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Graph Theory

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Contents

Information and Contacts

Personal notes and summaries collected as part of the *Graph Theory* course offered by the degree in Computer Science of the University of Rome "La Sapienza".

Further information and notes can be found at the following link:
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Suggested prerequisites:

- Progettazione degli Algoritmi

Licence:

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Basics of Graph Theory

In the 18th century, in the city of Königsberg (Prussia), a puzzle captured the imagination of the townspeople. Königsberg, nestled along the winding *Pregel River*, was divided into four land masses — two parts of the mainland and two islands, Kneiphof and Lomse. Connecting these regions were **seven bridges**, crisscrossing the river back and forth.

Over time, a curious question arose among the people of Königsberg: was it possible to take a *walk* through the city, crossing each of the seven bridges **exactly once**, without retracing any steps, and ending the walk in the same place where it started? This is known as the Seven Bridges of Königsberg problem.

It seemed simple enough, yet no one had managed to do it. The challenge became a favorite pastime, debated in marketplaces and whispered about in taverns. Some claimed it was possible with the right path, while others remained skeptical.

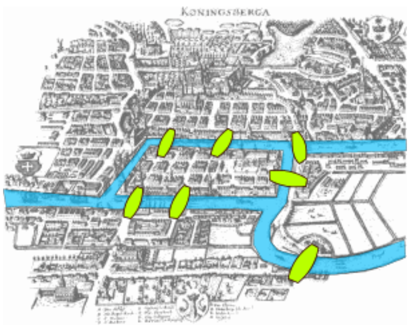


Figure 1.1: The map of Königsberg in Euler's time, showing the actual layout of the seven bridges [con25].

Word of this peculiar problem reached the brilliant Swiss mathematician Leonhard Euler, a man whose mind was always drawn to patterns and logic. Intrigued, Euler set out to solve the riddle — not by drawing endless maps or walking the streets himself, but by **abstracting** the problem into something entirely new.

Euler realized that the specific layout of the city was *irrelevant*. What truly mattered was the way the landmasses were connected by the bridges. He represented each landmass as a **dot** and each bridge as a **line** between them. In doing so, he stripped away unnecessary details and created a simple, elegant combinatorial structure that we now refer to as **graph**.

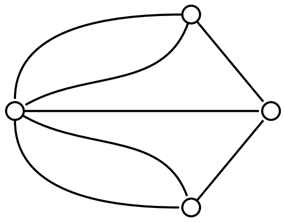


Figure 1.2: The graph drawn by Euler which models the *Seven Bridges of Königsberg* problem.

Through his analysis, Euler discovered a fundamental rule: for a walk to cross each bridge exactly once and return to the starting point, every landmass had to be connected by an **even** number of bridges. In Königsberg's case, however, each landmass had an odd number of bridges, making the task impossible.

Euler's proof, published in 1736, was groundbreaking — not just because he solved the Königsberg puzzle, but because he laid the foundation for an entirely new branch of mathematics: graph theory. His ideas would go on to shape the study of networks, from modern transportation systems to social media connections and even the vast web of the internet itself. And so, from a simple question about bridges in a small Prussian city, a whole new field of mathematics was born — one that continues to shape the world centuries later.

This chapter will discuss the basics of the field of **graph theory**, and will lay the foundation for later chapters.

1.1 Introduction

Definition 1.1: Graph

A **graph** is a pair $G = (V, E)$, where V is the — finite — set of **vertices** of the graph, and E is the set of **edges**.

For now, we will assume to be working with **simple** and **undirected** graphs, i.e. graphs in which the set of edges is defined as follows

$$E \subseteq \binom{V}{2} = \{\{x, y\} \mid x, y \in V \land x \neq v\}$$

where the notation $\{x, y\}$ will be used to indicate an edge between two nodes $x, y \in V$, and will be replaced with $xy = yx$ directly — the *set* notation for edges is used to highlight that edges have no direction.

We will indicate with n and m the cardinality of $|V|$ and $|E|$, respectively. Moreover, we will indicate with $V(G)$ and $E(G)$ the set of the vertices and edges of G respectively when there is ambiguity.

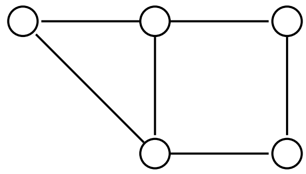


Figure 1.3: A simple graph.

Note that, in this definition, we are assuming that each edge has exactly 2 *distinct* endpoints — i.e. the graphs do not admit **loops** — and there cannot be two edges with the same endpoints — i.e. the graphs do not admit **parallel edges**. In fact, if we drop these assumptions, we obtain what is called a **multigraph**.

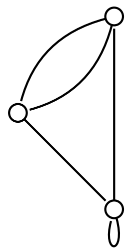


Figure 1.4: A multigraph.

Definition 1.2: Subgraph

Given a graph $G = (V, E)$, a **subgraph** $G' = (V', E')$ of G is a graph such that $V' \subseteq V$ and $E' \subseteq E$, and we write $G' \subseteq G$. If G' is a subgraph of G, then G is called **supergraph** of G'.

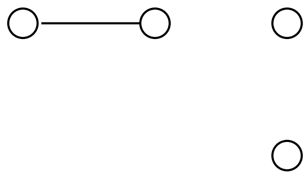


Figure 1.5: This is a subgraph of the graph shown in Figure 1.3.

Definition 1.3: Induced subgraph

Given a graph $G = (V, E)$, and a set of vertices $S \subseteq V$, then $G[S]$ represents the **subgraph induced by S on G**, obtained by removing from G all the nodes of $V - S$ and the edges incident on them.

In other words, a subgraph $G' = (V', E')$ of G is **induced** if every edge of G with both ends in V is an edge of V'. We observe that this definition is *stricter* than the definition of a *subgraph*; in fact, the last graph is *not* an example of an *induced subgraph*, but the following is:

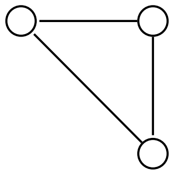
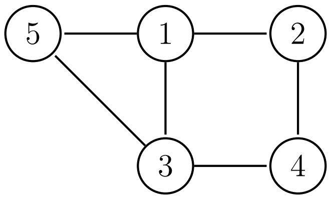


Figure 1.6: This is an *induced* subgraph of the graph shown in Figure 1.3.

Note that every induced subgraph of a graph is **unique** by definition, and we indicate each induced subgraph as follows: suppose that the graph in Figure 1.3 had the following *labeling* on the vertices



then, the induced subgraph in Figure 1.6 would have been referred to as $G[\{1, 3, 5\}]$. To clarify, when writing $G - A$

- if A is a set of *vertices* we are referring to $G[V(G) - A]$ — a vertex cannot be removed from a graph without removing all the edges incident to it
- if A is a set of *edges* we are refering to a graph G without the edges in A — note that this graph has still $V(G)$ as vertex set

Definition 1.4: Non-separating

Given a connected graph G, a subgraph H of G is said to be **non-separating** if $G - H$ is still connected.

Definition 1.5: Graph union

Given two graphs G and G', we define the **union** $G \cup G'$ as the following graph:

- $V(G \cup G') := V(G) \cup V(G')$
- $E(G \cup G') := E(G) \cup E(G')$

Intuitively, two vertices $x, y \in V$ are said to be **adjacent** if there is an edge $xy \in E$, and we write $x \sim y$. If there is no such edge, we write $x \nsim y$ for non-adjacency and we say that xy is an **anti-edge**. The **neighborhood** of a vertex $x \in V$ is the set of vertices that are adjacent to x, and it will be indicated as follows

$$\mathcal{N}(x) := \{y \in V \mid x \sim y\}$$

Similarly, the neighborhood of a set of vertices will be defined as follows

$$\forall S \subseteq V \quad \mathcal{N}(S) := \bigcup_{v \in S} \mathcal{N}(v)$$

The **degree** of a vertex $x \in V$, denoted with $\deg(x)$, is exactly $|\mathcal{N}(x)|$. We will use the following notation for the **minimum** and **maximum** degree of a graph, respectively

$$\delta := \min_{x \in V} \deg(x) \qquad \Delta := \max_{x \in V} \deg(x)$$

Lemma 1.1: Handshaking lemma

Given a graph $G = (V, E)$, it holds that

$$\sum_{x \in V} \deg(x) = 2\,|E|$$

Proof. Trivially, the sum of the degrees counts every edge in E exactly twice, once for each of the 2 endpoints. □

Corollary 1.1

Given a graph $G = (V, E)$, it holds that $|E| \geq \frac{\delta \cdot n}{2}$.

Proof. By the Handshaking lemma, we have that

$$2\,|E| = \sum_{x \in V} \deg(x) \geq \delta \cdot n \iff |E| \geq \frac{\delta \cdot n}{2}$$

□

Definition 1.6: k-regular graph

A graph G is said to be k**-regular** if every vertex of G has degree k.

Note that in a k-regular graph it holds that

$$\sum_{x \in V} \deg(x) = k \cdot n$$

Proposition 1.1

There are no k-regular graphs with k odd and an odd number of vertices.

Proof. By way of contradiction, suppose that there exists a k-regular graph $G = (V, E)$ such that both k and n are odd; however, by the Handshaking lemma we would get that

$$2|E| = \sum_{x \in V} \deg(x) = k \cdot n$$

but the product of two odd numbers, namely k and n, is still an odd number, while $2|E|$ must be even $\lightning$. $\square$

1.2 Important structures

In this section, we are going to explore some of the most important graphs that we will encounter throughout all later chapters.

Definition 1.7: Path

A **path** is a *graph* with vertex set $x_0, \ldots, x_n$ and edge set $e_1, \ldots, e_n$ such that $e_i = x_{i-1}x_i$.

The **length** of a path is the number of edges between x_0 and x_n, i.e. $|\{e_1, \ldots, e_n\}|$, namely n in this case. A path of length 0 is called *trivial* path. P_n is the path graph having n vertices.

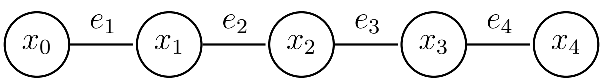


Figure 1.7: A path graph of length 4 that links x_0 and x_4.

Through *paths* we can provide the definition of **distance** between two nodes of a graph.

Definition 1.8: Distance

Given a graph $G = (V, E)$, and two vertices $x, y \in V$, the **distance** between x and y in G, denoted with $\text{dist}_G(x, y)$, is defined as the length of the *shortest* path between x and y in G.

If there is no ambiguity, we will simply write $\text{dist}(x, y)$ instead of $\text{dist}_G(x, y)$. Finally, given a path P and two vertices $u, v \in V(P)$, we will denote with $u\ P\ v$ the *subpath* of P between u and v. Now, consider the following definition.

Definition 1.9: Walk

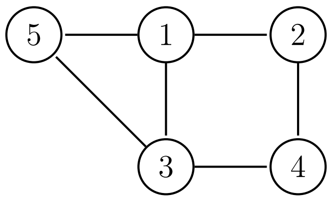
Given a graph $G = (V, E)$, a **walk** is a *sequence* of vertices and edges

$$x_0 \; e_1 \; x_1 \; \ldots \; x_{k-1} \; e_k \; x_k$$

where $x_0, \ldots, x_k \in V$, $e_1, \ldots, e_k \in E$ and $e_i = x_{i-1}x_i$.

The **length** of a walk is the number of edges between x_0 and x_k, i.e. $|\{e_1, \ldots, e_k\}|$, namely k in this case. If $x_0 = x_k$ we say that the walk is **closed**.

If there is a path – or a walk — between two vertices $x, y \in V$, we say that the path — or the walk — **links** x and y, and we write this as $x \to y$. Any vertex of the path that is different from x and y is called *internal node*. For instance, given the previous graph labeled as follows



an example of a walk over this graph is given by the following sequence

$$1 \; \{1,2\} \; 2 \; \{2,4\} \; 4 \; \{4,3\} \; 3 \; \{3,1\} \; 1 \; \{1,5\} \; 5$$

that *links* 1 and 5, i.e. the walk is of the form $1 \to 5$.

Note that there is a subtle difference between the definitions of **path** and **walk**: the definition of a path implies that this is always a *graph* on its own, while a walk is defined as a *sequence*. Nonetheless, we will treat *paths* as if they where *sequences* as well. This assumption holds for the following structures that will be discussed as well.

However, by definition of path, not every alternating sequence of vertices and edges is a valid path, in fact:

- in a *walk* it is possible to repeat both vertices and edges
- in a *path* there can be no repetition of vertices nor edges (note that *edge* repetition implies *vertex* repetition)

For instance, the previous example of *walk* is not a valid *path*, because the vertex 1 is repeated.

Theorem 1.1

Given a graph $G = (V, E)$ and two vertices $x, y \in V$, in G there is a path $x \to y$ if and only if there is a walk $x \to y$.

Proof. By definition, every path is a walk, thus the direct implication is trivially true. To prove the converse implication, consider two vertices x and y for which there is at least one walk $x \to y$ in G. Now, out of all the possible walks $x \to y$ in G, consider the *shortest* one, i.e. the one with the least amount of edges, and let it be the following sequence

$$x \ e_1 \ x_1 \ \ldots \ x_{k-1} \ e_k \ y$$

By way of contradiction, assume that this walk is not a path. Therefore, there must be either one vertex or one edge repeated, but since edge repetition always implies vertex repetition, we just need to take this case into account. Assume that there are two indices $i, j \in [k-1]$ such that $i \neq j$ and $x_i = x_j$; however this implies that

$$x \ e_1 \ \ldots \ x_{i-1} \ e_i \ x_i \ e_{j+1} \ x_{j+1} \ \ldots \ x_{k-1} \ e_k \ y$$

is still a walk $x \to y$ of strictly shorter length, but we chose the original sequence to be the *shortest* possible walk $x \to y$ $\lightning$. □

Proposition 1.2

The longest path in any graph has a length of at least δ.

Proof. Consider a graph $G = (V, E)$, and let P be a longest path in G, labeled as follows

$$x_0 \ e_1 \ x_1 \ \ldots \ x_{k-1} \ e_k \ x_k$$

and assume that its length is k. Since P is a longest path in G, x_k cannot have neighbors outside P itself, otherwise P would not have been the longest path of G — it could have been extended by one of x_k's neighbors. This implies that

$$\mathcal{N}(x_k) \subseteq \{x_0, \ldots, x_{k-1}\}$$

and since $\delta \leq \deg(x_k) := |\mathcal{N}(x_k)|$ by definition of δ, this implies that

$$\delta \leq |\{x_0, \ldots, x_{k-1}\}| = k$$

□

Definition 1.10: Cycle

A **cycle** is a *graph* with vertex set $x_1, \ldots, x_n$ and edge set $x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1$.

The **length** of a cycle is the number of edges between x_1 and x_n, namely n in this case. C_n is the cycle graph having n vertices.

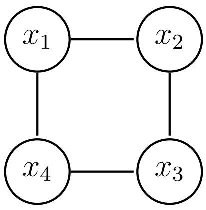


Figure 1.8: A cycle graph of length 4.

A graph that does not admit cycle subgraphs — or *cycles*, for short — is said to be **acyclic**.

Given a graph G that has a cycle C, such cycle is said to be **induced** if there is no pair of vertices $x, y \in V(C)$ such that $x \nsim y$ in C but $x \sim y$ in G. In other words, it must hold that $G[C]$ is a cycle graph.

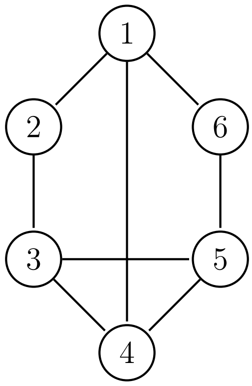


Figure 1.9: For instance, in this graph G we have that $G[\{3, 4, 5\}]$ is an induced cycle but $G[\{1, 2, 3, 4, 5, 6\}]$ is not, because $3 \sim 5$.

We observe that, by definition, C_3 is always induced.

Definition 1.11: Chord

Given a cycle C, a **chord** on C is an edge between two vertices $x, y \in V(G)$ such that $xy \notin E(C)$.

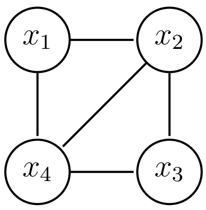


Figure 1.10: For instance, the edge $\{x_2, x_4\}$ is a *chord* of this cycle graph.

Proposition 1.3

Every graph with $\delta \geq 2$ has a cycle of length at least $\delta + 1$.

Proof. Consider the proof of Proposition 1.2; by applying the same reasoning, we know that x_k cannot have neighbors outside P itself. However, since $\delta \geq 2$, and $x_k \sim x_{k-1}$, there must be at least one vertex in x_k's neighborhood that lies in P. Therefore, let x_i be the first vertex of P — w.r.t. our labeling of P — that is adjacent to x_k; hence, we have

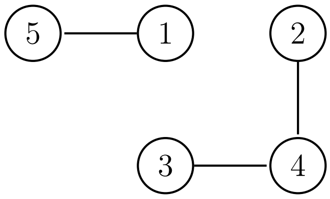
$$\mathcal{N}(x_k) \subseteq \{x_i, \ldots, x_{k-1}\} \implies \delta \leq |\{x_i, \ldots, x_{k-1}\}|$$

which implies that $x_i, \dots, x_{k-1}, x_k, x_i$ is a cycle of length at least $\delta + 1$. □

Definition 1.12: Connected graph

An undirected graph $G = (V, E)$ is said to be **connected** if and only if for each vertex pair $x, y \in V$ there is a path $x \to y$.

All the graphs that we presented so far are *connected*, thus the following figure provides an example of an **disconnected** graph.



Definition 1.13: Component

Given a graph G, a **component** of G is a maximal connected subgraph of G.

For instance, the graph of the previous example is made up of 2 components, namely the following two subgraphs

$$C_1 = (\{1, 5\}, \{\{1, 5\}\})$$
$$C_2 = (\{2, 3, 4\}, \{\{2, 4\}, \{4, 3\}\})$$

Proposition 1.4

If G is a connected graph, and C is a cycle in G, then for any edge $e \in C$ it holds that $G - \{e\}$ is still connected.

Proof. Consider a graph $G = (V, E)$ that has a cycle C, and any two vertices $x, y \in V$; in particular, since G is connected, there must be a path $x \to y$ in G, and let this path be

$$P = x \ e_1 \ x_1 \ \dots \ x_{k-1} \ e_k \ y$$

Consider an edge $e \in C$; if P does not traverse e, trivially $G - \{e\}$ will still contain P.

Now let

$$C = z_1 \ f_2 \ z_2 \ \dots \ z_{l-1} \ f_l \ z_l \ f_{l+1} \ z_1$$

and without loss of generality assume that $e = f_2 = z_1 z_2 = x_i x_{i+1} = e_{i+1}$ for some $i \in [k-1]$. Thus we can construct the following walk

$$x \ e_1 \ x_1 \ \dots \ x_i \ f_{l+1} \ z_l \ \dots \ f_3 \ z_2 \ e_{i+2} \ x_{i+2} \ \dots \ x_{k-1} \ e_k \ y$$

from x to y, and by Theorem 1.1 we have that there is a path from $x \to y$, which proves that $G - \{e\}$ is still connected. □

1.2.1 Trees

Definition 1.14: Tree

A **tree** is a connected acyclic graph. Usually, but not necessarily, there is a fixed vertex called **root**, and any vertex that has degree 1 in the tree is called **leaf**.

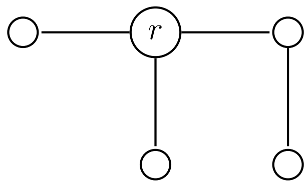


Figure 1.11: A tree with tree leaves, rooted in r.

A **forest** is an disconnected graph in which each component is a *tree*, as in the following example.

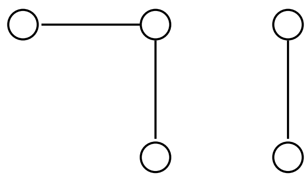


Figure 1.12: A forest.

Given a tree T rooted in some node $r \in V(T)$, and two vertices $x, y \in V(T)$, consider the paths P_x of the form $x \to r$ and P_y of the form $y \to r$, respectively. The first vertex of P_y that is encountered by tracing P_x from x to r is called **lowest common ancestor** (LCA) of x and y.

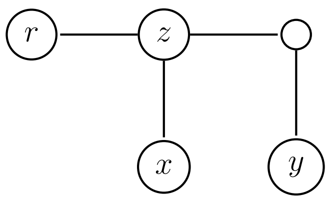


Figure 1.13: For instance, in this tree — rooted in r — the LCA of x and y is the vertex labeled with z.

Note that the LCA between any two vertices of a tree is *always defined*, since in the "worst case" it is the root r itself.

Theorem 1.2: Alternative definitions of tree

Given a graph $T = (V, E)$, the following statements are equivalent:

1. T is a tree
2. every vertex pair of T is connected by a unique path
3. T is *minimally connected*, i.e. T is connected and $\forall e \in E$ it holds that $T - \{e\}$ is disconnected
4. T is *maximally acyclic*, i.e. T is acyclic and $\forall x, y \in V$ such that $x \nsim y$, it holds that $T \cup \{xy\}$ has a cycle

Proof. We will prove the statements cyclically.

- $1 \implies 2$. By contrapositive, assume that in T there exist two vertices $x, y \in V$ for which there are two distinct paths P and Q of the form $x \to y$. If P and Q are edge-disjoint, then $P \cup Q$ is a cycle, which implies that T is not a tree by definition.

  Otherwise, assume that P and Q are not edge-disjoint. If we start say in x, and we follow Q edge by edge, since P and Q are distinct at some point we will encounter an edge $\{u, v\}$ such that $u \in P \cap Q$ and $v \in Q - P$ — possibly, $u = x$ itself. Moreover, since both paths lead to y, if we keep following Q we will encounter a vertex $z \in P \cap Q$ — possibly, $z = y$ itself — from which the two paths will coincide. Let Q' be the subpath of P starting with u and ending in z; then, $Q' \cup (Q - P)$ is a cycle in T, which implies that T is not a tree by definition.

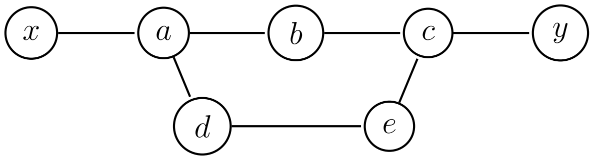


Figure 1.14: For instance, applying the argument of the proof in this graph we would get that $P = \{x, a, b, c, y\}$, $Q = \{x, a, d, e, c, y\}$, $Q - P = \{d, e\}$, $u = a$, $z = c$ and $Q' = \{a, b, c\}$, in fact $Q' \cup (Q - P) = \{a, b, c, e, d\}$ which is a cycle.

- $2 \implies 3$. Consider an edge $xy \in E$; this edge itself is a path $x \to y$, and if we assume statement 2 this implies that it is the *only* path from x to y. This implies that $T - \{xy\}$ cannot contain a path from x to y, therefore $T - \{xy\}$ is disconnected.
- $3 \implies 4$. Since statement 3 implies that T is connected, by Proposition 1.4 we have that T is acyclic. Now, pick $x, y \in V$ such that $x \nsim y$; by connectivity of T there must be a path $x \to y$ in T, and let this path be P. Lastly, since $x \nsim y$, we have that $P \cup \{xy\}$ is a cycle in T.
- $4 \implies 1$ By contrapositive, we want to prove that if T is not a tree, then T is not maximally acyclic. Note that if T is not a tree, we have two options:
  - if T is connected but contains a cycle, then T is clearly not maximally acyclic

– if T is acyclic but disconnected, then by definition there must be two vertices $x, y \in V$ such that there is no path $x \to y$, which implies that $T \cup \{xy\}$ still does not contain any cycle

$\square$

Lemma 1.2

Every tree with at least 2 vertices has a leaf.

Proof. By way of contradiction, assume T is a tree with at least 2 vertices that does not contain any leaves; then $\delta \geq 2$ in T, which implies that T contains a cycle of length at least $\delta + 1$ by Proposition 1.3, contradicting the fact that T was a tree $\lightning$. $\square$

Lemma 1.3

Given a tree T, and a leaf v of T, it holds that $T - \{v\}$ is still a tree.

Proof. Since T is acyclic by definition, $T - \{v\}$ is still acyclic, so we just need to prove that $T - \{v\}$ is still connected. By way of contradiction, assume that in $T - \{v\}$ there exist two vertices x and y such that there is no path between them. However, since T is connected, there is a path P of the form $x \to y$ in T.

Note that, if by removing v from T we disconnect x and y, it must be that v lies in P. Moreover, since v is in T but not in $T - \{v\}$, while both x and y are also in $T - \{v\}$, it must be that v is an *internal* node of P, i.e. $v \neq x, y$, which implies that $\deg(v) \geq 2$ by definition of path, contradicting the hypothesis for which v was a leaf $\lightning$. $\square$

Proposition 1.5

If T is a tree, then $m = n - 1$.

Proof. We will prove the statement by induction on n

Base case. When $n = 1$, there are no edges in the tree, and $0 = m = 1 - 1$.

Inductive hypothesis. Assume that for a tree that has $n = k - 1$ nodes the statement holds.

Inductive step. We will prove the statement for a tree T that has $n = k$ nodes. Note that, since $n = 1$ is the base case, we can assume that $n = k \geq 2$, hence by Lemma 1.2 T contains at least one leaf. Let this leaf be v; then, by Lemma 1.3 it holds that $T - \{v\}$ is still a tree, and clearly $T - \{v\}$ has $k - 1$ nodes, which implies that we can apply the inductive hypothesis on $T - \{v\}$, i.e.

$$|E(T - \{v\})| = |V(T - \{v\})| - 1 = k - 1 - 1 = k - 2$$

However, note that v is a leaf, concluding that

$$m = |E(T)| = |E(T - \{v\})| + 1 = k - 2 + 1 = k - 1 = n - 1$$

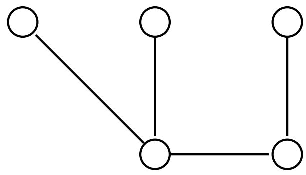
□

Definition 1.15: Spanning tree

Given a graph $G = (V, E)$, a **spanning tree** T of G is a subgraph of G such that

- T is a tree
- $V(T) = V(G)$, i.e. T *spans* every vertex of G

For instance, given the graph in Figure 1.3, a possible spanning tree is the following:



Lemma 1.4

Any connected graph has a spanning tree.

Proof. Consider a connected graph G, and keep removing edges from $E(G)$ — and their relative endpoints from $V(G)$ — one by one, as long as G is still connected. If no other edge can be removed from G without violating connectivity, we will end up with a graph that must be a tree by statement 3 of Theorem 1.2. □

Thanks to this last proposition, we can actually prove a stronger version of the Proposition 1.5, which is the following.

Theorem 1.3

T is a tree if and only if T is connected and $m = n - 1$.

Proof. The direct implication is proved in Proposition 1.5, so we just need to prove the converse implication. Consider a connected graph T such that $m = n - 1$; by Lemma 1.4 T must have a spanning tree T', and by Proposition 1.5 itself it holds that $|E(T')| = |V(T')| - 1$. However, T' is a spanning tree of T, therefore

$$V(T) = V(T') \implies |V(T)| = |V(T')| \implies |E(T')| = |V(T)| - 1 = |E(T)|$$

which implies $E(T) = E(T')$ because T' is a subgraph of T, therefore $T = T'$, concluding that T must be a tree since T' is a tree. □

1.2.2 Bipartite graphs

Definition 1.16: Bipartite graph

A graph $G = (V, E)$ is said to be **bipartite** if there exists a *proper subset* $X \subseteq V$ such that every edge of G has exactly one endpoint in X and one in $V - X$. If such a set X exists, we say that $(X, V - X)$ is a **bipartition** of G. If G has the maximum number of edges that preserve the bipartition, we say that it is **complete bipartite**.

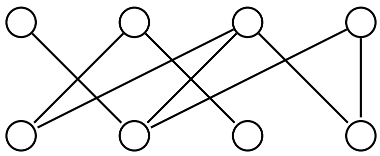


Figure 1.15: An example of a *bipartite graph*. In particular, if we call the uppermost set of nodes A and the lowermost one B, then (A, B) is a bipartition of the graph.

The graph $K_{a,b}$ is the *complete bipartite* graph bipartitioned through (A, B) such that $|A| = a$ and $|B| = b$.

There are various types of graphs that can be bipartitioned. For example, any **tree** T can be bipartitioned through a bipartition $(X, V(T) - X)$ by considering the following set of vertices

$$X := \{v \in V(T) \mid \mathrm{dist}(r, v) \text{ is even}\}$$

where r is T's root.

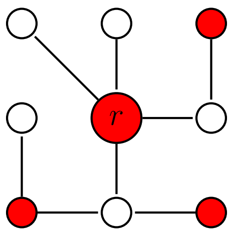


Figure 1.16: The set of red vertices X defines a bipartition $(X, V(T) - X)$ of this tree T.

However, *not every type* of graph can be bipartitioned. For instance, consider the following type.

Definition 1.17: Clique

A **clique** is a *graph* in which each vertex is adjacent to any other vertex of the graph. A clique that has n vertices — referred to as k-clique — is denoted as K_n.

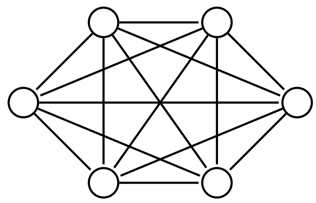


Figure 1.17: The clique K_6.

It is easy to see that no clique K_n can be bipartitioned, since there is an edge between any pair of vertices of the graph. However, this is not the only type of graph that cannot be bipartitioned.

Lemma 1.5

If G is a bipartite graph, and H is a subgraph of G, then H must be bipartite.

Proof. Given a bipartite graph $G = (V, E)$, assume that $(X, V - X)$ is a bipartition of G, and let H be a subgraph of G; then, it is easy to see that $(X \cap V(H), V(H) - X)$ is a bipartition for H. □

Note that this lemma implies that G is bipartite if and only if every connected component of G is bipartite: in fact, the direct implication follows from this lemma, and the following figure provides an intuition for the converse implication.

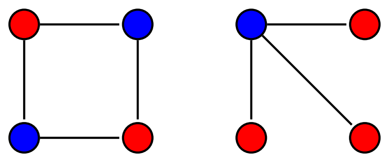


Figure 1.18: Consider the following disconnected graph G made of these two connected components, C_1 and C_2 respectively. Say that C_1 has a bipartition $(X_1, V(C_1) - X_1)$ where X_1 is the red set, and C_2 has a bipartition $(X_2, V(C_2) - X_2)$ where X_2 is the red set; thus, $(X_1 \cup X_2, V(C_1 \cup C_2) - X_1 - X_2)$ is clearly a bipartition of G. This process may be repeated for all the connected components of any disconnected graph.

The following theorem provides a *complete characterization* of bipartite graphs, and it will be used extensively throughout the rest of the notes.

Theorem 1.4

G is bipartite if and only if G has no odd-length cycle.

Proof.

Direct implication. We will prove the contrapositive, i.e. if G has an odd-length cycle, then G cannot be bipartitioned. Consider a graph G with an odd-length cycle

C_{2k+1} of vertices $x_1, \ldots, x_{2k+1}$; by way of contradiction, assume that G is bipartite through a bipartition $(X, V(G) - X)$ for some $X \subseteq V(G)$. Without loss of generality assume that $x_1 \in X$; then, since X defines a bipartition of G it must be that $x_2 \notin X$, and $x_3 \in X$ and so on and so forth. In particular, for any odd value of i we will have that $x_i \in X$, but this implies that both x_1 and x_{2k+1} must be inside X, which means that the edge $x_1 x_{2k+1}$ violates the bipartition induced by X $\lightning$.

Converse implication. Again, we will prove the contrapositive, i.e. if G is not bipartite it must contain an odd-length cycle. By the previous observation, G is not bipartite if and only if at least one connected component of G is not bipartite, and let this component be G'. Note that, since G' is connected, it must contain a spannning tree T by Lemma 1.4. Moreover, as previously described, we can always define a bipartition on a tree, namely $(X, V(T) - X)$ where

$$X := \{v \in V(T) \mid \mathrm{dist}_T(r, v) \text{ even}\}$$

for some root node $r \in V(T)$.

Now, since T is a spanning tree of G', which is now bipartite by hypothesis, there must be an edge $xy \in V(G')$ such that either $x, y \in X$ or $x, y \in V(T) - X$, i.e. the edge xy must have both endpoints in the same set of T's bipartition. Let z be the LCA between x and y in T, and P_x and P_y be the paths of the form $x \to r$ and $y \to r$, respectively. Note that, since xy has both endpoints in the same set, it must be that the lengths of P_x and P_y have the same parity by definition of X. Lastly, by statement 2 of Theorem 1.2 we have that $r\ P_x\ z = r\ P_y\ z$, which implies that the lengths of $z\ P_x\ x$ and $z\ P_y\ y$ must have the same parity. This concludes that

$$z\ P_x\ x \cup z\ P_y\ y \cup xy$$

is an odd-length cycle of G.

□

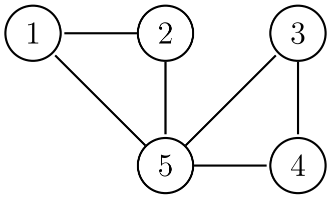
1.2.3 Eulerian tours

At the start of this chapter, we introduced the *Seven Bridges of Königsberg* problem, which led to the emergence of graph theory as a branch of combinatorics. Over time, as the field developed, this problem was formalized into the following definition.

Definition 1.18: Eulerian tour

An **Eulerian tour** over a graph G is a closed walk that traverses every edge of G exactly once.

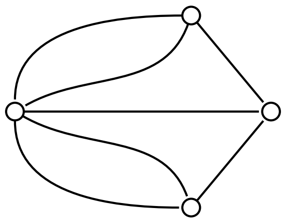
For instance, consider the following graph



After some trial an error, it is easy to find an Eulerian tour over this graph, for instance

$$1\ \{1,2\}\ 2\ \{2,5\}\ 5\ \{5,3\}\ 3\ \{3,4\}\ 4\ \{4,5\}\ 5\ \{5,1\}\ 1$$

Note that this is a valid Eulerian tour because there is no edge repetition, and vertex repetition is allowed by definition. On the counter side, the graph — or, more precisely, the *multigraph* — that the bridges of Königsberg define, which is the following



does not admit any Eulerian tour. But, given a graph, how can we determine with certainty whether it contains an Eulerian tour? The following theorem, proved by Euler himself in his original paper [Eul41], answers this question. Note that this theorem holds both for graphs and multigraphs.

Theorem 1.5: Euler's theorem

A graph G admits an Eulerian tour if and only G is connected and every vertex of G has even degree.

Proof.

Direct implication. Consider the contrapositive of the direct implication, and assume that G contains at least one odd-degree vertex v. Recall that an Eulerian tour is a closed walk that does not allow edge repetition, which implies that the *starting* point of the walk is actually not relevant. Therefore, we can assume without loss of generality that any possible Eulerian tour defined on G starts on v itself, but then to be *closed* it must end on v as well. Moreover, each time any Eulerian tour passes through v it must use 2 distinct edges, however v is an odd-degree vertex, therefore at the end of the Eulerian tour there is no way we can come back to v and close the walk.

Converse implication. Consider a graph G in which every vertex has even degree, and let W be the *longest* walk of G that does not repeat edges, and let it be labeled as follows

$$x_0\ e_1\ x_1\ \ldots\ x_{k-1}\ e_k\ x_k$$

Claim: $x_0 = x_k$, i.e. W is closed.

Proof of the Claim. By way of contradiction, assume that W $x_0 \neq x_k$. If this is the case, then $x_k = v$ for some vertex v that is not x_0. Note that W is a *walk*, therefore v may be repeated multiple times inside W, i.e.

$$x_0 \; e_1 \; x_1 \; \ldots \; v \; \ldots \; v \; \ldots \; x_{k-1} \; e_k \; v$$

If l is the number of times v occurs in W *without counting* x_k, then clearly in W there are $2l + 1$ edges incident to v, namely e_k and 2 edges each other time v appears in W. However, since $2l+1$ is odd and we assumed that G has no odd-degree vertices, there must be at least one edge $vu \in E(G)$ such that $u \notin V(W)$. This implies that

$$x_0 \; e_1 \; x_1 \; \ldots \; x_{k-1} \; e_k \; v \; vu \; u$$

is a longer walk than W, and it does not repeat any edges because $uv \notin E(W)$, contradicting the definition of W $\lightning$. $\square$

This claim proves that W is closed, but we still need to prove that it traverses every edge to claim that it is indeed an Eulerian tour. By way of contradiction, assume that there exists at least one edge $e \notin E(W)$ not used by W. Consider an vertex x_i of W; by connectivity of G, there must be a path P beween x_i and each of the endpoints of e. Let $x_i u$ be the first edge of P, for some $u \notin E(W)$. This implies that

$$u \; ux_i \; x_i \; e_{i+1} \; x_{i+1} \quad \ldots \; x_{k-1} \; e_k \; x_k \; e_1 \; x_1 \; \ldots \; x_{i-1} \; e_i \; x_i$$

is a longer walk than W that does not repeat any edge, since $ux_i \notin E(W)$, again contradicting the definition of W $\lightning$.

$\square$

Note that this theorem proves that it is not possible to describe an Eulerian tour over the bridges and landmasses of Königsberg, because every vertex of the multigraph has odd degree.

1.2.4 Hamiltonian cycles

Eulerian tours are closed walks that traverse every edge of the graph exactly once, but what if we are interested in traversing each *vertex* exactly once instead?

Definition 1.19: Hamiltonian paths and cycles

A **Hamiltonian path** over a graph G is a subgraph P_n of G. A **Hamiltonian cycle** over a graph G is a subgraph C_n of G.

Hamiltonian paths and cycles are named after W. R. Hamilton. Note that the notation P_n (or C_n) implies that the length of the path (or cycle) is n, hence this definition matches our requirements.

As for the case of Eulerian tours, when some conditions are met, Hamiltonian cycles are guaranteed to exist, as discussed in the following theorem proved by Dirac [Dir52].

Theorem 1.6: Dirac's theorem

A graph G such that $\delta \geq \frac{n}{2}$ contains a Hamiltonian cycle.

Proof. First, we will prove that the condition of the statement implies that G is connected.

Claim: G is connected.

Proof of the Claim. By way of contradiction, suppose that G is not connected; therefore G has at least two connected components. Let H be the smallest connected component of G; then, clearly $|V(H)| \leq \frac{n}{2}$. However, note that for any $x \in V(H)$ it holds that $|\mathcal{N}(x)| \geq \delta \geq \frac{n}{2}$ and since $\{x\} \cup \mathcal{N}(x) \subseteq V(H)$, we get that $V(H)$ must have at least $\frac{n}{2} + 1$ nodes $\lightning$. □

Let P be the longest path of G, and let $x_0, \ldots, x_k$ be its vertices.

Claim: There exists an index ℓ such that $x_0\ x_\ell\ x_{\ell+1}\ \ldots\ x_{k-1}\ x_k\ x_{\ell-1}\ x_{\ell-2}\ \ldots\ x_1\ x_0$ is a cycle.

Proof of the Claim. By the same argument used in the proof of Proposition 1.2, we know that

$$\mathcal{N}(x_0), \mathcal{N}(x_k) \subseteq \{x_0, \ldots, x_k\}$$

Let I_0 and I_k be the following two sets

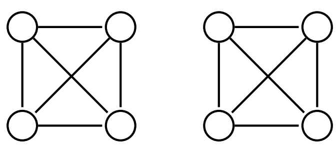
$$I_0 := \{i \mid i \in [1, k], x_i \in \mathcal{N}(x_0)\} \implies |I_0| = |\mathcal{N}(x_0)|$$

$$I_k := \{i \mid i \in [1, k], x_{i-1} \in \mathcal{N}(x_k)\} \implies |I_k| = |\mathcal{N}(x_k)|$$

Since $\delta \geq \frac{n}{2}$, we have that $|I_0|, |I_k| \geq \frac{n}{2}$. However, note that $k \leq n-1$ — since we started counting at 0 — hence by the pigeonhole principle there must be at least one index $\ell \in I_0 \cap I_k$, meaning that $x_0 \sim x_\ell$ and $x_k \sim x_{\ell-1}$, defining a cycle as described in the statement of the claim. □

This means that we found a cycle C in the graph that uses all the vertices of P, namely $x_0, \ldots, x_k$. By way of contradiction, assume that $k < n-1$, i.e. C has less than n nodes, meaning that C is a non-Hamiltonian cycle. In particular, if $k < n-1$, we have that $|V(G)| - |V(C)| \neq \varnothing$, thus let $y \in V(G) - V(C)$. By the previous claim, we know that G is connected, there must be an edge $xy \in E(G)$ such that $x \in V(C)$. However, this would imply that $P \cup \{xy\}$ is a path of longer path than P $\lightning$. □

Note that the statement of this theorem cannot be improved, even by 1; for instance consider the following graph



composed of two disconnected K_4. Here, we have that

$$\delta = 4 - 1 = 3 \geq \frac{2 \cdot 4}{2} - 1 = 4 - 1 = 3 = \frac{n}{2} - 1$$

Therefore, $\delta \geq \frac{n}{2} - 1$ is not sufficient to guarantee connectivity.

1.3 Exercises

Problem 1.1

Let $G = (V, E)$ be a graph of n vertices, where $n \geq 2$. Show that there must be two vertices $x, y \in V$ such that $\deg(x) = \deg(y)$.

Solution. By definition, the range of the possible degrees for any node of G is $[0, n-1]$. By way of contradiction, assume that for any two vertices $x, y \in V$ it holds that $\deg(x) \neq \deg(y)$; hence, since the graph has n nodes, it must be that each node is assigned a different degree, and that we use all the possible degrees in $[0, n-1]$. In particular, this implies that there are two vertices $u, v \in V$ such that $\deg(u) = 0$ and $\deg(v) = n - 1$, but this is a contradiction because if the degree of v is $n-1$, it must be adjacent to all the other nodes of V, including u, and $\deg(u) = 0$ $\lightning$. $\square$

Problem 1.2

Let G be a graph containing a cycle C, and assume that G contains a path of length at least k between two vertices of C. Show that G contains a cycle of length at least $\sqrt{k}$. Can this bound be improved?

Solution. Consider a graph G that contains a cycle C, and let P be a path between two vertices $x, y \in V(C)$ such that $|E(P)| \geq k$ for some k. Starting from x, let $x_1, \ldots, x_t$ be the vertices through which P "leaves" C, and $y_1, \ldots, y_t$ be the vertices through which P "joins back" C. We observe that if $2t \geq \sqrt{k}$, it means that there are at least $\sqrt{k}$ vertices in C, thus the theorem trivially holds by simply considering C itself, so we may assume that $2t < \sqrt{k}$.

Let $P_i := x_i \; P \; y_i$ and $\overline{P_i} := y_i \; P \; x_{i+1}$, and let C_i be the shortest subpath of C of the form $x_i \; C \; y_i$ — there are two subpaths of this form. We observe that there are t subpaths of the form P_i, and at most $t+1$ subpaths of the form $\overline{P_i}$, meaning that P can be partitioned in at most $2t+1$ subpaths of the forms described. Now, let P^* be the subpath of P that maximizes its length — P^* may be of the form P_i or $\overline{P}_i$. Hence, we get that

$$k \leq |E(P)| \leq (2t+1)\,|E(P^*)| < (\sqrt{k}+1)\,|E(P^*)|$$

and in particular

$$|E(P^*)|\,(\sqrt{k}+1) > k \iff |E(P^*)| > \frac{k}{\sqrt{k}+1} \iff |E(P^*)| \geq \frac{k}{\sqrt{k}+1} + 1 \geq \sqrt{k}$$

where the last inequality can be proved as follows

$$\frac{k}{\sqrt{k}+1}+1 \geq \sqrt{k} \iff \frac{k+\sqrt{k}+1}{\sqrt{k}+1} \geq \sqrt{k} \iff k+\sqrt{k}+1 \geq \sqrt{k}(\sqrt{k}+1) = k+\sqrt{k}$$

Now, if the subpath P^* lies inside C, since $|E(P^*)| \geq \sqrt{k}$ the theorem holds by considering the cycle C itself, so we may assume that there exists $i^* \in [t]$ such that $P_{i^*} = P^*$.

Lastly, since P_i and C_i do not intersect for each $i \in [t]$, we have that

$$|E(P_i \cup C_i)| = |E(P_i)| + |E(C_i)| \geq |E(P_i)| + 1$$

and in particular

$$|E(P_{i^*} \cup C_{i^*})| \geq \sqrt{k} + 1 \geq \sqrt{k}$$

which means that

- $P_{i^*} \cup C_{i^*}$ is the cycle that proves the statement
- we can improve the bound of the statement to be at least $\sqrt{k}+1$

$\square$

Problem 1.3: Tree-order

Consider a tree T, and let $r \in V(T)$ be its root. We define the **tree-order** $\leq_r$ associated with r as follows:

$$\forall x, y \in V(T) \quad x \leq_r y \iff x \in V(r\ T\ y)$$

Prove that

1. r is the least element in this partial order.
2. Every leaf different from r is a maximal element.
3. the endpoints of any edge of T are comparable.
4. For any $y \in V(T)$, every set of the form $\{x \in V(T) \mid x \leq_r y\}$ is a *chain*, i.e. a set of pairwise comparable elements.

Solution. We can prove the statements by using the properties of trees.

1. By way of contradiction, suppose that there is an element m such that $m \leq_r r$ and $m \neq r$; by definition, this happens if and only if $m \in V(r\ T\ r) = \{r\}$, meaning that $m = r$ $\lightning$.

2. By way of contradiction, let $l \neq r$ be a leaf of T that is not maximal, i.e. there is a node $v \in V(T)$ such that $l \leq_r v$ and $l \neq v$. By definition, this can happen if and only if $l \in V(r\ T\ v)$, but if $l \neq v$ and l is in a path $r \to v$, then $\deg(l) \geq 2$ contradicting the fact that l was a leaf of T $\lightning$.

3. Fix an edge $e \in E(T)$; we can label the endpoints of e with x and y such $\mathrm{dist}(r, y) = \mathrm{dist}(r, x) + 1$ — i.e. x "comes before" y in the path $r \to y$ — and this suffices to show that $x \in V(r\ T\ y) \iff x \leq_r y$.

4. Let $y \in V(T)$, and fix a set $S_y := \{x \in V(T) \mid x \leq_r y\}$. We observe that $x \leq_r y \iff x \in V(r\ T\ y)$, which implies that $x \in S_y \iff x \in V(r\ T\ y)$ meaning that $S_y = V(r\ T\ y)$. Then, fix a pair of distinct elements $a, b \in S_y$, and without loss of generality suppose that b is the element that maximizes the distance from r. We observe that $S_y = V(r\ T\ y)$ implies that for any $x \in S_y$ the path $r\ T\ y$ contains $r\ T\ x$, and in particular it contains both $r\ T\ a$ and $r\ T\ b$. Moreover, since b is further from r than a, the path $r\ T\ a$ is contained in $r\ T\ b$, which means that $a \in V(r\ T\ b) \iff a \leq_r b$.

□

Problem 1.4

Show that every connected graph G contains a path of length at least $\min\{2\delta, n-1\}$.

Solution. Let P be the longest path of vertices $x_1\ \dots\ x_k$. If $|V(P)| = n$ then $|E(P)| = n - 1 \geq \min\{2\delta, n-1\}$, so we may assume that $|V(P)| \leq n - 1$.

Claim: If $x_i \in \mathcal{N}(x_1)$, then $x_{i-1} \notin \mathcal{N}(x_k)$.

Proof of the Claim. By way of contradiction, suppose that there is an index j such that $x_1 \sim x_j$ and $x_{j-1} \sim x_k$; then

$$x_1\ \dots\ x_{j-2}\ x_{j-1}\ x_k\ x_{k-1}\ \dots\ x_{j+1}\ x_j\ x_1$$

is a cycle — call this cycle C. Since we are assuming that $|V(P)| \leq n-1$, there must be at least another vertex $z \in V(G) - V(P)$. By connectivity of G, we know that there is a path Q from z to some vertex $v_i \in V(C)$; hence, we have that

$$Q \cup x_i\ x_{i+1}\ \dots\ x_k\ x_1\ \dots\ x_{i-1}$$

is a path containing more vertices than the vertices of P, contradicting the maximality of $|E(P)|$ $\lightning$. □

We observe that $\mathcal{N}(x_1), \mathcal{N}(x_k) \subseteq V(P) = \{x_1, \dots, x_k\}$, otherwise we would contradict the maximality of P. Let $\mathcal{N}(x_1) = \{x_{i_1}, \dots, x_{i_\ell}\}$; by the previous claim, we get that

$$\mathcal{N}(x_k) \subseteq \{x_2, \dots, x_k\} - \{x_{i_1 - 1}, \dots, x_{i_\ell - 1}\}$$

therefore

$$\delta \leq |\mathcal{N}(x_k)| \leq k - 1 - |\mathcal{N}(x_1)| \leq k - 1 - \delta$$

meaning that $k \geq 2\delta + 1 \implies |E(P)| \geq 2\delta \geq \min\{2\delta, n-1\}$. □

Problem 1.5

Show that every tree T has at least $\Delta(T)$ leaves.

Solution. Consider a vertex v of maximum degree, i.e. $\deg(v) = \Delta(T)$.

Claim: There are $\Delta(T)$ connected components in $T - \{v\}$.

Proof of the Claim. By way of contradiction suppose that there are less than $\Delta(T)$ components in this subgraph. However, by pigeonhole principle, since there are $\Delta(T)$ edges incident to v, and less than $\Delta(T)$ components, there must be two neighbors of v — say x and y — lying in the same component of $T - \{v\}$. However, since the component is connected by definition, there is a path between x and y, say P, which implies that $v\ \{v, x\}\ P\ y\ \{y, v\}\ v$ is a cycle, contradicting the fact that T was a tree.

Finally, if there are more than $\Delta(T)$ components when removing v from T, it meant that at least one component was already present in T, meaning that T had at least 2 different components, contradicting the fact that T was a tree. □

Moreover, since T is a tree, each of the $\Delta(T)$ components in $T - \{v\}$ must be both connected and acyclic, implying that all of these components are trees themselves. Lastly, for each subtree we have two cases:

- if the subtree has 1 vertex, the subtree was a leaf of T
- else, if the subtree has at least 2 vertices, by Lemma 1.2 the subtree contains at least one leaf

and in both cases each subtree contains at least one leaf, meaning that T contains at least $\Delta(T)$ leaves. □

Problem 1.6

Show that any non-trivial tree without a vertex of degree 2 has more leaves than other vertices. Can you find a very short proof that does not use induction?

Solution. Let T be a non-trivial tree that does not contain vertices of degree 2, and let $L = \{v \in V(T) \mid \deg(v) = 1\}$ the set of its leaves. By the Handshaking lemma, we know that

$$\begin{aligned} 2\,|E(T)| &= \sum_{v \in V(T)} \deg(v) \\ &= \sum_{v \in V(T) - L} \deg(v) + \sum_{v \in L} \deg(v) \\ &\geq |L| + 3\,|V(T) - L| \end{aligned}$$

Moreover, since T is a tree, we have that

$$2\,|E(T)| = 2(n-1) = 2(|L| + |V(T) - L| - 1) = 2\,|L| + 2\,|V(T) - L| - 2$$

Therefore, we have that

$$2\,|L| + 2\,|V(T) - L| - 2 \geq |L| + 3\,|V(T) - L|$$

implying that $|L| \geq |V(T) - L| + 2 \iff |L| > |V(T) - L| + 1$. □

Problem 1.7

Let G be a connected graph that does not contain paths longer than t, and let P_1 and P_2 be two paths of G of length t. Prove that $V(P_1) \cap V(P_2) \neq \varnothing$.

Solution. By way of contradiction, suppose that $V(P_1) \cap V(P_2) = \varnothing$, and fix two vertices $v_1 \in V(P_1)$ and $v_2 \in V(P_2)$. By connectivity of G there must be a path $v_1 \in v_2$ inside G — call it R. For $i \in \{1, 2\}$ let $P_i = P_i' \cup P_i''$ such that $P_i' \cap P_i''$ is the endpoint of R in P_i, and assume $|V(P_i')| \geq |V(P_i'')|$. However, since $|V(P_i')| \geq \frac{t}{2}$ then we have that $P_1' \cup P_2' \cup R$ is a path of length at least $\frac{t}{2} + \frac{t}{2} + 1 = t + 1$, contradicting the definition of G. ☐

Problem 1.8

Prove that if a graph G is such that $\delta \geq 3$, then G has a cycle with a chord.

Solution. Consider a maximum length path $P = x_1 \ \ldots \ x_k$ of G; then, since it has maximum length, each neighbor of x_k must lie inside $V(P)$, and because $\delta \geq 3$ there must exist at least two vertices $x_i, x_j \in V(P)$ such that $x_i, x_j \sim x_k$ and $i, j \neq k - 1$ — without loss of generality suppose that $i < j$. This implies that

- $C := x_k \ x_i \ \ldots \ x_j \ \ldots \ x_k$ is a cycle of G
- $\{x_j, x_k\}$ is a chord of C

☐

Problem 1.9

Let T be a tree, and $T_1, \ldots, T_k$ be a collection of subrees of T such that for any $i, j \in [k]$ if $i \neq j$ then $V(T_i) \cap V(T_j) \neq \varnothing$. Prove that $\bigcap_{i=1}^{k} V(T_i) \neq \varnothing$.

Solution. We proceed by induction on n. Consider a tree T such that $n = 1$: then the collection of subtrees must be such that for all $i \in [k]$ it holds that $V(T_i) = \{v\}$ where v is the only node in T; therefore trivially $\bigcap_{i=1}^{k} V(T_i) = \{v\} \neq \varnothing$.

Assume the statement holds for a tree of size $n - 1$, and consider a tree T of size n, and let $T_1, \ldots, T_k$ a the collection of subtrees of T such that they have pairwise non-empty intersection. Moreover, fix a leaf ℓ, and by Lemma 1.3 we know that $T - \{\ell\}$ is still a tree. Therefore, if ℓ is not contained by any of the trees of the collection, the statement follows trivially by induction on $T - \{\ell\}$, so we may assume that ℓ is contained in at least one of the trees of the collection. Furthermore, we may assume that there is no tree T_i of the collection such that $V(T_i) = \{\ell\}$, otherwise by the fact that it must be non-pairwise nondisjoint with all the other trees it must hold that $\forall j \in [k] \quad j \neq i \implies V(T_i) \cap V(T_j) = \{\ell\}$ which trivially implies that $\bigcap_{i=1}^{k} V(T_i) = \{\ell\}$.

Therefore, we are assuming that there is at least a tree of the collection containing ℓ, and each such tree must contain at least another vertex. Thus, by connectivity of each subtree we know that there is one vertex — say u — such that $\ell \sim u$ which in turn implies that

$\forall i \in [k] \quad \ell \in V(T_i) \implies u \in V(T_i)$. Now, for each $i \in [k]$ let T_i' indicate the subtree $T_i - \{\ell\}$; the previous observation implies that $\forall i, j \in [k] \quad i \neq j \wedge \ell \in V(T_i) \cap V(T_j) \implies u \in V(T_i) \cap V(T_j) \implies u \in V(T_i') \cap V(T_j')$. Now, consider the collection $T_1', \ldots, T_k'$; this observation implies that for each $i, j \in [k]$, if $i \neq j$ then

- if $\ell \in V(T_i) \cap V(T_j)$ then $u \in V(T_i') \cap V(T_j') \implies V(T_i') \cap V(T_j') \neq \varnothing$
- if $\ell \notin V(T_i) \cap V(T_j)$ then ℓ is not present in at least one between T_i and T_j, but this must imply that there is some node such that $V(T_i') \cap V(T_j')$ since $V(T_i) \cap V(T_j) \neq \varnothing$ by assumption on $T_1, \ldots, T_k$

Therefore, this is a collection of non-pairwise disjoint trees over $T' = T - \{\ell\}$, which is a tree having $n-1$ nodes. Hence, by inductive hypothesis we know that $\bigcap_{i=1}^{k} V(T_i') \neq \varnothing$ which implies that $\bigcap_{i=1}^{k} V(T_i) \neq \varnothing$ since we did not remove any vertex — in fact, we just added ℓ back in the trees where it was present. $\square$

2

Matchings

In graph theory, a **matching** is a set of edges chosen such that no two edges share a common vertex. In other words, a matching is a subgraph where each node has either zero or one edge incident to it. Matchings are fundamental in various applications, including flow networks, scheduling and planning, chemical bond modeling, graph coloring, the stable marriage problem, and even neural networks in artificial intelligence.

Definition 2.1: Matching

Given a graph $G = (V, E)$, a **matching** of G is a set of edges $M \subseteq E$ such that

$$\forall e, e' \in M \quad e \cap e' = \varnothing$$

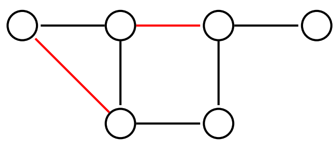


Figure 2.1: A *matching* of the previous graph.

As shown in figure, a matching is nothing more than a set of edges that must not share endpoints with each other — for this reason, in literature it is often referred to as **independent edge set**.

Given a matching M in a graph G, we say that a vertex $v \in V(G)$ is **free** w.r.t. M if there are no edges $e \in M$ such that $e \cap v \neq \varnothing$. An edge $xy \in E(G)$ is said to be *disjoint* from M if both x and y are free w.r.t. M.

In graph theory we are often interested in the matching that has the largest possible cardinality of a graph. For this purpose, we often distinguish the two following concepts, namely *maximal* and *maximum* matching.

Definition 2.2: Maximal matching

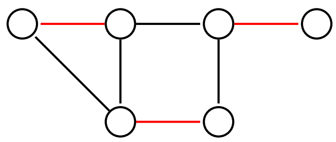
A **maximal matching** is a matching that cannot be extended any further.

For instance, the matching shown in Figure 2.1 is actually a **maximal matching**, because no other edge in E can be added to che current set of edges M of the matching without breaking the matching condition.

Definition 2.3: Maximum matching

A **maximum matching** is a matching that has the largest cardinality.

Clearly, the previous example does not repreent a **maximum matching**, because the following set of edges



is still a valid matching for the graph, but has a larger cardinality.

2.1 Augmenting paths

Given a matching M in a graph G, what conditions must be met to increase its cardinality? Trivially, if there exists an edge e disjoint from M, clearly $M \cup \{e\}$ is a larger matching than M — implying that M was not *maximal* in G. However, this is not the only situation in which the cardinality of M can be extended.

Definition 2.4: Alternating path

Given a graph G, and a matching M on G, an M**-alternating path** is a path of G that starts at a free node w.r.t. M, and is composed of edges that alternate between M and $E(G) - M$.

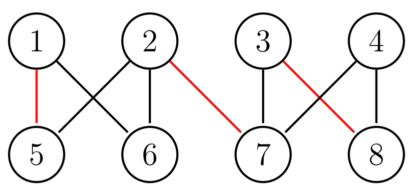


Figure 2.2: For instance, if M is the set of *red* edges — which forms a matching of the graph — then 6 $\{6, 2\}$ 2 $\{2, 7\}$ 7 $\{7, 3\}$ 3 $\{3, 8\}$ 8 is an M-alternating path.

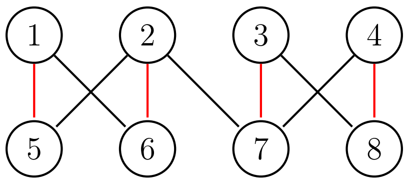
Definition 2.5: Augmenting path

Given a graph G, and a matching M on G, an M**-augmenting path** is an M-alternating path that ends at a free vertex w.r.t. M.

For example, if we consider the path

$$6\ \{6,2\}\ 2\ \{2,7\}\ 7\ \{7,3\}\ 3\ \{3,8\}\ 8\ \{8,4\}\ 4$$

this is actually an M-augmenting path of the previous graph. Augmenting paths are very useful because they can be used to *expand* the cardinality of an initial matching. In fact, in the previous graph we can actually define a *larger* matching by **swapping** the edges of this augmenting path, as shown below



2.1.1 Berge's theorem

This suggests that the absence of augmenting paths in a graph is a *necessary* condition for a matching *not* to be maximum, but we can actually prove that it is also *sufficient*, as stated in the following theorem, proved by Berge [Ber57] in 1957.

Theorem 2.1: Berge's theorem

Given a graph G, M is a maximum matching of G if and only if in G there are no M-augmenting paths.

Proof.

Direct implication. By contrapositive, consider a graph G and a matching M such that there is an M-augmenting path P in G. Moreover, by way of contradiction assume that M is maximum; however $M \Delta E(P)$ is a larger matching than M — here, Δ is the symmetric difference, therefore the operation $M \Delta E(P)$ has the same effect of *swapping* the edges of P between the ones in M and in $E(P) - M$.

Converse implication. By contrapositive, consider a graph G and a matching M of G that is not maximum, i.e. there exists a matching M^* of G such that $|M| < |M^*|$. Consider the subgraph of G that has the vertices of $V(G)$ and the edges described by $M \Delta M^*$; the symmetric difference of these two sets will yield the set of edges that are either in M or in M^*, but not in $M \cap M^*$, therefore this subgraph is *not* a multigraph. Moreover, since M and M^* are both matchings, we have that

(1) the degrees of the vertices of this subgraph can be either 0, 1 or 2

(2) in each component of the subgraph the edges *must* alternate between M and M^*

By the observation (1), we have that each component of the subgraph can be either

- an isolated vertex
- a cycle
- a path

and by observation (2), we have that all the cycle components must have *even* length, which implies that they have the same number of edges of M and M^*. On the other hand, path components may have either even or odd length; in particular, even-length paths must have the same number of edges of M and M^* — as for cycle components — while odd-length paths have a different number of edges of M and M^*. However, since $|M| < |M^*|$, there must be at least one path component of this subgraph such that its edges of M are less than the edges of M^*, and this is clearly an M-augmenting path.

□

2.1.2 Kőnig's theorem

Given a graph G, and a matching M of G, what is the maximum possible value for $|M|$? To answer this question, we need to introduce the following combinatorial structure.

Definition 2.6: Vertex cover

Given a graph G, a **vertex cover** for G is a set of vertices $C \subseteq V(G)$ such that every edge in G is incident to at least one vertex in C. Using symbols

$$\forall (u, v) \in E(G) \quad u \in C \lor v \in C$$

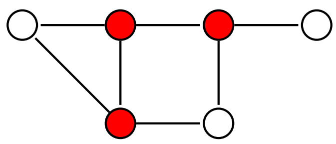


Figure 2.3: An example of a vertex cover.

As shown in figure, a vertex cover is simply a set of vertices that must *cover* all the edges of the graph. For vertex covers, we are interested in the *minimum* possible cardinality — the concepts of *minimal* and *minimum* are defined analogously as for *maximal* and *maximum*.

For our purposes, we are interested in vertex covers because through such combinatorial structures we can bound the size of any matching of a graph.

Theorem 2.2

Given a graph G, a matching M, and a vertex cover S of G, it holds that $|M| \leq |S|$.

Proof. By definition, any vertex cover S of $G = (V, E)$ is also a vertex cover for $G^B = (V, B)$, for any set of edges $B \subseteq E$, and in particular this is true for $G^M = (V, M)$.

Now consider G^M, and a vertex cover C on it: by construction we have that $\Delta \leq 1$, therefore any vertex in C will cover at most 1 edge of M. This implies that if $|C| = k$, then C will cover at most k edges of G^M.

Lastly, since G^M has $|M|$ edges by definition, any vertex cover defined on G^M has to contain at least $|M|$ vertices. This implies that no vertex cover S of G smaller than $|M|$ can exist, because S will have to cover at least the edges in M. $\square$

Corollary 2.1

Given a graph G, a maximum matching M^* and a minimum vertex cover S^*, it holds that $|M^*| \leq |S^*|$.

Moreover, if the graph is bipartite this theorem is actually *stronger*, as proved by Konig [Kon] in 1931.

Theorem 2.3: Kőnig's theorem

Given a bipartite graph G, a maximum matching M^* and a minimum vertex cover S^*, it holds that $|M^*| = |S^*|$.

Proof. Consider a graph G, a maximum matching M^* and a minimum vertex cover S^* of G; by the previous corollary, it follows that to prove the statement it suffices to show that there exists a vertex cover S such that $|S| = |M^*|$, because

$$|M^*| \leq |S^*| \leq |S| = |M^*| \implies |M^*| = |S^*|$$

Hence, we are going to construct the following vertex cover. Let G be bipartitioned through (A, B); then, for each edge $ab \in M^*$ such that $a \in A$ and $b \in B$, we place $b \in S$ if and only if there exists an M^*-alternating path that starts in A and ends at b, otherwise we place $a \in S$.

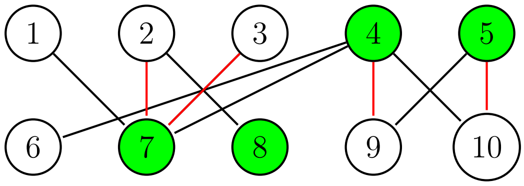


Figure 2.4: For instance, given this graph bipartitioned into (A, B) — where A is the uppermost row of vertices — and the *red* matching, we would construct the *green* vertex cover.

Note that, by definition, it holds that $|S| = |M^*|$.

Claim: S is a vertex cover for G.

Proof of the Claim. Consider an edge $ab \in E(G)$ such that $a \in A$ and $b \in B$; note that, by definition of S all the edges in M^* are already covered, hence we may assume that $ab \notin M^*$. We have two cases.

- a is free, i.e. $\nexists ab' \in M^*$ for $b' \in B$. Note that b cannot be free, otherwise $M^* \cup \{ab\}$ would still be a matching of G but greater than M^*. Hence, there must be an edge $a'b \in M^*$ for some $a' \in A$. Therefore, since a is free, the edge ab is a trivial M^*-alternating path ending at $b \in B$, meaning that $b \in S$ by definition, implying that ab is covered by S.
- a is matched, i.e. $\exists ab' \in M^*$ for $b' \in B$. Therefore, by definition of S, either a or b' lies in S. In particular, if $a \in S$, then ab is trivially covered by S, hence suppose that $b' \in S$.

  Observe that, by definition of S this implies that there must be an M^*-alternating path P that starts in A and ends at b'.

  - If $ab, ab' \notin E(P)$, then $P \cup \{b'a\} \cup ab$ is an M^*-augmenting path, which would contradict the fact that M^* is maximum by Theorem 2.1 $\lightning$
  - If $ab \notin E(P)$ but $ab' \in E(P)$, then P could not have been an M^*-alternating path starting at a vertex in A $\lightning$.
  - If $ab \in E(P)$ but $ab' \notin E(P)$, then P must have had the following form

    $$\ldots\ b''\ \{b'', a\}\ a\ \{a, b\}\ b\ \{b, a'\}\ a'\ \{a', b'\}\ b'$$

    where $a' \in A$, $b'' \in B$. However, since $ab' \in M^*$ and $ab \notin M^*$, and the edges of P must alternate w.r.t. M^*, ab'' must lie inside M^*, contradicting $ab' \in M^*$ by definition of matching $\lightning$.

  This implies that $ab', ab \in E(P)$, meaning that P encounters b "before" b'. However, this implies that $P - \{ab'\}$ is an M^*-alternating path that starts in A and ends at b, thus $b \in S$ by definition, concluding that ab is still covered by S.

$\square$

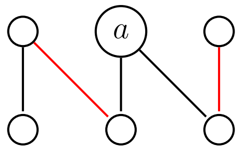
Hence, we have constructed a vertex cover S such that $|S| = |M^*|$, meaning that the statement holds because of the previous observation. $\square$

2.1.3 Finding maximum matchings

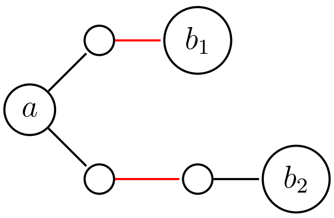
Consider a matching M of a graph G, and an M-augmenting path P; the idea of *swapping* the edges of P between M and $E(G) - M$ is very useful when G is **bipartite**. In fact, we can actually describe a procedure which is able to return a *maximum matching* of a bipartite graph, by swapping the edges of the augmenting paths present in G. However, for this algorithm to work, we first need a procedure capable of *finding* augmenting paths in bipartite graphs, which is defined down below.

1. Assume that the considered graph G is bipartite through (A, B), and consider a matching M of G
2. Starting from a node $a \in A$ free w.r.t. M, compute a *modified* BFS such that the edges of its tree alternate between $E(G) - M$ and M
3. If the tree of the BFS contains a free leaf $b \in B$, then the path $v \to b$ is M-augmenting

For instance, given the following bipartite graph G, and a matching M of G — outlined in *red*



the *modified* BFS rooted in a would produce the following tree



and we observe that the path $a \to b_1$ is M-alternating, while the path $a \to b_2$ is M-augmenting. The next proposition guarantees that if there are M-augmenting paths that start in a, our *modified* BFS will find at least one of them.

Proposition 2.1

Given a bipartite graph G, bipartitioned into (A, B), and a matching M of G, if there exists an M-augmenting path in G that starts in a vertex $a \in A$ free w.r.t. M, then there exists an M-augmenting path in the tree T of the *modified* BFS.

Proof sketch. Let P be an M-augmenting path that starts in a and minimizes the edges in $E(P) - E(T)$ and, by way of contradiction, assume that $E(P) - E(T) \neq \varnothing$, i.e. P is not completely contained in T. Therefore, let xy be the first edge in $E(P) - E(T)$

encountered while traversing P, starting at a, and without loss of generality assume that $x \in V(P)$.

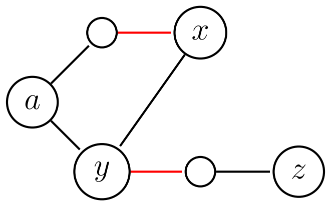
Claim: $x \in A$.

Proof of the Claim. By way of contradiction, assume that $x \in B$.

- Assume that x is not a leaf of T. Since the BFS starts at $a \in A$, and the edges of T alternate between M and $E(G) - M$, if $x \in B$ then the next edge xy' in T is an edge in M. Moreover — by the same reasoning — since P is M-augmenting, it must be that $xy \in M$, meaning that $xy, xy' \in M$ contradicting the definition of matching $\lightning$
- Now, assume that x is a leaf of T. By the same reasoning, xy must be in M because P is M-augmenting, but $xy \notin E(T)$ would imply that the BFS stopped before adding the edge xy to T $\lightning$.

□

For instance, given the following setting



a possible path for P would be $a \to x \to y \to z$. However, the path $a \to y \to z$ is still an M-augmenting path that starts at a but has one fewer edge not in T w.r.t. P, contradicting the definition of P $\lightning$.

Note that, in the general case we would consider the path $P' := a\ T\ y \cup y\ P$, however this is not guaranteed to be a path. The complete proof leverages the fact that G is bipartite in order to prove that P' is indeed a path, but it is very technical and outside the scope of these notes. □

Finally, now that we have a procedure which is guaranteed to find an augmenting path in a given bipartite graph, to return a maximum matching it suffices to run the following algorithm.

Algorithm 2.1: Maximum matching (bipartite graphs)

Given a bipartite graph G, the algorithm returns a maximum matching for G.

---

function $\textsc{MaximumMatchingBipGraphs}(G)$
    $M := \varnothing$
    do
        $P := \textsc{findAugmentingPath}(G)$ $\triangleright$ the previous procedure
        Swap the edges between M and $E(G) - M$ in P
    while $P \neq$ `None`
    return M
end function

In particular, the output of this algorithm is guaranteed to be a *maximum* matching thanks to Theorem 2.1, since the algorithm terminates when there are no more augmenting paths left in the graph G.

2.2 Perfect matching

By definition, a matching is *not* forced to cover all the vertices of a graph. However, if this happens the matching is called **perfect matching**.

Definition 2.7: Perfect matching

Given a graph G, a **perfect matching** of G is a matching that covers all the vertices of G, i.e. M is a perfect matching if and only if

$$\forall v \in V(G) \quad \exists e \in M \quad v \cap e \neq \varnothing$$

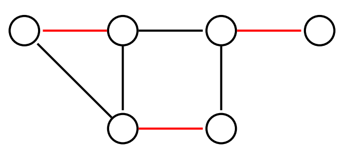


Figure 2.5: An example of a perfect matching.

Perfect matchings are an interesting topic of study when related to *bipartite graphs*. We observe that if a bipartite graph G, bipartitioned into (A, B), admits a perfect matching M, it must be that $|A| = |B|$. This is because every matched edge must connect one vertex from A to one vertex from B, and by definition M matches *each* vertex exactly once. However, the converse is not true.

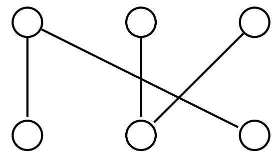


Figure 2.6: For instance, this bipartite graph, bipartitioned into (A, B) such that A is the uppermost row of vertices, even if $|A| = |B|$ this graph does not admit a perfect matching.

2.2.1 Hall's theorem

Because of this characterization on bipartite graphs, in addition to the concept of *perfect matching*, if $|A| \neq |B|$ we can define a *weaker* version of "perfect".

Definition 2.8: A-perfect matching

Given a bipartite graph G, bipartitioned through (A, B) such that $|A| \leq |B|$, we say that a matching M is an **A-perfect matching** if it covers all the vertices in A.

We will always assume that $|A| \leq |B|$, without loss of generality. The following theorem, known as the Hall's marriage theorem — proved by Hall [Hal35] in 1935 — shows the conditions that guarantee that an A-perfect matching exists.

Theorem 2.4: Hall's marriage theorem

Given a bipartite graph G, bipartitioned into (A, B), then G admits an A-perfect matching if and only if

$$\forall S \subseteq A \quad |S| \leq |\mathcal{N}(S)|$$

Proof. The direct implication is trivially true by definition of matching. We will prove the converse implication by contrapositive. Therefore, suppose that the bipartite graph G does not admit any A-perfect matching, and consider a minimum vertex cover V^* of G. Then, by Theorem 2.3, since G is bipartite we know that for any maximum matching M^* of G it holds that $|M^*| = |V^*|$. However, since we are assuming that there are no A-perfect matchings of G, any maximum matching must have size strictly less than $|A|$, meaning that $|V^*| = |M^*| < |A|$.

Now, consider the set $S := A - V^*$; since G is bipartite, it must hold that

$$\mathcal{N}(S) \subseteq V^* \cap B \implies |\mathcal{N}(S)| \leq |V^* \cap B| = |V^*| - |A \cap V^*|$$

Moreover, observe that

$$V^* \cap A = A - (A - V^*) = A - S \implies |V^* \cap A| = |A| - |S|$$

therefore, we have that

$$|\mathcal{N}(S)| \leq |V^*| - |V^* \cap A| = |V^*| - |A| + |S|$$

Lastly, since $|V^*| < |A| \iff |V^*| - |A| < 0$, we conclude that

$$|\mathcal{N}(S)| \le |V^*| - |A| + |S| < 0 + |S| = |S| \implies |\mathcal{N}(S)| < |S|$$

which proves that there is at least one set S for which the *Hall condition* does not hold. $\square$

2.2.2 Tutte's theorem

What about the general case? Consider any graph G on n vertices: clearly, if n is odd, the graph does not admit perfect matchings, since each edge matches exactly two nodes, meaning that there will always be a free vertex in the graph.

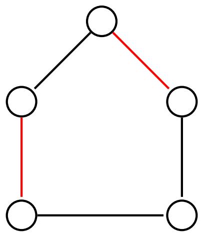
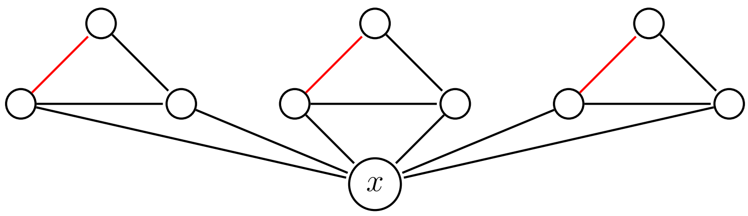


Figure 2.7: For instance, no matching of C_5 can be perfect.

For the same reasoning, if G has a connected component with an odd number of vertices, G does not admit perfect matchings. Hence, if G admits a perfect matching, there cannot be any connected component with an odd number of vertices. But is the converse true as well? Consider the following graph



This graph is connected, and has 10 vertices, but it does not admit perfect matchings, meaning that the converse does not hold. Can we find a condition that guarantees that a given graph always admits a perfect matching?

Given a graph G, let $\mathcal{O}(G)$ be the number of components with an odd number of vertices. The next theorem, proved by Tutte [LP09] in 1947 shows that the following property

$$\forall S \subseteq V(G) \quad \mathcal{O}(G[V(G) - S]) \le |S|$$

which we will refer to as **Tutte condition** — is a *necessary* and *sufficient* condition to guarantee that a graph admits a perfect matching. In fact, in the example of C_5 we observe that the set that violates the Tutte condition is $S = \varnothing$, and in the second example is $S = \{x\}$.

Lemma 2.1

Given a graph G, and a supergraph G' of G, if G satisfies the Tutte condition, then G' satisfies the Tutte condition as well.

Proof. By contrapositive, suppose that G' fails the Tutte condition, meaning that there exists a set $S \subseteq V(G') = V(G)$ such that $\mathcal{O}(G[V(G') - S]) > |S|$. Since G' is obtained by adding edges to G, the number of odd components in G' can either remain the same, or decrease if two different components having an odd number of vertices became connected in G'. Therefore, we have that

$$|S| < \mathcal{O}(G[V(G') - S]) \leq \mathcal{O}(G[V(G) - S])$$

meaning that G fails the Tutte condition as well. □

We are now ready to prove Tutte's theorem.

Theorem 2.5: Tutte's theorem

A graph G admits a perfect matching if and only if for any $S \subseteq V(G)$ it holds that $\mathcal{O}(G[V(G) - S]) \leq |S|$ — meaning that it satisfies the Tutte condition.

Proof.

Direct implication. For the direct implication, consider a graph G that admits a perfect matching M, and by way of contradiction assume that there exists a set $S \subseteq V(G)$ that violates the Tutte condition. For the previous observation, we know that each component of $G[V(G) - S]$ that has an even number of vertices will not have free nodes w.r.t. M, and each component of $G[V(G) - S]$ that has an odd number of vertices will have one free node w.r.t. M. In particular, these free vertices must be covered by M, since M is perfect, and the only way they can be covered by M is through the vertices of S. However, since $\mathcal{O}(G[V(G) - S]) > |S|$, there exists at least one free vertex in $G[V(G) - S]$ w.r.t. M that cannot be matched to any of the vertices of S, meaning that M is not perfect $\lightning$.

Converse implication. We will prove the contrapositive of the converse implication. Therefore, suppose that G does not admit any perfect matching, and we will prove that there exists a set of vertices for which the Tutte condition does not hold. Let G' be the maximal supergraph of G that still does not admit perfect matchings. We say that a set X satisfies the *clique-adjacency* condition if every component of $G'[V(G') - X]$ is a clique, and every vertex $u \in X$ is adjacent to every vertex $v \notin X$.

Claim: If there is a subset $S' \subseteq V(G')$ that satisfies the clique-adjacency condition, then G does not satisfy the Tutte condition.

Proof of the Claim. Note that $S' \subseteq V(G') = V(G)$, hence if S' violates the Tutte condition on G, the claim is trivially true. Therefore, we will assume that S' satisfies both the clique-adjacency condition, and the Tutte condition on G.

Let S' be a subset that satisfies the clique-adjacency condition; hence, by definition every component of $G'[V(G') - S']$ is a clique. Thus, consider a clique, and let n be the number of its vertices.

- If n is even, we can always find a perfect matching restricted on it.
- If n is odd, we can always find a perfect matching restricted to $n-1$ vertices, but since we are assuming that S' satisfies the Tutte condition on G, we know that $\mathcal{O}(G[V(G) - S']) \leq |S'|$. Therefore, by the clique-adjacency condition we know that we can always match the n-th vertex of the clique with a vertex of S'.

TODO □

da finire

Consider the set

$$S := \{v \in V(G') \mid \deg_{G'}(v) = n-1\}$$

By way of contradiction, suppose that S violates the clique-adjacency condition. This happens if at least one of the following holds:

(1) there are two vertices $u \in S$ and $v \notin S$ such that $u \notin S$

(2) there is a component of $G'[V(G') - S]$ that is not a clique

However, by definition of S, each vertex $v \in S$ has $\deg_{G'} = n-1$, hence (1) cannot be true, meaning that (2) must hold. Hence, consider a connected component of $G'[V(G') - S]$ that is not a clique, i.e. there are at least two vertices x and y such that $x \nsim y$.

Let P be the shortest path from x to y in $G'[V(G') - S]$, and let a, b and c be the first three vertices of P — note that $a, b, c \notin S$. We observe that $a \nsim c$, otherwise P would not be the shortest path. Note that $b \notin S$, therefore $\deg_{G'}(b) < n-1$, i.e. there exists a vertex d such that $b \nsim d$.

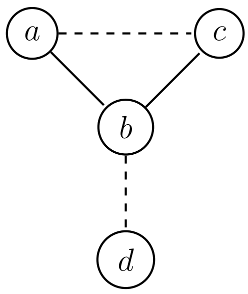


Figure 2.8: The "kite-shaped" figure we are considering — the dashed lines represent the *missing edges.*

By maximality of G', we know that $G' \cup \{ac\}$ must have a perfect matching M_1, and $G' \cup \{bd\}$ must have a perfect matching M_2. Moreover, $ac \in M_1$ and $bd \in M_2$, otherwise M_1 and M_2 would be perfect matchings on G'.

Claim: $(M_1 - \{ac\}) \cup (M_2 - \{bd\})$ contains a perfect matching for G'.

Proof of the Claim. Let $\overline{M} = (M_1 - \{ac\})\Delta(M_2 - \{bd\})$. By the same reasoning applied for Theorem 2.1, since M_1 and M_2 are two matchings, every vertex of $\overline{M}$ has degree at most 2. Hence, consider a vertex z in the subgraph induced by $\overline{M}$

- if $\deg_{\overline{M}}(z)$ is 0 or 2, z is matched both by M_1 and M_2
- since M_1 is a perfect matching, each vertex of G' is matched by $M_1 - \{ac\}$, except for a and c
- likewise, since M_2 is a perfect matching, each vertex of G' is matched by $M_2 - \{bd\}$, except for b and d

which implies that the only vertices z that have $\deg_{\overline{M}}(z) = 1$ are precisely a, b, c and d.

Moreover, each component of the subgraph induced by $\overline{M}$ is either an isolated vertex, a cycle or a path, and the edges of the components must alternate between M_1 and M_2. Therefore, because of the degrees of a, b, c and d, it must be that these vertices are endpoints of two alternating paths P_1 and P_2. Moreover, since $ac \in M_1$ and $bd \in M_2$, one of these paths must be an M_1-alternating path, while other must be M_2 alternating — and without loss of generality let P_1 be the M_1-alternating. Note that the set

$$M' := (M_1 \cup M_2) - (E(P_1) \cup E(P_2))$$

is a perfect matching on $G[V(G) - (V(P_1) \cup V(P_2))]$. We have three cases for P_1 and P_2:

(a) P_1 is a path $a \to c$, and P_2 is a path $b \to d$; then

$$(M_1 \cap E(P_2)) \cup (M_2 \cap E(P_1)) \cup M'$$

is a perfect matching on G'

(b) P_1 is a path $a \to b$, and P_2 is a path $c \to d$; then

$$(M_1 \cap E(P_2)) \cup (M_2 \cap E(P_1)) \cup M' \cup \{bd\}$$

is a perfect matching on G'

(c) P_1 is a path $a \to d$, and P_2 is a path $b \to c$; then

$$(M_1 \cap E(P_2)) \cup (M_2 \cap E(P_1)) \cup M' \cup \{ac\}$$

is a perfect matching on G'

□

Therefore, because of this claim G' always contains a perfect matching, contradicting the definition of G'. This implies that S must satisfy the clique-adjacency condition. Hence, by the first claim we have that G satisfies the Tutte condition first claim is wrong?

□

2.3 Stable matching

Matchings in bipartite graphs are particularly useful as they can be applied to model various types of problems across different fields. One well-known example is the stable matching problem, which arises in scenarios like job assignments, college admissions, and matchmaking systems. In this problem, the goal is to find a *stable pairing* between two sets of entities — such as students and universities — where no two unmatched entities would prefer each other over their current assignments.

Definition 2.9: Stable matching

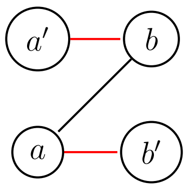
Given a graph G, bipartitioned through (A, B), and a matching M of G, consider a family of *preference* functions $\{w_v\}_{v \in V(G)}$ that for each vertex $v \in V(G)$, assign a value to the edges $vu \in E(G)$, for all $u \in \mathcal{N}(v)$

$$\forall v \in V(G) \quad w_v : \mathcal{N}(v) \to \mathbb{R}$$

We say that M is **stable** if for each $ab \in E(G)$ it does not happen that

$$(a \text{ free } \vee (\exists ab' \in M \quad w_a(b) > w_a(b')))$$
$$\wedge$$
$$(b \text{ free } \vee (\exists a'b \in M \quad w_b(a) > w_b(a')))$$

For instance, consider the following bipartite graph G, and a matching M — outlined in *red*:



If the family of preference functions $\{w_v\}_{v \in V}$ is such that

$$w_a(b) > w_a(b') \wedge w_b(a) > w_b(a')$$

which implies that both a and b would prefer to *switch* their current matched vertex — then M is *not* stable. Note that the empty matching is *not* stable, by definition.

The following theorem, proved by Gale and Shapley [GS62] in 1962, proves that a stable matching can be always constructed in a bipartite graph, regardless of the preference functions.

Theorem 2.6: Gale-Shapley theorem

Given a bipartite graph G, and a family of preference functions $\{w_v\}_{v \in V(G)}$, there exists a stable matching of G.

Proof. Before proving the theorem, we need to introduce some definitions.

Consider a bipartite graph G, bipartitioned through (A, B), and consider a matching M. Given two vertices $a \in A$ and $b \in B$, we say that a is **acceptable to** b if

- b is free, or
- b is matched to a vertex a', and $w_b(a) > w_b(a')$

Moreover, we say that a vertex $a \in A$ is **happy** if either

- a is free, or
- $ab \in M$ and for each b' such that a is acceptable to b', it holds that $w_a(b) \geq w_a(b')$

Observe that in the empty matching every vertex is happy.

Claim: Consider a matching M such that each vertex is happy; if M is not stable, then there exists a vertex $a \in A$ such that a is free and acceptable to some $b \in B$.

Proof of the Claim. By instability of M, there must be an edge $ab \notin M$ such that a is free, or it prefers b to its current partner, and vice versa. By way of contradiction, suppose that a is matched to some $b' \in B$, hence by instability of M through ab we know that $w_a(b) > w_a(b')$

- If b is free, then by definition a is acceptable to b; however, by happiness of a it must be that $w_a(b') \geq w_a(b)$ $\lightning$
- If b is matched by some edge $a'b \in M$, by instability of M through ab we know that $w_b(a) > w_b(a')$, hence a is acceptable to b, and by happiness of a it must be that $w_a(b') \geq w_a(b)$ $\lightning$

This implies that a must be free; therefore, we have that

- If b is free as well, then by definition a is acceptable to b
- If b is matched by some edge $a'b \in M$, by instability of M through ab we know that $w_b(a) > w_b(a')$, hence a is still acceptable to b

$\square$

Now, consider another matching M' of G; we say that M is **better than** M' if

- $\forall a'b \in M' \quad \exists ab \in M \quad w_b(a) \geq w_b(a')$, meaning that every vertex $b \in B$ prefers its match in M at least as much as its match in M', and
- $\exists a'b \in M', ab \in M \quad w_b(a) > w_b(a')$, meaning that there is at least one vertex b that strictly perfers its match in M over its match in M'

In other words, M is better than M' if no match in M is worse than in M', and at least one match is strictly better.

Let M_k be an unstable matching such that every vertex is happy; therefore, by the previous claim we know that there exists a vertex $a \in A$ such that a is free and and acceptable to some $b \in B$. We will construct a matching M_{k+1} as follows:

$$b^* \in \underset{\substack{b \in \mathcal{N}(a): \\ a \text{ acceptable to } b}}{\arg\max} \quad w_a(b) \implies M_{k+1} := (M_k \cup \{ab^*\}) - \{a'b^* \in M_k \mid a' \in A\}$$

meaning that M_{k+1} is obtained from M_k by adding the edge ab^*, where b^* maximizes $w_a(b^*)$, and removing the edge $a'b^*$ from M_k, if present — the last set is either $\{a'b^*\}$ or $\varnothing$.

Claim: M_{k+1} is better than M_k, and if M_k ensures that every vertex is happy, M_{k+1} does as well.

Proof of the Claim. First, we prove that M_{k+1} is better than M_k. The first condition that M_{k+1} has to satisfy in order to be better than M_k is true simply because we removed the edge $a'b^*$ if it was present in M_k, and the rest of the matching has not been altered. Moreover, if b^* was free then the second condition is vacuously true, otherwise if the edge $a'b^*$ was present in M_k, in M_{k+1} there we added the edge ab^* and we know that $w_{b^*}(a) > w_{b^*}(a')$ because a is acceptable to b^* by definition.

Now, assume that M_k is such that every vertex is happy. Since b^* is the neighbor of a that maximizes $w_a(b^*)$, a is happy by definition. Now, if b^* was free in M_k, then we only added ab^* to M_{k+1}, hence every vertex is happy w.r.t. M_{k+1}. Otherwise, if b^* was not free in M_k, i.e. there was an edge $a'b^* \in M_k$, by definition M_{k+1} will not contain $a'b^*$, meaning that a' is free w.r.t. M_{k+1}, hence a' is happy by definition. □

Now, consider the empty matching $M_0 := \varnothing$; we already observed that M_0 is not stable, but is such that each vertex is happy, therefore by the previous claim we can extend M_0 into M_1 through some vertex $a \in A$ that satisfied the condition of the first claim. In particular, since we proved that this process preserves the happiness of the vertices, we can repeat this process as long as the current matching M_i is still unstable.

Observe that every matching in the sequence is better than the previous ones, therefore such sequence cannot cycle. Moreover, since there is a finite number of possible matching of G, the sequence will eventually reach a matching where every vertex of A is matched — if $|A| \neq |B|$ we can assume that $|A| < |B|$ without loss of generality; call this matching $\hat{M}$. Hence, by contrapositive of the first claim, we know that $\hat{M}$ is either unstable, or has an unhappy vertex. However, by construction of our sequence, every vertex of $\hat{M}$ is happy, therefore it must be that $\hat{M}$ is stable. □

2.4 Exercises

Problem 2.1

Let G be a k-regular bipartite graph, bipartitioned through (A, B). Prove that

1. $|A| = |B|$
2. G has a perfect matching

Solution. Since G is k-regular, it holds that the number of edges that have an endpoint in A is precisely $k\,|A|$, and the number of edges that have an endpoint in B is exactly $k\,|B|$; moreover, since G is bipartite, we have that

$$k\,|A| = k\,|B| \implies |A| = |B|$$

We will prove the second statement by using Theorem 2.4. By way of contradiction, assume that there is a set $S \subseteq V(G)$ such that $|S| > |\mathcal{N}(S)| \implies k\,|S| > k\,|\mathcal{N}(S)|$. Applying the same argument as before, the number of edges ab such that $a \in S, b \in \mathcal{N}(S)$ is exactly $k\,|S|$, hence by the pigeonhole principle there must be at least one vertex in $\mathcal{N}(S)$ that has more than k vertices, contradicting the k-regularity of G. □

Problem 2.2

Let k be an integer. Show that any two partitions of a finite set into k-sets admit a common choice of representatives.

Solution. Let S be a finite set, and let $\mathcal{P} := \{P_1, \ldots, P_k\}$ and $\mathcal{P}' := \{P'_1, \ldots, P'_k\}$ be two partitions over S into k-sets — we observe that $|\mathcal{P}| = |\mathcal{P}'| = k$ and for any $Q_1, Q_2 \in \mathcal{P} \cup \mathcal{P}'$ we have that $|Q_1| = |Q_2| = \frac{|S|}{k}$.

Now, construct a graph G as follows:

- add a vertex for each k-set of the two partitions — hence $V(G) = \mathcal{P} \cup \mathcal{P}'$
- $\forall Q_1, Q_2 \in \mathcal{P} \cup \mathcal{P}' \quad Q_1 \sim Q_2 \iff Q_1 \cap Q_2 \neq \varnothing$

We observe that G is a bipartite graph, bipartitioned through $(\mathcal{P}, \mathcal{P}')$ itself: in fact, since $\mathcal{P}$ is a partition of S, its k-sets cannot intersect, hence $\mathcal{P}$ induces an independent set — and the same reasoning can be applied for $\mathcal{P}'$ as well.

Claim: G admits a perfect matching.

Proof of the Claim. Fix $X \subseteq \mathcal{P}$. Since $\mathcal{P}$ is a partition of S into k-sets, we have that $\left|\bigcup_{P \in X}\right| P = k\,|X|$ since the k-sets cannot overlap. By a similar argument, since $\mathcal{P}'$ is a partition of S into k-sets as well, we have that $\left|\bigcup_{P' \in \mathcal{N}(X)} P'\right| = k\,|\mathcal{N}(X)|$.

By way of contradiction, suppose that there is an element $x \in \bigcup_{P \in X} P - \bigcup_{P' \in \mathcal{N}(X)} P'$, meaning that there is an element $x \in P$ for a k-set $P \in X$ such that for any $P' \in \mathcal{N}(X)$

it holds that $x \notin \mathcal{P}'$. However, $\mathcal{P}$ and $\mathcal{P}'$ partition S, and in particular they cover S, meaning that there exists a k-set in $\mathcal{P}$ containing x, say P'. Moreover, by construction of G if $x \in P \cap P'$ then G contains the edge $\{P, P'\}$, meaning that $P' \in \mathcal{N}(X)$, but we assumed that no k-set of $\mathcal{N}(X)$ contained x $\lightning$. This proves that $\bigcup_{P \in X} P \subseteq \bigcup_{P' \in \mathcal{N}(X)} P'$.

Finally, together with the previous observation, this implies that

$$k\,|S| = \left|\bigcup_{P \in X} P\right| \leq \left|\bigcup_{P' \in \mathcal{N}(X)} P'\right| = k\,|\mathcal{N}(X)|$$

and in particular, we have that

$$k\,|S| \leq k\,|\mathcal{N}(X)| \iff |X| \leq |\mathcal{N}(X)|$$

meaning that the Hall condition is satisfied for any set $X \subseteq \mathcal{P}$. Hence, by Theorem 2.4 we get that G admits a $\mathcal{P}$-perfect matching, but because $|\mathcal{P}| = |\mathcal{P}'|$ then any $\mathcal{P}$-perfect matching of G is also a perfect matching of G. $\square$

Now, let M be a perfect matching of G, and consider the set R constructed as follows: for each edge $\{P, P'\} \in M$, pick one element $r \in P \cap P'$ and add it to R.

Claim: R is a set of representatives for both $\mathcal{P}$ and $\mathcal{P}'$.

Proof of the Claim. Since M is a matching, every element of R will represent exactly two partitions, namely one in $\mathcal{P}$ and the other in $\mathcal{P}'$. Moreover, since M is perfect, every partition of $\mathcal{P}$ and $\mathcal{P}'$ will be covered by at least one element of R. Finally, no element of R can represent more than one k-set in the same partition since $\mathcal{P}$ and $\mathcal{P}'$ are independent sets in G. $\square$

This last claim concludes the proof. $\square$

3

Graph packing

Connectivity is one of the most important aspects in graph theory, capturing the *robustness of the networks*. In this chapter, we delve into one of the foundational results in this area, namely **Menger's theorem**, which provides a deep and elegant characterization of *vertex and edge connectivity* in graphs.

Definition 3.1: A-B **paths**

Given a graph G, and two sets $A, B \subseteq V(G)$, an A-B **path** is a path that has one end in A, one end in B, and no internal vertices in $A \cup B$.

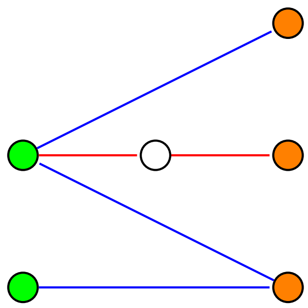


Figure 3.1: For instance, if A is the set of *green* vertices, and B is the set of *orange* vertices, then the *red* is an A-B path but the *blue* one is not.

Note that if $A \cap B \neq \varnothing$, any vertex in $A \cap B$ is a trivial A-B path.

Definition 3.2: A-B **hitting set**

Given a graph G, and two sets $A, B \subseteq V(G)$, an A-B **hitting** set is a set $X \subseteq V(G)$ such that every A-B path has a vertex in X.

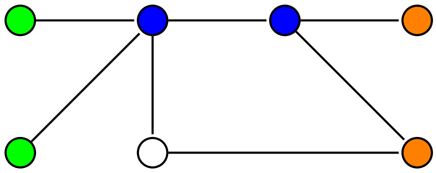


Figure 3.2: For example, if A is the set of *green* vertices, and B is the set of *orange* vertices, then the *blue* is an A-B hitting set.

Note that if X is an A-B hitting set, then $A \cap B \subseteq X$, otherwise any single vertex in $A \cap B$ would represent an A-B trivial path that does not pass through the hitting set X.

Proposition 3.1

Given a graph G, and two sets $A, B \subseteq V(G)$, it there exists an A-B hitting set X of size k then there are at most k vertex-disjoint A-B paths.

Proof. Trivially, by definition all the A-B paths must have at least one vertex in X, therefore there cannot be more than k vertex-disjoint A-B paths. □

This proposition clearly implies that the maximum number of disjoint A-B paths is upper bounded by the cardinality of the minimum A-B hitting set. However, hitting sets are not easy to work with, and a more convenient way to reason about A-B paths are **graph separations**.

Definition 3.3: A-B separation

Given a graph G, and two sets $A, B \subseteq V(G)$, two sets $X, Y \subseteq V(G)$ describe an A-B **separation** (X, Y) of G if and only if

- $A \subseteq X$
- $B \subseteq Y$
- $X \cup Y = V(G)$
- no edge has one end in $X - Y$ and the other in $Y - X$

We say that a separation (X, Y) has **order** $|X \cap Y|$.

TODO example

From the very definition, we see that any A-B path must pass through $X \cap Y$, since there are no edges "across" X and Y. Moreover, given a graph G, two sets $A, B \subseteq V(G)$, and an A-B separation (X, Y) of G, we observe that $G - (X \cap Y)$ is disconnected.

Now, consider the following property.

Proposition 3.2

Given a graph G, and two sets $A, B \subseteq V(G)$, there exists an A-B hitting set of size k in G if and only if there exists an A-B separation of order k.

Proof. The converse implication is trivially true, because if (X, Y) is an A-B separation of G, then $X \cap Y$ itself is an A-B hitting set of G by definition; therefore, we just need to prove the direct implication.

Let Z be an A-B hitting set of G such that $|Z| = k$, and consider the connected components $C_1, \ldots, C_\ell$ of $G - Z$. Note that there cannot be any component C_i that has a vertex $a \in V(G) \cap (A - Z)$ and a vertex $b \in V(C) \cap (B - Z)$, because otherwise by connectivity of C_i there would be a path $a \to b$ — which is an A-B path — that avoids the hitting set Z. Hence, we can define the two following sets

$$X := Z \cup \bigcup_{\substack{C \text{ component} \\ \text{of } G-Z \\ \text{such that } A \cap C \neq \varnothing}} V(C)$$

$$Y := Z \cup \bigcup_{\substack{C \text{ component} \\ \text{of } G-Z \\ \text{such that } A \cap C = \varnothing}} V(C)$$

We observe that, by definition

- $A \subseteq X$
- Y contains all the vertices in $B - Z$, therefore if $B \cap Z = \varnothing$ then trivially $B \subseteq Y$, and if $B \cap Z \neq \varnothing$ we observe that $Z \subseteq Y$ by definition, hence B is always completely covered by Y
- $X \cap Y = Z$

and for the previous observation there cannot be edges with one end in $X - Y$ and the other in $Y - X$. Hence, we conclude that (X, Y) is a separation of G, and $X \cap Y = Z \implies |X \cap Y| = |Z| = k$. □

This property directly implies that hitting sets and graphs separations can be treated *equivalently*. Hence, because of the previous proposition we derive the following observation.

Proposition 3.3

Given a graph G, two sets $A, B \subseteq V(G)$, and an A-B separation (X, Y) of minimum order, there are at most $|X \cap Y|$ vertex-disjoint A-B paths.

Proof. By Proposition 3.1 the maximum number of disjoint A-B is upper bounded by the minimum size of an A-B hitting set H, and by Proposition 3.2 there is an A-B separation having H's size as order. □

Finally, in 1927 Menger [Men27] proved that the lower bound also holds. To prove his result, we are going to prove a stronger version of his theorem.

Theorem 3.1: Menger's theorem (stronger version)

Given a graph G, and two sets $A, B \subseteq V(G)$, either there are at least k vertex-disjoint A-B paths or there is an A-B separation of order less than k.

Proof. We proceed by storng induction on m.

Base case. If $m = 0$, the only possible A-B paths in G are trivial paths, i.e. isolated vertices in $A \cap B$. Hence, if $|A \cap B| \geq k$ the base case holds; instead, if $|A \cap B| \leq k - 1$, we observe that $A \cap B$ itself is an A-B hitting set of G of size at most $k - 1$, which means that there exists an A-B separation of order at most $k - 1$ by Proposition 3.2.

Strong inductive hypothesis. Assume that the statement holds on a graph that has at most $m - 1$ edges.

Inductive step. We will prove that the statement holds for a graph G that has m edges as well. In particular, fix an edge $xy \in E(G)$, and note that the graph $G - \{xy\}$ has $m - 1$ edges, meaning that the inductive hypothesis holds for $G - \{xy\}$. However, if $G - \{xy\}$ has k disjoint A-B paths, then G also has k disjoint A-B paths, therefore we may assume that the inductive hypothesis holds for $G - \{xy\}$ precisely because the latter has an A-B separation of order at most $k - 1$.

Hence, let (X, Y) be an A-B separation of $G - \{xy\}$ of order at most $k - 1$; we observe that if $x, y \in X$, or $x, y \in Y$, then (X, Y) is also a separation of order at most $k - 1$ for G, which would conclude the proof, so we may assume otherwise. Thus, suppose that $x \in X - Y$ and $y \in Y - X$; then $(X, Y \cup \{x\})$ is an A-B separation of G that either has still order at most $k - 1$, or it becomes a separation of order exactly k. Since the first case concludes the proof, we may assume the second one to hold.

Consider the induced subgraphs $G[X]$ and $G[Y]$, and apply the inductive hypothesis on the sets $A, (X \cap Y) \cup \{x\} \subseteq X$ and $B, (X \cap Y) \cup \{y\}$ respectively. If both $G[X]$ and $G[Y]$ have k disjoint such paths, then in G there are k disjoint A-B paths, so without loss of generality we may assume that $G[X]$ does not have k such paths. Therefore, for the inductive hypothesis to hold this implies that $G[X]$ must have a (X', Y') separation of order at most $k - 1$; this implies that $A \subseteq X'$ and $(X \cap Y) \cup \{x\} \subseteq Y'$. Therefore, X' and $Y \cup Y'$ is a separation of G of order less than k.

□

Corollary 3.1: Menger's theorem (original version)

Given a graph G, and two sets $A, B \subseteq V(G)$, the maximum number of disjoint A-B paths is equal to the minimum order of an A-B separation of G.

Proof. Let M be the maximum numebr of vertex-disjoint A-B paths, and let m be the minum order of an A-B separation. Since each separation upper bounds the number of A-B paths by Proposition 3.3, we know that $M \leq m$. Now, by way of contradiction suppose that $M < m$; then, since there are less than m A-B paths there must exist a separation of order less than m, contradicting the minimality of m. This proves that $M \geq m$. $\square$

We will use Menger's theorem and its implications *extensively* in the following chapters.

3.1 k-connectivity

Definition 3.4: k-connectivity

A graph G is said to be k**-connected** if $|V(G)| \geq k+1$, and for each $X \subseteq V(G)$ such that $|X| \leq k-1$ it holds that $G[V(G) - X]$ is connected.

In other words, a graph is k-connected if it has at least $k+1$ vertices, and it remains connected whenever fewer than k vertices are removed. Therefore, by definition we observe that if a graph is *not* k-connected, it must admit a separation of order at most $k-1$.

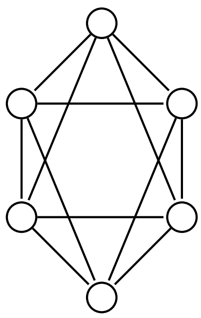


Figure 3.3: A 4-connected graph.

Clearly, any connected graph is 1-connected; moreover, any K_{t+1} clique is t-connected.

Proposition 3.4

If G is k-connected, then $\delta \geq k$.

Proof. By way of contradiction, assume that $\delta < k$, i.e. there is a vertex $x \in V(G)$ such that $\deg(x) < k \implies |\mathcal{N}(x)| < k$. Now, consider $G[V(G) - \mathcal{N}(x)]$: this subgraph is disconnected because it has x isolated, but $|\mathcal{N}(x)| \leq k-1$, meaning that this set contradicts the k-connectivity of G $\lightning$. $\square$

Proposition 3.5

If G is k-connected, then for any $xy \in E(G)$ it holds that $G - \{xy\}$ is $(k-1)$-connected.

Proof. By way of contradiction, suppose that $G - \{xy\}$ is not $(k-1)$-connected, i.e. there exists a set $X \subseteq V(G)$ such that $|X| \leq k-2$ and $(G - \{xy\})[V(G) - X]$ is disconnected. By k-connectivity of G, since $|X| \leq k-2$, we know that $G[V(G) - X]$ is connected, therefore if $(G - \{xy\})[V(G) - X]$ is disconnected it must imply that by removing xy we disconnect $G[V(G) - X]$. Now, since $|X| \leq k-2$, we know that $|X \cup \{y\}| \leq k-1$, but $G[V(G) - (X \cup \{y\})]$ does not contain the edge xy, hence it is disconnected, contradicting the k-connectivity of G $\lightning$. $\square$

Definition 3.5: Internal disjointness

Given a graph G, two paths P_1 and P_2 of G are said to be **internally disjoint** if every vertex of $V(P_1) \cap V(P_2)$ is an endpoint of P_1 and P_2.

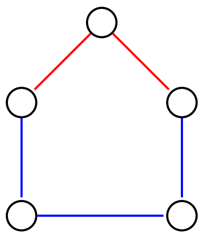


Figure 3.4: For instance, the *red* and *blue* paths are internally disjoint.

Proposition 3.6

Given a k-connected graph, and two sets $A, B \subseteq V(G)$, if $|A|, |B| \geq k$ then there exist k disjoint A-B paths in G.

Proof. By way of contradiction, if G does not admit k disjoint A-B paths, therefore by Theorem 3.1 the minimum order of an A-B separation is at most $k-1$; hence, let (X, Y) be a minimum order A-B separation of G, i.e. $|X \cap Y| \leq k-1$. Observe that $|A| \geq k$ and $A \subseteq X$, therefore $|X| \geq k$ which means that $X - Y \neq \varnothing$, and analogously $Y - X \neq \varnothing$. However, by definition of A-B separation we observe that $G[V(G) - (X \cap Y)]$ is disconnected, and we removed less than k vertices, contradicting the k-connectivity of G $\lightning$. $\square$

Theorem 3.2

A graph is k-connected graph if and only if for any pair of vertices $x, y \in V(G)$ such that $x \neq y$ there exist k $\{x\}$-$\{y\}$ internally disjoint paths.

Proof.

Direct implication. Fix a pair of distinct vertices $x, y \in V(G)$; we have two cases.

- The two vertices are not adjacent, i.e. $x \nsim y$. By Proposition 3.4, we know that $|\mathcal{N}(x)|, |\mathcal{N}(y)| \geq k$, hence by the previous proposition we there exist k disjoint $\mathcal{N}(x)$-$\mathcal{N}(y)$ paths, which will form k *internally disjoint* $\{x\}$-$\{y\}$ paths along with the edges to x and y.
- The two vertices are not adjacent, i.e. $x \nsim y$. By k-connectivity of G, we know that $G - \{xy\}$ is $(k-1)$-connected by Proposition 3.5, therefore by the same argument of the previous case we have $k-1$ internally disjoint $\{x\}$-$\{y\}$ path, therefore G will have k internally disjoint $\{x\}$-$\{y\}$ paths along with the edge xy.

Converse implication. Fix two distinct vertices $x, y \in V(G)$; since there are k $\{x\}$-$\{y\}$ internally disjoint paths, no set of $k-1$ — or less — vertices of G can disconnect x and y, implying that G is k-connected by definition.

□

Corollary 3.2

Given a k-connected graph G, a vertex $x \in V(G)$, and a subset $Y \subseteq V(G)$ such that $|Y| \geq k$, there exist k $\{x\}$-Y paths $P_1, \ldots, P_k$ such that $P_i \cap P_j = \{x\}$ — for all $i, j \in [k]$ such that $i \neq j$.

Proof. Add a vertex z to the vertices of G, and for each $y \in Y$ add an edge yz; we observe that $G[V(G) \cup \{z\}]$ is still k-connected, because $|Y| \geq k$. Now, since $x \neq z$, by the previous proposition there exist k $\{x\}$-$\{z\}$ internally disjoint paths, and these paths must traverse Y, therefore each path P_i must have the form $x \to y_i \to z$ for some $y_i \in Y$. Hence, if we consider the subpaths $x \to y_i$ for each i, in G these are precisely k $\{x\}$-Y paths intersects only in x. □

Theorem 3.3

Given a k-connected graph G such that $k \geq 2$, and k vertices $x_1, \ldots, x_k \in V(G)$, there exist a cycle in G that contains all the vertices $x_1, \ldots, x_k$.

Proof. By Proposition 3.4 we know that $\delta \geq k$, thus by Proposition 1.3 we know that there exists a cycle in G such that contains at least $k+1$ vertices. Let C be the cycle of G of length at least $k+1$ that contains as many vertices x_i as possible; if $x_1, \ldots, x_k \in V(C)$, the theorem is trivially verified, hence we may assume that at least one vertex x_i is not in $V(C)$, and without loss of generality assume that $x_k \notin V(C)$.

Now, since there are at most $k-1$ vertices x_i in $V(C)$, and there are k $\{x_k\}$-$V(C)$ paths, by the pigeonhole principle there must be a subpath Q of C of the form $x_i \to x_j$ — for $i, j \in [k-1]$ distinct — that has no internal vertex in $\{x_1, \ldots, x_{k-1}\}$, and such that there are two paths P_1 and P_2 that x_k as one endpoint and the other lies in Q.

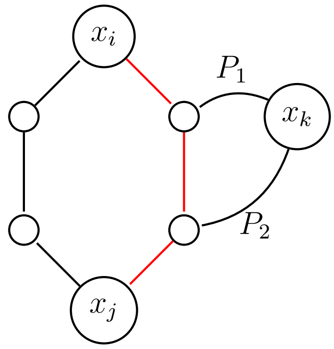


Figure 3.5: In the figure, the *red* segments compose the subpath Q.

Therefore, we can reroute Q by passing through P_1 and P_2 in order to find a cycle $C \cup \{x_k\}$ that has a larger number of vertices x_i, contradicting the definition of C $\lightning$. $\square$

Note that this theorem *does not* guarantee the order in which the vertices are present in the cycle. In fact, asking to preserve the order of the vertices $x_1, \ldots, x_k$ given is a *much harder* question, and the only result we currently know is the following.

Theorem 3.4

Given a $10k$-connected graph G, and k vertices $x_1, \ldots, x_k \in V(G)$, there exist a cycle in G that contains all the vertices $x_1, \ldots, x_k$ in the same order.

3.2 Feedback vertex set

Consider the following combinatorial structure.

Definition 3.6: Feedback vertex set

Given a graph G, a subset $X \subseteq V(G)$ is said to be a **feedback vertex set** (FVS) if $G - X$ is acyclic; equivalently, X is an FVS if it intersects all cycles of G.

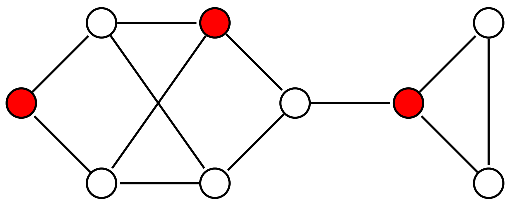
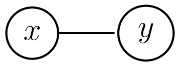


Figure 3.6: For instance, the *red* set of vertices is an FVS of this graph.

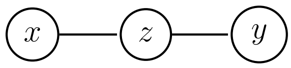
In many applications, it is important to find a *minimum* FVS; however, this problem is known to be NP-complete — as proved by Karp [Kar72] in 1972. Therefore, we often focus on estimating the size of this set by computing lower and upper bounds. In particular, in this section we will present a result proved by Erdös and Pósa [EP65] in 1965. But before introducing their result about FVSs, we must discuss some combinatorial structures first.

3.2.1 Topological minors

Consider a graph G, and an edge $xy \in E(G)$; to **subdivide** xy means to *remove* the edge xy from $E(G)$ and *replacing* it with a 2-edge path $x\ z\ y$ with some new vertex z. For instance, if G contains the edge xy



to *subdivide* the edge xy means to replace xy in G with the following 2-edge path



Definition 3.7: Subdivision

Given a graph G, a **subdivision** of G is a graph obtained from G by repeatedly subdividing its edges.

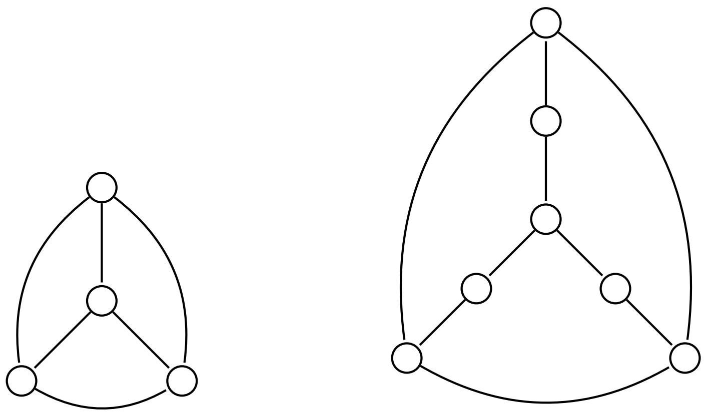


Figure 3.7: On the left: K_4. On the right: a subdivision of K_4.

Trivially, any graph is a subdivision of itself.

Definition 3.8: Topological minor

Given a graph G, and a graph H, we say that G contains H as **topological minor** if G has a subgraph which is a subdivision of H. Similarly, we say that H is a *topological minor* of G.

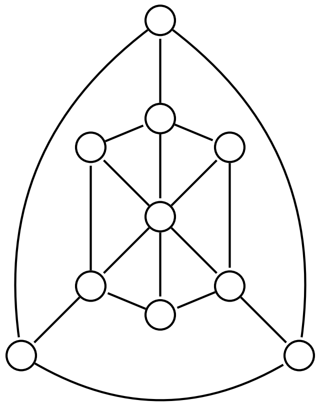


Figure 3.8: For instance, this graph has K_4 as topological minor, because it contains a subdivision of K_4 as subgraph.

Proposition 3.7

If G is 3-connected, then G has K_4 as topological minor.

Proof. If G is 3-connected, then by Proposition 3.4 we know that $\delta(G) \geq 3$, and by Proposition 1.3 this means that in G there is a cycle C. Moreover, by 3-connectivity of G, we know that G cannot be a cycle itself, therefore there must be at least one vertex $v \notin V(C)$. Now, by 3-connectivity agin we can apply Corollary 3.2, obtaining 3 $\{v\}$-$V(C')$ paths P_1, P_2 and P_3 that only intersect in v, and finally $G[V(C) \cup V(P_1) \cup V(P_2) \cup V(P_3)]$ is a subdivision of K_4. □

3.2.2 Erdős-Pósa theorem

Proposition 3.8

If G is 3-regular multigraph, then G has a cycle of length at most $2\lceil \log n \rceil$.

Proof. Clearly, if G admits loops or parallel edges, the statement is trivially true, therefore we may assume that G is a simple graph.

Fix a vertex $v \in V(G)$, and grow a BFS tree T from x as long no cycles of length at most $2\lceil \log n \rceil$ are encountered. Since G is 3-regular, T will be a tree in which the root x has 3 children, and all the other nodes that are not leaves will have 2 children. If the BFS stopped before visiting all the vertices of G, the statement holds, so we may assume that T covers all $V(G)$.

By 3-regularity of G, we have that

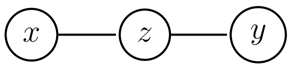
- $\delta \geq 3$, hence by Proposition 1.3 we know that G must contain a cycle
- the only possible cycle that can be formed in G is by connecting two leaves of T through an edge

and lastly, since $V(T) = V(G) = n$, the height of T is $\lceil \log n \rceil - 1$, which means that such a cycle must have length

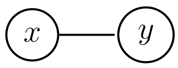
$$\lceil \log n \rceil - 1 + \lceil \log n \rceil - 1 + 1 = 2 \lceil \log n \rceil - 1$$

□

Along with the *subdivision* operation that we introduced previously, it naturally follows to define the *inverse* operation of the subdivision, i.e. the **suppression**. Given a graph G, and two edges $xz, zy \in E(G)$ such that $\deg(z) = 2$, to *suppress* z means to *remove* the vertex z from $V(G)$ along with the edges xz and zy, and *replacing* them with an edge xy in $E(G)$. For instance, if G contains the edges xz and zy



to *suppress* the vertex z means to replace xz and zy in G with the following edge xy



Now, consider the following definition.

Definition 3.9: Cut

Given an undirected graph $G = (V, E)$, and two subsets $S, T \subseteq V$, the **cut** induced by S and T on G is defined as follows

$$\mathrm{cut}(S, T) = \{e \in E : |S \cap e| = |T \cap e| = 1\}$$

In the case in which $T = \overline{S} = V - S$ we will simply write $\mathrm{cut}(S) = \mathrm{cut}(S, \overline{S})$.

In other words, a *cut* induced by a two sets S and T of vertices is the set of edges that have one endpoint in S and the other one in T.

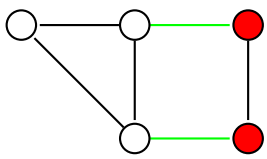


Figure 3.9: For instance, if the set S is described by the *red* vertices, then $\mathrm{cut}(S)$ is the set of green edges.

We will use the definition of cut in order to prove the following lemma.

Lemma 3.1

Given a 3-regular graph H such that $|V(H)| \geq c'k \log(k+1)$ for some constant c', then H contains k disjoint cycles.

Proof. We proceed by induction on k.

Base case. For $k = 1$, if H is 3-regular then $\delta(H) = 3$, therefore by Proposition 1.3 it must contain a cycle.

Inductive hypothesis. Assume that the statement holds for $k - 1$.

Inductive step. Let C be a cycle of H, and consider the graph $H - E(C)$.

Claim: $H - E(C)$ has a 3-regular graph H' as topological minor, such that $|V(H')| \geq |V(H)| - 2\,|V(C)|$.

Proof of the Claim. Starting with $V := V(C)$, as long as $H - V$ contains vertices of degree at most 1, move them into V; let M be the number of moved vertices. Then, since we are adding vertices to V, we have that $|\text{cut}(V)| \leq |\text{cut}(C)| - M$. We observe that, by construction $\delta(H - V) \geq 2$, and the number of vertices that have degree 2 in $H - V$ is at most $|\text{cut}(V)| \leq |\text{cut}(C)| - M$.

Lastly, let H' be the graph obtained from $H - V$ by suppressing all the vertices of degree 2; then we have that H' contains at least the vertices of H, without the vertices of C, the M vertices that we moved during the procedure, and the vertices of degree 2 that we suppressed. In other words, we have that

$$|V(H')| \geq |V(H)| - |V(C)| - M - (|\text{cut}(C)| - M) \geq |V(H)| - 2\,|V(C)|$$

Note that the last inquality comes from the observation that $|\text{cut}(C)| \leq |V(C)|$, which follows by 3-regularity of H.

Finally, since H' is 3-regular by construction, and $H - V$ is a subdivision of H' that is contained in $H - E(C)$, this concludes the claim. $\square$

Now, since $|V(H)| \geq c'k \log(k+1)$, and by the previous proposition we know that

$$|V(C)| \leq 2\log(|V(H)|) + 2 \iff -2\,|V(C)| \geq -4\log(|V(H)|) - 4$$

where the added term 2 takes care of the rounding error from the ceiling operation — we obtain the following

$$\begin{aligned} |V(H')| &\geq |V(H)| - 2\,|V(C)| \\ &\geq |V(H)| - 4\log(\log V(H)) - 4 \\ &\geq c'k\log(k+1) - 4\log(c'k\log(k+1)) - 4 \\ &\geq c'k\log(k+1) - 4[\log c' + \log k + \log(\log(k+1))] - 4 \end{aligned}$$

which is at least $c'(k-1)\log k$ for sufficiently big values of c'. In particular, since $|V(H')| \geq c'(k-1)\log k$, and we know that H' is 3-regular, we can apply the inductive hypothesis on H', which means that H' contains $k - 1$ disjoint cycles.

Finally, if H' contains k disjoint cycles, and $H - E(C)$ has H' as topological minor, i.e. $H - E(C)$ contains a subdivision H'' of H', by definition of the subdivision operation H'' will contain $k-1$ disjoint cycles as well. Therefore, since $H - E(C)$ contains H'', H contains k disjoint cycles along with the cycle C.

□

We are now ready to prove the theorem that we introduced at the beginning of this section.

Theorem 3.5: Erdős-Pósa theorem

There is a constant c such that for any graph G, and any $k \in \mathbb{N}$

- either G has k vertex-disjoint cycles or
- there is an FVS X of G such that $|X| \leq ck \log k$

Proof. If G contains k disjoint cycles, the theorem is trivially true, hence we may assume that G does not have k disjoint cycles. Fix H to be the larget subgraph of G such that $\forall v \in V(H) \quad 2 \leq \deg_H(v) \leq 3$ — note that H always exists because we may assume that G contains at least 1 cycle, since if G is acyclic the theorem is trivially true (and we would pick the cycle as H itself). Moreover, fix $\overline{H}$ to be the subgraph of G without all the "cycle components" of G — i.e. components of G that are cycles — and let $U := \{v \in V(\overline{H}) \mid \deg_{\overline{H}}(v) = 3\}$.

By TODO we know that $|U| < c'k \log(k+1)$, for some constant c. Let W be a set of vertices composed of one vertex for each cycle component — the ones not present in $\overline{H}$. discrepanze lemma

Claim: Every cycle of G is in H.

Proof of the Claim. If there was a cycle of G not present in H, this cycle would have contained vertices of degree at least 2, contradicting the maximality of H. □

Claim: There are no paths P in $G[V(G) - (U \cup W)]$ of length at least 1 such that

- both ends of P are in $H[V(H) - (U \cup W)]$
- no internal vertex of P is in H
- no edge of P is in H

Proof of the Claim. If such a path P exists, then $H \cup P$ is a bigger graph than H that has every vertex of degree 2 or 3, contradicting the maximality of H. □

Claim: Every cycle of $G - (U \cup W)$ intersects H in exactly one vertex.

Proof of the Claim. TODO □ da finire

TODO □ da finire

3.3 Directed graphs

Up until this point, we only discussed *undirected graphs*. In this section we are going to present briefly some results regarding **directed graphs** (or *digraphs*, for short), which are graph in which the edges have an **orientation** — implying that the edge (x, y) is different from the edge (y, x).

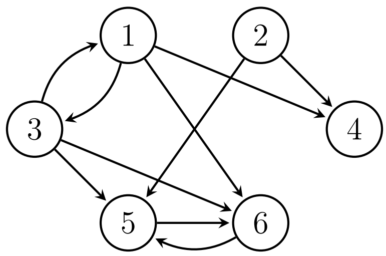


Figure 3.10: An example of a *digraph*.

Definition 3.10: Path cover

Let G be a digraph; a **path cover** for G is a set $\mathcal{P}$ of disjoint paths such that

$$V(G) = \bigcup_{P \in \mathcal{P}} V(P)$$

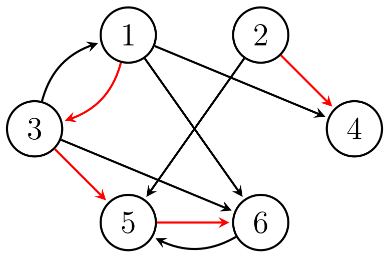


Figure 3.11: For instance, if we consider the digraph of the previous example, the set of *red* vertices form a *path cover*.

We observe that any digraph contains a trivial path cover, i.e. the one formed by the sef of every trivial path of the graph. In fact, we are interested in finding the path cover having *minimum cardinality*. Gallai and Milgram [GM60] proved that path covers are strictly related to independent sets, through the following theorem.

Theorem 3.6: Gallai-Milgram theorem

Given a digraph G, there are a path cover $\mathcal{P}$ of G and an independent set $X \subseteq V(G)$ such that

$$\forall P \in \mathcal{P} \quad |X \cap V(P)| = 1$$

We are going to prove a slightly stronger result that directly implies the above theorem. First, consider the following definitions.

Definition 3.11: Terminal nodes

Given a digraph G, and a path cover $\mathcal{P}$ of G, the set of **terminal nodes** of the paths of $\mathcal{P}$ is defined as follows:

$$\mathrm{ter}(\mathcal{P}) := \{v \in V(P) \mid P \in \mathcal{P} \land \deg_P^{out}(v) = 0\}$$

For example, in Figure 3.11 the terminal nodes are 4 and 6.

Definition 3.12: Minimality by inclusion

Given a digraph G, and a path cover $\mathcal{P}$ of G, we say that $\mathcal{P}$ is **minimal by inclusion** if there is no other path cover $\mathcal{P}'$ of G such that $\mathrm{ter}(\mathcal{P}') \subset \mathrm{ter}(\mathcal{P})$.

For instance, the path cover $\mathcal{P}$ in Figure 3.11 is clearly minimal by inclusion, because no directed edge can be removed from $\mathcal{P}$ such that the latter still covers G.

Proposition 3.9

Given a graph G, and a path cover $\mathcal{P}$ of G that is minimal by inclusion, there is an independent set $X \subseteq V(G)$ such that

$$\forall P \in \mathcal{P} \quad |X \cap V(P)| = 1$$

Proof. TODO □ da fare

Corollary 3.3

Given a digraph G, the minimum cardinality of a path cover for G is at most the maximum cardinality of an independent set for G.

Proof. By way of contradiction, suppose that the minimum cardinality of a path cover $\mathcal{P}$ for G is strictly greater than the maximum cardinality of an independent set X for G. Then, by the previous proposition there must be an independent set X' such that $\forall P \in \mathcal{P} \quad |X' \cap V(P)| = 1$. However, since every path in $\mathcal{P}$ is disjoint by definition we get that X' must have at leat $|\mathcal{P}|$ vertices, implying that $|X| < |\mathcal{P}| \leq |X'|$, thus contradicting the choice of X. □

In particular, this corollary provides a lower bound for the **maximum independent set** problem, which is notoriously NP-Hard.

Moreover, since a graph contains a Hamiltonian cycle if and only if there is a path cover of size 1, therefore the former problem can be reduced to the latter, implying that the minimum path cover problem is also NP-Hard.

Nonetheless, the Gallai-Milgram theorem still has applications in various branches of mathematics. For instance, it can be used to prove a result first derived by Dilworth [Dil50] in 1950 involving **posets**.

Definition 3.13: Poset

Given a set S and a relation $\prec$, the algebraic structure $(S, \prec)$ is called **partially ordered set** (or *poset*, for short) if $\prec$ is a binary relation that is *reflexive*, *antisymmetric* and *transitive*.

For instance, the structure $(\mathbb{R}, \leq)$ is clearly a partially ordered set, where the properties derive directly from the "$\leq$" relation.

Definition 3.14: Chain

Given a poset $(S, \prec)$, a **chain** is a sequence of pairwise comparatble elements of S w.r.t. $\prec$. In other words, $C \subseteq S$ is an anti-chain if it holds that

$$\forall s, s' \in C \quad s \prec s' \land s' \prec s$$

Definition 3.15: Anti-chain

Given a poset $(S, \prec)$, an **anti-chain** is a set of pairwise uncomparable elements of S w.r.t. $\prec$. In other words, $A \subseteq S$ is an anti-chain if it holds that

$$\forall s, s' \in A \quad s \not\prec s' \land s' \not\prec s$$

TODO drawing

Theorem 3.7: Dilworth's theorem

Given a poset $(S, \prec)$, the minimum number of chains covering S is equal to the maximum cardinality of an anti-chain.

Proof. Let $c(S)$ be the minimum number of chains that cover S, and let $a(S)$ be the maximum cardinality of an anti-chain in S. We construct a graph G_S such that

- $V(G_S) = S$
- $E(G_S) = \{(x, y) \mid x, y \in V(G_S) : x \prec y\}$

Then, by construction each chain in S corresponds to a path in G_S, and similarly each anti-chain in S corresponds to a set of disconnected nodes, i.e. an independent set in G_S. Therefore, by Theorem 3.6 we get that

$$c(S) = |\mathcal{P}^*| \leq |X^*| \leq a(S)$$

where $\mathcal{P}^*$ is a minimum path cover and X^* is a maximum independent set of G_S. Finally, for each path cover there must be at least one path for each node of any independent set of G_S, concluding that $c(S) \geq a(S)$ also holds — we observe that this does not hold in general, but in posets $\prec$ is transitive. $\square$

3.4 Exercises

Problem 3.1

Given a k-connected graph G with $k \geq 2$ such that $|V(G)| \geq 2k$, show that G contains a cycle of length at least $2k$.

Solution. By Proposition 3.4 we know that $\delta \geq k \geq 2$, therefore G contains at least one cycle. Let C be the longest cycle of G, and by way of contradiction suppose that $|V(C)| \leq 2k-1$. However, since $|V(G)| \geq 2k$ at least one vertex $v \in V(G) - V(C)$ must exist. By Proposition 1.3 we know that G has a cycle of length at least $\delta + 1 \geq k+1$, therefore C must also have length at least $k+1$, otherwise it would not be the longest cycle. Hence, by Corollary 3.2 there are k internally disjoint $\{v\}$-$V(C)$ paths that only intersect in v. However, since C only has at most $2k-1$ vertices, and there are k such paths, at least two paths P and P' must have adjacent endpoints in C by the pigeonhole principle. This implies that C can be extended through P and P' such that it also includes the vertex v and still contains all the vertices that it originally contained, in order to construct a cycle longer than C, contradicting its definition $\lightning$. $\square$

Problem 3.2

Prove that any t-connected subgraph G with $n \geq t+2$ contains $K_{2,t}$ as a topological minor.

Solution. Fix two vertices $x, y \in V(G)$; since $n \geq t+2$, we have that $|V(G) - \{x,y\}| \geq t$, and since G is t-connected, by Corollary 3.2

- there are t $\{x\}$-$V(G) - \{x,y\}$ paths that only intersect in x — let X be the set of the first vertices that these paths encounter from x to $V(G) - \{x,y\}$
- there are t $\{y\}$-$V(G) - \{x,y\}$ paths that only intersect in y — let Y be the set of the first vertices that these paths encounter from y to $V(G) - \{x,y\}$

and in particular, we observe that $|X|, |Y| \geq t$.

Fix two vertices $a \in X$ and $b \in Y$; by Theorem 3.2, a and b must be connected by at least one path, say $P_{b,a}$. Now, if we consider the path $P_{y,b}$ that connects y and a, we get a path $P_{y,b} \cup P_{b,a}$ that connects y and a.

The same reasoning can be applied to any pair of vertices in X and Y, which means that all the vertices in X are connected to x by definition, and they are also connected to y thanks to the paths passing through Y. This implies that G contains a subgraph that is a subdivision of $K_{2,t}$, meaning that $K_{2,t}$ is a topological minor of G. $\square$

Problem 3.3

Prove that every 3-connected graph has a cycle of even length.

Solution. Consider a 3-connected graph G, and by way of contradiction suppose that G has no even length cycles. Fix two adjacent vertices $u, v \in V(G)$; by Theorem 3.2 we know that there are at least 3 $\{u\}$-$\{v\}$ paths in G, and since the edge $\{u, v\}$ is already a path, we may assume there are at least two other internally disjoint paths P_1 and P_2 that have u and v as endpoints.

However, observe that $P_1 \cup \{u, v\}$ and $P_2 \cup \{u, v\}$ are cycles, therefore they must have odd length for the sake of contradiction — say that $2k + 1$ and $2h + 1$ are the length of the first and the second cycle, respectively. Then, consider $P_1 \cup P_2$: since P_1 and P_2 are internally disjoint, this is another cycle of G, that has length

$$(2k + 1) - 1 + (2h + 1) - 1 = 2k + 2h - 2$$

which is even, contradicting the fact that G only had odd-length cycles $\lightning$. $\square$

Problem 3.4

Prove that if a graph G has three edges $e, f, g \in E(G)$ such that e and f are contained in a common cycle, and f and g are also contained in a common cycle, then e and g are contained in a common cycle as well.

Solution. Let C_1 and C_2 be cycles of G such that $e, f \in E(C_1)$ and $f, g \in E(C_2)$ respectively, and by way of contradiction suppose that there is no common cycle of e and g. In particular, this implies that there is not pair of vertex-disjoint paths from the endpoints of e to the endpoints of g, hence by Theorem 3.1 G must admit an e-g separation of order at most 1 — call this separation (X, Y) and without loss of generality suppose that $e \subseteq X$ and $g \subseteq Y$. Hence, since $|X \cap Y| \leq 1$, it must be that either $e, f \subseteq X$ or $f, g \subseteq Y$. Then, in the former case there cannot be any common cycle for f and g, and in the latter there cannot be any common cycle for e and f $\lightning$. $\square$

4

Extremal graph theory

Extremal graph theory is a field of mathematics that investigates how *global properties* of a graph influence the **presence or absence** of specific substructures. Broadly speaking, it seeks to understand how large or dense a graph can be while avoiding certain configurations, or conversely, what constraints guarantee the appearance of particular patterns.

The central theme of the discipline lies in *exploring thresholds*: determining the minimal or maximal values of graph parameters — such as the number of edges, degrees, or vertices — beyond which a given substructure must necessarily exist.

This area plays a crucial role in understanding the fundamental behavior of graphs under constraints, offering insights that are both theoretically significant and applicable in various areas such as network design, data analysis, and algorithm optimization.

4.1 Edge bounds

In the first section of this chapter, we will explore the bounds on the number of **edges** that force any graph to contain interesting substructures. In particular, we define $\mathrm{ex}(n, H)$ to be the minimum number of edges in an n-vertex graph that guarantees the existence of H as subgraph.

4.1.1 Cliques

We will start our discussion of extremal graph theory with the question of **determining the existence** of cliques within a given graph. In general, finding the largest clique in a graph is an NP-Hard problem. This leads to a natural question: under what conditions can we guarantee the existence of a K_n subgraph?

Pál Turán addressed this by providing an exact upper bound on the number of edges a graph can have without containing a clique of a given size. But before discussing his result, let us first introduce some important definitions.

Definition 4.1: Edge maximum graph without K_r subgraphs

A graph G is said to be **edge maximum without K_r subgraph** — denoted as EMK_r, for short — if for any G' such that $|V(G')| = |V(G)|$ and $|E(G')| > |E(G)|$, G' has K_r as subgraph.

In other words, G is edge maximum without K_r if by adding an edge to G we obtain a graph G' that contains K_r as subgraph.

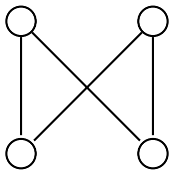


Figure 4.1: For instance, in this graph clearly there are no cliques as subgraph, but if we add one edge we construct K_3, meaning that this graph is EMK_3.

Definition 4.2: Independent set

Given a graph G, a subset $X \subseteq V(G)$ is said to be an **independent set** if for all u and v in X, it holds that $u \nsim v$.

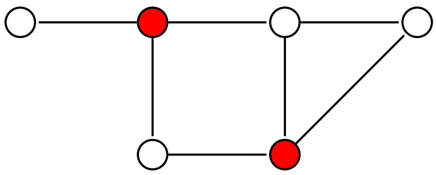


Figure 4.2: For instance, the *red* set of vertices is an independent set.

Note that we already encountered independent sets when discussing bipartite graphs: in fact, if G is bipartite through the bipartition (A, B), we observe that both A and B must be independent sets.

Now, consider the following type of graphs.

Definition 4.3: r-partite graph

A graph is said to be **r-partite** if it can be partitioned into $X_1, \ldots, X_r$ sets of vertices such that $X_1, \ldots, X_r$ are independent sets.

Moreover, the graph is said to be **complete r-partite** if for all $i, j \in [r]$ such that $i \neq j$, it holds that if $x \in X_i$ and $y \in X_j$ then $x \sim y$.

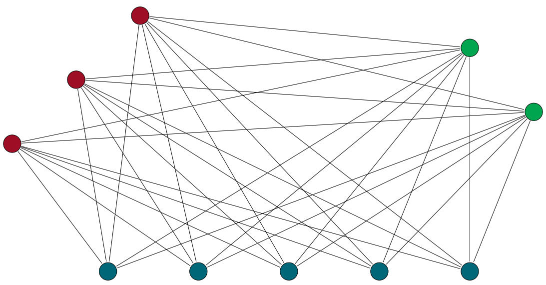


Figure 4.3: A complete 3-partite graph.

The following proposition shows that such graphs cannot contain cliques as subgraphs.

Proposition 4.1

An $(r-1)$-partite graph cannot contain K_r as subgraph.

Proof. Let G by an $(r-1)$-partite graph; by definition, G is composed of $r-1$ independent sets, therefore any subgraph of G that has r vertices $x_1, \ldots, x_r$ must have 2 vertices lying in the same independent set by the pigeonhole principle, meaning that any subgraph of vertices $x_1, \ldots, x_r$ will have two non-adjacent vertices. $\square$

From this proposition, it follows that the number of edges of a *complete* $(r-1)$-partite graph provides a *lower bound* for the maximum number of edges that a graph can have before inducing K_r as subgraph. Therefore, consider the following definition.

Definition 4.4: Turán graph

The Turán graph $T(n, r)$ is the complete r-partite graph that has n nodes, partitioned into $X_1, \ldots, X_r$, such that for all $i, j \in [r]$, if $i \neq j$ then $||X_i| - |X_j|| \leq 1$ — i.e. the cardinality of the partitions are as close to equal as possible.

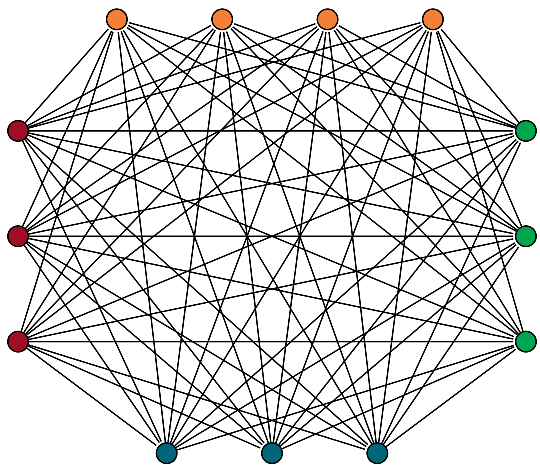


Figure 4.4: The Turán graph $T(13, 4)$.

We will refer to the number of edges of $T(n,r)$ as follows

$$t(n,r) := |E(T(n,r))| \sim \binom{n}{2}\frac{r-1}{r}$$

Note that, by definition, if $n \leq r$ then $T(n,r)$ can be partitioned into n independent sets that contain exactly 1 vertex each, and $r-n$ empty sets, meaning that $T(n,r) = K_n$.

Proposition 4.2

If G is EMK_r, then $t(n, r-1) \leq |E(G)|$.

Proof. We observe that $T(n, r-1)$ is $(r-1)$-partite in particular, therefore it does not contain K_r as subgraph by Proposition 4.1. However, since G has n vertices as well, and G is EMK_r, it must be that $t(n, r-1) \leq |E(G)|$. □

Turán graphs are important not only because of the *lower bound* we discussed for Proposition 4.1, but also because Turán was able to prove in 1941 [Tur41] that any EMK_r graph *must be* exactly $T(n, r-1)$.

Theorem 4.1: Turán's theorem

If G is EMK_r, then $G = T(n, r-1)$.

Proof. We proceed by strong induction on n.

Base case. For the base case, we consider $n < r$; in fact, G is EMK_r, and in particular it is edge maximum, therefore for $n < r$ we have that G is K_n. Hence, $G = T(n, r-1)$ — some of the $r-1$ partitions may be empty.

Strong inductive hypothesis. Assume that the statament is true for an EMK_r graph of $r \leq k \leq n-1$ vertices.

Inductive step. We are going to prove the statement for $n \geq r$. Note that G is EMK_r, therefore by adding an edge between two non-adjacent vertices we obtain K_r as subgraph of G; this must imply that K_{r-1} must be a subgraph of G. Hence, let $Y := \{y_1, \ldots, y_{r-1}\}$ be the vertices such that $G[Y] = K_{r-1}$.

Now, since G is EMK_r, then $G - Y$ does not contain K_r, therefore it must be that $|E(G-Y)|$ is at most the number of edges of a graph H that is EMK_r of $n-(r-1)$ vertices. However, by strong inductive hypothesis we know that $H = T(n-(r-1), r-1)$, therefore

$$|E(G-Y)| \leq |E(H)| = t(n-(r-1), r-1)$$

Claim: If $n \geq r$, then $t(n,r) = t(n-r, r) + (n-r)(r-1) + \binom{r}{2}$.

Proof of the Claim. Let $X_1, \ldots, X_r$ be a partition of $G' = T(n,r)$. Since $n \geq r$, we know that the independent sets are not empty by the pigeonhole principle, thus fix $x_i \in X_i$ for each X_i of the partition, and let $Z := \{x_1, \ldots, x_r\}$. Now, we observe the following.

- $G'[Z] = K_r$, hence there are $\binom{r}{2}$ edges in $G'[Z]$.
- $G' - Z = T(n-r, r)$ since we only removed one vertex per independent set, thus there are $t(n-r, r)$ edges in $G' - Z$.
- Fix an independent set X_i, and consider a vertex $y \in X_i$; this vertex has $r-1$ neighbors in Z, i.e. $|N(y) \cap Z| = r-1$, because $T(n, r)$ is a complete r-partite graph, and the r-th node of Z is in X_i itself. Therefore, because there are $n-r$ vertices in $G' - Z$, there are $(r-1)(n-r)$ edges between $G'[Z]$ and $G' - Z$.

The claim follows from the above statements. □

Now, similar to what we did in the claim, we observe the following.

- $G[Y] = K_{r-1}$, hence there are $\binom{r-1}{2}$ edges in $G[Y]$.
- $|E(G - Y)| \leq t(n - (r-1), r-1)$ for the previous observation.
- Any vertex $z \notin Y$ must have at most $r-2$ neighbors in Y — if z had $r-1$ neighbors, $G[Y \cup \{z\}]$ would be a K_r, contradicting the definition of G. Hence, there are at most $(r-2)(n-(r-1))$ edges between $G[Y]$ and $G - Y$.

Therefore, we get that

$$|E(G)| \leq \binom{r-1}{2} + (r-2)(n-(r-1)) + t(n-(r-1), r-1) = t(n, r-1)$$

where the last inequality follows from the previous claim. Moreover, for Proposition 4.2 this upper bound is tight. However, if this bound is tight, it must be that the vertices $z \notin Y$ have *exactly* $r-2$ neighbors in Y; hence, for each $i \in [r-1]$ let $X_i := \{z \in V(G) - Y \mid z \nsim y_i\} \cup \{y_i\}$. In particular, since z has exactly $r-2$ neighbors in Y, it is adjacent to every vertex in Y except for y_i, meaning that for any vertex $z \in V(G)$ it holds that z is in *exactly* one set between $X_1, \ldots, X_{r-1}$. Therefore, these sets describe a *partition* of G.

Claim: For all $i \in [r-1]$, it holds that X_i is an independent set.

Proof of the Claim. Fix $i \in [r-1]$, and by way of contradiction suppose that X_i is not an independent set, i.e. $\exists z, z' \in X_i \quad z \sim z'$. However, $z \in X_i \implies z \notin Y$, and by the previous observation we know that z has $r-2$ neighbors in Y, meaning that z is adjacent to every vertex in $Y - \{y_i\}$ — and the same reasoning can be applied on z'. Therefore, we have that $G[(Y - y_i) \cup \{z, z'\}]$ is K_r because $z \sim z'$, contradicting the definition of G $\lightning$. □

Therefore, G is $(r-1)$-partite through the partition $X_1, \ldots, X_{r-1}$.

Claim: If G is EMK_r and $(r-1)$-partite, then $G = T(n, r-1)$.

Proof of the Claim. Since no edges can be added to G without creating K_r as subgraph of G, it must be that G is complete $(r-1)$-partite. Let $X_1, \ldots, X_r$ be the partition that defines G, and by way of contradiction suppose that $G \neq T(n, r-1)$. By definition of Turán graph, this implies that for distinct $i, j \in [r]$ it holds that $||X_i| - |X_j|| \geq 2$, and without loss of generality suppose that $|X_i| < |X_j|$. Fix a vertex $y \in X_j$, and construct the graph G' starting from G as follows: remove all the edges $\{y, x_i\}$ for $x_i \in X_i$, move y into X_i, and add all the edges $\{y, x_j\}$ for $x_j \in X_j - \{y\}$. We observe that G' is still $(r-1)$-partite by construction, and by Proposition 4.1 we know that G' cannot contain K_r as subgraph. However, we observe that

$$|E(G')| = |E(G)| - |X_i| + |X_j| - 1 > |E(G)|$$

contradicting the fact that G was EMK_r $\lightning$. □

Finally, the statement follows from this last claim.

□

As a corollary of this theorem, we have the answer to our original question.

Corollary 4.1

$\mathrm{ex}(n, K_r) = t(n, r-1) + 1$

Proof. Clearly, if a graph G has n vertices and more than $t(n, r-1)$ edges, it cannot be $T(n, r-1)$, therefore by the contrapositive of Turan's theorem it must contain K_r as subgraph. □

4.1.2 k-connected subgraphs

After examining extremal edge conditions that guarantee the presence of a clique as subgraph, we now turn our attention to extremal conditions that ensure the existence of **k-connected subgraphs**. These types of conditions are particularly significant — unsurprisingly so, given that we have already seen how k-connectivity plays a key role in the emergence of structural properties within graphs. In particular, the following theorem — proved by Mader [Mad72] in 1972 — provides some conditions that guarantee the existence of a k-connected subgraph in a given graph.

Theorem 4.2: Mader's theorem

If G is such that $|V(G)| \geq 2k$ and $|E(G)| \geq (2k-1)(n-k)$, then G contains a k-connected subgraph.

Proof. We proceed by strong induction on n.

Base case. Since $n \geq 2k$, the base case of the induction is $n = 2k$, thus we have that

$$|E(G)| \geq (2k-1)k = \frac{(2k-1)2k}{2} = \binom{2k}{2}$$

which must imply that $G = K_{2k}$, which is $(2k-1)$-connected by definition.

Strong inductive hypothesis. Assume that the statement holds for any graph of h vertices such that $n - 1 \geq h > 2k$ vertices.

Inductive step. We are going to prove the statement for a graph G of n vertices such that $n > 2k$, i.e. $n \geq 2k + 1$.

Suppose that $\delta < 2k$, i.e. $\delta \leq 2k - 1$, implying that there exists a vertex v such that $\deg(v) \leq 2k - 1$, and consider $G - \{v\}$. Since this subgraph has $n - 1$ vertices, and we are assuming that $|E(G)| \geq (2k-1)(n-k)$ then we have that

$$\begin{aligned}
|E(G[V(G) - \{v\}])| &\geq |E(G)| - (2k-1) \\
&\geq (2k-1)(n-k) - (2k-1) \\
&\geq (2k-1)(n-k-1) \\
&\geq (2k-1)((n-1)-k)
\end{aligned}$$

Moreover, since $n \geq 2k+1 \iff n-1 \geq 2k$ and $G - \{v\}$ has $n - 1$ vertices, the latter has at least $2k$ vertices, hence we can apply the inductive hypothesis on it and obtaining a k-connected subgraph to prove the statement trivially. Therefore, we may assume that $\delta \geq 2k$.

To recap, we are now assuming that

(a) G is not k-connected — otherwise the statement is trivially true

(b) $|E(G)| \geq (2k-1)(n-k)$

(c) $\delta \geq 2k$

and in particular, assumption (a) implies that we may assume there is an A-B separation (X, Y) of G having order $|X \cap Y| \leq k - 1$, where $X - Y, Y - X \neq \varnothing$ — for some subsets $A, B \subseteq V(G)$.

Claim: It holds that $|E(G[X])| \geq (2k-1)(|X|-k)$ or $|E(G[Y])| \geq (2k-1)(|Y|-k)$.

Proof of the Claim. By way of contradiction, suppose that $|E(G[X])| \leq (2k-1)(|X|-k) - 1$ and $|E(G[Y])| \leq (2k-1)(|Y|-k) - 1$. Thus, we have that

$$\begin{aligned}
|E(G)| &\leq |E(G[X])| + |E(G[Y])| \\
&\leq (2k-1)(|X|-k) - 1 + (2k-1)(|Y|-k) - 1 \\
&= (2k-1)(|X| + |Y| - 2k) - 2 \\
&= (2k-1)(|X \cup Y| + |X \cap Y| - 2k) - 2 \\
&\leq (2k-1)(n + k - 1 - 2k) - 2 \\
&= (2k-1)(n-k-1) - 2 \\
&< (2k-1)(n-k)
\end{aligned}$$

which contradicts assumption (b) $\lightning$. $\square$

This claim implies that at least one between $G[X]$ or $G[Y]$ satisfies the edge bound, hence if we can prove that they also satisfy the vertex bound we can apply the inductive hypothesis and conclude the proof.

Now consider assumption (c): if $\delta \geq 2k$ then for each vertex $v \in V(G)$ its degree must be at least $2k$, and in particular this is true for any $v \in X - Y$. However, (X, Y) is a separation of G, meaning that there are not edges in $\mathrm{cut}(X, Y)$; therefore, all the neighbors of v must lie inside X, meaning that $|X| \geq |\mathcal{N}(v) \cup \{v\}| \geq 2k + 1 > 2k$. This last condition implies that the vertex bound of the inductive hypothesis is satisfied, so by applying the latter on one between $G[X]$ or $G[Y]$ the theorem holds.

□

It is not known if the edge bound provided by Mader's theorem is optimal, i.e. we do not know if there are graphs with fewer than $(2k-1)(n-k)$ edges that do not contain k-connected subgraphs. In fact, this is an open question in the field of extremal graph theory: given $k \in \mathbb{N}$, what is the smallest value c_k such that all graphs with n vertices and $|E(G)| \geq c_k n + f(k)$ — for some function $f(k)$ — contain a k-connected subgraph?

By Mader's theorem, we know that c_k is *upper bounded* by $c_k \leq 2k - 1$, and some type of graphs are known to provide *lower bounds*, such as the following type.

Definition 4.5: Split graph

A **split graph** is a graph whose vertices can be partitioned into a *clique* and an independent set. A **complete split graph** is a split graph where the independent set is fully connected to the clique.

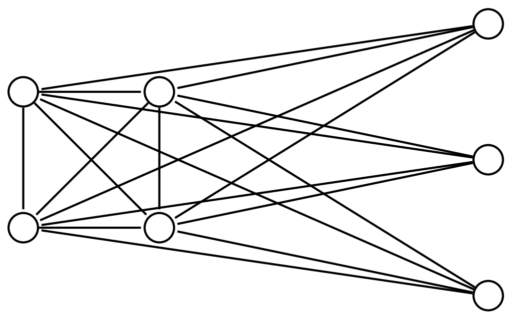


Figure 4.5: The complit split graph constructed from K_4 and an independent set of 3 vertices.

In fact, any complete split graph constructed from a $(k-1)$-clique and an independent set of $n - (k-1)$ vertices has the following number of edges

$$|E(G)| = \binom{k-1}{2} + (k-1)(n-(k-1))$$

and such split graphs are clearly not k-connected — removing K_{k-1} disconnects the graph by definition. Therefore, this implies that $c_k \geq k - 1$.

Lastly, as a corollary of Mader's theorem and Proposition 3.7, we get the following proposition.

Corollary 4.2

A graph G such that $|V(G)| \geq 6$ and $|E(G)| \geq 5n$ contains K_4 as topological minor.

Moreover, in 1996 Bollobás and Thomason [BT96] proved that the above corollary can be extended to any k-clique with a relatively small blow-up on the edge bound.

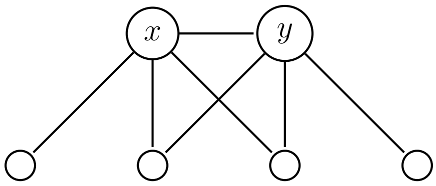
Theorem 4.3: Bollobás-Thomason theorem

There exists a constant $c > 0$ such that if G has at least ck^2n edges, then G has K_k as topological minor.

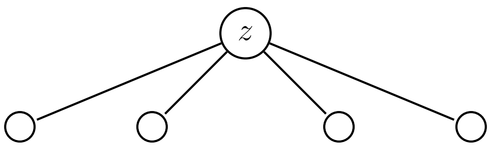
The topic of finding cliques as substructures of a given graph will be discussed in greater detail in the following section.

4.1.3 Cliques as minors

Consider a graph G, and an edge $xy \in E(G)$; to **contract** xy means to *identify* the vertices x and y and delete any parralel edge that formed afterwards. For instance, if G contains the edge xy



to *contract* the edge xy means to identify x and y in a new vertex z, and removing the 2 parallel edges that formed

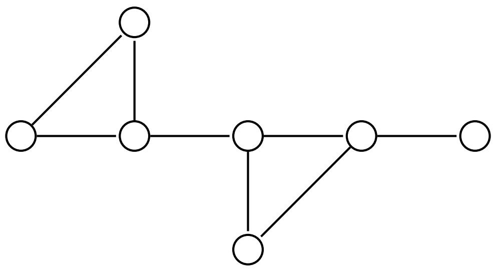


If we contracted the edge e in a graph G, we will denote the resulting graph as G/e.

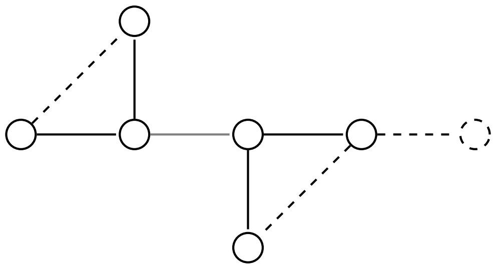
Definition 4.6: Minor

Given a graph G, and a graph H, we say that G contains H as **minor** if H can be obtained from G by a series of *vertex deletions*, *edge deletions* and *edge contractions*. Similarly, we say that H is a *minor* of G.

For instance, consider the following graph G

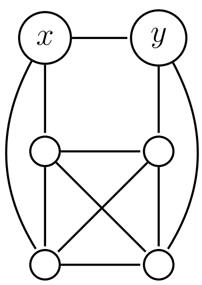


then clearly the following graph H is a minor of G — the *dashed* edges and vertices are the removed ones, and the *gray* edge is the contracted edge



Consider a graph G that has H as topological minor, meaning that it contains a graph H' that is a subdivision of H; if we remove from G all the edges and vertices that are *not* part of H', we are left with a subdivision of H, and through the *edge contraction* operation we can actually reverse the *edge subdivision* in order to obtain H from H'. In fact, for any 2-edge path $x\ y\ z$, we can contract $x\ y$ obtaining an edge $y\ z$, and no parallel edge can form because we previously removed all the other edges that are not in H', i.e. edges that were not constructed through subdivision. This means that if G contains H as topological minor, then G contains H as minor as well.

However, the opposite is not true in general: for instance, consider the following graph G



If we contract the edge xy, we obtain K_5, therefore G contains K_5 as minor, even if K_5 is not a topological minor of G: in fact, every subdivision of K_5 has 5 vertices of degree 4, but G has only 4 vertices of degree 4.

Theorem 4.4

If p is an integer such that $2 \leq p \leq 7$, then for any graph G such that $n \geq p$ and

$$|E(G)| \geq (p-2)n - \binom{p-1}{2} + 1$$

it holds that G contains K_p as minor.

Proof. We will prove each value of p individually.

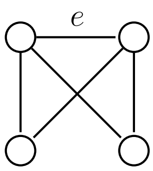
- For $p = 2$, we see that $|E(G)| \geq (2-2)n - \binom{2-1}{2} + 1 = 1$, therefore G contains at least 1 edge, and in fact K_2 is nothing more than a single edge.
- For $p = 3$, we have that $|E(G)| \geq (3-2)n - \binom{3-1}{2} + 1 = n - 1 + 1 = n$. Thus, by Proposition 1.5 this implies that G is not a tree (nor a forest), and in particular it is not acyclic, meaning that it contains at least one cycle C. Hence, if we contract all but 3 edges on C we obtain K_3.
- For $p = 4$, we have that $|E(G)| \geq (4-2)n - \binom{4-1}{2} + 1 = 2n - 3 + 1 = 2n - 2$. We proceed by induction on $n + m$.

  For $n = p = 4$, we have that $|E(G)| \geq 2 \cdot 4 - 2 = 6$, and thus $G = K_4$. Now, for the inductive hypothesis, we need to be careful: since we are applying the induction on $n + m$, then

  - if the graph G we are considering has $m - 1$ edges, we are going to inductively assume that the property holds for G such that $|E(G)| \geq 2n - 2$
  - if the graph G we are considering has $n-1$ vertices, we are going to inductively assume that the property holds for G such that $|E(G)| \geq 2(n-1) - 2$

  Now, because of the base case, consider a graph G of n vertices such that $n \geq 5$; moreover, suppose that $|E(G)| > 2n - 2$, and fix an edge $e \in E(G)$. Now, consider the graph $G - \{e\}$: clearly, this graph has still n vertices but since we removed an edge it holds that $|E(G - \{e\})| > 2n - 2 - 1 = 2n - 3 \iff |E(G - \{e\})| \geq 2n - 2$, therefore we can apply the inductive hypothesis as we described, and the theorem holds.

  Therefore, we may assume that $|E(G)| = 2n - 2$. Fix an edge $e \in E(G)$, and consider the contracted graph G/e: this graph has $n-1$ vertices, hence if $|E(G/e)| \geq 2(n-1) - 2$ by the inductive hypothesis the theorem holds trivially. Thus, we may assume that $|E(G/e)| < 2(n-1) - 2 = 2n - 4 \iff |E(G/e)| \leq 2n - 5$, and since we are assuming that $|E(G)| = 2n - 2$, this means that by contracting e we removed at least 3 edges from G — one of which is e itself. Therefore, for each edge e of G we have the following structure



and in particular, the endpoints of e must have at least two common neighbors. This implies that e is contained in at least two distinct K_3's. By applying the same argument on every edge e of G, we get that $\delta \geq 3$ because the degree of the endpoints of e must be at least 3. Moreover, if $\delta \geq 4$, by Corollary 1.1 we have that $|E(G)| \geq \frac{4n}{2} = 2n$ which is greater than $2n-2$, contradicting our assumption that $|E(G)| = 2n-2$. This implies that $\delta = 3$. Finally, fix a vertex v of degree 3.

Claim: $G[\mathcal{N}(v) \cup \{v\}] = K_4$.

Proof of the Claim. By way of contradiction, suppose that there are two vertices $x, y \in \mathcal{N}(v)$ such that $x \nsim y$. Note that each edge of G must be contained in at least two distinct 3-cliques, and in particular this must be true for vx. However, since we are assuming $xy \notin E(G)$ for the sake of contradiction, there must be at least one other vertex z such that v, x and z form a K_3, which would violate the degree of v $\lightning$. $\square$

- For $p = 5$, we have that $|E(G)| \geq (5-2)n - \binom{5-1}{2} + 1 = 3n - 6 + 1 = 3n - 5$. We proceed by induction on $n + m$ in the same fashion we did for the case of $p = 4$.

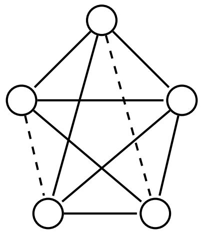
  For $n = p = 5$, we have that $|E(G)| \geq 3 \cdot 5 - 5 = 15 - 5 = 10$, and thus $G = K_5$. By applying the same reasoning of the case for $p = 4$, we can consider a graph G such that $n \geq 6$, and fix an edge $e \in E(G)$; now, if we assume that $|E(G)| > 3n - 5$ then $|E(G - \{e\})| > 3n - 5 - 1 = 3n - 6 \iff |E(G - \{e\})| \geq 3n - 5$ therefore we can apply the inductive hypothesis and theorem holds.

  Thus, we may assume that $|E(G)| = 3n - 5$. Fix an edge $e \in E(G)$, and consider G/e: this graph has $n - 1$ vertices, and by the same reasoning as the case for $p = 4$ we may assume that $|E(G/e)| < 3(n-1) - 5 = 3n - 3 - 5 = 3n - 8 \iff |E(G/e)| \leq$ $|E(G/e)| \leq 3n - 9$, and since we are assuming $|E(G)| = 3n - 5$, this means that by contracting e we removed at least 4 edges from G — one of which is e itself. This implies that the endpoints of e must have at least three distinct common neighbors, i.e. e is contained in at least three distinct K_3's.

  Thus, by applying the same argument on every edge e of G, we get that $\delta \geq 4$. Moreover, suppose that $\delta \geq 6$: then by Corollary 1.1 we have that $|E(G)| \geq \frac{6n}{2} = 3n$ which is greater than $3n - 5$, contradicting our assumption. This implies that δ is either 4 or 5. If $\delta = 4$, then because every edge must be contained in at least three distinct 3-cliques, we can apply the same reasoning as the case for $p = 4$ for any vertex v of degree 4, obtaining that $G[\mathcal{N}(v) \cup \{v\}] = K_5$. Otherwise, if $\delta = 5$, we have the following claim.

  Claim: If $\delta = 5$, for any $v \in V(G)$ such that $\deg(v) = 5$ it holds that K_5 is a minor of $G[\mathcal{N}(v) \cup \{v\}]$.

Proof of the Claim. Consider $G[\mathcal{N}(v)]$: by the same argument applied in the final claim of the case for $p = 4$, we know that each vertex x in $G[\mathcal{N}(v)]$ must have $\deg_{G[\mathcal{N}(v)]}(x) \geq 3$, otherwise the edges connecting v to $\mathcal{N}(v)$ would not be contained in at least three distinct 3-cliques. Now, if $G[\mathcal{N}(v)]$ is fully connected, we have that $G[\mathcal{N}(v)] = K_5$ hence the claim trivially holds, so we may assume the opposite. In fact, the degree bound we derived does not guarantee that $G[\mathcal{N}(v)]$ is fully connected; however, there cannot be more than 2 anti-edges, as shown in the figure below — the *dashed* edges are possible anti-edges of $G[\mathcal{N}(v)]$



Hence, we have only two cases:

- if $G[\mathcal{N}(v)]$ has one anti-edge, consider an edge $e \in E(G[\mathcal{N}(v)])$ incident to the anti-edge; then $G[\mathcal{N}(v)]/e = K_4$ — we observe that the contraction will form one parallel edge to remove
- if $G[\mathcal{N}(v)]$ has two anti-edges, consider an edge e incident to both the anti-edges; then $G[\mathcal{N}(v)]/e = K_4$ — we observe that the contraction does not form any parallel edges to remove

Therefore, in both cases we have that $G[\mathcal{N}(v) \cup \{v\}]/e = K_5$, where e was chosen accordingly. □

- The cases for $p = 6$ and $p = 7$ are analogous to the ones for $p = 4$ and $p = 5$.

□

The cases for $2 \leq p \leq 5$ were initially established by Dirac [Dir64], and later independently by Gyori [Gyo82]. Mader [Mad68] extended the result to $p = 6$ and $p = 7$. For $p = 8$, Jorgensen [Jor94] showed that the theorem holds, except for an infinite family of counterexamples. Similarly, Song and Thomas [ST06] proved the result for $p = 9$, identifying two infinite families of exceptions. More recently, Zhu [Zhu21] demonstrated that the theorem also extends to $p = 10$, aside from a few infinite families. The proofs for $p = 9$ and $p = 10$ both involved computer assistance.

In a series of papers, Kostochka [Kos82] and Thomason [Tho01] independently proved that there exists a constant c such that every graph with at least $cp\sqrt{\log p}n$ edges contains a K_p minor. Moreover, thomason further showed that this bound is asymptotically tight: there exist Erdős–Rényi random graphs with only $c'p\sqrt{\log p}n$ edges and no K_p minor, for some constant $c' < c$. Notably, in these examples, the number of vertices n is also a function of p.

Theorem 4.5: Kostochka-Thomason theorem

There is a constant $c > 0$ such that any graph G with $|E(G)| \geq cp\sqrt{\log p}n$ contains K_p as minor.

Seymour and Thomas conjectured that the missing condition for Theorem 4.4 to hold for every $p \geq 2$ is the $(p-2)$-connectivity of the graph.

Conjecture 4.1: Seymour-Thomas conjecture

For all $p \in \mathbb{N}$ there is a constant $N_p > 0$ such that any $(p-2)$-connected graph G with $n \geq N_p$ and $|E(G)| \geq (p-2)n - \binom{p-1}{2} + 1$ contains K_p as minor.

The closest result for this conjecture was proven by Böhme, Kawarabayashi, Maharry, et al. [BKM+09].

Theorem 4.6

There is a constant $c > 0$ such that for all $p \in \mathbb{N}$ there is another constant $N_p > 0$ for which any $c(p+1)$-connected graph G with $n \geq N_p$ contains K_p as minor.

4.2 Vertex bounds

Up until now, we focused on finding conditions on the number of *edges* which guaranteed the existence of particular structures inside graphs. As mentioned before, we are now going to focus on bounds on the number of **vertices** that guarantee the existence of substructures in graphs.

4.2.1 Ramsey numbers

In 1917, Ramsey [Ram87] published a groundbreaking paper in which he proved the existence of the so-called **Ramsey number**.

Definition 4.7: Ramsey number

Given a value $t \in \mathbb{N}$, the **Ramsey number** $R(t)$ is the minimum number of vertices such that every graph with at least $R(t)$ vertices has K_t as subgraph or contains an independent set of size t.

Trivially, $R(1) = 1$ since a single vertex is an independent set of size 1, and $R(2) = 2$ since a graph with two vertices can only be either K_2 or an independent set of size 2.

For the case when $t = 3$, clearly $R(3) \geq 3$ otherwise the graph cannot contain K_3 nor an independent set of size 3. Moreover, we can actually construct the following three graphs which lower-bound its value to be at least 6.

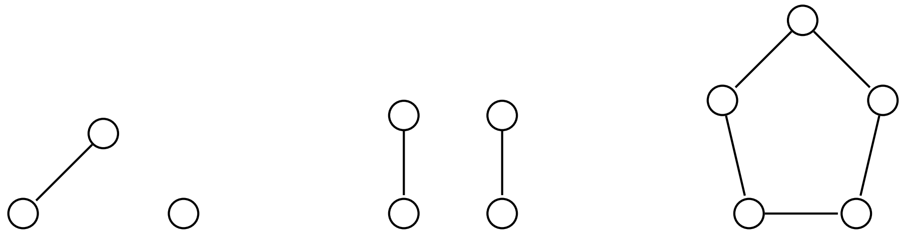


Figure 4.6: From left to right: the counterexamples with 3, 4 and 5 vertices that give the $R(3) > 5$ lower bound.

In fact, we can actually prove that this lower bound is tight.

Theorem 4.7

$R(3) = 6$

Proof. By way of contradiction, suppose that there is a graph G with 6 vertices that does not contain K_3 as subgraph, nor any independent set of size 3.

Claim: $\Delta(G) \leq 2$.

Proof of the Claim. By way of contradiction, suppose that there is a vertex v of degree 3; however, if there was an edge with two endpoints in $\mathcal{N}(v)$ then there would be K_3 as subgraph of G. Therefore, any three vertices in $\mathcal{N}(v)$ cannot have edges between them, implying that $\mathcal{N}(v)$ forms an independent set of size 3 $\lightning$. □

Fix a vertex v with at least 3 non-neighbors — its existence is guaranteed by the previous claim, since $n = 6$ — and let the latter be x, y and z. If two of the 3 non-neighbors of v are not adjacent, say x and y, then $\{v, x, y\}$ is an independent set of size 3. This implies that all of the non-neighbors of v must be pairwise adjacent, meaning that x, y and z form a K_3 in G $\lightning$. □

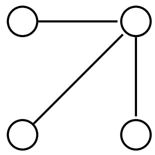
What about $t = 4$? In 1965 Kalbfleisch [Kal65] proved that $R(4) > 17$ through the so-called *Kalbfleisch's graph*. We are going to show that this lower bound is actually tight, i.e. $R(4) = 18$. But first, we need to introduce the following definitions.

Definition 4.8: Complementary graph

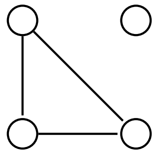
Given a graph G, the **complement of** G — denoted as $\overline{G}$ — is defined as follows:

- $V(\overline{G}) := V(G)$
- $E(\overline{G}) := \overline{E(G)} = V(G) \times V(G) - E(G)$

For instance, if we consider the following graph G



the following is $\overline{G}$, its complement



By definition, we observe that a graph G contains an independent set of size s if and only if $\overline{G}$ contains K_s as subgraph.

We are going to use this definition in order to introduce a **generalized** version of the Ramsey number, in which the clique subgraph and the independent set can have different sizes.

Definition 4.9: Generalized Ramsey number

Given two values $t, s \in \mathbb{N}$, the **generalized Ramsey number** $R(t, s)$ is the minimum number of vertices such that every graph G with at least $R(t, s)$ vertices has K_t as subgraph, or $\overline{G}$ has K_s as subgraph.

By definition we observe that $R(t, t) = R(t)$. Moreover, we note that $R(t, s) = R(s, t)$ since $G = \overline{\overline{G}}$.

Lemma 4.1

$R(4, 3) \leq 9$

Proof. By way of contradiction, let G be a graph having 9 vertices that has no K_4 as subgraph nor any independent set of size 3.

By the same reasoning applied for the proof of Theorem 4.7, we know that $\Delta(G) \leq 5$, and that no vertex of G can have more than 3 non-neighbors. Moreover, since $n = 9$, this implies that the degree of every vertex of G is *exactly* 5, i.e. G is 5-regular. However, we observe that $\sum_{x \in V(G)} \deg(x) = 5 \cdot 9 = 45$, which is impossible because 45 is odd and the sum of the degrees of G must be even by the Handshaking lemma $\lightning$.

Alternatively, this result can be proven as follows. Since $R(4, 3) = R(3, 4)$, by way of contradiction suppose there exists a graph G on 9 verices that has no K_3 as subgraph nor any independent set of size 3, and fix a vertex $v \in V(G)$. Now consider $G - \{v\}$: by the pigeonhole principle, since each edge can be either present or absent, there are at least $\left\lceil \frac{9}{2} \right\rceil = 4$ neighbors or non-neighbors of v — without loss of generality suppose that

$|\mathcal{N}(v)| \geq 4$. Since G has no K_3 as subgraph, any pair of vertices inside $\mathcal{N}(v)$ must be the endpoints of an anti-edge, otherwise such two vertices would form K_3 together with v itself. This implies that $\overline{G}$ contains and independent set of size 4, contradicting the definition of G $\lightning$. □

This idea can be actually generalized as follows.

Proposition 4.3

For any $t \in \mathbb{N}$ it holds that $R(t) \leq 2R(t-1,t)$.

Proof. By way of contradiction, suppose that there exists a graph on $2R(t-1,t)$ vertices without K_t as subgraph in G and $\overline{G}$. Fix a vertex v; then, by the pigeonhole principle v must have at least $R(t-1,t)$ neighbors or non-neighbors — assume that $|\mathcal{N}(v)| \geq R(t-1,t)$ without loss of generality. Then, by definition of $R(t-1,t)$ this implies that $G[\mathcal{N}(v)]$ contains K_{t-1} as subgraph, or K_t in its complement. However, its complement cannot contain K_t otherwise G would contain a K_t, contradicting its definition, hence the only possibility is that $K_{t-1} \subseteq G[\mathcal{N}(v)]$. However, this implies that $G[\mathcal{N}(v) \cup \{v\}]$ then contains a K_t, again contradicting the definition of G $\lightning$. □

Thanks to the generalized version of the Ramsey number, we can prove the value of $R(4)$ as follows.

Corollary 4.3

$R(4) = 18$

Proof. Since we already discussed that $R(4) \geq 18$, the result follows immediately from the previous proposition because $R(4) \leq 2 \cdot R(3,4) \leq 2 \cdot 9 = 18$ — and $R(3,4) \leq 9$ by the previous lemma. □

What about other values of t? Using a similar idea of this result, it can be proven that $R(5,4) \leq 25$, which immediately provides an upper bound for $R(5) \leq 50$. However, we still don't know the exact value of $R(5)$: in 1989 Exoo [Exo89] proved that $R(5) \geq 43$, and in 2024 Angeltveit and McKay [AM24] proved that $R(5) \leq 46$.

What about the general case? At the beginning of this section, we mentioned that Ramsey proved the existence of $R(t)$ for any $t \geq 1$. The following is the theorem through which he proved this result, which also provides an upper bound for any $R(t)$.

Theorem 4.8: Ramsey's theorem

For any $t \geq 1$, $R(t)$ exists. Moreover, if $t \neq 1$ then $R(t) \leq 2^{2t-3}$.

Proof. We already know the values for $R(1)$ and $R(2)$, therefore they exist. Hence, fix $t \geq 2$ and let G be a graph with 2^{2t-3} vertices.

Claim: There are a sequence of sets $X_1, \dots, X_{2t-2}$ and a sequence of vertices $x_1, \dots, x_{2t-2}$ such that for each $i \in [2t-2]$ it holds that

1. $x_i \in X_i$ and $|X_i| \geq 2^{2t-2-i}$
2. $X_{i+1} \subseteq X_i - \{x_i\}$
3. x_i is either adjacent to every vertex of X_{i+1} or non-adjacent to all vertices of X_{i+1}

Proof of the Claim. We are going to construct the sequences inductively. First, let $X_1 := V(G)$, and fix $x_1 \in X_1$ arbitrarily. Now, let X_2 to be the larger between $\mathcal{N}(x_1)$ and $V(G) - (\mathcal{N}(x_1) \cup \{x_1\})$; then, fix $x_2 \in X_2$ arbitrarily. We observe that the last two properties are satisfied by construction. For the first property, we observe that

$$\mathcal{N}(x_1) \cap (V(G) - (\mathcal{N}(x_1) \cup \{x_1\})) = X_1 - \{x_1\} = V(G) - \{x_1\}$$

meaning that the cardinality of their union is $|V(G) - \{x_1\}| = 2^{2t-3} - 1$, therefore

$$\begin{aligned}
|X_2| &= \max(|\mathcal{N}(x_1)|, |V(G) - (\mathcal{N}(x_1) \cup \{x_1\}|)) \\
&\geq \left\lceil \frac{|\mathcal{N}(x_1) \cup (V(G) - (\mathcal{N}(x_1) \cup \{x_1\}))|}{2} \right\rceil \\
&= \left\lceil \frac{2^{2t-3} - 1}{2} \right\rceil \\
&= 2^{2t-4} \\
&= 2^{2t-2-2} \qquad (i = 2)
\end{aligned}$$

In general, suppose that we have defined X_i and x_i for some $i < 2t-2$; then, let X_{i+1} be the larger between $\mathcal{N}(x_i) \cap X_i$ and $X_i - (\mathcal{N}(x_i) \cup \{x_i\})$. Again, we observe that the last two properties of the claim are satisfied by construction of X_{i+1}. For the first property, we observe that

$$(\mathcal{N}(x_1) \cup X_i) \cup (X_i - (\mathcal{N}(x_i) \cup \{x_i\})) = X_i - \{x_i\}$$

meaning that the cardinality of their union is $|X_i - \{x_i\}| \geq 2^{2t-2-i} - 1$ by induction, therefore

$$\begin{aligned}
|X_{i+1}| &= \max(|\mathcal{N}(x_1) \cap X_i|, |X_i - (\mathcal{N}(x_i) \cup \{x_i\}|)) \\
&\geq \left\lceil \frac{|(\mathcal{N}(x_1) \cup X_i) \cup (X_i - (\mathcal{N}(x_i) \cup \{x_i\}))|}{2} \right\rceil \\
&\geq \left\lceil \frac{2^{2t-2-i} - 1}{2} \right\rceil \\
&= 2^{2t-2-(i+1)}
\end{aligned}$$

□

Now, consider the sequence of sets and the sequence of vertices that we constructed.

Claim: For each $i, j \in [2t-2]$ such that $i < j$ it holds that:

- $x_i \sim x_j \implies \forall j' \in [i+1, 2t-2] \quad x_i \sim x_{j'}$
- $x_i \nsim x_j \implies \forall j' \in [i+1, 2t-2] \quad x_i \nsim x_{j'}$

Proof of the Claim. Fix $i, j \in [2t-2]$ such that $i < j$, and $j' \in [i+1, 2t-2]$. By construction of the sequence, since $j' \geq i+1$ we have that $x_{j'} \in X_{j'} \subseteq X_{i+1}$. Moreover, by construction of X_{i+1} we know that x_i is either adjacent to all the vertices in X_{i+1}, or non-adjacent to all the vertices in X_{i+1}. Thus, if $x_i \sim x_j$ then $x_i \sim x_{j'}$, otherwise $x_i \nsim x_{j'}$. □

In particular, this claim implies that if a vertex x_i is adjacent (or non-adjacent) to a vertex x_j that comes "after" x_i, then x_i is actually adjacent (or non-adjacent) to *all* the vertices $x_{j'}$ that come "after" x_i.

Now, consider the set $\mathcal{U} := \mathcal{N}(x_{2t-2}) \cap \{x_1, \ldots, x_{2t-3}\}$. If $|\mathcal{U}| \geq t-1$, by the second claim $\mathcal{U} \cup \{x_{2t-2}\}$ contains a t-clique; otherwise, if $|\mathcal{U}| \leq t-2$, again by the second claim $\{x_1, \ldots, x_{2t-3}\} - \mathcal{U}$ contains an independent set of size $2t-2-|\mathcal{U}| \geq 2t-2-(t-2) = t$. □

4.3 Exercises

Problem 4.1

Prove that $\mathrm{ex}(n, P_4) \leq n+1$.

Solution. Let G be a graph such that $|E(G)| = n+1$, and by way of contradiction suppose that G does not contain P_4 as subgraph. By the number of edges, we know that G cannot be a forest, implying that at least one of the components of G contains a cycle. Moreover, we observe that

- no cycle can be longer than 3, otherwise that cycle would contain P_4 as subgraph trivially
- no component containing a cycle can have more than 3 vertices, otherwise it would contain P_4 as subgraph trivially — by connectivity of the components

Therefore, we get that every component of G is either K_3 or a tree. Let $C_1, \ldots, C_\ell$ be the K_3-components of G, and let $T_1, \ldots, T_h$ be its tree components. Then, we have that

$$\begin{aligned}
|E(G)| &= \sum_{i=1}^{\ell} |E(C_i)| + \sum_{i=1}^{h} |E(T_i)| \\
&= \sum_{i=1}^{\ell} |V(C_i)| + \sum_{i=1}^{h} |V(T_i)| - 1 \\
&= \sum_{i=1}^{\ell} |V(C_i)| + \sum_{i=1}^{h} |V(T_i)| - h \\
&= 3\ell + n - 3\ell - h \\
&= n - h
\end{aligned}$$

However, this implies that

$$n + 1 = |E(G)| = n - h \iff h = -1$$

which is impossible because $h \geq 0$ $\lightning$. $\square$

Problem 4.2

Show that $\mathrm{ex}(n, K_{1,3}) = n + 1$.

Solution. First, we observe that any C_n cycle graph on n vertices has n edges but does not contain $K_{1,3}$ as subgraph, therefore $\mathrm{ex}(n, K_{1,3}) \geq n + 1$.

Consider a graph of n vertices and $n + 1$ edges — in particular, this implies that $\delta \geq 2$. By way of contradiction suppose that $\Delta(G) \leq 2$. If this is the case, we observe that

$$\forall v \in V(G) \quad 2 \leq \delta \leq \deg(v) \leq \Delta \leq 2 \implies \deg(v) = 2$$

However, by the Handshaking lemma we know that

$$\sum_{v \in V(G)} \deg(v) = 2\,|E(G)| = 2(n + 1)$$

which means that the number of vertices of G is given by $\frac{2(n+1)}{2} = n + 1$ contradicting the fact that G had n vertices $\lightning$.

In particular, if $\Delta(G) \geq 3$, it implies that $K_{1,3}$ must be a subgraph of G, concluding that $\mathrm{ex}(n, K_{1,3}) \leq n + 1$. $\square$

Problem 4.3

Find the value of $\mathrm{ex}(n, K_{1,t})$ for all $t \geq 4$ and prove it.

Solution. We observe that if $n < t + 1 \iff n \leq t$ it is impossible to have $K_{1,t}$ as subgraph since it requires at least $t + 1$ vertices in the graph. Therefore, we may assume that $n \geq t + 1$.

Note that to guarantee that $K_{1,t}$ is a subgraph of G, it suffices to guarantee that G contains a vertex of degree at least t. Hence, the largest number of edges G can have must be such that every vertex has degree $t - 1$, i.e. the sum of the degrees of G is $n(t - 1)$. Hence, by the Handshaking lemma we get that

$$2\,|E(G)| = \sum_{v \in V(G)} \deg(v) = n(t - 1) \iff |E(G)| = \frac{n}{2}(t - 1)$$

We are now going to prove that is indeed $\mathrm{ex}(n, K_{1,t}) = \frac{n}{2}(t - 1) + 1$.

For the lower bound, consider the graph $K_{t-1,t-1}$. Clearly, each vertex has degree $t - 1$ meaning that it cannot contain $K_{1,t}$ as subgraph, and by the Handshaking lemma the number of edges is given by

$$2\,|E(G)| = \sum_{v \in V(G)} \deg(v) = n(t-1) \iff |E(G)| = \frac{n}{2}(t-1)$$

Claim: For $t \geq 4$ it holds that $\mathrm{ex}(n, K_{1,t}) \leq \frac{n}{2}(t-1) + 1$.

Proof of the Claim. Let G be a graph of n vertices and $\frac{n}{2}(t-1)+1$ edges, and by way of contradiction suppose that $\Delta(G) \leq t-1$. By the Handshaking lemma, we have that

$$\sum_{v \in V(G)} \deg(v) = 2\,|E(G)| = 2\left[\tfrac{n}{2}(t-1)+1\right] = n(t-1)+2$$

meaning that the number of vertices of G is given by

$$\frac{n(t-1)+2}{t-1} \geq n + \frac{2}{t-1} > n$$

since $t \geq 4 \implies \frac{2}{t-1} \geq \frac{2}{3}$, contradicting the fact that G had n vertices $\lightning$. □

Together with the previous counterexample, we get that

$$n \geq t+1 \implies \mathrm{ex}(n, K_{1,t}) = \frac{n}{2}(t-1)+1$$

□

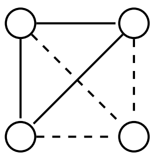
Definition 4.10

Given a graph H, let $R(H)$ be the minimum number of vertices such that for every graph G of $R(H)$ vertices either $H \subseteq G$ or $H \subseteq \overline{G}$.

Problem 4.4

Find the value of $R(P_4)$.

Solution. Trivially, the following graph shows that $R(P_4) \geq 5$ — the anti-edges are represented with dashed lines



Claim: $R(P_4) \leq 5$.

Proof of the Claim. Let G be a graph of 5 vertices; then $|E(G)| + \left|E(\overline{G})\right| = |E(K_5)| = \binom{5}{2} = 10$, which means that the largest between G and $\overline{G}$ contains at least 5 edges —

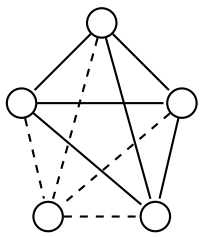
without loss of generality, suppose that G is larger. However, a graph with 5 vertices and at least 5 edges must contain a cycle, hence G contains a cycle. Pick the largest cycle in G: if the cycle has length 4 or 5, the theorem trivially holds, so we may assume that G contains C_3 — let x and y be the two vertices not contained in C_3. Then, since G must contain at least 5 edges on 5 vertices, and 3 edges form a C_3, there must be at least one edge connecting either x or y to C_3, meaning that P_4 is still contained in G. □

The claim, together with the previous graph, proves that $R(P_4) = 5$. □

Problem 4.5

Find the value of $R(P_5)$.

Solution. The following graph shows that $R(P_5) \geq 6$ — observe that the graph consists of a K_4 with an isolated vertex.



Claim: $R(P_5) \leq 6$.

Proof of the Claim. Let G be a graph of 6 vertices; then $|E(G)| + \left|E(\overline{G})\right| = |E(K_6)| = \binom{6}{2} = 15$, which means that the largest between G and $\overline{G}$ contains at least 8 edges — without loss of generality, suppose that G is larger. However, a graph with 6 vertices and at least 8 edges must contain a cycle, hence G contains a cycle. Pick the largest cycle in G: if the cycle has length 5 or 6, the theorem trivially holds, so we may assume that our cycle is either a C_3 or a C_4.

- Suppose that $C_3 \subsetneq G$; we have two cases.
  - There are at least two vertices $x, y \in V(C_3)$ that have neighbors outside C_3. If there was a vertex $z \notin V(C_3)$ such that $x, y \sim z$, then G contains C_4 contradicting the fact that C_3 was the largest cycle in G. Therefore, we may assume that there are two distinct vertices $u, v \notin V(C_3)$ such that $x \sim u$ and $y \sim v$, but $x \nsim v$ and $y \nsim u$. Moreover, $u \nsim v$ otherwise together with C_3 there would be C_5 inside G, again contradicting the maximality of the cycle we picked. Let z be the third vertex outside C_3; if $x \sim z$ then $zx\ (x\ C\ y)\ yv$ is a P_5 in G, so we may assume that $x \nsim z$. Then $y \nsim u \nsim v \nsim x \nsim z$ is a P_5 in $\overline{G}$.
  - No vertex of C_3 has neighbors outside C_3; then $\overline{G}$ contains $K_{3,3}$ as subgraph, implying that $P_5 \subsetneq \overline{G}$.

- Suppose that $C_4 \subsetneq G$; if there is a vertex of $V(C_4)$ that has a neighbor outside C_4, there is a P_5 in G, so we may assume that no vertex of $V(C)$ has neighbors outside the cycle. However, this means that $\overline{G}$ contains $K_{2,4}$ as subgraph, implying that $P_5 \subsetneq \overline{G}$.

□

The claim, together with the previous graph, proves that $R(P_5) = 6$. □

Problem 4.6

For any n, find a lower bound for $R(P_n)$.

Solution. Trivially, the clique K_{n-1} does not contain P_n, because its longest path is P_{n-1}.

Claim: Any complete bipartite graph that has $\lfloor \frac{n}{2} \rfloor - 1$ nodes in one of its partitions does not contain P_n.

Proof of the Claim. Any path in a complete bipartite graph must alternate between its two partitions, therefore the longest path will have length

$$2\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) + 1 \leq n - 2 + 1 = n - 1 < n$$

□

Fix a number n, and consider the following graph $G := K_{n-1} \cup K_{\lfloor \frac{n}{2} \rfloor - 1}$. By the previous observation, K_{n-1} does not contain P_n, and clearly $K_{\lfloor \frac{n}{2} \rfloor - 1}$ cannot contain P_n either, since it contains even less vertices. Moreover, we observe that $\overline{G} = K_{n-1, \lfloor \frac{n}{2} \rfloor - 1}$ by construction of G, and by the claim we get that $\overline{G}$ does not contain P_n. Therefore, this graph concludes that

$$R(P_n) \geq |V(G)| + 1 = (n-1) + \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) + 1 = n + \left\lfloor \frac{n}{2} \right\rfloor - 1$$

□

Problem 4.7

Find the value of $R(P_6)$.

Solution. By Problem 4.6, we know that

$$R(P_6) \geq 6 + \left\lfloor \frac{6}{2} \right\rfloor - 1 = 6 + 3 - 1 = 8$$

Let G be a graph on 8 vertices, and by way of contradiction suppose that neither G nor $\overline{G}$ contain P_6. Since G has 8 vertices, then $|E(G)| + \left|E(\overline{G})\right| = |E(K_8)| = \binom{8}{2} = 28$, which means that the largest between G and $\overline{G}$ contains at least 14 edges. However, a graph with 8 vertices and at least 14 edges must contain a cycle, hence the largest between G and $\overline{G}$ contains a cycle. However, by the same argument, the smallest *may also* contain a cycle, hence let C be the *largest* cycle in both G and $\overline{G}$ — without loss of generality, suppose that C is in G. In particular, if the cycle has length 6 or more, C trivially contains P_6 contradicting our assumption, so we may assume that our cycle is either C_3, C_4 or C_5.

First, suppose that $C = C_3$, and let $\{c_i, c_j, c_k\}$ be the set of its vertices.

Claim: If $C = C_3$, there is no pair of distinct vertices $c_i, c_j \in V(C_3)$ such that for any $x_p, x_q \in V(G) - V(C_3)$ it holds that $c_i \sim x_p$ and $c_j \sim x_q$.

Proof of the Claim. Without loss of generality, suppose that $c_i \sim x_p$ and $c_j \sim x_q$ for the sake of contradiction. If $x_p = x_q$, then $x_p \sim c_i \sim c_k \sim c_j$ form C_4, contradicting the maximality of C $\lightning$. Vice versa, if $x_p \neq x_q$, then for all $x_\ell \in V(G) - (V(C_3) \cup \{x_p, x_q\})$ it holds that $x_p, x_q \nsim x_\ell$, otherwise there would be P_6 in G, but this implies that there is C_4 in $\overline{G}$, again contradicting the maximality of C $\lightning$. $\square$

Now, suppose that there is a vertex $c_i \in V(C_3)$ such that $c_i \sim x_p$ for some $x_p \in V(G) - V(C_3)$; then, by the previous claim, we know that for all $x_\ell \in V(G) - V(C_3)$ it holds that $c_j, c_k \nsim x_p$, i.e. $K_{2,5}$ is a subgraph of $\overline{G}[V(G) - \{c_i\}]$. This implies that for each vertex $x_\ell \neq x_p$ outside C_3 it holds that $c_i, x_p \sim x_\ell$, otherwise the anti-edge from x_ℓ to c_i or x_p together with $K_{2,5}$ would form P_6 in $\overline{G}$. In particular, this implies that there are two vertices $x_{\ell_1}, x_{\ell_2} \neq x_p$ outside C_3 such that $c_i \sim x_{\ell_1} \sim x_p \sim x_{\ell_2}$ form C_4, contradicting the maximality of G $\lightning$.

This implies that for all $c_i \in V(C_3)$ and $x_p \in V(G) - V(C_3)$ it holds that $c_i \nsim x_p$, meaning that $\overline{G}$ contains $K_{3,5}$ as subgraph, which in turn contains P_6. This concludes that $C \neq C_3$.

Now, suppose that $C = C_4$.

Claim: If $C = C_4$, then for all $c_i \in V(C_4)$ and $x_\ell \in V(G) - V(C_4)$ it holds that $c_i \nsim x_\ell$.

Proof of the Claim. By way of contradiction, suppose that there is a vertex $c_i \in V(C_4)$ such that $c_i \sim x_\ell$ for some $x_\ell \in V(G) - V(C_4)$; then for all $x_t \in V(G) - (V(C_4) \cup \{x_\ell\})$ it holds that $x_\ell \nsim x_t$ otherwise there would be P_6 in $\overline{G}$. Moreover, let $c_j, c_k \in V(C_4)$ be the vertices such that $c_j, c_k \sim c_i$; then for all $x_t \in V(G) - (V(C_4) \cup \{x_\ell\})$ it holds that $c_j, c_k \nsim x_t$ otherwise there would be P_6 in $\overline{G}$. However, this implies that the other 3 vertices $x_{t_1}, x_{t_2}, x_{t_3} \in V(G) - (V(C_4) \cup \{x_\ell\})$ are such that $x_{t_1} \nsim x_\ell \nsim x_{t_2} \nsim c_k \nsim x_{t_3} \nsim c_j$ form P_6 in $\overline{G}$ $\lightning$. $\square$

In particular, the previous claim concludes that $\overline{G}$ contains $K_{4,4}$, which trivially contains P_6. Therefore, $C \neq C_3, C_4$.

Lastly, suppose that $C = C_5$. Then, if any of the vertices $c_i \in V(C_5)$ is adjacent to a vertex $x_\ell \in V(G) - V(C_5)$, then there is P_6 in $\overline{G}$, therefore for all $c_i \in V(C_5)$ and $x_\ell \in V(G) - V(C_5)$ it holds that $c_i \nsim v_\ell$. In particular, this implies that $\overline{G}$ contains $K_{3,5}$, which clearly contains P_6 $\lightning$. $\square$

Problem 4.8

Consider the complete graph K_{17}; show that for any choice of the coloring function $c : E(K_{17}) \to \{\text{R, G, B}\}$ — where R, G and B stand for *red*, *green* and *blue* respectively — there are three vertices $x, y, z \in V(K_{17})$ such that $c(xy) = c(yz) = c(xz)$, i.e. there is a K_3 subgraph having the edges colored all colored by the same color.

Solution. Fix a vertex $u \in V(K_{17})$; since $\deg(u) = 16$, and there are three possible color for each edge, by the pigeonhole principle u "sees" at least $\lceil \frac{16}{3} \rceil = 6$ edges having the same color. Without loss of generality, suppose that u sees at least 6 red edges $ux_1, \ldots, ux_6$, and consider the vertices $x_1, \ldots, x_6$. We may assume that between any distinct pair x_i, x_j of such vertices there are no red edges, otherwise K_{17} trivially contains a red K_3 having u, x_i and x_j as vertices. Therefore, for each $i, j \in [6]$ distinct, it holds that $c(x_i x_j)$ is either green or blue.

We observe that the Ramsey function can be interpreted in terms of *colors* on the edges: in fact, given a coloring function that assigns colors G and B, $R(3) = R(3, 3)$ is the minimum number of vertices which guarantees that a graph on $R(3)$ vertices contains a K_3 either completely colored in green or in blue. Therefore, by considering $G[\{x_1, \ldots, x_6\}]$, since this graph contains 6 vertices and no red edges, and $R(3) = 6$ by Theorem 4.7, it follows that such induced subgraph must contain either a blue or a green K_3. $\square$

Problem 4.9

A set $\mathcal{I}$ of graphs is said to be *minor closed* if for any $G \in \mathcal{I}$ it holds that if H is a minor of G, then $H \in \mathcal{I}$. Prove that the set $\mathcal{F}$ of the forests is minor closed.

Solution. Let $F \in \mathcal{F}$ be a forest, and consider a minor H of F; by way of contradiction, suppose that $H \notin \mathcal{F}$, i.e. H is not a forest. Then H contains a cycle in one of its components, but since H can be obtained from F thorugh minor operations, this would imply that F also contains a cycle, contradicting the fact that F was a forest. $\square$

Problem 4.10

Given a set $\mathcal{I}$ of graphs, we say that $H_1, \ldots, H_k$ are *forbidden minors* for $\mathcal{I}$ when $G \in \mathcal{I}$ if and only if G does not have any H_i as minor. Find the forbidden minors for the set $\mathcal{F}$ of forests.

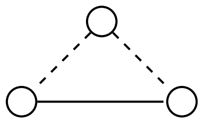
Solution. Let $F \in \mathcal{F}$ be a forest; we observe that F is a forest if and only if F has no cycles, which happens if and only if F has no C_3 as minor. This implies that C_3 is the only forbidden minor for $\mathcal{F}$. $\square$

Problem 4.11

A graph is said to be *complete multipartite* if there exists a partition $X_1, \ldots, X_k$ of $V(G)$ such that

- for each $i \in [k]$ it holds that X_i is an independent set
- for any $i, j \in [k]$ if $i \neq j$ then all the vertices of X_i are adjacent to all the vertices of X_j

Consider the following subgraph H — the *dashed* lines are anti-edges of H



Prove that if a graph G on $n \geq 3$ vertices has no H subgraph, then G is complete multipartite.

Solution. We proceed by induction on n. For the base case, if $n = 3$ and G has no H subgraph, then G can be one of the following:

- if G has 0 edges, then we can consider the partition $X_1 = V(G)$ and G is complete multipartite
- if G has 2 edges, then the endpoints of the anti-edge of G form a partition X_1 and the remaining vertex forms X_2, meaning that G is complete multipartite
- if G has 3 edges, then each vertex forms a partition on its own, thus G is again complete multipartite

Now, assume that the statement holds on a graph on $n-1$ vertices, and consider a graph on $n \geq 4$ vertices that does not contain H as subgraph. Fix a vertex x in G, and consider $G - \{x\}$: since G does not contain H as subgraph, then $G - \{x\}$ cannot contain H as subgraph either, and because $G - \{x\}$ has $n - 1$ vertices we can apply the inductive hypothesis. Hence $G - \{x\}$ is complete multipartite on a collection of independent sets $X_1, \ldots, X_k$.

Claim: If there is a vertex v such that $x \sim v$, and $v \in X_i$ for some $i \in [k]$ then $X_i \subseteq \mathcal{N}(x)$ — i.e. x is adjacent to all the vertices of the partition of v.

Proof of the Claim. Suppose that such v exists; then, since G does not contain H as subgrap, than X_i is an independent set, it must be that $X_i \subseteq \mathcal{N}(x)$ — otherwise any anti edge xv_j for some $v_j \in X$ would form an H subgraph because $v \sim x$ and $v_j \nsim v$. □

Hence, if x is adjacent to at least one vertex for every set X_i, then x is adjacent to *all* the vertices of all the partitions, thus $X_1, \ldots, X_k, \{v\}$ is a partition of $V(G)$ such that G is complete multipartite. So, we may assume that there exists at least one partition X_i such that $\forall v_j \in X_i$ it holds that $v_j \nsim x$. Then, $X_i \cup \{x\}$ is a partition of $V(G)$ such that G again complete multipartite. □

5

Graph decompositions

Graph decompositions are a way to *break down* a graph into simpler more structures in order to better understand its properties, make algorithms more efficient, or solve complex problems.

5.1 Blocks

We start our discussionon decompositions by introducing the **block decomposition**, which is used to find the "weak spots" of a graph.

Definition 5.1: Cut vertex

Given a graph G, a **cut vertex** of G is a vertex $v \in V(G)$ such that $G - \{v\}$ has more connected components than G.

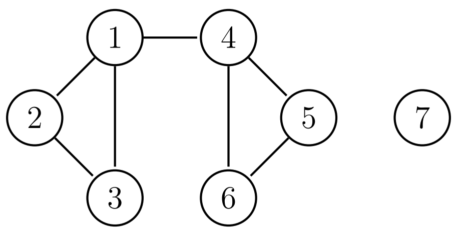


Figure 5.1: For instance, in this graph there are two cut vertices, namely 1 and 4.

Trivially, by definition we observe that no 2-connected graph can contain cut vertices.

Definition 5.2: Block

Given a graph G, a **block** is a *maximal* subgraph H with no cut vertex, i.e. there is no $H' \subseteq G$ without cut vertices such that $H \subsetneq H'$.

For instance, in the previous example we have exactly four blocks, and the set of vertices that describe them are the following:

$$B_1 := \{1, 2, 3\}$$

$$B_2 := \{4, 5, 6\}$$

$$B_3 := \{1, 4\}$$

$$B_4 := \{7\}$$

We observe that any edge of a graph is itself a subgraph without cut vertices, which implies that *all* the edges of a graph will belong to a block. Furthermore, we observe that any cut vertex of a graph must belong to at least two different blocks, by definition.

Moreover, consider the following definition.

Definition 5.3: Bridge

Given a graph G, an edge $e \in E(G)$ is called **bridge** if $G - \{e\}$ has more connected components than G.

For instance, in the previous example the edge $\{1, 4\}$ is clearly a bridge.

Bridges and cut vertices are similar and linked concepts; in fact, if an edge e is a bridge, clearly *at least one* of its endpoints must be a cut vertex. However, we observe that:

- the endpoints of a bridge may not be *both* cut vertices

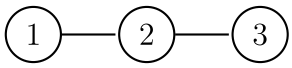


Figure 5.2: For instance, in this graph the edges $\{1, 2\}$ and $\{2, 3\}$ are clearly bridges, but the vertices 1 and 3 are not cut vertices.

- if an edge has a cut vertex as one of its endpoints, the edge *may not* be a bridge

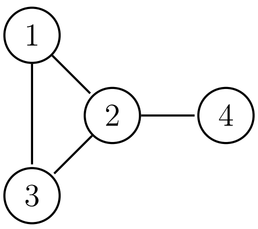


Figure 5.3: For instance, in this graph the vertex 2 is clearly a cut vertex, even if $\{1, 2\}$ and $\{2, 3\}$ are not bridges.

Proposition 5.1

A block of a graph can only be an isolated vertex, a bridge or a 2-connected subgraph.

Proof. If the block has only 1 vertex, clearly it must be an isolated vertex.

Consider a block B consisting of 1 edge uv: if this edge were not a bridge, then removing it would not create more components than before, meaning that there must be at least another path beteween u and v in the graph that does not cross the edge uv. However, this implies that uv is contained in a cycle, meaning that B could have been extended with the cycle, contradicting the maximality of B — we observe that a cycle cannot contain cut vertices.

Finally, consier a block B with more than two vertices: if B were not 2-connected, then by removing one vertex $v \in B$ we would get that $B - \{v\}$ is disconnected, meaning that v was a cut vertex in the first place, contradicting the definition of B $\lightning$. □

Lemma 5.1

Given a graph G, and two connected subgraphs $H, H' \subseteq G$ such that $V(H) \cap V(H') \neq \varnothing$, it holds that $H \cup H'$ is connected.

Proof. Let $z \in V(H) \cap V(H')$, which must exist by assumption, and two vertices $u \in V(H)$ and $v \in V(H')$:

- if $u, v \in V(H) \cap V(H')$, then there is a path $u \to v$ by connectivity of both H and H'
- without loss of generality, if $u \in V(H)$ and $v \in V(H') - V(H)$, then there is a path P of the form $u \to z$ by connectivity of H, and there is a path Q of the form $z \to v$ by connectivity of H', hence $P \cup Q$ is a path of the form $u \to v$

implying that $H \cup H'$ is still connected. □

Proposition 5.2

Given a graph G, and two distinct blocks H and H' of G, then $|V(H) \cap V(H')| \leq 1$.

Proof. By way of contradiction, suppose that $|V(H) \cap V(H')| \geq 2$. Let $z \in V(H \cup H')$; by Proposition 5.1 we get that both $H - \{z\}$ and $H' - \{z\}$ are still connected. Moreover, since $|V(H) \cap V(H')| \geq 2$, it holds that $|V(H)|, |V(H')| \geq 2$, meaning that

$$(H \cup H') - \{z\} = (H - \{z\}) \cup (H' - \{z\}) \neq \varnothing$$

which implies that $(H \cup H') - \{z\}$ is connected by the previous lemma.

Suppose that $|V(H \cup H')| = 2$; then we have only two possible cases:

- $H \cup H'$ consists of 2 isolated vertices, contradicting the fact that $|V(H) \cap V(H')| \geq 2$
- $H \cup H'$ consists of a single edge, meaning that the edge itself is one single block, hence $H = H'$, contradicting the fact that $H \neq H'$ by assumption

Hence, the only possibility is that $H \cup H'$ contains at least 3 vertices. Moreover, we observe that if $V(H) \cap V(H') \neq \varnothing$, then $H \cup H'$ is connected by Proposition 5.1.

However, we observe that if

- $H \cup H'$ is connected
- for any $z \in V(H \cup H')$ it holds that $(H \cup H') - \{z\}$ is connected
- $|V(H \cup H')| \geq 3$

it implies that $H \cup H'$ is 2-connected by definition. This means that $H \cup H'$ does not contain cut vertices, contradicting the maximality of the blocks $H, H' \subsetneq H \cup H'$ $\lightning$. ☐

Proposition 5.3

Given a graph G, and two distinct blocks H and H' of G, if $V(H) \cap V(H') \neq \varnothing$ then any vertex in $V(H) \cap V(H')$ is a cut vertex of G.

Proof. Since $V(H) \cap V(H') \neq \varnothing$, we know that neither H nor H' are isolated vertices — if both of them were isolated vertices, they would have empty intersection, and if one of them were an isolated vertex it would have been part of the other block, meaning that H and H' actually were a single block. Moreover, if at least one between H and H' is a bridge, the theorem trivially holds. Hence, we may assume that $|V(H)|, |V(H')| \geq 3$ — and in particular, they are 2-connected by Proposition 5.1.

By the previous theorem, if $V(H) \cap V(H') \neq \varnothing$ it holds that $|V(H) \cap V(H')| = 1$, hence let z be the vertex in the intersection of the two blocks, and by way of contradiction suppose that z is not a cut vertex. Then, by definition this means that for any two fixed vertices $u \in V(H - \{z\})$ and $v \in V(H' - \{z\})$ there is a path P that does not pass through z.

Claim: $H \cup H' \cup P$ is 2-connected.

Proof of the Claim. By Proposition 5.1, we know that both $H - \{z\}$ and $H' - \{z\}$ are connected, and since they both have a vertex in common with P, we get that

$$(H - \{z\}) \cup (H' - \{z\}) \cup P = (H \cup H' \cup P) - \{z\}$$

is still connected — since P is a path.

Moreover, by the same proposition we note that $H \cup H'$ is connecetd. Now, let p_1 and p_2 be the endpoints of P — without loss of generality, assume $p_1 \in V(H)$ and $p_2 \in V(H')$ — and fix a vertex $x \in V(H \cup H' \cup P)$; we have some cases to handle

- In the first case, we have that $x \in V(H \cup H') - \{z\}$. Fix two vertices $a, b \in V(H \cup H' \cup P) - \{x\}$
  - if $a, b \in V(P)$, they are trivially connected since P is a path
  - if $a, b \in V(H) - \{z, x\}$, or $a, b \in V(H') - \{z, x\}$, then a and b are connected by 2-connectivity of H and H' — even if a path $a \to b$ passes through x, there

must exist at least another path of the same form by Theorem 3.2 that does not pass through x

- without loss of generality, if $a \in V(H) - \{z, x\}$ and $b \in V(H') - \{z, x\}$, then by 2-connectivity of H and H' there are paths $a \to p_1$ and $p_2 \to b$ that do not pass through x, meaning that $(a \to p_1) \cup P \cup (p_2 \to b)$ is a path $a \to b$
- without loss of generality, if $a \in V(H) - \{z, x\}$ and $b \in V(P) - \{p_1, p_2\}$, again by 2-connectivity of H there is at least one path $a \to p_1$ that does not pass through x, meaning that $(a \to p_1) \cup (p_1 \ P \ b)$ is a path $a \to b$

This proves that $(H \cup H' \cup P) - \{x\}$ is connected, for $x \in V(H \cup H') - \{z\}$.

- In the second case, we have that $x \in V(P) - V(H \cup H')$. Fix two vertices $a, b \in V(H \cup H' \cup P) - \{x\}$
  - if $a, b \in V(H \cup H') - V(P)$, then a and b are trivially connected by connectivity of $H \cup H'$
  - without loss of generality, if $a \in V(H) - V(P)$ and $b \in V(P) - \{x\}$ then
    * if by removing x from P we did not disconnect a and b, there still is a path $a \to b$
    * otherwise, by connectivity of $H \cup H'$ there will be a path $a \to p_2$, hence $(a \to p_2) \cup (p_2 \ P \ b)$ is a path $a \to b$
  - similarly, if $a, b \in V(P) - \{x\}$, then
    * if by removing x from P we did not disconnect a and b, there still is a path $a \to b$
    * otherwise, by connectivity of $H \cup H'$ there will be a path $p_1 \to p_2$, hence $(a \ P \ p_1) \cup (p_1 \to p_2) \cup (p_2 \ P \ b)$ is a path $a \to b$

  This proves that $(H \cup H' \cup P) - \{x\}$ is connected, for $x \in V(P) - V(H \cup H')$.

Finally, this means that no matter the choice of x, we have that $H \cup H' \cup P - \{x\}$ is still connected, i.e. $H \cup H' \cup P$ is 2-connected by definition. □

In particular, since $H \cup H' \cup P$ is 2-connected by the claim, it cannot contain any cut vertices, implying that $H \cup H' \cup P$ contradicts the maximality of the blocks $H, H' \subsetneq H \cup H' \cup P$ ↯. □

Definition 5.4

Given a connected graph G, let $B_1, \ldots, B_k$ be its blocks, and let $z_1, \ldots, z_\ell$ be the set of its cut vertices. We define $\mathcal{B}(G)$ as the **block graph** of G through the following sets of nodes and edges:

- $V(\mathcal{B}(G)) = \{b_1, \ldots, b_k\} \cup \{z_1, \ldots, z_\ell\}$, where each b_i is a vertex corresponding to a block B_i of G
- there is an edge $\{b_i, z_j\} \in E(\mathcal{B}(G))$ if and only if $z_j \in V(B_i)$

TODO example

We observe that, by definition the only edges present in $\mathcal{B}(G)$ are between nodes representing blocks and nodes representing cut vertices of G. This means that $\{b_1, \dots, b_k\}$ and $\{z_1, \dots, z_\ell\}$ are independent sets, i.e. $\mathcal{B}(G)$ is bipartite.

Theorem 5.1

Given a connected graph G, then $\mathcal{B}(G)$ is a tree.

Proof. Since G is connected, $\mathcal{B}(G)$ is trivially connected as well. By way of contradiction, suppose that it is not a tree, i.e. there is an *induced* cycle

$$C := b_{i_1}\ z_{j_1}\ b_{i_2}\ \dots\ b_{i_{h-1}}\ z_{j_h}\ b_{i_h}$$

in $\mathcal{B}(G)$, where $z_{j_l} \in V(B_{i_l}) \cap V(B_{i_l+1})$ by definition. Moreover, since the block graph is bipartite by definition, we get that $|V(C)|$ must be even by Theorem 1.4. Additionally, since C is induced, we have that for $i_l \notin \{j_l, j_l+1\}$ it holds that $z_{j_l} \notin V(B_{i_l})$.

Claim: $\displaystyle\bigcup_{l=1}^{h} B_{i_l}$ has no cut vertex.

Proof of the Claim. Let $x \in V\left(\displaystyle\bigcup_{l=1}^{h} B_{i_l}\right)$, and without loss of generality suppose that $x \in V(B_{i_1})$. TODO da finire □

In particular, this claim contradicts the maximality of the blocks $b_{i_1}, \dots, b_{i_h}$ $\lightning$. □

Proposition 5.4

Given a connected graph G, every leaf of $\mathcal{B}(G)$ represents a block of G.

Proof. By way of contradiction, suppose that there is a leaf $v \in V(\mathcal{B}(G))$ that represents a cut vertex of G. Since v is a leaf, it must have degree 1 in $\mathcal{B}(G)$, and by definition of the block graph we know that its only neighbor must be a vertex b representing a block B of G. Therefore, by definition of $\mathcal{B}(G)$ there we know that $v \in V(B)$, but if v is a cut vertex of G there must be at least another block B' — represented by a vertex $b' \in V(\mathcal{B}(G))$ — in G that contains v, meaning that $v \sim b, b'$, contradicting the fact that v was a leaf of the block graph $\lightning$. □

5.2 Known decompositions

In this brief section, we are going to present characterizations of 2-connected and 3-connected graphs. In fact, these type of graphs can be decomposed using specific types of structural decompositions.

Definition 5.5: H-path

Given a graph G, and a subgraph $H \subseteq G$, an **H-path** P is a non-trivial path with both ends in $V(H)$, no internal vertex in $V(H)$, and no edge in $E(H)$.

We observe that the last condition of the definition implies that *single edges* are not H-paths.

TODO example

Definition 5.6: Ear decomposition

Given a graph G, an **ear decomposition** of G is a sequence $C, P_1, \ldots, P_k$ such that

- C is a cycle
- each P_i is a $\left(\bigcup_{j=1}^{i-1} P_j\right)$-path
- $C \cup P_1 \cup \ldots \cup P_k = G$

The **size** of an ear decomposition is the number of its paths — in this case k.

TODO example

The following theorem characterizes 2-connected graphs through ear decompositions.

Theorem 5.2

G is 2-connected if and only if it has an ear decomposition.

Proof.

Direct implication. If $|V(G)| = 3$, and G is 2-connected, then $G = C_3$ and the theorem trivially holds, so we may assume that $|G| \geq 4$.

If G is 2-connected, then $\delta \geq 2$ by Proposition 3.4, i.e. G has a cycle C. Fix $H \subseteq G$ as large as possible — i.e. *maximal by subgraph containment* — such that H has an ear decomposition.

By way of contradiction, suppose that $H \neq G$, meaning that there must be at least an edge $xy \in E(G) - E(H)$ — note that the vertex set of H might still be $V(G)$. Moreover, since $|V(G)| \geq 4$ and $xy \in E(G) - E(H)$, then $|V(H)| \geq 2$. Then, since G is 2-connected, by Corollary 3.2 there are at least two paths P and P' that connect x to $V(H)$ such that $V(P) \cap V(P') = \{x\}$, and two paths Q and Q' that connect y to $V(H)$ such that $V(Q) \cap V(Q') = \{y\}$. We have two cases.

- In the first case, all the paths intersect $V(P) \cap V(Q), V(P) \cap V(Q') \neq \varnothing$ and $V(P') \cap V(Q), V(P') \cap V(Q') \neq \varnothing$. Let $a \in V(P) \cap V(Q)$ and $b \in V(P') \cap V(Q')$;

then $Q\ a\ x\ y\ b\ P'$ is an H-path.

- At least two paths do not intersect, and without loss of generality suppose that $V(P) \cap V(Q) = \varnothing$. Then $P\ x\ y\ Q$ is an H-path.

However, in both cases the H-path contradicts the maximality of H $\lightning$.

Converse implication. We procced by induction on the size of the ear decomposition.

For the base case, if $k = 0$ then $G = C$, which implies that G is 2-connected. Now, assume inductively that if graph has an ear decomposition of size k it is 2-connected, and consider a graph G that has an ear decomposition of size $k + 1$.

By inductive hypothesis, we know that $H := C \cup P_1 \cup \ldots \cup P_k$ is 2-connected. Moreover, since $C, P_1, \ldots, P_{k+1}$ is an ear decomposition, we know that P_{k+1} is an H-path. Fix $x \in V(G)$; we observe that

- H is 2-connected, hence $H - \{x\}$ is still connected by definition
- every components of $P_{k+1} - \{x\}$ has a vertex in common with $H - \{x\}$

meaning that $(H - \{x\}) \cup (P_{k+1} - \{x\}) = (H \cup P_{k+1}) - \{x\}$ is connected. Lastly, since $C \subsetneq H$, then $|V(H)| \geq 3 \implies |V(H \cup P_{k+1})| \geq 3$, which concludes that $H \cup P_{k+1} = G$ is 2-connected.

$\square$

The other type of graphs that we are going to discuss are 3-connected graphs. First, consider the following theorem — proved by Tutte [Tut61] in 1961.

Theorem 5.3

If a graph G is 3-connected, and $n \geq 5$, then there exists an edge $e \in E(G)$ such that G/e is still 3-connected.

Proof. By way of contradiction, suppose that there is no edge $e \in E(G)$ such that G/e is still 3-connected. However, since $n \geq 5$, and G/e is not 3-connected, it must admit a separation (A, B) or order at most 2 by definition.

Claim 1: For any edge $e := xy \in E(G)$, there is a separation (A_e, B_e) of order *exactly* 3 such that $x, y \in A_e \cap B_e$.

Proof of the Claim. Consider a separation (A, B) of G/e of *exactly* 2, and let $v_e \in V(G/e)$ be the vertex constructed after contracting e. If $v_e \in B - A$, then $(A, (B - \{v_e\}) \cup \{x, y\})$ would be a separation of G of order 2, contradicting the 3-connectivity of G; this implies that $v_e \in A \cap B$. Moreover, it cannot hold that $|A \cap B| < 2$, otherwise by expanding v_e we get that $((A - v_e) \cup \{x, y\}, (B - v_e) \cup \{x, y\})$ is a separation of order 2 in G. Thus, it must hold that $A \cap B = \{v_e, z\}$ for some third vertex z, concluding that $((A - v_e) \cup \{x, y\}, (B - v_e) \cup \{x, y\})$ is a separation of order 3 of G — where $A \cap B = \{x, y, z\}$. $\square$

Thus, let (A_e, B_e) be the separation that minimizes $|B_e|$, and let $z \in (A_e \cap B_e) - \{x, y\}$ be the third vertex of $A_e \cap B_e$; if z has no neighbors in $B_e - A_e$, then by removing e from G we would disconnect the graph, contradicting the assumption of 3-connectivity. This means that there is a vertex $u \in B_e - A_e$ such that $z \sim u$. Hence, let $e' := zu$, and consider a non-trivial separation $(A_{e'}, B_{e'})$ of G of order exactly 3 such that $z, u \in A_{e'} \cap B_{e'}$ — which exists by Claim 1.

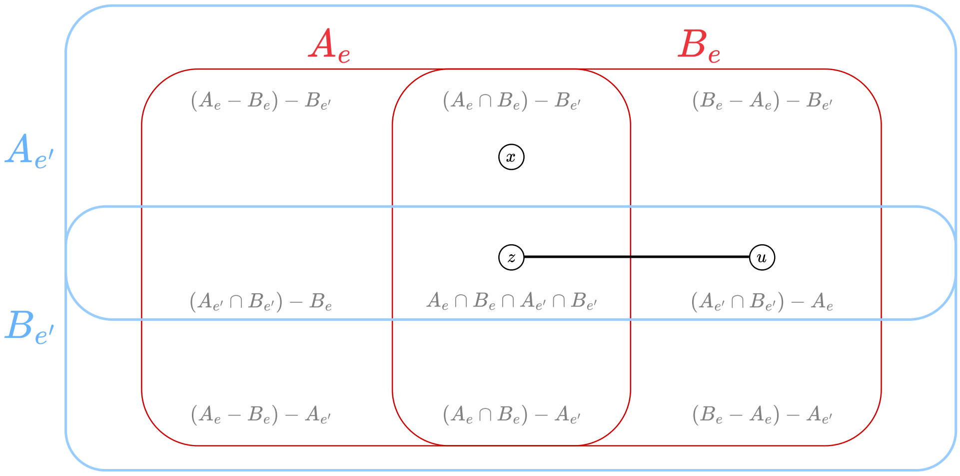


Figure 5.4: The two separations of G and the various regions they induce.

Claim 2: $(A_e \cap B_e) - A_{e'} = \varnothing$.

Proof of the Claim. By definition of $(A_{e'}, B_{e'})$, we have that $z \in (A_e \cap B_e) \cap (A_{e'} \cap B_{e'})$. However, if both x and y are contained in $(A_e \cap B_e) \cap (A_{e'} \cap B_{e'})$, then $\{x, y, z, u\} \subseteq A_{e'} \cap B_{e'}$ contradicting the order of $(A_{e'}, B_{e'})$.

Then, without loss of generality suppose that $x \in A_{e'} - B_{e'}$; hence, $y \notin B_{e'} - A_{e'}$ otherwise the edge xy would be an edge *across* $A_{e'} - B_{e'}$ and $B_{e'} - A_{e'}$, violating the fact that $(A_{e'}, B_{e'})$ is a separation. This implies that $(A_e \cap B_e) - A_{e'} = \varnothing$. $\square$

Claim 3: $(A_e - B_e) - A_{e'} = \varnothing$.

Proof of the Claim. By way of contradiction, suppose that $(A_e - B_e) - A_{e'} \neq \varnothing$, and fix a vertex $w \in (A_e - B_e) - A_{e'}$. Since (A_e, B_e) is a separation, we know that $w \nsim u$. Moreover, since $(A_{e'}, B_{e'})$ is a separation, w can only be adjacent to vertices in $(A_{e'} \cap B_{e'}) - (B_e - A_e)$. Therefore, by removing $(A_{e'} \cap B_{e'}) - (B_e - A_e)$ from G we disconnect w from u necessarily — since $(A_e \cap B_e) - A_{e'} = \varnothing$ by Claim 2. Moreover, since $(A_{e'} \cap B_{e'}) - (B_e - A_e)$ contains two vertices by choice on the order of $(A_{e'}, B_{e'})$, we get a contradiction on the 3-connectivity of G $\lightning$. $\square$

In particular, Claim 2 and Claim 3 imply that

$$((A_e \cap B_e) - A_{e'}) \cup ((A_e - B_e) - A_{e'}) = A_e - A_{e'} = \varnothing$$

Claim 4: $(A_{e'} \cap B_{e'}) - B_e = \varnothing$.

Proof of the Claim. Consider the set $B_e - (A_e \cup A_{e'})$; since we picked $(A_{e'}, B_{e'})$ to be a non-trivial separation, we know that $B_{e'} - A_{e'} \neq \varnothing$. Therefore, Claim 2 and Claim 3 imply that $B_e - (A_e \cup A_{e'}) \neq \varnothing$. Hence, let v be a vertex in such set; since (A_e, B_e) is a separation, we know that $v \nsim x$ because $x \in (A_{e'} - B_{e'}) \cap (A_e \cap B_e)$.

Now, by way of contradiction suppose that $(A_{e'} \cap B_{e'}) - B_e \neq \varnothing$, and by choice of $(A_{e'}, B_{e'})$ we know that $A_{e'} \cap B_{e'}$ only contains 3 vertices. Therefore, let w be the third vertex inside $(A_{e'} \cap B_{e'}) - B_e$; since $v \nsim x$ by the previous observation, we know that v can only be connected to w through z or u — since $A_e - A_{e'} = \varnothing$ by Claim 2 and Claim 3.

This implies that by removing $\{z, u\}$ from G we disconnect v from w, contradicting the 3-connectivity of G $\lightning$. $\square$

Therefore, by Claim 2, 3 and 4 we get that $B_{e'} \subseteq B_e$, but because $x \in (A_{e'} - B_{e'}) \cap (A_e \cap B_e)$ then $B_{e'} \subsetneq B_e$, which is a contradiction on the choice of the separation (A_e, B_e) that minimized $|B_e|$ $\lightning$. $\square$

From this theorem, we can immediately derive the direct implication of the following corollary, which gives the characterization of 3-connected graphs that we previously anticipated — the converse implication was also proved by Tutte. Unformally, we could say that every 3-connected graph can be constructed by repeatedly "un-contracting" edges.

Corollary 5.1

A graph G such that $n \geq 5$ is 3-connected if and only if there exists a sequence $G_1, \ldots, G_k$ such that

- for each $i \in [k-1]$ it holds that $G_i = G_{i+1}/e_{i+1}$ for some $e_{i+1} \in E(G_{i+1})$
- $G_1 = K_4$ and $G_k = G$
- for each $i \in [k]$, G_i is 3-connected

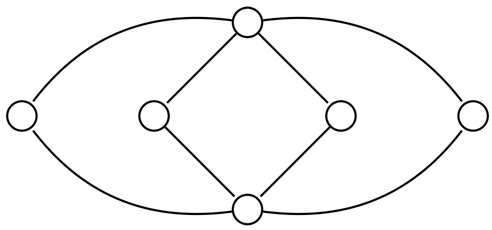
5.3 Exercises

Problem 5.1

Solve the following exercises on k-connectivity.

- Give an example of a 2-connected graph G such that for any $C \subseteq G$ it holds that $G - C$ is disconnected.
- Prove that every 3-connected graph G has a non-separating cycle

Solution. For the first exercise, a minimal example is given by the following graph — we observe that this graph is $K_{2,4}$:



For the second exercise, we proceed by induction on n. Since G is 3-connected, the base case is for $n \geq 4$, which implies that $G = K_4$ and can be easily checked that any cycle of K_4 is non-separating; now, assume the inductive hypothesis and consider a 3-connected graph on $n \geq 5$ vertices.

By Theorem 5.3 we know that there exists an edge $e \in E(G)$ such that G/e is 3-connected, thus let e be such edge and consider G/e. This graph has $n - 1$ vertices, therefore by induction we know that G/e has a non-separating cycle C. Now, consider the following cases:

- if $|e \cap V(C)| = 0$, then C is trivially also non-separating cycle of G, so we may assume otherwise
- if $|e \cap V(C)| = 2$, then e lies on C with both endpoints in G, meaning that C is still a non-separating cycle of G, so again we may assume otherwise

This implies that the only case left to handle is if $|e \cap V(C)| = 1$. Without loss of generality assume that $e = xy$, and that $x \in V(C)$ and $y \notin V(C)$. By 3-connectivity of G, since C is a cycle then $|V(C)| \geq 3$, implying that by Corollary 3.2 there are 3 internally disjoint $\{y\}$-$V(C)$ paths — one such path may be the edge xy itself, so we know there are at least other two paths P_1 and P_2, which do not intersect. Let p_1 and p_2 be the vertices of $V(C)$ in which these two paths end in C: we observe that P_1 and P_3 induce 3 sections of C, namely $p_1 \ C \ x$, then $x \ C \ p_2$ and lastly $p_2 \ C \ p_1$.

Now consider $G - C$; if it does not contain any vertices the statement trivially holds, so we may assume that $|V(G - C)| \geq 1$, thus fix one vertex $v \in V(G - C)$. Then, again by Corollary 3.2 we know that there are three $\{v\}$-$V(C)$ paths. Without loss of generality, consider the section $p_1 \ C \ p_2$: by the pigeonhole principle, at least two such paths from v will end in either $p_1 \ C \ p_2$ or its complement w.r.t. C — without loss of generality suppose that two paths lie inside $p_1 \ C \ p_2$. Finally, at least one between $\{xy\} \cup x \ C \ p_1 \cup P_1$ and $\{xy\} \cup x \ C \ p_2 \cup P_2$ is a non-separating cycle of G. $\square$

Problem 5.2

Prove that if G is 2-connected, and $n \geq 4$, then for any edge $e \in E(G)$ either G/e or $G - e$ is 2-connected.

Solution. By way of contradiction, suppose that there exists an edge $xy \in E(G)$ such that nor G/xy neither $G - \{xy\}$ is 2-connected. This implies that both of such graph admit a separation of order 0 or 1.

Claim 1: G/xy and $G - \{xy\}$ are not disconnected.

Proof of the Claim. If G/xy is disconnected, then G was disconnected as well, since contracting xy cannot disconnect G. However, G cannot be 0-connected since we are assuming it is 2-connected. Similarly, if $G - \{xy\}$ is disconnected, then at least one between x and y must be a cut vertex of G, implying that G is 1-connected, again contradicting the assumption of 2-connectivity of G $\lightning$. □

This claim implies that if the statement is assumed false for the sake of contradiction, it must hold that both G/xy and $G - \{xy\}$ are 1-connected. Let v_{xy} be the vertex of G/xy obtained by contracting the edge xy.

Claim 2: If G/xy is 1-connected, then for each separation (A, B) of G/xy oder 1 of G/xy it holds that $A \cap B = \{v_{xy}\}$.

Proof of the Claim. Since G/xy is 1-connected, it must admit a separation of order 1; by way of contradiction, suppose there exists a separation (A, B) such that $A \cap B \neq \{v_{xy}\}$, where v_{xy} is the vertex obtained after contracting the edge xy in G/xy. This implies that in G/xy either $v_{xy} \in A - B$ or $v_{xy} \in B - A$ holds, and without loss of generality suppose the former holds; then $((A - \{v_{xy}\} \cup \{x, y\}), B)$ is a separation of G of order 1, contradicting the fact that G is 2-connected. □

Claim 3: If G/xy is 1-connected, then $G - \{xy\}$ is 2-connected.

Proof of the Claim. Let $e := xy$, and consider a separation (A, B) of order 1 of G/e such that $v_e \in A \cap B$ — which exists by Claim 2; we observe that this is a separation of order 2 in G, such that $A \cap B = \{x, y\}$ — call this separation (A_e, B_e).

By way of contradiction, suppose that there is no edge e such that (A_e, B_e) is a non-trivial separation, i.e. each possible separation of G is trivial; then G is a clique on at least 4 vertices, implying that G/xy is still 2-connected $\lightning$. Therefore, we may assume that there exists at least one such non-trivial separation, thus we may assume that (A_e, B_e) is a non-trivial separation of G of order 2 such that $A_e \cap B_e = \{x, y\}$.

Now, by way of contradiction suppose that $G - e$ is not 2-connected, and by Claim 1 it must hold that $G - e$ has a separation of order exactly 1; let (A_z, B_z) be a separation of order 1 of $G - e$ such that $A_z \cap B_z = \{z\}$ and $x \in A_z - B_z$ and $y \in B_z - A_z$ — x and y are not adjacent in $G - e$ so such a separation always exists.

Now, we are going to overlap (A_e, B_e) and (A_z, B_z) inside $G - e$ as we did for Theorem 5.3. In particular, without loss of generality suppose that $z \in (B_e - A_e) \cap A_z \cap B_z$, $x \in A_e \cap B_e \cap A_z$ and $y \in A_e \cap B_e \cap B_z$. We observe the following.

1. Since $A_z \cap B_z = \{z\}$, it must be that $(A_z \cap B_z) - B_e = \varnothing$.
2. Suppose there exists a vertex $w \in (A_e \cap B_e) - B_z$; since $x \nsim y$ in $G - e$, and G is 2-connected, there must be some path connecting w and y, however such path cannot have edges across the (A_z, B_z) separation, therefore they must pass all through x. This implies that by removing x we disconnect w from y in G as well, meaning that x is a cut vertex of G, contradicting its 2-connectivity. This implies that

$(A_e \cap B_e) - B_z = \varnothing$

3. By repeating the same argument as before, we also get that $(A_e - B_e) - A_z = \varnothing$

Putting everything together, we get that $A_e - B_e = \varnothing$, i.e. (A_e, B_e) is a trivial separation of G, contradicting the choice of the separation $\lightning$. $\square$

This claim implies that if G/xy is 1-connected, then $G - \{xy\}$ is 2-connected, contradicting the fact that $G - \{xy\}$ was assumed not to be so. Moreover, by contrapositive, if $G - \{xy\}$ is not 2-connected then G/xy is not 1-connected, i.e. G/xy is 0-connected, contradicting Claim 1 $\lightning$. $\square$

6

Planar graphs

TODO

intro

TODO

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Definition 6.1: Straight line segment

Given two points $p, q \in \mathbb{R}^2$, the **straight line segment** (SLS) between p and q is the set

$$\{\alpha p + (1-\alpha)q \mid \alpha \in [0,1] \subset \mathbb{R}\}$$

Definition 6.2: Polygon

A **polygon** is a subset of $\mathbb{R}^2$ formed by the unions of finitely many SLSs *homeomorphic* to the *unit circle* $S^1 := \{x \in \mathbb{R}^2 \mid ||x|| = 1\}$.

Definition 6.3: Polygonal arc

A **polygonal arc** is a subset of $\mathbb{R}^2$ formed by the union of finitely many SLSs homeomorphic to the *unit interval* $[0,1] \subset \mathbb{R}$.

Here, the term **homeomorphic** refers to the existence of an *homeomorphism* between the two sets, i.e. a continuous function between them whoch inverse is also continuous. In particular, for the case of polygonal arcs the images of 0 and 1 under such homeomorphism are the **endpoints** of the arc.

If the ends of a polygonal arc P are $p, q \in \mathbb{R}^2$, we say that P **links** p and q, and $\mathring{P} := P - \{p, q\}$ is the **interior** of P.

Definition 6.4: Open ball

Given a point $x \in \mathbb{R}^2$, and a radius $\varepsilon \in \mathbb{R}$, the **open ball** centered at x having radius ε is the following set of points

$$B_\varepsilon(x) := \{y \in \mathbb{R}^2 \mid ||x - y|| < \varepsilon\}$$

Definition 6.5: Open and closed set

A set $X \subseteq \mathbb{R}^2$ is said to be

- **open** if for all $x \in X$ there is an $\varepsilon \in \mathbb{R}$ such that $B_\varepsilon(x) \subseteq X$
- **closed** if it is the complement of an open set in $\mathbb{R}^2$

We observe that, by definition, the union of two open sets is still open. Instead, the union of closed sets is guaranteed to be closed only if the sets are *finitely many*.

Let $O \subseteq \mathbb{R}^2$ be an open set; we observe that the property of "being linked by a polygonal arc that lies on O" induces an *equivalence relation* on O — i.e. two points are related if they can be linked through a polygonal arc that lies inside O. Thus, we have the following definition.

Definition 6.6: Region

Given $O \subseteq \mathbb{R}^2$, the equvialence classes of the equilvance relation induced by the property of "being linked by a polygonal arc that lies in O" are called **regions** of O.

Given an open set $O \subseteq \mathbb{R}^2$ and a closed set $X \subseteq \mathbb{R}^2$, we say that X **separates** O if $O - X$ has more regions than O.

Definition 6.7: Frontier

Given a region R, the **frontier** of R — written as $\text{front}(R)$ — is the set of points $x \in \mathbb{R}^2$ such that every ball centered at x intersects both X and $\mathbb{R}^2 - X$.

Now that we discussed some preliminary definitions, we are ready to define **planar graphs**.

Definition 6.8: Plane graph

A **plane graph** is a pair $G = (V, E)$ such that

- $V \subseteq \mathbb{R}^2$ is a set of points, called *vertices*
- E is a set of polygonal arcs between points in V, called *edges*
- there are no pair of edges of E that share both endpoints
- for each $e \in E$ it holds that $\mathring{e}$ is disjoint from $V \cup (E - e)$

In particular, the last condition implies that plane graphs do not admits edges that intersect.

We observe that, by definition, every plane graph G is a closed set of points of $\mathbb{R}^2$ formed by the union of its vertices and its edges — implying that $\mathbb{R}^2 - G$ is open.

Definition 6.9: Face

Given a plane graph G, a **face** of G is a region of $\mathbb{R}^2 - G$. The set of faces of G is denoted as $F(G)$.

We observe that there is exactly 1 unbounded face, which is called **outer face**, while all the other faces are called **inner faces**.

Lemma 6.1: Handshaking lemma (faces)

Given a plane graph G, the sum of the number of edges seen by each face of G is at most $2\,|E(G)|$.

Definition 6.10: Planar graph

A graph G is said to be **planar** if there exists a plane graph — called *drawing of G* — isomorphic to G.

In other words, a graph is said to be *planar* if it can be represented as a plane graph.

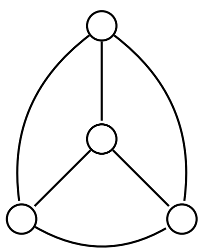

Figure 6.1: For instance, this is a *plane* representation of K_4, which implies that K_4 is *planar*.

6.1 Topological properties

Theorem 6.1: Jordan curve theorem

For every polygon $P \subseteq \mathbb{R}^2$, the set of points $\mathbb{R}^{-}P$ has exactly 2 regions, each of which has P as its frontier.

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Lemma 6.2

Let P_1, P_2 and P_3 be three polygonal arcs linking two points $x, y \in \mathbb{R}^2$ such that for each distinct $i, j \in [3]$ it holds that $\mathring{P}_i \cap \mathring{P}_j = \varnothing$. Then

1. $\mathbb{R}^2 - (P_1 \cup P_2 \cup P_3)$ has 3 regions with frontiers $P_1 \cup P_2$, $P_2 \cup P_3$ and $P_3 \cup P_1$
2. if P is a polygonal arc between a point in $\mathring{P}_1$ and $\mathring{P}_3$, and $\mathring{P}$ lies in the region of $\mathbb{R}^2 - (P_1 \cup P_3)$ that contains $\mathring{P}_2$, then $\mathring{P} \cap \mathring{P}_2 \neq \varnothing$.

TODO image

We observe that the intersection may contain more than one point, even an infinite amount.

Lemma 6.3

Let $X_1, X_2 \subseteq \mathbb{R}^2$ be two disjoint sets, each made up of a union of finitely many polygonal arcs and isolated points. Let O be a region of $\mathbb{R}^2 - (X_1 \cup X_2)$; then, for any polygonal arc $P \subseteq O$ linking a point in X_1 and a point in X_2, it holds that $O - P$ is a region of $\mathbb{R}^2 - (X_1 \cup X_2 \cup P)$, and the number of total regions does not change.

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Proposition 6.1

Let G be a plane graph; then, given a face f of G and a subplane graph $H \subseteq G$, it holds that H has a face f' containing f. Moreover, if the frontier of f lies in H, then $f = f'$.

Proof. Let $x, y \in \mathbb{R}^2$ be two points of f; then, by definition of face there is a polygonal arc linking x and y that does not intersect G. However, since $H \subseteq G$, they cannot intersect H either, which in turn implies that x and y lie in a common face of H by definition of face. Hence, there is a face f' of H that contains both x and y.

TODO □ controlla simone come l'ha scritto

Theorem 6.2

A plane forest has exactly one face.

Proof. We proceed by induction on the number of edges.

Base case. For the base case, if a plane forest G is such that $|E(G)| = 0$ then the plane graph considered is a finites set $V(G)$ of isolated points; hence, $R - V(G)$ has only one region.

Inductive hypothesis. Assume that a plane forest of k edges has exactly one face.

Inductive step. Consider a plane forest G of $k+1$ edges; then, fix an edge $e \in E(G)$, and let $G' := G - e$. Clearly, G' is still a plane forest but on k edges, which implies that has exactly one face by inductive hypothesis — call this face O. Now, say that e is a polygonal arc linking two vertices $x, y \in V(G)$; then, by definition of forest x and y must belong to different components of G'. Hence, by Lemma 6.3 we have that G still has exactly one face, namely $O - \mathring{e}$.

$\square$

Lemma 6.4

Given a plane graph G, let $e \in E(G)$ be one of its edges; then, it holds that

1. given a face f of G, either $e \subseteq \text{front}(f)$ or $\mathring{e} \cap \text{front}(f) = \varnothing$
2. if e lies on a cycle of G, e is in the frontier of exactly 2 faces of G
3. if e does not lie on a cycle of G, e is in the frontier of exactly 1 face

Proof. Fix an edge $e \in E(G)$, and a point $x_0 \in \mathring{e}$.

Claim: If e lies on a cycle of G, x_0 lies in the frontier of exactly 2 faces of G, otherwise it lies in the frontier of exactly 1 face of G.

Proof of the Claim. By definition, G is the union of finitely many SLSs and points; therefore, this implies that there exists a disc D_0 centered at x_0 $\square$

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Definition 6.11: Boundary

Given a plane graph G, and a face f of G, the **boundary** of f — denoted as $\text{bound}(f)$ — is the plane subgraph of G that forms $\text{front}(f)$.

Lemma 6.5

If G is a plane graph that has two distinct faces f_1 and f_2 such that $\mathrm{bound}(f_1) = \mathrm{bound}(f_2)$, then G is a cycle and $G = \mathrm{bound}(f_1) = \mathrm{bound}(f_2)$.

Proof. Let $H = \mathrm{bound}(f_1) = \mathrm{bound}(f_2) \subseteq G$; in particular, since $H \subseteq G$ by both statements of Proposition 6.1 f_1 and f_2 are also faces of H. In particular, this implies that H has more than one face, meaning that by the contrapositive of Theorem 6.2 H is not a plane forest, i.e. H contains a cycle C. Let F_1 and F_2 be the faces of C; since $\mathrm{bound}(f_1) = \mathrm{bound}(f_2)$ it must be the case that f_1 and f_2 lie in distinct faces of C — without loss of generality suppose that f_1 and f_2 lie in F_1 and F_2, respectively. Moreover, since H is the boundary of both f_1 and f_2, this implies that no point of H can lie inside F_1 or F_2. However, this can happen only if $H = C$, $f_1 = F_1$ and $f_2 = F_2$. Finally, by definition we have that $C \cup F_1 \cup F_2 = \mathbb{R}^2$, therefore all points of G must lie on G, concluding that $G = C$. □

Proposition 6.2

Every face boundary of a 2-connected plane graph is a cycle.

Proof. Consider a 2-connected plane graph G; we proceed by induction on n.

Base case. If G is 2-connected, then $n \geq 3$, therefore the base case is $n = 3$, for which we have that $G = K_3$ so the theorem trivially holds.

Inductive hypothesis. Assume the statement holds for 2-connected planar graphs on at most $n - 1$ vertices.

Inductive step. Suppose G has n vertices; since G is 2-connected, by Theorem 5.2 G admits an ear decomposition $C, P_1, \ldots, P_k$. If $k = 0$ then $G = C$ and the theorem trivially holds, so we may assume that $k \neq 0$; hence, consider $H = C \cup P_1 \cup \ldots \cup P_{k-1}$. We observe that H is the union of an ear decomposition, therefore by the same theorem H is 2-connected. Moreover, H cannot have more than $n-1$ vertices since P_k is an H-path, therefore we can apply the inductive hypothesis and assume that each face boundary of H is a cycle.

Now consider P_k; since each face boundary of H is a cycle we know that there must be a face f of H that contains $\mathring{P}_k$ bounded by a cycle C'. Hence, by Lemma 6.2 this implies that P_k divides f into 2 faces f' and f'' each bounded by a cycle — formed by P_k and a subpath of C'.

□

Proposition 6.3

Let G be a 3-connected plane graph, and let C be a cycle subgraph of G; then C is the boundary of a face of G is and only if C is induced and non-separating.

Proof.

Direct implication. Assume that a cycle C of a 3-connected graph G is the boundary of a face f of G.

By way of contradiction, suppose that C is not induced, which implies that there are two vertices $x, y \in V(C)$ such that $x \sim y$ in G. We observe that $\mathring{xy}$ must lie outside f, since we are assuming that $C = \text{bound}(f)$. Let $z_1, z_2 \in V(C)$ such that $x \to z_1 \to y \to z_2$ occur in this order while going around C. Let $P_1 := x \to z_1 \to y$, $P_2 := x \to z_2 \to y$; together with the edge xy, they form 3 pairwise disjoint polygonal arcs linking x and y, therefore by Lemma 6.2 we know that they form 3 regions, and this must imply that there is no path connecting z_1 and z_2 inside $G - \{x, y\}$.

To prove that C also non-separating, fix two vertices $x, y \in G - C$; since G is 3-connected, by Theorem 3.2 there are 3 internally disjoint paths between x and y — say P_1, P_2 and P_3. Hence, by Lemma 6.2 we know that they induce 3 regions. Without loss of generality, suppose that f lies inside the region bounded by $P_1 \cup P_2$; then $C \cap P_3 = \varnothing$ since f is completely inside the region formed by the other two paths, meaning that x and y are connected through P_3 in $G - C$. Therefore, we conclude that $G - C$ is still connected, i.e. C is non-separating.

Converse implication. Assume that C is induced and non-separating, and let f_1 be the outer face and f_2 be the inner face of C — considered as a subgraph of G. If both f_1 and f_2 contained a vertex of $G - C$, by removing the cycle we would disconnect such vertices (since C is induced), contradicting the fact that C is non-separating. This implies that at least one between f_1 and f_2 does not contain vertices of $G - C$; without loss of generality, suppose that f_2 does not contain vertices of $G - C$.

By way of contradiction, suppose that C is not the boundary of f_2; this implies that there is a portion of $G - C$ that lies inside f_2. We observe that f_2 does not contain vertices of $G - C$, therefore the only way for a portion of the latter to lie in f_2 is through an edge of $G - E(C)$ — linking two vertices of C. However, the existence of such an edge would contradict the fact that C is induced $\lightning$.

$\square$

Theorem 6.3: Euler's formula

If G is a connected plane graph on ℓ faces, then

$$n - m + \ell = 2$$

Proof. We proceed by induction on m.

Base case. Since G has to be connected, for the base case we have $m = n - 1$; in particular, this implies that G is a tree, and by Theorem 6.2 we know that $\ell = 1$, and in fact

$$n - m + \ell = n - (n - 1) + 1 = 2$$

Inductive hypothesis. Assume that Euler's formula holds for connected plane graphs having $m-1 \geq n$ edges.

Inductive step. Let G be a connected plane graph on $m \geq n$ edges; in particular, this implies that G has a cycle C. Fix an edge $e \in E(C)$; then, since C is a cycle we know that $G-e$ is still connected, and since it has $m-1 \geq n$ edges we can apply the inductive hypothesis, meaning that $n-(m-1)+\ell' = 2$ — where ℓ' is the number of faces of $G-e$. Moreover, since e lies in a cycle of G, by Lemma 6.4 we know that e is in the frontier of exactly 2 faces of G. Let these two faces be f_1 and f_2 in G. This implies that $f_1 \cup f_2 \cup \mathring{e}$ is a face of $G-e$, and all the other faces are unchanged in $G-e$, meaning that if $g \neq f_1, f_2$ is a face of G, then g is also a face of $G-e$. This impleis that $\ell' = \ell - 1$, therefore

$$n-(m-1)+\ell' = n-m+1+\ell-1 = 2 \iff n-m+\ell = 2$$

□

6.2 Planarity conditions

Definition 6.12: Maximally plane graph

A plane graph G is said to be **maximally plane** if no edge $e \notin E(G)$ can be added to G such that $G \cup e$ is still plane.

Definition 6.13: Triangulation

A **triangulation** is a plane graph in which every face is a triangle.

We observe that even the outer face of a triangulation is bounded by a triangle.

Unsurprisingly, every maximally plane graph *must* be a triangulation, obtained by simply partition each face into a finite number of triangles.

Proposition 6.4

A plane graph G such that $n \geq 3$ is maximally plane if and only G is a triangulation.

Proof.

Direct implication. Let G be maximally plane graph, and let f be a face of G. We observe that $n \geq 3$ meaning that $G \neq K_2$; moreover, $\mathrm{bound}(f)$ must be a complete graph, otherwise an edge could be added to G violating the maximally plane condition of G. This implies that $\mathrm{bound}(f) = K_t$. If $t = 3$ for each face f the theorem trivially holds, so we may assume that there is a face such that $t \geq 4$ — and since $K_t \subseteq K_{t+1}$, it suffices to consider the case for $\mathrm{bound}(f) = K_4$.

We observe that K_4 is 3-connected, therefore for any pair $x, y \in \text{bound}(f)$ by Theorem 3.2 there are 3 internally disjoint paths P_1, P_2 and P_3 connecting x to y. Hence, by Lemma 6.2 f lies inside one of the 3 regions induced $P_1 \cup P_2 \cup P_2$, implying that one of the three paths is not on the boundary of f, contradicting the definition of boundary.

Converse implication. Trivially, if every face is a triangle, there is no pair $x, y \in V(G)$ with $x \nsim y$ that lie common on face such that they can be connected, implying that G is maximally plane by definition.

□

Theorem 6.4

If G is a plane graph such that $n \geq 3$, then $m \leq 3n-6$. Moreover, if G is a triangulation then $m = 3n - 6$.

Proof. By the previous proposition, if G is maximally plane G is a triangulation; moreover, every plane graph is the subgraph of a maximally plane graph defined on the same vertices. Therefore, it suffices to show that every triangulation has $3n - 6$ edges.

Let G be a triangulation, and let ℓ be the number of faces of G; we observe that

- every face of G sees 3 edges, by the Handshaking lemma for the faces we have that $2m \geq 3\ell$
- every face is a triangle, hence every edge of G lies in 2 faces by Lemma 6.4.

This implies that $3\ell = 2m \iff \ell = \frac{2}{3}m$, and by Euler's formula we get that

$$n - m + \ell = n - m + \frac{2}{3}m = 2 \iff n - 2 = \frac{1}{3}m \iff m = 3n - 6$$

□

Corollary 6.1

If G is a plane graph such that $n \geq 3$, then G contains a vertex v such that $\deg(v) \leq 5$.

Proof. By way of contradiction, suppose that $\delta \geq 6$, i.e. G does not contain vertices of degree at most 5. Since G is a plane graph on at least 3 vertices, by the previous theorem we know that $m \leq 3n - 6$, but if $\delta \geq 6$ then $2m \geq 6n$ by the Handshaking lemma, which implies that $m \geq 3n$ ↯. □

In particular, the contrapositive of this theorem is enough to **certify** that a given graph is *not* planar. For instance, consider K_5; then

$$|E(K_5)| = \binom{5}{2} = \frac{5 \cdot 4}{2} = 10 > 9 = 3 \cdot 5 - 6 = 3\,|V(K_5)| - 6$$

Is this theorem enough to prove that our original graph $K_{3,3}$ is not planar? Unfortunately, we have that

$$|E(K_{3,3})| = 3 \cdot 3 = 9 \leq 12 = 3 \cdot 6 - 6 = 3 \cdot |V(K_{3,3})| - 6$$

which is not enough to prove that $K_{3,3}$ is not planar. However, consider the following proposition.

Proposition 6.5

Given a 2-connected bipartite plane graph G, it holds that $m \leq 2n - 4$.

Proof. Let ℓ be the number of faces of G; we observe that

- G is 2-connected and planar, hence by Theorem 5.2 is has an ear decomposition, therefore every edge must lie inside a cycles
- G is bipartite, hence by Theorem 1.4 G does not admit odd-length cycles, and in particular there is no C_3 inside G

Therefore, we get that every edge of G must lie inside a cycle with at least 4 edges, therefore the sum on the number of edges of each face of G is at least 4ℓ. Hence, by the Handshaking lemma on the faces we get that $2m \geq 4\ell \iff \ell \leq \frac{1}{2}m$. Lastly, by Euler's formula we conclude that

$$0 = n - m + \ell - 2 \leq n - m + \frac{1}{2}m - 2 \iff 1 \leq n - \frac{1}{2}m \iff m \leq 2m - 4$$

□

Finally, the contrapositive of this last property is enough to conclude that $K_{3,3}$ is *not* planar, because

$$|E(K_{3,3})| = 3 \cdot 3 = 9 > 8 = 2 \cdot 6 - 4 = 2 \cdot |V(K_{3,3})| - 4$$

Corollary 6.2

K_5 and $K_{3,3}$ are not planar.

6.2.1 Kuratowski's theorem

Trivially, we observe that if G has a subgraph $H \subseteq G$ such that H is not planar, then G will cannot be planar either. Moreover, now that we know that nor K_5 nor $K_{3,3}$, a trivial conclusion is that any subdivision of these two graphs will not be planar. Therefore, from these two observation we get the following proposition.

Lemma 6.6

If G contains K_5 or $K_{3,3}$ as topological minor, G is not planar.

In a landmark paper published in 1930, Kuratowski [Kur30] proved that the converse is also true, thus providing a *complete* characterization to any planar graph. But before presenting the theorem, first we need to discuss some properties of minors and topological minors.

Proposition 6.6

A graph G has H as minor if and only if there exists a family of sets $X_v \subseteq V(G)$ for each $v \in V(H)$ such that

- $\forall u, v \in V(H) \quad u \neq v \implies X_u \cap X_v = \varnothing$
- for each $u \in V(H)$ it holds that $G[X_u]$ is connected
- for each $uv \in E(H)$ there is an edge $e_{uv} \in E(G)$ with one endpoint in X_u and the other in X_v

Proof.

Direct implication. If G has H as minor, there is a sequence of contractions and vertex or edge deletions that yields H starting from G. Let H' be the following subgraph of G

- $V(H')$ contains the vertices of G that were not removed to obtain H from G, together with the endpoints of the edges that were contracted to obtain H from G
- $E(G)$ contains the edges of G that were contracted to obtain H from G

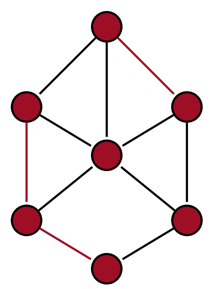


Figure 6.2: For instance, this graph has K_4 as minor, and we are considering as H' the subgraph colored in *red*.

We observe that, by construction, the number of connected components of H' is precisely $|V(H)|$; therefore, we can well-define $\{X_v \mid v \in V(H)\}$ where X_v is a component of H'. Hence, given such family of subsets we observe that

- for any pair of distinct u and v in $V(H)$ it holds that $X_u \cap X_v = \varnothing$ by construction of H'
- the edges of H obtained from G are edges e_{uv} connecting the connected components of H' which correspond to X_u and X_v, by construction of H'

Converse implication. For each $v \in V(H)$, fix a spanning tree T_v on $G[X_v]$, and for each $uv \in E(H)$ fix an edge $e_{uv} \in E(G)$; now, to obtain H we can simply

- remove all the edges *not* in $\bigcup_{v \in V(H)} E(T_v) \cup \bigcup_{uv \in E(H)} e_{uv}$ — i.e. the edges in each $G[X_v]$ outside the spanning tree T_v for each $v \in V(H)$, and the edges that connect the various sets that are not the edges e_{uv} that we fixed
- contract all the edges of T_v, for each $v \in V(H)$ — we observe that this contraction operation will leave one single vertex by connectivity of each tree T_v

□

In Section 4.1.3 we stated that if G has H as topological minor, then H is a minor of G. Moreover, we also provided an example to show that the converse does not always hold — i.e. it is not true in general that if G has H as minor, then H is topological minor of G as well. However, by adding an additional assumption on G we can get the converse implication, as stated below.

Proposition 6.7

If G contains H as minor, and $\Delta(H) = 3$, then G contains H as topological minor.

Proof. Suppose that G contains H as minor, which implies that there exists a collection $\{X_v \mid v \in V(H)\}$ that satisfies Proposition 6.6. Now, for each pair of distinct u and v of $V(H)$ fix an edge $e_{uv} \in E(G)$; since $\Delta(H) = 3$, for any set X_z there can be at most 3 edges of the form $e_{zv_i} \in E(G)$ for $1 \le i \le 3$. Now, if there are at most such 2 edges for all X_z, then H is a topological minor of G, so we may assume there is at least one set $X_{\hat{z}}$ which has 3 such edges.

Fix a spanning tree $T_{\hat{z}}$ of $G[X_{\hat{z}}]$; since there are three edges $e_{\hat{z}v_1}, e_{\hat{z}v_2}, e_{\hat{z}v_3}$ and $T_{\hat{z}}$ is connected, there must be a vertex r of degree 3 inside $T_{\hat{z}}$ such that there are 3 paths that start from r and end in the endpoints of such edges that lie insize $X_{\hat{z}}$. Therefore, by fixing a spanning tree T_u for each $G[X_u]$, we get that that $\bigcup_{u \in V(H)} T_u \cup \bigcup_{uv \in E(H)} e_{uv}$ is a subdivision of H inside G, i.e. H is a topological minor of G. □

We can leverage this property to prove the following characterization about the special cases of K_5 and $K_{3,3}$.

Lemma 6.7

A graph G has K_5 or $K_{3,3}$ as topological minor if and only if it has K_5 or $K_{3,3}$ as minor.

Proof. Since being a topological minor implies being a minor, the direct implication of the statement is trivial. Moreover, since $\Delta(K_{3,3}) = 3$, if G contains $K_{3,3}$ as minor, then it also contains $K_{3,3}$ as topological minor by the previous proposition. Therefore, to prove the statement it suffices to show that if K_5 is a minor of G, then K_5 or $K_{3,3}$ are topological minors of G.

Thus, suppose that G contains K_5 as minor, i.e. there are sets $\{X_v \mid v \in V(K_5)\}$ that satisfy Proposition 6.6. TODO $\square$ da finire

Finally, we are ready to discuss Kuratowski's theorem.

Theorem 6.5: Kuratowski's theorem

A graph G is planar if and only if G does not contain K_5 nor $K_{3,3}$ as topological minors.

Proof. The observation discussed in Lemma 6.6 trivially proves the direct implication of the theorem. Moreover, the previous lemma implies that the statement is equivalent of proving that G is planar if and only if G does not contain K_5 nor $K_{3,3}$ as minor. We are going to prove the converse implication of this statement only for the case in which G is 3-connected — the theorem can be extended to any graph G.

We proceed by induction on n. Since G is 3-connected, the base case is for $n = 4$, which implies that $G = K_4$, which is indeed planar — as shown in Figure 6.1.

Assume that if a 3-connected graph G on $n-1$ vertices does not contain K_5 nor $K_{3,3}$ as minors, then G is not planar. Now, consider a 3-connected graph G on $n \geq 5$ vertices that does not contain K_5 nor $K_{3,3}$ as minors. Since G is 3-connected, by Theorem 5.3 there exists an edge $xy \in E(G)$ such that G/xy is still 3-connected; moreover, we know that G/xy cannot contain K_5 nor $K_{3,3}$ as minors, otherwise they would be minors of G as well since G/xy is obtained from G by contracting an edge, which would contradict the definition of G. Furthermore, G/xy has $n-1$ vertices, which implies that we can apply the inductive hypothesis on it, thus G/xy is planar.

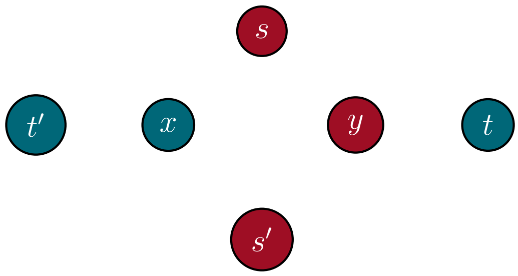
Let $v_{xy} \in V(G/xy)$ be the vertex obtained by contracting the edge xy, and consider $G/xy - \{v_{xy}\}$: since G/xy is 3-connected, such graph is at least still 2-connected — and surely still planar — hence by Proposition 6.2 we have that every face of such graph is bounded by a cycle. This implies that there must exists a plane graph isomorphic to $G/xy - v_{xy}$ that has a face f bounded by a cycle C such that $\mathcal{N}(v_{xy})$ lies completely on C. Let $\mathcal{N}(v_{xy}) = \{v_1, \ldots, v_k\}$; then, by definition of contraction, each $v_i \in \mathcal{N}(v_{xy})$ is adjecent to at least one between x and y. Now we are going to "un-contract" the edge xy and consider how G could be structured.

Claim 1: x and y do not have 3 common neighbors.

Proof of the Claim. By way of contradiction suppose there are v_1, v_2 and v_3 such that $v_1, v_2, v_3 \sim x, y$; then, by contracting C down to a triangle — which can always be done since C is a cycle — we get that G contains K_5 as minor, contradicting the definition of G $\lightning$. $\square$

Claim 2: There cannot exist vertices s, s', t and t' on C such that $x \sim s, s'$ and $y \sim t, t'$ and $s\ t\ s'\ t'$ occur on C on that cyclic order.

Proof of the Claim. If such vertices exist then G contains $K_{3,3}$ as minor, contradicting its definition $\lightning$



$\square$

TODO

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Claim 3: x and y have at least two neighbors on C.

Proof of the Claim. Since G is 3-connected, then $\deg(x), \deg(y) \geq 3$ by Proposition 3.4, which implies that they must have at least two neighbors on C — since all of their neighbors lie on C and they are one the neighbor of the other. $\square$

Claim 4: If there exist two internally disjoint paths P_1 and P_2 on C such that $\mathcal{N}(x) - \{y\} \subseteq V(P_1)$ and $\mathcal{N}(y) - \{x\} \subseteq P_2$, then G is planar.

Proof of the Claim. Trivially, such paths imply that there is always a drawing of G which allows to connect x to $\mathcal{N}(x) - \{y\}$ and y to $\mathcal{N}(y) - \{x\}$ without drawing overlapping edges — it may be necessary to "swap" the vertices x and y to obtain such drawing. $\square$

Claim 5: If x or y have exactly 2 neighbors on C, then G is planar.

Proof of the Claim. Without loss of generality, suppose that x has exactly two neighbors $v_1, v_2 \in V(C)$ — i.e. $\mathcal{N}(x) - \{y\} = \{v_1, v_2\} \subseteq V(C)$; we observe that v_1 and v_2 split C into two internally disjoint paths P_1 and P_2 having v_1 and v_2 as endpoints such that $C = P_1 \cup P_2$.

Again, consider $G[C \cup \{x\}]$: since $x \sim v_1, v_2$, such subplane graph has 3 faces — the outer face and two inner faces; without loss of generality, suppose that y lies in the face formed by P_2 and $v_1 \to x \to v_2$. Now, if $\mathcal{N}(y) - \{x\} \subseteq V(P_2)$, then by Claim 4 G is planar, so we may assume that there is at least one internal vertex of P_1 that is a neighbor of y. Then, by Claim 3 we know that y has at least another neighbor on C, and we know that such neighbor cannot be an internal vertex of P_2 otherwise we would contradict Claim 2. This implies that $\mathcal{N}(y) - \{x\} \subseteq P_1$, which means that G is planar by Claim 4. $\square$

Therefore,by this claim we may assume that both x and y have at least 3 neighbors on C. Moreover, by Claim 1 x and y cannot share 3 neighbors, implying that there exists at least one vertex $v \in V(C)$ such that $x \nsim v$ and $y \sim v$ without loss of generality. Let P_1 be the shortest path of $C - \{v\}$ that contains all the neighbors of x on C — and in particular, its endpoints v_i and v_j are both neighbors of x. Moreover, since $y \sim v$, if y had a neighbor z internal to P_1 then v, z, v_i and v_j would contradict Claim 2.

Lastly, let P_2 be such that $C = P_1 \cup P_2$ and P_1 and P_2 are internally disjoint; the last observation implies that $\mathcal{N}(y) - \{x\}$ lies completely on P_2, therefore G is planar by Claim 4. □

6.3 Exercises

Problem 6.1

Find an Euler formula for disconnected graphs.

Solution. We claim that if G is plane on ℓ faces and k connected components, then

$$n - m + \ell = k + 1$$

We prove our claim by induction on k.

If $k = 1$ then G is connected, and $n - m + \ell = 1 + 1 = 2$ which is true by Euler's formula.

Now, consider a graph G that has at least 2 components (i.e. $k \geq 2$), and assume our equation is true for a plane graph on $k - 1$ components. Fix a components C_i of G; then $G - C_i$ has $k - 1$ components, and it is plane since G is plane, therefore by induction we have that

$$|V(G - C_i)| - |E(G - C_i)| + |F(G - C_i)| = (k - 1) + 1 = k$$

where $F(H)$ is the number of faces of the graph H. Then, since C_i and $G - C_i$ have 1 face in common — i.e. the outer face — we have that

$$\begin{aligned} |V(G)| - |E(G)| + |F(G)| = {} & |V(G - C_i)| + |V(C_i)| - (|E(G - C_i)| + |E(C_i)|) + \\ & |F(G - C_i)| - 1 + |F(C_i)| \\ = {} & k + 2 - 1 \\ = {} & k + 1 \end{aligned}$$

□

7

Graph coloring

TODO introduction

Definition 7.1: Proper coloring

Given a graph G, a **proper coloring** of G is a function $c : V(G) \to \mathbb{N}$ that assigns a natural number to each vertex of G such that for any pair $x, y \in V(G)$ if $x \nsim y$ then $c(x) \neq c(y)$.

TODO example

We say that a graph is k-colorable if k colors suffice to provide a proper coloring of such graph. We observe that such a function always exists, i.e. the **trivial coloring**: it suffices to color each vertex with a different color. Therefore, we are interested in finding a proper coloring of a graph that *minimizes* the number of colors used.

Definition 7.2: Chromatic number

Given a graph G, the chromatic number $\chi(G)$ of G is the minimum number of colors such that there exists a proper coloring of G on $\chi(G)$ colors.

TODO example

This implies that if G is k-colorable, then $\chi(G) \leq k$ — since k-colors suffice to properly color G.

Proposition 7.1

Given a graph G, then G is bipartite if and only if $\chi(G) = 2$.

Proof. If G is bipartite through (A, B), then construct a coloring such that

$$\forall x \in V(G) \quad c(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \end{cases}$$

and since A and B are independent sets, this must be a proper coloring of G.

Viceversa, if $\chi(G) = 2$, then there is a proper coloring c of G that uses 2 colors — without loss of generality suppose that $c(x)$ is either 0 or 1 for all $x \in V(G)$. Hence, by definition of proper coloring the vertices with the same color cannot be adjacent, implying that G is bipartitioned through the vertices that are assigned 0 and the vertices that are assigned 1 by c. This implies that G is bipartite. □

Given that this fairly trivial proposition provides a complete characterization on 2-colorable graphs, how hard is colorability in general? Well, as proved by Karp [Kar72], 3-colorability is already NP-hard, in fact k-colorability is NP-hard for any $k \geq 2$. This implies that the best we can possibly achieve are bounds on the chromatic number.

7.1 Bounds on the chromatic number

7.1.1 Upper bounds

Thus, let us start with the basics: aside from the trivial coloring, which implies that $\chi(G) \leq |V(G)|$ always holds, is there a tighter *upper bound* for $\chi(G)$?

Proposition 7.2

Given a graph G, it holds that $\chi(G) \leq \Delta(G) + 1$.

Proof. We proceed by induction on n; for the base case, if $n = 1$ then clearly $\Delta(G) = 0$ and $1 = \chi(G) \leq 0 + 1$. Now, assume the claim for a graph on $n - 1$ nodes and consider a graph G on n nodes.

Fix a vertex $v \in V(G)$ and consider $G - \{v\}$; since we removed a vertex from G, the maximum degree of $G - \{v\}$ is at most the maximum degree of G itself, thus $\Delta(G - \{v\}) \leq \Delta(G)$. Moreover, since $G - \{v\}$ has $n - 1$ vertices, we know by induction that $\chi(G - \{v\}) \leq \Delta(G - \{v\}) + 1$. This implies that

$$\chi(G - \{v\}) \leq \Delta(G - \{v\}) + 1 \leq \Delta(G) + 1$$

which means that $G - \{v\}$ can be colored with at most $\Delta(G) + 1$ colors. Now, v has at most $\Delta(G)$, therefore $\mathcal{N}(v)$ requires at most $\Delta(G)$ colors, which means we are left with one more color to be used for v itself, thus the whole graph G can be colored with at most $\Delta(G) + 1$ colors. □

Can we do better if the our graph has some specific properties? For instance, consider the case in which G is a *planar graph*.

Proposition 7.3

Given a planar graph G, it holds that $\chi(G) \leq 6$.

Proof. We proceed by induction on n; for the base case, if $n = 1$ we know that $1 = \chi(G) \leq 6$, and if $n = 2$ then

- if the graph are two isolated vertices, then $1 = \chi(G) \leq 6$
- if the graph is K_2 then $2 = \chi(G) \leq 6$

Now, assume the claim for a graph on $n-1$ nodes where $n-1$ vertices, and consider a graph G on $n \geq 3$ nodes.

Since $n \geq 3$, by Corollary 6.1 we know that G contains a vertex of degree at most 5, therefore fix such a vertex $v \in V(G)$. We observe that if G is planar, then clearly $G - \{v\}$ is still planar, which means that by induction $\chi(G - \{v\}) \leq 6$. Lastly, since $\mathcal{N}(v)$ can use at most 5 different colors, we can still use one additional color to color v itself and still be able to 6-color the whole graph G. □

Is this the best result we can achieve through planarity? This upper bound on planar graphs can be actually improved thorugh the following proposition.

Proposition 7.4

Given a planar graph G, it holds that $\chi(G) \leq 5$.

Proof. We proceed by induction on n; since $\chi(G) \leq n$, if $n \leq 5$ the statement trivially holds, therefore assume the proposition for a graph on $n-1$ vertices and consider a graph of $n \geq 6$ nodes.

By Corollary 6.1 we know that G contains a vertex of degree at most 5; fix such vertex u, and inductively assume that $G - \{u\}$ is 5-colorable. If u can be colored with one of the colors we already used for $G - \{u\}$ the statement is trivially true. Hence, by way of contradiction suppose that G is not 5-colorable. We observe that the only way for G not to be 5-colorable is that $\deg(u) = 5$ and u sees every color used to 5-color $G - \{u\}$. Thus, fix a *cyclical* ordering $v_1, \ldots, v_5$ on the neighbors of u and suppose that each v_i gets color i on the 5-coloring of $G - \{u\}$ — such a coloring can be always constructed by permuting the colors if needed.

TODO add drawing of u

Now, let H be the subgraph of G induced from the set of vertices that have color 2 and 4 in $G - \{u\}$; clearly, since $\chi(H) = 2$, it must be that G is bipartite by Proposition 7.1. Let C be che component of H that contains v_2; we observe that inside C colors 2 and 4 can be swapped and still get a valid 5-coloring of $G - \{u\}$. Therefore, if $v_4 \notin C$ then by swapping 2 and 4 inside C we would get a 5-coloring of $G - \{u\}$ in which u would see both v_2 and v_4 colored with 4, meaning that u could be colored with 2 contradicting the fact that G was not 5-colorable. This implies that $v_4 \in C$, i.e. there is a path P inside C that connectes v_2 and v_4.

Now, let H' be the subgraph of G induced from the set of vertices that have color 3 and 5 in $G - \{u\}$, and let C' be the component of H' that contains v_3. Clearly C' must be disjoint from P, thus by choice of the *cyclical* order on the vertices v_5 cannot be contained inside C' because G is planar. This implies that colors 3 and 5 can be swapped inside C' such that both v_3 and v_5 get colored with 5, meaning that u can be colored with color 3, again contradicting the fact that G was not 5-colorable $\lightning$. □

Is *this* the best we can achieve through planarity then? In 1977 Appel and Haken [AH77] published the first *computer assisted proof* in mathematics, through which the notorious **Four Color Theorem** was proved.

Theorem 7.1: Four Colors theorem

Given a planar graph G, it holds that $\chi(G) \leq 4$.

7.1.2 Lower bounds

So far, we only discussed *upper bounds* on the chromatic number, but what about *lower bounds*? Consider the following definitions.

Definition 7.3: Independence number

Given a graph G, the **independence number** $\alpha(G)$ is the size of the largest independent set of G.

TODO example

The independence number provides a trivial lower bound on the chromatic number. In fact, if each independent set of a graph G has size exactly $\alpha(G)$, then each coloring of G will require at least k colors, where k is the number of independent sets of G, which is going to be $\left\lceil \frac{n}{\alpha(G)} \right\rceil$.

Proposition 7.5

Given a graph, it holds that $\chi(G) \geq \left\lceil \frac{n}{\alpha(G)} \right\rceil$.

Definition 7.4: Clique number

Given a graph G, the **clique number** $\omega(G)$ is the size of the largest clique subgraph of G.

TODO example

Since a t-clique always requires t-colors to be properly colored, the clique number *lower bounds* the chromatic number of a graph, because each graph containing a t-clique must require at least t colors to properly color its t-clique subgraph.

Proposition 7.6

Given a graph G, it holds that $\chi(G) \geq \omega(G)$.

However, $\omega(G)$ can be significantly lower than $\chi(G)$, in fact the next theorem that we are going to present proves that thera exists a family of graphs G_k for any $k \geq 1$ that are k-colorable but such that $\omega(G_k) = 2$. I nparticular, the theorem was first proven by Tutte (under the pseudonym of B. Descartes) [Des47], followed by easier constructions achieved by other researchers. The construction that we'll present is due to Erdős.

Definition 7.5: Color class

Given a graph G, and a proper coloring $c : V(G) \to [k]$ of G, for each $i \in [k]$ the **color class of** i is the set of vertices

$$\mathcal{C}(i) := \{v \in V(G) \mid c(v) = i\}$$

We observe that, by definition of proper coloring, for each $i \in [k]$ it holds that $G[\mathcal{C}(i)]$ is an independent set.

Proposition 7.7

Let G be a graph such that $\chi(G) = k$, and let $c : V(G) \to [k]$ be a proper coloring of G. Then for each $i \in [k]$ there exists $x \in \mathcal{C}(i)$ such that $\mathcal{N}(x)$ contains a vertex of every color except i.

Proof. By way of contradiction, suppose that there exists $i \in [k]$ such that for each $x \in \mathcal{C}(i)$ it holds that $\mathcal{N}(x)$ is missing at least one color in $[k] - \{i\}$; then, it is possible to recolor each $x \in \mathcal{C}(i)$ with the missing color in $\mathcal{N}(x)$ different from i itself, which implies that $\mathcal{C}(i)$ can be completely properly recolored without using i. Moreover, $G[\mathcal{C}(i)]$ is still properly colored because it is an independent set. This means that G is $(k-1)$-colorable, i.e. $\chi(G) \leq k - 1$, contradicting the assumption on $\chi(G)$ $\lightning$. □

Theorem 7.2

There exists a family of graphs $G_1, G_2, \ldots$ such that for each G_k it holds that $\chi(G_k) = k$ and $\omega(G_k) = 2$.

Proof. Let G_1 be a single vertex, and G_2 be a single edge, and consider a graph of such sequence G_k; we construct G_{k+1} inductively as follows:

- start with $V(G_{k+1}) = V(G_k)$ and $E(G_{k+1}) = E(G_k)$

- let $v_1, \ldots, v_n$ be the vertices of G_k
- add vertices $v'_1, \ldots, v'_n, u \in V(G_{k+1})$
- for each $\{v_i, v_j\} \in E(G_k)$ add an edge $\{v'_i, v_j\} \in E(G_{k+1})$
- for each $v'_i \in V(G_{k+1})$ add an edge $\{u, v'_i\} \in E(G_{k+1})$

TODO drawing

Trivially, we see that $\chi(G_1) = 1 = \omega(G_1)$ and $\chi(G_2) = 2 = \omega(G_2)$. Now, assume the property holds for G_k and consider the graph G_{k+1}.

Claim: $\omega(G_{k+1}) = 2$.

Proof of the Claim. By way of contradiction, suppose that $\omega(G_{k+1}) \geq 3$, i.e. G_{k+1} contains at least a K_3 as subgraph. Then, by construction of G_{k+1} we observe that there is no edge of the form $\{v'_i, v'_j\}$ in $E(G_{k+1})$, which implies that the K_3 subgraph of G_{k+1}

- cannot have u as one of its three vertices
- cannot contain two vertices of the form v'_i, v'_j

Moreover, since $\omega(G_k) = 2$ by induction, G_k does not contain any K_3 as subgraph, therefore the K_3 in G_{k+1} must have one vertex of the form v'_i, and the the other two vertices must be some v_j, v_k. However, if $v'_i \sim v_j, v_k$ then, by definition of G_{k+1}, this would imply that $v_i \sim v_j, v_k$ meaning that G_k contained a K_3 subgraph $\lightning$. □

Claim: $\chi(G_{k+1}) = k + 1$.

Proof of the Claim. By induction, we know that $\chi(G_k) = k$, and consider a proper coloring c of G_k; since for each $i \in [n]$ it holds that $v_i \nsim v'_i$, we can color v'_i with $c(v_i)$ itself, and then u will require one additional color to be properly colored. This proves that $\chi(G) \leq k + 1$.

Now, by way of contradiction suppose that there exists a proper coloring of G_{k+1} that requires k colors. By induction, we know that $\chi(G_k) = k$, thus fix a color $\ell \in [k]$ of a proper coloring of G_k. By the previous proposition we know that there exists $v_i \in \mathcal{C}(\ell)$ such that $\mathcal{N}(v_i)$ contains a vertex of every color except ℓ. This implies that in G_{k+1} we have that $\mathcal{N}(v'_i)$ contains a vertex of every color except ℓ, meaning that v'_i *must* be colored with ℓ. However, by repeating the same argument with every color class we get that each v'_i will be colored differently, meaning that u cannot be colored with any of the k available colors because of the edges $\{u, v'_i\}$ for each v'_i — we observe that $n \geq k$ always holds. □

□

TODO example of G4

7.2 Perfect graphs

Let's look deeper into the relationship between $\chi(G)$ and $\omega(G)$: for which graphs it holds that $\chi(G) = \omega(G)$? A rather uninteresting answer to this question would be any graph H on n vertices, together with a K_n disconnected from H itself. In fact, with this construction we would have that

$$n = \omega(G) \leq \chi(G) \leq n \implies \omega(G) = \chi(G)$$

This means that asking for a graph G such that its chromatic and clique number coincide does not provide interesting answers, hence we need a *strictier* definition.

Definition 7.6: Perfect graph

A graph G is said to be **perfect** if for each induced subgraph G' of G it holds that $\chi(G') = \omega(G')$.

What are examples of perfect graphs? A trivial class of perfect graphs are *cliques*, since each induced subgraph of a clique is still a clique.

Proposition 7.8

Cliques are perfect.

Another example of perfect graphs is given by *bipartite graphs*.

Proposition 7.9

Bipartite graphs are perfect.

Proof. First, consider the following property of bipartite graphs.

Claim: If G is bipartite, then $\chi(G) = \omega(G)$.

Proof of the Claim. Since G does not contain odd-length cycles by Theorem 1.4, it holds that $\omega(G) = 2$, and by Proposition 7.1 we know that $\chi(G) = 2$, thus $\chi(G) = 2 = \omega(G)$. □

Now, since each induced subgraph of a bipartite graph is still bipartite, from this claim we have that for any induced subgraph G' of a bipartite graph G it holds that $\chi(G') = 2 = \omega(G')$, thus G is perfect. □

Are there any other families of perfect graphs?

Definition 7.7: Chordal graph

A graph G is said to be **chordal** if every cycle C of G such that $|V(C)| \geq 4$ has a chord.

TODO example

A trivial example of chordal graphs are *cliques*.

Proposition 7.10

Chordal graphs are perfect.

Proof. First, consider the following property of chordal graphs.

Claim: Every induced subgraph of a chordal graph is chordal.

Proof of the Claim. Let G be a chordal graph, and G' be an induced subgraph of G; if G' does not contain any cycles of length at least 4, then G' is trivially chordal, so we may assume that there is at least one such cycle inside G'. Then, since G is chordal such cycle must contain a chord in G, and because G' is induced such chord must be contained inside G' as well. □

This implies that to prove the statement it suffices to show that if G is a chordal graph then $\chi(G) = \omega(G)$. We are going to prove this by induction on n.

For the base case, we have that $n = 1$, therefore the graph is an isolated vertex which is vacuously chordal and trivially perfect. Now, assume the statement for chordal graphs having $n - 1$ vertices, and consider a chordal graph G on n vertices.

If G is a clique, then G is both chordal and perfect, so we may assume that G is a graph on n vertices that is not a clique, i.e. there is a pair of vertices $x, y \in V(G)$ such that $x \nsim y$. Let (X, Y) be a separation of G of minimal order such that $x \in X - Y$ and $y \in Y - X$ — we observe that such a separation always exists, in fact in the "worst case" we have that $X = V(G) - \{y\}$ and $Y = V(G) - \{x\}$. Moreover, let C_x be the component of $G[X - Y]$ containing x, and similarly let C_y be the component of $G[Y - X]$ containing y.

Claim: $X \cap Y \subseteq \mathcal{N}(C_x)$. is this correct?

Proof of the Claim. TODO da finire □

□

Another family of known perfect graphs is the following.

Definition 7.8: Interval graph

A graph G is called **interval graph** if for each verterx $x \in V(G)$ there is an interval $[a_x, b_x]$ such that

$$\forall u, v \in V(G) \quad u \sim v \iff [a_u, b_u] \cap [a_v, b_v] \neq \varnothing$$

TODO example

We are going to prove that interval graphs are perfect by proving the following proposition.

Proposition 7.11

Interval graphs are chordal.

Proof. Let G be an interval graph, and way of contradiction suppose that G contains an induced cycle of length at least 4 that does not contain a chord. Let $v_1, \ldots, v_k$ be the vertices of such induced cycle and let $I_1, \ldots, I_k$ be the corresponding intervals. Fix I_i to be the interval that minimizes b_i, and let v_j and v_k the be neighbors of v_i on the cycle. Thus, by definition of I_i it holds that $b_i \leq b_j$, but since $v_i \sim v_j \iff [a_i, b_i] \cap [a_j, b_j] \neq \varnothing$ it must hold that $a_j \leq b_i$; this implies that $b_i \in I_j$, and by the same argument we have that $b_i \in I_k$ must hold as well. Therefore $b_i \in I_j \cap I_k \implies I_j \cap I_k \neq \varnothing \iff v_j \sim v_k$, and since v_j and v_k are not adjacent on the cycle, this implies that $\{v_j, v_k\}$ is a chord of the induced cycle $\lightning$. □

Corollary 7.1

Interval graphs are perfect.

Proof. Follows immediately from Proposition 7.10 and the previous proposition. □

7.2.1 Perfect graph theorems

So far we only provided examples of perfect graphs, but what about graphs that are *not perfect*? A trivial example is given by **odd-length cycles** of length at least 5: such graphs have clique number 2 and chromatic number 3 — by Proposition 7.1 — from which we derive the following proposition.

Proposition 7.12

Any graph that contains an induced odd-length cycle subgraph of length at least 5 is not perfect.

In reality, odd-length cycles are a very special type of graph. For instance, consider $\overline{C_7}$:

by definition we know that $\alpha(\overline{C_7}) = 2$, therefore by Proposition 7.5 we have that

$$\chi(\overline{C_7}) \geq \left\lceil \frac{n}{\alpha(\overline{C_7})} \right\rceil = \left\lceil \frac{7}{2} \right\rceil = 4$$

Moreover every clique contained inside this graph consists of non-sequential vertices on the cycle, meaning that the biggest size for a clique is 3 — to get a K_4 we would need at least 8 vertices since sequential vertices on the cycle are not adjacent. This means that $\omega(\overline{C_7}) = 3$, therefore $\overline{C_7}$ is not perfect either. By generalizing this idea on any odd-cycle of length at least 5, we get the following property.

Proposition 7.13

For any $k \geq 2$ it holds that $\chi(\overline{C_{2k+1}}) \geq \lceil \frac{2k+1}{2} \rceil = k + 1$ and $\omega(\overline{C_{2k+1}}) = k$.

Corollary 7.2

If a graph G has $\overline{C_{2k+1}}$ as induced subgraph, for some $k \geq 2$, then G is not perfect.

TODO buco lore

Consider a graph G, and a vertex $v \in V(G)$; to **expand** v means to add another vertex v' to $V(G)$ and add edges $\{v', x\}$ to $E(G)$ for each $x \in \mathcal{N}(v) \cup \{v\}$.

TODO example + intro?

Lemma 7.1

Given a perfect graph G, and a vertex $v \in V(G)$, it holds that G' obtained by expanding v is perfect.

Proof. We proceed by strong induction on n.

Base case. For the base case, consider a graph on having 1 vertex, and expand such vertex: the resulting graph is clearly K_2, which is trivially perfect since its a clique.

Strong inductive hypothesis. Assume that the statement holds for any graph G' on at most $n - 1$ vertices.

Inductive step. Consider a perfect graph G on $n - 1$ vertices, and expand one of its vertices $v \in V(G)$ into v' to obtain G' on n vertices. Now, fix an induced subgraph H of G; then

- if $v' \notin V(H)$ then $H \subseteq G$ therefore $\chi(H) = \omega(H)$ by perfection of G
- if $v' \in V(H)$ and $v \notin V(H)$ then H is isomorphic to an induced subgraph of G, hence $\chi(G) = \omega(H)$ by perfection of G

so we may assume that $v, v' \in V(H)$. Moreover, if $|V(H)| < |V(G')|$ then $H - \{v'\}$ is a *proper* induced subgraph of G, which implies that $|V(H - \{v'\})| < |V(H)|$.

Furthermore, since H can be equivalently constructed by expanding v' into v from $H - \{v'\}$, it means that we can apply the inductive hypothesis on $H - \{v'\}$ and therefore H is perfect, and in particular $\chi(H) = \omega(H)$. Therefore, we may assume that $|V(H)| = |V(G')|$, i.e. $H = G'$, and we need to prove that $\chi(G') = \chi(H) = \omega(H) = \omega(G')$.

In general, we observe that $\omega(G') \leq \omega(G) + 1$ since G and G' differ by one vertex, and $\omega(G') = \omega(G) + 1$ when there is a maximum-size clique of G' containing v' — which implies that it also must contain v since v' is an extension of v and it is fully connected to $\mathcal{N}(v) \cup \{v\}$. However, since G can be colored with $\omega(G)$ colors by perfection, if the equality holds we can use one more color for v' itself in order to properly color G', and this coloring uses $\chi(G) + 1 = \omega(G) + 1$ colors hence

$$\chi(G') = \chi(G) + 1 = \omega(G) + 1 = \omega(G')$$

Therefore, we may assume that $\omega(G') < \omega(G) + 1 \implies \omega(G') = \omega(G)$ — since $\omega(G') \geq \omega(G)$ must hold — which implies that v is not contained in any maximum-size clique of G. Let c be a proper $\omega(G)$-coloring of G — which exists by perfection of G — and consider the set $X := \{x \in V(G) \mid c(v) = c(x)\}$ (and in particular $v \in X$).

Claim: $\omega(G - (X - \{v\})) = \omega(G) - 1$.

Proof of the Claim. Fix an $\omega(G)$-clique K — a maximum-size clique of G; since K is a clique, each vertex of K must be assigned a different color from c and since it has maximum size it must use all the colors available, therefore $|K \cap X| = 1$. However, since we are assuming that v is not contained in any $\omega(G)$-clique, and in particular $v \notin V(K)$, it must hold that $|K \cap X| = 1 \implies |K \cap (X - \{v\})| = 1$. This implies that $X - \{v\}$ intersects all the $\omega(G)$-cliques of G, therefore in $G - (X - \{v\})$ we removed *exactly* one vertex per maximum-size clique, which implies that

$$\omega(G - (X - \{v\})) = \omega(G) - 1$$

□

Since $G - (X - \{v\})$ is an induced subgraph of G, and G is perfect, then $G - (X - \{v\})$ is perfect, therefore it can be $(\omega(G) - 1)$-colored by the claim. Finally, observe that $\mathcal{N}(v) \cap X = \varnothing \implies \mathcal{N}(v') \cap X = \varnothing$, which implies that $(X - \{v\}) \cup \{v'\}$ can be colored with one additional color — surely different from v since $v \sim v'$ — in order to properly $\omega(G')$-color G' entirely.

□

TODO

lore fulkerson

Theorem 7.3: Weak perfect graph theorem

A graph G is perfect if and only if $\overline{G}$ is perfect.

Proof. TODO □

da fare

TODO

interlude?

Theorem 7.4: Strong perfect graph theorem

A graph G is perfect if and only if G has no induced C_{2k+1} or $\overline{C_{2k+1}}$ subgraph, for any $k \geq 2$.

Corollary 7.2 is the direct implication of this theorem. We observe that this theorem is indeed stronger than the Weak perfect graph theorem: in fact, this theorem implies that if G is perfect then it does not contain C_{2k+1} or $\overline{C_{2k+1}}$ as induced subgraph, which means that $\overline{G}$ cannot contain such induced subgraph either, therefore $\overline{G}$ is perfect. Hence, by duality, we get that the Weak perfect graph theorem can be derived from the Strong perfect graph theorem.

7.3 Exercises

Problem 7.1

A graph G is said to be k-critical if

- $\chi(G) = k$
- $\forall v \in V(G) \quad \chi(G - \{v\}) \leq k - 1$

Give all the 3-critical graphs.

Solution. Consider a cycle graph C_{2h+1} of odd-length; we observe that 3 colors always suffice to color such a graph, since we can alternate between two different colors for $2h$ vertices until the last vertex, which will be assigned a third color. Moreover, if we remove any vertex v from such cycle, we see that the graph becomes a tree, and since trees are bipartite graph by Proposition 7.1 we have that $\chi(C_{2h+1} - \{v\}) = 2 \leq 3 - 1$. This proves that any odd-length cycle graph C_{2h+1} is 3-critical.

Now, consider a graph G which is *not* a cycle graph of odd-length; if G has no cycles or only even-length cycles then G is bipartite by Theorem 1.4, therefore by Proposition 7.1 $\chi(G) = 2$ implying that G is not 3-critical. Hence, we may assume that G has at least one odd-length cycle C, and since we are assuming that $G \neq C$, G may be such that

- it contains at least one vertex outside C, or
- it contains at least one edge outside C

In the first case, let v such vertex; then $G - \{v\}$ contains C as subgraph, and since it is an odd-length cycle it requires at least 3 colors to be proper colored, meaning that $\chi(G - \{v\}) \geq 3$ implying that G is not 3-critical. Therefore, we may assume that $V(C) =$

$V(G)$, but since $G \neq C$ there must be a chord on C — which also implies that $C \neq C_3$, i.e. $|V(G)| \geq 5$.

Then, such chord must split C into two subcycles C' and C'' such that $G = C' \cup C''$, and one of which must have odd length — without loss of generality suppose that C' is the odd-length subcycle. This means that by choosing any vertex $v \in V(C'') - V(C')$ — which exists because $|V(G)| \geq 5$ — we get a graph $G - \{v\}$ that still contains an odd-length cycle, meaning that $\chi(G - \{v\}) \geq 3$ implying that G is not 3-critical. This proves that if G is not an odd-length cycle then it is not 3-critical. $\square$

Problem 7.2

Prove that if G does not contain K_4 as minor, then $\chi(G) \leq 3$.

Solution. We prove the statement by *minimal counterexample*. Suppose that G has no K_4 as minor, which implies that it does not have K_4 as topological minor either, therefore by Proposition 3.7 we know that G is not 3-connected. This implies that G admits a separation (X, Y) of order at most 2. Let G be the *smallest* graph — w.r.t. the number of vertices — that does not contain K_4 as minor and that requires at least 4 colors to be colored — i.e. $\chi(G) \geq 4$.

Claim 1: $|X \cap Y| \neq 0$.

Proof of the Claim. Suppose that $|X \cap Y| = 0$; if G has no K_4, then nor $G[X]$ neither $G[Y]$ have K_4 as minor, however $|V(G[X])|, |V(G[Y])| < |V(G)|$ which implies that $G[X]$ and $G[Y]$ are 3-colorable by minimality of G, thus G is 3-colorable by the following proper coloring

$$\forall z \in V(G) \quad c(z) = \begin{cases} c_X(z) & z \in X \\ c_Y(z) & z \in Y \end{cases}$$

contradicting the fact that $\chi(G) \geq 4$ by definition $\lightning$. $\square$

Claim 2: $|X \cap Y| \neq 1$.

Proof of the Claim. Suppose that $X \cap Y = \{v\}$; by the same argument $G[X]$ and $G[Y]$ are 3-colorable by two proper coloring c_X and c_Y. If $c_X(v) = c_Y(v)$ then G is trivially 3-colorable, so we may assume that $c_X(v) \neq c_Y(v)$; then, it suffices to permute the colors of c_Y such that $c_X(v) = c_Y(c)$ and consider the same coloring function described in the previous case, to construct a 3-coloring for G $\lightning$. $\square$

Claim 3: $|X \cap Y| \neq 2$.

Proof of the Claim. Suppose that $X \cap Y = \{x, y\}$ and the proper colorings c_X and c_Y of $G[X]$ and $G[Y]$; if $x \sim y$, then both c_X and c_Y will assign different colors to x and y, meaning that it is always possible to permute the colors of c_Y such that $c_X(x) = c_Y(x)$ and $c_X(y) = c_Y(y)$, thus defining a proper 3-coloring of G and contradicting its definition. This implies that $x \nsim y$.

Similarly, if $x \nsim y$, but $G[X]$ and $G[Y]$ are such that there exists 3-colorings of them for which either $\begin{cases} c_X(x) = c_X(y) \\ c_Y(x) = c_Y(y) \end{cases}$ or $\begin{cases} c_X(x) \neq c_X(y) \\ c_Y(x) \neq c_Y(y) \end{cases}$, then it is always possible to consider such colorings and permute the colors to construct a 3-coloring of G. This implies that the only possibility is that — without loss of generality — every possible coloring of $G[X]$ forces x and y to have the same color, and every possible coloring of $G[Y]$ forces x and y to have different colors.

Now, consider $G[Y]$, and by way of contradiction suppose that there is no $x \to y$ path inside $G[Y]$; then, x and y must live in different components of $G[Y]$, which means that it is always possible to permute the colors in the component containing say y such that $c_Y(x) = c_Y(y)$ for any proper 3-coloring c_Y of $G[Y]$, contradicting the fact that every coloring of $G[Y]$ must force x and y to have the different colors. This implies that $G[Y]$ has an $x \to y$ path — call this path P.

Let H be a graph obtained by contracting P down to a single edge, together with $G[X]$ — in other words, $H = G[X] \cup \{xy\}$. This graph is clearly a minor of G, therefore it cannot contain K_4 as minor, however

- any coloring c_X of $G[X]$ forces x and y to have the same color on any proper 3-coloring of $G[X]$
- the edge xy forces x and y to have different colors

which means that 3 colors cannot suffice to properly color H. This means that H is a graph that contains less vertices than G, that has no K_4 as minor, but that requires at least 4 vertices to be properly colored, contradicting the minimality of G. □

Together, Claim 1, 2 and 3 contradict the fact that G was not 3-connected $\lightning$. □

Problem 7.3

Consider a graph G, and a minimum vertex cover V^* of G. Prove that

$$\chi(G) \leq |V^*| + 1$$

Solution. Consider the following coloring c of G: assign a different color to each vertex of V^*, and assign an additional color to all the vertices inside $V(G) - V^*$.

Claim: c is a proper coloring of G.

Proof of the Claim. Suppose that c is not proper, i.e. there exists two *adjacent* vertices $x, y \in V(G)$ such that $c(x) = c(y)$. By definition of c clearly x and y cannot be both inside V^* otherwise they would have been assigned different colors; moreover, it cannot happen that one of them is inside V^* and the other is not, because the vertices in $V(G) - V^*$ are assigned a color different from all the colors used for V^*. Hence, it must be that $x, y \in V(G) - V^*$, but since $x \sim y$ this implies that the edge xy is not covered by V^* $\lightning$. □

The claim concludes that $|V^*| + 1$ colors clearly suffice to properly color G. □

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