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Discrete Math

Lecture notes integrated with the book "TODO",
Author TODO, ...

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Information and Contacts

Personal notes and summaries collected as part of the *Discrete Math* course offered by the degree in Computer Science of the University of Rome "La Sapienza".

Further information and notes can be found at the following link:

<https://github.com/aflaag-notes>. Anyone can feel free to report inaccuracies, improvements or requests through the Issue system provided by GitHub itself or by contacting the author privately:

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The notes are constantly being updated, so please check if the changes have already been made in the most recent version.

Suggested prerequisites:

- Differential Calculus
- Integral Calculus

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1

TODO

1.1 TODO

1.1.1 TODO

Definition 1.1.1.1: Peano's axioms

The **Peano's axioms** are 5 axioms which define the set \mathbb{N} of the **natural numbers**, and they are the following:

- i) $0 \in \mathbb{N}$
- ii) $\exists \text{succ} : \mathbb{N} \rightarrow \mathbb{N}$, or equivalently, $\forall x \in \mathbb{N} \quad \text{succ}(x) \in \mathbb{N}$
- iii) $\forall x, y \in \mathbb{N} \quad x \neq y \implies \text{succ}(x) \neq \text{succ}(y)$
- iv) $\nexists x \in \mathbb{N} \mid \text{succ}(x) = 0$
- v) $\forall S \subseteq \mathbb{N} \quad (0 \in S \wedge (\forall x \in S \quad \text{succ}(x) \in S)) \implies S = \mathbb{N}$

Note: inside this notes, it will be assumed that $0 \in \mathbb{N}$.

Principle 1.1.1.1: Induction principle

Let P be a property which is true for $n = 0$, thus $P(0)$ is true; also, for every $n \in \mathbb{N}$ we have that $P(n) \implies P(n + 1)$; then $P(n)$ is true for every $n \in \mathbb{N}$.

Using symbols, using the formal logic notation, we have that

$$\frac{P(0) \quad P(n) \implies P(n + 1)}{\forall n \quad P(n)}$$

Observation 1.1.1.1: The fifth Peano's axiom

Note that the fifth Peano's axiom is equivalent to the induction principle, since, it states that for every subset S of \mathbb{N} containing 0 and closed under succ must be equal to \mathbb{N} itself.

Definition 1.1.1.2: Integers

TODO

Definition 1.1.1.3: Divisor

TODO

Example 1.1.1.1 (Divisors). TODO

Definition 1.1.1.4: \mathbb{P}

TODO

Proposition 1.1.1.1: \mathbb{P} is infinite

There are infinitely many primes. Using symbols

$$|\mathbb{P}| = +\infty$$

Proof. By way of contradiction, assume that \mathbb{P} is finite, thus

$$\exists n \in \mathbb{N} \mid \mathbb{P} = \{p_1, \dots, p_n\}$$

and let $x = p_1 \cdot \dots \cdot p_n$. Since $x \neq p_1, \dots, p_n$, then $x \notin \mathbb{P}$, so x is not a prime number; but x can't be divided by any of the p_1, \dots, p_n either, because the remainder will always be 1. This means that x is neither prime nor non-prime, which is a contradiction \nmid . \square

Definition 1.1.1.5: gcd

The **gcd** (*Greatest Common Divisor*) of two given numbers a, b is the greatest of the divisors which a and b have in common. Using symbols, we say that

$$d = \gcd(a, b) \iff \forall f \in \mathbb{N} : f \mid a \wedge f \mid b \implies f \mid d$$

If the gcd of two numbers is 1, they are said to be **coprime**.

Example 1.1.1.2 (gcd). Given 15 and 63, we have that $\gcd(15, 63) = 3$.

Algorithm 1.1.1.1: Euclid's algorithm**Input:** Two natural numbers a, b .**Output:** $\gcd(a, b)$.

```

1: function GCD( $a, b$ )
2:   TODO
3: end function

```

Idea. TODO

Example 1.1.1.3 (Euclid's algorithm). To compute the $\gcd(341, 527)$, using the [Algorithm 1.1.1.1](#), we get the following:

$$\begin{aligned}
 527 &= 341 \cdot 1 + 186 \\
 341 &= 186 \cdot 1 + 155 \\
 186 &= 155 \cdot 1 + 31 \\
 155 &= 31 \cdot 5 + 0
 \end{aligned}$$

hence we have that

$$\gcd(341, 527) = 31$$

Lemma 1.1.1.1: Bézout's identity

Given a pair of numbers $a, b \in \mathbb{Z}$, there exists $x, y \in \mathbb{Z}$ such that the $\gcd(a, b)$ is a [linear combination](#) of a and b . Using symbols

$$\forall a, b \in \mathbb{Z} \quad \exists x, y \in \mathbb{Z} \mid \gcd(a, b) = ax + by$$

Proof. Omitted. □

Example 1.1.1.4 (Bézout's identities). Using the [Example 1.1.1.3](#), in order to compute the Bézout's identity of 341 and 527, we need to do the following:

$$31 = 186 - 155 \cdot 1 = 186 - (341 - 186 \cdot 1) = 2 \cdot 186 - 341 = 2 \cdot (527 - 341) - 341 = 2 \cdot 527 - 3 \cdot 341$$

thus the Bézout's identity is

$$31 = 2 \cdot 527 - 3 \cdot 341$$

Corollary 1.1.1.1: Prime divisors

Given a natural number $n \in \mathbb{N}$ and a prime number $p \in \mathbb{P}$, it holds that

$$p \nmid n \iff \gcd(p, n) = 1$$

Proof.

First implication. Instead of proving that $p \nmid n \implies \gcd(p, n) = 1$, we will prove the contrapositive, namely that $\gcd(p, n) > 1 \implies p \mid n$. Hence, since $\gcd(p, n) \mid p$

by definition, because $p \in \mathbb{P}$ then $\gcd(p, n)$ must be either 1 or p itself, and we assumed that $\gcd(p, n) > 1$, $\gcd(p, n)$ must be p , which means that $p \mid n$.

Second implication. Note that $\gcd(p, n) = 1 \implies \exists x, y \in \mathbb{Z} \mid 1 = px + ny$ by the [Lemma 1.1.1.1](#), hence if $p \mid a$ then $p \mid 1$ by the [Definition 1.1.1.5](#), which is impossible because $p \in \mathbb{P}$ by the [Definition 1.1.1.4](#).

□

Lemma 1.1.1.2: Prime divisors

Given a pair of numbers $a, b \in \mathbb{N}$, and a prime number $p \in \mathbb{P}$ such that $p \mid ab$, then either $p \mid a$ or $p \mid b$. Using symbols

$$\forall a, b \in \mathbb{N} \quad \exists p \in \mathbb{P} : p \mid ab \implies p \mid a \vee p \mid b$$

Proof. Without loss of generality, assume that $p \nmid a$, thus $\gcd(p, a) = 1$ by the [Corollary 1.1.1.1](#); hence, for the [Lemma 1.1.1.1](#), we have that

$$\exists x, y \in \mathbb{Z} \mid 1 = px + ay \iff b = bpx + bay$$

Note that $p \mid ab \iff \exists k \in \mathbb{Z} \mid pk = ab$ which means that

$$b = bpx + pky = p(bx + ky) \iff p \mid b$$

The same argument can be used to show that $p \nmid b \implies p \mid a$.

□

Theorem 1.1.1.1: Fundamental theorem of arithmetic

The **fundamental theorem of arithmetic**, also known as the **UPF** theorem (*Unique Prime Factorization*) states that for every natural number $n \in \mathbb{N}$ there exists a unique prime factorization for n . Using symbols

$$\forall n \in \mathbb{N} \quad \exists! p_1, \dots, p_k \in \mathbb{P}, e_1, \dots, e_k \in \mathbb{N} \mid n = p_1^{e_1} \cdot \dots \cdot p_k^{e_k}$$

Proof. Omitted.

□

1.1.2 Continued fractions

Definition 1.1.2.1: Continued fraction

A **continued fraction** is an expression obtained through an iterative process of representing a number as the sum of its *integer part*, and the reciprocal of another number. Continued fractions can be both **finite** and **infinite**, and are represented with the following notation:

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}} = [a_0; a_1, a_2, \dots, a_n]$$

for finite continued fractions, and

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}} = [a_0; a_1, a_2, \dots]$$

for infinite continued fractions.

Example 1.1.2.1 (Finite continued fractions). Consider the [Example 1.1.1.3](#); note that the Euclid algorithm can be used to derive the finite continued fraction of $\frac{527}{341}$, as follows:

$$\begin{aligned} \frac{527}{341} &= 1 + \frac{186}{341} \\ \frac{341}{186} &= 1 + \frac{155}{186} \\ \frac{186}{155} &= 1 + \frac{31}{155} \\ \frac{155}{31} &= 5 \end{aligned}$$

and then, rearranging

$$\frac{527}{341} = 1 + \frac{1}{1 + \frac{155}{186}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{31}{155}}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{5}}} = [1; 1, 1, 5]$$

Example 1.1.2.2 (Infinite continued fractions). Assume that there exists an x such that

$$\sqrt{2} = 1 + \frac{1}{x}$$

then rearrange as follows:

$$\sqrt{2} = 1 + \frac{1}{x} \iff \sqrt{2} - 1 = \frac{1}{x} \iff x = \frac{1}{\sqrt{2} - 1} = \frac{\sqrt{2} + 1}{(\sqrt{2} - 1)(\sqrt{2} + 1)} = \sqrt{2} + 1$$

and now we can substitute $\sqrt{2}$ with $1 + \frac{1}{x}$, yielding the following:

$$x = 1 + \frac{1}{x} + 1 = 2 + \frac{1}{x}$$

Finally, this equation can be used to construct the infinite continued fraction of $\sqrt{2}$, like this:

$$x = 2 + \frac{1}{x} = 2 + \frac{1}{2 + \frac{1}{x}} = 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{\ddots}}}$$

implying that

$$\sqrt{2} = 1 + \frac{1}{x} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{\ddots}}} = [1; 2, 2, 2] = [1; \overline{2}]$$

Algorithm 1.1.2.1: Continued fractions

Given a continued fraction $[a_0; a_1, \dots, a_n]$, the corresponding number can be computed by constructing the following table (note that $N_0 := 1$ and $D_0 := 0$, meaning that a and N, D differ by 1 position at each column)

C.F.		a_0	a_1	a_2	\dots	a_n
N	1	a_0	$a_1 \cdot N_1 + N_0$	$a_2 \cdot N_2 + N_1$	\dots	$a_n \cdot N_n + N_{n-1}$
D	0	1	$a_1 \cdot D_1 + D_0$	$a_2 \cdot D_2 + D_1$	\dots	$a_n \cdot D_n + D_{n-1}$

then, the answer is

$$[a_0; a_1, \dots, a_n] = \frac{N_{n+1}}{D_{n+1}}$$

Idea. TODO

Example 1.1.2.3. To compute the number corresponding to the continued fraction $[2; 1, 3, 1, 5, 4]$, the following table can be constructed:

C.F.		2	1	3	1	5	4
N	1	2	3	11	14	81	338
D	0	1	1	4	5	29	121

meaning that

$$[2; 1, 3, 1, 5, 4] = \frac{338}{121}$$

Definition 1.1.2.2: The golden ratio

TODO

Example 1.1.2.4 (ICF of the Golden Ratio). TODO

1.1.3 Series

Definition 1.1.3.1: The harmonic series

The harmonic series is defined as follows:

$$\sum_{k=1}^{+\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

Proposition 1.1.3.1: Divergence of the harmonic series

The harmonic series diverges.

Proof. Suppose that the harmonic series converges, thus

$$\exists S \mid \sum_{k=1}^{+\infty} \frac{1}{k} = S$$

then we have that

$$\begin{aligned} S &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \dots = \\ &= \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) + \dots > \\ &> \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \dots = \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \dots = S \end{aligned}$$

implying that $S > S$. □

Definition 1.1.3.2: Geometric series

A **geometric series** is commonly written as

$$\sum_{k=0}^n ar^k$$

where $a \in \mathbb{R}$ is a coefficient and $r \in \mathbb{R}$ is the *ration between adjacent terms*.

Proposition 1.1.3.2: Convergence of geometric series

For $r \in \mathbb{R}$ such that $|r| < 1$, it holds that

$$\sum_{k=0}^{+\infty} ar^k = \frac{a}{1-r}$$

Proof. Omitted. □

Theorem 1.1.3.1: Reciprocal of primes

The sum of the reciprocal of the prime numbers diverges. Using symbols

$$\sum_{p \in \mathbb{P}} \frac{1}{p} = +\infty$$

Proof. Consider the following inequality:

$$\forall n \in \mathbb{N} \quad \prod_{p \in \mathbb{P} | p \leq n} \frac{p}{p-1} > \sum_{k=1}^n \frac{1}{k}$$

We can prove it with as follows:

- for any given $p \in \mathbb{P}$, the fraction $\frac{p}{p-1}$ can be rewritten as follows, by using the [Proposition 1.1.3.2](#):

$$\forall p \in \mathbb{P} \mid p \leq n \quad \frac{p}{p-1} = \frac{1}{\frac{p-1}{p}} = \frac{1}{1 - \frac{1}{p}} = \sum_{k=0}^{+\infty} \frac{1}{p^k} = 1 + \frac{1}{p} + \frac{1}{p^2} + \dots$$

where $a = 1$ and $r = \frac{1}{p^k} : \frac{1}{p^{k-1}} = \frac{1}{p}$ – which is the infinite sum of the reciprocal of the powers of some prime number p

- this means that

$$\exists p_1, \dots, p_j \in \mathbb{P} \mid \prod_{p \in \mathbb{P} | p \leq n} \frac{p}{p-1} = p_1 \cdot \dots \cdot p_j \implies \prod_{p \in \mathbb{P} | p \leq n} \frac{p}{p-1} = \sum_{k=0}^{+\infty} \frac{1}{p_1^k} \cdot \dots \cdot \sum_{k=0}^{+\infty} \frac{1}{p_j^k}$$

- thus, thanks to the [Theorem 1.1.1.1](#) this product expands to the sum of the reciprocal of every natural number that contains p_1, \dots, p_j in his prime factorization, namely

$$\exists e_1, \dots, e_j \in \mathbb{N} \mid \sum_{k=0}^{+\infty} \frac{1}{p_1^k} \cdot \dots \cdot \sum_{k=0}^{+\infty} \frac{1}{p_j^k} = \sum_{k=0}^{+\infty} \frac{1}{p_1^{e_1} \cdot \dots \cdot p_j^{e_j}}$$

- finally, since $p_1, \dots, p_j \leq n$ this summation must contain at least every term contained inside $\sum_{k=1}^n \frac{1}{k}$, which proves the inequality. □

Now consider the following:

$$\begin{aligned}
& \sum_{k=1}^{+\infty} \frac{1}{k} < \prod_{p \in \mathbb{P} | p \leq n} \frac{p}{p-1} \iff \\
& \iff \log \left(\sum_{k=1}^{+\infty} \frac{1}{k} \right) < \left(\prod_{p \in \mathbb{P} | p \leq n} \frac{p}{p-1} \right) = \\
& = \sum_{p \in \mathbb{P} | p \leq n} \log \left(\frac{p}{p-1} \right) = \sum_{p \in \mathbb{P} | p \leq n} (\log p - \log(p-1)) = \sum_{p \in \mathbb{P} | p \leq n} \int_{p-1}^p \frac{1}{x} dx
\end{aligned}$$

and consider the area under the curve $\frac{1}{x}$ within the $[p-1, p]$ interval, for some prime number p :

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since

$$\forall x_1, x_2 \in \mathbb{R} \quad x_1 < x_2 \iff \frac{1}{x_1} > \frac{1}{x_2}$$

the function $\frac{1}{x}$ is monotonically decreasing, and in particular

$$\forall p \in \mathbb{P} \mid p \leq n \quad p-1 < p \implies \frac{1}{p-1} > \frac{1}{p}$$

whic implies that the area under the curve $\frac{1}{x}$ within the $[\frac{1}{p}-1, p]$ must be smaller than the area of the rectangle that has a base of of $p - (p-1) = p - p + 1 = 1$ and an height of $\frac{1}{p-1}$, namely an area of $1 \cdot \frac{1}{p-1} = \frac{1}{1-p}$. This implies that

$$\sum_{p \in \mathbb{P} | p \leq n} \int_{p-1}^p \frac{1}{x} dx < \sum_{p \in \mathbb{P} | p \leq n} \frac{1}{p-1}$$

Suppose that

$$\frac{1}{p-1} > \frac{2}{p}$$

then we have that

$$\frac{1}{p-1} > \frac{2}{p} \iff p > 2 \cdot (p-1) \iff p > 2p-2 \iff 2 > 3p$$

which is not possibile because $p \in \mathbb{P} \mid p \leq n$. This implies that

$$\frac{1}{p-1} < \frac{2}{p} \implies \sum_{p \in \mathbb{P} | p \leq n} \frac{1}{p-1} < \sum_{p \in \mathbb{P} | p \leq n} \frac{2}{p}$$

and finally, this means that

$$\log \left(\sum_{k=1}^{+\infty} \frac{1}{k} \right) < \sum_{p \in \mathbb{P} | p \leq n} \frac{2}{p} \iff \frac{1}{2} \log \left(\sum_{k=1}^{+\infty} \frac{1}{k} \right) < \sum_{p \in \mathbb{P} | p \leq n} \frac{1}{p}$$

and because $\sum_{k=1}^{+\infty} \frac{1}{k}$ diverges, the left-hand side of the inequality diverges, thus the right-hand side must diverge too. This proves the statement, because the primes $p \in \mathbb{P}$ such that $p \leq n$ form a subset of \mathbb{P} .

1.2 Solved exercises

1.2.1 Number theory

Problem 1.2.1.1: $n^2 + n$ is even

Show that for every $n \in \mathbb{N}$, $n^2 + n$ is an even number.

Proof. Note that $n^2 + n = n \cdot (n + 1)$, hence:

- if n is even, then

$$\exists k \in \mathbb{N} \mid n = 2k \implies n(n + 1) = 2k(2k + 1) = 4k^2 + 2k = 2(k^2 + k)$$

which is an even number;

- if n is odd, then

$$\exists k \in \mathbb{N} \mid n = 2k + 1 \implies n(n + 1) = (2k + 1)(2k + 2) = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1)$$

which is an even number.

□

Problem 1.2.1.2: $4n - 1$ is not prime

Show that there are infinitely many numbers of the form $4n - 1$ that are not prime.

Proof. Note that

$$\forall x^2 \in \mathbb{N} - \{0\} \quad 4x^2 - 1 = (2x + 1)(2x - 1)$$

which is a proper factorization of $4x^2 - 1$, hence every perfect square yields a number of the form $4n - 1$ which is not a prime number. Note that the number of perfect squares is infinite since the set of perfect square has the same cardinality of \mathbb{N} since it's possible to construct a bijective function as follows:

$$f : \mathbb{N} \rightarrow \mathbb{N} : x \mapsto x^2$$

Also, note that this proof does not show *every non-prime number of the form $4n - 1$* , since that is outside the scope of the problem. □

Problem 1.2.1.3: The $4n - 3$ set

Consider the following set:

$$S := \{4n - 3 \mid n \in \mathbb{N}\}$$

1. Show that S closed under multiplication.
2. A number p is said to be S -prime if and only if p is the product of exactly two factors of S ; for example, even though $3^2 = 9 \notin \mathbb{P}$ we have that $9 = 1 \cdot 9$, and since $1 = 4 \cdot 1 - 3 \in S$ and $9 = 4 \cdot 3 - 3 \in S$, then 9 is S -prime. Is the set of S -prime numbers infinite?
3. TODO

Proof.

1. To show that S is closed under multiplication, it suffices to show that

$$\forall a, b \in \mathbb{N} \quad (4a - 3)(4b - 3) = 16ab - 12a - 12b + 9 = 4(4ab - 3a - 3b + 3) - 3 \in S$$

2. TODO

□

1.2.2 Induction**Problem 1.2.2.1: Cardinality of the power set**

Show that for every given set S such that $n := |S|$ it holds that $|\mathcal{P}(S)| = 2^n$.

Proof. The statement will be shown by induction over n , the number of elements contained into S .

Base case. $n = 0 \implies S = \emptyset \implies \mathcal{P}(S) = \mathcal{P}(\emptyset) = \{\emptyset\} \implies |\mathcal{P}(S)| = 1 = 2^0 = 2^n$.

Inductive hypothesis. Assume that the statement is true for some fixed integer n .

Inductive step. It must be shown that, for a given set of elements S such that $|S| = n + 1$, it holds true that $|\mathcal{P}(S)| = 2^{n+1}$. Consider a subset $S' \subseteq S$ such that $|S'| = |S| - 1 = n + 1 - n = n$, hence for the inductive hypothesis we have that $|\mathcal{P}(S')| = 2^n$. Thus, to get the cardinality of $\mathcal{P}(S)$ the $(n + 1)$ -th element inside $S - S'$ must be paired with every of the sets contained inside $\mathcal{P}(S')$, hence

$$|\mathcal{P}(S)| = 2 \cdot |\mathcal{P}(S')| = 2 \cdot 2^n = 2^{n+1}$$

□

1.2.3 Continued fractions

Problem 1.2.3.1: Limits of continued fractions

What is the value that the following limit approaches?

$$\lim_{n \rightarrow +\infty} [2; 1, 4, n]$$

Proof. By using the [Algorithm 1.1.2.1](#), we get the following table:

C.F.		2	1	4	n
N	1	2	3	14	$14 \cdot n + 3$
D	0	1	1	5	$5 \cdot n + 1$

which means that

$$[2; 1, 4, n] = \frac{14n + 3}{5n + 1} \implies \lim_{n \rightarrow +\infty} \frac{14n + 3}{5n + 1} = \frac{14}{5}$$

□