PROBLEM SET 1

16822 GEOMETRY-BASED METHODS IN VISION (FALL 2022)
https://piazza.com/cmu/fall2022/16822

OUT: Sep. 06, 2022
DUE: Sep. 13, 2022 11:59 PM
Instructor: Shubham Tulsiani
TAs: Mosam Dabhi, Kangle Deng, Jenny Nan

START HERE: Instructions

- **Collaboration policy:** All are encouraged to work together BUT you must do your own work (code and write up). If you work with someone, please include their name in your write up and cite any code that has been discussed. If we find highly identical write-ups or code without proper accreditation of collaborators, we will take action according to university policies, i.e. you will likely fail the course. See the Academic Integrity Section detailed in the initial lecture for more information.
- **Late Submission Policy:** There are **no** late days for Problem Set submissions.
- **Submitting your work:**
  - We will be using Gradescope (`https://gradescope.com/`) to submit the Problem Sets. Please use the provided template. Submissions can be written in LaTeX. Regrade requests can be made, however this gives the TA the opportunity to regrade your entire paper, meaning if additional mistakes are found then points will be deducted. Each derivation/proof should be completed on a separate page. For short answer questions you **should** include your work in your solution.
- **Materials:** The data that you will need in order to complete this assignment is posted along with the writeup and template on Piazza.

For multiple choice or select all that apply questions, replace `\choice` with `\CorrectChoice` to obtain a shaded box/circle, and don't change anything else.

Instructions for Specific Problem Types

For "Select One" questions, please fill in the appropriate bubble completely:

Select One: Who taught this course?

- ● Shubham Tulsiani
- ○ Deepak Pathak
- ○ Fernando De la Torre
- ○ Deva Ramanan

For "Select all that apply" questions, please fill in all appropriate squares completely:

Select all that apply: Which are scientists?

- ■ Stephen Hawking
- ■ Albert Einstein
- ■ Isaac Newton
- □ None of the above

For questions where you must fill in a blank, please make sure your final answer is fully included in the given space. You may cross out answers or parts of answers, but the final answer must still be within the given space.

Fill in the blank: What is the course number?

16-822

1 Vector Spaces [8pts]

1. **[2 pts]** Which of the following subsets of $\mathbb{R}^3$ are vector spaces? **Select all that are true: [2 pts]**

   - [x] The plane formed by the vector (v_1, v_2, v_3) such that $v_1 = v_2$
   - [ ] The plane formed by the vector (v_1, v_2, v_3) such that $v_1 = 1$
   - [ ] The plane formed by the vector (v_1, v_2, v_3) such that $v_1 v_2 v_3 = 0$
   - [x] All linear combinations of $\mathbf{v} = (1, 4, 0)$ and $\mathbf{w} = (2, 2, 2)$

2. **[2 pts]** For the following questions, consider a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$. Answer True or false (with a counterexample if false): **Select all that are true:**

   - [ ] The vectors b that are not in the column space $\mathbf{C}(\mathbf{A})$ form a subspace.
   - [x] If $\mathbf{C}(\mathbf{A})$ contains only zero vectors, then $\mathbf{A}$ is the zero matrix.
   - [x] The column space of the matrix $2\mathbf{A}$ equals the column space of $\mathbf{A}$
   - [ ] The column space of the matrix $\mathbf{A} - \mathbf{I}$ equals the column space of $\mathbf{A}$

3. **[2 pts]** Create a 3×4 matrix whose solution to $\mathbf{Ax} = 0$ is the $\mathbf{s}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{s}_2 = \begin{bmatrix} -2 \\ 0 \\ -6 \\ 0 \end{bmatrix}$

$$\begin{pmatrix} 3 & 9 & -1 & 1 \\ 3 & 9 & -1 & 2 \\ 3 & 9 & -1 & 3 \end{pmatrix}$$

4. **[2 pts]** If a 3×4 matrix has rank 3, what are its column space and left nullspace?

column space: $\text{rank}(A) = \dim(C(A)) = \dim(R(A)) = 3$

left nullspace: $\dim(N(A)) = \dim(\underset{A_{n \times m}}{m}) - \text{rank}(A) = 3 - 3 = 0$

2 Eigenvalues, Eigenvector, Singular Value Decomposition [16 pts]

1. **[2 pts]** Deduce the Eigenvalue and Eigenvectors of $\mathbf{A}$:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 5\lambda + 4 - 4 = 0$$

$$\lambda_1 = 0 \quad \lambda_2 = 5$$

for $\lambda_1 = 0$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \vec{e}_1 = 0$$

$$\vec{e}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

for $\lambda_2 = 5$

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \vec{e}_2 = 0$$

$$\vec{e}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

λ_1, λ_2 are eigen values

$\vec{e}_1$ $\vec{e}_2$ are eigen vectors

2. **[2 pts]** For

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Find the Eigenvalues and Eigenvectors of $\mathbf{A}$, $\mathbf{A}^2$ and $\mathbf{A}^{-1}$ and $\mathbf{A} + 4\mathbf{I}$

A:

$\lambda_1 = 3$, $\lambda_2 = 1$

$v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

A^2

$\lambda_1 = 9$, $\lambda_2 = 1$

$v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

A^{-1}

$\lambda_1 = \frac{1}{3}$, $\lambda_2 = 1$

$v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$A + 4I$

$\lambda_1 = 7$, $\lambda_2 = 5$

$v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

3. **[2 pts]** $\mathbf{A} \in \mathbb{R}^{m \times n}$ is positive definite if for any **non-zero** vector $\mathbf{x} \in \mathbb{R}^n$ we have $\mathbf{x}^\top \mathbf{A} \mathbf{x} > 0$:

$$\mathbf{x}^\top \mathbf{A} \mathbf{x} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Test matrices $\mathbf{C}$ and $\mathbf{D}$ for positive definitiveness

$$\mathbf{C} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}; \mathbf{D} = \begin{bmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{bmatrix}$$

Determinants of all upper-left sub-matrices are positive

$$|2| > 0 \quad \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0 \quad \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} > 0 \Rightarrow C \text{ is P.D.}$$

$$\begin{vmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{vmatrix} = -2b^2 + 2b + 4$$

$\Rightarrow \begin{cases} b > 2 \text{ or } b < -1 & D \text{ is N.D} \\ -1 < b < 2 & D \text{ is P.D.} \end{cases}$

4. **[3 pts]** Estimate the singular values σ_1 and σ_2 of the matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ C & 1 \end{bmatrix}$$

$A^T A = \begin{pmatrix} c^2+1 & c \\ c & 1 \end{pmatrix}$ $\lambda_1 = c^2 - \sqrt{c^2+4}\, c + 2 > 0$ for all c $\sigma_1 = \sqrt{\lambda_1}$

$\lambda_2 = c^2 + \sqrt{c^2+4}\, c + 2 > 0$ for all c $\sigma_2 = \sqrt{\lambda_2}$

5. **[3 pts]** Find the pseudoinverse of $\mathbf{A} \in \mathbb{R}^{m \times n}$

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

$A^+ = \begin{pmatrix} 0.2 & 0.1 \\ 0.2 & 0.1 \end{pmatrix}$

6. **[4 pts]** Suppose the following information is known about matrix $\mathbf{A}$:

$$\underset{①}{\mathbf{A} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}, \quad \underset{②}{\mathbf{A} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}, \quad \underset{③}{\mathbf{A} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}$$

③ − ② $A\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

I **[2 pts]** Find the eigenvalues of $\mathbf{A}$

II **[2 pts]** In each of the following subquestions, please justify with a reason (based on the theory of eigenvalues and eigenvectors).

(a) Is $\mathbf{A}$ a diagonalizable matrix?

(b) Is $\mathbf{A}$ an invertible matrix?

I. $Ax = \lambda x$.

$\lambda_1 = 6$, $\lambda_2 = 3$

$\lambda_3 = 0$

II. a) Yes, 3 eigen vectors $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are linearly independent.

b) No. A has Non-trival solution of $Ax = 0$

$x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Collaboration Questions Please answer the following:

1. Did you receive any help whatsoever from anyone in solving this assignment?

   ◯ Yes

   ◯ No

   - If you answered 'Yes', give full details:
   - (e.g. "Jane Doe explained to me what is asked in Question 3.4")

2. Did you give any help whatsoever to anyone in solving this assignment?

   ◯ Yes

   ◯ No

   - If you answered 'Yes', give full details:
   - (e.g. "I pointed Joe Smith to section 2.3 since he didn't know how to proceed with Question 2")

3. Did you find or come across code that implements any part of this assignment ?

   ◯ Yes

   ◯ No

   - If you answered 'Yes', give full details: No
   - (book & page, URL & location within the page, etc.).