in the Feynman rules for ϕ^4 theory might mean that we add a factor of $\frac{-i\lambda}{4!}$, however, taking into account the different permutations of the four ϕ fields, it's easier for us to just attach a $-i\lambda$ for each vertex then divide by a small number based on the look of the diagram. Our final goal here will be to compute amplitudes which contain terms like

$$i\mathcal{M} \sim \frac{-i\lambda}{4!}\int d^4x \, \left\langle p_1', p_2' \right| : \phi(x)\phi(x)\phi(x)\phi(x) \left| p_1, p_2 \right\rangle$$

4.11.1 Combinatoric factors

To motivate our discussion, we'll start by considering interactions in position, and not momentum space. We'll start by calculating a propagator-like object for a scattering system with m particles, which is given by

$$\langle 0| \mathcal{T}\{\phi_1 \dots \phi_n S\} |0\rangle$$

In our perturbation expansion for the case of ϕ^4 theory, we'll have terms like

$$\frac{1}{n!}\left(\frac{-i\lambda}{4!}\right)^n \int d^4y_1 \dots d^4y_n \, \langle 0| T\left\{\phi_1 \dots \phi_n \phi^4(y_1) \dots \phi^4(y_n)\right\} |0\rangle$$

To illustrate the idea of symmetry factors, we will consider the case of a first order expansion ($n = 1$), with four particles to consider. We expand our integral as

$$\dots = \frac{-i\lambda}{4}\int d^4x \, \langle 0| \mathcal{T}\left\{\phi_1 \dots \phi_4 \phi_x^4\right\} |0\rangle$$

Now, we apply Wick's theorem to determine the non-vanishing components in the integrand. One possible configuration we could have is that we could contract each of the ϕ_i term s with one of the ϕ_x terms, which gives several terms up to permutation. Or we could contract two of the ϕ_i ' s together and contract the rest with ϕ_x. Finally, we could have also just contracted the ϕ_i's amongst themselves, and t the ϕ_x 's amongst themselves. In total, this sum then looks like

$$
\begin{aligned}
\ldots = \frac{-i\lambda}{4!}\int d^4x\, &\phi_1\phi_2\phi_3\phi_4\phi_x\phi_x\phi_x\phi_x \\
+ \;&\text{similar terms gained from contracting all } \phi_i \text{ with } \phi_x \\
+ \frac{-i\lambda}{4!}\int d^4x\, &\phi_1\phi_2\phi_3\phi_4\phi_x\phi_x\phi_x\phi_x \\
+ \;&\text{similar contractions which only contract two } \phi_x \\
+ \frac{-i\lambda}{4!}\int d^4x\, &\phi_1\phi_2\phi_3\phi_4\phi_x\phi_x\phi_x\phi_x \\
+ \;&\text{similar contractions where no } \phi_x \text{ is contracted with } \phi_i
\end{aligned}
$$

Now, the upshot of this is that since contractions are $\mathbb{C}$ functions, Each of the terms in the same 'category' yield the same expression in terms of Feynman propagators. For the first term, we have a total of 24 permutations that yield the above. This is because we can contract ϕ_1 with any of the 4 ϕ_x 's. Then, we can pair ϕ_2 with the remaining 3 , and so on. So we have 4! permutations here. Each of these terms yields a factor of

$$\frac{-i\lambda}{4!}\int d^4x\, \Delta_F(x_1-x)\Delta_F(x_2-x)\Delta_F(x_3-x)\Delta_F(x_4-x)$$

and since we have 24 of them, our total contribution is

$$-i\lambda\int d^4x\, \Delta_F(x_1-x)\Delta_F(x_2-x)\Delta_F(x_3-x)\Delta_F(x_4-x)$$

Now, looking **specifically** at the terms where we have ϕ_3 and ϕ_4 contracted with ϕ_x, we have 12 ways to configure our contractions. More so, we could have equally included terms which contracted ϕ_1 and ϕ_2 with ϕ_x instead, which again yields a term with 12 contractions (we have 6 possible diagrams which give this configuration, each of them implcitly summing over 12 possibilities). Finally, in the term where we contract x_1 with x_2, and x_3 with x_4, we have 3 possible ways for the $\phi_x's$ to contract amongst themselves.

Thus, our total expansion in terms of Feynman propagators is given by

$$
\begin{aligned}
\langle 0|\, T\left\{\phi_1\phi_2\phi_3\phi_4\phi_x\phi_x\phi_\phi x\right\}|0\rangle = &-i\lambda\int d^4x\Delta_F(x_1-x)\Delta_F(x_2-x)\Delta_F(x_3-x)\Delta_F(x_4-x) \\
&\frac{-i\lambda}{2}\int d^x\,\Delta_F(x_1-x_2)\Delta_F(x_3-x)\Delta_F(x_4-x)\Delta_F(x-x) \\
&+\;5 \text{ similar permutations} \\
&-\frac{i\lambda}{8}\int d^4x\Delta_F(x_1-x_2)\Delta_F(x_3-x_4)\Delta_F(x-x)\Delta_F(x-x) \\
&+\;2 \text{ similar permuations}
\end{aligned}
$$

Now, if we expand to higher order, to $O(\lambda^2)$, we might get a term like

$$\frac{1}{2}\left(\frac{-i\lambda}{4!}\right)^2\int d^4xd^4y\,\langle 0|\, T\left\{\phi_1\phi_2\phi_3\phi_4\phi_x^4\phi_y^4\right\}|0\rangle$$

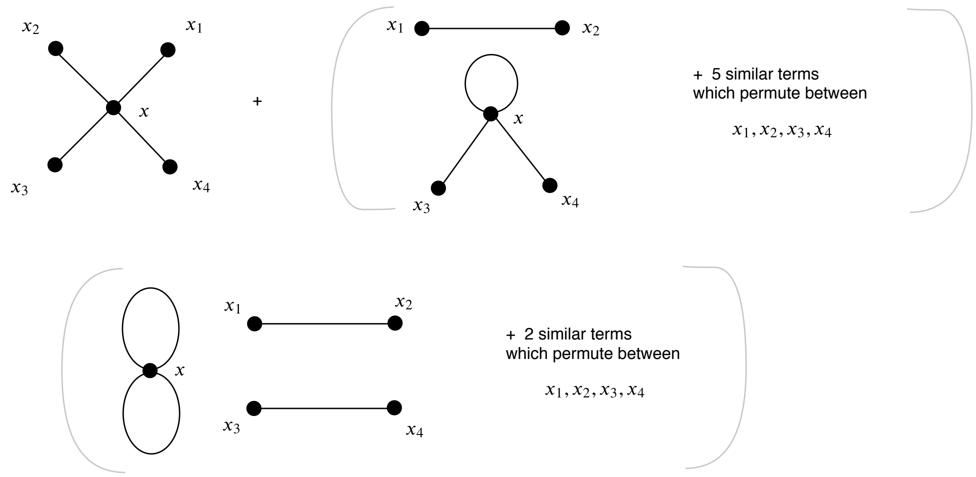


Figure 7: A diagrammatic expansion of our representation

Now, in this term, if we insist that x_1 and x_2 are contracted with ϕ_x and that ϕ_3, ϕ_4 are contracted with ϕ_y, then we could get a contraction that looks like

$$\phi_1\phi_2\phi_3\phi_4\phi_x\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y\phi_y$$

Now we ask how many terms of this type there are. We have a total of 12 possibilities from our possible contractions of ϕ_4, ϕ_3 with ϕ_y. And, we also have 12 times this from ϕ_1, ϕ_2 with ϕ_x. This means that our prefactor is going to be

$$\frac{1}{2} = \frac{1}{2}\frac{1}{4!}12 \times 12$$

Our associated diagram for this kind of object is shown in figure 8.

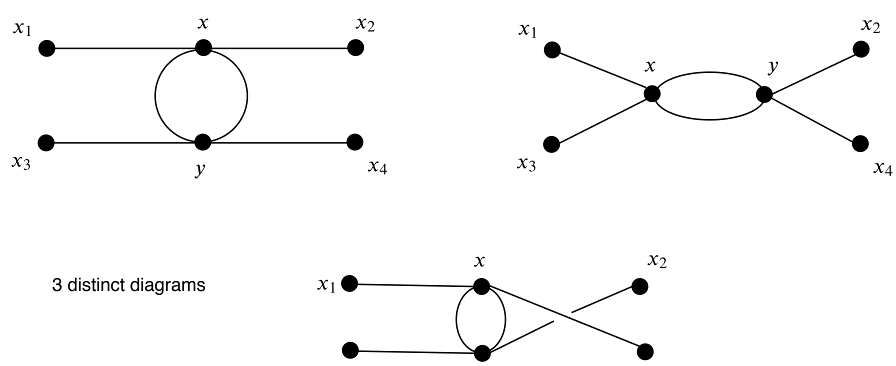


Figure 8: All diagrams of the above type with distinct contractions; each has a symmetry factor of 2

Our associated contribution to this scattering in terms of Feynman propagators is then

$$\frac{(-i\lambda)^2}{2}\int d^x\, d^4y\, \Delta_F(x_1 - x)\Delta_F(x_2 - x)\Delta_F(x_3 - y)\Delta_F(x_4 - y)(\Delta_F(x - y))^{@}$$

4.11.2 Deriving our Feynman rules

We can now use our analysis above to match a diagram with an integral expression. We conclude that the expression

$$\langle 0| T \left\{ \phi_1 \dots \phi_m \exp\left(\frac{-i\lambda}{4!} \int d^4x \phi_x^4 \right) \right\} |0\rangle$$

is the sum off all possible diagrams with m 'external points' which are just points with a single line coming out of them, along with any number of internal vertex points (which are points with four lines coming into them since we are dealing with ϕ^4 theory. Our routine is to then, for each diagram that exists, have the Feynman rules where we

1. Attach a propagator $\Delta_F(x-y)$ for any line connecting the points x and y, no matter whether they represent external or internal points.
2. For every vertex (a point where four lines meet), this represents an integral over our interaction space so we ascribe a factor of $(-i\lambda)\int d^4x$.
3. We divide by the appropriate symmetry factor of the diagram.

Now, we have a nice way to write down amplitudes in ϕ^4 theory in position space. The next thing to do would then be to transfer this into some momentum rules. Recall our momentum expansion of the Feynman propagator

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}$$

This integral has a nice interpretation. Since this represents a line going from y into x, we interpret $e^{ip\cdot y}$ and $e^{-ip\cdot x}$ as momentum factors we can attach at each vertex for momentum going out of y and into x. In addition, our $\frac{i}{p^2 - m_i^2 \epsilon}$ factor can be an expression attached to the line itself. At a vertex point, we'll have some propagators coming in to meet at a point. Due to the fact that we're integrating over $\int d^4x$, a factor like this corresponds to

$$\int d^4x e^{-ix\cdot(p_1+p_2-p_3-p_4)} = (2\pi)^4 \delta^4(p_1+p_2-p_3-p_4)$$

for example. Hence, in momentum space, our Feynman rules look like

1. For each line, assign a momentum p and attach a factor of $\frac{i}{p^2-m^2+i\epsilon}$ in the integral.
2. At each vertex, assign a factor of $i\lambda$, and also impose the conservation of momentum which comes from a delta function appearing.
3. Integrate over all other undetermined momenta $\int \frac{d^4k}{(2\pi)^4}$
4. Divide by the symmetry factor of the diagram

4.11.3 Vacuum Bubbles and Connected Diagrams

If we consider scattering both to and from our free theory vacuum state, we are describing the term $\langle 0| S |0\rangle$. This means however, that we have no external points. Our only valid diagrams are the ones where every point is an internal vertex (four lines going in). So, our total

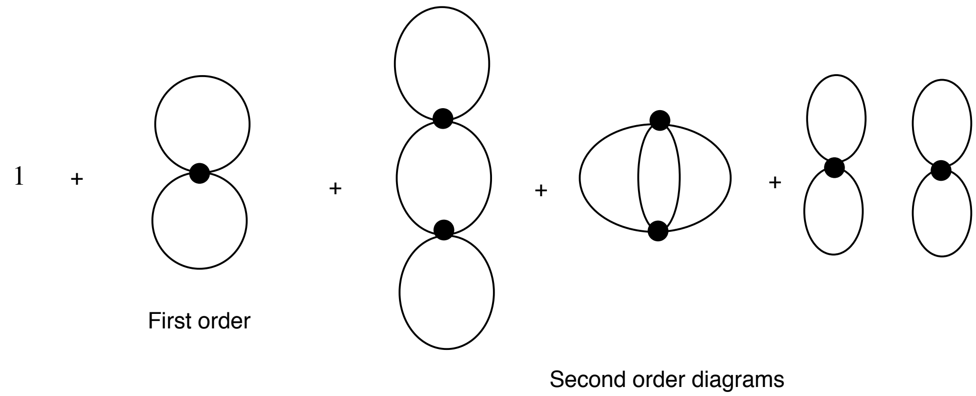


Figure 9: All vacuum bubble diagrams contributing to the scattering

amplitude is given by the sum of all vacuum bubble diagrams shown in figure 9. Now, one can convince themselves that we can write this sum as an exponential of just distinct vacuum bubble diagrams. This sum is shown diagrammatically in figure 10. In other words, we can write our

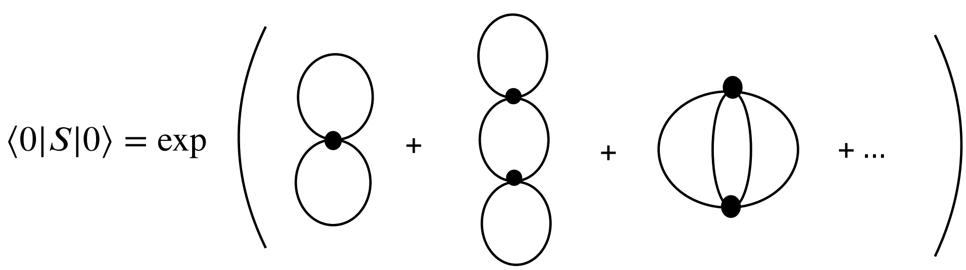


Figure 10: Combinatoric factors work out to give an exponential of distinct vacuum types

vacuum amplitude as

$$\langle 0| S |0\rangle = \exp\left(\text{Distinct vacuum bubble types}\right)$$

Now, from our previous discussion, we had that a term like

$$\langle 0| T\left\{\phi_1 \ldots \phi_n S\right\} |0\rangle = \sum \text{ diagrams with m external points}$$

but, amazingly, we have that this quantity **factorizes** into two parts, one factor being the exponential of distinct vacuum bubble types $\langle 0| S |0\rangle$, and the other factor the sum of just connected diagrams.

$$\langle 0| T\left\{\phi_1 \ldots \phi_m S\right\} |0\rangle = \left(\sum \text{ connected diagrams }\right)\left(\langle 0| S |0\rangle\right)$$

4.12 Green's function vacuum

Let's delve into the nuts and bolts of the vacuum state in our interacting theory. We denote vacuum state of our interacting theory as $|\Omega\rangle$. This is a different state from our vacuum state in our free theory, where we have that

$$H_0 |0\rangle = 0$$

However when we move to our interaction picture, we get that $H = H_0 + H_{\text{ int }}$ satisfies

$$H |\Omega\rangle = 0, \quad \langle\Omega| |\Omega\rangle$$

We define 'Green's functions' as

$$G^{(n)} = \langle\Omega| T \{\phi_H(x_1) \dots \phi_H(x_n)\} |\Omega\rangle$$

which can be interpreted as our n point correlator but this time, in our interaction picture. Notice that this is time dependent, and that our states are in the Heisenberg picture. Now, interaction picture vacuum states are difficult to deal with. The goal of this section is to be able to write them as a function of our free theory vacuum state $|0\rangle$. Our claim is that

$$\langle\Omega| T \{\phi_{1H} \dots \phi_{mH}\} |\Omega\rangle = \frac{\langle 0| T \{\phi_{1I} \dots \phi_{mI} S\} |0\rangle}{\langle 0| S |0\rangle}$$

Let's explain some of the notation here. Recall, that S is merely our time evolution operator and since we are considering initial and final state scattering this is

$$S = \lim_{T\to\infty} U(T, -T) = \lim_{T\to\infty} \exp\left(\int_{-T}^{T} dt H_I(x)\right)$$

We denote states in the Heisenberg picture (on the left hand side), as ϕ_{iH}, and states in the interaction picture as ϕ_{iI}. From the previous section, we showed that the numerator on the right has side is

$$\langle 0| T \{\phi_{1I} \dots \phi_{mI} S\} |0\rangle = \sum (\text{ connected diagrams with m external points }) \langle 0| S |0\rangle$$

In other words, our Green's function is **the sum of connected diagrams** with m external points. To prove this, we take, without loss of generality, that $x_1^0 > x_2^0 > \dots > x_m^0$. This means, we can employ a trick to parition the integral and reorder things in the expression.

$$\begin{aligned}\mathcal{T}\big(\phi_{1I}\phi_{2I} \dots \phi_{mI} \exp\left(-i \int_{-\infty}^{\infty} dt H_I(t)\right)\big) = \mathcal{T}\big\{&\phi_{1I}(x_1^0)\phi_{2I}(x_2^0) \dots \phi_{mI}(x_m^0) \\ &\exp\left(-i \int_{-\infty}^{x_m^0} dt H_I(x) - i \int_{x_m^0}^{x_{m-1}^0} dt H_I(x) - \dots - i \int_{x_1^0}^{\infty} dt H_I(t)\right)\big\}\end{aligned}$$

The uposhot of writing this that we can now we apply to time ordering operator, which shuffles around the terms in the exponential. Thus, the expression above reads

$$\begin{aligned}\dots = \exp\left(-i \int_{x_1^0}^{\infty} dt H_I(t)\right) \phi_{1I}(x_1) \exp\left(-i \int_{x_2^0}^{x_1^0} dt H_I(t)\right) \dots \\ \dots \exp\left(-i \int_{x_{n-1}^0}^{x_n^0} dt H_I(t)\right) \phi_{Im} \exp\left(-i \int_{-\infty}^{x_n^0} dt H_I(t)\right)\end{aligned}$$

Now, from our definition of $U(t,t')$, we have that finally

$$\mathcal{T}\{\phi_{1I}\ldots\phi_{mI}\} = U(\infty,t_1)\phi_{1I}U(t_1,t_2)\phi_{I2}\ldots U(t_{m-1},t_m)\phi_{Im}U(t_m,-\infty)$$

where we've cleaned up our notation above by writing $t_1 = x_i^0$.

Recall that $U(t,t_0) = e^{iH_0t}e^{iH(t-t_0)}e^{-iH_0t_0}$. From our discussion, we sandwich the above expression with $|0\rangle$, so we have that

$$\langle 0| T\{\phi_{1I}\ldots\phi_{mI}S\}|0\rangle = \langle 0| U(\infty,t_1)\phi_{I1}U(t_1,t_2)\phi_{2I}\ldots U(t_{m-1},t_n)\phi_{mI}U(t_m,-\infty)|0\rangle$$

Now, we split up the unitary operator terms to wrangle this expression to a form which is in the Heisenberg picture. The above term is

$$\cdots = \langle 0| U(\infty,0)U(0,t_1)\phi_{1I}U(t_1,0)U(0,t_2)\phi_{2I}\ldots U(0,t_m)\phi_{mI}U(t_m,0)U(0,\infty)|0\rangle$$

Now, we use this to switch to the Heisenberg picture by observing that the above expression is equal to

$$\langle 0| T\{\phi_{1I}\ldots\phi_{mI}S\}|0\rangle = \langle\psi| U(0,-\infty)|0\rangle = \langle 0| U(\infty,0)\phi_{1H}\ldots\phi_{mH}U(0,-\infty)|0\rangle$$

Here, we've defined the state $\langle\psi|$ as

$$\langle\psi| = \langle 0| U(\infty,0)\phi_{1H}\ldots\phi_{mH}$$

From the right hand side, this is just

$$\begin{aligned}
\ldots &= \lim_{t_0\to-\infty}\langle\psi| U(0,t_0)|0\rangle \\
&= \lim_{t_0\to-\infty}\langle\psi| e^{-iHt_0}|0\rangle\,,\ \text{since } U(0,t_0)|0\rangle = e^{-iHt_0}e^{-iH_0t_0}|0\rangle = e^{-iHt_0}|0\rangle \\
&= \lim_{t_0\to-\infty}\langle\psi| e^{iHt_0}\left[|\Omega\rangle\langle\Omega| + \sum_{n=1}^{\infty}\prod_{j=1}^{n}\int\frac{d^3p_j}{2E_{p_j}(2\pi)^3}|p_1,\ldots p_n\rangle\langle p_1,\ldots p_n|\right]|0\rangle \\
&= \langle\psi|\Omega\rangle\langle\Omega|0\rangle + \lim_{t_0\to-\infty}\sum_n\int\left(\prod_j\frac{d^3p_j}{(2\pi)^3 2E_{p_j}}\right)e^{-i\sum_{j=1}^{n}E_{p_j}t_0}\langle\psi|p_1,\ldots p_n\rangle\langle p_1,\ldots p_n|0\rangle
\end{aligned}$$

Going into the third line, we've applied our completeness relation for Fock space, where we sum over all momentum eigenstates for every number of particle we can have. Now, the second term in the last line goes to zero due to the Riemann-lebesgue lemma

$$\lim_{\mu\to\infty}\int_a^b f(x)e^{i\mu x}dx = 0$$

Hence, the above expression reads

$$\cdots = \langle 0| U(\infty,0)\phi_{1H}\ldots\phi_{mH}|\Omega\rangle\langle\Omega|0\rangle$$

Similar to the above, we haev that

$$\begin{aligned}
\langle 0| U(\infty,0)|\psi\rangle &= \lim_{t_0\to\infty}\langle 0| e^{iHt_0}|\psi\rangle \\
&= \langle\Omega|\phi_{1H},\ldots\phi_{mH}|\Omega\rangle\langle\Omega|0\rangle\langle 0|\Omega\rangle
\end{aligned}$$

Our denominator is $\langle 0| S|0\rangle = \langle 0||\Omega\rangle\langle\Omega||0\rangle$, by setting $m=0$. This means we (insert previous example) discounting connected diagrams

4.12.1 LSZ Reduction

To describe scattering in interacting theory, with external states (eg $|p_1, p_2\rangle$, should be from interacting theory. This means we exclude loops on external legs. THe above diagram is banned since it's absorbed into the definition of an initial interacting state. See AQFT or look up amputated diagrams.

5 Cross sections and decay rates

In this section, we'll look at calculating decay rates (the probability of particles undertaking a certain type of scattering due to the interaction potential).

5.0.1 Kinematics

Let's first consider the case of 2 particles scattering into two particles. For example, this could be a nucleon anti-nucleon pair scattering into two mesons ($\overline{\psi}\psi \to \phi\phi$.

To analyse this problem from a kinematic perspective, we analyse incoming and outgoing momenta. Suppose that p_1^μ and p_2^μ represent the 4 momenta of the two incoming particles, and that q_1^μ and q_2^μ represent their outgoing momenta. Due to conservation of momentum, we have that

$$p_1^\mu + p_2^\mu = q_1^\mu + q_2^\mu$$

if we define the Mandelstam variables

$$\begin{aligned} s &:= (p_1 + p_2)^2 \\ t &:= (p_1 - q_1)^2 \\ u &:= (p_1 - q_2)^2 \end{aligned}$$

The neat thing about these variables is that, when we add them up, we get the sum of the squares of the rest masses of our constituent particles.

$$\begin{aligned} s + t + u &= 3p_1^2 + p_2^2 + q_1^2 + q_2^2 + 2p_1 \cdot p_2 - 2p_1 \cdot q_1 - 2p_1 \cdot q_2 \\ &= 3p_1^2 + p_2^2 + q_1^2 + q_2^2 + 2p_1 \cdot p_2 - 2p_1 \cdot (p_1 + p_2) \\ &= 3p_1^2 + p_2^2 + q_1^2 + q_2^2 - 2p_1^2 \\ &= p_1^2 + p_2^2 + q_1^2 + q_2^2 \end{aligned}$$

However, due to the relativistic dispersion relation, we have that $p^2 = m^2$, so we have the result that

$$s + t + u = m_1^2 + m_2^2 + (m_1')^2 + (m_2')^2$$

We can write out scattering amplitude for $\psi\overline{\psi} \to \psi\overline{\psi}$ scattering we had earlier in terms of the

Mandelstam variables.

$$\begin{aligned} i\mathcal{M} &= (-ig)^2 \left\{ \frac{1}{(p_1 - q_1)^2 - m^2} + \frac{1}{(p_1 - q_2)^2 - m^2} \right\} \\ &= (-ig)^2 \left\{ \frac{1}{t - m^2} + \frac{1}{u - m^2} \right\} \end{aligned}$$

5.0.2 Decay rates

Recall that in our earlier section, we expressed the probability amplitude of scattering as a product of a Dirac delta function which imposed the conservation of momentum, and the factor $\mathcal{M}$ which we called the amplitude

$$\langle \phi | S - 1 | i \rangle = i\mathcal{M}(2\pi)^4 \delta^4(p_i - \sum_i q_i)$$

Note that p_i here is not an index, but a label for the total momentum of particles in the initial state!

Just to refresh our memory, we expressed momentum conservation in terms of a delta function which summed initial and final states in terms of of definite momenta. Now, we can ask ourselves, how do we convert this into a probability value that we can measure in experiment? Well, we take the modulus of the amplitude, and divide by $\langle f|f \rangle \langle i|i \rangle$ since these initial states necessarily already normalised. Our probability is given by

$$P = \frac{\| \langle f | (S-1) | i \rangle \|^2}{\langle f|f \rangle \langle i|i \rangle}$$

From this, we then define our differential cross section, which roughly is the incremental change in area we get when scattering particles through some interaction potential (see Peskin and Schroeder for a proof of this result).

$$d\sigma = \frac{(2\pi)^4 \delta^4 \left(p_1 + p_2 - \sum_i q_i\right) |\mathcal{M}|^2}{\mathcal{F}}$$

Here, $\mathcal{F}$ is our flux factor with

$$\mathcal{F} = 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}$$

To find our total cross section from $i \to f$, we integrate over our final state momentum in the usual Lorentz invariant way, that is, over integrals in momentum space with the measure $\int \frac{d^3p}{(2\pi)^3 2E_p}$. Thus, integrating our differential cross section over each momentum variable gives us

$$\sigma = \int \frac{1}{\mathcal{F}} dp_f |\mathcal{M}|^2$$

Our measure is given by

$$\int dp_f = (2\pi)^4 \int \left(\prod_{r=1}^{n} \frac{d^3 q_r}{(2\pi)^3 2E_{q_r}} \right) \delta^4 \left(p_i - \sum_i q_i \right)$$

Hence, in the case of $2 \to 2$ scattering for example, we set here that $n = 2$, so we only have two integrals.

5.0.3 $2 \to 2$ scattering

We will now present an instructive example to illustrate calculating decay rates. Our aim for this section will be to calculate the change in the cross section with respect to the Mandelstam variable t. We write out this variable as

$$t = (p_1 - q_1)^2 = m_1^2 + m_2^2 - 2E_{p_1}E_{q_1} + 2\vec{p}_1 \cdot \vec{q}_1$$

We have that

$$\frac{dt}{d\cos\theta} = 2|\vec{p}_1||\vec{q}_1|$$

where $\cos\theta$ is the angle between $\vec{q}_1$ and $\vec{p}_1$. It's convinient for us to work in the center of mass frame, the frame where our total momenum is 0. Our second Mandelstam variable is $s = (p_1 + p_2)^2$, which we can work out to be the centre of mass energy. This quantity is a constant of the scattering. In the first section, we showed that our Lorentz invariant measure could be written in terms of Lorentz invariant delta functions. For our purposes, we will return to that formalism for q_2 to aid computations. We set

$$\frac{d^3q_2}{2E_{q_2}} = d^4q_2\delta(q_2^2 - (m_2')^2)\theta(q_2^0)$$

where we've defined θ as the Heaviside step function. For the other measure d^3q_1, we rewrite the measure in terms of polar coordinates.

$$\frac{d^3q_1}{2E_{q_1}} = \frac{|\vec{q}_1|^2 d|\vec{q}_1| d(\cos\theta)d\phi}{2E_{q_1}} = \frac{1}{4|\vec{p}_1|}dE_{q_1}d\phi dt$$

With this in mind, we can write out the full Now, putting this together, we have an expression

$$\frac{d\sigma}{dt} = \frac{1}{8\phi\mathcal{F}|\vec{p}_1|}\int dE_{q_1}|\mathcal{M}|^2\delta(s - m_2^2 - (m_1')^2 - 2\vec{q}_1 \cdot (\vec{p}_1 + \vec{p}_2))$$

Now, if we boost back to the centre of mass fram, we have that

$$p_1^\mu = (\sqrt{|\vec{p}_1|^2 - m_1^2}, \vec{p}_1)$$
$$p_2^\mu = (\sqrt{|\vec{p}_1|^2 + m_2^2}, -\vec{p}_1)$$

This means we compute th s mandelstam variable

$$s = (\sqrt{|\vec{p}_1|^2 + m_1^2} + \sqrt{|\vec{p}_2|^2 + m_2^2})^2$$

This implies that we can rewrite

$$|\vec{p}_1| = \lambda^{\frac{1}{2}}(S, m_1^2, m_2^2)/2\sqrt{s}$$

where we have defined

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$$

From this, we have that

$$\mathcal{F} = 2\sqrt{\lambda(s, m_1^2, m_2^2)} \implies \frac{d\sigma}{dt} = \frac{|\mathcal{M}|^2}{16\pi\lambda(s, m_1^2, m_2^2)}$$

5.0.4 Decay rates

We defime a partial decay rate for $|i\rangle \to |f\rangle$, or partial width, as

$$\Gamma_f = \frac{1}{2E_p} \int dp_f |\mathcal{M}|^2$$

Note that this is not a Lorentz invariant quantity due to time dilation. Hence, our convention is to quote this in terms of the res frame of the initial particle, where the energy E_{p_i} is equal to the mass m_i. We define the total decay with whcih we write as Γ. This is the sum of all possible states

$$\Gamma = \sum_f \Gamma_f$$

The branching ratio for $|i\rangle \to |f\rangle$ is denoted as

$$BR(i \to f) = \frac{\Gamma_f}{\Gamma}$$

If we look at the units, we notice that the average lifetime is

$$\tau = 6.6 \times 10^{-25} \times (1\text{GeV })/\Gamma \text{ seconds}$$

6 The Dirac Equation

6.1 What does the Dirac equation give us?

The Dirac equation is important because it provides the framework for quantization of spin-$\frac{1}{2}$ particles, like electrons and other fermions. Another startling prediction that the Dirac equation gives us is the existence of anti-matter. This equation is an amazing combination of the geometry of spinors in Minkowski space-time, and the wave-function formalisation of quantum mechanics that we all know and love. In this section, we'll provide some motivation for constructing the Dirac equation, solve it, then quantise this object to give rise to particles (fermions).

Right now we'll try to analyse how fields with **multiple** components change under an underlying Lorentz transformation. Recall that when we have a Lorentz transformation in our coordinates, we induce a transform on our scalar field.

$$x^\mu \to (x')^\mu = \Lambda^\mu{}_\nu x^\nu, \quad \phi(x) \to \phi'(x) = \phi(\lambda^{-1}x)$$

Most particles have an intrinsic angular momentum-spin however, and so transform in a non-trivial way under boosts and rotations. For example a spin-1 vector field A^μ (for example, our familiar electromagnetic vector potential) transforms under a Lorentz boost as

$$A^\mu(x) \to (A')^\mu(x) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$$

There's no reason a priori why vector fields with multiple components need to transform with $\Lambda^\mu{}_\nu$ multiplying it out front. In general, we have that our field transforms as

$$\phi^a(x) \to D^a{}_b(\Lambda)\phi^b(\Lambda^{-1}x)$$

where our matrices D form a representation of our Lorentz group. Let's explain why this is. We treat ψ^a as a vector which is acted on by a symmetry. However, for this symmetry to **act** on an object like a vector, it needs to be represented in the same space (for example, as a matrix). Up until now, we've been representing a Lorentz transformation group in it's standard form as a matrix, but this isn't the only representation (recall the Lorentz group isn't defined in terms of matrices in the first place, but as members of the orthogonal group $O(1,3)$). Representations obey the properties that the group structure is preserved. This means that, if D is a representation, then

$$\begin{aligned} D(\Lambda_1)D(\Lambda_2) &= D(\Lambda_1\Lambda_2) \\ D(\Lambda^{-1}) &= D(\Lambda)^{-1} \\ D(I) &= 1 \end{aligned}$$

To find representations we look at the Lorentz algebra, considering infinitesimal transformations

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon\omega^\mu{}_\nu + O(\epsilon^2)$$

Working to infinitesimal order, the find that the generators obey the relation

$$\Lambda^\mu{}_\sigma\Lambda^\nu{}_\rho = \eta^{\mu\nu}$$

Substituting this into our relation we have that

$$(\delta^\mu{}_\sigma + \epsilon\omega^\mu{}_\sigma)(\delta^\nu{}_\rho + \epsilon\omega^\nu{}_\rho)\eta^{\sigma\rho} = \eta^{\mu\nu} + O(\epsilon^2)$$

This implies that our generator satisfies

$$\omega_{\mu\nu} = -\omega_{\nu\mu} \implies \omega_{\mu\nu} \text{ is anti symmetric}$$

This generator has 6 components, split up into 3 rotations and 3 boosts. Thus, we can introduce a basis of six 4×4 matrices

$$(\mathcal{M}^{\rho\sigma})^{\mu\nu} = \eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu}$$

We can lower the index here to give

$$(\mathcal{M}^{\rho\sigma})^\mu{}_\nu = \eta^{\rho\mu}\delta^\sigma{}_\nu - \eta^{\sigma\rho}\delta^\rho{}_\nu$$

For example, we have that

$$(\mathcal{M}^{01})^\mu{}_\nu = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This specific generator generates a boost in the x^1 direction. We can hence write our generator as a linear sum of these basis matrices.

$$\omega^\mu{}_\nu = \frac{1}{2}\left(\Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma}\right)^\mu{}_\nu$$

We call $\mathcal{M}^{\rho\sigma}$ the generators of our Lorentz group, and our corresponding $\Omega^{\rho\sigma}$ the anti symmetric parameters of the Lorentz transformation. Our Lorentz algebra, one can calcuate, is

$$[\mathcal{M}^{\rho\sigma}, \mathcal{M}^{\tau\nu}] = \eta^{\sigma\tau}\mathcal{M}^{\rho\nu} - \eta^{\rho\tau}\mathcal{M}^{\sigma\nu} - \eta^{\rho\nu}\mathcal{M}^{\sigma\tau} - \eta^{\sigma\nu}\mathcal{M}^{\rho\tau}$$

What we've done here is that we've calculated the Lie bracket of two elements in our representation. This means that any other representation of the Lorentz group necessarily satisfies this relation, since representations of a Lie algebra needs to preserve the Lie bracket. Our corresponding boost is $\exp\left(\frac{1}{2}\Omega_{\rho\sigma}\right)\mathcal{M}^{\rho\sigma}$. This is because when we are returning the infinitesimal transformation back into its full form by exponentiating.

6.2 The Spinor Representation

Now, the aim of the game is to try and find other representations which satisfy the Lorentz algebra. To help us find representations that satisfy the Lorentz algebra, we start by defining something called the Clifford algebra. This is an algebra whose elements satisfy the anti-commutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}I, \quad \{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$$

where γ^μ are a set of four matrices. We have that our matrices in the algebra obey the properties

$$\begin{aligned} &\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu, \quad \mu \neq \nu \\ &(\gamma^0)^2 = I, \quad (\gamma^i)^2 = -I, \quad i \in \{1, 2, 3\} \end{aligned}$$

Our simpliest representation which obeys these commutation relation are a set of 4×4 matrices, which we call the Chiral representation. These are given in block matrix form as

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

Here, σ^i are the Pauli matrices given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

These Pauli matrices obey the commutation and anti commutation relations

$$\left\{\sigma^i, \sigma^j\right\} = 2\delta^{ij} I_2, \quad [\sigma^j, \sigma^k] = 2i\epsilon^{jkl}\sigma^l$$

Given this representation, we can also generate a new representation by conjugating by a constant invertible matrix U, so $U\gamma^\mu U^{-1}$ works also as a representation.

If we define

$$S^{\rho\sigma} := \frac{1}{4}[\gamma^\rho, \gamma^\sigma] = \begin{cases} 0 & \rho = \sigma \\ \frac{1}{2}\gamma^\rho\gamma^\sigma & \rho \neq \sigma \end{cases} = \frac{1}{2}\gamma^\rho\gamma^\sigma - \frac{1}{2}\eta^{\rho\sigma} I$$

we aim to show that this satisfies the Lorentz algebra, whilst being philosphically different to the algebra we defined before. Now, expanding this whole thing out explicitly is difficult to do, so we prove a handful of useful relations to show this. We can

Claim. We have that

$$[S^{\mu\nu}, \gamma^\rho] = \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\rho\mu}$$

Proof. There's some steps we have to follow and it's not as straightforward as it seems from first glance. The basic idea is to commute things out so that we only have one γ factor per term.

$$\begin{aligned} [S^{\mu\nu}, \gamma^\rho] &= \frac{1}{2}[\gamma^\mu\gamma^\nu - \eta^{\mu\nu}, \gamma^\rho] \\ &= \frac{1}{2}[\gamma^\mu\gamma^\nu, \gamma^\rho] \\ &= \frac{1}{2}(\gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\rho\gamma^\mu\gamma^\nu) \end{aligned}$$

Now, this is a bit tricky to manage. The key is to anti-commute things in the right order. We anti-commute the last two matrices in the first term, and the first two matrices in the last term. The above is then equal to

$$\begin{aligned} \ldots &= \frac{1}{2}\gamma^\mu(\{\gamma^\nu, \gamma^\rho\} - \gamma^\rho\gamma^\nu) - \frac{1}{2}\left(\{\gamma^\rho, \gamma^\mu\} - \gamma^\mu\gamma^\rho\right)\gamma^\nu \\ &= \frac{1}{2}\gamma^\mu\left(2\eta^{\nu\rho} - \gamma^\rho\gamma^\nu\right) - \frac{1}{2}(2\eta^{\rho\mu} - \gamma^\mu g\gamma^\rho)\gamma^\nu \\ &= \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\rho\mu} \end{aligned}$$

$\square$

Using this, we can then prove that $S^{\rho\sigma}$ satisfies the Lorentz algebra! This means we would then be able to use this as a valid representation of the Lorentz group.

Claim.

$$[S^{\rho\sigma}, S^{\tau\nu}] = \eta^{\sigma\tau}S^{\rho\nu} - \eta^{\rho\tau}S^{\sigma\nu} + \eta^{\rho\nu}S^{\sigma\tau} - \eta^{\sigma\nu}S^{\rho\tau}$$

Proof. We can expand out one side and appeal to the relation we proved earlier.

$$\begin{aligned}
[S^{\rho\sigma}, S^{\tau\nu}] &= \frac{1}{2}[S^{\rho\sigma}, \gamma^\tau\gamma^\nu - \eta^{\tau\mu}I] \\
&= \frac{1}{2}[S^{\rho\sigma}, \gamma^\tau\gamma^\nu] \\
&= \frac{1}{2}[S^{\rho\sigma}, \gamma^\tau]\gamma^\nu + \frac{1}{2}\gamma^\tau[S^{\rho\sigma}, \gamma^\nu]
\end{aligned}$$

$$= \frac{1}{2}(\gamma^\rho\eta^{\sigma\tau} - \gamma^\sigma\eta^{\rho\tau})\gamma^\nu + \frac{1}{2}\gamma^\tau(\gamma^\rho\eta^{\sigma\nu} - \gamma^\sigma\eta^{\rho\nu})$$

Now, to recover what we had earlier, we need to sub out our expressions with two gammas for an expression in S. To do this, we invert our relation for S so that

$$\gamma^\mu\gamma^\nu = 2S^{\mu\nu} + \eta^{\mu\nu}I$$

We then get that the above is equal to

$$\begin{aligned}
\ldots &= \frac{1}{2}\left(2^{\rho\nu}\eta^{\sigma\tau} + \eta^{\sigma\tau}\eta^{\rho\nu}I\right) \\
&- \frac{1}{2}\left(2S^{\sigma\nu}\eta^{\rho\tau} + \eta^{\sigma\nu}\eta^{\rho\tau}I\right) \\
&+ \frac{1}{2}\left(2S^{\tau\rho}\eta^{\sigma\nu} + \eta^{\rho\tau}\eta^{\sigma\nu}I\right) \\
&- \frac{1}{2}\left(2S^{\tau\sigma}\eta^{\rho\nu} + \eta^{\rho\nu}\eta^{\tau\sigma}\right) \\
&= S^{\rho\nu}\eta^{\sigma\tau} - S^{\sigma\nu}\eta^{\tau\rho} + S^{\tau\rho}\eta^{\sigma\nu} - S^{\tau\sigma}\eta^{\rho\nu}
\end{aligned}$$

The terms with two η terms cancel out. The object that we're left with is precisely the Lorentz algebra.

□

Thus, S provides a representation of the Lorentz group. We now define a Dirac spinor, $\psi_\alpha(x)$which is a four component vector that satisfies the following transformation law

$$\psi^\alpha(x) \to S[\Lambda]^\alpha{}_\beta\psi^\beta(\Lambda^{-1}x)$$

where we've defined $\Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right)$, and our spinor representation $S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right)$. We need to check that the Spinor representation is not equivalent to the usual vector representation, in the sense that we need to check what we've written down gives something **different** than our standard Lorentz boost.

To check that what we have is different, we trial a particular Lorentz transformation (in this case, a rotation by 2π).

Example 2. (The Spinor representation is different) If we choose

$$S^{ij} = \frac{1}{4}\left[\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^j \\ 0 & \sigma^k \end{pmatrix}\right] = -\frac{i\epsilon^{ijk}}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \text{ from } \sigma^i \text{ algebra}$$

If we write as a paramater $\Omega_{ij} = -\epsilon_{ijk}\phi^k$. Exponentiating, we get that

$$S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right) = \begin{pmatrix} e^{i\phi\cdot\sigma/2} & 0 \\ 0 & e^{i\phi\cdot\sigma/2} \end{pmatrix}$$

Now, if we consider a rotation of 2π abou the x^3 axis, we get that our corresponding representation is

$$S[\Lambda] = \begin{pmatrix} e^{i\pi\sigma^3} & 0 \\ 0 & e^{i\pi\sigma^3} \end{pmatrix} = -I_4$$

In this notation, the exponential needs Thus, a rotation of 2π takes a spinor $\phi_\sigma(x) \to -\phi_\sigma(x)$, not like a vector, which goes like $\Lambda = I$ as expected (in our fundamental representation).

Now, what happens with boosts of spinors

$$S^{0i} = \frac{1}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

If we write our boost parameter as $\Omega_{0i} = \chi_i$, then our corresponding representation is

$$S[\Lambda] = \begin{pmatrix} e^{-\chi\cdot\sigma/2} & 0 \\ 0 & e^{\chi\cdot\sigma/2} \end{pmatrix}$$

Note that for rotations, $S[\Lambda]$ is unitary, since $S[\Lambda]^\dagger S[\Lambda] = I$, but for boosts, it's not.

Theorem. (No finite dimensional, unitary representations of the Lorentz group) There are no finite dimensional unitary representations of the Lorentz group!

Proof. $S[\Lambda] = \exp\left[\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right]$ is only unitary if $S^{\rho\sigma}$ are anti hermitian. Now, when we take the Hermitian conjugate of a commutator, not only does each element pick up a hermitian conjugate, but the arguments in the commutator switch around. This means we have to pick up a minus sign.

$$(S^{\rho\sigma})^\dagger = -\frac{1}{4}[(\gamma^\rho)^\dagger, (\gamma^\sigma)^\dagger]$$

can be anti-hermitian only if all γ^μ are hermitian or are all anti hermitian. However, this can't be arranged, and is impossible to do. This can't be arranged, because

$$(\gamma^0)^2 = 1$$

This means in particular that γ^0 has real eigenvalues, which means it can't be hermitian since hermitian matrices only have pure imaginary eigenvalues.

$$(\gamma^i)^2 = 1 \implies \gamma^i \text{ can't be hermitian}$$

In general, there's no way to pick γ^μ such that $S^{\mu\nu}$ are anti-Hermitian. □

6.3 Constructing a Lorentz Invariant action of ψ

Now, something of immediate interest is to see whether we can construct some sort of action from this. To carry this out, we should try to construct Lorentz invariant quantities. We write $\psi^\dagger(x) = (\psi^*)^T(x)$. One question that we might have is to ask whether $\psi^\dagger(x)\psi(x)$ is a Lorentz scalar. However, under a Lorentz transformation,

$$\psi^\dagger(x)\psi(x) \to \psi^\dagger(\Lambda^{-1}x)S[\Lambda]^\dagger S[\Lambda]\psi(\Lambda^{-1}x)$$

This is not invariant in general since $S[\Lambda]$ is not unitary. In the chiral representation, we have that

$$(\gamma^0)^\dagger = \gamma^0, \quad (\gamma^i)^\dagger = -(\gamma^i)$$

We can verify by checking for each index, that we can write out an identity to relate the hermitian conjugate of a gamma matrix to an expression which doesn't involve conjugates as

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$$

Thus, we can find the Hermitian conjugate of S as

$$(S^{\mu\nu})^\dagger = -\frac{1}{4}[(\gamma^\mu)^\dagger, (\gamma^\nu)^\dagger] = -\frac{1}{4}[\gamma^0\gamma^\mu\gamma^0, \gamma^0\gamma^\nu\gamma^0] = -\gamma^0 S^{\mu\nu}\gamma^0$$

We do this because, since $\gamma^0 = (\gamma^0)^{-1}$, when we exponentiate this object we can pull out the γ^0 on both sides. Hence, our exponential gives

$$S[\Lambda]^\dagger = \exp\left(\frac{1}{2}\Omega_{\mu\nu}(S^{\mu\nu})^\dagger\right) = \gamma^0 S[\Lambda]^{-1}\gamma^0$$

Definition. (The Dirac adjoint) The Dirac adjoint of ψ is defined as

$$\overline{\psi}(x) := \psi^\dagger(x)\gamma^0$$

We define this object in the spirit of creating scalars, vectors and other objects which transform nicely under Lorentz transformations.

Claim. (Lorentz invariance of $\overline{\psi}\psi$) Our claim is that $\overline{\psi}\psi$ is indeed a Lorentz scalar. To prove this observe that

$$\overline{\psi}\psi = \psi^\dagger\gamma^0\phi \to \psi^\dagger(\Lambda^{-1}x)S[\Lambda]^\dagger\gamma^0 S[\Lambda]\psi(\Lambda^{-1}x)$$

Using the identity above, we get that this is the same.

Claim. ($\overline{\psi}\gamma^\mu\psi$ transforms as a Lorentz vector) Things that transform nicely under Lorentz boosts are called Lorentz vectors (or more familiarly, 4-vectors). This means that they transform much like how a position vector transforms under a Lorentz boost:

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu$$

We claim that the object $\overline{\psi}\gamma^\mu\psi$ transforms in this way to, where under a Lorentz transform we have that

$$\overline{\psi}\gamma^\mu\psi \to \Lambda^\mu{}_\nu\overline{\psi}\gamma^\nu\psi$$

Proof. Our vector transforms as follows (note that the γ^μ matrix doesn't change since it's a constant object - only the spinors change.

$$\overline{\psi}\gamma^\mu\psi \to \overline{\psi}S[\Lambda]^{-1}\gamma^\mu S[\Lambda]\psi$$

For this to be a Lorentz vector, we require that

$$S[\Lambda]^{-1}\gamma^\mu S[\Lambda] = \Lambda^\mu{}_\nu\gamma^\nu$$

To go from here, we need to write out the transformations in terms of the exponents of their generators. In particular, we require that the coefficients of the generators $\Omega_{\rho\sigma}$ are the same on both sides - it's just the matrices $\mathcal{M}$ and S that are different since we're dealing with distinct representations.

$$\Lambda^\mu{}_\nu = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma}\right), \quad S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right)$$

So, we need to prove that

$$(\mathcal{M}^{\rho\sigma})^\mu{}_\nu\gamma^\nu = -[S^{\rho\sigma}, \gamma^\mu]$$

This can be shown to be true by observing that

$$(\eta^{\rho\mu}\delta^\sigma{}_\nu - \eta^{\sigma\mu}\delta^\rho{}_\nu)\,\gamma^\nu = \eta^{\rho\mu}\gamma^\rho - \gamma^\rho\eta^{\sigma\mu} = -[S^{\rho\sigma}, \gamma^\mu]$$

□

Now, the upshot of doing all this work is that we can now write down an action (which by right should be Lorentz invariant). We know that $\overline{\psi}\gamma^\mu\psi$ is a Lorentz four vector, which implies that $\overline{\psi}\gamma^\mu\partial_\mu\psi$ is a Lorentz invariant quantity. We also already know that $\overline{\psi}\psi$ is also Lorentz invariant on its own. Finally, we can construct a Lorentz invariant action from these objects, by setting

$$S = \int d^4x\,\overline{\psi}\,(\gamma^\mu\partial_\mu - m)\,\psi(x)$$

6.4 The Dirac equation

Varying $\overline{\psi}$ independently in the Euler Lagrange equations gives us the Dirac equation. Because there is no dependence on $\dot{\overline{\psi}}$, we have that our equation of motion is given by

$$\frac{\partial\mathcal{L}}{\partial\overline{\psi}} = 0$$

Specifically, this is the equation

$$(i\gamma^\mu\partial_\mu - m)\,\psi = 0$$

This is first order, not second order like KG. To save some ink, we write objects which are contracted with γ^μ with a slash. For example, we write $A_\mu\gamma^\mu = \not{A}$. In this notation, the Dirac equation reads

$$\left(i\not{\partial} - m\right)\psi = 0$$

Note that each component of ψ solves the KG equation.

$$\begin{aligned}(i\partial\!\!\!/ + m)(i\partial\!\!\!/ - m)\psi &= 0 \\ (\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2)\psi &= 0 \\ (\frac{1}{2}\left\{\gamma^\mu,\gamma^\nu\right\}\partial_\mu\partial_\nu + m^2)\psi &= 0 \\ -(\partial^2 + m^2)\psi &= 0\end{aligned}$$

6.5 Chiral Spinors

Since $S[\Lambda]$ is block diagonal in our Chiral representation, we say that it's reducible (since we can write the representation as a direct sum of two different representations). It decomposes into 2 irreducible representations. We write our spinor as

$$\psi = \begin{pmatrix} u_L \\ u_R \end{pmatrix}$$

where u_L, u_R are 2 $\mathbb{C}$ component objects which we call Weyl, or chiral, spinors. These objects transform identically under rotations, so that

$$u_{L,R} \to e^{\phi\cdot\sigma/2} u_{L,R}$$

but under boosts, they transform in the opposite fashion. This can be seen by just multiplying the matrix with the vector as $\psi \to S[\Lambda]_{\text{ rot }}\psi$, which gives us

$$u_L \to e^{-\chi\cdot\sigma/2}u_L, \quad u_R \to e^{\chi\cdot\sigma/2}u_R$$

Heuristically we write

$$u_L \sim (\frac{1}{2}, 0) \quad u_R \sim (0, \frac{1}{2}) \in (SU(2), SU(2))$$

we have that ψ is in $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$.

6.5.1 The Weyl equation

If we take our Dirac Lagrangian

$$\mathcal{L}_D = \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi$$

which is equal to

$$\cdots = iu_L^\dagger\overline{\sigma}^\mu\partial_\mu u_L + iu_R^\dagger\sigma^\mu\partial_\mu u_R - m\left(u_L^\dagger u_R + u_R^\dagger u_L\right)$$

where we define $\sigma^\mu = (I, \sigma), \overline{\sigma}^\mu = (I, -\sigma)$. This is derived simply by writing out the components explicitly, then multiplying by the required matrices. A massive fermion requires both u_L and u_R , but the massless limit $m = 0$ requires only u_L or u_R. The mass term is mixing up left handed and right handed spinors. In the case $m = 0$, we get the following EL equations

$$i\sigma^\mu\partial_\mu u_L = 0, \quad i\overline{\sigma}^\mu\partial_\mu u_R = 0$$

These are called Weyl's equations. In classical particle mechanics, the number of degrees of freedom is given by half the dimension of phase space. in field theory, we need to do things different and discuss the number of degrees of freedom per spacetime point. In field theory, we have a scalar ϕ and $\Pi_\phi = \dot{\phi}$, so our real degrees of freedom is $\frac{1}{2} \times 2 = 1$. However, for a spinor ψ_α we have 8 degrees of freedom since it's complex. For our conjugate momenta,

$$\Pi_\psi = i\psi^i$$

which doesn't add any degrees of freedom. This means that our real degrees of freedom amounts to $\frac{1}{2} \times 8 = 4$. We have 2 for the spin up and spin down particle, and 2 for the spin up and spin down anti-particle.

6.6 Dirac Field Bilinears

Throughout this section, we've only been using a specific representation of S (which we dubbed the chiral representation). However, note that in a different basis, $S[\Lambda]$ is not necessarily block diagonal! We can transform to a different basis by applying the map

$$\gamma^\mu \to U\gamma^\mu U^{-1}, \psi \to U\psi$$

We use

$$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3$$

to define Weyl spinors in a basis independent way. This object has very nice properties which relate agonistically with γ^μ for $\mu = 0, 1, 2, 3$.

Claim. (Commutation relations for γ^5) The γ^5 matrix satisfies the relations

$$\left\{\gamma^\mu, \gamma^5\right\} = 0, \quad \left(\gamma^5\right)^2 = I$$

Proof. To prove the first relation, it's instructive to set $\mu = 0$ just to see what's going on. We have that the anti-commutator is

$$\begin{aligned}
\left\{\gamma^\mu, \gamma^5\right\} &= i(\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 + \gamma^0\gamma^1\gamma^2\gamma^3\gamma^0) \\
&= i(\gamma^1\gamma^2\gamma^3 - \gamma^1\gamma^0\gamma^2\gamma^3\gamma^0) \\
&= i(\gamma^1\gamma^2\gamma^3 + \gamma^1\gamma^2\gamma^0\gamma^3\gamma^0) \\
&= i(\gamma^1\gamma^2\gamma^3 - \gamma^1\gamma^2\gamma^3(\gamma^0)^2) = 0
\end{aligned}$$

We've used liberally here the fact that

$$\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \text{ when } \nu \neq \mu$$

For the second relation, we use this fact as well. Note that when we pass a γ^j matrix past n matrices γ^i where i never equals j, we pick up a factor of $(-1)^n$. Thus, note that $(\gamma^5)^2$ is just

$$\begin{aligned}
(\gamma^5)^2 &= i^2\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 \\
&= (-1)^4\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 \\
&= (-1)^3\gamma^2\gamma^3\gamma^2\gamma^3 \\
&= (-1)^4 I = I
\end{aligned}$$

□

Now, we can write out γ^5 explicitly depending on the representation we choose. We check that in our chiral representation,

$$\gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

We define projection operators in the spirit of finding a way to extract the left-handed and right-handed spinors from the entire spinor. We define

$$P_L = \frac{1}{2}(I_4 - \gamma^5), \quad P_R = \frac{1}{2}(1 + \gamma^5)$$

These obey the properties that come with what we usually know as a projection operator, namely, that projecting twice is the same as projecting once, and that P_L and P_R project onto 'orthogonal' spaces

$$P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_L P_R = 0$$

We then define 'left handed' and 'right handed' spinors ψ_L and ψ_R, which are the projection spinors with these operators

$$\psi_L = P_L \psi, \quad \psi_R = P_R \psi$$

for left handed and right handed spinors respectively. Under a Lorentz transformation,

$$\overline{\psi}(x)\gamma^5\psi(x) \to \overline{\psi}(\Lambda^{-1}x)S[\Lambda]^{-1}\gamma^5 S[\Lambda]\psi$$

Checking that $[S_{\mu\nu}, \gamma^5]$, we have that this is

$$\cdots = \overline{\psi}(\Lambda^{-1}x)\gamma^5\psi(\Lambda^{-1}x)$$

This is called a pseudoscalar since it transforms under parity transformation. We can also define the axial vector, $\overline{\psi}\gamma^5\gamma^\mu\psi$ which transforms in the same way.

6.7 Parity

We'll now explore how spinors transform under discrete transformations. We'll look at a specific class of transformations which flips either our time coordinate or our spatial coordinate. The transformations are called parity transformations. Ultimately, we'll find that ψ_L and ψ_R are related by a parity trasnformation. In the Lorentz group $O(1,3)$, we've dealt with boosts and rotations, which are a continuous group of transformations. However, there are other discrete Lorentz transformations which are not part of the Lorentz group's component connected to the identity. These transformations are

$$\text{Time Reversal } T : x^0 \to -x^0, \quad x^i \to x^i$$

The other one is parity.

$$\text{Parity } P : x^0 \to x^0, \quad x^i \to -x^i$$

Other transformations like $x^2 \to -x^2$ are connected to the identity, because something like this can be written as a rotation. Under P, rotations don't change sign, but boosts do.

$$u_{L,R} \to e^{\phi\cdot\sigma/2}u_{L,R}, \quad u_{L,R} \to e^{\mp\chi\cdot\sigma/2}u_{L,R}$$

This is equivalent to writing, in our new notations with $\psi_{L,R} = \frac{1}{2}(I \mp \gamma^5)\psi$, that P exchanges LH and RH spinors.

$$P: \psi_{L,R}(\vec{x},t) \to \psi_{R,L}(-\vec{x},t)$$

Since our parity transformation switches around the left and right handed components of the spinor, it is completely equivalent to applying the γ^0 matrix to the spinor, (but also we have to remember to change the $\vec{x}$ sign)

$$P: \psi(\vec{x},t) \to \gamma^0\psi(-\vec{x},t)$$

How do our terms in our $\mathcal{L}$ density change? We look at each field one by one Firstly,

$$\overline{\psi}\psi(\vec{x},t) \to \overline{\psi}\psi(-\vec{x},t)$$

Also,

$$\overline{\psi}\gamma^\mu\psi(\vec{x},t) \to \begin{cases} \mu = 0 & \overline{\psi}\gamma^0\psi(-\vec{x},t) \\ \mu = i & \overline{\psi}\gamma^0\gamma^i\gamma^0\psi(-\vec{x},t) = -\overline{\psi}\gamma^i\psi(-\vec{x},t) \end{cases}$$

On the other hand, we have that under parity,

$$\overline{\psi}\gamma^5\gamma(\vec{x},t) \to \overline{\psi}\gamma^0\gamma^5\gamma^0\psi(-\vec{x},t) = -\overline{\psi}\gamma^5\psi$$

In addition,

$$\overline{\psi}\gamma^5\gamma^\mu\psi \to \overline{\psi}\gamma^0\gamma^5\gamma^0\psi = \begin{cases} \mu = 0 & -\overline{\psi}\gamma^5\gamma^0\psi \\ \mu = i & +\overline{\psi}\gamma^5\gamma^i\psi \end{cases}$$

To summarise, we write down the number of components as

$$\begin{aligned} \overline{\psi}\psi 1 \\ \overline{\psi}\gamma^\mu\psi 4 \\ \overline{\psi}S^{\mu\nu}6 \\ \overline{\psi}\gamma^5\psi 1 \\ \overline{\psi}\gamma^5\gamma^\mu\psi 4 \end{aligned} \tag{1}$$

this has a total of 16 components. Now we add extra terms to $\mathcal{L}$ involving γ^5. This can break P invariance. For example,

$$\mathcal{L} = gW_\mu\overline{\psi}\frac{\gamma^\mu(1-\gamma^5)\psi}{2}$$

This represents a W boson vector field coupling only to LH ψ ' s. If $\mathcal{L}$ treats ψ_L and ψ_R equally it is called vector like. If they appear differently they're called chiral.

6.8 Symmetries and Currents of Spinors

If we have a space time translation $x^\mu \to x^\mu - \epsilon^\mu$, then $\Delta\psi = \epsilon^\mu\partial_\mu\psi$. This gives us the stress energy tensor

$$T^{\mu\nu} = i\overline{\psi}\gamma^\mu\partial^\nu\psi - \eta^{\mu\nu}\mathcal{L}$$

We get a conserved current when equations of motion are obeyed, do we can impose them in $T^{\mu\nu}$.

$$\mathcal{L}_D = \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi = 0$$

This implies $\mathcal{L} = 0$ in $T^{\mu\nu}$ thus

$$T^{\mu\nu} = i\overline{\psi}\gamma^\mu\partial^\nu\psi$$

$$S = \int d^4x \frac{1}{2}\overline{\psi}\left(i\partial\!\!\!/^{\rightarrow} - m\right)\psi + \frac{1}{2}\overline{\psi}(-i\partial\!\!\!/^{\leftarrow} - m)\psi$$

This is

$$\int d^4x \frac{1}{2}\overline{\psi}(i\partial\!\!\!/^{\leftrightarrow} - 2m)\psi$$

where $\psi\!\!\!/^{\leftrightarrow} = \partial\!\!\!/^{\rightarrow} - \partial\!\!\!/^{\leftarrow}$. So

$$T^{\mu\nu} = \frac{i}{2}\overline{\psi}(\partial^\mu\partial^\nu - \partial^\nu\partial^\mu)\psi$$

6.9 Lorentz transformations

We know how a spinor is supposed to transform. We have that infinitesimally,

$$\psi \to S[\Lambda]^\alpha{}_\beta\psi^\beta(x^\mu - \omega^\mu{}_\nu x^\nu)$$

where we have that ω is generated as

$$\omega^\mu{}_\nu = \frac{1}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu$$

Since we defined earlier that

$$(M^{\rho\sigma})^\mu{}_\nu = \eta^{\rho\mu}\delta^\sigma{}_\nu - \eta^{\sigma\mu}\delta^\rho{}_\nu$$

This implies that $\omega^{\mu\nu} = \Omega^{\mu\nu}$. Our infinitesimal change in our spinor is

$$\delta\psi^\alpha = i\omega^{\mu\nu}\left[x^\nu\partial_\mu\psi^\alpha - \frac{1}{2}(S^{\rho\sigma})^\alpha{}_\beta\psi^\beta\right]$$

Our corresponding conjugate change is

$$\delta\overline{\psi}_\alpha = -\omega^{\mu\nu}\left[x_\nu\partial_\mu\overline{\psi}_\alpha + \frac{1}{2}\overline{\psi}(S_{\mu\nu})^\beta{}_\alpha\right]$$

where the last term comes from $\overline{\psi}_\beta S[\Lambda]^{-1}$. Our conserved current

$$(J^\mu)^{\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho} - i\overline{\psi}\gamma^\mu S^{\rho\sigma}\psi$$

THe first term looks like a real scalar stress energy tensor. The new piece will give us spin $\frac{1}{2}$ after we quantise the field. For example, looking at the last term $(J^0)^{ij} = -i\overline{\psi}\gamma^0 S^{ij}\psi$. Using the representation in Pauli matrices, we get that the above is equal to

$$(J^0)^{ij} = \frac{1}{2}\epsilon^{ijk}\psi^\dagger\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}\psi$$

6.10 Internal vector symmetry

We get that under the phase rotation

$$\psi \to e^{i\alpha}\psi \implies \delta\psi = i\sigma\psi$$

This gives us a conserved quantitiy

$$j^{\mu\nu} = \overline{\psi}\gamma^\mu\psi \text{ a vector current}$$

This gives us the conserved charge

$$Q = \int d^3x\, \overline{\psi}\gamma^0\psi = \int d^3x\, \psi^\dagger\psi$$

in other words, an electric charge.

6.11 Axial Symmetry

For massless spinors in the $m = 0$ limit, we transform

$$\psi_\alpha \to \left(e^{i\alpha\gamma^5}\right)_\alpha^{\ \beta}\psi_\beta$$

This rotates left handed and right handed spinors in the opposite direction. This leads to the conserved axial vector current

$$j_A^\mu = \overline{\psi}\gamma^\mu\gamma^5\psi$$

6.12 Plane-Wave Solutions

Now, let's actually start writing down solutions to the Dirac equation. We want to solve $(i\partial\!\!\!/ - m)\psi = 0$. Even though this is a first order differential equation and typically we wouldn't think of using wave solutions, we try this anyway and make the ansatz

$$\psi = u_{\vec{p}}e^{-ip\cdot x}$$

where $u_{\vec{p}}$ is a constant spinor depending on $\vec{p}$. Substituting this in, using the chiral representation of γ^μ, we have that

$$(\gamma^\mu p_\mu - m)u_{\vec{p}} = 0 = \begin{pmatrix} -m & p_\mu\sigma^\mu \\ p_\mu\overline{\sigma}^\mu & -m \end{pmatrix} u_{\vec{p}} = 0$$

Here, we are again using our notation that $\sigma^\mu = (1, \sigma)$, and $\overline{\sigma}^\mu = (1, -\sigma)$.

Claim. (The Spinor plane-wave solution)

Our claim is that we have a solution given by

$$u_{\vec{p}} = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\overline{\sigma}}\xi \end{pmatrix}$$

for any two component spinor ξ such that $\xi^\dagger \xi = 1$. To prove this, we let $u_{\vec{p}} = (u_1, u_2)$, and substitute into the above such that

$$(p \cdot \sigma) u_2 = m u_1, \quad (p \cdot \overline{\sigma}) u_1 = m u_2$$

Either of these equations implies the other, since

$$\begin{aligned}
(p \cdot \sigma)(p \cdot \overline{\sigma}) &= p_0^2 - p_i p_j \sigma^i \sigma^j \\
&= p_0^2 - p_i p_j \frac{1}{2} \left\{ \sigma^i, \sigma^j \right\} \\
&= p_\mu p^\mu = m^2
\end{aligned}$$

Now, we try the ansatz $u_1 = (p \cdot \sigma)\xi$ for the two component spinor ξ'. Substituting this into the above, we have that

$$u_2 = \frac{1}{m} (p \cdot \overline{\sigma})(p \cdot \sigma) \xi = m \xi'$$

Hence, any vector of the form $u_{\vec{p}} = A((p \cdot \sigma)\xi', m\xi')$. To make this look more symmetric, choose $A = \frac{1}{m}$ and $\xi' = \sqrt{p \cdot \overline{\sigma}}\xi$ with ξ const. Then, we have that

$$u_{\vec{p}} = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \overline{\sigma}} \xi \end{pmatrix}$$

Examples. Let's take ı = 0, then we have that our solution looks like

$$u_{\vec{p}=0} = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \text{ for any } \xi$$

Under spatial rotations, we have that

$$\xi \to e^{i \sigma \cdot \phi / 2} \xi$$

After quantisation, ξ describes the spin of the spinor. For example, we have that

$$\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ represents spin up along } x^3 \text{ axis}$$

Let's consider a particle boosted along the x^3 direction. Now, the momentum is not going to be zero. Now what we have is that

$$u_{\vec{p}} = \begin{pmatrix} \sqrt{E - p^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{E + p_3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

which, in the $m \to 0$ or $E \to p^3$ limit, converges to

$$\sqrt{2E} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Now, with instead $\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, when we act on this we instead get that

$$u_{\vec{p}} = \begin{pmatrix} \sqrt{E+p^3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \sqrt{E-p^3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

which tends to

$$\sqrt{2E} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

6.13 Helicity

The Helicity operator projects angular momentum along the direction of motion. If we define

$$h = \hat{\vec{p}} \cdot \vec{s} = \frac{1}{2} \hat{p}_k \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

A massless spin up operator has $h = \frac{1}{2}$, and a massless spin down particle has $h = -\frac{1}{2}$.

6.13.1 Negative frequency solutions

If we define $\psi = v_{\vec{p}} e^{ip \cdot x}$. The Dirac equation then gives

$$v_{\vec{p}} = \begin{pmatrix} \sqrt{p \cdot \sigma} \eta \\ -\sqrt{p \cdot \bar{\sigma}} \eta \end{pmatrix}$$

where we have as well that $\eta^\dagger \eta = 1$ and η is a constant 2 -spinor.

6.14 A canonical Basis for normalised Spinors

In what follows, we'll construct a basis for the two component spinors in a way such that some nice identities will hold when we're doing calculations. It will be convenient for us to choose the basis

$$\xi^1 = \eta^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi^2 = \eta^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

This gives us the sweet property that these things are normalised, so

$$(\xi^r)^\dagger \xi^s = \delta^{rs}, \quad (\eta^r)^\dagger \eta^s = \delta^{rs}$$

In this basis, we then get two independent solutions for both $u(\vec{p})$ and $v(\vec{p})$, which we will index as $u^s(\vec{p})$ and $v^s(\vec{p})$. We get these solutions by just plugging in ξ^s or η^s into the respective spinors - these linearly independent solutions are given by

$$u^s(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \bar{\sigma}} \xi \end{pmatrix}, \quad v^s(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma} \eta \\ -\sqrt{p \cdot \bar{\sigma}} \eta \end{pmatrix}$$

The neat thing is that, now with our choice of basis for the two component spinors, we get that u^s and v^s are themselves normalised depending on how we contract them. We can contract them in two ways. The first thing that we can do is take the hermitian dot product, so something like $u^{s\dagger}u^r$. Secondly, we could've also equally defined $\overline{u}^s = u^{s\dagger}\gamma^0$, and we also check the normalisation $\overline{u}^s u^r$. It turns out, both will be useful.

Let's do the first contraction.

$$\begin{aligned}
u^{s\dagger}u^r &= \begin{pmatrix} \xi^{s\dagger}\sqrt{p\cdot\sigma} & \xi^{s\dagger}\sqrt{p\cdot\overline{\sigma}} \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^r \\ \sqrt{p\cdot\overline{\sigma}}\xi^r \end{pmatrix} \\
&= \xi^{s\dagger}(p\cdot\sigma)\xi^r + \xi^{s\dagger}(p\cdot\overline{\sigma})\xi^r \\
&= 2p_0\xi^{s\dagger}\xi^r \\
&= 2p_0\delta^{rs}
\end{aligned}$$

We can see clearly that this is not a necessarily Lorentz invariant contraction since p_0 changes under boosts. Now, lets do the other type of contraction which is indeed Lorentz invariant! We have that

$$\begin{aligned}
\overline{u}^{s\dagger}u^r &= \begin{pmatrix} \xi^{s\dagger}\sqrt{p\cdot\sigma} & \xi^{s\dagger}\sqrt{p\cdot\overline{\sigma}} \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^r \\ \sqrt{p\cdot\overline{\sigma}}\xi^r \end{pmatrix} \\
&= 2\xi^{s\dagger}\sqrt{(p\cdot\sigma)(p\cdot\overline{\sigma})} \\
&= 2m\delta^{rs}
\end{aligned}$$

In the above we've used the relation we proved earlier that $(p\cdot\sigma)(p\cdot\overline{\sigma}) = m^2$.

Now, what about the 'negative frequency' spinors $v^s(\vec{p})$? Well, it is easy to also check that these spinors obey similar normalisation conditions, up to a minus sign.

$$v^{s\dagger}v^r = 2p_0\delta^{rs}, \quad \overline{v}^s v^r = -2m\delta^{rs}$$

Finally, something we will also have to use frequently in the coming section is the outer product

6.15 Quantising the Dirac field

In conclusion, we've found two plane wave solutions of the Dirac equation, which, for cleanliness, we will write as $u^s(\vec{p})$ and $v^s(\vec{p})$. Recall, the s index let's us choose between ξ^1 or ξ^2 in the definition $u^s(\vec{p})$, or η^1 or η^2 in the definition of $v^s(\vec{p})$. When we take these solutions and write them out in Fourier modes, we have that

$$\begin{aligned}
\psi(\vec{x}) &= \sum_{i=1}^{2}\int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}}\left[b^s_{\vec{p}}u^s_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} + c^{s\dagger}_{\vec{p}}v^s_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right] \\
\psi^\dagger(\vec{x}) &= \sum_{i=1}^{2}\int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}}\left[b^{s\dagger}_{\vec{p}}u^{s\dagger}_{\vec{p}}e^{-\vec{p}\cdot\vec{x}} + c^s_{\vec{p}}v^{s\dagger}_{\vec{p}}e^{i\vec{p}\cdot\vec{x}}\right]
\end{aligned}$$

Analogous to our previous case in nucleon anti-nucleon quantisation, we have that $b^s_{\vec{p}}$ and $b^{s\dagger}_{\vec{p}}$ represent the annihilation and creation operators for $u(\vec{p})^s$, and that $c^s_{\vec{p}}, c^{s\dagger}_{\vec{p}}$ represent annihilation and creation for $v^s(\vec{p})$. In spinor quantisation, it turns out that we require anti commutations relations instead of commutation relations, with

$$\{A, B\} = AB + BA$$

We'll go into more detail about why we need anti-commutation relations later, but for now just acknowledge that this is due to the fact that if we were to impose commutation relations, we'll come across the problem of unbounded negative energy. The specific anti-commutation relations are:

$$\{\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})\} = 0 = \left\{\psi^\dagger_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})\right\}, \quad \left\{\psi_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})\right\} = \delta_{\alpha\beta}\delta^3(\vec{x}-\vec{y})$$

In the same vein as the case we had with scalar fields and nucleons, we construct a set of commutation relations on the creation and annihilation operators which are logically equivalent to our anti-commutation relations. We can prove that these are equivalent to

$$\left\{b^r_{\vec{q}}, b^{s\dagger}_{\vec{q}}\right\} = (2\pi)^3\delta^3(\vec{p}-\vec{q})\delta^{rs} = \left\{c^r_{\vec{p}}, c^{s\dagger}_{\vec{q}}\right\}$$

We do the Legendre transform to get our Hamiltonian density. It's easy to see that our conjugate momenta π is $\pi = i\psi^\dagger$. This means that our Hamiltonian looks like

$$\begin{aligned}\mathcal{H} &= \pi\dot{\psi} - \mathcal{L} \\ &= i\psi^\dagger\dot{\psi} - i\overline{\psi}\gamma^0\partial_0\psi - i\overline{\psi}\gamma^i\partial_i\psi + m\overline{\psi}\psi \\ &= \overline{\psi}(-i\gamma^i\partial_i + m)\psi -\end{aligned}$$

If we plug in $\psi, \overline{\psi}$ from the above, we use anti commutation relations and some reuslts on inner products of spinors to get that

$$u^{r\dagger}_{\vec{p}}u^s_{\vec{p}} = v^{r\dagger}_{\vec{p}}v^s_{\vec{p}} = 2p_0\delta^{rs}, \quad u^{r\dagger}_{\vec{p}}v^s_{\vec{p}} = 0 = v^{r\dagger}_{\vec{p}}u^s_{\vec{p}}$$

This means that for our Hamiltonian, we fortunately get a positive definite quantity

$$H = \int \frac{d^3p}{(2\pi)^3}E_p\sum_{s=1}^{2}\left(b^{s\dagger}_{\vec{p}}b^s_{\vec{p}} + c^{s\dagger}_{\vec{p}}c^s_{\vec{p}}\right)$$

If we used commutation relations, we get a minus sign which means that we have unboudned lower energy and therefore unstable theory.

6.16 The Dirac Hole Interpretation

We can write the Dirac equation as

$$i\frac{\partial\psi}{\partial t} = (-i\alpha\cdot\nabla + m\beta)\psi, \quad \alpha = -\gamma^0\gamma, \beta = \gamma^0$$

The term in the brackets is considered as the 1-particle Hamiltonian $\hat{H}$. This has positive and negative energy solutions. There's no consistent way of realising the Dirac equation to one particle states.

6.17 Fermi-Dirac Statistics

We expanded our spinor field with operators b and c, which had a spin index. These operators annihilate the vacuum

$$b^s_{\vec{p}}\,|0\rangle = 0 = c^s_{\vec{p}}\,|0\rangle$$

Recall that these operators obey anti-commutation relations. We can check that these commute with the Hamiltonian as follows

$$[H, b^{r\dagger}_{\vec{p}}] = E_p b^{r\dagger}_{\vec{p}}$$
$$\left[H, b^r_{\vec{p}}\right] = -E_p b^r_{\vec{p}}$$

If we index this state, for example, as $|\vec{p}_1, r_1\rangle := b^{r_1\dagger}_{\vec{p}_1}\,|0\rangle$. Then the anti-commutation relations give us a sign change when we swap two things around. In particular, we have that

$$|\vec{p}_1, r_1; \vec{p}_2, r_2\rangle = -\,|\vec{p}_2, r_2; \vec{p}_1, r_1\rangle$$

6.17.1 Going into the Heisenberg picture

To study propagators in the theory, we make the operators ψ and $\overline{\psi}$ time dependent by moving into the Heisenberg picture. Now, we have a time dependent operator $\psi(x)$ satisfying $\frac{\partial \psi}{\partial t} = i\,[H, \psi]$ which is solved by the Heisenberg picture expansion

$$\psi_\alpha(x) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left(b^s_{\vec{p}} u^s_{\vec{p}_\alpha} e^{-ip\cdot x} + c^{s\dagger}_{\vec{p}} v^s_{\vec{p}_\alpha} e^{ip\cdot x}\right)$$

with an analogous expression for $\psi^\dagger_\alpha$. Honestly, the only thing we're doing here is attaching a time dependence to the exponential previously. Now, we are in a position to define the Feynman propagaator but in the case of these fermionic fields. We then define, in analogy with $\Delta(x-y) = [\phi(x), \phi(y)]$, that for $\mathbb{R}$ scalars,

$$iS_{\alpha\beta}(x-y) = \left\{\psi_\alpha(x), \overline{\psi}_\beta(y)\right\}$$

where $S_{\alpha\beta}$ is a four by four matrix. For brevity, we'll now drop the α, β for brevity. Substituting in our results from previously, we have that

$$iS(x-y) = \sum_{r,s} \int \frac{d^3p d^3q}{(2\pi)^6\sqrt{4E_pE_q}} \left[\left\{b^s_{\vec{p}}, b^{r\dagger}_{\vec{q}}\right\} e^{-i(p\cdot x - q\cdot y)} u^s_{\vec{p}} \overline{u}^r_{\vec{q}} + \left\{c^{s\dagger}_{\vec{p}}, c^r_{\vec{q}}\right\} v^s_{\vec{p}} \overline{v}^r_{\vec{q}} e^{i(p\cdot x - q\cdot y)}\right]$$

The anti commutators in this expression yield delta functions. So, this simplifies to the expression

$$iS_{\alpha\beta}(x-y) = \int \frac{d^3p}{(2\pi)^e 2E_p} \left[\sum_s u^s_{\vec{p}_\alpha} \overline{u}^s_{\vec{p}_\beta} e^{-ip\cdot(x-y)} + \sum_s v^s_{\vec{p}_\alpha} \overline{v}^s_{\vec{p}_\beta} e^{ip\cdot(x-y)}\right]$$

Earlier we showed that the outer products of the spinors u and v summed over the spin indices give $\not{p} + m$ and $\not{p} - m$. This thing then simplifies to

$$= \left(i\not{\partial}_x + m\right) D(x-y) - \left(i\not{\partial}_x + m\right) D(y-x) = \left(i\not{\partial}_x + m\right)\left(D(x-y) - D(y-x)\right)$$

Note that for $(x-y)^2 < 0$, we have that $D(x-y) - D(y-x) = 0$. We now have $\left\{\psi_\alpha(x), \overline{\psi}_\beta(y)\right\} = 0 \forall (x-y)^2 < 0$. So what about causality? Our observables are bilinear in fermions. They do commute at space-like separations so the theory is causal.

Away from singularities, we have that

$$\begin{aligned}
(i\not{p}_x - m) iS(x-y) = 0 & \\
&= (i\not{\partial}_x - m)(i\not{\partial} + m)\left[D(x-y) - D(y-x)\right] \\
&= -(\partial_x^2 + m^2)\left[D(x-y) - D(y-x)\right] \\
&= 0 \text{ using } p^\mu p_\mu = m^2
\end{aligned}$$

6.17.2 The Feynman Propagator

A similar calculation gives us us that the two point propagator between x and y are

$$\langle 0|\, \psi_\alpha(x)\overline{\psi}_\beta(y)\, |0\rangle = \int \frac{d^3p}{(2\pi)^3 2E_p}(\not{p} + m)_{\alpha\beta} e^{-ip\cdot(x-y)}$$
$$\langle 0|\, \overline{\psi}_\beta(y)\psi_\alpha(x)\, |0\rangle = \int \frac{d^3p}{(2\pi)^3 2E_p}(\not{p} - m)_{\alpha\beta} e^{ip\cdot(x-y)}$$

If we define $S_{F\alpha\beta} = \langle 0|\, T\psi_\alpha(x)\overline{\psi}_\beta(y)\, |0\rangle$ as the object

$$\begin{cases} \langle 0|\, \psi_\alpha(x)\overline{\psi}_\beta(y)\, |0\rangle & x^0 > y^0 \\ -\langle 0|\, \overline{\psi}_\beta(y)\psi_\alpha(x)\, |0\rangle & y^0 x^0 \end{cases}$$

The negative sign in the second term is required for Lorentz invariance when $(x-y)^2 < 0$, since there exists no Lorentz invariant way to determine whether $x^0 < y^0$ or the other way around. In this case, $\left\{\psi(x), \overline{\psi}(y)\right\} = 0$ and so T as defined is lorentz invariant. This is the same for strings of fermionic operators in T - they anti commute. We see the same behaviour for normal ordering, we get that

$$:\psi_1\psi_2: = -:\psi_2\psi_1:$$

We can define our Feynman propagator as

$$\overbracket{\psi(x)\psi^*}(y)$$

From this we get that

$$T(\psi(x), \overline{\psi}(y) =: \psi(x)\overline{\psi(y)}$$

6.18 Momentum Space Feynman Rules for Fermion Amplitudes

Let's review the diagrams for the Yukawa interaction, but with the view of Fermionic quantisation. We have the following diagrams. The important thing is that operators need to remain the same order, unless we decide to anti-commute them and pick up a minus sign. Hence, for a closed fermionic loop which looks like : $\overline{\psi}_\alpha(x)\psi_\alpha(x)\psi^*_\beta(y)\psi_\beta(y)$: where we contract the middle two fields, to contract the other two fields we need to pick up a minus sign (show here).

Question: what if $\mathcal{L}_{\text{int}} = -\lambda\phi\overline{\psi}_\alpha(\gamma^5)_{\alpha\beta}\psi_\beta$? This interaction preserves parity if ϕ is a pseudo-scalar, in other words

$$P\phi(\vec{x},t) = -\phi(-\vec{x},t)$$

This interaction looks like the below (diagram here) this contributes $(-i\lambda)(\gamma^5)_{\alpha\beta}$.

Question: how do we deal with spin and $|\mathcal{M}|^2$, in the cross section calculation? In most experiments $|i\rangle$ spin states are random and so we average over them. For example, for $\psi\psi$, it would be $\frac{1}{4}\sum_{r,s=1}^{2}$, Also spins in $|f\rangle$, aren't observed - they're summed over. Specifically

$$\mathcal{M} = B - A, \text{ in } \psi\psi \to \psi\psi$$

where B, A are the different terms. We wrote appropriate spin sums averages with a line on top.

$$\overline{|\mathcal{M}|}^2 = |\overline{A}|^2 + |\overline{B}|^2 - \overline{A^\dagger B} - \overline{B^\dagger A}$$

$$A = \frac{\lambda^2\left[\overline{u}^s_{\vec{p}'}\cdot u^r_{\vec{q}}\right]\left[\overline{u}^{r'}_{\vec{q}'}\cdot u^s_{\vec{p}}\right]}{\mu^2 - \mu^2 + i\epsilon}$$

Note that we have $(\gamma^0)^\dagger = \gamma^0$.

$$|\overline{A}|^2 = \frac{|\lambda|^4}{4(\mu^2 - m^2)^2}\sum_{r,s,s',r'}\overline{u}^{s'}_{\vec{p}'_\alpha}u^r_{\vec{q}_\alpha}\overline{u}^r_{\vec{q}_\beta}u^{s'}_{\vec{p}'_\beta}\overline{u}^{r'}_{\vec{q}'_\beta}u^s_{\vec{p}_\gamma}\overline{u}^s_{\vec{p}_\delta}u^{r'}_{\vec{q}'_\delta}$$

This is equal to

$$\frac{|\lambda|^4}{4}\frac{(\not{p}' + m)_{\alpha\beta}(\not{q} + m)_{\beta\alpha}\operatorname{tr}\left[(\not{q}' + m)(\not{p} + m)\right]}{(\mu^2 - m^2)^2}$$

Often, we are in the high energy limit, where we may wish to neglect particle masses. In this case, we then find that

$$|\overline{A}|^2 = \frac{|\lambda|^4\operatorname{tr}(\not{p}'\not{q})\operatorname{tr}(\not{q}'\not{p})}{4\mu^2}$$

Similarly, we have that for $|\overline{B}|^2$, we get that

$$|\overline{B}|^2 = \frac{|\lambda|^2}{4t^2}\operatorname{tr}(\not{q}'\not{q})\operatorname{tr}(\not{p}'\not{p})$$

We also want $-\overline{A^\dagger B} - \overline{B^\dagger A} = -2Re\overline{(A^\dagger B)}$. We calculate this as

$$\begin{aligned}\overline{A^\dagger B} &= \frac{|\lambda|^4}{4ut}\sum_{r,r',s,s'}\overline{u}^r_{\vec{q}_\beta}u^{s'}_{\vec{p}'_\beta}\overline{u}^s_{\vec{p}_\alpha}u^{r'}_{\vec{q}'_\alpha}\overline{u}^{r'}_{\vec{q}'_\gamma}u^r_{\vec{q}_\gamma}\overline{u}^{s'}_{\vec{p}'_\delta}u^s_{\vec{p}_\delta}\\ &= \frac{|\lambda|^2}{4ut}\operatorname{tr}\left(\not{p}\not{p}'\not{q}\not{q}'\right)\end{aligned}$$

This gives us the Feynman rules for spin summed $\mathcal{M}^2$ diagrams. We have that complex conjugation switches $|i\rangle$ and $|f\rangle$ in the diagram. Fermion lines are joined with identical momentum on the LHS and RHS. Adter a spin sum, a closed fermion line in the $|\mathcal{M}|^2$ diagram is given by a trace over γ matrices, with appropriate γ^5 ' s etc in vertices at the correct posiition in the trace. trace follows fermion arrows backwards.

6.19 Warming up with spinors from the $SO(3)$ rotation group

Spinors are (complex) elements in a vector field which transform linearly when the underlying coordinate basis is rotated. In three dimensional Euclidean space, this transformation would take the form of an element in $SO(3)$, acting on three dimensional real vectors in $\mathbb{R}^3$. Our goal of this subsection is to build an equivalence between this and elements in $SU(2)$, where then these elements act on 2 component complex vectors in $\mathbb{C}^2$.

6.19.1 Rotations in Euclidean space

Let's revise quickly how to map rotations out in three dimensional space. We multiply a vector $\mathbf{v}$ by a matrix to give the map $\mathbf{v} \mapsto R\mathbf{v}$. In the case of a rotation by an angle θ about the z-axis, our rotation matrix R_z takes the form

$$R_z = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

In the terminology of Lie groups, this is an element of the Lie group $SO(3)$, and has an associated generator J_z, given by it's derivative at $\theta = 0$;

$$J_z = \frac{d}{d\theta} R_z(\theta)\bigg|_{\theta=0} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We can repeat this same procedure with rotations along the y and x axes, and differentiate with respect to the parameters as well to obtain the generators J_x and J_y. One finds that these rotation generators obey the following commutation relations, or 'algebra'

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

We can then write out a full transformation, which is a rotation by an angle θ about the normal vector $\mathbf{n}$, as

$$\mathbf{v} \mapsto e^{i\mathbf{n}\cdot\mathbf{J}\theta}\mathbf{v}$$

where the exponent is simply the expected series sum of the matrices.

6.19.2 Transformations with Unitary matrices

We now explore the group $SU(2)$, the group of 2 by 2 matrices which satisfy the relation $U^\dagger = U^{-1}$, and $\det U = 1$. One can show, that by comparing coefficients, that unitary matrices can be written in general of the form

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

where we have the condition that $|a|^2 + |b|^2 = 1$. Recall that a and b are complex, so over the reals are specified by 2 parameters each. Our condition that $|a|^2 + |b|^2 = 1$ reduces our degrees of freedom by 1, so we have 3 degrees of freedom in total. We would like to see how unitary matrices transform vectors of the form

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

We have that

$$\xi \mapsto \xi' = U\xi = \begin{pmatrix} a\xi_1 + b\xi_2 \\ -b^*\xi_1 + a^*\xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix}$$

Now we ask the question of whether we can find other vectors derived from this which can transform in the same way. Turns out, the vector $(-\xi_2^*, \xi_1^*)$ does as well. You can verify this yourself! We employ the notation that Ryder uses here, using the sign $\sim$ to denote 'transforms as', we've discovered that

$$(\xi_1, \xi_2) \sim (-\xi_2^*, \xi_1^*)$$

Or, setting

$$\chi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

we can alternatively write that $\xi \sim \chi\xi^*$. Now, this also means that, taking the hermitian conjugate of a vector, that $\xi^\dagger$ transforms as $(-\xi_2, \xi_1)$ which implies that $\xi\xi^\dagger$ transforms as

$$H = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} (-\xi_2, \xi_1) = \begin{pmatrix} -\xi_1\xi_2 & \xi_1^2 \\ -\xi_2^2 & \xi_1\xi_2 \end{pmatrix}$$

But we already know that since we have the transformation laws $\xi \mapsto U\xi$ and $\xi^\dagger \mapsto \xi U^\dagger$,

$$\xi\xi^\dagger \mapsto U\xi\xi^\dagger U^\dagger \implies H \mapsto UHU^\dagger = UHU^{-1}$$

The main reason why we've invested so much into finding this matrix which transforms under U is that H is indeed traceless. So, we can write out a matrix which transforms under U in the way we want by giving ourselves a complex traceless matrix, which we'll call h. Now, h can be written as

$$h = \begin{pmatrix} z & x + iy \\ x - iy & -z \end{pmatrix} = \sigma \cdot \mathbf{x}$$

where $\mathbf{x}$ is our original position vector we've rotated, and σ are the Pauli sigma matrices. So, we've taken a vector rotating in $SO(3)$ and have reduced it down to induced rotations by $SU(2)$. By contruction we've shown that $h \mapsto UhU^{-1}$, and so we've shown that unitary transformations acting on spinors in $SU(2)$ correspond to rotations acting on position vectors in $SO(3)$. We have a specifc map from the spinors to the position vector, given by comparing elements of h and H

$$x = \frac{1}{2}(\xi_1 + \xi_2), \quad y = \frac{1}{2i}(\xi_1^2 + \xi_2^2), \quad z = \xi_1\xi_2$$

6.20 Lorentz tranformations and spinors

To motivate the derivation of the Dirac equation, we need to first discuss in general, objects which transform sensibly with Lorentz transformations. If we have a coordinate system denoted by x^μ, we Lorentz transform it with the map $x^\mu \to \Lambda^\mu{}_\nu x^\nu$. If we had a scalar field $\psi(x)$, then this passive transformation would induce the transformation

$$\psi(x) \to \psi(\Lambda^{-1}x)$$

Now, instead of a scalar field, we might wish to consider what happens to, say, a vector field when we induce a Lorentz transformation. In the case that our vector field transforms linearly in response to Lorentz transformations, we have a spinor. More specifically, we have a complex vector field $\psi^a(x)$ which obeys the transformation law

$$\psi^a(x) \to \Lambda^a{}_b \psi^b(x)$$

Suppose we had some eaution which a scalar field ϕ satisfies. For example, we have the Klein-Gordon equation

$$\left(\partial_\mu \partial^\mu - m^2\right)\phi = 0$$

. If we were to perform a Lorentz boost which shifts our frame of reference, then it only makes physical sense that the equation still holds, because nothing about the actual physical system has changed. Suppose that our Lorentz boost is written as

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

Then our corresponding scalar field should transform as

$$\phi(x) \to \phi(\Lambda^{-1}x)$$

We can chack that, under this transformation, the Klein-Gordon equation still holds.

We can generalise this kind of transformation to a field with multiple components which we index as $\phi_a(x)$. As inspiration, we explore how some vector field V^i in three dimensions transforms under a rotation.

$$V^i \to V'^i = R^{ij} V^j$$

and so we'd expect a Lorentz boost to have a scalar field transform like

$$\phi_a(x) \to \phi'(x)_a = M(\Lambda)^b{}_a \phi_b(\Lambda^{-1}x)$$

In the above, we're representing the Lorentz transformation as a matrix $M(\Lambda)$. We call this a representation of our Lorentz transformation. Representations should obey the rule that

$$M(\Lambda)M(\Lambda') = M(\Lambda\Lambda')$$

or in other words the representation should respect the group structure of Lorentz transformations. Proverbially, we are taking a methematical 'encoding' of our transformation. How do we go about finding the representation M?

Again, we seek inspiration from rotation groups in 3 dimensions. A rotation in 3 dimensions is represented in real space by the matrices in $SO(3)$, which is isomorphic to $SU(2)$. This $SU(2)$

has a basis, which are the Pauli matrices σ_i. Given a quantum state $|\psi\rangle$, how do we represent its transformation?

Let $|\psi\rangle$ be a wavefunction representing a $\frac{1}{2}$ spin particle. Then, we can represent rotations with thefollowing transformation

$$|\psi\rangle \to |\psi'\rangle = \exp\{i\mathbf{n} \cdot \mathbf{J}\} |\psi\rangle$$

In this case, J_i are the generators of this transformation. We say that the Lie group of rotations in three dimensions are generated by a Lie algebra with commutation relations

$$J_k = i\epsilon_{ijk}[J_i, J_j]$$

In three dimensions for our rotation group, this operator is given by

$$\mathbf{J} = \mathbf{x} \times \mathbf{p} = \mathbf{x} \times (-i\nabla)$$

Where we can index this angular momentum object as

$$J^{\nu\mu} = i(x^\nu \partial^\mu - x^\mu \partial^n u)$$

This will precisely be our generalisation for our generatorfor our Lorentz group, except now the indices μ, ν are taken over $\mu, \nu = 0, 1, 2, 3$. This object obeys the Lorentz algebra.

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(-g^{\mu\rho}J^{\nu\sigma} + g^{\mu\sigma}J^{\nu\rho} - g^{\nu\sigma}J^{\mu\rho} + g^{\nu\rho}J^{\mu\sigma})$$

And in general, any expression for $J^{\mu\nu}$ which satisfies this equation is a valid representation of our Lie algebra. We denote the components of $J^{\mu\nu}$ by writing down the object with two additional indices $\alpha\beta$. Thus we denote our object by $(J^{\mu\nu})_{\alpha_beta}$. A simple representation that we pull out of the hat is

$$(J^{\mu\nu})_{\alpha\beta} = i\left(\delta^\mu{}_\alpha \delta^\nu{}_\beta - \delta^\mu{}_\beta \delta^\nu{}_\alpha\right)$$

Why is this choice intuitive? Well, our Lorentz transformation is obtained by exponentiating our generator that we have here. Thus, for an anti-symmetric 2nd rank tensor, our full transformation is given by

$$\exp\left(i\frac{\omega_{\mu\nu}}{2}J^{\mu\nu}\right)$$

and infinitesimally applied to a 4-vector, our transformation is

$$V^\mu \to \left(\delta^\mu{}_\nu - \frac{i}{2}\omega_{\alpha\beta}(J^{\alpha\beta})^\mu{}_\nu\right) V^\mu$$

6.21 Constructing the Chiral representation for the Dirac equation

You can show that if we have a set of matrices $\{\gamma^\mu\}_\mu$ which obey the identity

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$$

then the object

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

obeys the Lorentz algebra which we derived above.

Whilst this choice of representation is not unique, we choose a canonical one called the Weyl representation which is given by

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

I'll touch on this a bit more later, but the important this is that this object represents boosts in 3 different dimensions, given by S^{0i} and rotations in 3 dimensions given by S^{ij}. Thus, our field $\phi(x)$ transforms as, when we exponentiate S_{ij}, as

$$\phi(x) \to \phi'(x) = \exp\Big(i\frac{\omega_{\mu\nu}}{2}S^{\mu\nu}\Big)\phi(\Lambda^{-1})$$

where we've done and inverse transform to represent our change in frame of reference, and also done a map on the field itself.

6.22 Motivating Lorentz invariance of the Dirac equation

In this section, we'll present the Dirac equation and show that it's Lorentz invariant. The way to do this is slightly more convoluted than showing the Lorentz invariance of the Klein-Gordon equation, like we did in the previous section. It involves exploring how γ^μ transforms under both the $S_{\mu\nu}$ and $J_{\mu\nu}$ representations of the Lorentz transformations, and how these representations relate to one another.

But first, we present the Dirac equation. A field $\psi(x)$ satisfying the Dirac equation satisfies the equation

$$(i\gamma_\mu \partial^\mu - m)\psi(x) = 0$$

Note that, as opposed to the Klein-Gordon equation, we have a m as the constant term instead of m^2. Also notice the fact that we're not taking two derivatives in this equation, we instead contract the derivative ∂^μ with γ_μ. Now, under the Lorentz transformation where $x^\mu \to x^\nu \Lambda^\mu{}_\nu$, recall our field transforms as

$$\psi(x) \mapsto \Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x)$$

To show that the Dirac equation is invariant under this specific transformation, we require that (for reasons we will show later)

$$\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu\gamma^\nu$$

One can see that if this condition is satisfied, then, upon transforming the equation

$$\begin{aligned}
(i\gamma^\mu\partial_\mu - m)\psi \to (i\gamma^\mu\partial_\mu - m)&\Lambda_{\frac{1}{2}}(\phi\Lambda^{-1}x) \\
&= \Lambda_{\frac{1}{2}}\Lambda_{\frac{1}{2}}^{-1}(i\gamma^\mu(\Lambda^{-1})_\mu{}^\nu\partial_\nu - m)\Lambda_{\frac{1}{2}}\psi(\Lambda^{-1}x) \\
&= \Lambda_{\frac{1}{2}}(i\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}}(\Lambda^{-1})_\mu{}^\nu\partial_\nu - m)\psi(\Lambda^{-1}x) \\
&= \Lambda_{\frac{1}{2}}(i\gamma^\mu\partial_\mu - m)\psi(\Lambda^{-1}x) \\
&= 0
\end{aligned}$$

Let's review carefully what we've done here. In the first line, we've transformed $\psi(x)$ appropriately according to our Lorentz transformation. In the second line, we've done two things. First, we multiply on the left by $\Lambda_{\frac{1}{2}}\Lambda_{\frac{1}{2}}^{-1}$ since it's just the identity. In addition, we also pull out a factor of Λ^{-1} due to the chain rule. Now, in the third line, we used the identity we've derived above to switch out our expression for $\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}}$ with just $\Lambda\gamma^\nu$. In doing this, the factors of Λ have cancelled each other out. The final expression is zero, which means that we've successfully shown that the Dirac equation is Lorentz invariant.

Now, we'll prove the expression above, by using the infinitesimal generators instead. Observe that the above condition is equivalent to

$$(1+\frac{i\omega_{\rho\sigma}}{2}S^{\rho\sigma})\gamma^\mu(1-\frac{i\omega_{\rho\sigma}}{2}S^{\rho\sigma} = (1-\frac{i\omega_{\rho\sigma}}{2}(J^{\rho\sigma})_\mu{}^\nu\gamma_n u$$

So in turn, if we can show that

$$[\gamma^\mu, S^{\rho\sigma}] = \gamma^\nu(J^{\rho\sigma})^\mu{}_\nu$$

we've proven the identity above. We do this as follows

$$(J^{\mu\nu})_{\alpha\beta} = i\left(\delta^\mu{}_\alpha\delta^\nu{}_\beta - \delta^\mu{}_\beta\delta^\nu{}_\alpha\right)$$

When we raise the index on a delta function, we get our metric back. So our expression is

$$(J^{\mu\nu})_\alpha{}^\beta = i\left(\delta^\mu{}_\alpha g^{\nu\beta} - g^{\mu\beta}\delta^\nu{}_\alpha\right)$$

Thus

$$\gamma^\alpha(J^{\mu\nu})_\alpha{}^\beta = i\left(\gamma^\mu g^{\nu\beta} - \gamma^\nu g^{\mu\beta}\right)$$

As for the left hand side

$$\begin{aligned}[\gamma^\beta, S^{\mu\nu}] &= [\gamma^\beta, \frac{i}{4}[\gamma^\mu,\gamma^\nu]] \\ &= \frac{i}{4}\left(\gamma^\beta\gamma^\mu\gamma^\nu - \gamma^\beta\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu\gamma^\beta + \gamma^\nu\gamma^\mu\gamma^\beta\right)\end{aligned}$$

but we can use the identity that

$$\{\gamma^\mu,\gamma^\nu\} = -g^{\mu\nu}$$

and commute index pairs which are either (ν,β) or (μ,β), to give

$$\frac{i}{4}\left(-\gamma^\mu\gamma^\beta\gamma^\nu - 2g^{\mu\beta}\gamma^\nu + \gamma^\nu\gamma^\beta\gamma^\mu + 2g^{\beta\nu}\gamma^\mu + \gamma^\mu\gamma^\beta\gamma^\nu + 2g^{\nu\ beta}\gamma^\mu - \gamma^\nu\gamma^\beta\gamma^\mu - 2\gamma^\nu g^{\mu\beta}\right)$$

but the terms above cancel to give the correct expression. Hence we've proven the identity above, and have shown that Dirac's equation is Lorentz invariant.

So we've done all this work - but let's remind ourselves what it's for. This commutation relation is important because it gives us an equation relating the generators $J^{\mu\nu}$ and $S^{\mu\nu}$. This commutation relation is equivalent to saying, with $\omega_{\rho\sigma}$ as a parameter, that infinitesimally we have

$$\left(1+\frac{i\omega_{\rho\sigma}S^{\rho\sigma}}{2}\right)\gamma^\mu\left(1-\frac{i\omega_{\rho\sigma}S^{\rho\sigma}}{2}\right) = (1-i(J^{\rho\sigma})^\mu{}_\nu)\gamma^\nu \implies \Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu\gamma^\nu$$

And with this identity, we've shown that the Dirac equation is Lorentz invariant.

6.23 Constructing boosts and rotations in block diagonal form

This spinor representation is reducible, which means that we can write out our transformations in block diagonal form. For example, we write out the components of $S^{\mu\nu}$ explicitly in terms of boosts and rotations, and discover that both the boost and rotations are parametrised by three parameters (hence we can encode them as vectors). In the boost case, we have that

$$S^{0i} = \frac{i}{2}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma \\ -\sigma_i & 0 \end{pmatrix} = \frac{i}{2}\begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$$

But, as a tranformation, contracting with $\omega_{\mu\nu}$, we're only summing to get $\exp\left(\frac{i}{2}\omega_{0i}S^{0i}\right)$, and so reparametrising $\omega_{0i} = \alpha_i$, we find that our transformation takes the form

$$\Lambda_{\frac{1}{2}} = \exp\left(i\frac{\omega_{\mu\nu}S^{\mu\nu}}{2}\right) = \begin{pmatrix} e^{\alpha\cdot\frac{\sigma}{2}} & 0 \\ 0 & e^{-\alpha\cdot\frac{\sigma}{2}} \end{pmatrix}$$

Bear in mind that since we're summing over all the indices, we have an extra factor of 2. Now, notice that we have a switch in sign under boosts.

Now we explore what happens when we transform under rotations. Rotations are represented by S^{ij}, where $i, j = 0, 1, 2, 3$. Let's compute these matrix representations explicitly - this is also good revision for remembering our Pauli Sigma matrix commutation relations. We have that

$$\begin{aligned} S^{ij} &= \frac{i}{4}[\gamma^i, \gamma^j] \\ &= \frac{i}{2}\gamma^i\gamma^j \\ &= \frac{i}{2}\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \\ &= \frac{i}{2}\begin{pmatrix} -\sigma_i\sigma_j & 0 \\ 0 & -\sigma_i\sigma_j \end{pmatrix} \end{aligned}$$

Now, we make use of our Pauli-sigma relations to simplify this expression. One can easily verify that

$$\sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k \implies S^{ij} = \frac{1}{2}\epsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

Now, when we want to contract S^{ij} with ω_{ij} we have that

$$\omega_{ij}S^{ij} = \frac{1}{2}\epsilon_{ijk}\omega_{ij}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

But the cool thing abou this is that we can write $\beta_k = \epsilon_{ijk}\omega_{ij}$, which gives us the final result for a rotatation transformation, which is

$$\Lambda_{\frac{1}{2}} = \begin{pmatrix} \exp^{-\frac{i}{2}\sigma\cdot\beta} & 0 \\ 0 & \exp^{-\frac{i}{2}\sigma\cdot\beta} \end{pmatrix}$$

Where we have a noticable difference with respect to boosts in that there's no sign change between the block diagonal matrices. The fact that we've broken down this spinor representation of Lorentz transformations into two block diagonal matrices is super useful because now we can also split up the spinor vector ψ into two parts, which we call the left and right handed parts of our spinor.

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

So infinitesimally, we have that

$$\begin{aligned} \psi_L &\mapsto \left(1 + \alpha \cdot \frac{\sigma}{2} - i\beta \cdot \frac{\sigma}{2}\right) \psi_L \\ \psi_R &\mapsto \left(1 - \alpha \cdot \frac{\sigma}{2} - i\beta \cdot \frac{\sigma}{2}\right) \psi_R \end{aligned}$$

6.24 The Weyl equations

Let's go back to our Dirac equation, but this time let's view it in the context of our spinor which we split up. We have that

$$\left(i\gamma^0 \partial_0 - \gamma^i \partial_i - m\right) \psi = 0$$

Now, we can write this in the form of a matrix equation, bearing in mind that m acts as a scalar multiple of the identity. Thus, we have the matrix equation

$$\begin{pmatrix} -m & i(\partial_0 + \sigma \cdot \nabla) \\ i(\partial_0 - \sigma \cdot \nabla) & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

In this case, we've mixed the components ψ_R and ψ_L. However, we can separate out the mixing by exploring the massless case with $m = 0$, which reduces us to the equations

$$\begin{aligned} (\partial_0 + \sigma \cdot \nabla)\psi_R &= 0 \\ (\partial_0 - \sigma \cdot \nabla)\psi_L &= 0 \end{aligned}$$

We can simplify this notation even further by 'extending' out σ, and defining the objects

$$\sigma^\mu = (1, \sigma), \quad \bar{\sigma}^\mu = (1, -\sigma)$$

Which gives us a condensed form of the equations to read

$$\begin{aligned} \sigma^\mu \partial_\mu \psi_R &= 0 \\ \bar{\sigma}^\mu \partial_\mu \psi_L &= 0 \end{aligned}$$

6.25 Constructing Plane-Wave solutions to the Weyl equation

At it's essence, the Dirac equation is a wave equation. In this section, we'll explore solutions of Dirac's equation which have the form $\psi(p) = e^{-ipx}u(p)$, where p denotes momentum, and we have the relativistic dispersion relation where $p^\mu p_\mu = m^2$. We can make our life easier by reducing the problem to that of the rest frame, where $p^\mu = (E_{\mathbf{p}}, 0)$. Our dispersion relation then tells us that $p^\mu = (m, 0)$. Our Dirac equation in matrix form then reads

$$\begin{pmatrix} -m & i(\partial_0 + \sigma \cdot \nabla) \\ i(\partial_0 - \sigma \cdot \nabla) & -m \end{pmatrix} e^{-ip\cdot x} u(p) = \begin{pmatrix} -m & m \\ m & -m \end{pmatrix} e^{-ip\cdot x} u(p) = 0$$

This implies that our general solution in the rest frame can be written as

$$u(p) = \sqrt{m}\begin{pmatrix}\xi \\ \xi\end{pmatrix}$$

Now, it's a matter of boosting this solution to a non-rest frame. To do this, we need to do a Lorentz boost on our 4-momentum vector, which for simplicity we'll just consider a boost in the z-direction. We know from special relativity that a boost given by

$$\begin{pmatrix}E \\ p_3\end{pmatrix} = \left(1 + \nu\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\right)\begin{pmatrix}m \\ 0\end{pmatrix}$$

Notice that we're not using our spinor representation here; the spinor representation is not for transforming 4-vectors; we're just using our ordinary Lorentz boost representation. Exponentiating this gives us our full Lorentz boost j The associated spinor representation however for this boost is

$$\Lambda_{\frac{1}{2}} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)$$

For a boost in the z-direction, our boost parameter looks is zero everywhere except in the z direction, hence $\omega_{03} = -\omega_{30} = \mu$, and contracting this object with $S^{\mu\nu}$ (rembering to include an extra factor of two due to antisymmetry, gives our Lorentz boost

$$\Lambda_{\frac{1}{2}} = \exp\left(\frac{\mu}{2}\begin{pmatrix}\sigma_i & 0 \\ 0 & -\sigma_i\end{pmatrix}\right)$$

Now we're in good shape to multiply our spinor by the Lorentz boost object.

7 Example sheet 1

Solved exercises and notes from example sheet 1.

7.1 Question 1

In this question, we're asked to derive the momentum operator $P^\mu = \int d^3x\, T^{0\mu}$. The first thing we notice is that this object is derived from the momentum part of the energy-momentum tensor, and is simply integrating this over all of space. Hence, we expect it to be an operator on fields. The first thing we'll do is derive the form of the energy momentum tensor.

The energy-momentum tensor is a conserved object which arises from translational symmetry. In Minkowski spacetime, translational symmetries in time correspond to the conservation of energy, and translational symmetries in space correspond to conserved momemtum. Hence, we consider the translation

$$x^\mu \to x^\mu + \epsilon^\mu$$

But, since this is a passive transformation of our reference frame, we have that our scalar field $\psi(x)$ transforms as

$$\psi(x) \to \psi(x - \epsilon) = \psi(x) - \epsilon^\mu \partial_\mu \psi(x)$$

So our change in the scalar field $\delta(x) = \epsilon^\mu \partial_\mu \psi(x)$. Similarly, our Lagrangian L in general transforms in the same way, with $\delta L = \epsilon^\mu \partial_\mu L$. This is indeed a symmetry of the Lagrangian since $\delta L = \partial_\mu(F^\mu)$ where $F^\mu = \epsilon^\mu L$ Thus, by Noether's theorem we can write out conserved current as

$$j^\mu = \epsilon^\mu L - \epsilon^\nu \partial_\nu \psi \left(\frac{\partial L}{\partial(\partial_\mu \psi)} \right)$$

However, now we can 'factorise' out the ϵ^μ since it's arbitrary, and write our conserved current as

$$T^\mu{}_\nu = \delta^\mu{}_\nu L - \partial_\nu \psi \left(\frac{\partial L}{\partial(\partial_\mu \psi)} \right)$$

Now, substituting our expression for the Lagrangian (which is associated to the Klein-Gordon equation)

$$L = \frac{1}{2}\partial_\alpha \psi \partial^\alpha \psi - \frac{1}{2} m^2 \psi^2$$

Our expression for our energy momentum tensor becomes

$$T^{\mu\nu} = \eta^{\mu\nu} \left(\frac{1}{2}\partial_\alpha \psi \partial^\alpha \psi - \frac{1}{2} m^2 \psi^2 \right) - \partial^\nu \psi \partial^\mu \psi$$

Let's examine this term by term for $\int d^3x\, T^{0\mu}$. For starters, let's take the $\partial^0 \psi \partial^\mu \psi$ term. First, we Fourier expand this object to get

$$\psi(x) = \int d^3p \frac{1}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ip\cdot x} + a^\dagger_{\mathbf{p}} e^{ip\cdot x} \right)$$

If we differentiate this thing with ∂^0 and ∂^μ, we have that

$$\partial^0 \psi = \int d^3p \, \frac{p^0}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \left(-a_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)$$
$$\partial^\mu \psi = \int d^3p \, \frac{p^\mu}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \left(-a_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)$$

Now, amalgamating this all together, we have that the integral of this is

$$\int d^3x T^{0\mu} = \int d^3x \, d^3p \, d^3q \, \frac{p^0 q^\mu}{(2\pi)^6\sqrt{2E_{\mathbf{p}}E_{\mathbf{q}}}} \left(-a_{\mathbf{p}}a_{\mathbf{q}} e^{-i(p+q)\cdot x} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p-q)\cdot x} + a_{\mathbf{p}} a_{\mathbf{q}}^\dagger e^{i(p-q)\cdot x} - a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p+q)\cdot x}\right)$$

But here, we can use the identity that

$$\int d^3x \, e^{i\mathbf{x}\cdot(\mathbf{c}} = (2\pi)^3\delta(\mathbf{c})$$

In the spatial part to isolate out a delta function. In addition, we also use the fact that $p^0 = E_{\mathbf{p}}$ to cancel out with $E_{\mathbf{p}}$ in the denominator. Hence, we end up with the expression that the above is

$$\int \frac{d^3p}{2(2\pi)^3} p^\mu (a_{\mathbf{p}}a_{-\mathbf{p}} + a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}} a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger)$$

Now, the first and last terms disappear because under a change of variables $\mathbf{p} \to -\mathbf{p}$, p^μ is an odd function under the spatial integral, but $a_{\mathbf{p}}a_{-\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger$ are even functions, so these terms disappear. Now, if we commute the third expression with our standard relation and remove our delta function since it's an infinite term (like we do with our Hamiltonian), we get the final expression.

One can show that the rest of the terms in the energy momentum tensor disappear under integrating over $\int d^3x$.

Now, for the next part of the question, we'd like to show that in the Heisenberg picture

$$[P^\mu, \psi(x)] = -i\partial^\mu \psi(x)$$

This is achieved fairly easily by pulling the commutators to inside the integral and then using our standard commutation relations.

$$\begin{aligned}
[P^\mu, \psi(x)] &= [\int \frac{d^3p}{(2\pi)^3} \, p^\mu a_{\mathbf{p}}^\dagger a_{\mathbf{p}}, \psi] \\
&= \int \frac{d^3p}{(2\pi)^3} \, [p^\mu a_{\mathbf{p}}^\dagger a_{\mathbf{p}}, \psi] \\
&= \int \frac{d^3p}{(2\pi)^3} \, p^\mu [a_{\mathbf{p}}^\dagger a_{\mathbf{p}}, \psi]
\end{aligned}$$

Where, going into the third line, since p^μ is not an operator, we've just pulled it out of the commutator. By linearity of the integral, we can also pull the commutator inside the integral, as we did going into the second line. Fourier expanding out, the above expression reads

$$\begin{aligned}\int \frac{d^3pd^3q}{(2\pi)^6} p^\mu [a_{\mathbf{p}}^\dagger a_{\mathbf{p}}, a_{\mathbf{q}} e^{-iq\cdot x} + a_{\mathbf{q}}^\dagger e^{iq\cdot x}] &= \int \frac{d^3pd^3q}{(2\pi)^6}\, p^\mu \left(e^{-iq\cdot x}[a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}]a_{\mathbf{p}} + e^{iq\cdot x} a_{\mathbf{p}}^\dagger [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger]\right) \\ &= \int \frac{d^3p}{(2\pi)^3} \left(-p^\mu a_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger p^\mu e^{ip\cdot x}\right) \\ &= -i\partial^\mu \psi(x)\end{aligned}$$

7.2 Question 2

In this question we're trying to derive the Klein-Gordon equation, but via the Heisenberg picture for operators. This is when operators evolve in time instead of states. In the Heisenberg picture, our equation for time evolution for an operator $\mathcal{O}_H$ is

$$\frac{d\mathcal{O}_H}{dt} = i[H, \mathcal{O}_H]$$

First, let's look at computing

$$\dot{\phi}(x) = i[H, \phi(x)]$$

The Hamiltonian associated with our free field Lagrangian is

$$H = \int d^3y\, \frac{1}{2]}\pi(y)^2 + \frac{1}{2}(\nabla\phi(y))^2 + \frac{1}{2}m^2\phi(y)^2$$

So, to compute the commutator required, we need to remind ourselves of the commutation relations between the our scalar field $\phi(x)$ and our conjugate momentum field $\pi(x)$. This our entirely analogous to those we know from quantum mechanics:

$$\begin{aligned}[][\phi(x), \phi(y)] = [\pi(x), \pi(y)] = 0 \\ [\phi(x), \pi(y)] = \delta^3(\mathbf{x} - \mathbf{y})\end{aligned}$$

I just wanted to note that here, when we write $\phi(x)$, this is condensed notation for $\phi(\mathbf{x}, t)$, where $\mathbf{x}$. Now, when we bring the commutator inside the integral to calculate $[H, \phi(x)]$, all functions which are functions of $\phi(x)$ disappear under the commutator. The only expression we're left with is

$$\dot{\phi}(x) = \frac{i}{2}\int d^3y\, [\pi(y)^2, \phi(x)]$$

However, we can apply the 'product rule' for operator commutators which is

$$[AB, C] = A[B, C] + [A, C]B$$

so that the above expression reads as follows:

7.3 Question 3

Showing invariance

Our Lagrangian density is the Klein-Gordon field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

This question is about symmetries under **rotations**, transformations which infinitesimally take the form, to first order in θ,

$$\phi_a \to \phi_a + \theta\epsilon_{abc}\eta_b\phi_c$$

Our Lagrangian changes as

$$\mathcal{L} \to \mathcal{L} + \theta\epsilon_{abc}\eta_b\partial^\mu\phi_a\partial_\mu\phi_c - m^2\epsilon_{abc}\eta_b\phi_a\phi_c$$

The important thing to notice here is that the two terms added on go to zero, since ϵ_{abc} is antisymmetric in (a, c), yet $\partial^\mu\phi_a\partial_\mu\phi_c$ are symmetric in (a, c)! Antisymmetric objects contracted with symmetric objects go to zero.

Constructing conserved quantities

Since $\mathcal{L}$ is invariant under our transformation our Noether current is reduced to

$$j^\mu = -\delta\phi_a\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \propto \epsilon_{abc}\eta_b\phi_c\partial^\mu\phi_a$$

Now, to get a conserved quantity in integral form, we write out the space and time components separately. Our condition that $\partial_\mu j^\mu = 0$ implies that

$$\frac{\partial j^0}{\partial t} - \nabla\cdot\mathbf{j} = 0 \implies \frac{\partial}{\partial t}\int d^3x\, j^0 = \int d^3x\,\nabla\cdot\mathbf{j}$$

Where we've simply integrated the equation and moved the spatial part to the left, and took the partial derivative out of the integral. However, using divergence theorem, we have that

$$\int d^3\,\nabla\cdot\mathbf{j} = \int d\mathbf{S}\cdot\mathbf{j} \to 0$$

since currents should decay at infinity. Hence our conserved quantity is just

$$\int d^3x\, j^0 = \int d^3x\epsilon_{abc}\eta_b\phi_c\partial^0\phi_a = \eta_b\int d^3x\epsilon_{abc}\phi_c\dot{\phi}_a$$

However, there was freedom in our choice of ϵ_b this whole time, so we can 'strip' out this component such that $\int d^3x\,\epsilon_{abc}\phi_c\dot{\phi}_a$ is our conserved quantity, which is what the question asks for up to relabelling of indices.

Verifying our conserved quantity with our field equations

It's easy to check that the Euler-Lagrange equations yield the Klein-Gordon equation

$$(\partial_\mu \partial^\mu - m^2)\phi = 0$$

which with time and spatial derivatives gives

$$\ddot{\phi}_a - \nabla^2 \phi_a + m^2 \phi_a = 0$$

Differetiating our conserved quantity with respect to time, and then applying our Klein-Gordon equation gives

$$\begin{aligned}
\dot{Q}_a &= \int d^3x \epsilon_{abc} \dot{\phi}_b \dot{\phi}_c + \epsilon_{abc} \ddot{\phi}_b \phi_c \\
&= \int d^3x \, \epsilon_{abc} \ddot{\phi}_b \phi_c \\
&= \int d^3x \, \epsilon_{abc} (m^2 \phi_b - \nabla^2 \phi_b) \phi_c \\
&= \int d^3x \, \epsilon_{abc} m^2 \phi_b \phi_c + \epsilon_{abc} (\nabla \phi_b) \cdot (\nabla \phi_c) \\
&= 0
\end{aligned}$$

Going into the second line, we have symmetric time derivatives in (b, c). Going into the penultimate line, we've integrated by parts to shift one derivative in the Laplacian to the other side. The result is zero because both terms have ϕ_i expressions which are symmetric in (b, c).

7.4 Question 4

Minkowski metric is invariant under Lorentz transformations

We're also exploring invariance under Lorentz transformations $x^\mu \to x'^\mu \Lambda^\mu{}_\nu x^\nu$. Our most basic quantity that we can make Lorentz invariant is simply our 'length' of our 4-vector, which is our Minkwoski metric contracted with our vectors

$$L = \eta_{\mu\nu} x^\mu x^\nu = \eta_{\mu\nu} x'^\mu x'^\nu$$

This implies that, expanding out the Lorentz transforms with slightly different summation indices, that

$$\eta_{\mu\nu} x^\mu x^\nu = \eta_{\rho\sigma} \Lambda^\rho{}_\mu x^\mu \Lambda^\rho{}_\nu x^\nu x^\mu$$

However, since our choice for x^μ was arbitrary, we must have that identically

$$\eta_{\mu\nu} = \eta_{\rho\sigma} \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu$$

Conditions on an infinitesimal Lorentz transformation

Our condition that the Minkowski metric is invariant under Lorentz transformations infinitesimally yields the condition that

$$\begin{aligned}
\eta_{\rho\alpha} &= \eta_{\rho\alpha} + \alpha \left(\delta^\nu{}_\alpha \omega^\nu{}_\rho \eta_{\nu\mu} + \delta^\mu{}_\rho \omega^\nu{}_\alpha \eta_{\mu\nu} \right) \\
&= \eta_{\rho\alpha} + \alpha \left(\eta_{\mu\alpha} \omega^\mu{}_\rho + \eta_{\rho\nu} \omega^\mu{}_\alpha \right) \\
&= \eta_{\rho\alpha} + \alpha \left(\omega_{\rho\alpha} + \omega_{\alpha\rho} \right)
\end{aligned}$$

Thus the tensor ω is antisymmetric here. The matrix which corresponds to rotations about the x_3 direction is a basis element for antisymmetric 4 by 4 matrices, and is

$$\omega^\nu{}_\mu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Similary, for boosts, whilst $\omega_{\mu\nu}$ is antisymmetric, $\omega^\mu{}_\nu$ isn't since we're contracting with a Minkwoski metric. Thus, a boost in the x direction is given by

$$\omega^\nu{}_\mu = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

7.5 Question 5

Our corresponding active transformation in the field it given by

$$\begin{aligned}\psi(x^\mu) \to \psi(x^\mu - \alpha\omega^\mu{}_\nu x^\nu) &= \psi(x) - \alpha\omega^\mu{}_\nu x^\nu \partial_\mu \psi(x) \\ &= \psi(x) + 0 - \alpha\omega^\mu{}_\nu x^\nu \partial_\mu \psi(x) \\ &= \psi(x) - \omega_{\mu\nu}\eta^{\mu\nu}\psi(x) - \alpha\omega^\mu{}_\nu x^\nu \partial_\mu \psi(x) \\ &= \psi(x) - \alpha\omega^\mu{}_\nu \partial_\mu (x^\nu \psi(x))\end{aligned}$$

where in the last equality we're just performing a standard Taylor expansion. Since our Lagrangian is a function of $L(\phi) = L(\phi(x)) = L(x)$, our transformation on x induces exactly the same transformation on L, and since $\omega_{\mu\nu}$ is constant we can pull the derivative out:

$$L \to L - \alpha\partial_\mu(\omega^\mu{}_\nu x^\nu)L$$

Putting this together, this implies that our Noether current is

$$j^\mu = \omega^\mu{}_\nu x^\nu L - (\omega^\rho{}_\nu \partial_\rho x^\nu \phi)\frac{\partial L}{\partial(\partial_\mu \phi)}$$

But observe that this is

$$j^\mu = \omega^\rho{}_\nu \left(\delta^\mu{}_\rho x^\nu L - \partial_\rho x^\nu \frac{\partial L}{\partial(\partial_\mu \phi)}\right)$$

Our expression in the brackets is exactly the energy momentum tensor with one index contracted down, with an extra factor of x^ν. Hence, our Noether current is

$$j^\mu = \omega^\rho{}_\nu x^\nu T^\mu{}_\rho$$

Our conserved current is given by

$$Q = \int d^3x\, j^0 = \omega_{\rho\nu} \int d^3x\, T^{0\rho} x^\nu$$

Since we shown earlier that $\omega_{\mu\nu}$ is antisymmetric, if we consider indices only in the spatial part, we can write this thing as

$$\omega_{ij} = \epsilon_{ijk} n_k \implies Q = n_k \epsilon_{ijk} \int d^3x\, T^{0i} x^j$$

Since n_k is free, we can choose, relabelling indices, that

$$Q_i = \epsilon_{ijk} \int d^3x\, T^{0j} x^k = \frac{1}{2}\epsilon_{ijk} \int d^3x\, T^{0j} x^k - T^{0k} x^j$$

where we've added the extra term due to antisymmetry in j, k. Similarly, we can do this with boosts by looking at $\omega_{0i} = -\omega_{i0}$ components, which gives our conserved current

$$Q = \omega_{0i} \int d^3x\, T^{00} x^i - T^{0i} x^0$$

Again, since ω_{i0} is free, the quantity

$$Q_i = \int d^3x\, T^{00} x^i - T^{0i} x^0$$

is conserved. If we differentiate with respect to time, the LHS is zero and thus

$$\begin{aligned}
\frac{d}{dt}\left(\int d^3x\,T^{00}x^i\right) &= \frac{d}{dt}\left(\int d^3xx^0T^{0i}\right) \\
&= \int d^3T^{0i} + t\frac{d}{dt}\int d^3xT^{0i} \\
&= const
\end{aligned}$$

This is because we already know that $\int d^3xT^{0i}$ is already a conserved quantity, so is constant. The second term is zero since the derivative of a conserved quantity with respect to time is zero. (Also recall that $x^0 = t$.

7.6 Question 6

7.6.1 Gauge invariance

To show that $\mathcal{L}$ is invariant under gauge transforms, is suffices to show that $F_{\mu\nu}$ itself is invariant. The gauge transform we're doing is $A_\mu \to A_\mu + \partial_\mu \xi$, so $F_{\mu\nu}$ transforms as

$$\begin{aligned}
F_{\mu\nu} &\to \partial_\mu(A_\nu + \partial_\nu \xi) - \partial_\nu(A_\mu + \partial_\mu \xi) \\
&= \partial_\mu A_\nu + \partial_\mu \partial_\nu \xi - \partial_\nu A_\mu - \partial_\nu \partial_\mu \xi \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu
\end{aligned}$$

In the second line, we've applied symmetry of second partial derivatives.

7.6.2 Constructing the energy-momentum tensor

From our section earlier, we derived that our four conserved currents which satisfy $\partial_\mu T^{\mu\nu} = 0$, for a scalar field with multiple components ϕ_a, are given by

$$T^\mu{}_\nu = \mathcal{L}\delta^\mu{}_\nu - \partial_\nu \phi_a \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right)$$

In EM field theory, this scalar field is simply replced with our four-vector potential A_ρ! Also, we raise the indices so that the delta function in the first term becomes a Minkowski metric term $\eta^{\mu\nu}$. Our expression for the energy-momentum tensor is EM theory is therefore

$$T^{\mu\nu} = \mathcal{L}\eta^{\mu\nu} - \partial^\nu A_\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} \right)$$

Now, we have a bit of a tricky term to deal with here. How do we compute the term $\frac{\mathcal{L}}{\partial(\partial_\nu A_\rho)}$? We bear in mind that since we're in phase space, each of our degrees of freedom mean that $\partial_\mu A_\nu$ represents a separate variable for each index (so we have 16 variables here).Hence, we assert the following about derivatives with these terms

$$\frac{\partial(\partial_\mu A_\beta)}{\partial(\partial_\nu A_\alpha)} = \delta^\nu{}_\mu \delta^\alpha{}_\beta$$

To assist us as well, we apply product rule and antisymmetry. We'll now calculate this thing

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\rho)} &= -\frac{1}{4} \frac{\partial(F_{\mu\sigma}F^{\mu\sigma})}{\partial(\partial_\nu A_\rho)} \\
&= -\frac{1}{2} F^{\mu\sigma} \frac{\partial F_{\mu\sigma}}{\partial(\partial_\nu A_\rho)} \\
&= -F^{\mu\sigma} \frac{\partial(\partial_\mu A_\sigma)}{\partial(\partial_\nu A_\rho)} \\
&= -F^{\mu\sigma} \delta^\nu{}_\mu \delta^\rho{}_\sigma \\
&= -F^{\nu\rho}
\end{aligned}$$

From this, we get something for free. Since our Lagrangian has no dependence on A_μ on it's own, our Euler-Lagrange equations dictate that

$$\partial_\mu F^{\mu\nu} = 0$$

We'll use this fact later on. Now, it's just a matter of substituting this in to the expression for energy-momentum to get

$$T^{\mu\nu} = F^{\mu\rho}\partial^\nu A_\rho - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

Now, let's add on our extra component $-F^{\mu\rho}\partial_\rho A^\nu$. We check that this is conserved under differentiation:

$$\partial_\mu(F^{\mu\rho}\partial_\rho A^\nu) = (\partial_\mu F^{\mu\nu})\partial_\rho A^\nu + F^{\mu\rho}\partial_\mu\partial_\rho A^\nu = 0$$

the first term goes to zero as a result of the Euler-Lagrange equations. The second term goes to zero since we're contracting the antisymmetric $F^{\mu\rho}$ with the symmetric $\partial_\mu\partial_\rho$. Hence, we've shown that $\Omega^{\mu\nu}$ is a conserved current!

To show symmetry, observe that

$$\Omega^{\mu\nu} = -\frac{1}{4}\eta^{\mu\nu}F_{\sigma\rho}F^{\sigma\rho} + F^{\mu\rho}(\partial^\nu A_\rho - \partial_\rho A^\nu) = -\frac{1}{4}\eta^{\mu\nu}F_{\sigma\rho}F^{\sigma\rho} + F^{\mu\rho}F^\nu{}_\rho$$

But this is clearly symmetric in indices μ, ν! Furthermore, since this tensor is composed entirely from $F_{\mu\nu}$, it's gauge invariant.

Finally, taking the trace of this object, we have that the trace of $\eta^{\mu\nu} = 4$, so

$$\Omega^{\nu\nu} = -F_{\rho\sigma}F^{\rho\sigma} + F^{\nu\rho}F_{\nu\rho} = 0$$

Thus, this object is traceless! It's a much more natural form of the energy momentum tensor than what we had before.

7.7 Question 7

We want to derive the equations of motion for the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 C_\mu C^\mu$$

Instead of looking at the Euler-Lagrange equations, it's slightly easier to do this by varying the action directly instead. With the product rule, one can convince themselves pretty easily that the varied action is

$$\begin{aligned}
\delta S &= \int -\frac{1}{2}F_{\mu\nu}\delta F^{\mu\nu} + m^2 C_\mu \delta C^\mu \\
&= \int -\frac{1}{2}F_{\mu\nu}\delta\left(\partial^\mu C^\nu - \partial^\nu C^\mu\right) + m^2 C_\mu \delta C^\mu \\
&= \int -F_{\mu\nu}\delta\left(\partial^\mu C^\nu\right) + m^2 C_\mu \delta C^\mu \\
&= \int \delta C^\nu \left(\partial^\mu F_{\mu\nu} + m^2 C_\nu\right)
\end{aligned}$$

This implies that the equation of motion is

$$\partial^\mu F_{\mu\nu} + m^2 C_\nu = 0$$

When $m \neq 0$, if we differentiate both sides by ∂_ν, we have that

$$m^2 \partial_\nu C^\nu = \partial_\nu \partial_\mu F^{\nu\mu} = 0$$

Since, the second term as $F^{\mu\nu}$ which is antisymmetric in $\mu\nu$, and $\partial_\mu \partial \nu$ which is symmetric. Contracting this antisymmetric object with this symmetric one reduces the term to zero.

Our equation of motion dictates that, setting $\nu = 0$,

$$\partial_\mu F^{\mu 0} = m^2 C^0$$

The trick now is to expand this separately in terms of derivatives with respect to time and space. We have that

$$\begin{aligned}
\partial_\mu \partial^\mu C^0 - \partial_\mu \partial^0 C^\mu &= m^2 C^0 \\
\ddot{C}^0 - \partial_i \partial^i C^0 - \partial_\mu \dot{C}^\mu &= m^2 C^0 \\
\ddot{C}^0 - \partial_i \partial^i C^0 - \ddot{C}^0 + \partial_i \dot{C}^i &= m^2 C^0
\end{aligned}$$

Rearrangement gives our required result. Up to relabelling of indices, our Lagrangian is given by

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu C_\nu \partial^\nu C^\nu - \partial_\nu C_\mu \partial^\mu C^\nu)$$

To keep track of minus signs from contracting with up and down indices, we go slow and sum one index at a time. We find that

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}(\partial_0 C_\nu \partial^0 C^\nu - \partial_i C_\nu \partial^i C^\nu - \partial_\nu C_0 \partial^0 \partial^\nu + \partial_\nu C_i \partial^i C^\nu \\ &= -\frac{1}{2}(\dot{C}_0 \dot{C}^0 - \dot{C}_i \dot{C}^i - \partial_i C_0 \partial^i C^0 + \partial_i C_j \partial^i C^j - \dot{C}_0 \dot{C}^0 + \partial_i C_0 \dot{C}^i + \dot{C}_i \partial^i C^0 - \partial_j C_i \partial^i C^j)\end{aligned}$$

Note that here, the $\dot{C}_0 \dot{C}^0$ terms cancel. After some algebra, we get that

$$\mathcal{L} = \frac{1}{2}(\dot{C}^i \dot{C}_i + \partial_i C_0 \partial^i C^0 - (\partial_i C_j)^2 - 2\dot{C}^i \partial_i C_0 + \partial_j C_i \partial^i C^j) + \frac{1}{2} m^2 C_\mu C^\mu$$

Since there's no dependence on $\dot{C}^0$, our conjugate momentum to C^0 vanishes. Differentiating, we have that

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{C}^i} = \dot{C}_i - \partial_i C_0$$

Our Hamiltonian density is given by by

$$\mathcal{H} = \pi_i \dot{C}^i - \mathcal{L}$$

Since we have an expression for π_i, we substitute this expression in favour of $\dot{C}_i$ terms in both the first term in the expression as well as the Lagrangian. After a significant amount of algebra, we have that

$$\mathcal{H} = \frac{1}{2}(\pi_i \pi^i) + \frac{1}{2}(\partial_i C_j)^2 + \frac{1}{2}\partial_j C_i \partial^i C^j - \frac{1}{2} m^2 C^\mu C_\mu$$

This expression is positive definite!

7.8 Question 8

This question is a bit weird, because our discussion will include the effect of our measure d^4x that we have in our Lagrangian. Under the simultaneous scaling transformation, we'd like our action to transform, along with the transformation $x^\mu \to \lambda x^\mu$ as

$$\begin{aligned}\int d^4x\,\mathcal{L}(x) &\to \int d^4(x')^\mu\,\mathcal{L}'(x') \\ &= \lambda^4 \int d^4x\,\mathcal{L}'(x')\end{aligned}$$

Since we want our action to be invariant, the integrands here need be the same. Thus, our condition for invariance is

$$\mathcal{L}(x) = \lambda^4 \mathcal{L}'(x') \implies \mathcal{L}'(x') = \lambda^{-4}\mathcal{L}(x)$$

More generally, for $n+1$ spacetime dimensions, we have that our condition changes to

$$\mathcal{L}'(x') = \lambda^{-(n+1)}\mathcal{L}(x)$$

Since our field transforms like $\phi \to \lambda^{-D}\phi$, and our derivative transforms like

$$\partial'_\mu = \frac{\partial}{\partial(x')^\mu} = \frac{\partial}{\lambda\partial x^\mu} = \lambda^{-1}\partial_\mu$$

Our condition that the partial derivative terms are invariant imply that

$$\lambda^{-(n+1)}\partial_\mu\phi\partial^\mu\phi = \lambda^{-2D-2}\partial_\mu\phi\partial^\mu\phi$$

For invariance, we thus require that in $n+1$ space time dimensions, we have the condition

$$2D + 2 = n+1, \quad D = \frac{n+1}{2} - 1 \implies D = 1 \text{ for when } n = 3$$

Now, for our mass term to be invariant, using the relation that we have above for the Lagrangian, we have the requirement that upon our transformation, we have

$$\lambda^{-(n+1)}\frac{1}{2}m^2\phi^2 = \frac{1}{2}m^2\lambda^{-2D}\phi^2$$

which implies that $D = \frac{n+1}{2}$, which contradicts the above assertion for the derivative terms to be invariant. Hence, we have that $m = 0$ if we want a symmetry to appear. Similarly, comparing with our ϕ^p term we have the condition that $pD = n+1$, which upon substituting for our value of D, we yield the condition that

$$p = 2\left(\frac{n+1}{n-1}\right)$$

So in 4 spacetime dimensions, p = 4.

Constructing a Noether current

Working in 4 space time dimensions with $D = 1$, our field changes like $\phi \to \lambda^{-1}\phi = (1 - \log\lambda)\phi$ for λ close to 1. In the argument above, we asserted that $\mathcal{L} \to \lambda^{-4}\mathcal{L} = (1 - 4\log\lambda)\mathcal{L}$. Hence $\delta\phi = -\phi\log\lambda\delta$.

8 Example Sheet 2

In general, save ink by writing exponents with either $e^{ix\cdot(p-q)}$ together, or either $e^{ix\cdot(p+q)}$ together.

Save ink by not bothering to write the whole measure, and writing out just the operators we're concerned with. We can write out.

8.1 Question 1

Our stress energy tensor for $\mu = 1,2,3$ is given by $T^{0\mu} = \partial^\mu \phi \dot{\phi} = \eta^{\mu\nu}$.

We can write out time evolution for a_p explicitly by using the Heisenberg equation of motion

$$\frac{da_p}{dt} = i[H, a_p] = iE_p a_p$$

8.2 Question 4

Notes on this question.

We need to use the energy momentum tensor expansion. Tske the derivatives out.

Write out x^j by differentiating in the very beginning, then only differentiate by parts after that.

Use liberally the argument involving odd and even.

8.3 Question 7

Be careful with counting! Especially with the double 8 diagram. In this question, we wish to verify the bubble diagram expansion for ϕ^4 theory, for vacuum to vacuum scattering. Specifically, we're interested in the expansion to λ^2 order for $\langle 0| S |0\rangle$, where, due to Dyson's formula, we have that

$$S = 1 - \frac{i\lambda}{4!}\int_{-\infty}^{\infty} dt H_I(x) + \frac{1}{2}\frac{\lambda^2}{(4!)^2}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \mathcal{T}\left\{H_I(t_1, \vec{x}) H_I(t_2, \vec{x})\right\}$$

Writing this in terms of our Hamiltonian density instead, we have that this simplifies down to

$$S = 1 - \frac{i\lambda}{4!}\int d^4x\, \mathcal{H}_I(x) + \frac{1}{2}\frac{\lambda^2}{(4!)^2}\int d^4x_1 d^4x_2 \mathcal{T}\left\{\mathcal{H}_I(x_1) H_I(x_2)\right\}$$

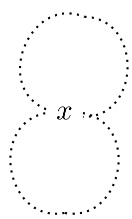
where we have our interaction picture Hamiltonian given by $\mathcal{H}_I(x) = \phi_I^4(x)$. Now,let's look at the contribution to the first order in λ. This contribution is given by

$$\frac{-i\lambda}{4!}\int d^4x\, \langle 0| \mathcal{T}\left\{\phi(x)\phi(x)\phi(x)\phi(x)\right\} |0\rangle$$

Now, due to Wick's theorem, the integrand can be expanded out in terms of all possible contractions of the two fields. There are 3 ways to do this. This is because we could've contracted the first field with 3 others, which leaves the last two fields to contract on their own. This means that our total contribution from first order from this field is

$$-\frac{i\lambda}{8}\int d^4x\, D(x-x)D(x-x)$$

This is represented by a figure 8 diagram. Now, we aim to show that this matches up with the diagram shown in the exponential. To first order in λ, our exponential is given by



We know that from this diagram that, by assigning propagators to each line, that this is proportional to $\int d^4x\, D(x-x)D(x-x)$. But, what about the symmetry factors? This diagram has a symmetry factor of 8 since we assign a factor of two for being able to switch the ends of the bottom and top loops. In addition, we assign another factor of 2 since we can rotate the diagram 180 degrees.

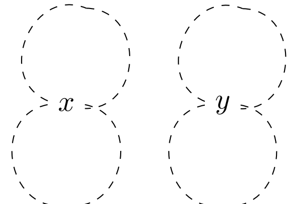
Now, looking to second order in λ^2, we'd like to find the amplitude given by the expression

$$\frac{\lambda^2}{2(4!)^2}\int d^4x\, d^y \mathcal{T}\left\{\phi_x\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y\phi_y\right\}$$

Let's examine the possible contractions that we get from this. We could have that the ϕ_x and ϕ_y terms are contracted internally amongst themselves. There are 3 ways to contract the ϕ_x amongst themselves and 3 ways to contract the ϕ_y amongst themselves, so we have 9 ways in total. This means that our contribution is given by

$$\frac{1}{2}\frac{\lambda^2}{8^2}\int d^4xd^4yD(x-x)D(x-x)D(y-y)D(y-y)$$

The factor of $\frac{1}{2}$ comes from Dyson's formula. As a diagram, this is represented by the diagram

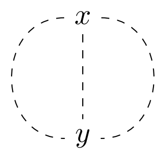


Our symmetry factor for this diagram is 2×8^2, since, as we discussed before, we have a symmetry of 8 for each diagram. In addition, we could've swapped the diagrams as well. This agrees with the above.

The next thing to do is that we could have contracted each of the ϕ_x terms with ϕ_y. Since we have four choices to contract the first ϕ_x with, then 3, then 2, we have that our contribution to this term is

$$\frac{\lambda^2 4!}{2(4!)^2}\int d^4x\, d^4y\, D(x-y)D(x-y)D(x-y)D(x-y) = \\ \frac{\lambda^2}{2(4!)}\int d^4x\, d^4y\, D(x-y)D(x-y)D(x-y)D(x-y)$$

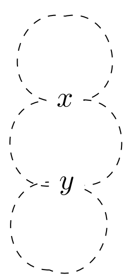
This corresponds to the diagram



Finally, our last possible type of contraction is what happens when we contract just two of the ϕ_x fields with two of the ϕ_y fields, and loop the rest. There are $\binom{4}{2} = 6$ ways to choose which to ϕ_x fields we contract with ϕ_y, and once we chose the ϕ_x fields, we have $12 = 4 \times 3$ options to which we can contract them. This means that the contribution gained from this is

$$-\frac{\lambda^2 6 \times 4 \times 3}{2(4!)^2}\int d^4x\, d^4y D(x-x)D(y-x)D(y-x)D(y-y) = \\ -\frac{\lambda^2}{16}\int d^4x\, d^4y D(x-x)D(y-x)D(y-x)D(y-y)$$

This is associated with the diagram



We can switch around the two loops as well as the two propagators in the centre. Hence, we've showed that to second order in λ, the combinatoric and symmetry factors work out for each diagram when we exponentiate the bubble diagrams.

8.4 Question 8

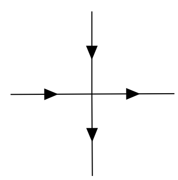
For the Yukawa term, we need only attach ig to each vertex. We need to take into account combinatoric factors for each vertex if the same time of propagator is going into them.

To find the mass dimensions of g, h, k, l, we need to first find the mass dimensions of the fields ϕ and ψ. Our scalar action needs to be dimensionless, so

$$\int d^4x\, \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$$

has mass dimension 0. This implies that $[\phi] = 1$. Similarly, we find that $[\psi] = 1$. This means that the mass dimensions of g, k are $[g] = [k] = 1$. We have that $[h] = 0$, and $[l] = -1$. This theory contains both relevant, marginal and irrelevant operators. Thus, the theory is non-renormalisable since the l dimension is irrelevant.

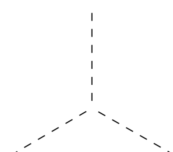
We have four interaction terms, which means that we get 4 distinct types of diagrams, each with their own set of Feynman rules. Let's tackle the Yukawa interaction first. We know the momentum rules for this. Now, for the interaction $h|\psi|^4 = h\psi\psi^*\psi\psi^*$, we have that our interaction vertex is given by



Now, with this kind of diagram, our propagator comes from contracting just ψ, ψ^* together, which gives us a propagator in momentum space which looks like

$$\frac{i}{p^2 - \mu^2 + i\epsilon}$$

For the term $k\phi^3$, we have that our vertices look like



We need to also add in combinatoric factors, so we attach a 6 to this. Each of these objects come with a propagator which comes from contracting two scalars together

$$\frac{i}{p^2 - m^2 + i\epsilon}$$

Finally, the non-trivial one is the propagator that comes from interactions with $\phi\partial_\mu\psi\partial^\mu\psi^*$. In this case, we have a different kind of propagator! Non-zero contributions come from contractions of the form

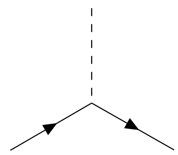
$$\phi\phi\psi^*\partial_\mu\psi\partial^\mu\psi^*\psi$$

Differentiating the contraction of ψ and ψ^* pulls out a factor of $i\pi^\mu$. Thus, since we'll always have an even number of propagators involving the anti-nucleons, for each pair we attach a factor of

$$\frac{-p \cdot q}{p^2 - m^2 + i\epsilon}$$

8.5 Question 9

We want to find the decay width for the scattering process $\phi \to \psi\overline{\psi}$. For studying the decay width up to order g^2, we need only consider the trivial Feynman diagram with one vertex (since $\Gamma \sim |\mathcal{M}|^2$.



Now, since this already imposes momentum conservation, our amplitude $\mathcal{M}$ is just ig. Thus, our decay width is given by the formula

$$\Gamma = \frac{1}{2m} g^2 \frac{1}{(2\pi)^2} \int \frac{d^3q_1 d^3q_2}{2E_{q_1} 2E_{q_2}} \delta^4(p_1 - q_1 - q_2)$$

The trick here is to rewrite the invariant measure $\frac{d^3q_2}{(2E_{q_2}}$ as $\delta^4\left((q_2^2 - \mu^2\right)$ (notice we're using μ^2 since this comes from the nucleon. Substituting this in, then integrating over q_2, we get that our decay width is

$$\Gamma = \frac{g^2}{(2\pi)^2 2m} \int \frac{d^3q_1}{2E_{q_1}} \delta^4\left((p_1 - q_1)^2 - \mu^2\right)$$

Now, we'd like to find the set of q_1 such that $(p_1 - q_1)^2 = \mu^2$. Our relativistic dispersion relation gives us that $p_1^2 = m^2$, and $q_1^2 = \mu^2$. Also, in the convention of calculating decay widths, we work in the rest frame of the incoming particle (in this case, the meson). This means that we take $p_1 = (m, 0)$.

$$\begin{aligned} p_1^2 + q_1^2 - 2p_1 \cdot q_1 &= \mu^2 \\ m^2 + \mu^2 - 2mE_{q_1} &= \mu^2 \\ m^2 - 2m\sqrt{\mu^2 + \vec{q}_1^{\,2}} &= 0 \\ \|\vec{q}_1\| &= \pm\sqrt{\frac{m^2}{4} - \mu^2} \end{aligned}$$

Our decay width is the integral

$$\Gamma = \frac{g^2}{16\pi^2 m} \int \frac{d^3q_1}{E_{q_1}} \delta^4\left(m^2 - 2m\sqrt{\mu^2 + \|\vec{q}_1\|^2}\right)$$

Selecting the positive root in the delta function, using our standard identity for a delta function of a function, this gives that the integral is

$$\begin{aligned} \Gamma &= \frac{g^2}{16\pi^2 m} \int \frac{d^3q_1}{E_{q_1}} \frac{\delta\left(|\vec{q}_1| - |\vec{q}_1^{\,*}|\right)}{\frac{2m|\vec{q}_1^{\,*}|}{\sqrt{\mu^2 + |\vec{q}_1^{\,*}|^2}}} \\ &= \frac{g^2}{16\pi^2 m} \int d|\vec{q}_1| \frac{4\pi|\vec{q}_1|^2 \delta(|\vec{q}_1| - |\vec{q}_1^{\,*}|)}{2mE_{q_1} \frac{|\vec{q}_1^{\,*}|}{\sqrt{\mu^2 + |\vec{q}_1^{\,*}|^2}}} \end{aligned}$$

In the last two lines we converted this integral into spherical coordinates. Integrating out the delta function, this is just

$$
\begin{aligned}
\Gamma &= \frac{g^2(2\pi)}{16\pi^2 m}\frac{1}{m}\frac{|\vec{q}_1^*||E_{q_1^*}}{|E_{q_1^*}|} \\
&= \frac{g^2}{8\pi m^2}\sqrt{\frac{m^2}{4}-\mu^2} \\
&= \frac{g^2}{16\pi m^2}\left(1-\frac{4\mu^2}{m^2}\right)^{\frac{1}{2}} \\
&= \frac{g^2}{16\pi m}\sqrt{1-(4\mu^2/m^2)}
\end{aligned}
$$

This has dimensions of mass since g has mass dimension 1, thus decay make sense since this is a rate.

8.6 Question 1

In this question, we're asked to derive the momentum operator $P^\mu = \int d^3x\, T^{0\mu}$. The first thing we notice is that this object is derived from the momentum part of the energy-momentum tensor, and is simply integrating this over all of space. Hence, we expect it to be an operator on fields. The first thing we'll do is derive the form of the energy momentum tensor.

The energy-momentum tensor is a conserved object which arises from translational symmetry. In Minkowski spacetime, translational symmetries in time correspond to the conservation of energy, and translational symmetries in space correspond to conserved momemtum. Hence, we consider the translation

$$x^\mu \to x^\mu + \epsilon^\mu$$

But, since this is a passive transformation of our reference frame, we have that our scalar field $\psi(x)$ transforms as

$$\psi(x) \to \psi(x-\epsilon) = \psi(x) - \epsilon^\mu \partial_\mu \psi(x)$$

So our change in the scalar field $\delta(x) = \epsilon^\mu \partial_\mu \psi(x)$. Similarly, our Lagrangian L in general transforms in the same way, with $\delta L = \epsilon^\mu \partial_\mu L$. This is indeed a symmetry of the Lagrangian since $\delta L = \partial_\mu(F^\mu)$ where $F^\mu = \epsilon^\mu L$ Thus, by Noether's theorem we can write out conserved current as

$$j^\mu = \epsilon^\mu L - \epsilon^\nu \partial_\nu \psi \left(\frac{\partial L}{\partial(\partial_\mu \psi)} \right)$$

However, now we can 'factorise' out the ϵ^μ since it's arbitrary, and write our conserved current as

$$T^\mu{}_\nu = \delta^\mu{}_\nu L - \partial_\nu \psi \left(\frac{\partial L}{\partial(\partial_\mu \psi)} \right)$$

Now, substituting our expression for the Lagrangian (which is associated to the Klein-Gordon equation)

$$L = \frac{1}{2}\partial_\alpha \psi \partial^\alpha \psi - \frac{1}{2} m^2 \psi^2$$

Our expression for our energy momentum tensor becomes

$$T^{\mu\nu} = \eta^{\mu\nu} \left(\frac{1}{2}\partial_\alpha \psi \partial^\alpha \psi - \frac{1}{2} m^2 \psi^2 \right) - \partial^\nu \psi \partial^\mu \psi$$

Let's examine this term by term for $\int d^3x\, T^{0\mu}$. For starters, let's take the $\partial^0\psi\partial^\mu\psi$ term. First, we Fourier expand this object to get

$$\psi(x) = \int d^3p \frac{1}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ip\cdot x} + a^\dagger_{\mathbf{p}} e^{ip\cdot x} \right)$$

If we differentiate this thing with ∂^0 and ∂^μ, we have that

$$\partial^0 \psi = \int d^3p\, \frac{p^0}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \left(-a_{\mathbf{p}} e^{-ip\cdot x} + a^\dagger_{\mathbf{p}} e^{ip\cdot x} \right)$$
$$\partial^\mu \psi = \int d^3p\, \frac{p^\mu}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \left(-a_{\mathbf{p}} e^{-ip\cdot x} + a^\dagger_{\mathbf{p}} e^{ip\cdot x} \right)$$

Now, amalgamating this all together, we have that the integral of this is

$$\int d^3xT^{0\mu} = \int d^3x\, d^3p\, d^3q\, \frac{p^0 q^\mu}{(2\pi)^6\sqrt{2E_{\mathbf{p}}E_{\mathbf{q}}}} \left(-a_{\mathbf{p}}a_{\mathbf{q}}e^{-i(p+q)\cdot x} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}}e^{i(p-q)\cdot x} + a_{\mathbf{p}}a_{\mathbf{q}}^\dagger e^{i(p-q)\cdot x} - a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p+q)\cdot x} \right)$$

But here, we can use the identity that

$$\int d^3x\, e^{i\mathbf{x}\cdot(\mathbf{c}} = (2\pi)^3\delta(\mathbf{c})$$

In the spatial part to isolate out a delta function. In addition, we also use the fact that $p^0 = E_{\mathbf{p}}$ to cancel out with $E_{\mathbf{p}}$ in the denominator. Hence, we end up with the expression that the above is

$$\int \frac{d^3p}{2(2\pi)^3} p^\mu (a_{\mathbf{p}}a_{-\mathbf{p}} + a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}}a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger)$$

Now, the first and last terms disappear because under a change of variables $\mathbf{p} \to -\mathbf{p}$, p^μ is an odd function under the spatial integral, but $a_{\mathbf{p}}a_{-\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger$ are even functions, so these terms disappear. Now, if we commute the third expression with our standard relation and remove our delta function since it's an infinite term (like we do with our Hamiltonian), we get the final expression.

One can show that the rest of the terms in the energy momentum tensor disappear under integrating over $\int d^3x$.

Now, for the next part of the question, we'd like to show that in the Heisenberg picture

$$[P^\mu, \psi(x)] = -i\partial^\mu \psi(x)$$

This is achieved fairly easily by pulling the commutators to inside the integral and then using our standard commutation relations.

$$\begin{aligned} [P^\mu, \psi(x)] &= [\int \frac{d^3p}{(2\pi)^3}\, p^\mu a_{\mathbf{p}}^\dagger a_{\mathbf{p}}, \psi] \\ &= \int \frac{d^3p}{(2\pi)^3}\, [p^\mu a_{\mathbf{p}}^\dagger a_{\mathbf{p}}, \psi] \\ &= \int \frac{d^3p}{(2\pi)^3}\, p^\mu [a_{\mathbf{p}}^\dagger a_{\mathbf{p}}, \psi] \end{aligned}$$

Where, going into the third line, since p^μ is not an operator, we've just pulled it out of the commutator. By linearity of the integral, we can also pull the commutator inside the integral, as we did going into the second line. Fourier expanding out, the above expression reads

$$\begin{aligned} \int \frac{d^3p d^3q}{(2\pi)^6} p^\mu [a_{\mathbf{p}}^\dagger a_{\mathbf{p}}, a_{\mathbf{q}}e^{-iq\cdot x} + a_{\mathbf{q}}^\dagger e^{iq\cdot x}] &= \int \frac{d^3p d^3q}{(2\pi)^6}\, p^\mu \left(e^{-iq\cdot x}[a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}]a_{\mathbf{p}} + e^{iq\cdot x} a_{\mathbf{p}}^\dagger [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] \right) \\ &= \int \frac{d^3p}{(2\pi)^3} \left(-p^\mu a_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger p^\mu e^{ip\cdot x} \right) \\ &= -i\partial^\mu \psi(x) \end{aligned}$$

8.7 Question 2

In this question we're trying to derive the Klein-Gordon equation, but via the Heisenberg picture for operators. This is when operators evolve in time instead of states. In the Heisenberg picture, our equation for time evolution for an operator $\mathcal{O}_H$ is

$$\frac{d\mathcal{O}_H}{dt} = i[H, \mathcal{O}_H]$$

First, let's look at computing

$$\dot{\phi}(x) = i[H, \phi(x)]$$

The Hamiltonian associated with our free field Lagrangian is

$$H = \int d^3y \, \frac{1}{2]}\pi(y)^2 + \frac{1}{2}(\nabla\phi(y))^2 + \frac{1}{2}m^2\phi(y)^2$$

So, to compute the commutator required, we need to remind ourselves of the commutation relations between the our scalar field $\phi(x)$ and our conjugate momentum field $\pi(x)$. This our entirely analogous to those we know from quantum mechanics:

$$[\phi(x), \phi(y)] = [\pi(x), \pi(y)] = 0$$
$$[\phi(x), \pi(y)] = \delta^3(\mathbf{x} - \mathbf{y})$$

I just wanted to note that here, when we write $\phi(x)$, this is condensed notation for $\phi(\mathbf{x}, t)$, where $\mathbf{x}$. Now, when we bring the commutator inside the integral to calculate $[H, \phi(x)]$, all functions which are functions of $\phi(x)$ disappear under the commutator. The only expression we're left with is

$$\dot{\phi}(x) = \frac{i}{2}\int d^3y \, [\pi(y)^2, \phi(x)]$$

However, we can apply the 'product rule' for operator commutators which is

$$[AB, C] = A[B, C] + [A, C]B$$

so that the above expression reads as follows: