Module 2 Rotational Motion

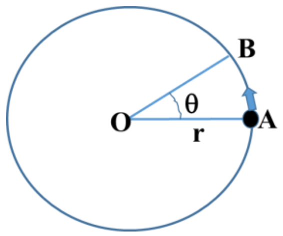
Motion is the change with time of the position or orientation of a body.

Types of Motion

a) **Translation motion** (Linear Motion)
   In linear motion, the object moves from one position to another in either a curved direction or a straight line.
   Rectilinear Motion - The route taken by an object is a straight line.
   Curvilinear Motion - The route taken by the object is curved.
b) Rotational Motion : The objects moves around an axis and their location do not change with time.
c) Oscillatory motion: TO and fro motion of an object about a fixed point.

Circular Motion:

An object which moves in a circle at constant speed is said to be executing uniform circular motion. Magnitude of the velocity remains constant, the direction of the velocity continuously changes.



During circular motion, the radius vector traces an angle at the centre. This angle is **angular displacement θ with unit** rad.
(1 radian is the angle subtended by an arc whose length is equal to the radius of the circle. 2л rad = 360)

1 radian = 360/2π = 57.3°

Angular velocity (ω) : It is defined as the angular displacement in unit time.
If θ is the angular displacement in a time t, **ω = θ/t** rad/s

Linear velocity: The displace ment of the radius vector along the circular path during time t is called linear velocity . i.e, **v=s/t where s is the displacement from A to B**

Relation between linear velocity and Angular velocity (v and ω)

We know that the linear velocity *v=s/t* where s is the length of the arc AB
From geometry of a circle, the angle subtended at the centre by an arc is given by angle= arc/radius I,e, $\theta = s/r$ or

$s = r\,\theta.$

Now substituting this value of s in v gives

$$v = \frac{s}{t} = \frac{r\theta}{t} - = r\omega$$ **where ω = θ/t**

Hence the relation between linear velocity and angular velocity is given by $\boldsymbol{v = r\omega}$

Relation between linear acceleration and angular acceleration

Angular Acceleration $\alpha = \frac{\omega_2 - \omega_1}{t}$ rad/s^2

Linear acceleration $a = \frac{v_2 - v_1}{t} = \frac{r\omega_2 - r\omega_1}{t} = \frac{r(\omega_2 - \omega_1)}{t} = r\alpha$

$\boldsymbol{a = r\alpha}$

Period

The time required to complete one revolution is called period(T). When one revolution is completed, angular displacement is of 2π radians. Then angular velocity is given by

$$\omega = \frac{2\pi}{T} \qquad \text{hence} \quad T = \frac{2\pi}{\omega}$$

Centripetal acceleration a_c

The acceleration of a particle moving along a circular path with uniform speed is always directed towards the centre of the circle. This acceleration is called **centripetal acceleration**.

Centripetal acceleration = change in velocity /time =vθ/t =vω
Since v =rω it can be also written as

$$a_c = v\omega = \frac{v^2}{r} = r\omega^2$$

Centripetal Force

The force which, acting along the radius towards the centre of the circular path, causes the body to move in a circle with constant speed is called centripetal force

$$F_c = \frac{mv^2}{r} = mr\omega^2 = mv\omega$$

The centripetal force is provided differently for different bodies and a few examples are given below:

a) If a body is attached at one end of a string and whirled round, the tension provides the centripetal force.

b) In the case of planets revolving around the sun, the necessary centripetal force is provided by the gravitational attraction between them.

c) When an electron moves around the nucleus of an atom, the centripetal force is provided by the electrostatic force of attraction between electron and proton.

d) When a vehicle moves along a curved path, the centripetal force is provided by the frictional force between the tyres and the road.

Banking of the roads

To avoid skidding, the outer edge of the road is raised above the level of the inner edge at the curves. This is known as the banking of roads.

The banking of roads avoids skidding and reduces wear and tear of the tyres.

In a banked road, the horizontal component of normal reaction will also contribute to centripetal force in addition to frictional force.

$\theta = tan^{-1} (v^2/ rg)$

<u>Derivation</u>

The angle of banking is the angle made by the elevated path with the horizontal.

Let AB and AC represent the horizontal and banked paths respectively as shown in Fig

Let θ be the angle of banking.

Consider a vehicle of mass m takes a curved path of radius r with a speed v.

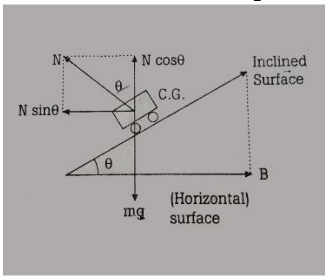
The weight of the vehicle mg acts vertically downwards.

The normal reaction N of the road on the vehicle will be perpendicular to the AC.

The normal reaction can be resolved into vertical and horizontal components.

The vertical component is equal to the weight of the body. $Ncos\theta = mg$

The horizontal component provides the centripetal force $Nsin\theta = mv^2/ r$



Dividing the second equation by the first gives $tan\theta = v^2/rg$

Or $= tan^{-1} (v^2/ rg)$

The angle of banking depends on the radius of the curve of the road and the speed of the vehicle.

Banking of railway tracks

When a fast-moving train takes a curved path, it tends to move away tangentially off the track. To avoid this, the outer rail is raised above the level of the inner rail. This is known as the banking of railway tracks. The banking of railway tracks avoids skidding and reduces the wear and tear of the wheels.

In the case of a curved railway track, the level of the outer rail is higher than that of the inner one. The height of the outer rail above the inner rail in the banked rail track is called superelevation (S).

If **d** is the distance between the rails of super elevation. Then $sin\theta = S\ d$ or $S = dsin\theta$

Since θ is usually small for banked rail tracks approximately equal to tanθ

$tan\theta = S/d$

But the equation for the angle of banking is given by

$tan\theta = v^2/rg$ i.e $S/d = v^2/rg$

Or $S = \frac{v^2 d}{rg}$

ROTATIONAL MOTION OF RIGID BODIES

Moment of Inertia of a rigid body

Moment of Inertia of a rigid body is the measure of an object's resistance to change its direction of rotation. The moment of inertia depends on the mass of the object and the distance between the axis of rotation and centre of mass.

Centre of mass is the point at which all the mass of a body is considered to be concentrated**.**

The moment of inertia of a particle about a given axis is defined as the product of the mass of the particle and the square of the distance of the body from the axis.

The formula of Moment of Inertia is $I = \sum_i m_i r_i^2$ =MK^2 where M is the total mass of the body and K is called radius of gyration

Radius of Gyration

Radius of Gyration is defined as the distance from the axis of rotation to the centre of mass of the body so that the moment of inertia will be equal to the moment of inertia of the actual body

If M is the total mass of the body and K is the radius of gyration of the body about the axis of rotation, then the moment of inertia is given by $I = MK^2$

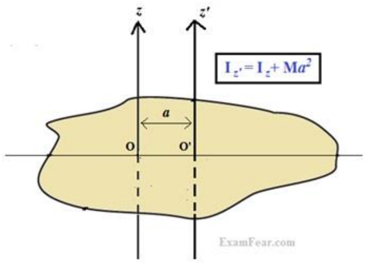
$$K = \sqrt{\frac{I}{M}}$$

The SI unit of the radius of gyration is meter. The radius of gyration depends on the distribution of mass from the axis of rotation and the position and direction of the axis of rotation.

Parallel axes theorem

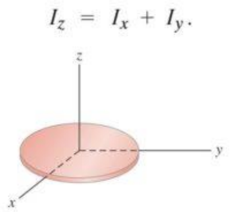
The moment of inertia of a body about an axis parallel to the axis passing through its center is the sum of moment of inertia of a body about the axis passing through the CM(I_{cm}) and product of the mass of the body times the square of the distance between these two axes. Consider a body of mass M. Let I_{cm} be the MI about the axis passing through CM. Then

$$I = I_{cm} + Md^2$$

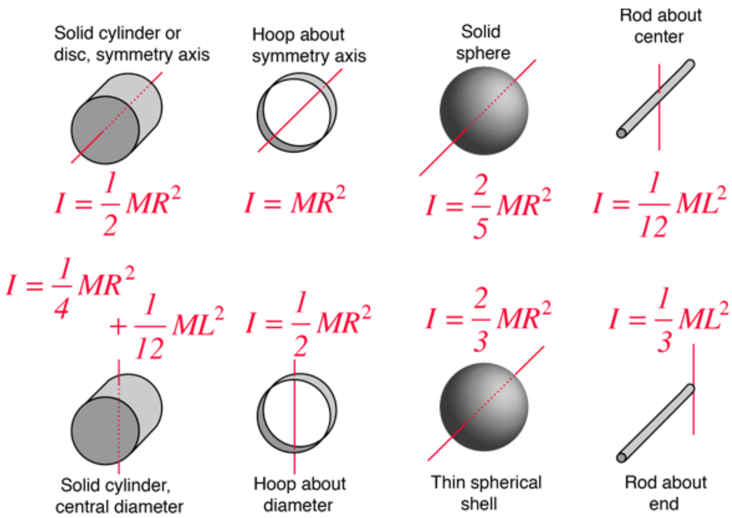


Perpendicular Axis theorem

The perpendicular axis theorem states that the moment of inertia of a planar lamina (i.e. 2-D body) about an axis perpendicular to the **plane** of the lamina is equal to the sum of the moments of inertia of the lamina about the two axes at right angles to each other, in its own **plane** intersecting each other at the point. $\boldsymbol{I_z = I_x + I_y}$



MI of different shapes



Angular momentum

Momentum of a body due to rotation. It is called moment of linear momentum
Consider a particle of linear momentum **P** (*mv*). Let r be the vector distance of the particle from the point.
Then the angular momentum $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{P} = mvr$. But $\boldsymbol{v} = r\omega$

Therefore angular momentum $\boldsymbol{L} = mr\omega r = mr^2\omega = I\omega$
That is $\boldsymbol{L = I\omega}$

Torque (Rotational Force)

The rotational force required to provide angular acceleration is called torque(τ)
Torque is a vector quantity.
Torque in rotational motion is equivalent to force in linear motion.

So, similar to the definition of force, **the rotational force(torque) is defined as the rate of change of angular momentum**

$$\tau = \frac{\partial L}{\partial t}$$

$$\frac{\partial L}{\partial t} = \frac{\partial (I\omega)}{\partial t} = I\frac{\partial \omega}{\partial t} = I\alpha$$

Hence $\tau = I\alpha$ where I is the moment of inertia. (Mass in linear motion is replaced with MI in rotation motion)

Linear force and torque are related using the equation

$$\tau = rF$$

So $rma = rmr\alpha = mr^2\alpha = I\alpha = \tau$