# Part III Symmetries, Fields and Particles

Notes by Afiq Hatta, based on lectures by Nick Dorey and notes by Nicholas Manton

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## Contents

# 1 What is a symmetry?

Let's ask ourselves what is a symmetry? A symmetry is an operator we can do on variables (either dynamical variables or the coordinate frame or otherwise), which leaves physical laws invariant. This is completely analogous to the idea of spatial symmetries, where operators in space leave structures invariant.

Let's analyse one of our most simple symmetries useful for physics - the rotation. We can motiavate what rotations look like based on how they act on vectors. A rotation would look like the map:

$$\mathbf{v} \to \mathbf{v'} = M\mathbf{v}$$

But, we have extra restrictions on this. Firstly a rotation should preserve our length of vectors and hence our squared legnth $\mathbf{v}^T\mathbf{v}$. This means that

$$\mathbf{v'}^T\mathbf{v'} = \mathbf{v}^T M^T M\mathbf{v} = \mathbf{v}^T\mathbf{v}$$

Which implies that $M^TM = I$. In addition, the orientation of vectors should be preserved, which means that we can't 'flip' vectors. This imposes the condition that $\det M = 1$. There's more reason to be had that $M^TM = 1$, owing to the fact that since physical laws are preserved, our Lagrangian should be preserved as well. This means that since our Lagrangian contains a kinetic term

$$\mathcal{L} = \frac{1}{2}m\mathbf{v}^T\mathbf{v} + V(|\mathbf{x}|^2)$$

we also have imposed the condition that $M^TM = I$, since this is the only way to make Lagrangians invariant. As a sanity check, physical laws of motion should transform under these rotations as well. This is trivial to check in the case of linear transforms (which includes rotations)

$$\mathbf{F} = m\ddot{\mathbf{x}} \implies \mathbf{F}' = m\ddot{x}'$$

This result is achieved by just multiplying both sides by $M$.

## 1.1 Groups to represent symmetries

Intuitively, symmetries should follow a group structure if we represent them as maps. For example, if we compose two symmetries together we should expect to obtain another symmetry. Similarly, we also expect symmetries to have inverses. This is where the notion of groups becomes useful for us. A group is a set $G$ with an operation $\_ \cdot \_ : G \times G \to G$ which obeys the following axioms

- **Existence of a group identity** $\exists e \in G$ such that for all $g \in G$, multiplication satisfies

$$e \cdot g = g \cdot e = g$$

- **Existence of unique group inverses** for all $g \in G$we have a unique group inverse denoted $g^{-1}$ such that

$$g^-1 \cdot g = g \cdot g^{-1} = e$$

- **Associativity** this means that we don't care about the order in which we multiply things things in the group. So, for all $g_1, g_2, g_3 \in G$, we have that

$$g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$$

If the ordering of multiplication doesn't matter, in other words if, for all elements $g, h \in G$ we have

$$g \cdot h = h \cdot g$$

then our group is called **Abelian**. Otherwise, it's called a non-abelian group. Our group of rotations is non-abelian, because the order in which you compose operations matters. For example, rotating something about the x-axis 90 degrees clockwise and then about the z-axis 90 degrees clockwise is different to rotating first about the z-axis, then rotating about the x-axis.

In the case of rotations in three dimensions, we have a group which we'll call the special orthogonal group in three dimensions, or $S0(3)$.

### 1.2 Symmetries correspond to conserved quantities

Physics, the existence of symmetries lead to conserved quantities. In the case of rotations, in classical mechanics, rotational invariance leads to conservation of angular momentum by Noether's theorem. This is a conserved quantity derived from a symmetry acting on $\mathbb{R}^3$. This manifests itself in the form of an angular momentum vector $\mathbf{L} = (L_1, L_2, L_3)$.

In quantum mechanics, instead of phase space, we can work on Hilbert space $\mathcal{H}$, and do something similar. In Hilbert space, we work with state vectors $|\phi\rangle \in \mathcal{H}$ and observable quantities are Hermitian operators $\hat{\mathcal{O}}$. Looking at generators of the rotation group, our analog of the angular momentum vector are angular momentum generators $\hat{L}_i$ which we obey a 'spin' algebra

$$[\hat{L}_i, \hat{L}_j] = i\epsilon_{ijk}\mathcal{L}_k$$

which, we'll see later, that our commutator of operators in quantum mechanics is precisely our Lie algebra in the context of $\mathcal{L}SO(3)$.

# 2 Manifolds and Tangent Spaces

## 2.1 Manifolds

A manifold $\mathcal{M}$ is a shape which looks like a 'flat plane' from the perspective of a creature standing on it. This means that at every neighbourhood on this manifold there exists a homeomorphism (a continuous function whose inverse is also continuous) from the neighbourhood to a subset of $\mathbb{R}^n$ for some $n$. This $n$ is the dimension of the manifold. An easy example to wrap one's head around is a sphere, which is a two dimensional manifold.

More specifically, at every point $p \in \mathcal{M}$ there exists an open set $\mathcal{P}$ with a homeomorphism $\phi_p : \mathcal{P} \to U \subset \mathbb{R}^n$. These maps are called coordinate charts, and our collection of open sets $\mathcal{P}_i$ should cover the manifold.

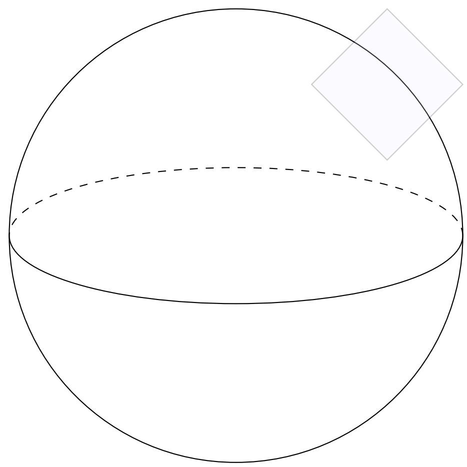


Figure 1: A sphere locally looks like a flat plane, at any given point on the surface

Since these homeomorphisms are from the manifold to $\mathbb{R}^n$, we can identify coordinates on an open set $\mathcal{P} \subset \mathcal{M}$, denoted by $\{\theta^i\}$, where $i$ ranges from $i = 1, 2, \ldots, n$, where $n$ is the dimension of the manifold.

## 2.2 Functions on manifolds

On this manifold, we can define objects like curves and scalar functions. A function on a manifold is like a 'heatmap',

$$f : \mathcal{M} \to \mathbb{R}.$$

This map $f$ can be smooth or not, of course. How do we define concepts like differentiability and continuity on functions on a manifold? Well, since the space is homeomorphic to $\mathbb{R}^n$, we can say a function $f : \mathcal{M} \to \mathbb{R}$ is differentiable provided that that its corresponding function $f' : \mathcal{U} \to \mathbb{R}^n$ is differentiable, where $U$ is the image of the homeomorphism from $\mathcal{M} \to \mathbb{R}^n$.

Similarly, we can define curves on this manifold. Think of a path an ant takes on this curved space. A curve $\lambda$ is precisely this kind of path. This is a smooth map from an interval $I$, which without loss of generality we set $I = (0,1)$, to the manifold.

$$\lambda : I \to \mathcal{M}.$$

## 2.3 Tangent vectors

If we compose a curve with a function on a manifold, we get a function from $I \to \mathbb{R}$. This of this composition as taking a 'slice' of the function along the manifold. What interests us is that we can then take the derivative at $x = 0$ along this curve. Think of

$$\frac{df(\lambda(t))}{dt} \,|_{t=0},$$

which makes sense since $f \circ \lambda : I \to \mathbb{R}$. So, we're not doing anything suspicious here. This action of taking a function on a manifold and then differentiating it is interesting. With every curve, we can associate a differential operator which corresponds to differentiating a function on that curve.

## 2.4 Embedding manifolds in $\mathbb{R}^n$

Let's think about the manifold $S^2$. We've already defined an appropriate coordinate chart in this object, but a natural way to think about this surface is also to parametrise this as a surface in $\mathbb{R}^3$, as a sphere with unit radius 1. So, we're taking a manifold $\mathcal{M}$ and then finding a way to express this in some higher dimensional $\mathbb{R}^k$ (this is called an embedding). We have that clearly

$$S^2 = \{\mathbf{x} \in \mathbb{R}^3 \mid |\mathbf{x}|^2 = 1\}$$

Note that here, the way we parametrise $S^2$ is by specifying an element in $\mathbb{R}^n$ and then imposing a condition on the surface. More generally, we can embed a general manifold in $\mathbb{R}^n$ by writing it as

$$\mathcal{M} = \{\mathbf{x} \in \mathbb{R}^n \mid F_\alpha(\mathbf{x}) = 0\}$$

where we have $l$ constraints given by $\alpha = 1, 2 \ldots, l$, where $F_\alpha : \mathbb{R}^n \to \mathbb{R}$, and is a smooth function. In the case of $S^2$, we have that $\alpha = 1$, and

$$F_1(\mathbf{x}) = |\mathbf{x}|^2 - 1$$

### 2.4.1 The Embedding Theorem

A natural question to ask ourselves is how we can relate our dimension of our manifold to the dimension of our embedded space and our number of constraints. This is where the embedding theorem comes in. If we have $\alpha = 1, 2 \ldots, l$, then the dimension of our manifold $\mathcal{M}$ is of dimension $m - l$, where $m$ is the dimension of the space we're embedding in, if and only if

$$J^\alpha{}_\beta = \frac{\partial F_\alpha}{\partial x^\beta}$$

is of full rank $l$ everywhere on our manifold. In the case of our 2-sphere, we had that in components

$$F(x,y,z) = x^2 + y^2 + z^2 - 1 \implies J = 2(x,y,z)$$

But in this case, on the manifold, $(x,y,z) \neq (0,0,0)$, so is of rank $1 = l$ everywhere. Hence the dimension of $S^2$ is $3 - 1 = 2$.

## 2.5 Classifying manifolds

We can classify manifolds by some of their topological properties. We'll outline them here.

- Connectedness is the property that we can construct a path from anywhere in the manifold to any other point in the manifold. For example, a sphere is connected, but a manifold whose set of points is two spheres is not.
- Simply connectedness is subtly different from connectedness. It comes from the concept of a homotopy group. For a space to be simply connected, it means that any loop on the surface can be continuously deformed to a point. In the language of algebraic topology, we require that

$$\pi_1(\mathcal{M}) \simeq \{e\}$$

  Somthing that's not simply connected is a solid torus; since we can 'tie a string' around the centre donut.
- Compactness is when we can cover a space with a finite amount of subsets. This condition is equivalent to the condition, that if we're embedding in $\mathbb{R}^n$, then the subset is closed and bounded. For example, a quadratic curve on $\mathbb{R}$ is not a compact manifold since it's not bounded.

# 3 Lie Groups

## 3.1 What is a Lie Group?

A Lie group is a group of continuous transformations, which depends smoothly on $n$ given parameters, say. Since it takes $n$ parameters to define a transformation in this group, we could also interpret this as an $n$ dimensional manifold. Thus, we define a Lie group as a group which is also a smooth manifold. Thus, multiplication between group elements:

$$G \times G \to G$$

must respect the manifold's structure and must be a smooth map. In addition, inverses of elements must also be smooth maps. Since an n dimensional Lie group is homeomorphic around $e$ to an open subset of $R^n$, we can re-parametrize the elements of the group $G$ to depend on $n$ different coordinates, say $\{\theta_i \mid i = 1, \ldots, n\}$.

Let's have an example of maps which respect group structure. On a Lie group (or equivalently, a manifold), we can construct a 'left action' induced by an element $g \in G$, which in the context of a Lie group is simply multiplicaiton by that element on the left

$$L_g : G \to G, \quad h \in G \mapsto gh \in G$$

However, in the context of viewing a Lie group as a manifold, this map represents 'translating' a member $g$ on a manifold to it's new place. It turns out, that due to this translation map we've defined above, we can actually characterise most of our Lie group based on what happens infinitesimally at $e \in G$. We'll go over this later, but infinitesimal behavour at defines a tangent space which we call a **Lie algebra**. This notion is useful because Lie groups and manifolds are complicated, but defining the tangent space is handy because it's a **vector space**. This tangent space, you guessed it, is exactly what we've defined abpve.

#### Smoothness of group multiplication on manifolds

Our condition that group multiplication and inversion should be smooth maps on the manifold is a statement about smoothness of coordinate maps. Since we can define a coordinate chart on an open set in the manifold, we can denote group elements $g \in G$ by their coordinates. So, we can write

$$g = g(\theta) \in G$$

where $\theta = (\theta_1, \ldots, \theta_n)$, and our identity $e \in G$ lies at $g(0)$. Group multiplication in $G$ is a map

$$G \times G \to G, \quad g \times g' \mapsto g''$$

but since we can assign coordinates to elements in a Lie group, we can write this multiplication as

$$g(\theta) \times g(\theta') \to g(\theta'')$$

where we assume that $\theta''$ lies in the image of the same coordinate path $\mathcal{P}$ that we used to assign coordinates to $\theta, \theta'$. This implies that we've induced a map between coordinates on the manifold, and we express the function $\theta''$ as a function of the other coordinates.

$$\theta'' = \theta''(\theta, \theta')$$

Thus, one of the conditions for a group to be a Lie group is that this induced map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ must be smooth (differentiable). More specifically, each of the components of this function $(\theta'')^i = (\theta'')^i(\theta, \theta')$ must be smooth.

**Smoothness of inversion of Lie Groups**

In a group, taking an inverse is the operation

$$G \to G, \quad g \mapsto g^{-1} \text{ where we have } gg^{-1} = e$$

Just as in the case of group multiplication, we can write in terms of coordinates on our manifold where we have

$$g(\theta) \to g(\hat{\theta}) = (g(\theta))^{-1}$$

This induces a map between coorinates

$$\theta \mapsto \hat{\theta}$$

Our condition that this group is a Lie group is that this operation between coordinates is also differentiable.

As an example, the additive group of vectors on $\mathbb{R}^d$, denoted $(\mathbb{R}^d, +)$, is a lie group. Group operations on this space between vectors are

- Addition: $\mathbf{x}_1 + \mathbf{x}_2 = \mathbf{x}_3$
- Inversion: $\mathbf{x} \mapsto -\mathbf{x}$

The corresponding coordinates are just their Euclidean coordinates $\mathbf{x} = (x_1, \ldots, x_n)$, and addition is represented by the usual way to add coordinates in $\mathbb{R}^n$, which is obviously continuous and differentiable. In addition, inversion is just attaching a minus sign to map $(x_1, \ldots, x_n) \mapsto -(x_1, \ldots, x_n)$, which is also a smooth map.

**A remark on classifying Lie algebras**

It turns out, we can classify all finite semi-simple Lie algebras over $\mathbb{C}$. into four infinite families denoted $A_n, B_n, C_n, D_n$, where $n \in \mathbb{N}$ with five exceptional cases: Lie algebras denoted $E_6, E_7, E_8, G_2$ and $G_4$. This is called the Cartan classification for Lie algebras and we'll cover it further later.

## 3.2 Matrix groups

Given some field $F$, which could be either $\mathbb{C}$ or $\mathbb{R}$, we define the set of $n \times n$ matrices over it as $Mat_n(F)$. Let's choose matrix multiplication as our group operation. Clearly, there exists a mutliplicative identity, the identity map, and we also have by the definition of matrix multiplication that its associative. However, this set doesn't form a group in it's own right since not all matrices are invertible. However, we can pick out the set of invertible matrices

$$GL_n(F) = \{A \in Mat_n(F) \mid detA \neq 0\}$$

which we refer to as the general linear group. In fact, $GL(n, F)$ is a lie group in its own right, because matrix multiplication and inversion is smooth. We will also prove shortly that $GL(n, F)$ is an open set of $Mat_n(F)$, so our coordinate chart into $\mathbb{R}^{n^2}$ is just the identity map, and hence we have an $n^2$ dimensional Lie group.

The special linear group is $SL_n(F)$ where our matrices have determinant 1, so

$$SL(n, F) = \{M \in GL(n, F) \mid detM = 1\}$$

This is a group by the multiplicative property of determinants where

$$det(M_1 M_2) = det(M_1)det(M_2)$$

Hence, multiplying two matrices with determinant 1 also gives a matrix with determinant 1. We now ask ourselves what is the dimension of this Lie group? Well, we've embedded this object in as

$$SL(n, F) = \{M \in Mat_n(\mathbb{R}) \simeq \mathbb{R}^{n^2} \mid detM = 1\}$$

Thus, if we can show that the function $F(M) = detM - 1$ has a non-negative Jacobian at each point on the manifold, then we have an $n^2 - 1$ dimensional Lie group. We reindex the variables to differentiate with as $M_{ij}$. This looks weird but we're merely reindexing each component of the matrix in the completely natural way. Our Jacobian element (due to a result which we will not prove), is

$$J_{ij} = \frac{\partial F}{\partial Mij} = \pm det(\hat{M}^{ij})$$

$\hat{M}^{ij}$ represents the matrix of minors, which is the $(n-1) \times (n-1)$ matrix that we get when we remove the $i$th row and $j$th column. It's a result that $det(\hat{M}^{ij}) = 0 \iff det(M) = 0, \quad \forall i, j = 1, 2, \ldots n$, hence this $J$ is non zero and hence is of rank one. Thus, the dimension of our manifold is $n^2 - 1$. Furthermore, matrix multiplication and matrix inversion are smooth maps, hence this is a lie group.

We can extend this argument to complex matrices where our condition that our determinant be 1 gives two constraint functions, since we're fixing a condition on both real and imaginary components. We have

$$\begin{aligned}
dim(GL(n, \mathbb{R}) &= n^2 \\
dim(SL(n, \mathbb{R}) &= n^2 - 1 \\
dim(GL(n, \mathbb{C}) &= 2n^2 \\
dim(SL(n, \mathbb{C}) &= 2n^2 - 2
\end{aligned}$$

### 3.2.1 Matrix groups in $GL(n, \mathbb{R})$

Our first matrix group that we'd like to examine is the matrix group whose action on vectors leaves their lengths unchanged.

$$\mathbf{v} \mapsto \mathbf{v'} = M\mathbf{v}$$

The condition that these maps leave lengths unchanged means that

$$|\mathbf{v'}|^2 = \mathbf{v}^T M^T M \mathbf{v} = \mathbf{v}^T \mathbf{v}$$

This implies that our matrix $M$ has to satisfy the condition that $M^M = I$. This motivates the definition of the orthogonal group

$$O(n) = \{M \in Mat_n(F) \mid M^T M = I\}$$

Taking determinants,

$$M^T M = I \implies det(M)^2 = 1 \implies detM = \pm 1$$

$O(n)$ defines a manifold since there's as chart to a subset of $\mathbb{R}^{n^2}$ (although we don't know what dimension it is yet), but we can see that immediately that it's not a connected manifold. This is because of the fact that $det(M) = \pm 1$ Taking a determinant is a smooth function of $M$, but there's no smooth path we can parametrise $M$ with on the manifold to swith the determinant from $+1$ to $-1$. This is becasue if there was, the composition of two smooth functions would be discontinuous, a contradiction.

So, we've essentially split $O(n)$ into two portions. If we focus on the part of the group where $detM = 1$, this gives us the special orthogonal group.

$$SO(n) = \{M \in O(n) \mid detM = 1\}$$

This has the property that we're preserving the orientation of a basis of $\mathbb{R}^n$ say. Suppose we had a basis

$$\mathcal{B} = \{\mathbf{v}^1, \dots \mathbf{v}^n\}$$

Then a linear map is orentation preserving if the sign of

$$\Omega = \epsilon^{i_1 i_2, \cdots i_n} v^1_{i_1} \dots v^n_{i_n}$$

is preserved after transformation of the vectors. Matrices with determinant 1 preserve the sign of this object. Hence $M \in SO(n)$ are considered as rotations, where as matrices in $O(n)$ without unit determinant are considered some composition of rotations and reflections.

The condition that $M^T M = I$ provides constraints only up to a transpose. This means that our constraints which are linearly independent only lie in the top triangle, which gives us $\frac{1}{2}(n)(n+1)$ constraints on our manifold. This means that, using the embedding theorem,

$$dim(O(n)) = n^2 - \frac{1}{2}(n)(n+1) = \frac{1}{2}(n)(n-1)$$

Since our condition already fixes that that $detM = \pm 1$ up to a sign, this imposes no additional contraints on our manifold and we have that $dimSO(n) = dimO(n)$.

Matrix groups are Lie groups - the can parametrise them through their matrix entries smoothly, where each matrix element is a coordinate. With this reasoning, the dimension of $GL_n(\mathbb{R})$ is $n^2$, for example.

### 3.2.2 General properties of $O(n)$

In this section, we'll look at some properties of some common orthogonal groups that will arise. To help us with this, we'll present some details about matrices in $O(n)$ to help us with our discussion. In particular, facts about eigenvectors and eigenvalues, which are basis invariant properties, are particularly useful for our discussion.

**Theorem.** If $\lambda \in \mathbb{C}$ is an eigenvalue of $M \in O(n)$, then so is $\lambda^* \in C$ Furthermore, we have that $|\lambda|^2 = 1$.

*Proof.* Suppose that $\lambda \in C$ is an eigenvalue. Then by definition, there exists a (complex) vector $\mathbf{v} \in \mathbb{C}^n$such that $M\mathbf{v}_\lambda = \lambda \mathbf{v}_\lambda$. However, recall that $M \in GL(n, \mathbb{R})$. This means we can take the complex conjugate (not transpose!) of this equation, and since the entries of $M$ are real, it doesn't change. This gives us

$$M\mathbf{v}_\lambda^* = \lambda^* \mathbf{v}_\lambda^*$$

However, this implies that $\mathbf{v}_\lambda^*$ is an eigenvector with eigenvalue $\lambda^*$. So we're done. Another way to phrase this is that since the entries of $M$ are real, then the coefficients of its characteristic polynomial are real. However, this means that any complex root must have it's complex conjugate as a root as well.

We can use the orthogonality property to our advantage here. Note that

$$|\mathbf{v}|^2 = \mathbf{v}^\dagger I \mathbf{v} = \mathbf{v}^\dagger M^T M \mathbf{v} = |\lambda|^2 |\mathbf{v}|^2 \implies |\lambda|^2 = 1$$

□

This fact is important, this means that if we have a complex eigenvalue, then it must be a phase of the form $e^{i\theta}$ and $e^{-i\theta}$ must also be an eigenvalue. Our condition that modulus squared also applies to real numbers. If we have $\lambda \in \mathbb{R}$, then $\lambda \in \{-1, +1\}$. However, these are also phases and so are included in the condition above.

### 3.2.3 SO(2)

Take the group $SO(2)$. This is the matrix group

$$SO(2) = \{A \in O(n) \mid \det A = 1\}$$

We have from our previous discussion that matrices in $O(n)$ have eigenvalues which must be $e^{i\theta}, e^{\cdot -i\theta}$, with $\theta \in \mathbb{R}$. However, phases are invariant by shifting $\theta \to \theta + 2\pi$, so we have the identification that

$$\theta \sim \theta + 2\pi$$

This is suggestive that elements in the group represent rotations in the 2 dimensional plane. We can thus use $\theta$ as a coordinate to parametrise our manifold as rotations in the plane as $M(\theta)$, with a general group element written as

$$g(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where we restrict $\theta \in S^1$. Thus, our corresponding manifold for this Lie group is the circle $S^1$, and $\mathcal{M}(SO(2)) \simeq S^1$. To reaffirm our confidence that this is a Lie group, one can check that $M(\theta_1)M(\theta_2) = M(\theta_2)M(\theta_1) = M(\theta_1 + \theta_2)$, so group multiplication acts smoothly on the manifold. Furthermore, this manifold is connected, but not simply connected because tying a string around a circle, we can't contract this down to a point. In fact the homotopy group of $S^1$ is

$$\pi_1(S^1) \simeq \mathbb{Z}$$

This is because we can loop a string around $S^1$ n times in an additive sense.

### 3.2.4 SO(3)

We increase the dimension a bit and explore what's going on with $SO(3)$.Before we go into the maths behind representing rotations in three dimensions, let's first go into some of the geometrical intuition. To rotate something in three dimensions, a systematic way one could go about doing this is to first choose an axis of rotation which we could parametrise as a unit vector $\hat{\mathbf{n}} \in S^2$ on our two-sphere. We can then rotate this. Naively, we'll say that we can rotate this by an angle $\theta \in [0, 2\pi)$, but more careful consideration tells us that if we parametrise rotations in this range, we're double counting. Rotating about the $\hat{\mathbf{z}}$ axis by an angle of $\theta = \frac{\pi}{2}$ is exactly the same as rotating about the $-\hat{\mathbf{z}}$ axis by an angle of $\theta = \frac{3\pi}{2}$. So, we have a **redundancy** in our initial idea of the range of $\theta \in [0, 2\pi)$, and it's sufficient to parametrise elements in SO(3) with

$$\hat{\mathbf{n}} \in S^2, \quad \theta \in [0, \pi)$$

Now we can go into more detail about representing objects in SO(3) as matrices. We first ask what our eigenvalues look like. Since the matrix elements of a matrix in $SO(3)$ are real, its characteristic polynomial has real coefficients. Thus, we must have one real eigenvalue and two complex eigenvalues. Due to the conditions we described earlier about orthogonal matrices, they eigenvalues are $1, e^{i\theta}, e^{-i\theta}$. Given that $\mathbf{n}$ is a unit modulus eigenvector of $M \in SO(3)$ with $M\mathbf{n} = \mathbf{n}$, a general element in this matrix can be written as

$$M(\theta, \mathbf{n}) = \cos\theta \delta_{ij} + (1 - \cos\theta) n_i n_j + \sin\theta \epsilon_{ijk} n_k.$$

We're interpreting $\mathbf{n}$ to be the axis of rotation, since it's an invariant vector. Notice that there's some ambiguity in our description here. For example, if we were to map $\theta \to 2\pi - \theta$, this gives us the same result as if were to map $\mathbf{n} \to -\mathbf{n}$, which we argued in the previous paragraph: In addition, rotating about a given axis by 0 degrees does nothing, so we have even more redundancy in this system. We have the following identifications on our manifold $M$:

$$M(-\mathbf{n}, \theta) \sim M(\mathbf{n}, 2\pi - \theta), \quad M(\mathbf{n}, 0) = I_3, \quad \theta \in [0, \pi]$$

Now let's ask ourselves, what are problematic values of $\theta$ and where can we identify what things with what things? Well, since we require that

$$\theta \in [0, \pi], 2\pi - \theta \in [0, \pi] \implies \pi \leq \theta \leq 2\pi \implies \theta = \pi$$

is where our things need to be identified. So, we have the identification that

$$(-\mathbf{n}, \pi) \sim (\mathbf{n}, \pi)$$

Let's think of this geometrically by defining a new mapping from our manifold to a ball. Let's now parametrise the manifold in terms of a vector

$$\mathbf{w} = \theta \hat{\mathbf{n}}, \quad \theta \in [0, \pi)$$

This is already consistent with our second identification since when $\theta = 0$, $\mathbf{w} = 0$, an identity element. Now, it takes some work to wrangle this in a manifold that's compatible with the first identification structure. We can define the ball with radius $\pi$.

$$B_3 = \{\mathbf{w} \in \mathbb{R}^3 \mid |\mathbf{w}|^2 \leq \pi\}$$

and its corresponding boundary

$$\partial B_3 = \{\mathbf{w} \in \mathbb{R}^3 \mid |\mathbf{w}|^2 = \pi\}.$$

However, because of our identification condition earlier, we have that antipodal points on this boundary sphere are considered to be 'identified' or 'the same' with one another. So in actual fact, the boundary of this manifold is $\partial B_3 / \sim$, where our equivalence relation quotients our antipodal points. This means that our manifold has no boundary since points on the boundary sphere aren't unique. It's easy to see that whilst this manifold is connected, it's not simply connected because we can take a straight line going through the origin through the sphere, but since antipodal points are identified, this constitutes a loop. We can't contract this however since rotating this line will always keep us on the boundary. On the other hand, we can easily construct a loop just in the interior of this sphere.

However, we can 'square' a loop by going straight through the sphere twice, and that curve is deformable to a point! This suggests that our homotopy group structure is akin to $\mathbb{Z}_2$, so

$$\pi_1(SO(3)) \simeq \mathbb{Z}_2 \simeq \{+1, -1\}$$

This manifold is compact since it's bounded, and closed since we're including points on the boundary.

#### 3.2.5 Non compact groups; the Lorentz group and more general metrics

We brought up orthogonal groups previously because they preserve the Euclidean metric

$$M^T M = M^T I M. = I$$

However, there are matrices which preserve more general metrics, ubiquitous in physics. A metric with signature $(p, q)$, is of the form

$$\eta = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

We denote the matrices which preserve this metric as

$$O(p, q) = \{M \in Mat_n(F) | M^T \eta M = \eta\}.$$

These are matrices which constitute the general orthogonal group. Our simplest example of a matrix in the class of general orthogonal matrices is the a general matrix in $SO(1, 1)$, which looks strikingly similar to a rotation matrix in $SO(2)$, expect that we've replaced our sin and cos terms with

$$M = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}, \quad \phi \in \mathbb{R}$$

One can easily check that this has determinant 1, and that it preserves the metric $diag(1, -1)$ However, since our coordinate parameter $\phi \in \mathbb{R}$, this group is not compact since $\mathbb{R}$ is not compact.

#### 3.2.6 Subgroups of $GL(n, \mathbb{C})$

Thus far, we've invested quite a bit of effort in looking at real matrices which have the interpretation of leaving vector norms unchanged. We can do the exact same with complex matrices. These matrices are analog of orthogonal matrices but in complex space instead, for $F = \mathbb{C}$. The set of matrices in this matrix group which leave the norm of complex vectors unchanged are the unitary matrices.

$$U(n) = \{M \in Mat_n(\mathbb{C}) \mid M^\dagger M = I\}.$$

In analog with what we had in the real case,

$$\mathbf{v}' \in \mathbb{C}^n \mapsto \mathbf{v}' = M\mathbf{v}, \implies |\mathbf{v}'|^2 = |\mathbf{v}|^2$$

Our condition for unitary implies that the determinant of matrices in the group is a phase, since

$$|det(M)|^2 = 1 \implies det(M) = e^{i\theta}, \quad M \in U(n)$$

Our key difference that we have from the real case with $O(n)$ is now that we no longer have discontinuous jumps when taking the determinant, since our determinants vary by a smooth phase! This means that $U(n)$ is a connected manifold! Just as before, we can restrict our matrices in $U(n)$ to those which have unit determinant,

$$SU(n) = \{M \in U(n) \mid detM = 1\}$$

By arguments that matrix multiplication in complex space is smooth, we have that $U(n)$ and $SU(n)$ are Lie groups. Thus, we can count their dimensions. If we consider complex numbers as being parameterised by 2 real numbers, our over-arching embedding space is $\mathbf{R}^{n^2}$. To figure out the dimension of $U(n)$, we need to impose the condition that

$$H = M^\dagger M = I, \text{ equivalently } H^\dagger = H$$

On the surface this looks like we have $2n^2$ constraints, but this is symmetric up to hermitian conjugation. Hence, we ask how many degrees of freedom we need to define a Hermitian matrix. On the diagonal of a hermitian matrix, we require real elements. On the upper triangular part of a Hermitian matrix, we have $n(n-1)/2$ place holders for complex numbers, but since each complex number has 2 degrees of freedom over the reals, we multiply this number by 2. Hence our total degrees of freedom is

$$2(n(n-1))/2 + n = n^2$$

Hence we have a set of $n^2$ quadratic constraint functions. This thing one can check is of full rank for non-zero matrices. Now, to find the dimension of $SU(n)$ we impose an additional constraint on the determinant, $detU = 1$. Since we're working over complex numbers, this is superficially looks like two conditions in real space. However, since our determinant is already a phase by the unitarity conditions, we actually just have one constraint that

$$e^{i\theta} = 1$$

So in summary we have that

$$dim(U(n)) = n^2, dim(SU(n)) = n^2 - 1$$

Two Lie groups $G$ and $G'$ are isomorphic ($G \simeq G'$) if there exists a one to one smooth map $\mathcal{I} : G \to G'$ such that for any pair of elements of the Lie group $G$, we have that for $\forall g_1, g_2 \in G$

$$\mathcal{I}(g_1 g_2) = \mathcal{I}(g_1)\mathcal{I}(g_2)$$

A general element $z = e^{i\theta}$ of $G = U(1)$, with

$$\theta \in \mathbb{R}, \quad \theta \sim \theta + 2\pi$$

corresponds to a unique element

$$M(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

of G' $= SO(2)$. We have the map

$$\xi z(\theta) = e^{i\theta} \to M(\theta), \quad \xi : U(1) \to SO(2)$$

We can easily check that $\xi$ is bijective, and in addition we have that

$$\xi(z(\theta_1)z(\theta_2)) = M(\theta_1 + \theta_2) = M(\theta_1)M(\theta_2) = \xi(z(\theta_1))\xi(z(\theta_2))$$

Therefore we have an isomorphism that $U(1) \simeq SO(2)$.

Now consider the Lie group $G = SU(2)$, with dimension 3. We can show that we can write

$$U = a_o I + i\mathbf{a}\cdot\sigma, \quad \sigma = (\sigma_1, \sigma_2, \sigma_3), \quad (a_0, a_1, a_2.a_3) \in \mathbb{R}^3$$

We have the condition that

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1 \implies \mathcal{M}(SU(2)) \simeq S^3 \subset \mathbb{R}^4$$

This defines a different manifold than that of $SO(3)$ We have that the corresponding manifolds are

$$\pi_1(SU(2)) \simeq \{1\}, \quad \pi_1(SO(3)) \simeq \mathbb{Z}_2 \implies SU(2) \neq SO(3)$$

Our most obvious example is

$$U(1) = \{z \in \mathbb{C} | |z|^2 = 1\},$$

thus, we can write

$$z \in U(1) \iff z = e^{i\theta}, \quad \theta \in [0, 2\pi).$$

We can now construct a bijective map $U(1) \to SO(2)$

$$e^{i\theta} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Thus we have that $U(1) \simeq SO(2)$. As for $SU(2)$, we can show that a general element is of the form

$$U = a_0 I + i\mathbf{a}\dot{\sigma}$$

where $\sigma$ represents the Pauli matrices. Our unitarity condition implies that $a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1$. Thus

$$\mathcal{M}(SU(2)) \simeq S^3.$$

# 4 Lie Algebras

Lie algebras are 'infinitesimal versions' of Lie groups, and they should be interpreted as such. A Lie algebra } is a vector space $\mathcal{G}$ (for example, a set of matrices) over some field $\mathbb{R}$ or $\mathbb{C}$. which has a special structure defined on it called a 'bracket', which is a bilinear map.

$$\_\dot{\_} : \mathcal{G} \times \mathcal{G} \to \mathcal{G}.$$

This should obey some special properties.

1. Antisymmetry: $[X, Y] = -[Y, X] \quad \forall X, Y \in \mathcal{G}$
2. Linearity: $[X, \alpha Y + \beta Z] = \alpha[X, Y] + \beta[X, Z] \quad \forall X, Y, Z \in \mathcal{G}, \quad \alpha, \beta \in F.$
3. Jacobi Identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad \forall X, Y, Z \in \mathcal{G}.$

Note, the Jacobi identity is a non trivial property inherited from the associativity of group multiplication in the underlying Lie group. For an example of a valid Lie algebra, we can define the vector field of all matrices $Mat_n(\mathbb{R})$, and simply define the Lie bracket as the anti-commutator of these matrices. It's easy to check that it satisfies the properties above.

If a vector space $V$ has an associated product

$$* : V \times V \to V$$

which satisfies

$$(X * Y) * Z = X * (Y * Z), \quad Z * (\alpha X + \beta Y) = \alpha Z * X + \beta Z * Y, \quad \forall X, YZ \in V, \forall \alpha, \beta \in F$$

We can get a Lie algebra straight from this by setting

$$[X, Y] = X * Y - Y * X, \quad \forall X, Y \in V$$

Lot of examples with $V$ being the vector space of matrices, with the operation $*$ being matrix multiplication. The dimension of $\mathcal{G}$ is the dimension of the vector space. We can choose a basis $\mathcal{B}$ for $\mathcal{G}$, denoted as the set

$$\mathcal{B} = \{T^a, a = 1, \ldots, n = dim(\})\}$$

Since we have a vector space, we have that for any $X \in \mathcal{G}$, we can write this as

$$X = X_a T^a := \sum_{a=1}^{n} X_a T^a, \quad X_a \in F$$

Now, if we take the Lie bracket of elements $X, Y \in \mathcal{G}$, by linearity we have that

$$[X, Y] = X_a Y_b [T^a, T^b]$$

Hence it's in our interest to know what the values of the brackets of the basis elements are. These characterise the Lie algebra, given by

$$[T^a, T^b] = f_c^{ab} T^c$$

If the structure constants are the same in two Lie algebras, we have an isomorphism. These structure constants are elements of the underlying field $F$. Our properties of the Lie bracket imposes conditions on the structure constants. We have that

- Antisymmetry $\implies f^{ab}_c = -f^{ba}_c$.
- The Jacobi identity obeys the quadratic like identity that

$$f^{ab}_c f^{cd}_e + f^{da}_c f^{cb}_e + f^{bd}_c f^{ca}_e = 0$$

Two Lie algebras are isomorphic if there's a bijective linear map $f : \mathcal{G} \to \mathcal{G}'$ such that our **Lie bracket structure is preserved**. In other words, we require the condition that

$$[f(X), f(Y)] = f([X, Y]), \quad X, Y \in \mathcal{G}$$

A subalgebra $h \subset \mathcal{G}$ is a vector subspace which itself is also a Lie algebra.

### 4.0.1 Properties of Lie algebras

Like groups in the usual sense, we have derived objects from Lie algebras. The most obvious of these would be a sub-algebra $\mathcal{H}$ We define the important concept, an ideal. This is a sub-algebra $\mathcal{I} \triangleleft \mathcal{G}$, where

$$[X, Y] \in \mathcal{I} \quad \forall Y \in \mathcal{I}, \forall X \in \mathcal{G}$$

This is roughy the same as a normal subgroup in a group. We'll now give some examples. Every $\mathcal{G}$ has two trivial ideals, which are

$$h = \{0\}, \quad h = \mathcal{G}$$

The next example is called the **derived algebra** which may not be a trivial ideal. This is

$$\mathcal{I} = [\mathcal{G}, \mathcal{G}] := span_F\{[X, Y] : X, Y \in \mathcal{G}\}$$

This is the linear sum of all possible Lie brackets of elements in our Lie algebra. **The Centre**

The centre $C(\mathcal{G})$of a Lie algebra is the ideal whose elements commute with all other elements.

$$C(\mathcal{G}) = \{X \in \mathcal{G} \mid [X, Y] = 0 \forall Y \in \mathcal{G}\}.$$

An abelian Lie algebra is such that

$$[X, Y] = 0, \quad \forall X, Y \in \mathcal{G}$$

In this case, the derived algebra is simple the set with identity, and the centre is the whole group. We say that a Lie algebra $\mathcal{G}$ is simple if it is non-abelian and has no non trivial ideals. For simple Lie algebras, the derived ideal is the whole group, and the centre is 0.

### 4.0.2 Cartan classification

We can classify all finite dimensional simple Lie algebras over the complex field.

## 4.1 Construction of Lie algebras from Lie groups

We show that every Lie group corresponds to a Lie algebra.

#### 4.1.1 Preliminaries

Let $\mathcal{M}$ be a smooth manifold of dimension D and $p \in \mathcal{M}$ a point. Introduce some coordinates $\left\{x^i\right\}, i = 1, \dots D$ in some region $p \subset \mathcal{M}$. Now, we choose coordinates such that our point $p$ corresponds to origin $x^i = 0, i = 1, \dots D$.

The tangent space $T_p(\mathcal{M})$ to $\mathcal{M}$ at $p$ is a D-dimensional vector space spanned by differential operators $\frac{\partial}{\partial x^i}, i = 1, \dots D$ which acts on functions $f : \mathcal{M} \to \mathbb{R}$. A tangent vector

$$V = v^i \frac{\partial}{\partial x^i} \in \mathcal{T}_p(\mathcal{M}), \quad v^i \in \mathbb{R}$$

This objects acts on functions $f = f(x)$ as

$$V \cdot f = v^i \frac{\partial f(x)}{\partial x^i} \mid_{x=0}$$

(Draw diagram of curve on manifold here. We now present a different way to think about tangent vectors. Consider a smooth curve from $\mathcal{I} \subset \mathbb{R}$.

$$C : \mathcal{I} \to \mathcal{M}$$

on $\mathcal{M}$ which passes through $p \in \mathcal{M}$;

$$C : t \in \mathcal{I} \subset \mathbb{R} \mapsto x^i(t) \in \mathbb{R}, \quad i = 1, \dots, D$$

We have that $\left\{x^i(t)\right\}$ are continuous, differentiable. For now, just consider curves which are differentiable once. Also we insist that

$$x^i(0) = 0 \quad \forall i = 1, \dots, D$$

( Draw diagram of tangent here ) Our tangent vector to $C$ at $p$ is

$$V_c = \dot{x}^i(0) \frac{\partial}{\partial x^i} \in \mathcal{T}_p(\mathcal{M}), \quad \dot{x}^i(t) = \frac{dx^i(t)}{dt}$$

The interpretation of this tangent vector is nothing but the velocity of a particle, say. Every smooth curve has a tangent vector at every point, even if the curve has closed endpoints, say

$$J = [0, L]$$

This tangent can be taken from derivatives in interior. Conversely, given an element of the tangent space. we can find a curve which yields that tangent vector by fixing the first term of its Taylor expansion.

## 4.2 The Lie Algebra $\mathcal{L}(G)$

We stated earlier that Lie groups are manifolds in themselves. Well, on a manifold, we can define the tangent space at some point of reference (in this case we'll set to be $e \in G$) as $\mathcal{T}_e(G)$. We know that tangent spaces are vector spaces with the basis $\{\partial_i\}_i$, so we could try making a Lie group out of this by trying to define the commutator. Let's proceed with constructing this relationship. Let $G$ be a Lie group of dimension $D$. Introduce coordinates $\left\{\theta^i\right\}, i = 1, \ldots D$ in some region $\mathcal{P}$ containing $e$, the identity.

$$\forall g \in \mathcal{P}, \quad g = g(\theta) \in G, \quad g(0) = e$$

(Draw diagram of tangent space here) The tangent space $\mathcal{T}_e(G)$ is a D dimensional vector space. We can define a bracket

$$[\,,\,] : \mathcal{T}_e(G) \times \mathcal{T}_e(G) \to \mathcal{T}_e(G)$$

such that we have $\mathcal{L}(G) = (\mathcal{T}_e(G), [\,,\,])$ is a Lie algebra.

### 4.2.1 Constructing the Lie Algebra for Matrix Lie Groups

There are two ways to think about this. A tangent vector is a differential operator of a function, which can be written like

$$V = v_i \partial_i \in \mathcal{T}_e(G).$$

We can represent this as a matrix as follows. An element in an n-dimensional Lie group can be represented as $g(\theta) \in G$, where $\theta$ is an n-vector. Thus, we can map $V$ like

$$v_i \partial_i \mapsto v_i \frac{\partial g(\theta)}{\partial \theta_i}\bigg|_{\theta=0} \in Mat_n(F).$$

This construction is easiest to show for **Matrix Lie groups** $G \subset Mat_n(F)$ for either a real or $\mathbb{C}$ fields. We can map tangent vectors to matrices.

$$\rho : \mathcal{T}_e(G) \to Mat_n(F)$$

given by the map

$$v^i \frac{\partial}{\partial \theta^i} \in \mathcal{T}_e(G) \mapsto v^i \frac{\partial g(\theta)}{\partial \theta^i} \; |_{\theta=0} \in Mat_n(F)$$

We see that this map $\rho$ is injective, but not necessarily surjective. This allows us to identify $\mathcal{T}_e(G)$ with the subspace of $Mat_n(F)$ spanned by

$$\left\{ \frac{\partial g(\theta)}{\partial \theta^i}\bigg|_{\theta=0} \right\}$$

We have an obvious candidate for our Lie bracket

$$[X, Y] := XY - YX, \quad X, Y \in \mathcal{T}_e(G)$$

We need to check that our properties for Lie algebras holds. This is clear since antisymmetry and Linearity are inherent straight from the definition of the matrix commutator. The Jacobi identity is easy to prove but requires a little bit of algebra. With this assignment, we are now free to define our commutator as the usual commutator for matrices. However, we need to show that the Lie algebra generated by the commutator is closed. There is a subtle point however, that we should check that the resulting object is still indeed in the tangent space. We do this by showing that there is a curve, that if we differentiate it at a point, we recover the commutator object itself. To do this, we reference what we said in our preliminaries about constructing curves that yield tangent vectors. In other words, we need to prove that,

$$[X, Y] \in \mathcal{L}(G), \quad \forall X, Y \in \mathcal{L}(G)$$

Let $c$ be a smooth curve on $G$ through the identity. Let

$$c : \mathcal{J} \subset \mathbb{R} \to G$$
$$c : t \mapsto g(t) \in G, \quad g(0) = I_n$$

Now, if we consider the derivative we have

$$\frac{dg(t)}{dt} = \frac{d\theta^{i(t)}}{dt} \cdot \frac{\partial g(\theta)}{\partial d\theta^i}$$

If we take

$$\dot{g}(0) = \left.\frac{dg(t)}{dt}\right|_{t=0} = \dot{\theta}^i(0) = \left.\frac{\partial g(\theta)}{\partial \theta^i}\right|_{\theta=0} \in \mathcal{T}_e(G)$$

This is a tangent vector to $c$ at $e$. We have that

$$\dot{g}(0) \in Mat_n(F) \text{ is not generally in } G$$

Observe that near to $t = 0$, we have that

$$g(t) = 1 + tX + O(t^2), \quad X = \dot{g}(0) \in \mathcal{L}(G)$$

Given 2 elements $X_1, X_2 \in \mathcal{L}(G)$, since all we need to do is fix the first term of our Taylor expansion, it is simple to find curves $c_1, c_2$ such that

$$c_1 : t \mapsto g_1(t) \in G, \quad g_1(0) = 1$$
$$c_2 : t \mapsto g_2(t) \in G, \quad g_2(0) = 1$$

where we have $\dot{g}_1(0) = X_1, \dot{g}_2(0) = X_2$. Near $t = 0$, we have

$$g_1(t) = 1 + X_1 t + W_1 t^2 + O(t^3)$$
$$g_2(t) = 1 + X_2 t + W_2 t^2 + O(t^3)$$

for some $W_1, W_2 \in Mat_n(F)$. We define a new curve

$$h(T) = g_1^{-1}(t) g_2^{-1}(t) g_1(t) g_2(t) \in G$$

This curve is smooth since this is a composition of multiplication and inversion, which are smooth maps. This is equivalent to

$$g_1(t) g_2(t) = g_2(t) g_1(t) h(t)$$

If we expand the curve about $t = 0$, we get that

$$g_1(t)g_2(t) = 1 + t(X_1 + X_2) + t^2(X_1X_2 + W_1 + W_2) + O(t^3)$$
$$g_2(t)g_1(t) = 1 + t(X_2 + X_1) + t^2(X_2X_1 + W_1 + W_2 + O(t^3)$$

Now, if we set

$$h(t) = 1 + h_1t + h_2t^2 + O(t^3)$$

, if we plug this into our relation we have that

$$h_1 = 0, \quad h_2 = (X_1X_2 - X_2X_1) = [X_1, X_2]$$

Thus, we have that

$$h(t) = 1 + t^2[X_1, X_2] + O(t^3)$$

We're not quite done yet! We need a curve whose linear part is the commutator. Define $s = t^2, s/geq0$ in other words a curve who's endpoint is at zero, then define a new curve

$$c_3 : s \mapsto g_3(s) = h(\sqrt{s}), \quad s \geq 0$$

We have

$$g_3(s) = 1 + s[X_1, X_2] + O(s^{3/2})$$

This is okay up to one derivative. If we have a second derivative this term is not analytic or defined at 0! But we have a $C^1$ curve so we're okay. We have, from this curve, that

$$\frac{dg_3(e)}{ds}\,|_{s=0} = [X_1, X_2]$$

This means that the commutator $[X_1, X_2]$ is an element of our tangent space $\mathcal{L}(G)$. Thus,

$$\mathcal{L}(G) = (\mathcal{T}_e(G), [\ ,\ ])$$

defines a Lie algebra.

## 4.3 Examples of derived Lie Algebras

Throughout this section examples, we will derive examples of the Lie algebras we can farm from Lie groups. We will follow the same recipe. We take our favourite matrix Lie group, and generate a smooth curve $g(t)$ in this group.

Then, we take a condition or equation from our definition of our Lie group, then differentiate this at $t = 0$. This should give us a sufficient condition on our corresponding Lie algebra.

Before we go into this, we'll prove an important identity. This is the fact that the derivative of the determinant function of a matrix is the trace. More explicitly, we prove the claim that

$$\det(I + tA) = I + t \operatorname{tr} A + O(t^2)$$

where $I, A \in \text{ Mat }_n(F)$. This can be shown by recalling the definition of the determinant, that

$$\det B = \sum_{\sigma \in S_n} \epsilon(\sigma) \prod_{i=1}^{n} A_{i\sigma(i)}$$

In components, we have that

$$(I + tA)_{ij} = \delta_{ij} + tA_{ij}$$

Substituting this into our expression above, we have that

$$\det(I + tA) = \sum_{\sigma \in S_n} \epsilon(\sigma) \prod_{i=1}^{n} (\delta_{i\sigma(i)} + tA_{i\sigma(i)})$$

The trick here is to now separate from the sum our identity permutation, and leave everything else. Hence, our expression reads

$$\det(I + tA) = \prod_{i=1}^{n} (1 + tA_{ii}) + \sum_{\sigma \neq e} \prod_{i=1}^{n} (\delta_{i\sigma(i)} + tA_{i\sigma(i)})$$

Now, in the latter sum, we have that if a permutation is not the identity, it must contain a transposition. That means at least two terms in the product have $\delta_{i\sigma(i)}$ vanishing. This means that everything in this product is at least $O(t^2)$. Furthermore, expanding out the first product, our only term of order $t$ is $\sum A_{ii}$. Hence we have that

$$\det(I + tA) = I + t \operatorname{tr} A + O(t^2)$$

### 4.3.1 $SO(n)$ to $\mathcal{L}(SO(n))$

Consider the group $SO(n)$. Take a curve

$$g(t) = R(t) \in SO(n), \quad \forall t \in \mathcal{I} \subset \mathbb{R}, \quad R(0) = I_n$$

A thing to note: everywhere on this smooth curve, $R(t)$, through our Lie group, we have that $\det(R(t)) = 1$. This is because of continuity. We have that $\det(R(0)) = 1$. Hence, if there exists a $t^*$ such that $\det(R(t^*)) = -1$, this means that the smooth curve

$$\det \circ R : I \to \{-1, +1\}$$

jumps from 1 at $t = 0$ to $-1$ at $t = t^*$. This is a contradiction since det and $R$ are smooth.

By our orthogonality condition we have that

$$R(t)^T R(t) = 1, \quad \forall t \in \mathcal{I}$$

Now, the trick to get the Lie algebra from this is to differentiate with respect to our parameter $t$.

$$\dot{R}^T(t)R(t) + R^T(t)\dot{R}(t) = 0, \quad \forall t \in \mathcal{I}$$

If we set this equation at $t = 0$, and $X = \dot{R}(0)$, then we get

$$X^T + X = 0$$

Notice that the only condition we've used so far is that of orthogonality. There's no further constraint that from $\det(R) = 1$, by continuity since we're guaranteed that this would hold true anyway from our above argument. Another way to see this is that if we set expand out

$$\det\Big(R(0) + t\dot{R}(0) + \dots\Big) = 1 + t\,\mathrm{tr}\,X + \dots$$

Then, we have that $\mathrm{tr}\,X = 0$, but this is automatically satisfied since our anti-symmetry condition automatically ensures that a matrix is traceless.

Now, let's do the associated case for $O(n)$. Our condition for orthogonality yields the same anti-symmetric condition on our Lie algebra. In addition, by continuity, we also have that matrices on $R(t)$ have determinant 1. Hence, our Lie algebras are the same.

Thus,

$$\mathcal{L}(O(n)) \simeq \mathcal{L}(SO(n)) = \left\{X \in Mat_n(\mathbb{R}) \mid X^T = -X\right\}$$

The dimension of $\mathcal{L}(SO(n)) = \frac{1}{2}n(n-1)$, since this is the number of parameters to uniquely define an antisymmetric matrix (diagonal elements are set to zero, and we are free to define the elements on the upper triangle). Now, notice that this is the same as $\dim SO(n)$. This of course is not a coincidence: the dimension of a tangent space will always be the same as the dimension of the underlying manifold. This is because our coordinates induce a basis of partial derivatives that we work with in the tangent space of the same size.

### 4.3.2 $SU(n)$ to $\mathcal{L}(U(n))$

Let's repeat this form of analysis with matrices in $U(n)$ and $SU(n)$. If we look at $G = SU(n)$, again we define a curve

$$g(t) = U(t) \in SU(n), \quad U(0) = I_n$$

Differentiating again, our condition that

$$U(t)^\dagger U(t) = I_n \implies Z + Z^\dagger = 0, \quad Z = \dot{U}(0) \in \mathcal{L}(SU(n))$$

We require that $det(U) = 1, \quad \forall t \in \mathbb{R}$. This time, it's perfectly possible for elements in $R(t)$ to have $\det R(t) \neq 1$, since our determinant operation is a phase. This means that for the Lie algebra of $SU(n)$, we require

$$\det U(t) = 1 + t\,\mathrm{tr}\,Z + O(t^2) = 1, \quad \forall t \in \mathcal{I}$$

The higher order terms in $O(t^2)$ don't place any extra additional conditions because we have a lot of freedom. For example. expanding out our terms and setting $t = 1$, we have that

$$1 + \operatorname{tr} Z + A_2 + \dots A_n + \dots = 1$$

We can set the coefficients $A_i$ to 0 by freedom in choice of our curves. Thus, we have an extra condition on our Lie algebra element: our first order term $TrZ$ is forced to be zero. Our Lie algebra for $SU(n)$ is therefore

$$\mathcal{L}(SU(n)) = \left\{ z \in Mat_n(\mathbb{C}) \mid Z^\dagger = -Z, trZ \right\}$$

Our dimension, counting matrices of this type, comes from anti hermitian and traceless conditions. First we count the dimension of anti-hermitian matrices. We can freely choose our elements in our upper left triangle of this matrix not including diagonal. Since we have complex numbers, we need to double the amount of parameters. The number of placeholders on our diagonal is $\frac{1}{2}n(n-1)$, so since they can be complex, this our number of free parameters are $n^2 - n$. Now, in an anti-Hermitian matrix our elements on the diagonal can be imaginary, so this adds another $n$ degrees of freedom. Taking into account that our matrix need be traceless, we subtract a degree of freedom as well

$$\dim \mathcal{L}(SU(n)) = n^2 - n + n - 1 = n^2 - 1 = \dim(SU(n))$$

Notice that in this case, we have that

$$\dim(SU(2)) = \dim(SO(3)) = 3$$

We can exhibit a basis for $SU(2)$ and $SO(3)$, which both have 3 elements. These bases are useful to remember as well. Let's examine this more closely If we consider $G = SU(2)$, then

$$\mathcal{SU}(\in) = \{2 \text{ by } 2 \text{ traceless antihermitian matrices}\}$$

We know that our Pauli matrices $\sigma_a$ obeys $\sigma_a = \sigma_a^\dagger$, $tr\sigma_a = 0$. Thus, we can provide a basis from

$$T^a = -\frac{1}{2} i\sigma_a, \quad a = 1, 2, 3$$

We can now compute the structure constants. The Pauli matrices obey the algebra

$$\sigma_a \sigma_b = \delta_{ab} I + i\epsilon_{abc}\sigma_c$$

This means that

$$[T^a, T^b] = -\frac{1}{4}[\sigma_a, \sigma_b] = -\frac{1}{2} i\epsilon_{abc}\sigma_c = f^{ab}{}_c T^c$$

Hence, we have that $f^{ab}{}_c = \epsilon_{abc}$.

Now, for $G = SO(3)$, we have that

$$\mathcal{B}(SO(3)) = \{ \text{ 3 by 3 real anti Hermitian matrices } \}$$

We can take the basis

$$\tilde{T}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tilde{T}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \tilde{T}^3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We have that

$$[\tilde{T}^a, \tilde{T}^b] = f^{ab}_{\ \ c}, \quad f^{ab}_{\ \ c} = \epsilon_{abc}$$

Thus we have that, since the structure constants are same, these vector spaces are isomorphic (they're also isomorphic by the fact that vector spaces with the same dimension are isomorphic).

$$\mathcal{L}(SO(3)) \simeq \mathcal{L}(SU(2)), \quad SO(3) \neq SU(2)$$

We've written here that the actual Lie groups underlying them are not isomorphic, because for one $SO(3)$ is not a connected manifold. $SU(2)$ however, is connected. Thus, different lie groups can correspond to the same Lie algebra. In fact, $SO(3) \simeq SU(2)/\mathbb{Z}_2$

## 4.4 Reconstructing Lie Groups from Lie Algebras

### 4.4.1 Actions: a natural maps on Lie groups

So far, we've learnt about defining tangent spaces at $e \in G$, or if we're working in a $D$ dimensional Lie group, we can define a set of coordinates $\{\theta_i\}_i$ for $i = 1, \ldots, D$, we would identify this $e \in G$ with $\theta = 0$. But, what about tangent spaces at other points on the manifold?

For this reason, we define left and right actions. A Lie group is a very special type of manifold in that we can define smooth maps which correspond to group mutliplication by an element. This is akin to having the Lie group $G$ act on itself by shuffling the group elements around by a fixed $h \in G$. This can be done either from the left or from the right. Define

$$\begin{aligned} L_h : \quad & G \to G \\ & g \mapsto hg \\ R_h : \quad & G \to G \\ & g \mapsto gh \end{aligned}$$

We will now show that these maps are diffeomorphisms on our group manifold. First we show surjectivity. That is, we wish to show that

$$\forall g' \in G, \exists g \in G, \text{such that } L_h(g) = g'$$

Clearly, setting $g = h^{-1}g'$, satisfies this condition. These maps are also injective because

$$\forall g, g' \in G, L_h(G) = L_h(g') \implies g = g'$$

This is because $L_h(g) = L_h(g') \implies hg = hg' \implies g = g$. Surjectivity and injectivity implies that this map is bijective. This means there's an inverse map

$$(L_h)^{-1} = L_{h^{-1}}$$

which exists. Now, we appeal to the fact that group multiplication on a Lie group is a smooth operation to show that left actions are a diffeomorphism on $\mathcal{M}(G)$ . Now, this can be shown by introducing coordinates $\left\{\theta^i, \quad i = 1, \ldots D\right\}$ If we set $g = g(\theta) \subset G$, and $g(0) = e$, Let

$$g' = L_h(g) = h \cdot g(\theta) = g(\theta')$$

assuming the same coordinate patch for $g'$. In coordinates, $L_h$ is specified by

$$\theta'^i = \theta'^i(\theta), \quad i = 1, \ldots D, \mathbb{R}^D \to \mathbb{R}^D$$

But this is just group multiplication on our Lie group, which was one of our starting axioms. Thus, we have a diffeomorphism. The fact that our left action has an inverse also implies that the Jacobian matrix

$$J^i_{\ j}(\theta) = \frac{\partial \theta'^i}{\partial \theta^j}$$

is invertible. The inverse $J^{-1}$ exists, so $det J \neq 0$. There's more stuff which we an get out of this.

### 4.4.2 Actions move us between tangent spaces

This diffeomorphism also induces another map $L_h^*$ from $\mathcal{T}_g(G)$ to another tangent space $\mathcal{T}_{hg}(G)$. So, we are 'moving' our tangent space along this line. This map is given by

$$L_h^* : \mathcal{T}_g(G) \to \mathcal{T}_{hg}(G)$$
$$L_h^* : v = v^i \frac{\partial}{\partial \theta^i} \in \mathcal{T}_g(G) \mapsto v' = {v'}^i \frac{\partial}{\partial \theta'^i} \in \mathcal{T}_{hg}(G)$$

In the above, we have from the chain rule that

$${v'}^i = \mathcal{J}(\theta)^i_j v^j$$

We refer to the map $L_h^*$ as the differential map of $L_h$. We identify a left action diffeomorphism as a **change of coordinates**, and hence the motivation from this map is a chain rule transformation.

From this, we define an object called a **vector field**. A vector field on $G$ specifies a tangent vector

$$v(g) \in \mathcal{T}_g(G), \quad g = g(\theta)$$

So, given a point on the manifold we get a tangent vector. In coordinates, we write $v$ as an object parametrised with $\theta$ so that

$$v(\theta) = v^i(\theta) \frac{\partial}{\partial \theta^i} \in \mathcal{T}_{g(\theta)}(G)$$

Now, $v$ is smooth if $v^i(\theta)$ are continuous and differentiable: (when we move across our Lie group manifold, our tangent vector $v$ doesn't change abruptly. We can take a vector $w$ in our Lie algebra, $w \in \mathcal{T}_e(G)$, and from this we can define a vector field

$$v(g) = L_g^*(w) \in \mathcal{T}_g(G), \quad \forall g \in G$$

As $L_g^*$ is smooth and invertible, we have that $v(g)$ is smooth and invertible, and non vanishing. Our matrix $\mathcal{J}$ has no zero eigenvalues and so $v(g)$ is non vanishing. Starting from a basis $\{\omega_a\}, a = 1, \ldots D$ for $\mathcal{T}_e(G)$, we have $D$ independent, non vanishing vector fields

$$v_a(g) = L_g^*(\omega_a)$$

The Lefschetz fixed point theorem, or 'hairy ball theorem' states that any smooth vector field on $S^2$ has at least 2 zeros. Thus for our manifold, we have $\mathcal{M}(G) \neq S^2$. In fact, out of two manifolds which are compact, the only manifold possible for a Lie group is that

$$\mathcal{M}(G) = T^2, \quad G = U(1) \times U(1)$$

In fact, this technology of left invariant of vector fields is enough technology to define Lie groups, without resorting to matrix groups as we have done previously.

#### 4.4.3 Left actions on Matrix Lie Algebras

Now let's think about a matrix Lie group, and apply this technology of left and right multiplication maps. Suppose we had a matrix Lie group $G \subset Mat_n(F)$ for $n \in \mathbb{N}$ with our field $F = \mathbb{R}$ or $\mathbb{C}$, then our left multiplcation map induces a differential map which is matrix multiplication. So, $\forall h \in G, X \in \mathcal{L}(G)$, we have that

$$L_h^*(X) = hX \in \mathcal{T}_h(G)$$

Note that this only works in the language of matrix Lie groups and Lie algebras because everything is a matrix, so $hX$ makes sense. To show the above, we apply our usual recipe of finding a curve whose tangent vector gives $X$. So, take any curve

$$C : t \in \mathcal{J} \subset \mathbb{R} \mapsto g(t) \in G, \quad g(t) = g(0) + t\dot{g}(0) + O(t^2)$$

By definition, we have that

$$\dot{g}(0) = \left.\frac{dg(t)}{dt}\right|_{t=0} \in \mathcal{T}_{g(0)}(G)$$

We have by construction that $g(0) = e = I$, and that $\dot{g}(0) = X$, with $X \in \mathcal{L}(G) \simeq \mathcal{T}_e(G)$, since our Lie algebra is defined to be the set of tangent vectors at the origin. Now we can define a new curve, which is our original curve but with the left action $L_h$ applied to it.

$$C' : t \in \mathbb{R} \mapsto h(t) = h \cdot g(t) \in G$$

which is given by our left multiplication map. So, taking a Taylor expansion here we have that near $t = 0$,

$$h(t) = h + thX + O(t^2) \implies hX \in \mathcal{T}_h(G)$$

This is the tangent space at $h$ because our leading order term is $h \in G$. Hence, since we've exhibited a curve whose vector is $hX$ at $h$, we've shown that

$$hX \in \mathcal{T}_h(G)$$

We can apply this idea to a smooth curve in general. Equivalently, we have that a smooth curve in general

$$C : t \in \mathbb{R} \mapsto g(t) \subset G$$

for matrix lie groups we have that

$$\dot{g}(t) \in \mathcal{T}_{g(t)}(G) \implies g^{-1}(t)\dot{g}(t) = L^*_{g^{-1}(t)}(\dot{g}(t)) \in \mathcal{T}_{g^{-1}(t)g(t)} = \mathcal{T}_e(G) \simeq \mathcal{L}(G)$$

We use this fact to show that we can reconstruct Lie groups from Lie algebras. Working in the opposite direction, if we have a vector $X \in \mathcal{L}(G)$, we can reconstruct the curve

$$C : \mathcal{J} \subset \mathbb{R} \to G$$

by solving the ordinary differential equation that we derived earlier

$$g^{-1}(t)\frac{dg(t)}{dt} = X, \quad g(0) = I$$

To solve this, we define the exponential of a matrix $M \in Mat_n(F)$ as

$$\text{Exp}M := \sum_{n=0}^{\infty} \frac{1}{l!} M^l \in Mat_n(F)$$

We solve our differential equation above by setting

$$g(t) = \text{Exp }(tX)$$

We check that this indeed solves the equation. We have that $g(0) = \text{ Exp }(0) = 1$.

$$\frac{dg(t)}{dt} = \sum_{n=1}^{\infty} \frac{1}{(l-1)!} t^{l-1} X^l = \text{ Exp }(tX) \cdot X = g(t)X$$

Thus, it must be the case that $\text{ Exp }(tX) \in G, \forall t \in \mathbb{R}, \forall X \in \mathcal{L}(G)$. So we know a group determines a Lie algebra, but to what extent does a Lie algebra determine a Lie group? An exercise, we should check that if $X \in \mathcal{L}(SU(n))$, we check that

$$\text{Exp }(tX) \in SU(N) \forall t \in \mathbb{R}$$

### 4.4.4 Reconstructing $G$ from $\mathcal{L}(G)$

If we set $t = 1$, we have that

$$\text{Exp }: \mathcal{L}(G) \to G$$

This means that if we have a map $X \in \mathcal{L}(G)$ , exponentiating a this thing gives us an element in the Lie group, so $\text{Exp}(X) \in G$. If we think of exponential map as a map on the complex numbers, the inverse is the logarithm function which has branch points. So, we can't expect this map to be globally bijective! However, we can make a slightly weaker statement that the exponential map

$$\text{Exp} L\mathcal{L}(G) \to G$$

is bijective in some neighbourhood of the identity element $e$ in our Lie group. The point is that given some knowledge of our Lie bracket, we can then locally reconstruct the Lie group. Hence, given $X, Y \in \mathcal{L}(G)$, we can construct

$$g_X = \text{Exp}(X), g_Y = \text{Exp}(Y) \in G$$

We have that

$$g_X g_Y = g_Z = \text{Exp}(Z) \in G, \text{for some } Z \in \mathcal{L}(G)$$

near the identity. By the Baker-Campbell-Hausdorff formula, we have that $Z$ is related to our vectors $X, Y$ by

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}\left([X, [X, Y]] - [Y, [X, Y]]\right)$$

Under a rescaling, we have that $X \to \epsilon X, Y \to \epsilon Y$ gives a small series in which our formula above is valid. Note that the formula is given purely in terms of sums of nested commutators, and the sum of these expressions. This means that $Z$ **will be** in the Lie algebra. This reflects the fact that the formula is given in terms of the Lie algebra. This means we can apply this to all Lie algebras, not just matrix Lie algebras. Using the jacobi identity, we have that the BCH formula is not a unique expression.

Now, what about away from the identity? Our conclusion from the previous part was that $\mathcal{L}(G)$ only determines $G$ in the identity, in other words within the radius of convergence of our BCH formula. An example that the exponential map is not surjective is that Exp clearly is not surjective when $G$ is not connected. As an example, consider $G = O(n)$. Our Lie algebra is

$$\mathcal{L}(G) = \left\{ X \in \mathrm{Mat}_m(\mathbb{R}) \mid X^T + X = 0 \right\}$$

Now, $X \in \mathcal{L}O(n) \implies \operatorname{tr} X = 0$. Now examine $\mathrm{Exp}(X)$ for $X \in \mathcal{L}(O(n))$. We have that

$$\det(\mathrm{Exp}(X)) = \exp(\operatorname{tr} X) = +1$$

This means that Exp $(X) \in SO(n)$. In full generality, when $G$ is compact, the image of the lie algebra $\mathcal{L}(G)$ under the exponential map is the **connected component** of the identity. This breaks down when $G$ is not compact.

Also, the exponential map is not injective either. This is always the case when $G$ has a $U(1)$ subgroup. Our trivial example is $U(1)$ itself. Consider

$$\mathcal{L}(U(1)) = \{X \in ix \in \mathbb{C} \mid x \in \mathbb{R}\}$$

We have that $g = \mathrm{Exp}(X) = \exp(ix) \in U(1)$. Our Lie algebra elements $ix$ and $ix + 2\pi i$ yield the same group element.

Let's explore the relationship between $SU(2)$ and $SO(3)$ some more. We've shown that their Lie algebras are isomorphic.

$$\mathcal{L}(SU(2)) \simeq \mathcal{L}(SO(3))$$

Now, even though $SU(2)$ is not isomorphic to $SO(3)$, We can construct a double-covering, or a globally 2: 1 map. We call this map $d : SU(2) \to SO(3)$. It's constructed by

$$A \in SU(2) \mapsto d(A) \in SO(3)$$

We represent $A$ as a matrix so that

$$d(A)_{ij} = \frac{1}{2} \operatorname{tr}_2(\sigma_i A \sigma_j A^\dagger$$

We need to first establish that our right hand side is indeed in $SO(3)$. We also need to establish that $d(A) = d(-A)$, therefore establishing that this is indeed a double cover.

This map provides an isomorphism between $SO(3)$ and our quotient

$$SO(3) \simeq SU(2)/\mathbb{Z}_2$$

We quotient by the centre $\mathbb{Z}_2 = \{I_2, -I_2\}$, which is the set of matrices which commute with everything.

Let's think about this in terms of group manifolds. We identify $\mathcal{M}(SU(2)) \sim S^3$, where we write this out as

$$S^3 \sim \left\{ \vec{x} \in \mathbb{R}^4 \mid \|x\|^2 = 1, x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \right\}$$

Our corresponding manifold $\mathcal{M}(SO(3))$ is $S^3$ on the surface with antipodal points on the three sphere identified, $\vec{x} \sim -\vec{x}$. This is the same thing as $S^3_+$, which is our upper hemisphere with $x_4 \geq 0$, along with our equator with antipodal points in our equator. So, we have that

$$\mathcal{M}(SO(3)) \simeq S^3_+ \cup \text{Equator with antipodal points identitified}$$

However, this is the same thing as our three-ball with points on the boundary identified.

( Insert figure here )

# 5 Representations

A representation is a map $D$ from a group to our set of matrices, which satisfies the property that

$$D : G \to GL_n(F) \subset \mathrm{Mat}_n(F), \quad n \in \mathbb{N}$$

such that the image matrices are non singular. In addition, this map needs to be a homomorphism, so we have that

$$D(g_1)D(g_2) = D(g_1 g_2), \quad \forall g_1, g_2 \in G$$

A representation is **faithful** if $D$ is injective. A Lie group $G$ representation $D$ is a group representation but our map $D$ **must be smooth!**.

The fact that this is a group homomorphism means that

$$D(g)D(e) = D(g), \quad \forall g \in G \quad \implies D(e) = I_n$$

In addition, the representation of a group inverse is thae same as the inverse of its representation. This is because

$$\forall g \in G, D(g^{-1})D(g) = I \implies D(g^{-1}) = D(g)^{-1}$$

Let's go into a bit more detail about the representation of a matrix Lie group. If $G$ is a matrix Lie group, $G \subset \mathrm{Mat}_n(F)$, let $D$ be a representation of $G$, with $\dim(D) = n \neq m \neq \dim(G)$. Now, construct our representation of the Lie algebra. For each $X \in \mathcal{L}(G)$, construct a curve $C : t \in \mathcal{I} \subset \mathbb{R} \to g(t) \in G$. We have the expansion

$$g(t) = I_m + Xt + O(t^2)$$

Now, we can apply our representation and construct an image curve

$$D(g(t)) \in GL(n, F) \subset \ \mathrm{Mat}\ _n(F)$$

By Taylor's theorem, we have that

$$D(g(t)) = D(I_m) + \frac{d}{dt} D(g(t)) \mid_{t=0} + O(t^2)$$

Where in this case, $D(I_m) = I_n$ in representation space. This motivates our definition for a representation of the Lie algebra of $G$. We define

$$d(X) := \frac{d}{dt} D(g(t)) \mid_{t=0}, \quad \forall X \in \mathcal{L}(G)$$

**Claim.** This is a valid representation of the Lie algebra.

*Proof.* For any $X_1, X_2 \in \mathcal{L}(G)$, construct curves

$$c_1 : t \to g_1(t) \subset G \quad g_1(0) = I_m, \dot{g}_1(0) = X_1$$
$$c_2 : t \mapsto g_2(t) \subset G, \quad g_2(0) = I_m, \dot{g}_2(0) = X_2$$

We can then define a curve which we considered previously. We have that

$$h(t) = g_1^{-1}(t) g_2^{-1}(t) g_1(t) g_2(t) \in G, \quad h(t) = I_n + t^2 [X_1, X_2] + O(t^3)$$

Now, we need to use the fact that $D$ is a representation of $G$, which means we can factor things out smoothly. So, we have that

$$D(h) = D(g_1^{-1}g_2^{-1}g_1g_2) = D(g_1)^{-1}D(g_2^{-1}D(g_1)D(g_2)\forall t$$

We now use the representation expression which we derived earlier, that

$$D(g_1(t)) = D(I_m + tX_1 + \dots) = D(I_m) + td(X_1) + O(t^2)$$

We compare this with

$$\begin{aligned} D(h(t)) &= D(I_m + t^2[X_1, X_2] + O(t^3)) \\ &= D(I_m) + t^2\frac{d}{dt^2}D(h(t)) \mid_{t=0} \\ &= I_n + d([X_1, X_2])t^2 + \dots \end{aligned}$$

We have that since

$$D(h) = D(g_1)^{-1}D(g_2)^{-1}D(g_1)D(g_2)$$

multiplying this term out, and comparing sides, we get that

$$d([X_1, X_2]) = [d(X_1), d(X_2)]$$

Also, we require that $d$ is linearity. □

As an exercise, given a representation $d$ of $\mathcal{L}(G), \forall g \in G$ of from

$$g = \text{ Exp }(X), X \in \mathcal{L}(G)$$

If we define

$$D(g) = D(\text{Exp } X); = \text{ Exp }(d(X))$$

This is a valid representation.

## 5.1 Representations of Lie algebras

If we let $\mathcal{G}$ be a lie algebra of dimension $D$, we can pull out some obvious representations.

**Definition.** (Trivial Representation). The trivial representation $d_0$ is the representation

$$d_0(X) = 0 \quad \forall X \in \mathcal{G} \implies \dim(d_0) = 1$$

**Definition.** (Fundamental representation) If $g = \mathcal{L}(G)$ for some matrix Lie group $G \subset \text{Mat}_n(F)$, we define our fundamental representation

$$d_f(X) = X, \quad \forall X \in \mathcal{G} \implies \dim(d_f) = n$$

All Lie algebras have an adjoint representation, $d_{\text{Adj}}$. Our dimension of this representation is

$$\dim(d_{\text{ Adj }}) = \dim(\mathcal{G}) = D$$

For all $X \in \mathcal{G}$ we define a linear map

$$\text{Ad}_X : \mathcal{G} \to \mathcal{G}, \quad Y \in \mathcal{G} \mapsto \text{Ad}_X(Y) = [X, Y] \in \mathcal{G}$$

We have that $\text{Ad}_X$ is equivalent to a $D \times D$ matrix. We can choose a basis

$$\mathcal{B} = \{T^a, a = 1, \dots d\}$$

Expanding out Lie algebra components in terms of this basis,

$$X = X_a T^a, \quad Y = Y_a T^a$$

Expanding out our Lie algebra components in terms of structure constants gives us

$$[X, Y] = X_a Y_b [T^a, T^b] = X_a Y_b f^{ab}_c T^c$$

then, we have that

$$[\text{Ad}_X(Y)]_c = (R_X)^b_c Y_b \implies (R_X)^b_c = X_a f^{ab}_c$$

Our adjoint representation is hence defined by

$$d_{\text{ adj }}(X) = \text{Ad}_X \forall X \in \mathcal{G}$$

This is given by components

$$[d_{\text{ Adj }}(X)]^b_c = (R_X)^b_c \forall X \in \mathcal{G}$$

For a Lie algebra $\}$ is a representation $d$

$$d : \} \to \text{Mat}_n(F)$$

such that

1. $[d(X_1), d(X_2)] = d([X_1, X_2]), \quad \forall X_1, X_2 \in \}$
2. linearity such that $d(\alpha X_1 + \beta X_2) = \alpha d(X_1) + \beta d(X_2), \quad \forall X_1, X_2 \in \}, \alpha, \beta \in F$

The dimension of our representation is just the dimension of our matrices involved. As above, they'll both be $n$ dimensional representations. Our representations (matrices) act as linear maps on a vector space

$$V \simeq F^n$$

and this is known as our representation space. Every representation comes with a representation space. Now, if we have a representation of our Lie group, there is a direct relation between representations of $G$ and of $\mathcal{L}(G)$. Take a representation $\mathcal{D}$of a matrix Lie group $G$, and for $X in \mathcal{L}(G)$, define a curve

$$C_X : \mathcal{J} \subset \mathbb{R} \to G, t \in \mathbb{R} \mapsto g_X(t)$$

with

$$g_X(t) = I + tX + \dots$$

Using this,

$$d(X) = \frac{d}{dt}(D(g(t)) \mid_{t=0} \in \mathrm{Mat}_n(F)$$

Representations are maps from groups (in this context we'll be talking about Lie groups), to some set of maps which act on a vector space. Representations respect group structure, and can be viewed as group actions on a vector space, but they have the additional condition that these maps must be **linear** maps on the vector space. Representations are important because they allow us to write down possible matrix representations of Lie groups or Lie algebras, giving us a conduit to study their properties further. We denote a representation $D$ of some group $G$ as $D(G)$, and for elements $g \in G$ their corresponding representation is denoted $D(g) \in D(G)$.

Since representations are group actions, we have that $D(G)V = V$ for any given vector space $V$. More importantly, we require that representations respect group structure by asserting that

$$D(g_1 g_2) = D(g_1)D(g_2), \quad \forall g_1, g_2 \in G$$

With this property once can show that, as is true of group actions, that $D(g)^{-1} = D(g^{-1})$ and that $D(e) = I$, the identity matrix acting on the vector space. The fact that a representations of a group element are linear maps on a given vector space $V$, imposes the condition that

$$D(g)(\alpha \mathbf{v} + \beta \mathbf{u}) = \alpha D(g)\mathbf{v} + \beta D(g)\mathbf{u} \quad \forall g \in G, \forall \mathbf{u}, \mathbf{v} \in V$$

We call the dimension of $V$ the dimension of the representation. Thus if a representation is $N$ dimensional, then $D(G) \leq GL(N)$, the group of invertible matrices over a vector field.

A representation can also be interpreted as a group homomorphism from the original group $G$ to the space of linear maps on the vector space $GL(N)$. If this map is injective, then we call the representation a **faithful** representation. In other words, a group representation is faithful when $D(g) = I \iff g = e \in G$.

To summarise, a representation is a linear group action on a vector space. It has a dimension (which may be finite or infinite), corresponding to the dimension of the associated vector space $V$.

### 5.1.1 Examples of representations of the additive group on the reals $(\mathbb{R}, +)$

We'll now cover some examples of representations of the group of real numbers $\mathbb{R}$, under addition. Since it's a group action, we require that $D$, the representation, satisfies $D(\alpha + \beta) = D(\alpha) \cdot D(\beta)$ $\forall \alpha, \beta \in \mathbb{R}$. Our first example would be the representation

$$\begin{aligned} D : \mathbb{R} &\to \mathbb{R} \\ \alpha &\mapsto \exp(\alpha) \end{aligned}$$

This is a 1-dimensional representation because we've respected the group structure, and the vector space $\mathbb{R}$ is one dimensional. The action is also faithful since $\exp(\alpha) = 1$ (the multiplicative identity) when $\alpha = 0$. On the other hand, one can check that even though

$$\begin{aligned} D : \mathbb{R} &\to SL_2(\mathbf{R}) \\ \alpha &\to \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \end{aligned}$$

is a valid representation (which one can check using double angle formulae), it's not faithful because $D(2\pi n) = I_2 \; \forall n \in \mathbb{Z}$. This representation is two dimensional.

## 5.2 A canonical representation of Lie Groups with Lie Algebras

The Lie algebra $L(G)$ of a Lie group $G$ is a vector space, as we've shown earlier. Since it's a vector space, we can construct a natural represntation of the Lie group $G$, that acts as a linear group action on it's associated Lie algebra $L(G)$. This representation is called the adjoint representation for Lie groups and is given by sending a group element $g \in G$ to a map on the Lie algebra as follows.

If we have a group element $g$, the adjoint representation of that element, denoted $\mathrm{Ad}_g$, is

$$\begin{aligned} \mathrm{Ad}_g : L(G) &\to L(G) \\ X &\mapsto gXg^{-1} \end{aligned}$$

For this to a valid representation, we need to check that the term is indeed $gXg^{-1} \in L(G)$. For this, observe that since $X \in L(G)$, there is a curve $h(t) \in G$ where $t$ is a real parameter, that when expanded infinitesimally around 1 yields the tangent vector $X$

$$h(t) = 1 + tX + \frac{t^2 X^2}{2} \dots$$

Since we have Lie group, **conjugating** this group on either side by $g$, to construct the curve $gh(t)g^{-1} \in L(G)$, gives us a new smooth curve, which has infinitesimal expansion

$$gh(t)g^{-1} = gg^{-1} + tgXg^{-1} + \dots = I + tgXg^{-1} + \dots$$

We also need to verify that this is a representation, and that it's linear. We have to show that

$$\mathrm{Ad}_{g_1 g_2} = \mathrm{Ad}_{g_1}\mathrm{Ad}_{g_2}$$

. To do this, we apply this map to an arbitrary vector $X$ and show that

$$\begin{aligned} \mathrm{Ad}_{g_1 g_2} X &= (g_1 g_2) X (g_1 g_2)^{-1} \\ &= g_1 g_2 X g_2^{-1} g_1^{-1} \\ &= g_1 (g_2 X g_2^{-1}) g_1^{-1} \\ &= \mathrm{Ad}_{g_1} \; \mathrm{Ad}_{g_2} X \end{aligned}$$

## 5.3 Classifying representations

Representations of groups could have properaties of interest, for example unitarity or orthogonality. We say that an $n$ dimensional representation is orthogonal or unitary if the maps lie in $O(n)$ or $U(n)$ respectively. For example, our representation of the additive group on the reals has an orthogonal representation

$$\alpha \mapsto \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

We can also cook up a unitary 1 dimensional representation by sending

$$\alpha \to \exp(i\alpha)$$

, for example. More strictly, we can have Hermitian or Anti-Hermitian representations if the corresponding matrices are Hermitian or Anti-Hermitian.

## 5.4 Invariant subspaces and reducibility

### 5.4.1 Invariant subspaces

Since we think of representations group actions on a vector space, we are interested in portions of this vector space which are permuted just amongst themselves. In other words, orbits which are not the entire vector space. With this in mind, we define the concept of a **invariant subspace** $W \subset V$ associated with a representation $D(G)$ and the associated vector space $V$. An invariant subspace is a subspace which satisfies

$$D(G)W = W$$

Think of this as a vector subspace which doesn't 'care' about other vector subspaces when acted on by representations. Suppose we have an invariant subspace $W$ satisfying $D(G)W = w$, where $D(G)$ is some finite orthogonal representaiton. Then, $W^\perp$ is also an invariant subspace. To show this, observe that since $D(G)$ is a finite representation, we can represent any element $D(g)$ as a matrix. Take an arbitrary element of $D(G)$ and call this matrix $A$. Now take vectors $w, v$ in $W$ and $W^\perp$ respectively. In index notation,

$$\begin{aligned}(Av) \cdot w &= A_{ij} v_j w_i \\ &= A_i j w_i v_j \\ &= (A^T)_{ji} w_i v_j \\ &= (A^T)_{ji} w_i v_j \\ &= (A^{-1})_{ji} w_i v_j\end{aligned}$$

where in the last line we used the fact that we're using an orthogonal representation. Since $A^{-1} \in D(G)$ by properties of representations, and since $W$ is an invariant subspace, the last line can be read as $u \cdot v$ where $u \in W$. And since $v \in W^\perp$, the above line is equal to zero, for arbitrary $A$. Hence, for an orthogonal representation, $W^\perp$ is an invariant subspace. If we have an invariant subspace, we can 'break' this off from the rest of representation and package it as a separate representation.

It is a natural thing to do for us to construct a basis around which these vector subspaces are written. For example, if $W \in V$ was an invariant subspace of finite dimension $M$ say, then we could construct the basis $\{w_1, w_2, \ldots, w_m\}$ which is a basis for $W$ and then extend this to the whole vector space with vectors to make the basis

$$\mathcal{B} = \{w_1, \ldots, w_M, v_{M+1}, \ldots v_N\}$$

#### 5.4.2 Irreducible representations and total irreducibility

With this in mind, we define the concept of an irreducible representation. An irreducible representation is a representation with no invariant subspaces. In other words, the orbit of any given element under $D(G)$ is the entire vector space itself.

### 5.5 A canonical representation of Lie algebras

For a Lie algebra $\mathcal{L}(G)$, we imbue more rules with regards to a representation $d$, and impose that

1. Linearity: we have that

$$d(\alpha X + \beta Y) = \alpha d(X) + \beta d(Y), \quad \forall \alpha, \beta \in F, \forall X, Y \in \mathcal{L}(G)$$

2. The representation should respect our Lie algebra, so we require that

$$d([X, Y]) = [d(X), d(Y)], \quad \forall X, Y \in \mathcal{L}(G)$$

### 5.6 Constructing a natural map of between representations of Lie groups to representations of Lie algebras

Suppose we're given a representation $D$ of a particular Lie group $G$. Then, given an element $X \in \mathcal{L}(G)$, what's a natural way to construct a representation of the corresponding Lie algebra? If we let $g(t)$ be the associated curve which gives rise to $X$, then we let

$$d(X) = \left. \frac{D(g(t))}{dt} \right|_{t=0}$$

Does this obey property 2? If we let $g(t), h(t)$ be curves corresponding to the vectors $X, Y$, then by linearity of representations we have that

$$\begin{aligned} D(g(t)) &= I + tD(\dot{g}(0)) + \dots \\ &= I + td(X) + \dots \end{aligned}$$

and similarly

$$D(h(t)) = I + tD(\dot{h}(0)) + \dots$$

If we set

$$f(t) = ghg^{-1}h^{-1}(t) \implies D(f) = D(g)D(h)D(g)^{-1}D(h)^{-1}$$

If we substitute our above expressions for $D(g), D(h)$, then our expansion is

$$D(f) = 1 + t^2[d(X), d(Y)] + \dots$$

However, from our previous analysis we know that

$$f(t) = 1 + t^2[X, Y] + \dots$$

and applying our representation here, as well as a change of variables, we have

$$Df(t) = 1 + t^2 \left. \frac{d}{d(t^2)} D([X,Y]) \right|_{t=0} + \dots$$

Comparing coefficients gives us the identity

$$d([X,Y]) = [d(X), d(Y)]$$

### 5.6.1 Remark on alternate representations

For a given Lie algebra, we have a range of possible representations which satisfy our required properties. We have the trivial representation

$$d_T(X) = 0, \quad \forall X \in \mathcal{L}(G)$$

If the Lie algebra is already a matrix vector space, we can just leave things as they are. This is called the fundamental representation

$$d_f(X) = X, \quad \forall X \in \mathcal{L}(G)$$

# 6 Representations of SU(2) in Quantum Mechanics

## 6.1 Our choice of basis

In quantum mechanics, particles have a property called spin which is quantised in units of $\mathbb{N}/2$. In this section, we learn how this property is related to the representations of the Lie Group $SU(2)$, and how this gives rise to the familiar concept of ladder operators in angular momentum. First, let's figure out a sensible basis for $\mathcal{L}(SU(2))$. We know already that the Pauli Sigma matrices

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

make a good basis, when we set

$$\mathcal{B} = \left\{ T_a \mid T_a = -\frac{i}{2}\sigma_a, a = 1, 2, 3 \right\}$$

Recall that this yields the structure constants

$$[T_a, T_b] = \epsilon_{abc} T_c$$

This basis is over the field of the reals, to construct elements of $\mathcal{L}(SU(2))$, which we can perhaps write in a more clear manner as $\mathcal{L}_{\mathbb{R}}(SU(2))$ (we're writing out the base field explictly here. However, we can choose instead the following basis which has some nice properties which we'll make use of later. We will **complexify** our basis of $\mathcal{L}(SU(2))$, and denote this complxification explicitly by writing $\mathcal{L}_{\mathbb{C}}(SU(2))$. This basis we call the Cartan-Weyl basis of the complexfied $\mathcal{L}_{\mathbb{C}}(SU(2))$ and is denoted by

$$\begin{aligned} H &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\sigma_z \\ E_+ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(\sigma_x + i\sigma_y) \\ E_- &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}(\sigma_x - i\sigma_y) \end{aligned}$$

These matrices obey the commutation relations

$$\begin{aligned} [h, e_+] &= e_+ \\ [H, E_-] &= -E_- \\ [E_+, E_-] &= 2H \end{aligned}$$

We have in interesting change of picture here. Previously, we wrote $\mathcal{L}(SU(2))$ in terms of complex matrices but with real vector coefficients. In this case, we are writing things in terms of real matrices but with complex coefficients!

$$\mathcal{L}_{\mathbb{C}}(SU(2)) = span_{\mathbb{C}} \{T^a, a = 1, 2, 3\}$$

For a general Lie algebra element, we can expand this thing as

$$X = X_H H + X_+ E_+ + X_- E^-$$

Now, the fact that $X$ is anti-Hermitian imposes conditions on these coefficients,

$$X_H = i\mathbb{R}, \quad X_+ = \overline{X}_-$$

For reasons we will discover later, we have that $H$ is a little special in this basis and we will dub it the 'Cartan' element.

The fact that we computed our commmutators means that we can view things a little differently, in terms of our Ad map. We have that

$$\begin{aligned}\mathrm{Ad}_H(E_\pm) &= \pm 2E_\pm \\ \mathrm{Ad}_H(H) &= 0\end{aligned}$$

Hence, we can view the Cartan basis as an eigenbasis of $\mathrm{Ad}_H$, with eigenvalues $0, -2, 2$.

We we ewhich are reminiscent of ladder operators we use in quantum mechanics in representing angular momentum states. We want to analyse this problem from a representation theory point of view. Our task that we want to do now is come up with a finite $N$-dimensional representation of this Lie group.

## 6.2 Roots and Weights

In representation theory, for our purposes, roots and weights will be terms to describe the eigenvalues associated with a certain basis element in our Lie algebra and its corresponding representation.

### 6.2.1 Roots

So, in our previous discussion we found that the Cartan basis diagonalised the map $\mathrm{Ad}_H$, with eigenvalues $0, -2, +2$. We shall refer to this as the roots of this system.

### 6.2.2 Weights

Now, let's consider a finite representation of $\mathcal{L}(SU(2))$ with the Cartan basis as a backdrop. Recall that a finite representation is a representation whose vector space we're acting on is of finite dimension. For this analysis, we will make a big simplifying assumption that our Cartan element $H$ has a **diagonalisable** representation. This means, when we map $H$ into representation space as a $\dim R \times \dim R$ matrix, then $R(H)$ is diagonalisable.

In other words, there's a basis set of eigenvectors such that

$$R(H)\vec{v}_\lambda = \lambda \vec{v}_\lambda, \lambda \in \mathbb{C}$$

We have that $\lambda \in \mathbb{C}$ since we're working in complexified space. In addition, these set of eigenvectors $\vec{v}_\lambda$ should also serve as a basis of our representation space $V$.

Now, the fact that representations of Lie algebras preserve the Lie bracket becomes very useful here. Since we have already worked out our commutation relations, we can deduce that

$$\begin{aligned} R(H)R(E_\pm)\vec{v}_\lambda &= \left(R(H)R(E_\pm) + [R(H), R(E_\pm)]\right)\vec{v}_\lambda \\ &= (\lambda \pm 2)\vec{v}_\lambda \end{aligned}$$

This means that given a weight, we can step up or down to get a new eigenvector $R(E_\pm)$ with a new weight $\lambda \pm 2$.

### 6.2.3 Construction of an irreducible representation

Our next thing to do in building a representation of $\mathcal{L}(SU(2))$ is to then use our finite dimensionality constraint to construct our whole set of eigenvectors. A quick outline of these steps is

1. Declare a highest weight vector.
2. Step down with $E_-$
3. Show that we have no other vectors the same eigenvalue

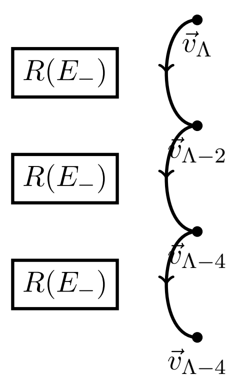
Since we have a finite dimensional representation, there must be a point where raising our vector with $R(E_+)$ terminates. This implies that there exists a vector $\vec{v}_\Lambda$ with $\Lambda$ our highest weight, such that

$$R(H)\vec{v}_\Lambda = \Lambda\vec{v}_\Lambda, \quad R(E_+)\vec{v}_\Lambda = 0$$

Now, from this object, let's see what we can do. Well, we can repeatedly apply a lowering operator on $\vec{v}_\Lambda$, and we use this procedure to build a set of eigenstates which we index as

$$\vec{v}_{\Lambda-2k} = [R(E-)]^k\vec{v}_\Lambda$$

This procedure is shown in the figure.




Our final goal for this section will be to show that these states are non-degenerate in their weights. This means that, for every given weight, we only have one associated vector. To show this, we use the assumption that our representation is **irreducible**. This assumption is important because it implies that every state in our representation space can be obtained by composing strings of $R(E_-), R(E_+)$ and $R(H)$. Using this assumption, if we prove that

$$R(E_+)\vec{v}_{\Lambda-2k} \propto \vec{v}_{\Lambda-2k+2}$$

then we are done, because this means that at we cannot access any other states other than the set $\{\vec{v}_{\Lambda-2k}, k \in \mathbb{Z}\}$. We can prove this via induction. The base case is straight forward. If we assume that the above relation holds, then extending to the case $k \to k+1$, we get

$$\begin{aligned}
R(E_+)\vec{v}_{\Lambda-2k-2} &= R(E_+)R(E_-)\vec{v}_{\Lambda-2k} \\
&= \left([R(E_+), R(E_-)] + R(E_-)R(E_+)\right)\vec{v}_{\Lambda-2k} \\
&= \left(R(H) + R(E_-)R(E_+)\right)\vec{v}_{\Lambda-2k} \\
&= R(H)\vec{v}_{\Lambda-2k} + R(E_-)K\vec{v}_{\Lambda-2k+2} \\
&= R(H)\vec{v}_{\Lambda-2k} + K\vec{v}_{\Lambda-2k} \\
&= \left(\Lambda - 2k + K\right)\vec{v}_{\Lambda-2k}
\end{aligned}$$

In this induction step we used $K$ as the constant of proportionality here. We can do even better than this and determine this constant of proportionality. Let's define our constant of proportionality recursively as

$$\vec{v}_{\Lambda-2k} = r_k \vec{v}_{\Lambda-2k+2}$$

Now, to get to the next term in the series, we apply our lowering operator to both sides. Once we apply the appropriate commutation relations, we can obtain the relations as follows

$$\begin{aligned}
R(E_-)R(E_+)\vec{v}_{\Lambda-2k} &= r_k R(E_-)\vec{v}_{\Lambda-2k+2} \\
\left([R(E_-), R(E_+)] + R(E_+)R(E_-)\right)\vec{v}_{\Lambda-2k} &= r_k\vec{v}_{\Lambda-2k} \\
\left(-R(H)\vec{v}_{\Lambda-2k} + R(E_+)\vec{v}_{\Lambda-2k-2}\right) &= r_k\vec{v}_{\Lambda-2k}
\end{aligned}$$

Now, substituting our expression for our recursion constant with $k+1$,

$$-(\Lambda - 2k)\vec{v}_{\Lambda-2k} + r_{k+1}\vec{v}_{\Lambda-2k} = r_k\vec{v}_{\Lambda-2k}$$

This yields the recursion relation that

$$r_{k+1} = r_k + \Lambda - 2k, \quad r_0 = 0$$

The boundary condition for $r_0$ can be obtained by just considering what happens at $k = 0$, we have that $R(E_+)$ annihilates $\vec{v}_\Lambda$. Writing out the first few terms explicitly, one can easily be convinced that

$$r_k = k(\Lambda - k + 1)$$

Now, our condition that $R$ is a finite dimensional representation gives us another condition that $R(E_)\vec{v}_{\Lambda-2N} = 0$ for some non-zero $\vec{v}_{\Lambda-2N}$. This implies that, going one step down, that we have

$$0 = R(E_+)\vec{v}_{\Lambda-2(N+1)} = r_{N+1}\vec{v}_{\Lambda-2N}$$

This implies that $r_{N+1} = 0$. Using our previous recursion relation

$$r_{N+1} = (N+1)(\Lambda - N - 1 + 1) = 0 \implies \Lambda = N, N \in \mathbb{N}$$

## 6.3 The structure of our representation

To quickly recap what we did before, we'll just go back over our argument on why our highest weight $\Lambda$ is indeed some non-negative integer. Since we assumed that our representation had finite dimension, this means that there exists a lowest weight $\Lambda - 2N$, for some $N \in \mathbb{N}$. Let's assign the vector which gives this lowest weight as $\vec{v}_{\Lambda-2N}$, with the property that

$$R(H)\vec{v}_{\Lambda-2N} = (\Lambda - 2N)\vec{v}_{\Lambda-2N}, \quad \vec{v}_{\Lambda-2N} \neq 0$$

Here we've made an important qualification that the lowest weight vector is assumed of course, to be non-zero. When we act on this lowest weight vector with our lowering operator, we have that

$$R(E_-)\vec{v}_{\Lambda-2N} = 0 \implies \vec{v}_{\Lambda-2N-2} = 0$$

This means that when we act on this state by the raising operator again, we have that

$$R(E_+)\vec{v}_{\Lambda-2N-2} = r_{N+1}\vec{v}_{\Lambda-2N}$$

But this implies that we must have $r_{N+1} = 0$. Our formula for our recursion coefficients thus gives us the condition which fixes $\Lambda$, since

$$r_{N+1} = (\Lambda - N)(N+1) = 0$$

Thus, this means that our highest weight $\Lambda$ must be an integer, $\Lambda = N$.

## 6.4 Conclusion

We have thus found finite dimensional irreducible representations which we call $R_\Lambda$, which are labelled by our highest weight $\Lambda \in \mathbb{N}$. In each of these representations, our weights in $R_\Lambda$ are given by

$$S_\Lambda = \{-\Lambda, -\Lambda + 2, \dots, \Lambda - 2\Lambda\} \subset \mathbb{Z}$$

This implies that the dimension of our representation $\dim(R_\Lambda) = \Lambda + 1$. Note that, when deriving the structure of these representations, we made no other assumptions of how these representations were derived. Thus, for each representation of a given dimension, it must be unique. Hence, by uniqueness of these representations, we can relate what we did here to some of the canonical representations we mentioned earlier.

- $R_0 = d_0$, our trivial representation with dimension 1
- $R_1 = d_f$, our fundamental representation with dimension 2
- $R_2 = d_{\text{adj}}$, our adjoint representation with dimension 3 (where $\mathrm{Ad}_H$ is represented by diag $(2, 0, -2)$, from acting on $H, E_\pm$ with the commutator.

For each representation, our Cartan element is represented by a diagonal matrix of weights

$$R_\Lambda(H) = \mathrm{diag}(\Lambda, \Lambda - 2, \dots, -\Lambda + 2, -\Lambda)$$

### 6.4.1 Linking this to Angular momentum in Quantum Mechanics

We can apply this to what we know by identifying objects in representation theory with operators in quantum mechanics. Let's review some basic angular momentum theory. Our total angular momentum is given by the sum of our orbital angular momentum and our spin.

Our angular momentum operator $\mathbf{J} = (J_1, J_2, J_3)$ has eigenstates labelled by $j \in \mathbb{Z}/2, j \geq 0$. This set of eigenstates is the representation space we're playing with. We denote them in their familiar form of bra and ket vectors.

$$J^2 |j; m\rangle = j(j+1) |j; m\rangle$$
$$J_3 |j; m\rangle = m |j; m\rangle$$

In the language or representation theory. we can identify this with

$$J_3 = \frac{1}{2} R(H), \quad J_\pm = J_1 \pm i J_2 = R(E_\pm)$$

Note that as a matter of convention, we're dividing our representation matrix by 2. This is because, in this case our highest weight is $\Lambda = 2j \in \mathbb{Z}_{\geq 0}$, and our other weights are given by $\lambda = 2m \in \mathbb{Z}$. Our vectors are given by $\vec{v}_\Lambda \sim |j; j\rangle$, and $\vec{v}_\lambda \sim |j; m\rangle$. Hence, when we apply $J_3$ to our eigenstate, we recover

$$J_3 |j; m\rangle = \frac{1}{2} R(H) |j; m\rangle = m |j; m\rangle$$

## 6.5 $SU(2)$ representations from $\mathcal{L}(SU(2))$

In this section, we'll go over how to obtain representations of our Lie group from representations of our Lie algebra. If we'd like to recover our lie group representation, we can locally parametrise $A \in SU(2)$ from our Lie algebra as

$$A = \text{Exp}(X), \quad X \in \mathcal{L}(SU(2)), A \in SU(2)$$

Now, motivated by this definition, we can define a representation of $SU(2)$ itself by exponentiating the representation of the Lie algebra. If we start from an irreducible representation $R_\Lambda$ of $\mathcal{L}(SU(2))$, we define the corresponding representation of the Lie group $D_\Lambda$ as

$$D_\Lambda(A) := \text{Exp}(R_\Lambda(X)), \quad \Lambda \in \mathbb{Z}_{\geq 0}$$

Now, we ask how we might obtain a representation of $SO(3) \sim SU(2)/\mathbb{Z}_2$ from this. $SO(3)$ is just $SU(2)$ but with antipodal points identified, meaning that

$$A \sim -A, \quad \forall A \in SU(2)$$

. Thus, to obtain a valid representation of $SO(3)$ that works, naturally we need that our representation satisfies

$$D_\Lambda(A) = D_\Lambda(-A), \quad \forall A \in SU(2)$$

The aim of the game now is to check that this actually works for the representation we've already presented here for $SU(2)$. There's a simple trick to check this. Notice that by the linearity property of representations, the condition above is equivalent to just checking that

$$D_\Lambda(I) = D_\Lambda(-I), \Lambda \in \mathbb{Z}_{\geq 0}$$

Let's check this. What follows is a clever proof that hinges on looking at what happens when we exponentiate $H$. From our definition of $H$, we have that

$$\text{Exp}(i\pi H) = \text{Exp}(\text{diag}(e^{i\pi}, e^{-i\pi})) = -I_2$$

This implies that

$$D_\Lambda(-I_2) = \text{Exp}(i\pi R_\Lambda(H)) = \text{Exp}(\text{diag}(e^{i\pi\Lambda}, e^{i\pi(\Lambda-2)}, \ldots, e^{-i\pi\Lambda}))$$

This implies that $D_\Lambda(-I_2)$ has eigenvalues $\exp(i\pi\lambda) = (-1)^\lambda = (-1)^\Lambda$. Hence, we have that

$$D_\Lambda(-I_2) = D_\Lambda(I_2) = I_{\Lambda+1}, \quad \Lambda \in 2\mathbb{Z}$$

The last equality comes from the property of representations where the identity element in the base space is sent to the identity element in the representation space. Since, we need $D_\Lambda(-I_2)$ to be the identity element, we must have that all of its eigenvalues are 1. This means that we have a representation of $SO(3)$ only under certain conditions. If $\Lambda$ is even, we have a representation of both $SU(2)$ and $SO(3)$. But otherwise, we only have a representation of $SU(2)$ ! This is and interesting fundamental fact.

## 6.6 New representations from old

Given $R$, a representation of a real lie algebra $\mathcal{G}$, we can define a conjugate representation by conjugating our existing representation.

$$\overline{R}(X) = R(X)^*, \forall X \in \mathcal{G}$$

Sometimes, we have that $\overline{R} \simeq R$. We can also glue two representations together. Given representations $R_1$ and $R_2$, any representation spaces $V_1, V_2$ of dimensions $d_1$ and $d_2$, we can define the direct sum of representations. We denote this as $R_1 \oplus R_2$.

$$V_1 \oplus V_2 = \{v_1 \oplus v_2 \mid v_1 \in V_1, v_2 \in V_2\}$$

In this formalism, we have that acting on a Lie algebra gives us

$$(R_1 \oplus R_2)(X)(v_1 \oplus v_2) = (R_1(X)) \oplus (R_2(X))$$

This is represented by 2 block diagonal matrices. In the representation space of our direct sum, our matrix is

$$C = \left[\begin{array}{c|c} R_1(X) & \mathbf{0} \\ \hline \mathbf{0} & R_2(X) \end{array}\right]$$

Now let's break this down for a physical system of multiparticle states. Suppose that we have two particles represented as

$$|\uparrow\rangle_1 , |\downarrow\rangle_1 \in \mathcal{H}_1$$
$$|\uparrow\rangle_2 , |\downarrow\rangle_2 \in \mathcal{H}_2$$

We can compose this to get two particle states, for example

$$|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2$$

We discussed earlier that if we had two representations $R_1$ and $R_2$ of $\mathcal{G}$, with representation spaces $V_1$ and $V_2$ , and dimensions $d_1$ and $d_2$ , we can represent an element in the direct sum representation as $(R_1 \oplus R_2)(X)$, represented as a block diagonal matrix with dimension $d_1 + d_2$.

## 6.7 Tensor products

In addition to doing a direct sum to combine two representations of a Lie algebra, we can also do a tensor product. Given vector spaces $V_1$ and $V_2$, we can define the tensor product space as

$$V_1 \otimes V_2 = span_F \{v_1 \otimes v_2 : v_1 \in V_1, v_2 \in V_2\}$$

Formally, we have the rules of linearity in both arguments, as well as scalar multiplcation, giving the properties

$$(v_1 + w_1) \otimes (v_2 + w_2) = v_1 \otimes v_2 + w_1 \otimes v_2 + v_1 \otimes w_2 + w_1 \otimes w_2$$
$$\alpha(v_1 \otimes v_2) = (\alpha v_1) \otimes v_2 + v_1 \otimes (\alpha v_2)$$

Just note that the scalar multiplcation property may look a bit weird, but this is really the only definition that we can think of which makes senses with regards to symmetry. Where the above properties hold for $v_i, w_i \in V_i$. We can also define the tensor products of two linear maps, $M_1 : V_1 \to V_1, M_2 : V_2 \to V_2$, we can define a new map, the tensor product map

$$M_1 \otimes M_2 : V_1 \otimes V_2 \to V_1 \otimes V_2$$

which is defined by acting the linear map on each of their respective spaces

$$(M_1 \otimes M_2)(v_1 \otimes v_2) = (M_1 v_1) \otimes (M_2 v_2) \in V_1 \otimes V_2$$

where we can calcukate this thing explicitly by extending it via linearity. Now, we can do this given representations $R_1$ and $R_2$ of $\mathcal{G}$ with representations $V_1$ and $V_2$ . If we have, for all $X \in \mathcal{G}$,

$$R_1(X) : V_1 \to V_1$$
$$R_2(X) : V_2 \to V_2$$

we can now define a new representation $R_1 \otimes (R_2)$ with representation space $V_1 \otimes V_2$ so that, for all $X \in \mathcal{G}$

$$(R_1 \otimes R_2)(X) : V_1 \otimes V_2 \to V_1 \otimes V_2$$
$$(R_1 \otimes R_2)(X) = R_1(X) \otimes I_2 + I_1 \otimes R_2(X)$$

Note, this is not equal to the map $R_1(X) \otimes R_2(X)$. We can start calculating things explicity. If we have two bases

$$B_1 = \left\{ V_1^j : j = 1, \dots d_1 \right\}, \quad B_2 = \{ V_2^\alpha, \alpha = 1, \dots d_2 \}$$

we can represent these things as a matrix for $R_1 \otimes R_2$, with $i, j = 1, \dots d_1$, and $\alpha, \beta = 1, \dots d_2$. Our matrix index is given by

$$(R_1 \otimes R_2)_{ij,\alpha\beta}(X) = R_1(X)_{ij} I_{\alpha\beta} + I_{ij} R_2(X)_{\alpha\beta}$$

This representation $R_1 \otimes R_2$ has dimension $d_1 d_2$ . We should check that this is actually a valid representation, and one can prove this easily by verifying that the Lie bracket is preserved. The only non-trivial thing to recall is that in the proof, $R_1 \otimes I$ and $I \otimes R_2$ commute since they act on different spaces.

## 6.8 Reducibility

A representation $R$ with representation space $V$ has an invariant subspace $R(X)u \in U, \quad \forall X \in \mathcal{G}, u \in U, U \subset V$. An irreducible representation has no non trivial subspaces. We define a **fully reducible** representation as a representation which can be written as the direct sum of irreducible representations.

On other words, if $R$ has a non trivial invariant subspace, then we can find a basis such that

$$R(X) = \begin{pmatrix} A(X) & B(X) \\ 0 & C(X) \end{pmatrix}, \quad \forall X \in \mathcal{G}$$

In this case, elements of $U$ correspond do vectors of the form $(\vec{u}, 0)$. We can see that when we act on a vector of this type by the matrix above, it stays in this form. If $R$ is fully reducible, then $R = R_1 \oplus R_2 \cdots \oplus R_l$. This means that we have a basis where, $\forall X \in \mathcal{G}$, $R(X)$ is represented by a block diagonal matrix where $R(X) = diag(R_1(X), R_2(X), \dots R_l(X)$, where $R_i(X)$ are the matrix representations of each.

**Theorem 1.** (Tensor product irreducible representations of a simple Lie algebra) Now, we will state an important fact, which we will not prove. If $R_i = 1, \dots m$ are finite dimensional irreducible representations of a simple Lie algebra, then the tensor product $(R_1 \otimes \cdots \otimes R_m)$ is fully reducible (we can express this as the direct sum of In other words, we have that

$$R_1 \otimes R_2 \otimes \cdots \otimes R_m \simeq \tilde{R}_1 \oplus \tilde{R}_2 \oplus \dots \tilde{R}_{m'}$$

## 6.9 Tensor products of the $\mathcal{L}(SU(2))$ Representations

Let $R_\Lambda$ and $R_{\Lambda'}$ be irreducible representations of $L(SU(2))$, where $\Lambda, \Lambda'$ are the highest weights in $\mathbb{N}$. Hence our dimensions of these representations are $\dim(R_\Lambda) = \Lambda + 1$, and $\dim(R_{\Lambda'}) = \Lambda' + 1$, with representations $V_\Lambda$ and $V'_\Lambda$. We can form tensor product representation space $R_\Lambda \otimes R_{\Lambda'}$

$$V_\Lambda \otimes V_{\Lambda'} = \text{span}_{\mathbb{C}} \left\{ v \otimes v' : v \in V_\Lambda, v' \in V_{\Lambda'} \right\}, \quad X \in \mathcal{L}(SU(2))$$

Our representations act on $X$ as

$$R_\Lambda \otimes R_{\Lambda'}(X)(v \otimes v') = (R_\Lambda(X)v) \otimes v' + v \otimes (R_{\Lambda'}(X)v')$$

This yields a fully reducible representation of $\mathcal{L}(SU(2))$. Since our basis for this tensor product representation was every possible tensor product of $v_\lambda$ and $v_{\lambda'}$ in the weight set, we have that our dimension of this representation is

$$\dim(R_\Lambda \otimes R_{\Lambda'}) = (\Lambda + 1)(\Lambda' + 1)$$

We appeal to the theorem that we mentioned in the previous section. Since $R_\Lambda$ and $R_{\Lambda'}$ are, by construction, irreducible representations, we can then say that

$$R_\Lambda \otimes R_{\Lambda'} = \bigoplus_{\Lambda'' \in \mathbb{Z}_{\geq 0}} \mathcal{L}^{\Lambda''}_{\Lambda\Lambda'} R_{\Lambda''}$$

we call $\mathcal{L}^{\Lambda''}_{\Lambda,\Lambda'} \in \mathbb{Z}$ the Littlewood coefficients. We have that $v_\Lambda$ has a basis $\{v_\Lambda\}$ which are all eigenvectors of $R_\Lambda(H)$, this has weights in the weight set $S_\Lambda = \{-\Lambda \dots \Lambda\}$ In addition, we denote the weight set for $V_{\Lambda'}$ as $S_{\Lambda'} = \{-\Lambda', \dots, \Lambda'\}$ Our aim is to now construct a basis of our tensor product space $V_\Lambda \otimes V_{\Lambda'}$. This basis is just the tensor product of the basis vectors from each individual space.

$$\mathcal{B} = \left\{ v_\lambda \otimes v'_\lambda : \quad \lambda \in S_\Lambda, \lambda' \in S_{\Lambda'} \right\}$$

To find the weights, we apply the representation matrix of our Cartan element.

$$\begin{aligned}(R_\Lambda \otimes R_{\Lambda'})(H)(v_\lambda \otimes v'_\lambda) &= (R_\Lambda(H)) \otimes v'_{\lambda'} + v_\lambda \otimes (R_{\Lambda'}(H)v'_{\lambda'}) \\ &= (\lambda + \lambda')(v_\lambda \otimes v_{\lambda'})\end{aligned}$$

Thus, the tensor product of the eigenvectors is also an eigenvector of the tensor product of the representations. We thus deduce that the tensor product representation $R_\Lambda \otimes R_{\Lambda'}$ has weight set $S_{\Lambda,\Lambda'} = \{\lambda + \lambda' : \lambda \in S_\Lambda, \lambda' \in S_{\Lambda'}\}$. This notation is a little funny, because we don't strictly mean the **set** of these objects, but the set including multiplicities. We now aim to decompose this weight set into the irreducible weight sets of lower dimensional representations. We can see here that the highest weight need be $\Lambda + \Lambda'$, (it's just the sum of the highest weights). This has multiplicity one since this is the only way to add the two things to get the weight. This implies that $R_{\Lambda+\Lambda'}$ appears only once in our irreducible representation. This means that our Littlewood coefficient

$$L^{\Lambda+\Lambda'}_{\Lambda,\Lambda} = 1$$

Thus, we can factor this out to get

$$R_\Lambda \otimes R_{\Lambda'} = R_{\Lambda+\Lambda'} \oplus \tilde{R}_{\Lambda,\Lambda'}$$

The remainder representation $\tilde{R}_{\Lambda,\Lambda'}$ has weight set $\tilde{S}_{\Lambda,\Lambda'}$ where the whole weight set is the union of this and the highest weight set

$$\tilde{S}_{\Lambda,\Lambda'} = S_{\Lambda+\Lambda'} \cup \tilde{S}_{\Lambda,\Lambda'}$$

where we have that the weight set

$$S_{\Lambda+\Lambda'} = \left\{ -\Lambda - \Lambda', \dots, \Lambda + \Lambda' \right\}$$

Now we find the highest weight of $\tilde{R}_{\Lambda,\Lambda'}$, and so on.

**Example.** Consider adding the representations $\Lambda = 1, \Lambda' = 1$ , with $S_1 = \{-1, +1\}$. When we add these weight sets together, we get that

$$S_{1,1,} = \{-1, +1\} + \{-1, +1\} = \{-2, 0, 0, 2\}$$

Generating the weight set associated with the highest weight representation, we decompose this thing as

$$\cdots = \{-2, 0, 2\} \cup \{0\} \implies R_1 \otimes R_1 = R_2 \oplus R_0$$

As exercise, try to decompose general $\Lambda' = M, \Lambda = N$ ,

# 7 The Killing Form

We will now create a structure on our Lie algebra which is somewhat analogous to a metric on vectors (which we covered in the general relativity notes). Given a vector-space like Lie algebras, it is of interest for us to define a scalar product on the Lie algebra, which takes two vectors and returns a scalar.

Let's first define an inner product first.

**Definition.** (Inner product) Given a vector space $V$ over a field $F$, an inner product is a bilinear map

$$i : V \times V \to F$$

where this map is symmetric. This is notion is analogous to the usual dot product we all know and love, but we've made things a bit more general.

Right now, we'll define the inner product in this way. However, we'll want to make an extra definition, motivated by the fact that we don't care about inner products where, when we contract a specific element with all others in a lie algebra, we get zero.

**Definition.** (Non-degeneracy) We say that the inner product $i$ is non-degenerate if for all $v \in V (v \neq 0)$, there is a $w \in V$ such that $i(v, w) \neq 0$. With this, our concept of an inner product looks more like a metric.

Now, a natural question to pose is if there is a natural inner product which we can write down for a Lie algebra? What does the term 'natural' even mean? We'll answer this question later. For the first question The answer is yes; our answer is the **killing form**.

**Definition.** (Killing form) Our Killing form is a map from the Lie algebra $\mathcal{G}$

$$\kappa : \mathcal{G} \times \mathcal{G} \to F$$

defined as the map which takes two $X, Y \in \mathcal{G}$, which takes

$$K(X, Y) = \operatorname{tr}(\operatorname{Ad}_X \circ \operatorname{Ad}_Y)$$

So, we are taking the trace of the composition of tow adjoint representations. By the cyclic property of trace, we have that this object is symmetric since we can switch the adjoint maps around and still have the same map. Hence, it's still and inner product. In addition, since the adjoin maps are linear in both arguments, we have that this map is linear.

Now, what does this map look like in terms of a basis of our Lie algebra? Let's look at the map

$$(\operatorname{Ad}_X \circ \operatorname{Ad}_Y) : \mathcal{G} \to \mathcal{G}$$

we write out the definition of the adjoint map explicitly, which is the commutator. Composing two commutator operations means that some $Z$ is mapped as

$$Z \in \mathcal{G} \to [X, [Y, Z]] \in \mathcal{G}$$

What is the matrix representation of this map? To do this, we construct a basis $\{T^a : a = 1, \ldots, D\}$ for $\mathcal{G}$. Writing out our components, and our structure constants explicitly,

$$X = X_a T^a, \quad Y = Y_a Y^a, \quad Z = Z_a T^a, \quad [T^a, T^b] = f^{ab}_c T^c$$

Multiplying components out, we find that

$$\begin{aligned} [X.[Y,Z]] &= X_a Y_b Z_c [T^a, [T^b, T^c]] \\ &= X_a Y_b Z_c f^{ad}_e f^{bc}_d T^e \\ &= M(X,Y)^c_e Z_c T^e \end{aligned}$$

In this expression, we have that

$$M(X,Y)^c_e = X_a Y_b f^{ad}_e f^{bc}_d$$

Taking the trace of this map, we find the components explicitly by taking the trace

$$K(X,Y) = \mathrm{tr}_D[M(X,Y)] = K^{ab} X_a Y_b$$

where $\kappa^{ab} = f^{ad}_c f^{bc}_d$.

What does the term natural mean? It means that the map $\kappa$ should be invariant under the adjoint action $\mathcal{G}$. This action condition is given by

$$\kappa([Z,X],Y) + \kappa(X,[Z,Y]) = 0, \forall X, Y, Z \in \mathcal{G}$$

We now prove this invariance condition.

**Theorem.** ($\kappa$ is invariant under the adjoint action ) Writing out the term on the left explicitly,

$$\kappa([Z,X],Y]) = \mathrm{tr}\left[\mathrm{Ad}_{[Z,X]} \circ \mathrm{Ad}_Y\right]$$

the defining property of our adjoint representation is that

$$\mathrm{Ad}_{[Z,X]} = (\mathrm{Ad}_Z \circ \mathrm{Ad}_X - \mathrm{Ad}_X \circ \mathrm{Ad}_Z)$$

This means that the above reads

$$\cdots = \mathrm{tr}[\mathrm{Ad}_Z \circ \mathrm{Ad}_X \circ \mathrm{Ad}_Y] - \mathrm{tr}[\mathrm{Ad}_X \circ \mathrm{Ad}_Z \circ \mathrm{Ad}_y]$$

Similarly, applying this to the second term, we find that

$$\kappa(X,[Z,y]) = \mathrm{tr}[\mathrm{Ad}_X \circ \mathrm{Ad}_Z \circ \mathrm{Ad}_Y] - \mathrm{tr}[\mathrm{Ad}_X \circ \mathrm{Ad}_Y \circ \mathrm{Ad}_Z]$$

By cyclicity of the trace, this evaluates to zero.

Given a matrix Lie group $G$, we can define the adjoint action of $G$ on $\mathcal{L}(G)$. For all $g \in G$, since we're dealing with matrix Lie groups, we can do inverses and multiply Lie algebra elements with $g \in G$. The action is defined as

$$X \in \mathcal{L}(G) \mapsto gXg^{-1} \in \mathcal{L}(G)$$

Our Killing form is invariant of this action. In particular, we have that

$$K(X,Y) = \kappa(gXg^{-1}, gYg^{-1}), \quad \forall X, Y \in \mathcal{L}(G), \forall g \in G$$

We can recover the invariance property for Killing forms by writing $g = \text{Exp}(-tZ)$. Expanding out, we get the invariance property.

**Definition.** (Semi-Simple) A Lie algebra $\mathcal{G}$ is semi-simple if it has no Abelian ideals.

**Theorem.** (Cartan) We have that $\kappa$ is a non-degenerate Killing form $\iff$ $\mathcal{G}$ is semi-simple. We will prove later on that any semi-simple $\mathcal{G}$ is a direct sum of simple Lie algebras, so we can decompose $\mathcal{G}$ is

$$\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \cdots \oplus \mathcal{G}_m$$

*Proof.* We will show first that $\mathcal{G}$ is not semi-simple implies that $\kappa$ is degenerate. If $\mathcal{G}$ is not semi-simple, then it has an Abelian ideal, which we will call $\mathcal{J} \subset \mathcal{G}$, with $\dim(\mathcal{G}) = D$ and $\dim(\mathcal{J}) = d \leq D$. Construct a basis of the algebra $\{T^a\}$ with $a = 1, \dots D$, which we write as

$$\mathcal{B} = \{i = 1, \dots d\} \cup \{T^\alpha, \alpha = 1, \dots D - d\}$$

where we have $T^i$ spanning $\mathcal{J}$. As $\mathcal{J}$ is Abelian, we have by definition that

$$[T^i, T^j] = 0, \forall i, j$$

We also have that since it's an ideal,

$$[T^\alpha, T^j] = f^{\alpha j}_k T^k \in \mathcal{J}$$

We thus have

$$f^{ij}_a = 0, \quad f^{\alpha j}_\beta = 0$$

Now suppose that we have $X = X_a T^a \in \mathcal{G}$,and $Y = Y_i T^i \in \mathcal{J}$. If we write out our Killing form

$$K[X,Y] = K^{ai} X_a Y_i$$

then we have that

$$\begin{aligned} \kappa^{ai} &= f^{ad}_c f^{ic}_d \\ &= f^{aj}_\alpha f^{i\alpha}_j \\ &= 0 \end{aligned}$$

Thus, for all $Y \in \mathcal{J}$ we have that

$$\kappa(Y, X) = 0, \quad \forall X \in \mathcal{G}$$

which implies that this killing form is degenerate. □

# 8 The Cartan Classification

in this section, we will classify all finite dimensional, simple, complex $\mathcal{G}$. This is based on work done by Cartan in 1894. When we were looking at the Lie algebra $\mathcal{L}(SU(2))$, we chose the convenient basis $\{H, E_-, E_+\}$ because $H$ was nice and diagonal. But, there's a deeper reason why we chose this basis - it's part of a wider class of bases which have really nice properties.

**Definition.** (Ad-diagonalisability) We say that a Lie algebra element $X \in \mathcal{G}$ is ad-diagonalisable (AD) if the map

$$\mathrm{Ad}_X : \mathcal{G} \to \mathcal{G} \text{ is diagonalisable}$$

In the case of $\mathcal{L}(SU(2))$, this was our Cartan element $H$.

We can turn the set of ad-diagonalisable elements into a subalgebra of our Lie algebra. This is called the Cartan subalgebra.

**Definition.** (Cartan Subalgebra) A Cartan subalgebra $h \in \mathcal{G}$ is a maximal abelian subalgebra consisting of AD elements. Recall that what we mean by abelian is that the vectors in the Lie algebra commute under the Lie bracket. This is defined as follows.

1. If $H \in h \implies H$ is ad-diagonalisable by definition.
2. $H, H' \in h \implies [H, H'] = 0$.
3. If $X \in \mathcal{G}$ and $[X, H] = 0, \forall H \in h$, then we necessarily have that $X \in h$.

In fact, we have that all possible Cartan subalgebras of $\mathcal{G}$ are isomorphic and have the same dimension.

$$r = \dim[h] \in \mathbb{N} \text{ defined as the rank of } \mathcal{G}$$

**Example.** Consider the Lie algebra $\mathcal{G} = \mathcal{L}_{\mathbb{C}}(SU(2)) = \mathrm{span}_{\mathbb{C}} \{H, E_-, E_+\}$. We only have one diagonal element, where $H = \sigma_3$ is AD. Specifically,

$$[H, E_\pm] = \pm 2E_\pm, [H, H] = 0$$

Observe that however, $E_\pm$ are not diagonalisable. These are not AD.

Let's consider another example.

**Example.** Suppose that $h = \mathrm{span}_{\mathbb{C}} \{H\}$ is a choice of a Cartan subalgebra. Choose a basis $\{H^i, i = 1, \dots r\}$, such that $[H^i, H^j] = 0 \forall i, j$. Consider $\mathcal{L}_{\mathbb{C}}(SU(N))$, the set of traceless $n \times n$ matrices. Now, diagonal elements of } provide a choice of Cartan subalgebra! Take $\alpha, \beta = 1, \dots N$. We can construct this basis by choosing

$$\left(H^i\right)_{\alpha\beta} = \delta_{\alpha i}\delta_{\beta i} - \delta_{\alpha, i+1}\delta_{\beta, i+1}, \quad i = 1, \dots, N-1$$

This shows that our rank $[\mathcal{L}_{\mathbb{C}}(SU(N)) = N - 1$. We have that our basis elements $[H^i, H^j] = 0, \forall i, j = 1, \dots r$. By the property of the adjoint representation, we have that

$$(\mathrm{Ad}_{H^i} \circ \mathrm{Ad}_{H^j} - \mathrm{Ad}_{H^j} \circ \mathrm{Ad}_{H^i}) = 0$$

This implies $r$ linear maps $\text{Ad}_{H^i} : \mathcal{G} \to \mathcal{G}$ are simultaneously diagonalisable. So $\mathcal{G}$ is spanned by the simultaneous eigenvectors of $\text{Ad}_{H_i}$.

$$\text{Ad}_{H^i}(E^\alpha) = [H^i, E^\alpha] = \alpha^i E^\alpha, \quad i = 1, \dots, r$$

This allows us to define a basis called the **Cartan Weyl** basis, which is a basis constructed from the diagonal elements as well as the step down and step up operators.

For example for $\mathcal{G} = \mathcal{L}_{\mathbb{C}}(SU(2))$, our Cartan-Weyl basis was the set $\{H, E_-, E_+\}$, with the diagonalisable element being $H$, with the step operators being $E_\pm$.

We define something more general called a **Cartan Subalgebra**. The Cartan subalgebra is a subalgebra $h \subset \mathcal{G}$ , of dimension $r = \text{Rank}[\mathcal{G}]$. We construct a basis of $h$ as

$$\left\{H^i, i = 1 \dots r\right\}, \quad [H^i, H^j] = 0 \quad \forall i, j$$

In this case, we can try to diagonalise the adjoint action for each of these $H^i$. This is possible, by the defining property of representations. $\mathcal{G}$ is spanned by the simultaneous eigenvectors of the ad maps

$$\text{Ad}_{H^i} : \mathcal{G} \to \mathcal{G}$$

We can see that this is true for $\mathcal{L}(SU(2))$. Starting off with zero eigenvalues, the eigenvectors are just in the Cartan subalgebra themselves, in other words the set $\left\{H^i, i = 1, \dots, r\right\}$. This is because by construction, we have that

$$\text{Ad}_{H^i}(H^j) = [H^i, H^j] = 0 \quad \forall i, j$$

Now, what about non-zero eigenvectors? Consider the indexed set $\{E^\alpha, \alpha \in \Phi\}$. Because this is as eigenvector, we have that

$$\text{Ad}_{H^i}(E^\alpha) = [H^i, E^\alpha] = \alpha^i E^\alpha$$

Since we're working in a complex basis, we must have that $\alpha^i \in \mathbb{C}$, which are not all zero (otherwise it would lie in the Cartan subalgebra). We call the objects $\alpha$ as **roots** of the Lie algebra.

Now lets say a little bit more about the roots. For a general element of the Cartan subalgebra $h$, $H \in h$. As we have a basis of the Cartan subalgebra, we can write

$$H = e_i H^i, \quad e_i \in \mathbb{C}$$

By linearity, we have that the bracket of $H$ with a step generator can be expanded, as follows.

$$[H, E^\alpha] = \alpha(H) E^\alpha, \quad \alpha(H) = e_i \alpha^i \in \mathbb{C}$$

This is because, upon expanding $H$, we get that

$$\begin{aligned}
[H, E^\alpha] &= [e_i H^i, E^\alpha] \\
&= e_i [H^i, E^\alpha] \\
&= e_i \alpha^i E^\alpha \\
&= e_i \alpha^i E^\alpha \quad \text{define } e_i \alpha^i = \alpha(H) \in \mathbb{C}
\end{aligned}$$

Thus, a more sophisticated way to think about roots is to think of them as $\alpha : h \to \mathbb{C}$. Thus, they're elements of the dual vector space $\alpha \in h^*$, where we write $h^*$ as the dual space. Now we may have degeneracy in this case, but we can prove that the roots are non-degenerate if $\mathcal{G}$ is simple. However, we will not prove this in these notes. This means that the set of roots $\Phi$ consist of $d - r$ distinct vectors by non-degeneracy, which are distinct elements of $h^*$. This is for $d = \dim[\mathcal{G}]$, $r = \dim[h]$.

**Definition.** (Cartan Weyl Basis) We can write the basis of the Lie algebra in terms of ad-diagonalisable elements and the step operators.

$$\mathcal{B} = \left\{H^i, i = 1, \dots r\right\} \cup \{E^\alpha, \alpha \in \Phi\}$$

So far, we haven't really used the assumption that the Lie algebra is simple ( other than for the non-degeneracy condition). Using Cartan's theorem, we remind ourselves that $\mathcal{G}$ being simple implies that the Killing form

$$\kappa(X, Y) = \frac{1}{N} \operatorname{tr}[\mathrm{Ad}_X \circ \mathrm{Ad}_Y]$$

is non-degenerate. With this, we can play around with the structure of the Killing form in the Cartan-Weyl basis. We'll start by proving two things.

**Theorem.**

1. $\forall H \in h, \alpha \in \Phi$, we have that the Killing form vanishes for

$$\kappa(H, E^\alpha) = 0$$

This is somewhat of a nice statement because we can interpret this as the vectors $E^\alpha$ being orthogonal to the vectors in our Cartan subalgebra $H$.

2. $\forall \alpha, \beta \in \Phi$, and $\alpha + \beta \neq 0$, we have that

$$\kappa(E^\alpha, E^\beta) = 0$$

This is somewhat of a non-obvious condition!

*Proof.* Throughout this proof, we'll use Jacobi identity, the adjoint representation, and Killing form invariance. We'll prove the first statement.

1. Consider $\alpha(H')\kappa(H, E^\alpha)$ for some arbitrary $H'$ in the Cartan subalgebra. Note that, we can't possibly have $\alpha(H') = 0$ for all $H'$ in the subalgebra because, by maximality, this would then imply that $[H', E^\alpha] = 0$ for all $H'$ , which implies that $E^\alpha$ was in the subalgebra (false by assumption). By linearity, we pull the root side the killing form in the first line. We have that

$$\begin{aligned}
\alpha(H')\kappa(H, E^\alpha) &= \kappa(H, [H', E^\alpha]) \\
&= -\kappa([H', H], E^\alpha) \text{ by the invariance property of the Killing form} \\
&= -\kappa(0, E^\alpha) \text{ by the Abelian property of the Cartan subalgebra} \\
&= 0
\end{aligned}$$

Now, $\alpha(H') \neq 0$ from $H'$, implies that

$$\kappa(H, E^\alpha) = 0$$

This proves the first statement.

2. Again, consider $H' \in h$. Now, consider

$$(\alpha(H') + \beta(H'))\kappa(E^\alpha, E^\beta) = \kappa([H', E^\alpha], E^\beta) + \kappa(E^\alpha, [H', E^\beta])$$
$$= 0 \text{ by the invariance property}$$

Now, provided we assume that $\forall \alpha, \beta \in \Phi$, with $\alpha + \beta \neq 0$ then by maximality that $\alpha(H') + \beta(H') \neq 0$ for some $H'$. This in turn then means that the other factor must be zero and hence we have that

$$\kappa(E^\alpha, E^\beta) = 0$$

□

**Theorem.** (Non-degenracy of the Killing form result) We have that for all $H \in h$, there exists a $H'$ such that $\kappa(H, H') \neq 0$.

*Proof.* For some $H \in h$, assume the converse that $\kappa(H, H') = 0, \quad \forall H' \in h$. From i), we have that $\kappa(H, E^\alpha) = 0, \quad \forall \alpha \in \Phi$. This means that $\kappa(H, X) = 0$, for all $X \in \mathcal{G}$, which implies that $\kappa$ is degenerate. Contradiction. □

The significance of this theorem is that we've shown that $\kappa$ is not only non-degenerate on the whole space, but is also a non-degenerate inner product on $h$. In particular, if we write the inner product for any two elements $H, H'$ as

$$\kappa(H, H') = \kappa^{ij} e_i e'_j, \quad H = e_i H^i, H' = e'_i H^i$$

The condition of non-degeneracy implies that $\kappa^{ij} = \kappa(H^i, H^j)$ is an invertible $n \times n$ matrix. This means that we can find an inverse.

$$\exists (\kappa^{-1})_{ij} \implies (\kappa^{-1})_{ij}\kappa^{jk} = \delta_i^k$$

Thus, like in general relativity, it's natural to think about lowering indices and transferring things over to the dual space.

Now, $\kappa^{-1}$ defines a non-degenerate inner product on the dual space $h^*$. Suppose we have a root $\alpha$ so that

$$[H^i, E^\alpha] = \alpha^i E^\alpha, \quad [H^i, E^\beta] = \beta^i E^\beta$$

We can now define the inner product $(\alpha, \beta)$, as

$$(\alpha, \beta) = (\kappa^{-1})_{ij}\alpha^i \beta^j$$

**Theorem.** (A result on the roots) If we have $\alpha \in \Phi$, then necessarily we have that $-\alpha \in \Phi$ , and that $\kappa(E^\alpha, E^-\alpha) \neq 0$.

*Proof.* From i), we have that $\kappa(E^\alpha, H) = 0, \forall H \in h$. From ii), we have that $\kappa(E^\alpha, E^\beta) = 0, \forall \beta \in \Phi$ with $\alpha \neq -\beta$. Hence, unless $-\alpha \in \Phi$, and $\kappa(E^\alpha, E^{-\alpha}) \neq 0$, we have that $\kappa(E^\alpha, X) = 0, \quad \forall X \in \mathcal{G}$. This implies that $\kappa$ is degenerate. □

## 8.1 Algebra in the Cartan Weyl basis

### 8.1.1 Working out the Bracket of $\left[E^\alpha, E^\beta\right]$

In this section, we'll work on fully fleshing out the Lie algebra relations in our Cartan Weyl basis. Let's summarise some of the relations which we we have so far. By construction, elements in the Cartan subalgebra commute, and we've defined eigenvectors with weights.

$$[H^i, H^j] = 0, \quad \forall i, j1, , \ldots, r$$
$$[H^i, E^\alpha] = \alpha^i E^\alpha \quad \forall \alpha \in \Phi$$

The only thing we've yet to do is explore what happens when we commute two of the eigenvectors, say $E^\alpha$ and $E^\beta$. It remains to evaluate $[E^\alpha, E^\beta], \forall \alpha, \beta \in \Phi$. This is an eigenvector of elements in the Cartan subalgebra. Using the Jacobi identity,

$$[H^i, [E^\alpha, E^\beta]] = -[E^\alpha, [E^\beta, H^i]] - [E^\beta, [H^i, E^\alpha]] = (\alpha^i + \beta^i)[E^\alpha, E^\beta]$$

This means that we get one extra bracket to our set of relations. Now, if $\alpha + \beta \neq = 0$, then we have a non-zero eigenvalue here, and by non-degeneracy of eigenvalues this implies that $\left[H^I, \left[E^\alpha, E^\beta\right]\right]$ is proportional to the $\alpha + \beta$ eigenvector. This, If $\alpha + \beta \neq 0$,

$$[E^\alpha, E^\beta] = N_{\alpha, \beta} E^{\alpha + \beta}, \quad \text{if } \alpha + \beta \in \Phi$$

or is 0 otherwise. By the way, we could also have that $N_{\alpha,\beta}$ is also zero. If This means that putting a bracket around two step operators gives the summed step operator only if the roots are in the right place. Now, what happens when $\alpha + \beta = 0$? Then, we learn that

$$[H^i, [E^\alpha, E^{-\alpha}]] = 0, \quad \forall i \in 1, \ldots r$$

This implies that actually $[E^\alpha, E^{-\alpha}] \in h$, since our Cartan subalgebra is defined to be the maximal subalgebra of vectors which commute with each other.

**Example.** (Relating this back to $\mathcal{L}(SU(2))$ We have shown that when we commute two vectors with eigenvalues that are negatives of each other, we get an element that's right back in the Cartan subalgebra. Note that this is exactly what we had in the case of $\mathcal{L}_{\mathbb{C}}(SU(2))$ We can compare this with the structure of the complexified Lie algebra $\mathcal{L}_{\mathbb{C}}(SU(2))$, where commuting $E_+$ and $E_-$ gives us back $H$.

So, we've found that $[E^\alpha, E^{-\alpha}]$ is in our Cartan subalgebra, but what exactly is it? Define the following normalised element

$$H^\alpha = \frac{[E^\alpha, E^{-\alpha}]}{\kappa(E^\alpha, E^{-\alpha})}$$

To figure out $H^\alpha$, plug it into the Killing form

$$\begin{aligned}
\kappa(H^\alpha, H) &= \frac{1}{\kappa(E^\alpha, E^{-\alpha})} \kappa([E^\alpha, E^{-\alpha}], H) \\
&= \frac{1}{\kappa(E^\alpha, E^{-\alpha})} \kappa(E^\alpha, [E^{-\alpha}, H]) \\
&= \alpha(H) \frac{\kappa\left(E^\alpha, E^{-\alpha}\right)}{\kappa\left(E^\alpha E^{-\alpha}\right)}
\end{aligned}$$

This means that ultimately we have that

$$\kappa\left(H^{\alpha}, H\right)=\alpha\left(H\right)$$

From our result that $\kappa$ is a non-degenerate Killing form on the Cartan subalgebra, the can try to invert this equation. In components, we have that

$$H^{\alpha},=e_i^{\alpha}H^i, \quad H=e_iH^i\in h$$

This implies that, in components,

$$\kappa^{ij}e_i^{\alpha}e_j=\alpha^je_j, \implies \kappa^{ij}e_i^{\alpha}=\alpha^j \implies e_i^{\alpha}=(\kappa^{-1})_{ij}\alpha^j$$

This means that, putting everything back together,

$$H^{\alpha}=e_i^{\alpha}H^i=\left(\kappa^{-1}\right)_{ij}\alpha^jH^i$$

Let's summarise what's going on here.

$$[E^{\alpha}, E^{\beta}]=\begin{cases} N_{\alpha,\beta}E^{\alpha+\beta} & \alpha+\beta\in\Phi \\ \kappa(E^{\alpha}, E^{-\alpha})H^{\alpha} & \alpha+\beta=0 \\ 0 & \text{otherwise}\end{cases}$$

### 8.1.2 Connections with $\mathcal{L}(SU(2))$

Now, to go further with this, and to restrict the structure even further, we need to exploit next our understanding of the representation theory of $\mathcal{L}(SU(2))$. There's actually an $\mathcal{L}(SU(2))$ subalgebra hiding in the structure of the roots, and using this we can start to pin down the root systems. Consider the bracket $H^{\alpha}\in h$, for all $\alpha,\beta\in\Phi$

$$\begin{aligned}[H^{\alpha}, E^{\beta}] &= \left(\kappa^{-1}\right)_{ij}\alpha^j[H^i, E^{\beta}] \\ &= \left(\kappa^{-1}\right)_{ij}\alpha^i\beta^jE^{\beta} \\ &= (\alpha,\beta)E^{\beta}\end{aligned}$$

In the last line we have an 'inner product' given by our Killing form. We now define a rescaling $\forall\alpha\in\Phi$,

$$e^{\alpha}=\frac{\sqrt{2}}{\left((\alpha,\alpha)\kappa\left(E^{\alpha}, E^{-\alpha}\right)\right)^{\frac{1}{2}}}, \quad h^{\alpha}=\frac{2}{(\alpha,\alpha)}H^{\alpha}$$

here we have not proven that $(\alpha,\alpha)\neq 0$. See Fuchs, page 87 for a proof of this. For all $\alpha,\beta\in\Phi$, we have that using our normalised vectors,

$$\begin{aligned}[h^{\alpha}, h^{\beta}] &= 0 \\ [h^{\alpha}, e^{\beta}] &= \frac{2\left(\alpha,\beta\right)}{(\alpha,\alpha)}e^{\beta} \\ [e^{\alpha}, e^{\beta}] &= n_{\alpha,\beta}e^{\alpha+\beta} \quad \alpha+\beta\in\Phi \\ &= h^{\alpha} \quad \alpha+\beta=0 \\ &= 0 \text{ otherwise}\end{aligned}$$

We've shown that $\alpha \in \Phi \implies -\alpha \in \Phi$. We can look at the triplet of elements $\{h^\alpha, e^\alpha, e^{-\alpha}\}$. Now, in the above equation, if we set $\beta = -\alpha$, we find that

$$[h^\alpha, e^{\pm\alpha}] = \pm e^{\pm\alpha}$$

The point is, if we specialise the brackets to these three elements, they take a very special form. We also have that

$$\left[e^\alpha, e^{-\alpha}\right] = h^\alpha$$

Hence, we have an $\mathcal{L}_{\mathbb{C}}(SU(2))$ subalgebra for each root! Now note, than in general, the generators for the $SU(2)$ Lie algebra may not commute with each other - they're not commuting subalgebras. We will call the subalgebra generated by a given root $Sl(2)_\alpha$. In fact, we can use our knowledge about simple finite dimensional representations of $\mathcal{L}(SU(2))$ to constrain values of the root.

## 8.2 Consequences

For any $\alpha, \beta \in \Phi$ where $\alpha \neq \pm\beta$, we define the '$\alpha$-string passing through $\beta$ ' as the set of all roots from the form $\beta + e\alpha$ for some $e \in \mathbb{Z}$. This is the set

$$S_{\alpha,\beta} = \{\beta + e\alpha \in \Phi, e \in \mathbb{Z}\}$$

and we define the corresponding vector subspace of $\mathcal{G}$,

$$V_{\alpha,\beta} = \text{span}_{\mathbb{C}} \left\{ e^{\beta+e\alpha}; \beta + e\alpha \in S_{\alpha,\beta} \right\}$$

Now consider the action of $Sl(2)_\alpha$ on $V_{\alpha,\beta}$

$$\begin{aligned}
[h^\alpha, e^{\beta+e\alpha}] &= \frac{2(\alpha, \beta + e\alpha)}{(\alpha,\alpha)} e^{\beta+e\alpha} \\
&= \left( \frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2e \right) e^{\beta+e\alpha}
\end{aligned}$$

In addition, we have that

$$[e^{\pm\alpha}, e^{\beta+e\alpha}] \begin{cases} e^{\beta+(e\pm1)\alpha} & \text{if } \beta + (e\pm1)\alpha \in \Phi \\ 0 & \text{otherwise} \end{cases}$$

Hence, we find that $V_{\alpha,\beta}$ is a representation space for some representation $R$ of $Sl(2)_\alpha$. Specifically, our weight set of $R$ is given by

$$S_R = \left\{ \frac{2(\alpha,\beta)}{(\alpha,\alpha 0)} + 2e; \beta + e\alpha \in \Phi \right\}$$

R is finite dimensional, and irreducible. This means that $R = R_\Lambda$ for some highest weight $\Lambda \in \mathbb{Z}$, and $S_\Lambda = \{-\Lambda, \dots, \Lambda\}$. This means that we must have $R \simeq R_\Lambda$, for some $\Lambda \in \mathbb{Z}_{\geq 0}$. Hence, we must have $e = n \in \mathbb{Z}$, with $n_- \leq n \leq n_+$, and $n_\pm \in \mathbb{Z}$. We can match up our weights to give

$$-\Lambda = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} + 2n_-$$
$$\Lambda = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} + 2n_+$$

Adding this gives $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = -(n_+ + n_-)$

## 8.3 More on roots

Recall that we can think of our roots of our Lie algebra as members of the dual space of the Cartan subalgebra, $\alpha \in h^*$, with $\dim \Phi = d - r$. Throughout this section, we'll be showing that roots form a real vector subspace of our roots space. We've already shown that $2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$, but we'll show that both the numerator and denominator are real.

Recall that in the Cartan-Weyl basis, we have that

$$\left[H^i, E^\delta\right] = \delta^i E^\delta, \quad \forall \delta \in \Phi, i = 1, \dots r$$

If we restrict our Killing form to the Cartan Weyl basis, we define

$$\kappa^{ij} = \kappa(H^i, H^j) = \frac{1}{\mathcal{N}} \operatorname{tr}\left[\mathrm{Ad}_{H^I} \circ \mathrm{Ad}_{H^j}\right] = \frac{1}{\mathcal{N}} \sum_{\delta \in \Phi} \delta^i \delta^j$$

It can be easily shown that $\mathrm{Ad}_H$ is represented by a diagonal matrix of $\delta^i$ in the first diagonals, and then a diagonal of zeros for the rest for the elements in the Cartan subalgebra. Hence, for $\alpha, \beta \in \Phi$, we have that

$$(\alpha, \beta) = \alpha^i \beta^j \left(\kappa^{-1}\right)_{ij} = \alpha_i \beta^i = \alpha_i \beta_j \kappa^{ij}, \quad \alpha_i = \left(\kappa^{-1}\right)_{ij} \alpha^j$$

Applying our previous formula for the Killing form, we get that this is expressed much more explicitly as

$$\dots = \frac{1}{\mathcal{N}} \sum_{\delta \in \Phi} \alpha_i \delta^i \delta^j \beta_j = \frac{1}{\mathcal{N}} \sum_{\delta \in \Phi} (\alpha, \delta)(\beta, \delta)$$

Now, from the fact that our ratio of inner product

$$R_{\alpha,\beta} \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z} \subset \mathbb{R}$$

dividing through by the appropriate factor we get that

$$\frac{2}{(\beta,\beta)} R_{\alpha,\beta} = \frac{1}{\mathcal{N}} \sum_{\delta \in \Phi} R_{\alpha,\delta} R_{\beta,\delta} \in \mathbb{R}$$

we take $\mathcal{N} \in \mathbb{R}$. This implies that the left hand side is real. So, the right hand side is real. So, $(\beta, \beta)$ is real for all $\beta \in \Phi$, so $(\alpha, \beta)$ is real for all $\alpha, \beta$ in $\Phi$.

### 8.3.1 Real geometry of roots

We will not prove, but use, the result that the dual space of the Cartan subalgebra $h$ is spanned by the roots.

$$h^* = \text{span}_{\mathbb{C}} \{\alpha \in \Phi\}$$

Hence, we can find $r$ roots which we will index as $\alpha_{(i)}$, $\left\{\alpha_{(i)} \in \Phi, i = 1, \ldots, r\right\}$ which provide a basis for $h^*$. We define a real subspace $h^*_{\mathbb{R}} \subset h^*$, as the real span of elements in this basis.

$$h^* = \text{span}\left\{\alpha_{(i)} \in \Phi, i = 1, \ldots, r\right\}$$

By the fact that $\alpha^{(i)}$ spans $h^*$, we write any roots $\beta \in \Phi$ as $\beta = \sum_{i=1}^{r} \beta^i \alpha_{(i)} \beta^i \in \mathbb{C}$. The coefficients $\beta^i$ solve

$$\left(\beta, \alpha_{(j)} = \sum_{i=1}^{r} \beta^i \left(\alpha_{(i)}, \alpha_{(j)}\right)\right)$$

Hence, by our previous result, we have that $(\alpha, \beta) \in$, for all $\alpha, \beta \in \Phi$. This means that $\beta_i \in$, and that $\beta \in h^*$. Now, let's think about the inner product of any two vectors in $h^*$. We write them in components in our basis as

$$\lambda = \sum_{i=1}^{r} \lambda^i \alpha_{(i)} \in h^*, \quad \mu = \sum_{i=1}^{r} \mu^i \alpha_{(i)} \in h^*$$

where by construction we have that $\lambda^i, \mu^i \in$. This means that our inner product is given by

$$(\lambda, \mu) = \sum_{i=1}^{r} \lambda^i \mu^j \left(\alpha_{(i)}, \alpha_{(j)}\right) \in$$

From our identity for the Killing form in terms of the Cartan-Weyl basis, we get that

$$(\lambda, \lambda) = \frac{1}{\mathcal{N}} \sum \lambda_i \delta^i \delta^j \lambda_j = \frac{1}{\mathcal{N}} \sum_{\delta \in \Phi} (, \delta)^2 \geq 0$$

Thus, the inner product in this space is positive, and we only have equality if and only if $(\lambda, \delta) = 0$, for all $\delta \in \Phi$, if and only if $\lambda = 0$, by the non-degeneracy of $\kappa$. The key point here is that a simple Lie algebra has a non-degenerate inner product, and this gives us a lot of information.

Accumulating all of these facts, we can conclude that the roots $\alpha$ live in real vector space $h^* \simeq^r$, where $r = \text{Rank}(\mathcal{G})$, with a Euclidean inner product. Specifically, this means that for all $\lambda, \mu \in h^*$, we have that

1. $(\lambda, \mu) \in$
2. $(\lambda, \lambda) \geq 0$
3. $(\lambda, \lambda) = 0 \iff \lambda = 0$

This are the characteristic properties of a standard inner product. Now, we can draw our vectors in two dimensional real vector space, and apply geometric intuition.

# 9 Example sheet 1

## 9.1 Question 1

Our group axioms are identity, closure, inverses and associativity. We go about proving the first three, and in the case of matrix groups we can say that associativity is inherited from matrix multiplication.

We check closure, since associativty is inherited from matrix multiplication.

$$C^T C = (AB)^T AB = B^T A^T AB = B^T B = I.$$

Finally, if $A \in O(n)$, then by inverting both sides,

$$A^T A = I \implies A^{-1}(A^T)^{-1} = A^{-1}(A^{-1})^T = I.$$

The case for the unitary group is entirely similar except that we replace transposition by Hermitian conjugation. To show that $O(n)$ is a subgroup of $U(n)$, we must first show that it's a subset of $U(n)$. Since $O(n)$ is defined for just matrices over the reals, Hermitian conjugation is equivalent to transposition over the reals. Thus, matrices in $O(n)$ are also unitary. Since $O(n)$ is a group in its own right, it's a subgroup.

$SO(n)$ consists of matrices in $O(n)$ which have determinant 1. Thus, we have that $SO(n) \subset O(n) \subset U(n)$. Thus, $SO(n) \subset U(n)$. But since its matrices have determinant 1, $SO(n) \subset SU(n)$.

**Showing that $U(n)$ is a subgroup of $SO(2n)$**

A complex vector $\mathbf{c} \in \mathbb{C}^n$ and be represented as the sum of a real and imaginary vector, where

$$\mathbf{c} = \mathbf{a} + i\mathbf{b}, \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^n$$

Similarly, a unitary matrix $U \in U(n)$ can be represented as the sum of a real and imaginary matrix.

$$U = A + iB, \quad A, B \in Mat_n(\mathbb{R})$$

Our condition of unitarity however imposes conditions on $A, B$. By comparting real and imaginary parts, we can extract these conditions from

$$U^\dagger U = (A^\dagger - iB^\dagger)(A + iB) = (A^\dagger A + B^\dagger B) + i(A^\dagger B - B^\dagger A) = I \implies A^\dagger A + B^\dagger B = I, \quad A^\dagger B - B^\dagger A = 0$$

In addition, we can act on vectors in this notation, still treating the real and imaginary components separately. We find that

$$(A + iB)(\mathbf{a} + i\mathbf{b}) = A\mathbf{a} - B\mathbf{b} + i(B\mathbf{a} + A\mathbf{b})$$

This whole system can be represented as a map on $\mathbb{R}^{2n}$ by stacking up $\mathbf{a}$ and $\mathbf{b}$.

$$\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \mapsto \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}$$

In this step, we're representing the matrix $U$ as an operator on $\mathbb{R}^{2n}$ instead of the usual, fundamental representation. All that's left to show is that a matrix of this form is in $SO(2n)$. We compute the sum and use the conditions above to show that

$$\begin{pmatrix} A^T & B^T \\ -B^T & A^T \end{pmatrix} \begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \begin{pmatrix} A^T A + B^T B & B^T A - A^T B \\ -B^T A + A^T B & A^T A + B^T B \end{pmatrix} = I$$

We've replaced the daggers in the above conditions with transposes since the matrices are real!

More simply, we could argue from the fact that if $v = (v_1, \ldots, v_n) \in \mathbb{C}^n$, then $U(n)$ preserves our norm $\|v\|^2 = \sum_i \|v_i\|^2$. If we write $v_i = x_i + iy_i$, then $|v|^2 = \sum_i x_i^2 + y_i^2$, which is preserved by $O(2n)$. Thus, $U(n) \leq O(2n)$. Thus, we've shown $U(n) \leq O(2n)$. Note that $U(n)$ is connected, so it must be embedded in the connected component of the identity as $I$ is sent to $I$. $U(n)$ is connected as we can diagonalise it as $diag(e^{i\theta_1}, \ldots e^{i\theta_n})$, and take a path from any element to any element by parametrising $\theta_i$ with $t$.

## 9.2 Question 2

To third order in $X/Y$, we can expand the BCH formula first as

$$\begin{aligned}\mathrm{Exp}(X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X.Y]]-\frac{1}{12}[Y,[X,Y]]) = \\ \left(X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]-\frac{1}{12}[Y,[X,Y]]\right)+\frac{1}{2}\left(X+Y+\frac{1}{2}[X,Y]\right)^2+\frac{1}{6}\left(X+Y\right)^3\end{aligned}$$

Note that in the squared and cubed terms, we truncated terms of higher order since we're just working to third order in $X, Y$. Our first term in the sum is

$$\begin{aligned}X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]-\frac{1}{12}[Y,[X,Y]] &= X+Y+\frac{1}{2}(XY-YX) \\ &+\frac{1}{12}(X^2Y+YX^2+Y^2X+XY^2)-\frac{1}{6}(XYX+YXY)\end{aligned}$$

Our second term in the sum is

$$\begin{aligned}\frac{1}{2}(X+Y+\frac{1}{2}[X,Y])^2 &= \frac{1}{2}(X^2+XY+\frac{1}{2}X^2Y-\frac{1}{2}XYX \\ &+YX+Y^2+\frac{1}{2}YXY-\frac{1}{2}Y^2X+\frac{1}{2}XYX \\ &+\frac{1}{2}XY^2-\frac{1}{2}YX^2-\frac{1}{2}YXY) \\ &= \frac{1}{2}(X^2+XY+\frac{1}{2}X^2Y+YX+Y^2-\frac{1}{2}Y^2X+\frac{1}{2}XY^2-\frac{1}{2}YX^2)\end{aligned}$$

Our final term in the sum is

$$\frac{1}{6}(X+Y)^3 = \frac{1}{6}(X^3+X^2Y+XYX+YX^2+Y^2X+YXY+XY^2+Y^3)$$

Pairing terms together, we have that this is

$$\begin{aligned}\mathrm{Exp}\ (X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]-\frac{1}{12}[Y,[X,Y]] &= X+Y+XY+\frac{1}{2}(X^2Y+YX^2) \\ &+\frac{1}{6}(X^3+Y^3)+\dots\end{aligned}$$

However, this agrees completely with the expansion

$$\mathrm{Exp}\ X\cdot\ \mathrm{Exp}\ Y = \left(1+X+\frac{X^2}{2}+\frac{X^3}{6}+\dots\right)\left(1+Y+\frac{Y^2}{2}+\frac{Y^3}{6}+\dots\right)$$

## 9.3 Question 3

In this question we verify why

$$R(\mathbf{n},\theta)_{ij} = \cos\theta\delta_{ij}+(1-\cos\theta)n_in_j+\sin\theta\epsilon_{ijk}n_k$$

for a chosen unit vector $\mathbf{n}$ and some $\theta \in [0, 2\pi)$ is a geometrically sensible element of $SO(3)$.

First we construct a natural basis for this mnatrix. $\mathbf{n}$ itself is an eigenvector with eigenvalue 1 because

$$\begin{aligned} R_{ij} n_j &= \cos\theta n_i + (1 - \cos\theta) n_j n_j n_i + \sin\theta \epsilon_{ijk} n_k n_j \\ &= n_i \end{aligned}$$

where the last term vanishes due to symmetric and antisymmetric contraction in the $j, k$ indices, and $n_j n_j = 1$ since $\mathbf{n}$ is a unit vector.

To construct an orthonornal basis from this, we do a Grahm-Schmidt like procedure with $\mathbf{n}$ and $\mathbf{e}_1$, the standard $x$ coordinate. It's not hard to see that

$$\mathbf{m} = \frac{\mathbf{e}_1 - \mathbf{n}(\mathbf{e}_1 \cdot \mathbf{n})}{|\mathbf{e}_1 - \mathbf{n}(\mathbf{e}_1 \cdot \mathbf{n})|}$$

satisfies our condition that $\mathbf{m} \cdot \mathbf{n} = 0$.

To find the third vector which is perpendicualr to both of thesebasis vectors, just observe that both are in the span of $\{\mathbf{n}, \mathbf{e}_1\}$ so we can simply take the cross product of these objects to find

$$\mathbf{k} = \frac{\mathbf{n} \times \mathbf{e}_1}{|\mathbf{n} \times \mathbf{e}_1|}$$

is our third basis vector.

A straight forward calculation with index notation shows that

$$R_{ij} k_j = \cos\theta k_i + \sin\theta m_i$$

and similarly we have that

$$R_{ij} m_j = \cos\theta m_i - \sin\theta k_j$$

which implies that this is indeed a rotation of the angle $\theta$ about the $\mathbf{n}$ axis!

## 9.4 Question 4

To find the manifold of this thing, we define

$$F(\vec{x}) = |\alpha|^2 - \ \beta|^2 = \alpha_1^2 + \alpha_2^2 - \beta_1^2 - \beta_2^2$$

Our group is defined by the surface $F = 1$. Our Jacobian is

$$\frac{\partial F}{\partial x^i} = (2\alpha_1, 2\alpha_2, -2\beta_1, -2\beta_2) \neq 0 \text{ on } F = 1$$

Hence our dimension is $4 - 1 = 3$. To show that our group is unbounded, we can take explicitly that $\alpha_2 = \beta_2 = 0$, and $\alpha_1^2 = \beta_2^+ 1$, then the modulus of this object is

$$|\vec{x}|^2 = 2\beta_1^2 + 1 \to \infty, \text{ as } \beta_1 \to \infty$$

So the manifold is unbounded.

Closure is the property we would like to check here. The other group properties follow easily from the definition. We set

$$U_i = \begin{pmatrix} \alpha_i & \beta_i \\ \beta_i^* & \alpha_i^* \end{pmatrix}$$

It's easy to check that

$$U_1 U_2 = \begin{pmatrix} \alpha_1\alpha_2 + \beta_1\beta_2^* & \alpha_1\beta_2 + \beta_1\alpha_2^* \\ \beta_1^*\alpha_2 + \alpha_1^*\beta_2^* & \beta_1^*\beta_2 + \alpha_1^*\alpha_1^* \end{pmatrix}$$

To fulfil the condition above we then require that

$$|\alpha_1\alpha_2 + \beta_1\beta_2^*|^2 - |\alpha_1\beta_2 + \beta_1\alpha_2^*|^2 = 1$$

Cancelling out the cross terms, we write this in terms of its conjugates and we're left with the above equalling

$$\left(|\alpha_1|^2 - |\beta_1^2|^2\right)\left(|\alpha_2|^2 - |\beta_2|^2\right) = 1$$

This is a Lie group because clearly matrix multiplication and matrix inversion depend smoothly on the defining parameters $\alpha, \beta$. since addition, multiplcation, division and complex conjugation are smooth operations. The dimension of a general $U$ over the reals is 3. This is becasue $\mathbb{C}$ is a two dimensional vector space over the reals, and writing $\alpha = a + ib, \beta = c + id$, our constraint that $a^2 + b^2 - c^2 - d^2 = 1$ gives us a 3 dimensional manifold.

Thus, we can map this object into $\mathbb{R}^4$ as

$$U \mapsto \begin{pmatrix} a \\ b \\ c \\ \pm\sqrt{a^2 + b^2 - c^2 - 1} \end{pmatrix}$$

The norm of this vector is $\sqrt{2a^2 + 2b^2 - 1}, a, b \in \mathbb{R}$. This is clearly unbounded and therefore the corresponding manifold is non compact.

## 9.5 Question 5

Given a matrix $A \in SU(2)$ we have that $\det U = 1$, so we have a simplified expression for our matrix inverse.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad a, b, c, d \in \mathbb{C}$$

However, the condition that $A$ is unitary means that $A^\dagger = A^{-1}$. Thus, comparing coefficients we have that

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \implies d = a^*, c = -b^*$$

Thus, a general unitary matrix looks like

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1$$

The condition on $\alpha$ and $\beta$ comes from the fact that the determinant of this matrix should equal 1.

Since $\alpha, \beta \in \mathbb{C}$, we can write them as two real numbers. We set $\alpha = a_0 + ia_3$, and set $\beta = a_2 + ia_3$. This measn we can expand

$$\begin{aligned} U &= \begin{pmatrix} a_0 + ia_3 & a_2 + ia_1 \\ -a_2 + ia_1 & a_0 - ia_3 \end{pmatrix} \\ &= a_0 I + ia_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + ia_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + ia_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= Ia_0 + i(a_1\sigma_x + a_2\sigma_y + a_3\sigma_z) \\ &= Ia_0 + i(\mathbf{a} \cdot \sigma) \end{aligned}$$

Our condition that $|\alpha|^2 + |\beta|^2 = 1 \implies a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1$.

If we set

$$\begin{aligned} U_1 &= a_0 I + i\mathbf{a} \cdot \sigma \\ U_2 &= b_0 I + i\mathbf{b} \cdot \sigma \end{aligned}$$

Multiplying these terms out gives

$$\begin{aligned} U_1 U_2 &= (a_0 I + i\mathbf{a} \cdot \sigma)(b_0 I + i\mathbf{b} \cdot \sigma) \\ &= a_0 b_0 I + ib_0 \mathbf{a} \cdot \sigma + ia_0 \mathbf{b} \cdot \sigma - (\mathbf{a} \cdot \sigma)(\mathbf{b} \cdot \sigma) \\ &= a_0 b_0 I + i(b_0 \mathbf{a} \cdot \sigma + a_0 \mathbf{b} \cdot \sigma) - (\mathbf{a} \cdot \sigma)(\mathbf{b} \cdot \sigma) \end{aligned}$$

Let's take a close look at what $(\mathbf{a} \cdot \sigma)(\mathbf{b} \cdot \sigma)$ is

$$\begin{aligned} (\mathbf{a} \cdot \sigma)(\mathbf{b} \cdot \sigma) = a_1 b_1 \sigma_x^2 + a_2 b_2 \sigma_y^2 + a_3 b_3 \sigma_z^2 \\ + a_1 b_2 \sigma_x \sigma_y + a_1 b_3 \sigma_x \sigma_y + a_2 b_1 \sigma_y \sigma_x \\ + a_2 b_3 \sigma_y \sigma_z + a_3 b_1 \sigma_z \sigma_x + a_3 b_2 \sigma_z \sigma_y \end{aligned}$$

Now, we use the fact that $\sigma_i^2 = I$ and our algebra $[\sigma_i, \sigma_j] = i\epsilon_{ijk}\sigma_k$. Hence, our term simplifies to

$$(\mathbf{a} \cdot \sigma)(\mathbf{b} \cdot \sigma) = I\mathbf{a} \cdot \mathbf{b} + i(a_1 b_2 \sigma_z - a_1 b_3 \sigma_y - a_2 b_1 \sigma_z + a_2 b_3 \sigma_x + a_3 b_1 \sigma_y - a_3 b_2 \sigma_x)$$

This however is just

$$= I(\mathbf{a} \cdot \mathbf{b}) + i(\mathbf{a} \times \mathbf{b}) \cdot \sigma$$

Hence, our final expression for $U_1 U_2$ is

$$U_1 U_2 = I(a_0 b_0 - \mathbf{a} \cdot \mathbf{b}) + i(b_0 \mathbf{a} + a_0 \mathbf{b} - \mathbf{a} \times \mathbf{b}) \cdot \sigma$$

A slightly easier way to have shown this would be via use of the Pauli matrix identity

$$\sigma_i \sigma_j = \delta_{ij} I + \epsilon_{ijk} \sigma_k$$

## 9.6 Question 6

To verify that this constitutes a Lie algebra, since our operation $*$ is associative, we need to verify that the Lie bracket is antisymmetric, linear and obeys the Jacobi identity. Antisymmetry is simple

$$[Y, X] = Y * X - X * Y = -(X * Y - Y * X) = -[X, Y]$$

Now, to show linearity, it's a matter of just expanding out the algebra.

$$\begin{aligned}
[X, \alpha Y + \beta Z] &= X * (\alpha Y + \beta Z) - (\alpha Y + \beta Z) * X \\
&= \alpha(X * Y) + \beta X * Z - \alpha Y * X - \beta Z * X \\
&= \alpha[X, Y] + \beta[X, Z]
\end{aligned}$$

Thus, we have linearity in the second argument. The final this to do is to prove the Jacobi identity.

$$\begin{aligned}
&[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] \\
&\qquad = X * [Y, Z] - [Y, Z] * Z + Y * [Z, X] - [Z, X] * Y + Z * [X, Y] - [X, Y] * Z \\
&\qquad = X * (Y * Z - Z * Y) - (Y * Z - Z * Y) * X + Y * (Z * X - X * Z) - (Z * X - X * Z) * Y \\
&\qquad + Z * (X * Y - Y * Z) - (X * Y - Y * X) * Z \\
&\qquad = X * (Y * Z) - X * (Z * Y) - (Y * Z) * Z + (Z * Y) * X + Y * (Z * X) - Y * (X * Z) \\
&\qquad - (Z * X) * Y + (X * Y) * Z + Z * (X * Y) - Z * (Y * X) - (X * Y) * Z + (Y * X) * Z \\
&\qquad = 0
\end{aligned}$$

Our terms cancel in the last term since associativity allows us to move around the respective brackets.

## 9.7 Question 7

**Showing that the Heisenberg matrices form a group**

Group multiplication is an associative operation, so the associative property for this group holds by inheritance. To show closure, we multiply two arbitrary matrices $A, B \in G$, of this form in the group;

$$AB = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & d+a & e+af+b \\ 0 & 1 & f+c \\ 0 & 0 & 1 \end{pmatrix}$$

We observe that the resulting product is of the form required of group elements. By solving for the identity matrix, one can verify that the matrix inverse is given by

$$a = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \quad a^{-1} = \begin{pmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}$$

Now, this is also in the form required of matrices in the group. Finally we have that $I \in G$, so we are done. To show that this is a Lie group, we argue that we can simply parametrise an element in $G$ as an element of $\mathbb{R}^3$, the underlying manifold. Moreover, as shown above, inversion and multiplication are smooth operations on this space. This group is not abelian. Consider the matrices

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad [A, B] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

**The corresponding Lie algebra**

Suppose we can take a smooth curve parameterised by $t \in \mathbb{R}$, giving the group element

$$g(t) = \begin{pmatrix} 1 & a(t) & b(t) \\ 0 & 1 & c(t) \\ 0 & 0 & 1 \end{pmatrix}$$

Then, performing a Taylor expansion, we can write this group element infinitesimally as

$$g(t) = g(0) + tX + \dots, \quad X = \begin{pmatrix} 0 & \dot{a}(0) & \dot{b}(0) \\ 0 & 0 & \dot{c}(0) \\ 0 & 0 & 0 \end{pmatrix}$$

However, it's easy to see that we can pick an arbitrary curve to give rise to any strict upper triangle we want in the Lie algebra. Hence $L(G)$ is the set of matrices generated by

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

We can calculate the bracket of two general elements in this Lie algebra as

$$[\begin{pmatrix} 0 & a_1 & b_1 \\ 0 & 0 & c_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_2 & b_2 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{pmatrix}] = \begin{pmatrix} 0 & 0 & a_1c_2 - a_2c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We can generate a proper ideal from this. Consider the proper subspace of $L(G)$ generated by matrices of the form

$$I = \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid a \in \mathbb{R} \right\}$$

This is clearly a vector subspace under matrix addition, and is isomorphic to $(\mathbb{R}, +)$ by identifying the top corner element as just a real number on its own . From the multiplication formula above, we have that

$$[X, Y] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in I \quad \forall X \in I, Y \in L(G)$$

This subgroup is in the kernel of our Lie bracket, and hence is an ideal. But, clearly it is not the whole Lie algebra, so it is a proper ideal. Hence, the Lie algebra is not simple. Also, a slicker way to say this is that since elements of this form commute with $\mathcal{L}(G)$, its a non-trivial ideal.

A simplified argument to show that its a Lie group is that our corresponding action is smooth, ie the map

$$g \in G, g_L : G \times G, g_l(h) = gh \text{ is smooth, bijective with a smooth inverse}$$

## 9.8 Question 8

Given the matrices $\{T^{ij}\}_{i,j=1,2..n}$ where the components of $T^{ij}$ is simply an 1 in the ith row and jth column (with a zero elsewhere), we want to find a set of constants $g_m^{ij,kl}n$ such that

$$[T^{ij}, T^{kl}] = \sum_m \sum_n g_{mn}^{ij,kl} T_{mn}$$

where in the left hand side, this is simply the matrix commutator. We have that in components,

$$(T^{ij})_{\alpha\beta} = \delta^i_\alpha \delta^j_\beta$$

So, taking the components of the equation above we have the condition that

$$(T^{ij})_{\alpha\gamma}(T^{kl})_{\gamma\beta} - (T^{kl})_{\alpha\gamma}(T^{ij})_{\gamma\beta} = \sum_m \sum_n g_{mn}^{ij,kl}(T^{mn})_{\alpha\beta}$$

In this question, we don't care about whether indices are raised or lowered, so in what follows we'll use the Einstein summation convention with all lowered indices. We find that the right hand side is just

$$\begin{aligned} \delta_{i\alpha}\delta_{j\gamma}\delta_{k\gamma}\delta_{l\beta} - \delta k\alpha\delta_{l\gamma}\delta_{i\gamma}\delta_{j\beta} &= \delta_{i\alpha}\delta_{jk}\delta_{l\beta} - \delta_{k\alpha}\delta_{li}\delta_{j\beta} \\ &= (T^{il})_{\alpha\beta}\delta^{jk} - (T^{kl})_{\alpha\beta}\delta^{li} \end{aligned}$$

Going into the first line on the left hand side, we've summed over $\gamma$ to contract coefficients. Comparing this with our expression for the structure constants, we find that

$$g_{mn}^{ij,kl} = \delta_{im}\delta_{ln}\delta_{jk} - \delta_{km}\delta_{jn}\delta_{li} = (T^{il})_{mn}\delta^{jk} - (T^{jk})_{mn}\delta^{li}$$

Note: we can't re-express the structure constants in terms of $T$!

## 9.9 Question 9

For the first part of the question, we appeal to the exact definition of what the exponential map is of a matrix.

$$\text{Exp } (iH) = \sum_{n=1}^{\infty} \frac{(iH)^n}{n!}$$

When we take the Hermitian conjugate of this object, we pick up a minus sign from the $i$, but since $H$ is Hermitian that part remains unchanged. Hence, we have that

$$\begin{aligned}
(\text{Exp} iH)^\dagger &= \sum_{n \geq 0} \frac{(-iH^\dagger)^n}{n!} \\
&= \sum_{n \geq 0} \frac{(-iH)^n}{n!} \\
&= \text{Exp } (-iH) \\
&= (\text{ Exp } iH)^{-1}
\end{aligned}$$

Thus we have that $U^\dagger = U^{-1}$. Since $H$ is Hermitian, its diagonalisable. So, we can write $H = \text{diag}(\lambda_1, \dots \lambda_n)$ where $\lambda_i$ are eigenvalues. But then, we have that since $U = \text{Exp}) iH)$, we have that $U = \text{diag}(e^{i\lambda_1}, \dots, e^{i\lambda_n})$. Hence, since our determinant is just the product of this, we have that

$$\det U = \prod_j e^{i\lambda_j} = e^{i \sum \lambda_j} = 1$$

since traceless-ness implies $\sum \lambda_j = 0$. Since $iH$ is anti-Hermitian and traceless, we have that the set

$$\mathcal{L} = \left\{ iH \mid H = H^\dagger, \operatorname{tr} H = 0 \right\}$$

forms the Lie algebra $\mathcal{L}(SU(n))$. Moreover, the exponential map maps it to a subset of $SU(n)$. Thus, the inverse map takes our Lie algebra to a subset of $SU(n)$. To relate this to our exponential map, we have that $\text{Exp } : \mathcal{L}(U(n))$ is okay, since,

$$\begin{aligned}
\mathcal{L}(U(n)) &\to U(n) \\
\mathcal{L}(SL(n)) &\to SL(n)
\end{aligned}$$

Also: we can't pile up more conditions from our det condition for tracelssness since our dimensions line up; so we're done.

# 10 Example Sheet 2

## 10.1 Question 4

We need to invert the expression explicitly for the last part!

## 10.2 Question 9

Assume vector space decomposition here!

## 10.3 Question 8

The tensor product of our irreducible representations $R_N \otimes R_M$ has weight set

$$S_{N,M} = \left\{ \lambda + \lambda' \mid \lambda \in S_N, \lambda' \in S_M \right\}$$

Now, including multiplicities, we can write this weight set explicitly in rows and columns as

$$\begin{array}{llll}
M+N & & & \\
(M-2)+N & M+(N-2) & & \\
(M-4)+N & (M-2)+(N-2)M+(N-4) & & \\
(M-6)+N & \dots & & M+(N-6) \\
\vdots & & & \\
(-M-2N+2)+N & M+(-N-2M+2) & & \\
-M-2N+N & & &
\end{array}$$

The way to do this is to keep subtracting until we hit lowest weights. Then, write out a formula for our degeneracy. Then count.

## 10.4 Question 2

To third order in $X/Y$, we can expand the BCH formula first as

$$\text{Exp}(X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X.Y]]-\frac{1}{12}[Y,[X,Y]]) =$$
$$\left(X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]-\frac{1}{12}[Y,[X,Y]]\right)+\frac{1}{2}\left(X+Y+\frac{1}{2}[X,Y]\right)^2+\frac{1}{6}(X+Y)^3$$

Note that in the squared and cubed terms, we truncated terms of higher order since we're just working to third order in $X, Y$. Our first term in the sum is

$$\begin{aligned} X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]-\frac{1}{12}[Y,[X,Y]] &= X+Y+\frac{1}{2}(XY-YX) \\ &+\frac{1}{12}(X^2Y+YX^2+Y^2X+XY^2)-\frac{1}{6}(XYX+YXY) \end{aligned}$$

Our second term in the sum is

$$\begin{aligned}\frac{1}{2}(X+Y+\frac{1}{2}[X,Y])^2 = \frac{1}{2}(&X^2+XY+\frac{1}{2}X^2Y-\frac{1}{2}XYX \\ &+YX+Y^2+\frac{1}{2}YXY-\frac{1}{2}Y^2X+\frac{1}{2}XYX \\ &+\frac{1}{2}XY^2-\frac{1}{2}YX^2-\frac{1}{2}YXY) \\ &=\frac{1}{2}(X^2+XY+\frac{1}{2}X^2Y+YX+Y^2-\frac{1}{2}Y^2X+\frac{1}{2}XY^2-\frac{1}{2}YX^2)\end{aligned}$$

Our final term in the sum is

$$\frac{1}{6}(X+Y)^3 = \frac{1}{6}(X^3+X^2Y+XYX+YX^2+Y^2X+YXY+XY^2+Y^3)$$

Pairing terms together, we have that this is

$$\begin{aligned}\text{Exp }(X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]-\frac{1}{12}[Y,[X,Y]] &= X+Y+XY+\frac{1}{2}(X^2Y+YX^2) \\ &+\frac{1}{6}(X^3+Y^3)+\dots\end{aligned}$$

However, this agrees completely with the expansion

$$\text{Exp } X\cdot \text{ Exp } Y = \left(1+X+\frac{X^2}{2}+\frac{X^3}{6}+\dots\right)\left(1+Y+\frac{Y^2}{2}+\frac{Y^3}{6}+\dots\right)$$

# 11 Problems left to contribute

- Deriving the most general form of a matrix element in $SO(3)$.