PHYS/ENGP 3170/6170
Computational Physics and Engineering
Spring 2021

Assignment #6: Integrating Newton's Laws
Due: Fri, April 16 11:59 pm

Readings: LPB 8.1-8.6 (8.8-8.10 may also be helpful)

You will start with the code rk4.py, which is set up to integrate the equations of motion for a harmonic oscillator using the 4th order Runge-Kutta method. The equations solved are $dx/dt=v$ and $dv/dt=-(k/m)\,x$, where we take $k=m=1$, and the initial conditions are $x=1$ and $v=0$ at $t=0$.

(1) Plot position x and energy $E = (1/2)mv^2 + (1/2)kx^2$ as functions of time over at least 10 periods of oscillation, and compare with the analytic results. How many steps do you need to use to obtain good agreement?

(2) Obtain the period of the motion as accurately as you can by starting with $x=x_0$, $v=0$, and finding the time T when the velocity v again crosses 0 for positive x. Since you are using a fixed step size in time, and no step will exactly correspond to $v=0$, you will probably want to use some kind of interpolation to find the best value for T. For example, if (t_1,v_1) and (t_2,v_2) are two time steps that "bracket" your solution ($v_1<0<v_2$), then you may think of t as a function of v, and use linear interpolation $t = ((v_2-v)t_1+(v-v_1)t_2)/(v_2-v_1)$ to find t that corresponds to $v=0$. Using a higher-order interpolation might be even better, depending on your time step. Check that the period T is independent of the amplitude x_0. Compare the period you obtain with the analytic value.

(3) Modify your program to solve the equations of motion for a simple pendulum: $d\theta/dt=\omega$ and $d\omega/dt=-(g/L)\sin\theta$. Just as above, you may without loss of generality choose units where $g=L=1$. Start with $\theta=\theta_0$, $\omega=0$, and obtain the period of motion T as a function of the amplitude θ_0, for $0 < \theta_0 < \pi$. Check that for small θ_0, the period is the same as for the harmonic oscillator in part (2). Compare the dependence of T on θ_0 with the analytic prediction for small θ_0: $T=2\pi[1+(1/4)\sin^2(\theta_0/2)+(9/64)\sin^4(\theta_0/2)+\ldots]$. What happens as θ_0 approaches π, and why?

(4) Add an external driving force and damping to your equations: $d\omega/dt= -\sin\theta + F_{dr}\sin(\omega_{dr}t) - \gamma\omega$, where driving amplitude F_{dr}, driving frequency ω_{dr}, and damping rate γ are constants. Pick a small damping constant, e.g. $\gamma = 0.1$, and a driving frequency ω_{dr} different from the natural frequency $\omega_0 =1$ of the pendulum. Choose F_{dr} so that the amplitude of oscillations is reasonable, and check that you observe mode locking, where the period of motion equals the period of the driving after a sufficient time has elapsed. How does the phase of the motion compare with the phase of the driving force?

(5) [Required for 6170 students only, extra credit for 3170 students] Now turn off the damping and set the initial position and velocity to zero. Choose F_{dr} small, so that the oscillations are close to harmonic. For frequencies ω_{dr} close (but not equal) to the natural frequency $\omega_0 = 1$, identify the "beats" as energy flows into and out of the system. Plot the amplitude of oscillation vs. the driving frequency ω_{dr}, and comment on your results. What happens when $\omega_{dr} = \omega_0$?