1 2D Projective Geometry [18 pts]

1. (a) **[2 pts]** Calculate the line passing through 2 given points: (1) $\mathbf{p}_1 = [3, 4, 1]^T, \mathbf{p}_2 = [4, 3, 0]^T$, (2) $\mathbf{p}_1 = [3, 4, 2022]^T, \mathbf{p}_2 = [3, 4, -1967]^T$.

   (b) **[2 pts]** Calculate the intersection point between 2 given lines: (1) $\mathbf{l}_1 = [3, 4, 1]^T, \mathbf{l}_2 = [0, 0, 1]^T$, (2) $\mathbf{l}_1 = [3, 4, 1]^T, \mathbf{l}_2 = [3, 4, 2]^T$.

(a)(1)

$$l_1 = p_1 \times p_2 = \begin{vmatrix} i & j & k \\ 3 & 4 & 1 \\ 4 & 3 & 0 \end{vmatrix} = \begin{bmatrix} -3 \\ 4 \\ -7 \end{bmatrix}$$

(2)

$$l_2 = p_3 \times p_4 = \begin{vmatrix} i & j & k \\ 3 & 4 & 2022 \\ 3 & 4 & -1967 \end{vmatrix} = \begin{bmatrix} \cancel{-13954} \\ \cancel{13989} \\ \cancel{-7} \end{bmatrix} \begin{bmatrix} -15956 \\ 11967 \\ 0 \end{bmatrix}$$

b). $$p_1 = l_1 \times l_2 = \begin{vmatrix} i & j & k \\ 3 & 4 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$$

b). $$p_2 = l_1 \times l_2 = \begin{vmatrix} i & j & k \\ 3 & 4 & 1 \\ 3 & 4 & 2 \end{vmatrix} = \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$$

(point at infinity).

2. **[3 pts]** Suppose a conic in 2D projective space is given by $\mathbf{C} = \mathbf{l}\mathbf{m}^T + \mathbf{m}\mathbf{l}^T$, where $\mathbf{l}$ and $\mathbf{m}$ are 2 lines. Show that a point belongs to $\mathbf{C}$ if and only if it is on $\mathbf{m}$ or $\mathbf{l}$.

$x^T C x = 0$ & $C = lm^T + ml^T \Rightarrow x^T(lm^T + ml^T)x = 0$.

$\Rightarrow \cancel{x^T}\ x^T l m^T x + x^T m l^T x = 0$.

$m^T x = x^T m$, $x^T l = l^T x$.

$\Rightarrow 2(x^T l)(x^T m) = 0 \Rightarrow$ either $x^T l = 0$ or $x^T m = 0$

$\Rightarrow x$ is on m or l.

3. **[4 pts]** Given a transformation $\mathbf{H} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

   (a) transform a point $\mathbf{p} = [3, 4, 1]^T$,

   (b) transform a line $\mathbf{l} = [-4, 3, 0]$

   (c) transform a conic $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

   (d) does this transformation leaves the circular points at infinity unchanged? Explain the reason without calculation.

a). $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 1 \end{bmatrix}$

2) $H^{-T} l = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} \\ \frac{3}{2} \\ 0 \end{bmatrix}$

3). $C' = H^{-T} C H^{-1}$

$= \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{9} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 3 \end{bmatrix}$

4). ~~Yes, circular~~. No, the circularity is unpresered because of different scales in diagonal of H.

4. **[4 pts]** Are these 2D projective transformations?

(a) Reflection along a line,

(b) Doubling spherical coordinates: $(r, \theta) \rightarrow (2r, 2\theta)$,

(c) A picture hanging on a wall and its image taken by a camera,

(d) Transformation between these 2 world maps.

1) Yes

2) Yes

3). Yes

d). No.

5. **[3 pts]** Are these statements true or false?

(a) Given a line l, if both $\mathbf{H}_A$ and $\mathbf{H}_B$ map l to $[0, 0, 1]^T$, then $\mathbf{H}_A\mathbf{H}_B^{-1}$ is an affine transformation.

(b) Instead of annotating orthogonal lines, if we annotate multiple pairs of lines that form 45 degree angles in the metric space, we can still calculate $\mathbf{C}_\infty^{\star}{}'$.

(c) If we are allowed to annotate pairs of parallel and orthogonal lines, we need at least 5 pairs of them to calculate $\mathbf{C}_\infty^{\star}{}'$.

a) Yes

b) Yes

c). No.

2 3D Projective Geometry [12 pts]

6. **[3 pts]** Show that the Plucker Representation of a 3D line $\mathbf{L} = \mathbf{x}_1\mathbf{x}_2^T - \mathbf{x}_2\mathbf{x}_1^T$ is equivalent to representing the line as $(\tilde{\mathbf{d}}, \tilde{\mathbf{x}} \times \tilde{\mathbf{d}})$, i.e. show they have the same elements up to scale.

   Notations: $\tilde{\mathbf{d}}$ is the unit direction vector along the line, and $\tilde{\mathbf{x}}$ is any point on the line. Note that $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{d}}$ are 3-dim Euclidean coordinates while $\mathbf{x}_1$ and $\mathbf{x}_2$ are 4-dim homogeneous coordinates.

$$\tilde{d} \triangleq \frac{\tilde{X}_1 - \tilde{X}_2}{|\tilde{X}_1 - \tilde{X}_2|} = \alpha \begin{bmatrix} X_{1x} - X_{2x} \\ X_{1y} - X_{2y} \\ X_{1z} - X_{2z} \end{bmatrix}$$

$$\tilde{X} \triangleq \tilde{X}_1 + \beta\tilde{d} = \begin{bmatrix} X_{1x} + \beta(X_{1x} - X_{2x}) \\ X_{1y} + \beta(X_{1y} - X_{2y}) \\ X_{1z} + \beta(X_{1z} - X_{2z}) \end{bmatrix}$$

$$\tilde{X} \times \tilde{d} = \alpha \begin{bmatrix} X_{1x} - X_{2x} \\ X_{1y} - X_{2y} \\ X_{1z} - X_{2z} \end{bmatrix} \times \begin{bmatrix} X_{1x} + \beta(X_{1x} - X_{2x}) \\ X_{1y} + \beta(X_{1y} - X_{2y}) \\ X_{1z} + \beta(X_{1z} - X_{2z}) \end{bmatrix}$$

$$= [\tilde{X}_1 + \beta(\tilde{X}_1 - \tilde{X}_2)] \times \alpha(\tilde{X}_1 - \tilde{X}_2).$$

$$= [(1+\beta)\tilde{X}_1 - \beta\tilde{X}_2] \times \alpha(\tilde{X}_1 - \tilde{X}_2)$$

$$= -\alpha(1+\beta)\tilde{X}_1 \times \tilde{X}_2 - \alpha\beta\tilde{X}_2 \times \tilde{X}_1 = -\alpha(1+\beta)\tilde{X}_1 \times \tilde{X}_2 + \alpha\beta\tilde{X}_1 \times \tilde{X}_2 = -\alpha\tilde{X}_1 \times \tilde{X}_2$$

$$\Rightarrow \tilde{X} \times \tilde{d} = -\alpha\tilde{X}_1 \times \tilde{X}_2$$

so the plucker coordinates are

$$\left(\alpha(\tilde{X}_1 - \tilde{X}_2) \,;\, -\alpha\tilde{X}_1 \times \tilde{X}_2\right).$$

$$\left(\tilde{d} \,;\, \tilde{X} \times \tilde{d}\right).$$

7. **[2 pts]** Suppose $\mathbf{U}$ is a 4×4 matrix. $\mathbf{U}_{4\times4} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4]$ and $\mathbf{U}^T\mathbf{U} = \mathbf{I}$.

   (a) Suppose $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ represent 3 points in the 3D space. What is the plane passing through these 3 points?

   (b) Suppose $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ represent 4 points in the 3D space. Let $\mathbf{l}_1$ be the line passing through $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{l}_2$ be the line passing through $\mathbf{u}_3, \mathbf{u}_4$. Do $\mathbf{l}_1$ and $\mathbf{l}_2$ intersect or not? (Only consider real-number points.)

~~a) Yes~~ 1) 3 points define a plane. 2). No.

~~b) Yes~~. $\Rightarrow \begin{bmatrix} \vec{u}_1^T \\ \vec{u}_2^T \\ \vec{u}_3^T \end{bmatrix} \pi = \vec{0}$ $\quad \underline{\pi = \vec{u}_4}$ $\quad \begin{vmatrix} i & j & k \\ (\vec{u}_1 - \vec{u}_2) \\ \vec{u}_3 - \vec{u}_4 \end{vmatrix} = \vec{0}$.

~~c)~~

$U^T U = I \Rightarrow U$ is orthogonal

$\vec{u}_4^T \cdot \vec{u}_1 = \vec{u}_4^T \vec{u}_2 = \vec{u}_4^T \vec{u}_3 = 0$.

8. **[4 pts]** (a) Calculate the 3D transformation $\mathbf{H}$ that represents the projection onto a plane $\pi = [\mathbf{n}^T, 0]^T$, where $\mathbf{n} = [a, b, c]^T$ is a unit vector.

(b) Calculate the 3D transformation $\mathbf{H}$ that represents the reflection along a plane $\pi = [\mathbf{n}^T, 0]^T$, where $\mathbf{n} = [a, b, c]^T$ is a unit vector.

a). point transformation gives $X' = HX$

and all points on the plane can be represented as $X' = Mx$.

where M is a 4x3 matrix whose columns generate the rank 3 null-space of π^T

$$M^T = \begin{pmatrix} -b/a & 1 & 0 & 0 \\ -c/a & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad M = \begin{pmatrix} -b/a & -c/a & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$HX = Mx$ where X is any point in $\mathbb{R}^3$ in homogenous form (4x1).

x : all points on the plane (3x1).

$$\therefore H: \begin{pmatrix} -b/a & -c/a & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

b). reflection $\Rightarrow HX - X \propto \pi$

$$\therefore (H - I)X = \begin{pmatrix} a \\ b \\ c \\ 0 \end{pmatrix} \Rightarrow H - I = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for $X \in \mathbb{R}^3$.

9. **[3 pts]** In the lecture, we introduced an algorithm to compute homography between images from 4 pairs of point correspondences. Design an algorithm that instead uses pairs of line correspondences. Write the constraints provided by each correspondence, and how to compute the **H** that satisfies these.

To compute H^{-T} first:

$H^{-T} l = m$

$$H^{-T} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

$h_{11} l_1 + h_{12} l_2 + h_{13} l_3 = m_1$.

$h_{21} l_1 + h_{22} l_2 + h_{23} l_3 = m_2$

$h_{31} l_1 + h_{32} l_2 + h_{33} l_3 = m_3$.

$$\begin{bmatrix} l_1 & l_2 & l_3 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & l_1 & l_2 & l_3 & 0\ 0\ 0 \\ 0 & 0 & 0 & 0\ 0\ 0 & l_1 & l_2 & l_3 \\ \vdots & & \ddots & & & & \\ \vdots & & & & & & \ddots \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ \vdots \\ \vdots \end{bmatrix}$$

solve for $Ax = b$.

H^{-T} is same as H which has 8 DoF, remaining 1 for scalar

each correspondence from line pair gives 3 linear equations.

In total, we need 3 line corresspondances to compute H.