# Graph Theory

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# Contents

# Information and Contacts

Personal notes and summaries collected as part of the *Graph Theory* course offered by the degree in Computer Science of the University of Rome "La Sapienza".

Further information and notes can be found at the following link:
**https://github.com/aflaag-notes**. Anyone can feel free to report inaccuracies, improvements or requests through the Issue system provided by GitHub itself or by contacting the author privately:

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The notes are constantly being updated, so please check if the changes have already been made in the most recent version.

**Suggested prerequisites:**

- Progettazione degli Algoritmi

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# 1

# Basics of Graph Theory

In the 18th century, in the city of Königsberg (Prussia), a puzzle captured the imagination of the townspeople. Königsberg, nestled along the winding *Pregel River*, was divided into four land masses — two parts of the mainland and two islands, Kneiphof and Lomse. Connecting these regions were **seven bridges**, crisscrossing the river back and forth.

Over time, a curious question arose among the people of Königsberg: was it possible to take a *walk* through the city, crossing each of the seven bridges **exactly once**, without retracing any steps, and ending the walk in the same place where it started? This is known as the Seven Bridges of Königsberg problem.

It seemed simple enough, yet no one had managed to do it. The challenge became a favorite pastime, debated in marketplaces and whispered about in taverns. Some claimed it was possible with the right path, while others remained skeptical.



Figure 1.1: The map of Königsberg in Euler's time, showing the actual layout of the seven bridges [con25].

Word of this peculiar problem reached the brilliant Swiss mathematician Leonhard Euler, a man whose mind was always drawn to patterns and logic. Intrigued, Euler set out to solve the riddle — not by drawing endless maps or walking the streets himself, but by **abstracting** the problem into something entirely new.

Euler realized that the specific layout of the city was *irrelevant*. What truly mattered was the way the landmasses were connected by the bridges. He represented each landmass as a **dot** and each bridge as a **line** between them. In doing so, he stripped away unnecessary details and created a simple, elegant combinatorial structure that we now refer to as **graph**.
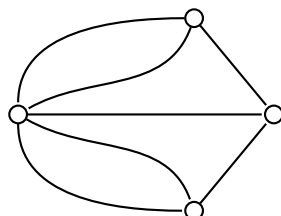


Figure 1.2: The graph drawn by Euler which models the *Seven Bridges of Königsberg* problem.

Through his analysis, Euler discovered a fundamental rule: for a walk to cross each bridge exactly once and return to the starting point, every landmass had to be connected by an **even** number of bridges. In Königsberg's case, however, each landmass had an odd number of bridges, making the task impossible.

Euler's proof, published in 1736, was groundbreaking — not just because he solved the Königsberg puzzle, but because he laid the foundation for an entirely new branch of mathematics: graph theory. His ideas would go on to shape the study of networks, from modern transportation systems to social media connections and even the vast web of the internet itself. And so, from a simple question about bridges in a small Prussian city, a whole new field of mathematics was born — one that continues to shape the world centuries later.

This chapter will discuss the basics of the field of **graph theory**, and will lay the foundation for later chapters.

## 1.1 Introduction

> **Definition 1.1: Graph**
>
> A **graph** is a pair $G = (V, E)$, where $V$ is the — finite — set of **vertices** of the graph, and $E$ is the set of **edges**.

For now, will assume to be working with **simple** and **undirected** graphs, i.e. graphs in which the set of edges is defined as follows

$$E \subseteq [V]^2 = \{\{x, y\} \mid x, y \in V \wedge x \neq v\}$$

where the notation $\{x, y\}$ will be used to indicate an edge between two nodes $x, y \in V$, and will be replaced with $xy = yx$ directly — the *set* notation for edges is used to highlight that edges have no direction.

We will indicate with $n$ and $m$ the cardinality of $|V|$ and $|E|$, respectively. Moreover, we will indicate with $V(G)$ and $E(G)$ the set of the vertices and edges of $G$ respectively when there is ambiguity.
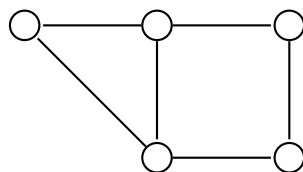


Figure 1.3: A simple graph.

Note that, in this definition, we are assuming that each edge has exactly 2 *distinct* endpoints — i.e. the graphs do not admit **loops** — and there cannot be two edges with the same endpoints — i.e. the graphs do not admit **parallel edges**. In fact, if we drop these assumptions, we obtain what is called a **multigraph**.



Figure 1.4: A multigraph.

**Definition 1.2: Subgraph**

Given a graph $G = (V, E)$, a **subgraph** $G' = (V', E')$ of $G$ is a graph such that $V' \subseteq V$ and $E' \subseteq E$, and we write $G' \subseteq G$. If $G'$ is a subgraph of $G$, then $G$ is called **supergraph** of $G'$.
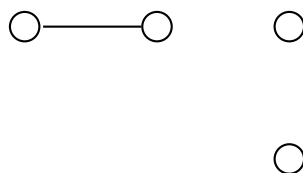


Figure 1.5: This is a subgraph of the graph shown in Figure 1.3.

**Definition 1.3: Induced subgraph**

Given a graph $G = (V, E)$, and a set of vertices $S \subseteq V$, then $G[S]$ represents the **subgraph induced by** $S$ **on** $G$, obtained by removing from $G$ all the nodes of $V - S$ and the edges incident on them.

In other words, a subgraph $G' = (V', E')$ of $G$ is **induced** if every edge of $G$ with both ends in $V$ is an edge of $V'$. We observe that this definition is *stricter* than the definition of a *subgraph*; in fact, the last graph is *not* an example of an *induced subgraph*, but the following is:
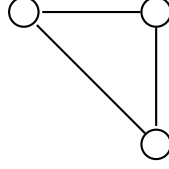


Figure 1.6: This is an *induced* subgraph of the graph shown in Figure 1.3.

Note that every induced subgraph of a graph is **unique** by definition, and we indicate each induced subgraph as follows: suppose that the graph in Figure 1.3 had the following *labeling* on the vertices



then, the induced subgraph in Figure 1.6 would have been referred to as $G[\{1, 3, 5\}]$. To clarify, when writing $G - A$

- if $A$ is a set of *vertices* we are referring to $G[V(G) - A]$ — a vertex cannot be removed from a graph without removing all the edges incident to it

- if $A$ is a set of *edges* we are refering to a graph $G$ without the edges in $A$ — note that this graph has still $V(G)$ as vertex set

### Definition 1.4: Non-separating

Given a connected graph $G$, a subgraph $H$ of $G$ is said to be **non-separating** if $G - H$ is still connected.

### Definition 1.5: Graph union

Given two graphs $G$ and $G'$, we define the **union** $G \cup G'$ as the following graph:

- $V(G \cup G') := V(G) \cup V(G')$

- $E(G \cup G') := E(G) \cup E(G')$

Intuitively, two vertices $x, y \in V$ are said to be **adjacent** if there is an edge $xy \in E$, and we write $x \sim y$. If there is no such edge, we write $x \nsim y$ for non-adjacency and we say

that $xy$ is an **anti-edge**. The **neighborhood** of a vertex $x \in V$ is the set of vertices that are adjacent to $x$, and it will be indicated as follows

$$\mathcal{N}(x) := \{y \in V \mid x \sim y\}$$

Similarly, the neighborhood of a set of vertices will be defined as follows

$$\forall S \subseteq V \quad \mathcal{N}(S) := \bigcup_{v \in S} \mathcal{N}(v)$$

The **degree** of a vertex $x \in V$, denoted with $\deg(x)$, is exactly $|\mathcal{N}(x)|$. We will use the following notation for the **minimum** and **maximum** degree of a graph, respectively

$$\delta := \min_{x \in V} \deg(x) \qquad \Delta := \max_{x \in V} \deg(x)$$

**Lemma 1.1: Handshaking lemma**

Given a graph $G = (V, E)$, it holds that

$$\sum_{x \in V} \deg(x) = 2 |E|$$

*Proof.* Trivially, the sum of the degrees counts every edge in $E$ exactly twice, once for each of the 2 endpoints. □

**Corollary 1.1**

Given a graph $G = (V, E)$, it holds that $|E| \geq \frac{\delta \cdot n}{2}$.

*Proof.* By the Handshaking lemma, we have that

$$2 |E| = \sum_{x \in V} \deg(x) \geq \delta \cdot n \iff |E| \geq \frac{\delta \cdot n}{2}$$

□

**Definition 1.6: $k$-regular graph**

A graph $G$ is said to be $k$-**regular** if every vertex of $G$ has degree $k$.

Note that in a $k$-regular graph it holds that

$$\sum_{x \in V} \deg(x) = k \cdot n$$

**Proposition 1.1**

There are no $k$-regular graphs with $k$ odd and an odd number of vertices.

*Proof.* By way of contradiction, suppose that there exists a $k$-regular graph $G = (V, E)$ such that both $k$ and $n$ are odd; however, by the Handshaking lemma we would get that

$$2\,|E| = \sum_{x \in V} \deg(x) = k \cdot n$$

but the product of two odd numbers, namely $k$ and $n$, is still an odd number, while $2\,|E|$ must be even $\notlightning$. $\square$

## 1.2 Important structures

> **Definition 1.7: Path**
>
> A **path** is a *graph* with vertex set $x_0, \ldots, x_n$ and edge set $e_1, \ldots, e_n$ such that $e_i = x_{i-1}x_i$.
>
> The **length** of a path is the number of edges between $x_0$ and $x_n$, i.e. $|\{e_1, \ldots, e_n\}|$, namely $n$ in this case. A path of length 1 is called *trivial* path. $P_n$ is the path graph having $n$ vertices.
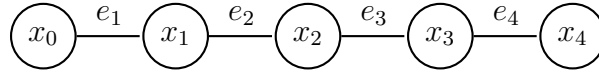


Figure 1.7: A path graph of length 4 that links $x_0$ and $x_4$.

Through *paths* we can provide the definition of **distance** between two nodes of a graph.

> **Definition 1.8: Distance**
>
> Given a graph $G = (V, E)$, and two vertices $x, y \in V$, the **distance** between $x$ and $y$ in $G$, denoted with $\mathrm{dist}_G(x, y)$, is defined as the length of the *shortest* path between $x$ and $y$ in $G$.

If there is no ambiguity, we will simply write $\mathrm{dist}(x, y)$ instead of $\mathrm{dist}_G(x, y)$. Finally, given a path $P$ and two vertices $u, v \in V(P)$, we will denote with $u\,P\,v$ the *subpath* of $P$ between $u$ and $v$. Now, consider the following definition.

> **Definition 1.9: Walk**
>
> Given a graph $G = (V, E)$, a **walk** is a *sequence* of vertices and edges
>
> $$x_0 \; e_1 \; x_1 \; \ldots \; x_{k-1} \; e_k \; x_k$$
>
> where $x_0, \ldots, x_k \in V$, $e_1, \ldots, e_k \in E$ and $e_i = x_{i-1}x_i$.
>
> The **length** of a walk is the number of edges between $x_0$ and $x_k$, i.e. $|\{e_1, \ldots, e_k\}|$, namely $k$ in this case. If $x_0 = x_k$ we say that the walk is **closed**.

If there is a path – or a walk — between two vertices $x, y \in V$, we say that the path — or the walk — **links** $x$ and $y$, and we write this as $x \to y$. Any vertex that is different from $x$ and $y$ is called *internal node*. For instance, given the previous graph labeled as follows



an example of a walk over this graph is given by the following sequence

$$1 \; \{1, 2\} \; 2 \; \{2, 4\} \; 4 \; \{4, 3\} \; 3 \; \{3, 1\} \; 1 \; \{1, 5\} \; 5$$

that *links* 1 and 5, i.e. the walk is of the form $1 \to 5$.

Note that there is a subtle difference between the definitions of **path** and **walk**: the definition of a path implies that this is always a *graph* on its own, while a walk is defined as a *sequence*. Nonetheless, we will treat *paths* as if they where *sequences* as well. This assumption holds for the following structures that will be discussed as well.

However, by definition of path, not every alternating sequence of vertices and edges is a valid path, in fact:

- in a *walk* it is possible to repeat both vertices and edges

- in a *path* there can be no repetition of vertices nor edges (note that *edge* repetition implies *vertex* repetition)

For instance, the previous example of *walk* is not a valid *path*, because the vertex 1 is repeated.

> **Theorem 1.1: Paths and walks**
>
> Given a graph $G = (V, E)$ and two vertices $x, y \in V$, in $G$ there is a path $x \to y$ if and only if there is a walk $x \to y$.

*Proof.* By definition, every path is a walk, thus the direct implication is trivially true. To prove the converse implication, consider two vertices $x$ and $y$ for which there is at least one walk $x \to y$ in $G$. Now, out of all the possible walks $x \to y$ in $G$, consider the *shortest* one, i.e. the one with the least amount of edges, and let it be the following sequence

$$x \; e_1 \; x_1 \; \ldots \; x_{k-1} \; e_k \; y$$

By way of contradiction, assume that this walk is not a path. Therefore, there must be either one vertex or one edge repeated, but since edge repetition always implies vertex repetition, we just need to take this case into account. Assume that there are two indices $i, j \in [k-1]$ such that $i \neq j$ and $x_i = x_j$; however this implies that

$$x \; e_1 \; \ldots \; x_{i-1} \; e_i \; x_i \; e_{j+1} \; x_{j+1} \; \ldots \; x_{k-1} \; e_k \; y$$

is still a walk $x \to y$ of strictly shorter length, but we chose the original sequence to be the *shortest* possible walk $x \to y$ ↯. □

> **Proposition 1.2**
>
> The longest path in any graph has a length of at least $\delta$.

*Proof.* Consider a graph $G = (V, E)$, and let $P$ be a longest path in $G$, labeled as follows

$$x_0 \; e_1 \; x_1 \; \ldots \; x_{k-1} \; e_k \; x_k$$

and assume that its length is $k$. Since $P$ is a longest path in $G$, $x_k$ cannot have neighbors outside $P$ itself, otherwise $P$ would not have been the longest path of $G$ — it could have been extended by one of $x_k$'s neighbors. This implies that

$$\mathcal{N}(x_k) \subseteq \{x_0, \ldots, x_{k-1}\}$$

and since $\delta \leq \deg(x_k) := |\mathcal{N}(x_k)|$ by definition of $\delta$, this implies that

$$\delta \leq |\{x_0, \ldots, x_{k-1}\}| = k$$

□

> **Definition 1.10: Cycle**
>
> A **cycle** is a *graph* with vertex set $x_1, \ldots, x_n$ and edge set $x_1 x_2, x_2 x_3, \ldots, x_{n-1} x_n, x_n x_1$.
>
> The **length** of a cycle is the number of edges between $x_1$ and $x_n$, namely $n$ in this case. $C_n$ is the cycle graph having $n$ vertices.
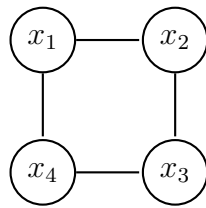


Figure 1.8: A cycle graph of length 4.

A graph that does not admit cycle subgraphs — or *cycles*, for short — is said to be **acyclic**.

Given a graph $G$ that has a cycle $C$, such cycle is said to be **induced** if there is no pair of vertices $x, y \in V(C)$ such that $x \nsim y$ in $C$ but $x \sim y$ in $G$. In other words, it must hold that $G[C]$ is a cycle graph.
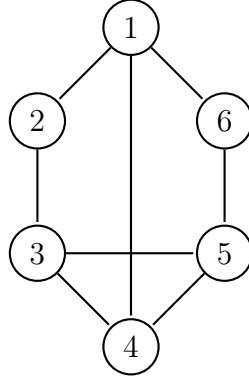
Figure 1.9: For instance, in this graph $G$ have that $G[\{3,4,5\}]$ is an induced cycle but $G[\{1,2,3,4,5,6\}]$ is not, because $1 \sim 6$.

We observe that, by definition, $C_3$ is always induced.

**Proposition 1.3**

Every graph with $\delta \geq 2$ has a cycle of length at least $\delta + 1$.

*Proof.* Consider the proof of Proposition 1.2; by applying the same reasoning, we know that $x_k$ cannot have neighbors outside $P$ itself. However, since $\delta \geq 2$, and $x_k \sim x_{k-1}$, there must be at least one vertex in $x_k$'s neighborhood that lies in $P$. Therefore, let $x_i$ be the first vertex of $P$ — w.r.t. our labeling of $P$ — that is adjacent to $x_k$; hence, we have

$$\mathcal{N}(x_k) \subseteq \{x_i, \ldots, x_{k-1}\} \implies \delta \leq |\{x_i, \ldots, x_{k-1}\}|$$

which implies that $x_i, \ldots, x_{k-1}, x_k, x_i$ is a cycle of length at least $\delta + 1$. $\qquad\square$

**Definition 1.11: Connected graph**

An undirected graph $G = (V, E)$ is said to be **connected** if and only if for each vertex pair $x, y \in V$ there is a path $x \to y$.

All the graphs that we presented so far are *connected*, thus the following figure provides an example of an **disconnected** graph.
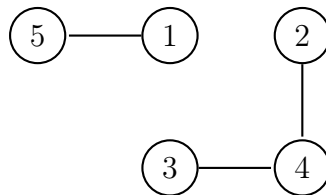


**Definition 1.12: Component**

Given a graph $G$, a **component** of $G$ is a maximal connected subgraph of $G$.

For instance, the graph of the previous example is made up of 2 components, namely the following two subgraphs

$$C_1 = (\{1, 5\}, \{\{1, 5\}\})$$
$$C_2 = (\{2, 3, 4\}, \{\{2, 4\}, \{4, 3\}\})$$

**Proposition 1.4**

If $G$ is a connected graph, and $C$ is a cycle in $G$, then for any edge $e \in C$ it holds that $G - \{e\}$ is still connected.

*Proof.* Consider a graph $G = (V, E)$ that has a cycle $C$, and any two vertices $x, y \in V$; in particular, since $G$ is connected, there must be a path $x \to y$ in $G$, and let this path be

$$P = x \; e_1 \; x_1 \; \ldots \; x_{k-1} \; e_k \; y$$

Consider an edge $e \in C$; if $P$ does not traverse $e$, trivially $G - \{e\}$ will still contain $P$.

Now let

$$C = z_1 \; f_2 \; z_2 \; \ldots \; z_{l-1} \; f_l \; z_l \; f_{l+1} \; z_1$$

and without loss of generality assume that $e = f_2 = z_1 z_2 = x_i x_{i+1} = e_{i+1}$ for some $i \in [k-1]$. Thus we can construct the following walk

$$x \; e_1 \; x_1 \; \ldots \; x_i \; f_{l+1} \; z_l \; \ldots \; f_3 \; z_2 \; e_{i+2} \; x_{i+2} \; \ldots \; x_{k-1} \; e_k \; y$$

from $x$ to $y$, and by Theorem 1.1 we have that there is a path from $x \to y$, which proves that $G - \{e\}$ is still connected. $\qquad\square$

### 1.2.1 Trees

**Definition 1.13: Tree**

A **tree** is a connected acyclic graph. Usually, but not necessarily, there is a fixed vertex called **root**, and any vertex that has degree 1 in the tree is called **leaf**.
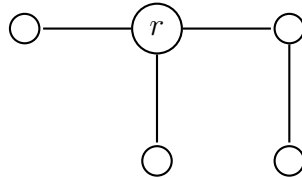


Figure 1.10: A tree with tree leaves, rooted in $r$.

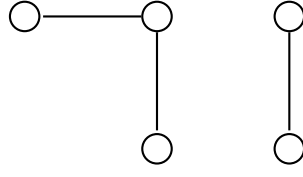A **forest** is an disconnected graph in which each component is a *tree*, as in the following example

Figure 1.11: A forest.

Given a tree $T$ rooted in some node $r \in V(T)$, and two vertices $x, y \in V(T)$, consider the paths $P_x$ of the form $x \to r$ and $P_y$ of the form $y \to r$, respectively. The first vertex of $P_y$ that is encountered by tracing $P_x$ from $x$ to $r$ is called **lowest common ancestor** (LCA) of $x$ and $y$.
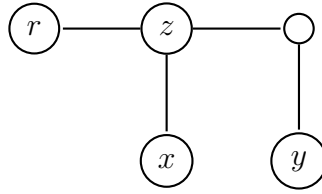


Figure 1.12: For instance, in this tree — rooted in $r$ — the LCA of $x$ and $y$ is the vertex labeled with $z$.

Note that the LCA between any two vertices of a tree is *always defined*, since in the "worst case" it is the root $r$ itself.

---

**Theorem 1.2: Alternative definitions of tree**

Given a graph $T = (V, E)$, the following statements are equivalent:

1. $T$ is a tree

2. every vertex pair of $T$ is connected by a unique path

3. $T$ is *minimally connected*, i.e. $T$ is connected and $\forall e \in E$ it holds that $T - \{e\}$ is disconnected

4. $T$ is *maximally acyclic*, i.e. $T$ is acyclic and $\forall x, y \in V$ such that $x \nsim y$, it holds that $T \cup \{xy\}$ has a cycle

---

*Proof.* We will prove the statements cyclically.

- 1 $\implies$ 2. By contrapositive, assume that in $T$ there exist two vertices $x, y \in V$ for which there are two distinct paths $P$ and $Q$ of the form $x \to y$. If $P$ and $Q$ are edge-disjoint, then $P \cup Q$ is a cycle, which implies that $T$ is not a tree by definition.

  Otherwise, assume that $P$ and $Q$ are not edge-disjoint. If we start say in $x$, and we follow $Q$ edge by edge since $P$ and $Q$ are distinct, at some point we will encounter an edge $\{u, v\}$ such that $u \in P \cap Q$ and $v \in Q - P$ — possibly, $u = x$ itself. Moreover, since both paths lead to $y$, if we keep following $Q$ we will encounter a

vertex $z \in P \cap Q$ — possibly, $z = y$ itself — from which the two paths will coincide. Let $Q'$ be the subpath of $P$ starting with $u$ and ending in $z$; then, $Q' \cup (Q - P)$ is a cycle in $T$, which implies that $T$ is not a tree by definition.



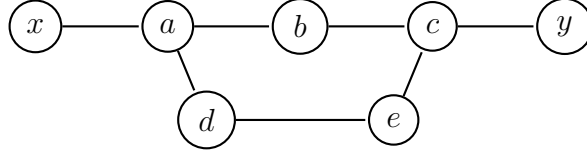Figure 1.13: For instance, applying the argument of the proof in this graph we would get that $P = \{x, a, b, c, y\}$, $Q = \{x, a, d, e, c, y\}$, $Q - P = \{d, e\}$, $u = a$, $z = c$ and $Q' = \{a, b, c\}$, in fact $Q' \cup (Q - P) = \{a, b, c, e, d\}$ which is a cycle.

- 2 $\implies$ 3. Consider an edge $xy \in E$; this edge itself is a path $x \to y$, and if we assume statement 2 this implies that it is the *only* path from $x$ to $y$. This implies that $T - \{xy\}$ cannot contain a path from $x$ to $y$, therefore $T - \{xy\}$ is disconnected.

- 3 $\implies$ 4. Since statement 3 implies that $T$ is connected, by Proposition 1.4 we have that $T$ is acyclic. Now, pick $x, y \in V$ such that $x \nsim y$; by connectivity of $T$ there must be a path $x \to y$ in $T$, and let this path be $P$. Lastly, since $x \nsim y$, we have that $P \cup \{xy\}$ is a cycle in $T$.

- 4 $\implies$ 1 By contrapositive, we want to prove that if $T$ is not a tree, then $T$ is not maximally acyclic. Note that if $T$ is not a tree, we have two options:
  - if $T$ is connected but contains a cycle, then $T$ is clearly not maximally acyclic
  - if $T$ is acyclic but disconnected, then by definition there must be two vertices $x, y \in V$ such that there is no path $x \to y$, which implies that $T \cup \{xy\}$ still does not contain any cycle

$\square$

### Lemma 1.2

Every tree with at least 2 vertices has a leaf.

*Proof.* By way of contradiction, assume $T$ is a tree with at least 2 vertices that does not contain any leaves; then $\delta \geq 2$ in $T$, which implies that $T$ contains a cycle of length at least $\delta + 1$ by Proposition 1.3. $\square$

### Lemma 1.3

Given a tree $T$, and a leaf $v$ of $T$, it holds that $T - \{v\}$ is still a tree.

*Proof.* Since $T$ is acyclic by definition, $T - \{v\}$ is still acyclic, we just need to prove that $T - \{v\}$ is still connected. By way of contradiction, assume that in $T - \{v\}$ there exist two

vertices $x$ and $y$ such that there is no path between them. However, since $T$ is connected, there is a path $P$ of the form $x \to y$ in $T$.

Note that, if by removing $v$ from $T$ we disconnect $x$ and $y$, it must be that $v$ lies in $P$. Moreover, since $v$ is in $T$ but not in $T - \{v\}$, while both $x$ and $y$ are also in $T - \{v\}$, it must be that $v$ is an *internal* node of $P$, i.e. $v \neq x, y$, which implies that $\deg(v) \geq 2$ by definition of path, contradicting the hypothesis for which $v$ was a leaf $\lightning$.  $\square$

> **Proposition 1.5**
>
> If $T$ is a tree, then $m = n - 1$.

*Proof.* We will prove the statement by induction on $n$

*Base case.* When $n = 1$, there are no edges in the tree, and $0 = m = 1 - 1$.

*Inductive hypothesis.* Assume that for a tree that has $n = k - 1$ nodes the statement holds.

*Inductive step.* We will prove the statement for a tree $T$ that has $n = k$ nodes. Note that, since $n = 1$ is the base case, we can assume that $n = k \geq 2$, hence by Lemma 1.2 $T$ contains at least one leaf. Let this leaf be $v$; then, by Lemma 1.3 it holds that $T - \{v\}$ is still a tree, and clearly $T - \{v\}$ has $k - 1$ nodes, which implies that we can apply the inductive hypothesis on $T - \{v\}$, i.e.

$$|E(T - \{v\})| = |V(T - \{v\})| - 1 = k - 1 - 1 = k - 2$$

However, note that $v$ is a leaf, concluding that

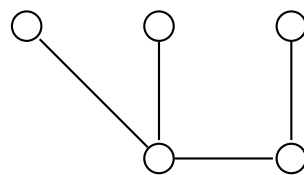$$m = |E(T)| = |E(T - \{v\})| + 1 = k - 2 + 1 = k - 1 = n - 1$$

$\square$

> **Definition 1.14: Spanning tree**
>
> Given a graph $G = (V, E)$, a **spanning tree** $T$ of $G$ is a subgraph of $G$ such that
>
> - $T$ is a tree
> - $V(T) = V(G)$, i.e. $T$ *spans* every vertex of $G$

For instance, given the graph in Figure 1.3, a possible spanning tree is the following:

**Lemma 1.4**

Any connected graph has a spanning tree.

*Proof.* Consider a connected graph $G$, and keep removing edges from $E(G)$ — and their relative endpoints from $V(G)$ — one by one, as long as $G$ is still connected. If no other edge can be removed from $G$ without violating connectivity, we will end up with a graph that must be a tree by statement 3 of Theorem 1.2. $\square$

Thanks to this last proposition, we can actually prove a stronger version of the Proposition 1.5, which is the following.

**Theorem 1.3**

$T$ is a tree if and only if $T$ is connected and $m = n - 1$.

*Proof.* The direct implication is proved in Proposition 1.5, so we just need to prove the converse implication. Consider a connected graph $T$ such that $m = n - 1$; by Lemma 1.4 $T$ must have a spanning tree $T'$, and by Proposition 1.5 itself it holds that $|E(T')| = |V(T')| - 1$. However, $T'$ is a spanning tree of $T$, therefore

$$V(T) = V(T') \implies |V(T)| = |V(T')| \implies |E(T')| = |V(T)| - 1 = |E(T)|$$

which implies $E(T) = E(T')$ because $T'$ is a subgraph of $T$, therefore $T = T'$, concluding that $T$ must be a tree since $T'$ is a tree. $\square$

### 1.2.2 Bipartite graphs

**Definition 1.15: Bipartite graph**

A graph $G = (V, E)$ is said to be **bipartite** if there exists a set $X \subseteq V$ such that every edge of $G$ has exactly one endpoint in $X$ and one in $V - X$. If such a set $X$ exists, we say that $(X, V - X)$ is a **bipartition** of $G$. If $G$ has the maximum number of edges that preserve the bipartition, we say that is **complete bipartite**.
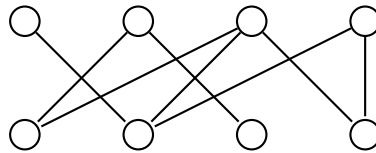


Figure 1.14: An example of a *bipartite graph*. In particular, if we call the uppermost set of nodes $A$ and the lowermost one $B$, then $(A, B)$ is a bipartition of the graph.

The graph $K_{a,b}$ is the *complete bipartite* graph bipartitioned through $(A, B)$ such that $|A| = a$ and $|B| = b$.

There are various types of graphs that can be bipartitioned. For example, any **tree** $T$ can be bipartitioned through a bipartition $(X, V(T) - X)$ by considering the following set of vertices

$$X := \{v \in V(T) \mid \text{dist}(r, v) \text{ even}\}$$

where $r$ is $T$'s root.
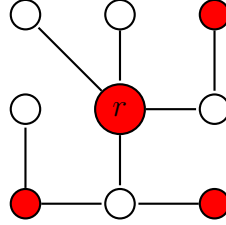


Figure 1.15: The set of red vertices $X$ defines a bipartition $(X, V(T) - X)$ of this tree $T$.

However, *not every type* of graph can be bipartitioned. For instance, consider the following type.

**Definition 1.16: Clique**

A **clique** is a *graph* in which each vertex is adjacent to any other vertex of the graph. A clique that has $n$ vertices — referred to as $k$-clique — is denoted as $K_n$.
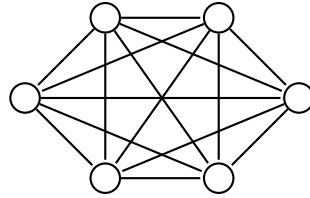


Figure 1.16: The clique $K_6$.

It is easy to see that no clique $K_n$ can be bipartitioned, since there is an edge between any pair of vertices of the graph. However, this is not the only type of graph that cannot be bipartitioned.

**Lemma 1.5**

If $G$ is a bipartite graph, and $H$ is a subgraph of $G$, then $H$ must be bipartite.

*Proof.* Given a bipartite graph $G = (V, E)$, assume that $(X, V - X)$ is a bipartition of $G$, and let $H$ be a subgraph of $G$; then, it is easy to see that $(X \cap V(H), V(H) - X)$ is a bipartition for $H$. □

Note that this lemma implies that $G$ is bipartite if and only if every connected component of $G$ is bipartite: in fact, the direct implication follows from this lemma, and the following figure provides an intuition for the converse implication.
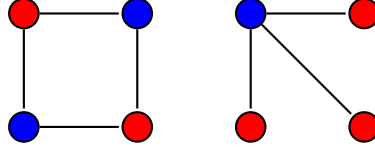


Figure 1.17: Consider the following disconnected graph $G$ made of these two connected components, $C_1$ and $C_2$ respectively. Say that $C_1$ has a bipartition $(X_1, V(C_1) - X_1)$ where $X_1$ is the red set, and $C_2$ has a bipartition $(X_2, V(C_2) - X_2)$ where $X_2$ is the red set; thus, $(X_1 \cup X_2, V(C_1 \cup C_2) - X_1 - X_2)$ is clearly a bipartition of $G$. This process may be repeated for all the connected components of any disconnected graph.

**Theorem 1.4**

$G$ is bipartite if and only if $G$ has no odd-length cycle.

*Proof.*

*Direct implication.* We will prove the contrapositive, i.e. if $G$ has an odd-length cycle, then $G$ cannot be bipartitioned. Consider a graph $G$ with an odd-length cycle $C_{2k+1}$ of vertices $x_1, \ldots, x_{2k+1}$; by way of contradiction, assume that $G$ is bipartite through a bipartition $(X, V(G) - X)$ for some $X \subseteq V(G)$. Without loss of generality assume that $x_1 \in X$; then, since $X$ defines a bipartition of $G$ it must be that $x_2 \notin X$, and $x_3 \in X$ and so on and so forth. In particular, for any odd value of $i$ we will have that $x_i \in X$, but this implies that both $x_1$ and $x_{2k+1}$ must be inside $X$, which means that the edge $x_1 x_{2k+1}$ violates the bipartition induced by $X$ ⨍.

*Converse implication.* Again, we will prove the contrapositive, i.e. if $G$ is not bipartite it must contain an odd-length cycle. By the previous observation, $G$ is not bipartite if and only if at least one connected component of $G$ is not bipartite, and let this component be $\overline{G}$. Note that, since $\overline{G}$ is connected, it must contain a spannning tree $T$ by Lemma 1.4. Moreover, as previously described, we can always define a bipartition on a tree, namely $(X, V(T) - X)$ where

$$X := \{v \in V(T) \mid \mathrm{dist}_T(r, v) \text{ even}\}$$

for some root node $r \in V(T)$.

Now, since $T$ is a spanning tree of $\overline{G}$, which is now bipartite by hypothesis, there must be an edge $xy \in V(\overline{G})$ such that either $x, y \in X$ or $x, y \in V(T) - X$, i.e. the edge $xy$ must have both endpoints in the same set of $T$'s bipartition. Let $z$ be the LCA between $x$ and $y$ in $T$, and $P_x$ and $P_y$ be the paths of the form $x \to r$ and $y \to r$, respectively. Note that, since $xy$ has both endpoints in the same set, it must be that the lengths of $P_x$ and $P_y$ have the same parity by definition of $X$. Lastly, by statement 2 of Theorem 1.2 we have that $r\ P_x\ z = r\ P_y\ z$, which implies that

the lengths of $z\ P_x\ x$ and $z\ P_y\ y$ must have the same parity. This concludes that

$$z\ P_x\ x \cup z\ P_y\ y \cup xy$$

is an odd-length cycle of $G$.

$\square$

### 1.2.3   Eulerian tours

At the start of this chapter, we introduced the *Seven Bridges of Königsberg* problem, which led to the emergence of graph theory as a branch of combinatorics. Over time, as the field developed, this problem was formalized into the following definition.

> **Definition 1.17: Eulerian tour**
>
> An **Eulerian tour** over a graph $G$ is a closed walk that traverses every edge of $G$ exactly once.

For instance, consider the following graph
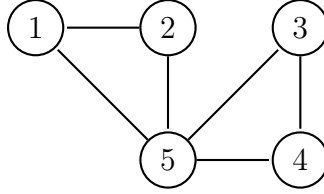


After some trial an error, it is easy to find an Eulerian tour over this graph, for instance

$$1\ \{1,2\}\ 2\ \{2,5\}\ 5\ \{5,3\}\ 3\ \{3,4\}\ 4\ \{4,5\}\ 5\ \{5,1\}\ 1$$

Note that this is a valid Eulerian tour because there is no edge repetition, and vertex repetition is allowed by definition. On the counter side, the graph — or, more precisely, the *multigraph* — that the bridges of Königsberg define, which is the following
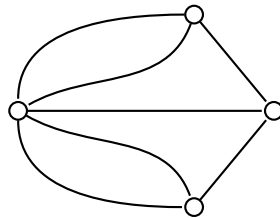


does not admit any Eulerian tour. But, given a graph, how can we determine with certainty whether it contains an Eulerian tour? The following theorem, proved by Euler himself in his original paper [Eul41], answers this question. Note that this theorem holds both for graphs and multigraph.

> ### Theorem 1.5: Eulerian tours
>
> A graph $G$ admits an Eulerian tour if and only $G$ is connected and every vertex of $G$ has even degree.

*Proof.*

*Direct implication.* Consider the contrapositive of the direct implication, and assume that $G$ contains at least one odd-degree vertex $v$. Recall that an Eulerian tour is a closed walk that does not allow edge repetition, which implies that the *starting* point of the walk is actually not relevant. Therefore, we can assume without loss of generality that any possible Eulerian tour defined on $G$ starts on $v$ itself, but then to be *closed* it must end on $v$ as well. Moreover, each time any Eulerian tour passes through $v$ it must use 2 distinct edges, however $v$ is an odd-degree vertex, therefore at the end of the Eulerian tour there is no way we can come back to $v$ and close the walk.

*Converse implication.* Consider a graph $G$ in which every vertex has even degree, and let $W$ be the *longest* walk of $G$ that does not repeat edges, and let it be labeled as follows

$$x_0 \ e_1 \ x_1 \ \ldots \ x_{k-1} \ e_k \ x_k$$

**Claim:** $x_0 = x_k$, i.e. $W$ is closed.

*Proof of the Claim.* By way of contradiction, assume that $W$ $x_0 \neq x_k$. If this is the case, then $x_k = v$ for some vertex $v$ that is not $x_0$. Note that $W$ is a *walk*, therefore $v$ may be repeated multiple times inside $W$, i.e.

$$x_0 \ e_1 \ x_1 \ \ldots \ v \ \ldots \ v \ \ldots \ x_{k-1} \ e_k \ v$$

If $l$ is the number of times $v$ occurs in $W$ *without counting* $x_k$, then clearly in $W$ there are $2l + 1$ edges incident to $v$, namely $e_k$ and 2 edges each other time $v$ appears in $W$. However, since $2l + 1$ is odd and we assumed that $G$ has no odd-degree vertices, there must be at least one edge $vu \in E(G)$ such that $u \notin V(W)$. This implies that

$$x_0 \ e_1 \ x_1 \ \ldots \ x_{k-1} \ e_k \ v \ vu \ u$$

is a longer walk than $W$, and it does not repeat any edges because $uv \notin E(W)$, contradicting the definition of $W$ $\lightning$. $\qquad\square$

This claim proves that $W$ is closed, but we still need to prove that it traverses every edge to claim that it is indeed an Eulerian tour. By way of contradiction, assume that there exists at least one edge $e \notin E(W)$ not used by $W$. Consider an vertex $x_i$ of $W$; by connectivity of $G$, there must be a path $P$ beween $x_i$ and each of the endpoints of $e$. Let $x_i u$ be the first edge of $P$, for some $u \notin E(W)$. This implies that

$$u \ ux_i \ x_i \ e_{i+1} \ x_{i+1} \ \ldots \ x_{k-1} \ e_k \ x_k \ e_1 \ x_1 \ \ldots \ x_{i-1} \ e_i \ x_i$$

is a longer walk than $W$ that does not repeat any edge, since $ux_i \notin E(W)$, again contradicting the definition of $W$ $\lightning$.

$\square$

Note that this theorem proves that it is not possible to describe an Eulerian tour over the bridges and landmasses of Königsberg, because every vertex of the multigraph has odd degree.

### 1.2.4 Hamiltonian cycles

Eulerian tours are closed walks that traverse every edge of the graph exactly once. But what if we are interested in traversing each *vertex* exactly once instead?

> **Definition 1.18: Hamiltonian paths and cycles**
>
> A **Hamiltonian path** over a graph $G$ is a subgraph $P_n$ of $G$. A **Hamiltonian cycle** over a graph $G$ is a subgraph $C_n$ of $G$.

Hamiltonian paths and cycles are named after W. R. Hamilton. Note that the notation $P_n$ (or $C_n$) implies that the length of the path (or cycle) is $n$, hence this definition matches our requirements.

As for the case of Eulerian tours, when some conditions are met, Hamiltonian cycles are guaranteed to exist, as discussed in the following theorem.

> **Theorem 1.6: Hamiltonian cycles**
>
> A graph $G$ such that $\delta \geq \frac{n}{2}$ contains a Hamiltonian cycle.

*Proof.* First, we will prove that the condition of the statement implies that $G$ is connected.

**Claim:** $G$ is connected.

*Proof of the Claim.* By way of contradiction, suppose that $G$ is not connected; therefore $G$ has at least two connected components. Let $H$ be the smallest connected component of $G$; then, clearly $|V(H)| \leq \frac{n}{2}$. However, note that $|\mathcal{N}(x)| \geq \delta \geq \frac{n}{2}$ and since $\{x\} \cup \mathcal{N}(x) \subseteq V(H)$, we get that $V(H)$ must have at least $\frac{n}{2} + 1$ nodes $\lightning$. $\square$

Let $P$ be the longest path of $G$, and let $x_0, \ldots, x_k$ be its vertices.

**Claim:** There exists an index $\ell$ such that $x_0 \; x_\ell \; x_{\ell+1} \; \ldots \; x_{k-1} \; x_k \; x_{\ell-1} \; x_{\ell-2} \; \ldots \; x_1 \; x_0$ is a cycle.

*Proof of the Claim.* By the same argument used in the proof of Proposition 1.2, we know that
$$\mathcal{N}(x_0), \mathcal{N}(x_k) \subseteq \{x_0, \ldots, x_k\}$$
Let $I_0$ and $I_k$ be the following two sets
$$I_0 := \{i \mid i \in [1, k], x_i \in \mathcal{N}(x_0)\} \implies |I_0| = |\mathcal{N}(x_0)|$$
$$I_k := \{i \mid i \in [1, k], x_{i-1} \in \mathcal{N}(x_k)\} \implies |I_k| = |\mathcal{N}(x_k)|$$

Since $\delta \geq \frac{n}{2}$, we have that $|I_0|, |I_k| \geq \frac{n}{2}$. However, note that $k \leq n - 1$ — since we started counting at $0$ — hence by the pigeonhole principle there must be at least one index $\ell \in I_0 \cap I_k$, meaning that $x_0 \sim x_\ell$ and $x_k \sim x_{\ell-1}$, defining a cycle as described in the statement of the claim. $\qquad\square$

This means that we found a cycle $C$ in the graph that uses all the vertices of $P$, namely $x_0, \ldots, x_k$. By way of contradiction, assume that $k < n - 1$, i.e. $C$ has less than $n$ nodes, meaning that $C$ is a non-Hamiltonian cycle. In particular, if $k < n - 1$, we have that $|V(G)| - |V(C)| \neq \varnothing$, thus let $y \in V(G) - V(C)$. By the previous claim, we know that $G$ is connected, there must be an edge $xy \in E(G)$ such that $x \in V(C)$. However, this would imply that $P \cup \{xy\}$ is a path of longer path than $P$ $\lightning$. $\qquad\square$

Note that the statement of this theorem cannot be improved, even by 1; for instance consider the following graph
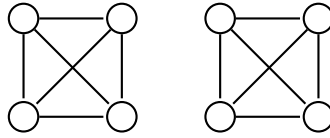


composed of two disconnected $K_4$. Here, we have that

$$\delta = 4 - 1 = 3 \geq \frac{2 \cdot 4}{2} - 1 = 4 - 1 = 3 = \frac{n}{2} - 1$$

Therefore, $\delta \geq \frac{n}{2} - 1$ is not sufficient to guarantee connectivity.

## 1.3 Exercises

### Problem 1.1

Let $G = (V, E)$ be a graph of $n$ vertices, where $n \geq 2$. Show that there must be two vertices $x, y \in V$ such that $\deg(x) = \deg(y)$.

*Solution.* By definition, the range of the possible degrees for any node of $G$ is $[0, n - 1]$. By way of contradiction, assume that for any two vertices $x, y \in V$ it holds that $\deg(x) \neq \deg(y)$; hence, since the graph has $n$ nodes, it must be that each node is assigned a different degree, and that we use all the possible degrees in $[0, n - 1]$. In particular, this implies that there are two vertices $u, v \in V$ such that $\deg(u) = 0$ and $\deg(v) = n - 1$, but this is a contradiction because if the degree of $v$ is $n - 1$, it must be adjacent to all the other nodes of $V$, including $u$, and $\deg(u) = 0$ $\lightning$. $\qquad\square$

### Problem 1.2

Let $G$ be a graph containing a cycle $C$, and assume that $G$ contains a path of length at least $k$ between two vertiecs of $C$. Show that $G$ contains a cycle of length at least $\sqrt{k}$. Can this bound be improved?

*Solution.* Consider a graph $G$ that contains a cycle $C$, and let $P$ be a path between two vertices $x, y \in V(C)$ such that $|E(P)| \geq k$ for some $k$. Starting from $x$, let $x_1, \ldots, x_t$ be the vertices through which $P$ "leaves" $C$, and $y_1, \ldots, y_t$ be the vertices through which $P$ "joins back" $C$. We observe that if $2t \geq \sqrt{k}$, it means that there are at least $\sqrt{k}$ vertices in $C$, thus the theorem trivially holds by simply considering $C$ itself, so we may assume that $2t < \sqrt{k}$.

Let $P_i := x_i \, P \, y_i$ and $\overline{P_i} := y_i \, P \, x_{i+1}$, and let $C_i$ be the shortest subpath of $C$ of the form $x_i \, C \, y_i$ — there are two subpaths of this form. We observe that there are $t$ subpaths of the form $P_i$, and at most $t+1$ subpaths of the form $\overline{P_i}$, meaning that $P$ can be partitioned in at most $2t + 1$ subpaths of the forms described. Now, let $P^*$ be the subpath of $P$ that maximizes its length — $P^*$ may be of the form $P_i$ or $\overline{P_i}$. Hence, we get that

$$k \leq |E(P)| \leq (2t + 1)\,|E(P^*)| < (\sqrt{k} + 1)\,|E(P^*)|$$

and in particular

$$|E(P^*)|\,(\sqrt{k} + 1) > k \iff |E(P^*)| > \frac{k}{\sqrt{k}+1} \iff |E(P^*)| \geq \frac{k}{\sqrt{k}+1} + 1 \geq \sqrt{k}$$

where the last inequality can be proved as follows

$$\frac{k}{\sqrt{k}+1} + 1 \geq \sqrt{k} \iff \frac{k + \sqrt{k} + 1}{\sqrt{k}+1} \geq \sqrt{k} \iff k + \sqrt{k} + 1 \geq \sqrt{k}(\sqrt{k} + 1) = k + \sqrt{k}$$

Now, if the subpath $P^*$ lies inside $C$, since $|E(P^*)| \geq \sqrt{k}$ the theorem holds by considering the cycle $C$ itself, so we may assume that there exists $i^* \in [t]$ such that $P_{i^*} = P^*$.

Lastly, since $P_i$ and $C_i$ do not intersect for each $i \in [t]$, we have that

$$|E(P_i \cup C_i)| = |E(P_i)| + |E(C_i)| \geq |E(P_i)| + 1$$

and in particular

$$|E(P_{i^*} \cup C_{i^*})| \geq \sqrt{k} + 1 \geq \sqrt{k}$$

which means that

- $P_{i^*} \cup C_{i^*}$ is the cycle that proves the statement
- we can improve the bound of the statement to be at least $\sqrt{k} + 1$

$\square$

## Problem 1.3: Tree-order

Consider a tree $T$, and let $r \in V(T)$ be its root. We define the **tree-order** $\leq_r$ associated with $r$ as follows:
$$\forall x, y \in V(T) \quad x \leq_r y \iff x \in V(r\ T\ y)$$
Prove that

1. $r$ is the least element in this partial order.

2. Every leaf different from $r$ is a maximal element.

3. the endpoints of any edge of $T$ are comparable.

4. For any $y \in V(T)$, every set of the form $\{x \in V(T) \mid x \leq_r y\}$ is a *chain*, i.e. a set of pairwise comparable elements.

*Solution.* We can prove the statements by using the properties of trees.

1. By way of contradiction, suppose that there is an element $m$ such that $m \leq_r r$ and $m \neq r$; by definition, this happens if and only if $m \in V(r\ T\ r) = \{r\}$, meaning that $m = r \nleq$.

2. By way of contradiction, let $l \neq r$ be a leaf of $T$ that is not maximal, i.e. there is a node $v \in V(T)$ such that $l \leq_r v$ and $l \neq v$. By definition, this can happen if and only if $l \in V(r\ T\ v)$, but if $l \neq v$ and $l$ is in a path $r \to v$, then $\deg(l) \geq 2$ contradicting the fact that $l$ was a leaf of $T \nleq$.

3. Fix an edge $e \in E(T)$; we can label the endpoints of $e$ with $x$ and $y$ such $\text{dist}(r, y) = \text{dist}(r, x) + 1$ — i.e. $x$ "comes before" $y$ in the path $r \to y$ — and this suffices to show that $x \in V(r\ T\ y) \iff x \leq_r y$.

4. Let $y \in V(T)$, and fix a set $S_y := \{x \in V(T) \mid x \leq_r y\}$. We observe that $x \leq_r y \iff x \in V(r\ T\ y)$, which implies that $x \in S_y \iff x \in V(r\ T\ y)$ meaning that $S_y = V(r\ T\ y)$. Then, fix a pair of distinct elements $a, b \in S_y$, and without loss of generality suppose that $b$ is the element that maximizes the distance from $r$. We observe that $S_y = V(r\ T\ y)$ implies that for any $x \in S_y$ the path $r\ T\ y$ contains $r\ T\ x$, and in particular it contains both $r\ T\ a$ and $r\ T\ b$. Moreover, since $b$ is further from $r$ than $a$, the path $r\ T\ a$ is contained in $r\ T\ b$, which means that $a \in V(r\ T\ b) \iff a \leq_r b$.

$\square$

## Problem 1.4

Show that every connected graph $G$ contains a path of length at least $\min\{2\delta, n-1\}$.

*Solution.* Let $P$ be the longest path of vertices $x_1 \ldots x_k$. If $|V(P)| = n$ then $|E(P)| = n - 1 \geq \min\{2\delta, n-1\}$, so we may assume that $|V(P)| \leq n - 1$.

**Claim:** If $x_i \in \mathcal{N}(x_1)$, then $x_{i-1} \notin \mathcal{N}(x_k)$.

*Proof of the Claim.* By way of contradiction, suppose that there is an index $j$ such that $x_1 \sim x_j$ and $x_{j-1} \sim x_k$; then

$$x_1 \ \ldots \ x_{j-2} \ x_{j-1} \ x_k \ x_{k-1} \ \ldots \ x_{j+1} \ x_j \ x_1$$

is a cycle — call this cycle $C$. Since we are assuming that $|V(P)| \le n - 1$, there must be at least another vertex $z \in V(G) - V(P)$. By connectivity of $G$, we know that there is a path $Q$ from $z$ to some vertex $v_i \in V(C)$; hence, we have that

$$Q \cup x_i \ x_{i+1} \ \ldots \ x_k \ x_1 \ \ldots \ x_{i-1}$$

is a path containing more vertices than the vertices of $P$, contradicting the maximality of $|E(P)|$ $\lightning$. $\qquad\square$

We observe that $\mathcal{N}(x_1), \mathcal{N}(x_k) \subseteq V(P) = \{x_1, \ldots, x_k\}$, otherwise we would contradict the maximality of $P$. Let $\mathcal{N}(x_1) = \{x_{i_1}, \ldots, x_{i_\ell}\}$; by the previous claim, we get that

$$\mathcal{N}(x_k) \subseteq \{x_2, \ldots, x_k\} - \{x_{i_1-1}, \ldots, x_{i_\ell-1}\}$$

therefore
$$\delta \le |\mathcal{N}(x_k)| \le k - 1 - |\mathcal{N}(x_1)| \le k - 1 - \delta$$

meaning that $k \ge 2\delta + 1 \implies |E(P)| \ge 2\delta \ge \min\{2\delta, n - 1\}$. $\qquad\square$

---

> ### Problem 1.5
>
> Show that every tree $T$ has at least $\Delta(T)$ leaves.

*Solution.* Consider a vertex $v$ of maximum degree, i.e. $\deg(v) = \Delta(T)$.

**Claim:** There are $\Delta(T)$ connected components in $T - \{v\}$.

*Proof of the Claim.* By way of contradiction suppose that there are less than $\Delta(T)$ components in this subgraph. However, by pigeonhole principle, since there are $\Delta(T)$ edges incident to $v$, and less than $\Delta(T)$ components, there must be two neighbors of $v$ — say $x$ and $y$ — lying in the same component of $T - \{v\}$. However, since the component is connected by definition, there is a path between $x$ and $y$, say $P$, which implies that $v \ \{v, x\} \ P \ y \ \{y, v\} \ v$ is a cycle, contradicting the fact that $T$ was a tree.

Finally, if there are more than $\Delta(T)$ components when removing $v$ from $T$, it meant that at least one component was already present in $T$, meaning that $T$ had at least 2 different components, contradicting the fact that $T$ was a tree. $\qquad\square$

Moreover, since $T$ is a tree, each of the $\Delta(T)$ components in $T - \{v\}$ must be both connected and acyclic, implying that all of these components are trees themselves. Lastly, for each subtree we have two cases:

- if the subtree has 1 vertex, the subtree was a leaf of $T$

---

- else, if the subtree has at least 2 vertices, by Lemma 1.2 the subtree contains at least one leaf

and in both cases each subtree contains at least one leaf, meaning that $T$ contains at least $\Delta(T)$ leaves. $\qquad\square$

### Problem 1.6

Show that any non-trivial tree without a vertex of degree 2 has more leaves than other vertices. Can you find a very short proof that does not use induction?

*Solution.* Let $T$ be a non-trivial tree that does not contain vertices of degree 2, and let $L = \{v \in V(T) \mid \deg(v) = 1\}$ the set of its leaves. By the Handshaking lemma, we know that

$$2\,|E(T)| = \sum_{v \in V(T)} \deg(v)$$
$$= \sum_{v \in V(T)-L} \deg(v) + \sum_{v \in L} \deg(v)$$
$$\geq |L| + 3\,|V(T) - L|$$

Moreover, since $T$ is a tree, we have that

$$2\,|E(T)| = 2(n-1) = 2(|L| + |V(T) - L| - 1) = 2\,|L| + 2\,|V(T) - L| - 2$$

Therefore, we have that

$$2\,|L| + 2\,|V(T) - L| - 2 \geq |L| + 3\,|V(T) - L|$$

implying that $|L| \geq |V(T) - L| + 2 \iff |L| > |V(T) - L| + 1.$ $\qquad\square$

### Problem 1.7

Let $G$ be a connected graph that does not contain paths longer than $t$, and let $P_1$ and $P_2$ be two paths of $G$ of length $t$. Prove that $V(P_1) \cap V(P_2) \neq \varnothing$.

*Solution.* By way of contradiction, suppose that $V(P_1) \cap V(P_2) = \varnothing$, and fix two vertices $v_1 \in V(P_1)$ and $v_2 \in V(P_2)$. By connectivity of $G$ there must be a path $v_1 \in v_2$ inside $G$ — call it $R$. For $i \in \{1,2\}$ let $P_i = P_i' \cup P_i''$ such that $P_i' \cap P_i''$ is the endpoint of $R$ in $P_i$, and assume $|V(P_i')| \geq |V(P_i'')|$. However, since $|V(P_i')| \geq \frac{t}{2}$ then we have that $P_1' \cup P_2' \cup R$ is a path of length at least $\frac{t}{2} + \frac{t}{2} + 1 = t + 1$, contradicting the definition of $G$. $\qquad\square$

# 2

# Matchings

**Definition 2.1: Matching**

Given a graph $G = (V, E)$, a **matching** of $G$ is a set of edges $M \subseteq E$ such that

$$\forall e, e' \in M \quad e \cap e' = \varnothing$$
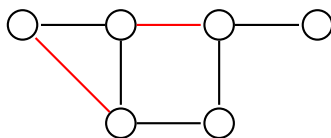


Figure 2.1: A *matching* of the previous graph.

As shown in figure, a matching is nothing more than a set of edges that must not share endpoints with each other — for this reason, in literature it is often referred to as **independent edge set**.

Given a matching $M$ in a graph $G$, we say that a vertex $v \in V(G)$ is **free** w.r.t. $M$ if there are no edges $e \in M$ such that $e \cap v \neq \varnothing$. An edge $xy \in E(G)$ is said to be *disjoint* from $M$ if both $x$ and $y$ are free w.r.t. $M$.

In graph theory we are often interested in the matching that has the largest possible cardinality of a graph. For this purpose, we often distinguish the two following concepts, namely *maximal* and *maximum* matching.

**Definition 2.2: Maximal matching**

A **maximal matching** is a matching that cannot be extended any further.

For instance, the matching shown in Figure 2.1 is actually a **maximal matching**, because

no other edge in $E$ can be added to che current set of edges $M$ of the matching without breaking the matching condition.

> **Definition 2.3: Maximum matching**
>
> A **maximum matching** is a matching that has the largest cardinality.

Clearly, the previous example does not repreent a **maximum matching**, because the following set of edges
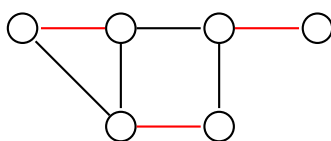
is still a valid matching for the graph, but has a larger cardinality than the previous set.

Given a matching $M$ in a graph $G$, what conditions must be met to increase its cardinality? Trivially, if there exists an edge $e$ disjoint from $M$, clearly $M \cup \{e\}$ is a larger matching than $M$ — implying that $M$ was not *maximal* in $G$. However, this is not the only situation in which the cardinality of $M$ can be extended. In fact, consider the following definitions.

## 2.1  Augmenting paths

> **Definition 2.4: Alternating path**
>
> Given a graph $G$, and a matching $M$ on $G$, an $M$**-alternating path** is a path of $G$ that starts at a free node w.r.t. $M$, and is composed of edges that alternate between $M$ and $E(G) - M$.

Figure 2.2: For instance, if $M$ is the set of *red* edges — which forms a matching of the graph — then 6 $\{6, 2\}$ 2 $\{2, 7\}$ 7 $\{7, 3\}$ 3 $\{3, 8\}$ 8 is an $M$-alternating path.

> **Definition 2.5: Augmenting path**
>
> Given a graph $G$, and a matching $M$ on $G$, an $M$**-augmenting path** is an $M$-alternating path that ends at a free vertex w.r.t. $M$.

For example, if we consider the path

$$6 \; \{6, 2\} \; 2 \; \{2, 7\} \; 7 \; \{7, 3\} \; 3 \; \{3, 8\} \; 8 \; \{8, 4\} \; 4$$

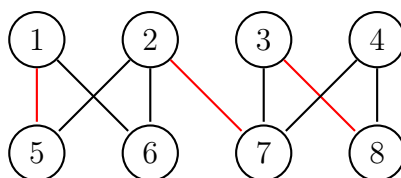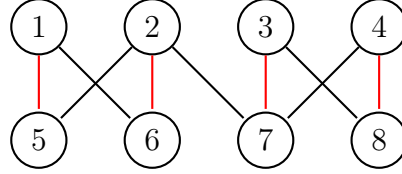this is actually an $M$-augmenting path of the previous graph. Augmenting paths are very useful because they can be used to *expand* the cardinality of an initial matching. In fact, in the previous graph we can actually define a *larger* matching by **swapping** the edges of this augmenting path, as shown below



### 2.1.1 Berge's theorem

This suggests that the presence of augmenting paths in a graph is a *sufficient* condition for a matching *not* to be maximum, but we can actually prove that it is also *necessary*, as stated in the following theorem, proved by Berge [Ber57] in 1957.

> **Theorem 2.1: Berge's theorem**
>
> Given a graph $G$, $M$ is a maximum matching of $G$ if and only if in $G$ there are no $M$-augmenting paths.

*Proof.*

*Direct implication.* By contrapositive, consider a graph $G$ and a matching $M$ such that there is an $M$-augmenting path $P$ in $G$. Moreover, by way of contradiction assume that $M$ is maximum; however $M\Delta E(P)$ is a larger matching than $M$ — here, $\Delta$ is the symmetric difference, therefore the operation $M\Delta E(P)$ has the same effect of *swapping* the edges of $P$ between the ones in $M$ and in $E(P) - M$.

*Converse implication.* By contrapositive, consider a graph $G$ and a matching $M$ of $G$ that is not maximum, i.e. there exists a matching $M^*$ of $G$ such that $|M| < |M^*|$. Consider the subgraph of $G$ that has the vertices of $V(G)$ and the edges described by $M\Delta M^*$; the symmetric difference of these two sets will yield the set of edges that are either in $M$ or in $M^*$, but not in $M \cap M^*$, therefore this subgraph is *not* a multigraph. Moreover, since $M$ and $M^*$ are both matchings, we have that

(1) the degrees of the vertices of this subgraph can be either 0, 1 or 2

(2) in each component of the subgraph the edges *must* alternate between $M$ and $M^*$

By the observation (1), we have that each component of the subgraph can be either

- a isolated vertex

- a cycle

- a path

and by observation (2), we have that all the cycle components must have *even* length, which implies that they have the same number of edges of $M$ and $M^*$. On the other hand, path components may have either even or odd length; in particular, even-length paths must have the same number of edges of $M$ and $M^*$ — as for cycle components — while odd-length paths have a different number of edges of $M$ and $M^*$. However, since $|M| < |M^*|$, there must be at least one path component of this subgraph such that its edges of $M$ are less than the edges of $M^*$, and this is clearly an $M$-augmenting path.

$\square$

Given a graph $G$, and a matching $M$ of $G$, what is the maximum possible value for $|M|$? To answer this question, we need to introduce the following combinatorial structure.

## 2.1.2 Kőnig's theorem

> **Definition 2.6: Vertex cover**
>
> Given a graph $G$, a **vertex cover** for $G$ is a set of vertices $C \subseteq V(G)$ such that every edge in $G$ is incident to at least one vertex in $C$. Using symbols
>
> $$\forall (u,v) \in E(G) \quad u \in C \lor v \in C$$
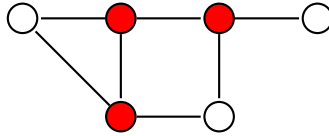


Figure 2.3: An example of a vertex cover.

As shown in figure, a vertex cover is simply a set of vertices that must *cover* all the edges of the graph. For vertex covers, we are interested in the *minimum* possible cardinality — the concepts of *minimal* and *minimum* are defined analogously.

As introduced before, through vertex covers we can bound the size of any matching of a graph.

> **Theorem 2.2: Matchings bound vertex covers**
>
> Given a graph $G$, a matching $M$, and a vertex cover $S$ of $G$, it holds that $|M| \leq |S|$.

*Proof.* By definition, any vertex cover $S$ of $G = (V, E)$ is also a vertex cover for $G^B = (V, B)$, for any set of edges $B \subseteq E$, and in particular this is true for $G^M = (V, M)$.

Now consider $G^M$, and a vertex cover $C$ on it: by construction we have that $\Delta \leq 1$, therefore any vertex in $C$ will cover at most 1 edge of $M$. This implies that if $|C| = k$, then $C$ will cover at most $k$ edges of $G^M$.

Lastly, since $G^M$ has $|M|$ edges by definition, any vertex cover defined on $G^M$ has to contain at least $|M|$ vertices. This implies that no vertex cover $S$ of $G$ smaller than $|M|$ can exist, because $S$ will have to cover at least the edges in $M$. $\qquad\square$

> **Corollary 2.1**
>
> Given a graph $G$, a maximum matching $M^*$ and a minimum vertex cover $S^*$, it holds that $|M^*| \leq |S^*|$.

Moreover, if the graph is bipartite this theorem is actually *stronger*, as proved by Konig [Kon] in 1931.

> **Theorem 2.3: Kőnig's theorem**
>
> Given a bipartite graph $G$, a maximum matching $M^*$ and a minimum vertex cover $S^*$, it holds that $|M^*| = |S^*|$.

*Proof.* Consider a graph $G$, a maximum matching $M^*$ and a minimum vertex cover $S^*$ of $G$; by the previous corollary, it follows that to prove the statement it suffices to show that there exists a vertex cover $S$ such that $|S| = |M^*|$, because

$$|M^*| \leq |S^*| \leq |S| = |M^*| \implies |M^*| = |S^*|$$

Hence, we are going to construct the following vertex cover. Let $G$ be bipartitioned through $(A, B)$; then, for each edge $ab \in M^*$ such that $a \in A$ and $b \in B$, we place $b \in S$ if and only if there exists an $M^*$-alternating path that starts in $A$ and ends at $b$, otherwise we place $a \in S$.
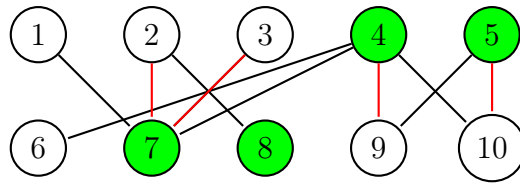


Figure 2.4: For instance, given this graph bipartitioned into $(A, B)$ — where $A$ is the uppermost row of vertices — and the *red* matching, we would construct the *green* vertex cover.

Note that, by definition, it holds that $|S| = |M^*|$.

**Claim:** $S$ is a vertex cover for $G$.

*Proof of the Claim.* Consider an edge $ab \in E(G)$ such that $a \in A$ and $b \in B$; note

that, by definition of $S$ all the edges in $M^*$ are already covered, hence we may assume that $ab \notin M^*$. We have two cases.

- $a$ is free, i.e. $\nexists ab' \in M^*$ for $b' \in B$. Note that $b$ cannot be free, otherwise $M^* \cup \{ab\}$ would still be a matching of $G$ but greater than $M^*$. Hence, there must be an edge $a'b \in M^*$ for some $a' \in A$. Therefore, since $a$ is free, the edge $ab$ is a trivial $M^*$-alternating path ending at $b \in B$, meaning that $b \in S$ by definition, implying that $ab$ is covered by $S$.

- $a$ is matched, i.e. $\exists ab' \in M^*$ for $b' \in B$. Therefore, by definition of $S$, either $a$ or $b'$ lies in $S$. In particular, if $a \in S$, then $ab$ is trivially covered by $S$, hence suppose that $b' \in S$.

  Observe that, by definition of $S$ this implies that there must be an $M^*$-alternating path $P$ that starts in $A$ and ends at $b'$.

  - If $ab, ab' \notin E(P)$, then $P \cup \{b'a\} \cup ab$ is an $M^*$-augmenting path, which would contradict the fact that $M^*$ is maximum by [Theorem 2.1](#) $\lightning$

  - If $ab \notin E(P)$ but $ab' \in E(P)$, then $P$ could not have been an $M^*$-alternating path starting at a vertex in $A$ $\lightning$.

  - If $ab \in E(P)$ but $ab' \notin E(P)$, then $P$ must have had the following form

    $$\ldots \; b'' \; \{b'', a\} \; a \; \{a, b\} \; b \; \{b, a'\} \; a' \; \{a', b'\} \; b'$$

    where $a' \in A$, $b'' \in B$. However, since $ab' \in M^*$ and $ab \notin M^*$, and the edges of $P$ must alternate w.r.t. $M^*$, $ab''$ must lie inside $M^*$, contradicting $ab' \in M^*$ by definition of matching $\lightning$.

  This implies that $ab', ab \in E(P)$, meaning that $P$ encounters $b$ "before" $b'$. However, this implies that $P - \{ab'\}$ is an $M^*$-alternating path that starts in $A$ and ends at $b$, thus $b \in S$ by definition, concluding that $ab$ is still covered by $S$.

  $\square$

Hence, we have constructed a vertex cover $S$ such that $|S| = |M^*|$, meaning that the statement holds because of the previous observation. $\square$

### 2.1.3 Finding maximum matchings

Consider a matching $M$ of a graph $G$, and an $M$-augmenting path $P$; the idea of *swapping* the edges of $P$ between $M$ and $E(G) - M$ is very useful when $G$ is **bipartite**. In fact, we can actually describe a procedure which is able to return a *maximum matching* of a bipartite graph, by swapping the edges of the augmenting paths present in $G$. However, for this algorithm to work, we first need a procedure capable of finding augmenting paths in bipartite graphs, which is defined down below:

1. Assume that the considered graph $G$ is bipartite through $(A, B)$, and consider a matching $M$ of $G$

2. Starting from a node $a \in A$ free w.r.t. $M$, compute a *modified* BFS such that the edges of its tree alternate between $E(G) - M$ and $M$

3. If the tree of the BFS contains a free leaf $b \in B$, then the path $v \to b$ is $M$-augmenting

For instance, given the following bipartite graph $G$, and a matching $M$ of $G$ — outlined in *red*
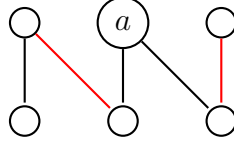


the *modified* BFS rooted in $a$ would produce the following tree
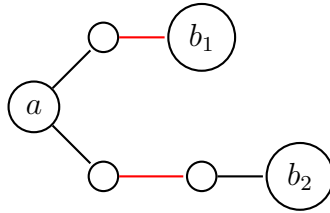


and we observe that the path $a \to b_1$ is $M$-alternating, while the path $a \to b_2$ is $M$-augmenting. The next proposition guarantees that if there are $M$-augmenting paths that start in $a$, our *modified* BFS will find at least one of them.

---

**Proposition 2.1**

Given a bipartite graph $G$, bipartitioned into $(A, B)$, and a matching $M$ of $G$, if there exists an $M$-augmenting path in $G$ that starts in a vertex $a \in A$ free w.r.t. $M$, then there exists an $M$-augmenting path in the tree $T$ of the *modified* BFS.

---

*Proof.* Let $P$ be an $M$-augmenting path that starts in $a$ and minimizes the edges in $E(P) - E(T)$ and, by way of contradiction, assume that $E(P) - E(T) \neq \varnothing$, i.e. $P$ is not completely contained in $T$. Therefore, let $xy$ be the first edge in $E(P) - E(T)$ encountered while traversing $P$, starting at $a$, and without loss of generality assume that $x \in V(P)$.

**Claim:** $x \in A$.

*Proof of the Claim.* By way of contradiction, assume that $x \in B$.

- Assume that $x$ is not a leaf of $T$. Since the BFS starts at $a \in A$, and the edges of $T$ alternate between $M$ and $E(G) - M$, if $x \in B$ then the next edge $xy'$ in $T$ is an edge in $M$. Moreover — by the same reasoning — since $P$ is $M$-augmenting, it must be that $xy \in M$, meaning that $xy, xy' \in M$ contradicting the definition of matching ⨮

- Now, assume that $x$ is a leaf of $T$. By the same reasoning, $xy$ must be in $M$ because $P$ is $M$-augmenting, but $xy \notin E(T)$ would imply that the BFS stopped before adding the edge $xy$ to $T$ ⨮.

---

□

For instance, given the following setting
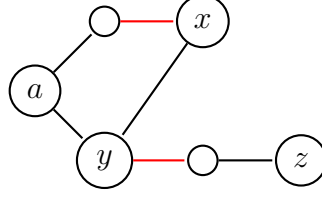


a possible path for $P$ would be $a \to x \to y \to z$. However, the path $a \to y \to z$ is still an $M$-augmenting path that starts at $a$ but has one fewer edge not in $T$ w.r.t. $P$, contradicting the definition of $P \nmid$.

Note that, in the general case we would consider the path $P' := a\ T\ y \cup y\ P$, however this is not guaranteed to be a path. The complete proof leverages the fact that $G$ is bipartite in order to prove that $P'$ is indeed a path, but it is very technical and outside the scope of these notes. □

Finally, now that we have a procedure which is guaranteed to find an augmenting path in a given bipartite graph, to return a maximum matching it suffices to run the following algorithm.

---

**Algorithm 2.1: Maximum matching (bipartite graphs)**

Given a bipartite graph $G$, the algorithm returns a maximum matching for $G$.

---

1: **function** MAXIMUMMATCHINGBIPGRAPHS($G$)
2:     $M := \varnothing$
3:     **do**
4:         $P := $ FINDAUGMENTINGPATH($G$)          ▷ the previous procedure
5:         Swap the edges between $M$ and $E(G) - M$ in $P$
6:     **while** $P \neq$ None
7:     **return** $M$
8: **end function**

---

In particular, the output of this algorithm is guaranteed to be a *maximum* matching thanks to Theorem 2.1, since the algorithm terminates when there are no more augmenting paths left in the graph $G$.

## 2.2  Perfect matching

By definition, a matching is *not* forced to cover all the vertices of a graph. However, if this happens the matching is called **perfect matching**.

> **Definition 2.7: Perfect matching**
>
> Given a graph $G$, a **perfect matching** of $G$ is a matching that covers all the vertices of $G$, i.e. $M$ is a perfect matching if and only if
>
> $$\forall v \in V(G) \quad \exists e \in M \quad v \cap e \neq \varnothing$$
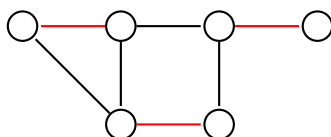


Figure 2.5: An example of a perfect matching.

Perfect matchings are an interesting topic of study when related to *bipartite graphs*. We observe that if a bipartite graph $G$, bipartitioned into $(A, B)$, admits a perfect matching $M$, it must be that $|A| = |B|$. This is because every matched edge must connect one vertex from $A$ to one vertex from $B$, and by definition $M$ matched *each* vertex exactly once. However, the converse is not true.
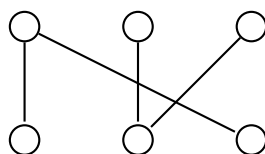


Figure 2.6: For instance, this bipartite graph, bipartitioned into $(A, B)$ such that $A$ is the uppermost row of vertices, even if $|A| = |B|$ this graph does not admit a perfect matching.

### 2.2.1 Hall's theorem

Because of this characterization on bipartite graphs, in addition to the concept of *perfect matching*, if $|A| \neq |B|$ we can define a *weaker* version of "perfect".

> **Definition 2.8: $A$-perfect matching**
>
> Given a bipartite graph $G$, bipartitioned through $(A, B)$ such that $|A| \leq |B|$, we say that a matching $M$ is an $A$-**perfect matching** if it covers all the vertices in $A$.

Note that we can always assume that $|A| \leq |B|$, without loss of generality. The following theorem, known as the Hall's marriage theorem — proved by Hall [Hal35] in 1935 — shows the conditions that guarantee that an $A$-perfect matching exists.

> **Theorem 2.4: Hall's marriage theorem**
>
> Given a bipartite graph $G$, bipartitioned into $(A, B)$, then $G$ admits an $A$-perfect matching if and only if
> $$\forall S \subseteq A \quad |S| \leq |\mathcal{N}(S)|$$

*Proof.* The direct implication is trivially true by definition of matching. We will prove the converse implication by contrapositive. Therefore, suppose that the bipartite graph $G$ does not admit any $A$-perfect matching, and consider a minimum vertex cover $V^*$ of $G$. Then, by Theorem 2.3, since $G$ is bipartite we know that for any maximum matching $M^*$ of $G$ it holds that $|M^*| = |V^*|$. However, since we are assuming that there are no $A$-perfect matchings of $G$, any maximum matching must have size strictly less than $|A|$, meaning that $|V^*| = |M^*| < |A|$.

Now, consider the set $S := A - V^*$; since $G$ is bipartite, it must hold that

$$\mathcal{N}(S) \subseteq V^* \cap B \implies |\mathcal{N}(S)| \leq |V^* \cap B| = |V^*| - |A \cap V^*|$$

Moreover, observe that

$$V^* \cap A = A - (A - V^*) = A - S \implies |V^* \cap A| = |A| - |S|$$

therefore, we have that

$$|\mathcal{N}(S)| \leq |V^*| - |V^* \cap A| = |V^*| - |A| + |S|$$

Lastly, since $|V^*| < |A| \iff |V^*| - |A| < 0$, we conclude that

$$|\mathcal{N}(S)| \leq |V^*| - |A| + |S| < 0 + |S| = |S| \implies |\mathcal{N}(S)| < |S|$$

which proves that there is at least one set $S$ for which the *Hall condition* does not hold. $\square$

### 2.2.2 Tutte's theorem

In the general case, what obstructs the existence of a perfect matching in a graph? Consider a graph $G$; clearly, if $n$ is odd, the graph does not admit perfect matchings, since each edge matches exactly two nodes, meaning that there will always be a free vertex in the graph.
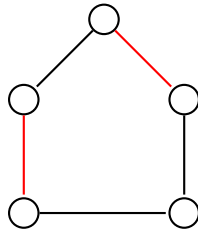


Figure 2.7: For instance, no matching of $C_5$ can be perfect.

For the same reasoning, if $G$ has a connected component with an odd number of vertices, $G$ does not admit perfect matchings. Hence, if $G$ admits a perfect matching, there cannot

be any connected component with an odd number of vertices. But is the converse true as well? Consider the following graph
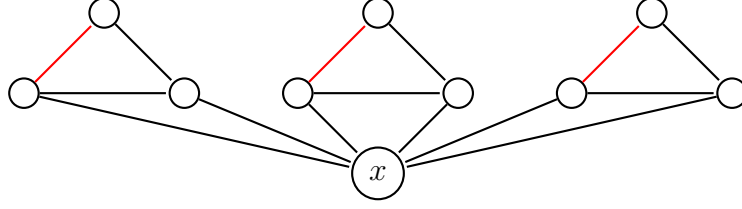


This graph is connected, and has 10 vertices, but it does not admit perfect matchings, meaning that the converse does not hold. Can we find a condition that guarantees that a given graph always admits a perfect matching?

Given a graph $G$, let $\mathcal{O}(G)$ be the number of components with an odd number of vertices. The next theorem, proved by Lovász and Plummer [LP09] in 1947 shows that the following property

$$\forall S \subseteq V(G) \quad \mathcal{O}(G[V(G) - S]) \leq |S|$$

which we will refer to as **Tutte condition** — is a *necessary* and *sufficient* condition to guarantee that a graph admits a perfect matching. In fact, in the example of $C_5$ we observe that the set that violates the Tutte condition is $S = \varnothing$, and in the second example is $S = \{x\}$.

> **Lemma 2.1**
>
> Given a graph $G$, and a supergraph $G'$ of $G$, if $G$ satisfies the Tutte condition, then $G'$ satisfies the Tutte condition as well.

*Proof.* By contrapositive, suppose that $G'$ fails the Tutte condition, meaning that there exists a set $S \subseteq V(G') = V(G)$ such that $\mathcal{O}(G[V(G') - S]) > |S|$. Since $G'$ is obtained by adding edges to $G$, the number of odd components in $G'$ can either remain the same, or decrease if two different components having an odd number of vertices became connected in $G'$. Therefore, we have that

$$|S| < \mathcal{O}(G[V(G') - S]) \leq \mathcal{O}(G[V(G) - S])$$

meaning that $G$ fails the Tutte condition as well. □

We are now ready to prove Tutte's theorem.

> **Theorem 2.5: Tutte's theorem**
>
> A graph $G$ admits a perfect matching if and only if for any $S \subseteq V(G)$ it holds that $\mathcal{O}(G[V(G) - S]) \leq |S|$ — meaning that it satisfies the Tutte condition.

*Proof.*

*Direct implication.* For the direct implication, consider a graph $G$ that admits a perfect matching $M$, and by way of contradiction assume that there exists a set $S \subseteq V(G)$ that violates the Tutte condition. For the previous observation, we know that each component of $G[V(G) - S]$ that has an even number of vertices will not have free nodes w.r.t. $M$, and each component of $G[V(G) - S]$ that has an odd number of vertices will have one free node w.r.t. $M$. In particular, these free vertices must be covered by $M$, since $M$ is perfect, and the only way they can be covered by $M$ is through the vertices of $S$. However, since $\mathcal{O}(G[V(G) - S]) > |S|$, there exists at least one free vertex in $G[V(G) - S]$ w.r.t. $M$ that cannot be matched to any of the vertices of $S$, meaning that $M$ is not perfect $\lightning$.

*Converse implication.* We will prove the contrapositive of the converse implication. Therefore, suppose that $G$ does not admit any perfect matching, and we will prove that there exists a set of vertices for which the Tutte condition does not hold. Let $G'$ be the maximal supergraph of $G$ that still does not admit perfect matchings. We say that a set $X$ satisfies the *clique-adjacency* condition if every component of $G'[V(G') - X]$ is a clique, and every vertex $u \in X$ is adjacent to every vertex $v \notin X$.

**Claim:** If there is a subset $S' \subseteq V(G')$ that satisfies the clique-adjacency condition, then $G$ does not satisfy the Tutte condition.

*Proof of the Claim.* Note that $S' \subseteq V(G') = V(G)$, hence if $S'$ violates the Tutte condition on $G$, the claim is trivially true. Therefore, we will assume that $S'$ satisfies both the clique-adjacency condition, and the Tutte condition on $G$.

Let $S'$ be a subset that satisfies the clique-adjacency condition; hence, by definition every component of $G'[V(G') - S']$ is a clique. Thus, consider a clique, and let $n$ be the number of its vertices.

- If $n$ is even, we can always find a perfect matching restricted on it.

- If $n$ is odd, we can always find a perfect matching restricted to $n - 1$ vertices, but since we are assuming that $S'$ satisfies the Tutte condition on $G$, we know that $\mathcal{O}(G[V(G) - S']) \leq |S'|$. Therefore, by the clique-adjacency condition we know that we can always match the $n$-th vertex of the clique with a vertex of $S'$.

TODO _____ □

> da
> finire

Consider the set

$$S := \{v \in V(G') \mid \deg_{G'}(v) = n - 1\}$$

By way of contradiction, suppose that $S$ violates the clique-adjacency condition. This happens if at least one of the following holds:

(1) there are two vertices $u \in S$ and $v \notin S$ such that $u \notin S$

(2) there is a component of $G'[V(G') - S]$ that is not a clique

However, by definition of $S$, each vertex $v \in S$ has $\deg_{G'} = n - 1$, hence (1) cannot be true, meaning that (2) must hold. Hence, consider a connected component of

$G'[V(G') - S]$ that is not a clique, i.e. there are at least two vertices $x$ and $y$ such that $x \nsim y$.

Let $P$ be the shortest path from $x$ to $y$ in $G'[V(G') - S]$, and let $a$, $b$ and $c$ be the first three vertices of $P$ — note that $a, b, c \notin S$. We observe that $a \nsim c$, otherwise $P$ would not be the shortest path. Note that $b \notin S$, therefore $\deg_{G'}(b) < n - 1$, i.e. there exists a vertex $d$ such that $b \nsim d$.
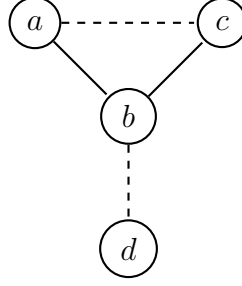


Figure 2.8: The "kite-shaped" figure we are considering — the dashed lines represent the *missing edges*.

By maximality of $G'$, we know that $G' \cup \{ac\}$ must have a perfect matching $M_1$, and $G' \cup \{bd\}$ must have a perfect matching $M_2$. Moreover, $ac \in M_1$ and $bd \in M_2$, otherwise $M_1$ and $M_2$ would be perfect matchings on $G'$.

**Claim:** $(M_1 - \{ac\}) \cup (M_2 - \{bd\})$ contains a perfect matching for $G'$.

*Proof of the Claim.* Let $\overline{M} = (M_1 - \{ac\}) \Delta (M_2 - \{bd\})$. By the same reasoning applied for Theorem 2.1, since $M_1$ and $M_2$ are two matchings, every vertex of $\overline{M}$ has degree at most 2. Hence, consider a vertex $z$ in the subgraph induced by $\overline{M}$

- if $\deg_{\overline{M}}(z)$ is 0 or 2, $z$ is matched both by $M_1$ and $M_2$

- since $M_1$ is a perfect matching, each vertex of $G'$ is matched by $M_1 - \{ac\}$, except for $a$ and $c$

- likewise, since $M_2$ is a perfect matching, each vertex of $G'$ is matched by $M_2 - \{bd\}$, except for $b$ and $d$

which implies that the only vertices $z$ that have $\deg_{\overline{M}}(z) = 1$ are precisely $a$, $b$, $c$ and $d$.

Moreover, each component of the subgraph induced by $\overline{M}$ is either an isolated vertex, a cycle or a path, and the edges of the components must alternate between $M_1$ and $M_2$. Therefore, because of the degrees of $a$, $b$, $c$ and $d$, it must be that these vertices are endpoints of two alternating paths $P_1$ and $P_2$. Moreover, since $ac \in M_1$ and $bd \in M_2$, one of these paths must be an $M_1$-alternating path, while other must be $M_2$ alternating — and without loss of generality let $P_1$ be the $M_1$-alternating. Note that the set

$$M' := (M_1 \cup M_2) - (E(P_1) \cup E(P_2))$$

is a perfect matching on $G[V(G) - (V(P_1) \cup V(P_2))]$. We have three cases for $P_1$ and $P_2$:

(a) $P_1$ is a path $a \to c$, and $P_2$ is a path $b \to d$; then

$$(M_1 \cap E(P_2)) \cup (M_2 \cap E(P_1)) \cup M'$$

is a perfect matching on $G'$

(b) $P_1$ is a path $a \to b$, and $P_2$ is a path $c \to d$; then

$$(M_1 \cap E(P_2)) \cup (M_2 \cap E(P_1)) \cup M' \cup \{bd\}$$

is a perfect matching on $G'$

(c) $P_1$ is a path $a \to d$, and $P_2$ is a path $b \to c$; then

$$(M_1 \cap E(P_2)) \cup (M_2 \cap E(P_1)) \cup M' \cup \{ac\}$$

is a perfect matching on $G'$

$\square$

Therefore, because of this claim $G'$ always contains a perfect matching, contradicting the definition of $G'$. This implies that $S$ must satisfy the clique-adjacency condition. Hence, by the first claim we have that $G$ satisfies the Tutte condition

first claim is wrong?

$\square$

## 2.3 Stable matching

Matchings in bipartite graphs are particularly useful as they can be applied to model various types of problems across different fields. One well-known example is the stable matching problem, which arises in scenarios like job assignments, college admissions, and matchmaking systems. In this problem, the goal is to find a *stable pairing* between two sets of entities — such as students and universities — where no two unmatched entities would prefer each other over their current assignments.

> **Definition 2.9: Stable matching**
>
> Given a graph $G$, bipartitioned through $(A, B)$, and a matching $M$ of $G$, consider a family of *preference* functions $\{w_v\}_{v \in V(G)}$ that for each vertex $v \in V(G)$, assign a value to the edges $vu \in E(G)$, for all $u \in \mathcal{N}(v)$
>
> $$\forall v \in V(G) \quad w_v : \mathcal{N}(v) \to \mathbb{R}$$
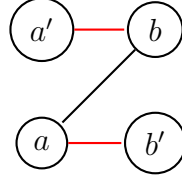>
> We say that $M$ is **stable** if for each $ab \in E(G)$ it does <u>not</u> happen that
>
> $$(a \text{ free } \vee (\exists ab' \in M \quad w_a(b) > w_a(b')))$$
>
> $$\wedge$$
>
> $$(b \text{ free } \vee (\exists a'b \in M \quad w_b(a) > w_b(a')))$$

For instance, consider the following bipartite graph $G$, and a matching $M$ — outlined in *red*:



If the family of preference functions $\{w_v\}_{v \in V}$ is such that

$$w_a(b) > w_b(b') \wedge w_b(a) > w_b(a')$$

which implies that <u>both</u> $a$ and $b$ would prefer to *switch* their current matched vertex — then $M$ is *not* stable. Note that the empty matching is *not* stable, by definition.

The following theorem, proved by Gale and Shapley [GS62] in 1962, proves that a stable matching can be always constructed in a bipartite graph, regardless of the preference functions.

---

**Theorem 2.6: Gale-Shapley theorem**

Given a bipartite graph $G$, and a family of preference functions $\{w_v\}_{v \in V(G)}$, there exists a stable matching of $G$.

---

*Proof.* Before proving the theorem, we need to introduce some definitions.

Consider a bipartite graph $G$, bipartitioned through $(A, B)$, and consider a matching $M$. Given two vertices $a \in A$ and $b \in B$, we say that $a$ is **acceptable to** $b$ if

- $b$ is free, or
- $b$ is matched to a vertex $a'$, and $w_b(a) > w_b(a')$

Moreover, we say that a vertex $a \in A$ is **happy** if either

- $a$ is free, or
- $ab \in M$ and for each $b'$ such that $a$ is acceptable to $b'$, it holds that $w_a(b) \geq w_a(b')$

Observe that in the empty matching every vertex is happy.

**Claim:** Consider a matching $M$ such that each vertex is happy; if $M$ is not stable, then there exists a vertex $a \in A$ such that $a$ is free and acceptable to some $b \in B$.

*Proof of the Claim.* By instability of $M$, there must be an edge $ab \notin M$ such that $a$ is free, or it prefers $b$ to its current partner, and vice versa. By way of contradiction, suppose that $a$ is matched to some $b' \in B$, hence by instability of $M$ through $ab$ we know that $w_a(b) > w_a(b')$

- If $b$ is free, then by definition $a$ is acceptable to $b$; however, by happiness of $a$ it must be that $w_a(b') \geq w_a(b)$ ↯

- If $b$ is matched by some edge $a'b \in M$, by instability of $M$ through $ab$ we know that $w_b(a) > w_b(a')$, hence $a$ is acceptable to $b$, and by happiness of $a$ it must be that $w_a(b') \geq w_a(b)$ ↯

This implies that $a$ must be free; therefore, we have that

- If $b$ is free as well, then by definition $a$ is acceptable to $b$

- If $b$ is matched by some edge $a'b \in M$, by instability of $M$ through $ab$ we know that $w_b(a) > w_b(a')$, hence $a$ is still acceptable to $b$

$\square$

Now, consider another matching $M'$ of $G$; we say that $M$ is **better than** $M'$ if

- $\forall a'b \in M' \quad \exists ab \in M \quad w_b(a) \geq w_b(a')$, meaning that every vertex $b \in B$ prefers its match in $M$ at least as much as its match in $M'$, and

- $\exists a'b \in M', ab \in M \quad w_b(a) > w_b(a')$, meaning that there is at least one vertex $b$ that strictly perfers its match in $M$ over its match in $M'$

In other words, $M$ is better than $M'$ if no match in $M$ is worse than in $M'$, and at least one match is strictly better.

Let $M_k$ be an unstable matching such that every vertex is happy; therefore, by the previous claim we know that there exists a vertex $a \in A$ such that $a$ is free and and acceptable to some $b \in B$. We will construct a matching $M_{k+1}$ as follows:

$$b^* \in \underset{\substack{b \in \mathcal{N}(a): \\ a \text{ acceptable to } b}}{\arg\max} \quad w_a(b) \implies M_{k+1} := (M_k \cup \{ab^*\}) - \{a'b^* \in M_k \mid a' \in A\}$$

meaning that $M_{k+1}$ is obtained from $M_k$ by adding the edge $ab^*$, where $b^*$ maximizes $w_a(b^*)$, and removing the edge $a'b^*$ from $M_k$, if present — the last set is either $\{a'b^*\}$ or $\varnothing$.

**Claim:** $M_{k+1}$ is better than $M_k$, and if $M_k$ is ensures that every vertex is happy, $M_{k+1}$ does as well.

*Proof of the Claim.* First, we prove that $M_{k+1}$ is better than $M_k$. The first condition that $M_{k+1}$ has to satisfy in order to be better than $M_k$ is true simply because we removed the edge $a'b^*$ if it was present in $M_k$, and the rest of the matching has not been altered. Moreover, if $b^*$ was free then the second condition is vacuously true, otherwise if the edge $a'b^*$ was present in $M_k$, in $M_{k+1}$ there we added the edge $ab^*$ and we know that $w_{b^*}(a) > w_{b^*}(a')$ because $a$ is acceptable to $b^*$ by definition.

Now, assume that $M_k$ is such that every vertex is happy. Since $b^*$ is the neighbor of $a$ that maximizes $w_a(b^*)$, $a$ is happy by definition. Now, if $b^*$ was free in $M_k$, then we only added $ab^*$ to $M_{k+1}$, hence every vertex is happy w.r.t. $M_{k+1}$. Otherwise, if $b^*$ was not free in $M_k$, i.e. there was an edge $a'b^* \in M_k$, by definition $M_{k+1}$ will not contain $a'b^*$, meaning that $a'$ is free w.r.t. $M_{k+1}$, hence $a'$ is happy by definition. $\square$

Now, consider the empty matching $M_0 := \varnothing$; we already observed that $M_0$ is not stable, but is such that each vertex is happy, therefore by the previous claim we can extend $M_0$

into $M_1$ through some vertex $a \in A$ that satisfied the condition of the first claim. In particular, since we proved that this process preserves the happiness of the vertices, we can repeat this process as long as the current matching $M_i$ is still unstable.

Observe that every matching in the sequence is better than the previous ones, therefore such sequence cannot cycle. Moreover, since there is a finite number of possible matching of $G$, the sequence will eventually reach a matching where every vertex of $A$ is matched — if $|A| \neq |B|$ we can assume that $|A| < |B|$ without loss of generality; call this matching $\hat{M}$. Hence, by contrapositive of the first claim, we know that $\hat{M}$ is either unstable, or has an unhappy vertex. However, by construction of our sequence, every vertex of $\hat{M}$ is happy, therefore it must be that $\hat{M}$ is stable. $\square$

## 2.4 Exercises

### Problem 2.1

Let $G$ be a $k$-regular bipartite graph, bipartitioned through $(A, B)$. Prove that

1. $|A| = |B|$

2. $G$ has a perfect matching

*Solution.* Since $G$ is $k$-regular, it holds that the number of edges that have an endpoint in $A$ is precisely $k|A|$, and the number of edges that have an endpoint in $B$ is exactly $k|B|$; moreover, since $G$ is bipartite, we have that

$$k|A| = k|B| \implies |A| = |B|$$

We will prove the second statement by using Theorem 2.4. By way of contradiction, assume that there is a set $S \subseteq V(G)$ such that $|S| > |\mathcal{N}(S)| \implies k|S| > k|\mathcal{N}(S)|$. Applying the same argument as before, the number of edges $ab$ such that $a \in S, b \in \mathcal{N}(S)$ is exactly $k|S|$, hence by the pigeonhole principle there must be at least one vertex in $\mathcal{N}(S)$ that has more than $k$ vertices, contradicting the $k$-regularity of $G$. $\square$

### Problem 2.2

Let $k$ be an integer. Show that any two partitions of a finite set into $k$-sets admit a common choice of representatives.

*Solution.* Let $S$ be a finite set, and let $\mathcal{P} := \{P_1, \ldots, P_k\}$ and $\mathcal{P}' := \{P'_1, \ldots, P'_k\}$ be two partitions over $S$ into $k$-sets — we observe that $|\mathcal{P}| = |\mathcal{P}'| = k$ and for any $Q_1, Q_2 \in \mathcal{P} \cup \mathcal{P}'$ we have that $|Q_1| = |Q_2| = \frac{|S|}{k}$.

Now, construct a graph $G$ as follows:

- add a vertex for each $k$-set of the two partitions — hence $V(G) = \mathcal{P} \cup \mathcal{P}'$

- $\forall Q_1, Q_2 \in \mathcal{P} \cup \mathcal{P}'$ $Q_1 \sim Q_2 \iff Q_1 \cap Q_2 \neq \varnothing$

We observe that $G$ is a bipartite graph, bipartitioned through $(\mathcal{P}, \mathcal{P}')$ itself: in fact, since $\mathcal{P}$ is a partition of $S$, its $k$-sets cannot intersect, hence $\mathcal{P}$ induces an independent set — and the same reasoning can be applied for $\mathcal{P}'$ as well.

**Claim:** $G$ admits a perfect matching.

*Proof of the Claim.* Fix $X \subseteq \mathcal{P}$. Since $\mathcal{P}$ is a partition of $S$ into $k$-sets, we have that $\left| \bigcup_{P \in X} \right| P = k\,|X|$ since the $k$-sets cannot overlap. By a similar argument, since $\mathcal{P}'$ is a partition of $S$ into $k$-sets as well, we have that $\left| \bigcup_{P' \in \mathcal{N}(X)} P' \right| = k\,|\mathcal{N}(X)|$.

By way of contradiction, suppose that there is an element $x \in \bigcup_{P \in X} P - \bigcup_{P' \in \mathcal{N}(X)} P'$, meaning that there is an element $x \in P$ for a $k$-set $P \in X$ such that for any $P' \in \mathcal{N}(X)$ it holds that $x \notin \mathcal{P}'$. However, $\mathcal{P}$ and $\mathcal{P}'$ partition $S$, and in particular they cover $S$, meaning that there exists a $k$-set in $\mathcal{P}$ containing $x$, say $P'$. Moreover, by construction of $G$ if $x \in P \cap P'$ then $G$ contains the edge $\{P, P'\}$, meaning that $P' \in \mathcal{N}(X)$, but we assumed that no $k$-set of $\mathcal{N}(X)$ contained $x$ $\lightning$. This proves that $\bigcup_{P \in X} P \subseteq \bigcup_{P' \in \mathcal{N}(X)} P'$.

Finally, together with the previous observation, this implies that

$$k\,|S| = \left| \bigcup_{P \in X} P \right| \leq \left| \bigcup_{P' \in \mathcal{N}(X)} P' \right| = k\,|\mathcal{N}(X)|$$

and in particular, we have that

$$k\,|S| \leq k\,|\mathcal{N}(X)| \iff |X| \leq |\mathcal{N}(X)|$$

meaning that the Hall condition is satisfied for any set $X \subseteq \mathcal{P}$. Hence, by Theorem 2.4 we get that $G$ admits a $\mathcal{P}$-perfect matching, but because $|\mathcal{P}| = |\mathcal{P}'|$ then any $\mathcal{P}$-perfect matching of $G$ is also a perfect matching of $G$. $\qquad \square$

Now, let $M$ be a perfect matching of $G$, and consider the set $R$ constructed as follows: for each edge $\{P, P'\} \in M$, pick one element $r \in P \cap P'$ and add it to $R$.

**Claim:** $R$ is a set of representatives for both $\mathcal{P}$ and $\mathcal{P}'$.

*Proof of the Claim.* Since $M$ is a matching, every element of $R$ will represent exactly two partitions, namely one in $\mathcal{P}$ and the other in $\mathcal{P}'$. Moreover, since $M$ is perfect, every partition of $\mathcal{P}$ and $\mathcal{P}'$ will be covered by at least one element of $R$. Finally, no element of $R$ can represent more than one $k$-set in the same partition since $\mathcal{P}$ and $\mathcal{P}'$ are independent sets in $G$. $\qquad \square$

This last claim concludes the proof. $\qquad \square$

# 3
# Graph packing

> **Definition 3.1:** *A-B* **paths**
>
> Given a graph $G$, and two sets $A, B \subseteq V(G)$, an *A-B* **path** is a path that has one end in $A$, one end in $B$, and no internal vertices in $A \cup B$.



Figure 3.1: For instance, if $A$ is the set of *green* vertices, and $B$ is the set of *orange* vertices, then the *red* is an *A-B* path but the *blue* one is not.

Note that if $A \cap B \neq \varnothing$, any vertex in $A \cap B$ is a trivial *A-B* path.

> **Definition 3.2:** *A-B* **hitting set**
>
> Given a graph $G$, and two sets $A, B \subseteq V(G)$, an *A-B* **hitting** set is a set $X \subseteq V(G)$ such that every *A-B* path has a vertex in $X$.
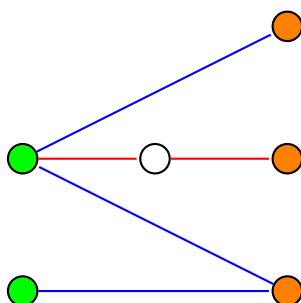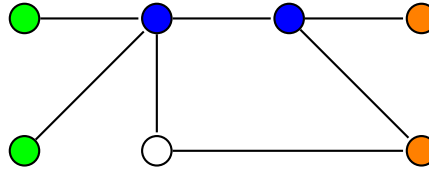
Figure 3.2: For example, if $A$ is the set of *green* vertices, and $B$ is the set of *orange* vertices, then the *blue* is an $A$-$B$ hitting set.

Note that if $X$ is an $A$-$B$ hitting set, then $A \cap B \subseteq X$, otherwise any single vertex in $A \cap B$ would represent an $A$-$B$ trivial path that does not pass through the hitting set $X$. Moreover, consider an $A$-$B$ hitting set $X$ of a graph $G$; by definition, all the $A$-$B$ paths must have at least one vertex in $X$, but since $A$-$B$ paths do not have internal vertices in $A \cup B$, there cannot be $|X| + 1$ disjoint $A$-$B$ paths.

### Definition 3.3: $A$-$B$ separation

Given a graph $G$, and two sets $A, B \subseteq V(G)$, two sets $X, Y \subseteq V(G)$ describe an $A$-$B$ **separation** $(X, Y)$ of $G$ if and only if

- $A \subseteq X$
- $B \subseteq Y$
- $X \cup Y = V(G)$
- no edge has one end in $X - Y$ and the other in $Y - X$

We say that a separation $(X, Y)$ has **order** $|X \cap Y|$.

Given a graph $G$, two sets $A, B \subseteq V(G)$, and an $A$-$B$ separation $(X, Y)$ of $G$, we observe that $G[V(G) - (X \cap Y)]$ is disconnected.

### Proposition 3.1

Given a graph $G$, and two sets $A, B \subseteq V(G)$, there exists an $A$-$B$ hitting set of size $k$ in $G$ if and only if there exists an $A$-$B$ separation of order $k$.

*Proof.* The converse implication is trivially true, because if $(X, Y)$ is an $A$-$B$ separation of $G$, then $X \cap Y$ itself is an $A$-$B$ hitting set of $G$ by definition; therefore, we just need to prove the direct implication.

Let $Z$ be an $A$-$B$ hitting set of $G$ such that $|Z| = k$, and consider the connected components $C_1, \ldots, C_\ell$ of $G[V(G) - Z]$. Note that there cannot be any component $C_i$ that has a vertex $a \in V(G) \cap (A - Z)$ and a vertex $b \in V(C) \cap (B - Z)$, because otherwise by connectivity of $C_i$ there would be a path $a \to b$ — which is an $A$-$B$ path — that avoids

the hitting set $Z$. Hence, we can define the two following sets

$$X := Z \cup \bigcup_{\substack{C \text{ component} \\ \text{of } G[V(G)-Z] \\ \text{such that } A \cap C \neq \varnothing}} V(C)$$

$$Y := Z \cup \bigcup_{\substack{C \text{ component} \\ \text{of } G[V(G)-Z] \\ \text{such that } A \cap C = \varnothing}} V(C)$$

We observe that, by definition

- $A \subseteq X$

- $Y$ contains all the vertices in $B - Z$, therefore if $B \cap Z = \varnothing$ then trivially $B \subseteq Y$, and if $B \cap Z \neq \varnothing$ we observe that $Z \subseteq Y$ by definition, hence $B$ is always completely covered by $Y$

- $X \cap Y = Z$

and for the previous observation there cannot be edges with one end in $X - Y$ and the other in $Y - X$. Hence, we conclude that $(X, Y)$ is a separation of $G$, and $X \cap Y = Z \implies |X \cap Y| = |Z| = k$. $\qquad \square$

The following theorem establishes a deep connection between *connectivity* and *separation* in graphs, and was proved by Menger [Men27] in 1927.

> **Theorem 3.1: Menger's theorem**
>
> Given a graph $G$, and two sets $A, B \subseteq V(G)$, the number of disjoint $A$-$B$ paths is equal to the minimum order of an $A$-$B$ separation of $G$.

*Proof.* Consider an $A$-$B$ separation $(X, Y)$ of minimum order of $G$, i.e. a separation that minimizes $|X \cap Y|$; observe that each $A$-$B$ path must traverse at least one vertex of $X \cap Y$, therefore the number of disjoint $A$-$B$ paths that can exist is at most $|X \cap Y|$. This proves the direct implication of the theorem.

To prove the converse implication, we would need to prove that the minimum order of an $A$-$B$ separation of $G$ is no more than the number of disjoint $A$-$B$ paths. Instead, we will prove that, for a fixed value $k$, in $G$ there are $k$ disjoint $A$-$B$ paths, or there is an $A$-$B$ separation of order at most $k - 1$, which is logically equivalent. In particular, we will proceed by induction on $m$, i.e. the number of edges of $G$.

> *Base case.* If $m = 0$, the only possible $A$-$B$ paths in $G$ are trivial paths, i.e. isolated vertices in $A \cap B$. Hence, if $|A \cap B| \geq k$ the base case holds; instead, if $|A \cap B| \leq k - 1$, we observe that $A \cap B$ itself is an $A$-$B$ hitting set of $G$ of size at most $k - 1$, which means that there exists an $A$-$B$ separation of order at most $k - 1$ by Proposition 3.1.
>
> *Inductive hypothesis.* Assume that the statement holds on a graph that has $m - 1$ edges.

*Inductive step.* We will prove that the statement holds for a graph $G$ that has $m$ edges as well. In particular, fix an edge $xy \in E(G)$, and note that the graph $G - \{xy\}$ has $m-1$ edges, meaning that the inductive hypothesis holds for $G - \{xy\}$. However, if $G - \{xy\}$ has $k$ disjoint $A$-$B$ paths, then $G$ also has $k$ disjoint $A$-$B$ paths, therefore we may assume that the inductive hypothesis holds for $G - \{xy\}$ precisely because the latter has an $A$-$B$ separation of order at most $k - 1$.

Hence, let $(X, Y)$ be an $A$-$B$ separation of $G - \{xy\}$ of order at most $k - 1$; we observe that if $x, y \in X$, or $x, y \in Y$, then $(X, Y)$ is also a separation for $G$, which would conclude the proof. Thus, suppose that $x \in X - Y$ and $y \in Y - X$; then $(X, Y \cup \{x\})$ is an $A$-$B$ separation of $G$ that either has still order at most $k - 1$, or it becomes a separation of order $k$ with $k - 1$ $A$-$B$ paths. Since the first case concludes the proof, we may assume the second one to hold.

Consider the induced subgraphs $G[X]$ and $G[Y]$ ──────────────────── da finire

□

## 3.1 $k$-connectivity

> **Definition 3.4: $k$-connectivity**
>
> A graph $G$ is said to be $k$-**connected** if $|V(G)| \geq k + 1$, and for each $X \subseteq V(G)$ such that $|X| \leq k - 1$ it holds that $G[V(G) - X]$ is connected.

In other words, a graph is $k$-connected if it has at least $k + 1$ vertices, and it remains connected whenever fewer than $k$ vertices are removed. Therefor, by definition we observe that if a graph is *not* $k$-connected, it must admit a separation of order at most $k - 1$.
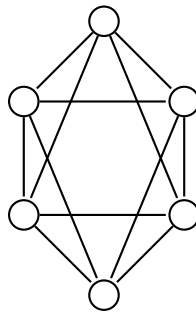


Figure 3.3: A 4-connected graph.

Clearly, any connected graph is 1-connected; moreover, any $K_{t+1}$ clique is $t$-connected.

> **Proposition 3.2**
>
> If $G$ is $k$-connected, then $\delta \geq k$.

*Proof.* By way of contradiction, assume that $\delta < k$, i.e. there is a vertex $x \in V(G)$ such that $\deg(x) < k \implies |\mathcal{N}(x)| < k$. Now, consider $G[V(G) - \mathcal{N}(x)]$: this subgraph is disconnected because it has $x$ isolated, but $|\mathcal{N}(x)| \leq k - 1$, meaning that this set contradicts the $k$-connectivity of $G$ ↯. $\square$

> ### Proposition 3.3
>
> If $G$ is $k$-connected, then for any $xy \in E(G)$ it holds that $G - \{xy\}$ is $(k - 1)$-connected.

*Proof.* By way of contradiction, suppose that $G - \{xy\}$ is not $(k-1)$-connected, i.e. there exists a set $X \subseteq V(G)$ such that $|X| \leq k - 2$ and $(G - \{xy\})[V(G) - X]$ is disconnected. By $k$-connectivity of $G$, since $|X| \leq k - 2$, we know that $G[V(G) - X]$ is connected, therefore if $(G - \{xy\})[V(G) - X]$ is disconnected it must imply that by removing $xy$ we disconnect $G[V(G) - X]$. Now, since $|X| \leq k - 2$, we know that $|X \cup \{y\}| \leq k - 1$, but $G[V(G) - (X \cup \{y\})]$ does not contain the edge $xy$, hence it is disconnected, contradicting the $k$-connectivity of $G$ ↯. $\square$

> ### Definition 3.5: Internal disjointness
>
> Given a graph $G$, two paths $P_1$ and $P_2$ of $G$ are said to be **internally disjoint** if every vertex of $V(P_1) \cap V(P_2)$ is an endpoint of $P_1$ and $P_2$.
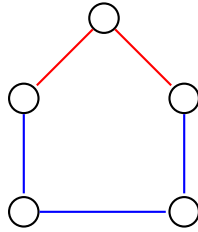


Figure 3.4: For instance, the *red* and *blue* paths are internally disjoint.

> ### Proposition 3.4
>
> Given a $k$-connected graph, and two sets $A, B \subseteq V(G)$, if $|A|, |B| \geq k$ then there exist $k$ disjoint $A$-$B$ paths in $G$.

*Proof.* By way of contradiction, if $G$ does not admit $k$ disjoint $A$-$B$ paths, therefore by Theorem 3.1 the minimum order of an $A$-$B$ separation is at most $k - 1$; hence, let $(X, Y)$ be a minimum order $A$-$B$ separation of $G$, i.e. $|X \cap Y| \leq k - 1$. Observe that $|A| \geq k$ and $A \subseteq X$, therefore $|X| \geq k$ which means that $X - Y \neq \varnothing$, and analogously $Y - X \neq \varnothing$. However, by definition of $A$-$B$ separation we observe that $G[V(G) - (X \cap Y)]$ is disconnected, and we removed less than $k$ vertices, contradicting the $k$-connectivity of $G$ ↯. $\square$

**Theorem 3.2**

A graph is $k$-connected graph if and only if for any pair of vertices $x, y \in V(G)$ such that $x \neq y$ there exist $k$ $\{x\}$-$\{y\}$ internally disjoint paths.

*Proof.*

*Direct implication.* Fix a pair of distinct vertices $x, y \in V(G)$; we have two cases.

- The two vertices are not adjacent, i.e. $x \nsim y$. By Proposition 3.2, we know that $|\mathcal{N}(x)|, |\mathcal{N}(y)| \geq k$, hence by the previous proposition we there exist $k$ disjoint $\mathcal{N}(x)$-$\mathcal{N}(y)$ paths, which will form $k$ *internally disjoint* $\{x\}$-$\{y\}$ paths along with the edges to $x$ and $y$.

- The two vertices are not adjacent, i.e. $x \nsim y$. By $k$-connectivity of $G$, we know that $G - \{xy\}$ is $(k-1)$-connected by Proposition 3.3, therefore by the same argument of the previous case we have $k - 1$ internally disjoint $\{x\}$-$\{y\}$ path, therefore $G$ will have $k$ internally disjoint $\{x\}$-$\{y\}$ paths along with the edge $xy$.

*Converse implication.* Fix two distinct vertices $x, y \in V(G)$; since there are $k$ $\{x\}$-$\{y\}$ internally disjoint paths, no set of $k-1$ — or less — vertices of $G$ can disconnect $x$ and $y$, implying that $G$ is $k$-connected by definition.

$\square$

**Corollary 3.1**

Given a $k$-connected graph $G$, a vertex $x \in V(G)$, and a subset $Y \subseteq V(G)$ such that $|Y| \geq k$, there exist $k$ $\{x\}$-$Y$ paths $P_1, \ldots, P_k$ such that $P_i \cap P_j = \{x\}$ — for all $i, j \in [k]$ such that $i \neq j$.

*Proof.* Add a vertex $z$ to the vertices of $G$, and for each $y \in Y$ add an edge $yz$; we observe that $G[V(G) \cup \{z\}]$ is still $k$-connected, because $|Y| \geq k$. Now, since $x \neq z$, by the previous proposition there exist $k$ $\{x\}$-$\{z\}$ internally disjoint paths, and these paths must traverse $Y$, therefore each path $P_i$ must have the form $x \to y_i \to z$ for some $y_i \in Y$. Hence, if we consider the subpaths $x \to y_i$ for each $i$, in $G$ these are precisely $k$ $\{x\}$-$Y$ paths intersects only in $x$. $\square$

**Theorem 3.3**

Given a $k$-connected graph $G$ such that $k \geq 2$, and $k$ vertices $x_1, \ldots, x_k \in V(G)$, there exist a cycle in $G$ that contains all the vertices $x_1, \ldots, x_k$.

*Proof.* By way of contradiction, let $C$ be the cycle that contains as many vertices $x_i$ as possible; if $x_1, \ldots, x_k \in V(C)$, the theorem is trivially verified, hence we may assume that at least one vertex $x_i$ is not in $V(C)$, and without loss of generality assume that $x_k \notin V(C)$. By Proposition 3.2 we know that $\delta \geq k$, and by Proposition 1.3 we know

that $|V(C)| \geq k+1$. Hence, by the previous corollary we know that exist $k$ $\{x_k\}$-$V(C)$ paths that only intersect in $x_k$.

Now, since there are at most $k-1$ vertices $x_i$ in $V(C)$, and there are $k$ $\{x_k\}$-$V(C)$ paths, by the pigeonhole principle there must be a subpath $Q$ of $C$ of the form $x_i \to x_j$ — for $i, j \in [k-1]$ distinct — that has no internal vertex in $\{x_1, \ldots, x_{k-1}\}$, and such that there are two paths $P_1$ and $P_2$ that $x_k$ as one endpoint and the other lies in $Q$.
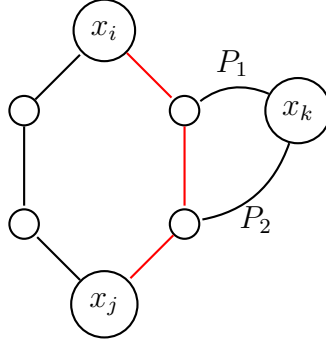


Figure 3.5: In the figure, the *red* segments compose the subpath $Q$.

Therefore, we can reroute $Q$ by passing through $P_1$ and $P_2$ in order to find a cycle $C \cup \{x_k\}$ that has a larger number of vertices $x_i$, contradicting the definition of $C$ $\lightning$. □

Note that this theorem *does not* guarantee the order in which the vertices are present in the cycle. In fact, asking to preserve the order of the vertices $x_1, \ldots, x_k$ given is a *much harder* question, and the only result we currently know is the following.

**Theorem 3.4**

Given a $10k$-connected graph $G$, and $k$ vertices $x_1, \ldots, x_k \in V(G)$, there exist a cycle in $G$ that contains all the vertices $x_1, \ldots, x_k$ in the same order.

## 3.2 Feedback vertex set

Consider the following combinatorial structure.

**Definition 3.6: Feedback vertex set**

Given a graph $G$, a subset $X \subseteq V(G)$ is said to be a **feedback vertex set** (FVS) if $G[V(G) - X]$ is acyclic; equivalently, $X$ is an FVS if it intersects all cycles of $G$.
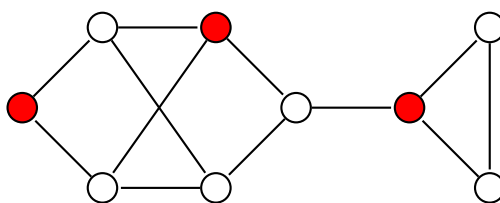
Figure 3.6: For instance, the *red* set of vertices is an FVS of this graph.

In many applications, it is important to find a **minimum** FVS; however, this problem is known to be NP-Complete — as proved by Karp [Kar72] in 1972. Therefore, we often focus on estimating the size of this set by computing lower and upper bounds. In particular, in this section we will present a result proved by Erdös and Pósa [EP65] in 1965. But before introducing their result about FVSs, we must discuss some combinatorial structures first.

### 3.2.1 Topological minors

Consider a graph $G$, and an edge $xy \in E(G)$; to **subdivide** $xy$ means to *remove* the edge $xy$ from $E(G)$ and *replacing* it with a 2-edge path $x\ z\ y$ for some new vertex $z$. For instance, if $G$ contains the edge $xy$
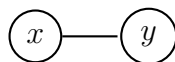


to *subdivide* the edge $xy$ means to replace $xy$ in $G$ with the following 2-edge path
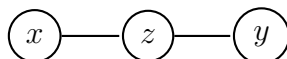


> **Definition 3.7: Subdivision**
>
> Given a graph $G$, a **subdivision** of $G$ is a graph obtained from $G$ by repeatedly subdividing its edges.
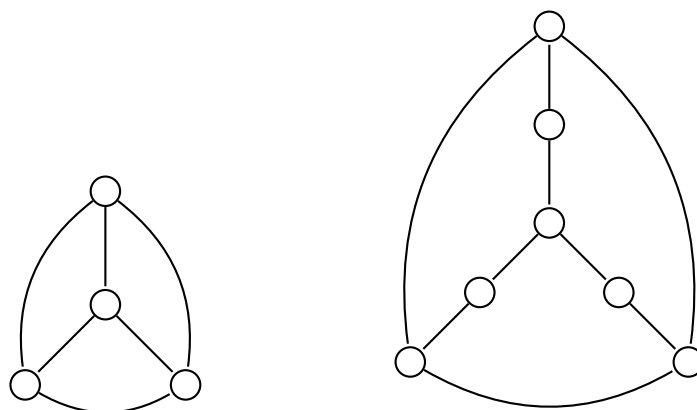


Figure 3.7: On the left: $K_4$. On the right: a subdivision of $K_4$.

Trivially, any graph is a subdivision of itself.

> **Definition 3.8: Topological minor**
>
> Given a graph $G$, and a graph $H$, we say that $G$ contains $H$ as **topological minor** if $G$ has a subgraph which is a subdivision of $H$. Similarly, we say that $H$ is a *topological minor* of $G$.
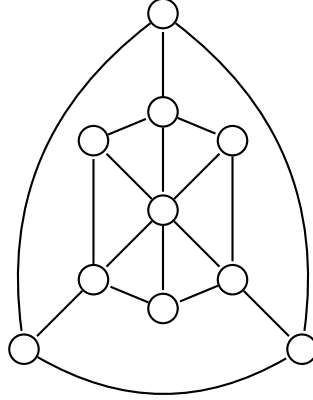


Figure 3.8: For instance, this graph has $K_4$ as topological minor, because it contains a subdivision of $K_4$ as subgraph.

> **Proposition 3.5**
>
> If $G$ is 3-connected, then $G$ has $K_4$ as topological minor.

*Proof.* If $G$ is 3-connected, then by Proposition 3.2 we know that $\delta(G) \geq 3$, and by Proposition 1.3 this means that in $G$ there is a cycle $C$. Moreover, by 3-connectivity of $G$, we know that $G$ cannot be a cycle itself, therefore there must be at least one vertex $v \notin V(C)$. Now, by 3-connectivity agin we can apply Corollary 3.1, obtaining 3 $\{v\}$-$V(C')$ paths $P_1$, $P_2$ and $P_3$ that only intersect in $v$, and finally $G[V(C) \cup V(P_1) \cup V(P_2) \cup V(P_3)]$ is a subdivision of $K_4$. $\square$

### 3.2.2 Erdős-Pósa theorem

> **Proposition 3.6**
>
> If $G$ is 3-regular multigraph, then $G$ has a cycle of length at most $2 \lceil \log n \rceil$.

*Proof.* Clearly, if $G$ admits loops or parallel edges, the statement is trivially true, therefore we may assume that $G$ is a simple graph.

Fix a vertex $v \in V(G)$, and grow a BFS tree $T$ from $x$ as long no cycles of length at most $2 \lceil \log n \rceil$ are encountered. Since $G$ is 3-regular, $T$ will be a tree in which the root $x$ has

3 children, and all the other nodes that are not leaves will have 2 children. If the BFS stopped before visiting all the vertices of $G$, the statement holds, so we may assume that $T$ covers all $V(G)$.

By 3-regularity of $G$, we have that

- $\delta \geq 3$, hence by Proposition 1.3 we know that $G$ must contain a cycle

- the only possible cycle that can be formed in $G$ is by connecting two leaves of $T$ through an edge

and lastly, since $V(T) = V(G) = n$, the height of $T$ is $\lceil \log n \rceil - 1$, which means that such a cycle must have length

$$\lceil \log n \rceil - 1 + \lceil \log n \rceil - 1 + 1 = 2 \lceil \log n \rceil - 1$$

$\square$

Along with the *subdivision* operation that we introduced previously, it naturally follows to defined the *inverse* operation of the subdivision, i.e. the **suppression**. Given a graph $G$, and two edges $xz, zy \in E(G)$ such that $\deg(z) = 2$, to *suppress* $z$ means to *remove* the vertex $z$ from $V(G)$ along with the edges $xz$ and $zy$, and *replacing* them with an edge $xy$ in $E(G)$. For instance, if $G$ contains the edges $xz$ and $zy$

$$x \!-\!\!-\!\! z \!-\!\!-\!\! y$$
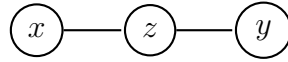
to *suppress* the vertex $z$ means to replace $xz$ and $zy$ in $G$ with the following edge $xy$

$$x \!-\!\!-\!\! y$$
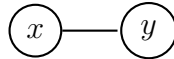
Now, consider the following definition.

> **Definition 3.9: Cut**
>
> Given an undirected graph $G = (V, E)$, and two subsets $S, T \subseteq V$, the **cut** induced by $S$ and $T$ on $G$ is defined as follows
>
> $$\mathrm{cut}(S, T) = \{e \in E : |S \cap e| = |T \cap e| = 1\}$$
>
> In the case in which $T = \overline{S} = V - S$ we will simply write $\mathrm{cut}(S) = \mathrm{cut}(S, \overline{S})$.

In other words, a *cut* induced by a two sets $S$ and $T$ of vertices is the set of edges that have one endpoint in $S$ and the other one in $T$.
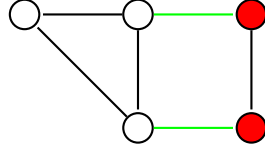
Figure 3.9: For instance, if the set $S$ is described by the *red* vertices, then cut$(S)$ is the set of green edges.

We will use the definition of cut in order to prove the following lemma.

**Lemma 3.1**

Given a 3-regular graph $H$ such that $|V(H)| \geq c'k \log(k+1)$ for some constant $c'$, then $H$ contains $k$ disjoint cycles.

*Proof.* We proceed by induction on $k$.

*Base case.* For $k = 1$, if $H$ is 3-regular then $\delta(H) = 3$, therefore by Proposition 1.3 it must contain a cycle.

*Inductive hypothesis.* Assume that the statement holds for $k - 1$.

*Inductive step.* Let $C$ be a cycle of $H$, and consider the graph $H - E(C)$.

**Claim:** $H - E(C)$ has a 3-regular graph $H'$ as topological minor, such that $|V(H')| \geq |V(H)| - 2\,|V(C)|$.

*Proof of the Claim.* Starting with $V := V(C)$, as long as $H - V$ contains vertices of degree at most 1, move them into $V$; let $M$ be the number of moved vertices. Then, since we are adding vertices to $V$, we have that $|\text{cut}(V)| \leq |\text{cut}(C)| - M$. We observe that, by construction $\delta(H - V) \geq 2$, and the number of vertices that have degree 2 in $H - V$ is at most $|\text{cut}(V)| \leq |\text{cut}(C)| - M$.

Lastly, let $H'$ be the graph obtained from $H - V$ by suppressing all the vertices of degree 2; then we have that $H'$ contains at least the vertices of $H$, without the vertices of $C$, the $M$ vertices that we moved during the procedure, and the vertices of degree 2 that we suppressed. In other words, we have that

$$|V(H')| \geq |V(H)| - |V(C)| - M - (|\text{cut}(C)| - M) \geq |V(H)| - 2\,|V(C)|$$

Note that the last inquality comes from the observation that $|\text{cut}(C)| \leq |V(C)|$, which follows by 3-regularity of $H$

Finally, since $H'$ is 3-regular by construction, and $H - V$ is a subdivision of $H'$ that is contained in $H - E(C)$, this concludes the claim. $\square$

Now, since $|V(H)| \geq c'k \log(k+1)$, and by the previous proposition we know that

$$|V(C)| \leq 2 \log(|V(H)|) + 2 \iff -2\,|V(C)| \geq -4 \log(|V(H)|) - 4$$

where the added term 2 takes care of the rounding error from the ceiling operation — we obtain the following

$$\begin{aligned}
|V(H')| &\geq |V(H)| - 2\,|V(C)| \\
&\geq |V(H)| - 4\log(\log V(H)) - 4 \\
&\geq c'k\log(k+1) - 4\log(c'k\log(k+1)) - 4 \\
&\geq c'k\log(k+1) - 4[\log c' + \log k + \log(\log(k+1))] - 4
\end{aligned}$$

which is at least $c'(k-1)\log k$ for sufficiently big values of $c'$. In particular, since $|V(H')| \geq c'(k-1)\log k$, and we know that $H'$ is 3-regular, we can apply the inductive hypothesis on $H'$, which means that $H'$ contains $k-1$ disjoint cycles.

Finally, if $H'$ contains $k$ disjoint cycles, and $H - E(C)$ has $H'$ as topological minor, i.e. $H - E(C)$ contains a subdivision $H''$ of $H'$, by definition of the subdivision operation $H''$ will contain $k-1$ disjoint cycles as well. Therefore, since $H - E(C)$ contains $H''$, $H$ contains $k$ disjoint cycles along with the cycle $C$.

$\square$

We are now ready to prove the theorem that we introduced at the beginning of this section.

> **Theorem 3.5: Erdős-Pósa theorem**
>
> There is a constant $c$ such that for any graph $G$, and any $k \in \mathbb{N}$
>
> - either $G$ has $k$ vertex-disjoint cycles or
> - there is an FVS $X$ of $G$ such that $|X| \leq ck\log k$

*Proof.* If $G$ contains $k$ disjoint cycles, the theorem is trivially true, hence we may assume that $G$ does not have $k$ disjoint cycles. Fix $H$ to be the largest subgraph of $G$ such that $\forall v \in V(H) \quad 2 \leq \deg_H(v) \leq 3$ — note that $H$ always exists because we may assume that $G$ contains at least 1 cycle, since if $G$ is acyclic the theorem is trivially true (and we would pick the cycle as $H$ itself). Moreover, fix $\overline{H}$ to be the subgraph of $G$ without all the "cycle components" of $G$ — i.e. components of $G$ that are cycles — and let $U := \{v \in V(\overline{H}) \mid \deg_{\overline{H}}(v) = 3\}$.

By TODO we know that $|U| < c'k\log(k+1)$, for some constant $c$. Let $W$ be a set of vertices composed of one vertex for each cycle component — the ones not present in $\overline{H}$.

discrepanze lemma

**Claim:** Every cycle of $G$ is in $H$.

*Proof of the Claim.* If there was a cycle of $G$ not present in $H$, this cycle would have contained vertices of degree at least 2, contradicting the maximality of $H$. $\square$

**Claim:** There are no paths $P$ in $G[V(G) - (U \cup W)]$ of length at least 1 such that

- both ends of $P$ are in $H[V(H) - (U \cup W)]$
- no internal vertex of $P$ is in $H$

- no edge of $P$ is in $H$

*Proof of the Claim.* If such a path $P$ exists, then $H \cup P$ is a bigger graph than $H$ that has every vertex of degree 2 or 3, contradicting the maximality of $H$. □

**Claim:** Every cycle of $G - (U \cup W)$ intersects $H$ in exactly one vertex.

*Proof of the Claim.* TODO _____ □

TODO _____ □

## 3.3 Directed graphs

Up until this point, we only discussed *undirected graphs*. In this section we are going briefly presents some results regarding **directed graphs** (or *digraphs*, for short), in which the edges have an **orientation** — implying that the edge $(x, y)$ is different from the edge $(y, x)$.
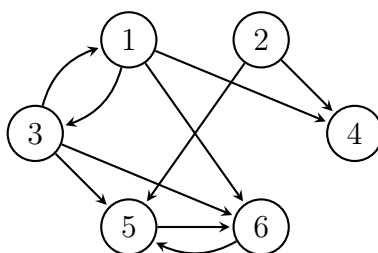


Figure 3.10: An example of a *digraph*.

> **Definition 3.10: Path cover**
>
> Let $G$ be a digraph; a **path cover** for $G$ is a set $\mathcal{P}$ of disjoint paths such that
>
> $$V(G) = \bigcup_{P \in \mathcal{P}} V(P)$$
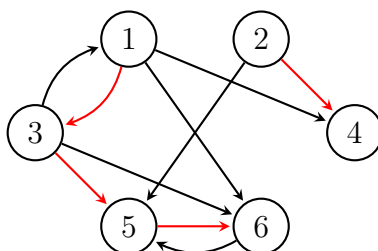


Figure 3.11: For instance, if we consider the digraph of the previous example, the set of *red* vertices form a *path cover*.

We observe that any digraph contains a trivial path cover, i.e. the one formed by the sef of every trivial path of the graph. In fact, we are interested in finding the path cover

having *minimum cardinality*. Gallai and Milgram [GM60] proved that path covers are strictly related to independent sets, through the following theorem.

> ### Theorem 3.6: Gallai-Milgram theorem
>
> Given a digraph $G$, there are a path cover $\mathcal{P}$ of $G$ and an independent set $X \subseteq V(G)$ such that
> $$\forall P \in \mathcal{P} \quad |X \cap V(P)| = 1$$

We are going to prove a slightly stronger result that directly implies the above theorem. First, consider the following definitions.

> ### Definition 3.11: Terminal nodes
>
> Given a digraph $G$, and a path cover $\mathcal{P}$ of $G$, the set of **terminal nodes** of the paths of $\mathcal{P}$ is defined as follows:
> $$\text{ter}(\mathcal{P}) := \{v \in V(P) \mid P \in \mathcal{P} \wedge \deg_P^{out}(v) = 0\}$$

For example, in Figure 3.11 the terminal nodes are 4 and 6.

> ### Definition 3.12: Minimality by inclusion
>
> Given a digraph $G$, and a path cover $\mathcal{P}$ of $G$, we say that $\mathcal{P}$ is **minimal by inclusion** if there is no other path cover $\mathcal{P}'$ of $G$ such that $\text{ter}(\mathcal{P}') \subset \text{ter}(\mathcal{P})$.

For instance, the path cover $\mathcal{P}$ in Figure 3.11 is clearly minimal by inclusion, because no directed edge can be removed from $\mathcal{P}$ such that the latter still covers $G$.

> ### Proposition 3.7
>
> Given a graph $G$, and a path cover $\mathcal{P}$ of $G$ that is minimal by inclusion, there is an independent set $X \subseteq V(G)$ such that
> $$\forall P \in \mathcal{P} \quad |X \cap V(P)| = 1$$

*Proof.* TODO ☐ da fare

## 3.4 Exercises

> ### Problem 3.1
>
> Given a $k$-connected graph $G$ such that $|V(G)| \geq 2k$, show that $G$ contains a cycle of length at least $2k$.

*Solution.* Since $|V(G)| \geq 2k$, we can split $G$ into two sets $A$ and $B$ such that $|A|, |B| \geq k$ and $A \cap B = \varnothing$. Fix a vertex $a \in A$; since $G$ is $k$-connected and $|A| \geq k$, by Corollary 3.1 there exists at least one $\{a\}$-$B$ path, and let $b \in B$ be the other endpoint of this path. By the same argument, since $|B| \geq k$ there exists at least one $\{b\} - A$ path different from $a \to b$, and let $a' \in A$ be the other endpoint of this path.

We can repeat the same argument — starting from $a'$ — going "back and forth" from $A$ to $B$, and since $|A|, |B| \geq k$ this process can be repeated at least $2k - 1$ times. Hence, at the end of this procedure we found a path $P$ such that $|E(P)| \geq 2k - 1$ that starts in $a$ and ends in some vertex $x \in B$.

Lastly, consider the vertices $a$ and $x$; clearly $a \neq x$, and since $G$ is $k$-connected by Theorem 3.2 there exists at least one $\{x\}$-$\{a\}$ path $P_{x,a}$, meaning that $P \cup P_{x,a}$ forms a cycle in $G$ that must have length at least $2k - 1 + 1 = 2k$. $\qquad\square$

> ### Problem 3.2
>
> Prove that any $t$-connected subgraph $G$ with $n \geq t + 2$ contains $K_{2,t}$ as a topological minor.

*Solution.* Fix two vertices $x, y \in V(G)$; since $n \geq t + 2$, we have that $|V(G) - \{x, y\}| \geq t$, and since $G$ is $t$-connected, by Corollary 3.1

- there are $t$ $\{x\}$-$V(G) - \{x, y\}$ paths that only intersect in $x$ — let $X$ be the set of the first vertices that these paths encounter from $x$ to $V(G) - \{x, y\}$

- there are $t$ $\{y\}$-$V(G) - \{x, y\}$ paths that only intersect in $y$ — let $Y$ be the set of the first vertices that these paths encounter from $y$ to $V(G) - \{x, y\}$

and in particular, we observe that $|X|, |Y| \geq t$.

Fix two vertices $a \in X$ and $b \in Y$; by Theorem 3.2, $a$ and $b$ must be connected by at least one path, say $P_{b,a}$. Now, if we consider the path $P_{y,b}$ that connects $y$ and $a$, we get a path $P_{y,b} \cup P_{b,a}$ that connects $y$ and $a$.

The same reasoning can be applied to any pair of vertices in $X$ and $Y$, which means that all the vertices in $X$ are connected to $x$ by definition, and they are also connected to $y$ thanks to the paths passing through $Y$. This implies that $G$ contains a subgraph that is a subdivision of $K_{2,t}$, meaning that $K_{2,t}$ is a topological minor of $G$. $\qquad\square$

> ### Problem 3.3
>
> Show, without using Menger's theorem, that any two vertices of a 2-connected graph lie on a common cycle.

*Solution.* Let $G$ be a 2-connected graph, and by way of contradiction suppose that there are two vertices $x, y \in V(G)$ that do not lie on a common cycle of $G$. By Proposition 3.2, we know that $\delta \geq 2$, hence there must be vertices $x_1, x_2$ and $y_1, y_2$ such that $x \sim x_1, x_2$ and $y \sim y_1, y_2$.

If $x$ and $y$ do not lie on a common cycle, without of loss of generality we may assume that $x_1$ and $y_1$ are connected by only one path — there must be at least one otherwise $G$ would be disconnected. However, since $x_1 \sim x \sim x_2$ and $y_1 \sim y \sim y_2$, such path must pass through $x$, $x_2$, $y_2$ and $y$. In particular, this implies that $G - \{x_2\}$ is disconnected because $x_1$ and $y_1$ are not connected anymore. Therefore, we can find a set of 1 vertex that disconnects $G$, meaning that $G$ is not 2-connected $\lightning$. $\qquad\square$

> **Problem 3.4**
>
> Solve the following exercises on $k$-connectivity.
>
> - Give an example of a 2-connected graph $G$ such that for any $C \subseteq G$ it holds that $G - C$ is disconnected.
>
> - Prove that every 3-connected graph $G$ has a cycle $C$ such that $G - C$ is connected.

*Solution.* For the first exercise, a minimal example is given by the following graph:
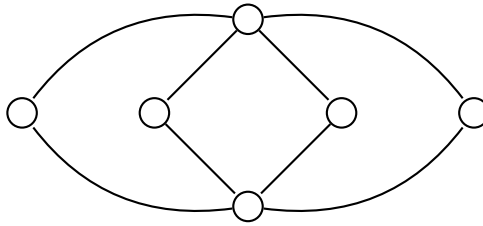


TODO $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$ da finire

> **Problem 3.5**
>
> Prove that every 3-connected graph has a cycle of even length.

*Solution.* Consider a 3-connected graph $G$, and by way of contradiction suppose that $G$ has no even length cycles. Fix two adjacent vertices $u, v \in V(G)$; by Theorem 3.2 we know that there are at least 3 $\{u\}$-$\{v\}$ paths in $G$, and since the edge $\{u, v\}$ is already a path, we may assume there are at least two other internally disjoint paths $P_1$ and $P_2$ that have $u$ and $v$ as endpoints.

However, observe that $P_1 \cup \{u, v\}$ and $P_2 \cup \{u, v\}$ are cycles, therefore they must have odd length for the sake of contradiction — say that $2k + 1$ and $2h + 1$ are the length of the first and the second cycle, respectively. Then, consider $P_1 \cup P_2$: since $P_1$ and $P_2$ are internally disjoint, this is another cycle of $G$, that has length

$$(2k + 1) - 1 + (2h + 1) - 1 = 2k + 2h - 2$$

which is even, contradicting the fact that $G$ only had odd-length cycles $\lightning$. $\qquad\square$

# 4

# Extremal graph theory

Extremal graph theory is a field of mathematics that investigates how *global properties* of a graph influence the **presence or absence** of specific substructures. Broadly speaking, it seeks to understand how large or dense a graph can be while avoiding certain configurations, or conversely, what constraints guarantee the appearance of particular patterns.

The central theme of the discipline lies in *exploring thresholds*: determining the minimal or maximal values of graph parameters — such as the number of edges, degrees, or vertices — beyond which a given substructure must necessarily exist. These substructures can range from simple motifs like *paths* and *cycles* to more complex or irregular patterns.

This area plays a crucial role in understanding the fundamental behavior of graphs under constraints, offering insights that are both theoretically significant and applicable in various areas such as network design, data analysis, and algorithm optimization.

## 4.1 Edge bounds

In the first section of this chapter, we will explore the bounds on the number of **edges** that force any graph to contain interesting substructures. In particular, we define $ex(n, H)$ to be the minimum number of edges in an $n$-vertex graph that guarantees the existence of $H$ as subgraph.

### 4.1.1 Cliques

We will start our discussion of extremal graph theory with the question of **determining the existence** of cliques within a given graph. In general, finding the largest clique in a graph is an NP-Hard problem. This leads to a natural question: under what conditions can we guarantee the existence of a $K_n$ subgraph?

Pál Turán addressed this by providing an exact upper bound on the number of edges a graph can have without containing a clique of a given size. But before discussing his

results, let us first introduce some important definitions.

## Definition 4.1: Edge maximum graph without $K_r$ subgraphs

A graph $G$ is said to be **edge maximum without $K_r$ subgraph** — denoted as $\mathrm{EM}K_r$, for short — if for any $G'$ such that $|V(G')| = |V(G)|$ and $|E(G')| > |E(G)|$, $G'$ has $K_r$ as subgraph.

In other words, $G$ is edge maximum without $K_r$ if by adding an edge to $G$ we obtain a graph $G'$ that contains $K_r$ as subgraph.
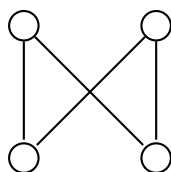


Figure 4.1: For instance, in this graph clearly there are no cliques as subgraph, and if we add one edge we construct $K_3$ as subgraph.

## Definition 4.2: Independent set

Given a graph $G$, a subset $X \subseteq V(G)$ is said to be an **independent set** if for all $u$ and $v$ in $X$, it holds that $u \nsim v$.
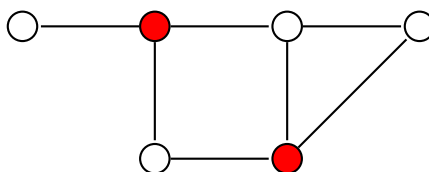


Figure 4.2: For instance, the *red* set of vertices is an independent set.

Note that we already encountered independent sets when discussing bipartite graphs: in fact, if $G$ is bipartite through the bipartition $(A, B)$, we observe that both $A$ and $B$ must be independent sets.

Now, consider the following type of graphs.

## Definition 4.3: $r$-partite graph

A graph is sait to be $r$-**partite** if it can be partitioned into $X_1, \ldots, X_r$ sets of vertices such that $X_1, \ldots, X_r$ are independent sets.

Moreover, the graph is said to be **complete $r$-partite** if forall $i, j \in [r]$ such that $i \neq j$, if $x \in X_i$ and $y \in X_j$ then $x \sim y$.
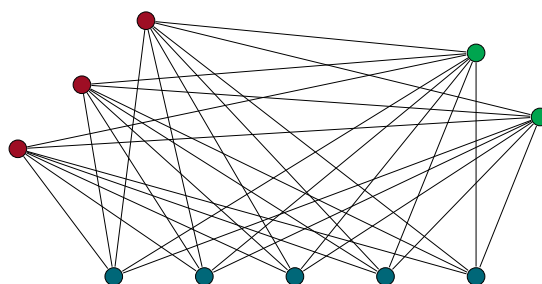
Figure 4.3: A complete 3-partite graph.

The following proposition shows that such graphs cannot contain cliques as subgraphs.

**Proposition 4.1**

An $(r-1)$-partite graph cannot contain $K_r$ as subgraph.

*Proof.* Let $G$ by an $(r-1)$-partite graph; by definition, $G$ is composed of $r-1$ independent sets, therefore any subgraph of $G$ that has $r$ vertices $x_1, \ldots, x_r$ must have 2 vertices lying in the same independent set by the pigeonhole principle, meaning that any subgraph of vertices $x_1, \ldots, x_r$ will have two non-adjacent vertices. $\qquad\square$

From this proposition, it follows that the number of edges of a *complete* $(r-1)$-partite graph provides a *lower bound* for the maximum number of edges that a graph can have before inducing $K_r$ as subgraph. Therefore, consider the following definition.

**Definition 4.4: Turán graph**

The Turán graph $T(n, r)$ is the complete $r$-partite graph that has $n$ nodes, partitioned into $X_1, \ldots, X_r$, such that for all $i, j \in [r]$, if $i \neq j$ then $||X_i| - |X_j|| \leq 1$ — i.e. the cardinality of the partitions are as close to equal as possible.
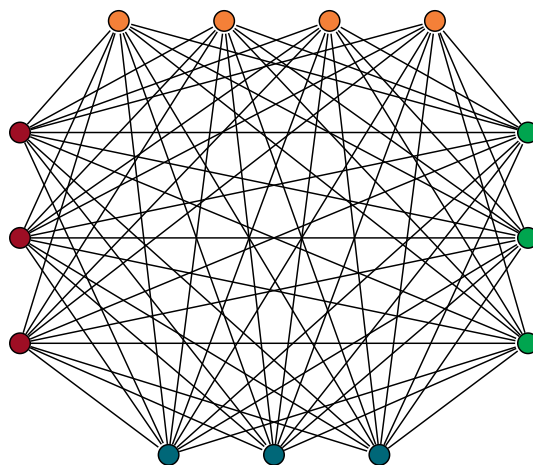


Figure 4.4: The Turán graph $T(13, 4)$.

We will refer to the number of edges of $T(n, r)$ as follows

$$t(n, r) := |E(T(n, r))| \sim \binom{n}{2} \frac{r-1}{r}$$

Note that, by definition, if $n \leq r$ then $T(n, r)$ can be partitioned into $n$ independent sets that contain exactly 1 vertex each, and $r - n$ empty sets, meaning that $T(n, r) = K_n$.

**Proposition 4.2**

If $G$ is EM$K_r$, then $t(n, r-1) \leq |E(G)|$.

*Proof.* We observe that $T(n, r-1)$ is $(r-1)$-partite in particular, therefore it does not contain $K_r$ as subgraph by Proposition 4.1. However, since $G$ has $n$ vertices as well, and $G$ is EM$K_r$, it must be that $t(n, r-1) \leq |E(G)|$. $\qquad\square$

Turán graphs are important not only because of the *lower bound* we discussed for Proposition 4.1, but also because Turán was able to prove in 1941 [Tur41] that any EM$K_r$ graph *must be* exactly $T(n, r-1)$.

**Theorem 4.1: Turán's theorem**

If $G$ is EM$K_r$, then $G = T(n, r-1)$.

*Proof.* We proceed by strong induction on $n$.

*Base case.* For the base case, we consider $n < r$; in fact, $G$ is EM$K_r$, and in particular it is edge maximum, therefore for $n < r$ we have that $G$ is $K_n$. Hence, $G = T(n, r-1)$ — some of the $r - 1$ partitions may be empty.

*Strong inductive hypothesis.* Assume that the statment is true for an EM$K_r$ graph of $r \leq k \leq n - 1$ vertices.

*Inductive step.* We are going to prove the statement for $n \geq r$. Note that $G$ is EM$K_r$, therefore by adding an edge between two non-adjacent vertices we obtain $K_r$ as subgraph of $G$; this must imply that $K_{r-1}$ must be a subgraph of $G$. Hence, let $Y := \{y_1, \ldots, y_{r-1}\}$ be the vertices such that $G[Y] = K_{r-1}$.

Now, since $G$ is EM$K_r$, then $G[V(G) - Y]$ does not contain $K_r$, therefore it must be that $|E(G[V(G) - Y])|$ is at most the number of edges of a graph $H$ that is EM$K_r$ of $n - (r-1)$ vertices. However, by strong inductive hypothesis we know that $H = T(n - (r-1), r-1)$, therefore

$$|E(G[V(G) - Y])| \leq |E(H)| = t(n - (r-1), r-1)$$

**Claim:** If $n \geq r$, then $t(n, r) = t(n-r, r) + (n-r)(r-1) + \binom{r}{2}$.

*Proof of the Claim.* Let $X_1, \ldots, X_r$ be a partition of $T(n, r)$. Since $n \geq r$, we

know that the independent sets are not empty, thus fix $x_i \in X_i$ for each $X_i$ of the partition, and let $Z := \{x_1, \ldots, x_r\}$. Now, we observe the following.

- $G[Z] = K_r$, hence there are $\binom{r}{2}$ edges in $G[Z]$.

- $G[V(G) - Z] = T(n - r, r)$ since we only removed one vertex per independent set, thus there are $t(n - r, r)$ edges in $G[V(G) - Z]$.

- Fix an independent set $X_i$, and consider a vertex $y \in X_i$; this vertex has $r - 1$ neighbors in $Z$, i.e. $|N(y) \cap Z| = r - 1$, because $T(n, r)$ is a complete $r$-partite graph, and the $r$-th node of $Z$ is in $X_i$ itself. Therefore, because there are $n - r$ vertices outside $G$, there are $(r - 1)(n - r)$ edges between $G[Z]$ and $G[V(G) - Z]$.

The claim follows from the above statements. $\qquad\square$

Now, similar to what we did in the claim, we observe the following.

- $G[Y] = K_{r-1}$, hence there are $\binom{r-1}{2}$ edges in $G[Y]$.

- $|E(G[V(G) - Y])| \leq t(n - (r - 1), r - 1)$ for the previous observation.

- Any vertex $z \notin Y$ must have at most $r - 2$ neighbors in $Y$ — if $z$ had $r - 1$ neighbors, $G[Y \cup \{z\}]$ would be a $K_r$, contradicting the definition of $G$. Hence, there are $(r - 2)(n - (r - 1))$ edges between $G[Y]$ and $G[V(G) - Y]$.

Therefore, we get that

$$|E(G)| \leq \binom{r - 1}{2} + (r - 2)(n - (r - 1)) + t(n - (r - 1), r - 1) = t(n, r - 1)$$

where the last inequality follows from the previous claim. Moreover, for Proposition 4.2 this upper bound is tight. However, if this bound is tight, it must be that the vertices $z \notin Y$ have *exactly* $r - 2$ neighbors in $Y$; hence, for each $i \in [r - 1]$ let $X_i := \{z \in V(G) - Y \mid z \nsim y_i\} \cup \{y_i\}$. In particular, since $z$ has exactly $r - 2$ neighbors in $Y$, it is adjacent to every vertex in $Y$ except for $y_i$, meaning that for any vertex $z \in V(G)$ it holds that $z$ is in *exactly* one set between $X_1, \ldots, X_{r-1}$. Therefore, these sets describe a *partition* of $G$.

**Claim:** For all $i \in [r - 1]$, it holds that $X_i$ is an independent set.

*Proof of the Claim.* Fix $i \in [r - 1]$, and by way of contradiction suppose that $X_i$ is not an independent set, i.e. $\exists z, z' \in X_i \quad z \sim z'$. However, $z \in X_i \implies z \notin Y$, and by the previous observation we know that $z$ has $r - 2$ neighbors in $Y$, meaning that $z$ is adjacent to every vertex in $Y - \{y_i\}$ — and the same reasoning can be applied on $z'$. Therefore, we have that $G[(Y - y_i) \cup \{z, z'\}]$ is $K_r$ because $z \sim z'$, contradicting the definition of $G \nleq$. $\qquad\square$

Therefore, $G$ is $(r - 1)$-partite through the partition $X_1, \ldots, X_{r-1}$.

**Claim:** If $G$ is EM$K_r$ and $(r - 1)$-partite, then $G = T(n, r - 1)$.

*Proof of the Claim.* Since no edges can be added to $G$ without creating $K_r$ as

subgraph of $G$, it must be that $G$ is complete $(r-1)$-partite. Let $X_1, \ldots, X_r$ be the partition that defines $G$, and by way of contradiction suppose that $G \neq T(n, r-1)$. By definition of Turán graph, this implies that for distinct $i, j \in [r]$ it holds that $||X_i| - |X_j|| \geq 2$, and without loss of generality suppose that $|X_i| < |X_j|$. Fix a vertex $y \in X_j$, and construct the graph $G'$ starting from $G$ as follows: remove all the edges $\{y, x_i\}$ for $x_i \in X_i$, move $y$ into $X_i$, and add all the edges $\{y, x_j\}$ for $x_j \in X_j - \{y\}$. We observe that $G'$ is still $(r-1)$-partite by construction, and by Proposition 4.1 we know that $G'$ cannot contain $K_r$ as subgraph. However, we observe that

$$|E(G')| = |E(G)| - |X_i| + |X_j| - 1 > |E(G)|$$

contradicting the fact that $G$ was EM$K_r$ $\not{\natural}$. $\qquad\square$

Finally, the statement follows from this last claim.

$\qquad\square$

As a corollary of this theorem, we have the answer to our original question.

**Corollary 4.1**

$$\mathrm{ex}(n, K_r) = t(n, r-1) + 1$$

*Proof.* Clearly, if a graph $G$ has $n$ vertices and more than $t(n, r-1)$ edges, it cannot be $T(n, r-1)$, therefore by the contrapositive of Turan's theorem it must contain $K_r$ as subgraph. $\qquad\square$

## 4.1.2 $k$-connected subgraphs

After examining extremal edge conditions that guarantee the presence of a clique as subgraph, we now turn our attention to extremal conditions that ensure the existence of a $k$-connected subgraph. These types of conditions are particularly significant — unsurprisingly so, given that we have already seen how $k$-connectivity plays a key role in the emergence of structural properties within graphs. In particular, the following theorem — proved by Mader [Mad72] in 1972 — provides some conditions that guarantee the existence of a $k$-connected subgraph in a given graph.

**Theorem 4.2: Mader's theorem**

If $G$ is such that $|V(G)| \geq 2k$ and $|E(G)| \geq (2k-1)(n-k)$, then $G$ contains a $k$-connected subgraph.

*Proof.* We proceed by strong induction on $n$.

*Base case.* Since $n \geq 2k$, the base case of the induction is $n = 2k$, thus we have that

$$|E(G)| \geq (2k-1)k = \frac{(2k-1)2k}{2} = \binom{2k}{2}$$

which must imply that $G = K_{2k}$, which is $(2k-1)$-connected by definition.

*Strong inductive hypothesis.* Assume that the statement holds for any graph of $h$ vertices such that $n - 1 \geq h > 2k$ vertices.

*Inductive step.* We are going to prove the statement for a graph $G$ of $n$ vertices such that $n > 2k$, i.e. $n \geq 2k + 1$.

Suppose that $\delta < 2k$, i.e. $\delta \leq 2k - 1$, implying that there exists a vertex $v$ such that $\deg(v) \leq 2k - 1$, and consider $G[V(G) - \{v\}]$. Since this subgraph has $n - 1$ vertices, and we are assuming that $|E(G)| \geq (2k-1)(n-k)$ then we have that

$$\begin{aligned}
|E(G[V(G) - \{v\}])| &\geq |E(G)| - (2k-1) \\
&\geq (2k-1)(n-k) - (2k-1) \\
&\geq (2k-1)(n-k-1) \\
&\geq (2k-1)((n-1)-k)
\end{aligned}$$

Moreover, since $n \geq 2k + 1 \iff n - 1 \geq 2k$ and $G[V(G) - \{v\}]$ has $n - 1$ vertices, the latter has at least $2k$ vertices, hence we can apply the inductive hypothesis on it and obtaining a $k$-connected subgraph to prove the statement trivially. Therefore, we may assume that $\delta \geq 2k$.

To recap, we are now assuming that

(a) $G$ does not contain a $k$-connected graph — otherwise the statement is trivially true

(b) $|E(G)| \geq (2k-1)(n-k)$

(c) $\delta \geq 2k$

and in particular, assumption (a) implies that we may assume there is an $A$-$B$ separation $(X, Y)$ of $G$ having order $|X \cap Y| \leq k - 1$, where $X - Y, Y - X \neq \varnothing$ — for some subsets $A, B \subseteq V(G)$.

**Claim:** It holds that $|E(G[X])| \geq (2k-1)(|X|-k)$ or $|E(G[Y])| \geq (2k-1)(|Y|-k)$.

*Proof of the Claim.* By way of contradiction, suppose that $|E(G[X])| \leq (2k-1)(|X|-k) - 1$ and $|E(G[Y])| \leq (2k-1)(|Y|-k) - 1$. Thus, we have that

$$\begin{aligned}
|E(G)| &\leq |E(G[X])| + |E(G[Y])| \\
&\leq (2k-1)(|X|-k) - 1 + (2k-1)(|Y|-k) - 1 \\
&= (2k-1)(|X|+|Y|-2k) - 2 \\
&= (2k-1)(|X \cup Y|+|X \cap Y|-2k) - 2 \\
&\leq (2k-1)(n+k-1-2k) - 2 \\
&= (2k-1)(n-k-1) - 2 \\
&< (2k-1)(n-k)
\end{aligned}$$

which contradicts assumption (b) $\frac{1}{4}$. $\qquad\qquad\square$

This claim implies that at least one between $G[X]$ or $G[Y]$ satisfies the edge bound, hence if we can prove that they also satisfy the vertex bound we can apply the inductive hypothesis and conclude the proof.

Now consider assumption (c): if $\delta \geq 2k$ then for each vertex $v \in V(G)$ its degree must be at least $2k$, and in particular this is true for any $v \in X - Y$. However, $(X, Y)$ is a separation of $G$, meaning that there are not edges in $\text{cut}(X, Y)$; therefore, all the neighbors of $v$ must lie inside $X$, meaning that $|X| \geq |\mathcal{N}(v) \cup \{v\}| \geq 2k + 1 > 2k$. This last condition implies that the vertex bound of the inductive hypothesis is satisfied, so by applying the latter on one between $G[X]$ or $G[Y]$ the theorem holds.

$\square$

It is not known if the edge bound provided by Mader's theorem is optimal, i.e. we do not know if there are graph with fewer than $(2k - 1)(n - k)$ edges that do not contain $k$-connected subgraphs. In fact, this is an open question in the field of extremal graph theory: given $k \in \mathbb{N}$, what is the smallest value $c_k$ such that all graphs with $n$ vertices and $|E(G)| \geq c_k n + f(k)$ — for some function $f(k)$ — contain a $k$-connected subgraph?

By Mader's theorem we that $c_k$ is *upper bounded* by $c_k \leq 2k - 1$, and some type of graphs are known to provide *lower bounds*, such as the following type.

> **Definition 4.5: Split graph**
>
> A **split graph** is a graph whose vertices can be partitioned into a *clique* and an independent set. A **complete split graph** is a split graph where the independent set is fully connected to the clique.
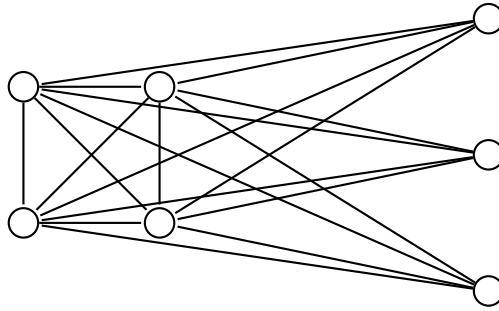


Figure 4.5: The complit split graph constructed from $K_4$ and an independent set of 3 vertices.

In fact, any complete split graph constructed from a $(k - 1)$-clique and an independent set of $n - (k - 1)$ vertices has the following number of edges

$$|E(G)| = \binom{k - 1}{2} + (k - 1)(n - (k - 1))$$

and such split graphs are clearly not $k$-connected — removing $K_{k-1}$ disconnect the graph by definition. Therefore, this implies that $c_k \geq k - 1$.

Lastly, as a corollary of Mader's theorem and Proposition 3.5, we get the following proposition.

**Corollary 4.2**

A graph $G$ such that $|V(G)| \geq 6$ and $|E(G)| \geq 5n$ contains $K_4$ as topological minor.

Moreover, in 1996 Bollobás and Thomason [BT96] proved that the above corollary can be extended to any $k$-clique with a relatively small blow-up on the edge bound.

**Theorem 4.3: Bollobás-Thomason theorem**

There exists a constant $c > 0$ such that if $G$ has at least $ck^2n$ edges, then $G$ has $K_k$ as topological minor.

The topic of finding cliques as substructures of a given graph will be discussed in greater detail in the following section.

### 4.1.3  Cliques as minors

Consider a graph $G$, and an edge $xy \in E(G)$; to **contract** $xy$ means to *identify* the vertices $x$ and $y$ and delete any parralel edge that formed afterwards. For instance, if $G$ contains the edge $xy$
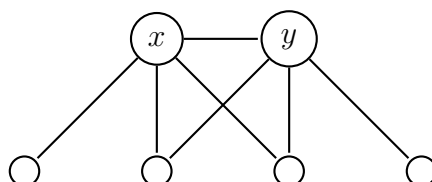


to *contract* the edge $xy$ means to identify $x$ and $y$ in a new vertex $z$, and removing the 2 parallel edges that formed
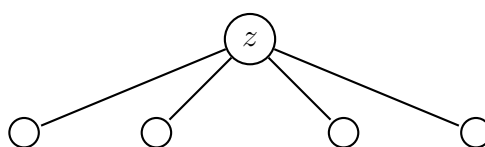


If we contracted the edge $e$ in a graph $G$, we will denote the resulting graph as $G/e$.

**Definition 4.6: Minor**

Given a graph $G$, and a graph $H$, we say that $G$ contains $H$ as **minor** if $H$ can be obtained from $G$ by a series of *vertex deletions*, *edge deletions* and *edge contractions*. Similarly, we say that $H$ is a *minor* of $G$.

For instance, consider the following graph $G$
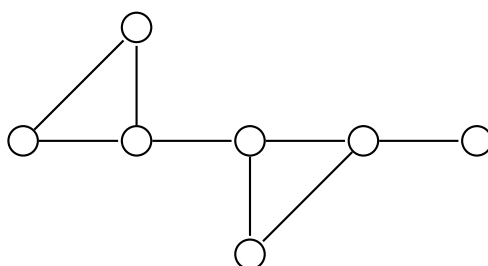
then clearly the following graph $H$ is a minor of $G$ — the *dashed* edges and vertices are the removed ones, and the *gray* edge is the contracted edge
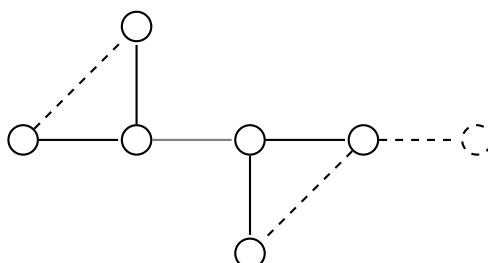
Consider a graph $G$ that has $H$ as topological minor, meaning that it contains a graph $H'$ that is a subdivision of $H$; if we remove from $G$ all the edges and vertices that are *not* part of $H'$, we are left with a subdivision of $H$, and through the *edge contraction* operation we can actually reverse the *edge subdivision* in order to obtain $H$ from $H'$. In fact, for any 2-edge path $x\ y\ z$, we can contract $x\ y$ obtaining an edge $y\ z$, and no parallel edge can form because we previously removed all the other edges that are not in $H'$, i.e. edges that were not constructed through subdivision. This means that if $G$ contains $H$ as topological minor, then $G$ contains $H$ as minor as well. However, the opposite is not true in general: for instance, consider the following graph $G$
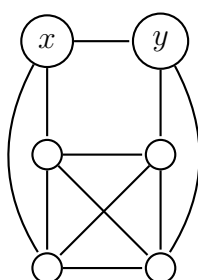
If we contract the edge $xy$, we obtain $K_5$, therefore $G$ contains $K_5$ as minor, even if $K_5$ is not a topological minor of $G$: in fact, every subdivision of $K_5$ has 5 vertices of degree 4, but $G$ has only 4 vertices of degree 4.

> **Theorem 4.4**
>
> If $p$ is an integer such that $2 \leq p \leq 7$, then for any graph $G$ such that $n \geq p$ and
>
> $$|E(G)| \geq (p-2)n - \binom{p-1}{2} + 1$$
>
> it holds that $G$ contains $K_p$ as minor.

*Proof.* We will prove each value of $p$ individually.

- For $p = 2$, we see that $|E(G)| \geq (2-2)n - \binom{2-1}{2} + 1 = 1$, therefore $G$ contains at least 1 edge, and in fact $K_2$ is nothing more than a single edge.

- For $p = 3$, we have that $|E(G)| \geq (3-2)n - \binom{3-1}{2} + 1 = n - 1 + 1 = n$. Thus, by Proposition 1.5 this implies that $G$ is not a tree (nor a forest), and in particular it is not acyclic, meaning that it contains at least one cycle $C$. Hence, if we contract all but 3 edges on $C$ we obtain $K_3$.

- For $p = 4$, we have that $|E(G)| \geq (4-2)n - \binom{4-1}{2} + 1 = 2n - 3 + 1 = 2n - 2$. We proceed by induction on $n + m$.

  For $n = p = 4$, we have that $|E(G)| \geq 2 \cdot 4 - 2 = 6$, and thus $G = K_4$. Now, for the inductive hypothesis, we need to be careful: since we are applying the induction on $n + m$, then

  - if the graph $G$ we are considering has $m - 1$ edges, we are going to inductively assume that the property holds for $G$ such that $|E(G)| \geq 2n - 2$

  - if the graph $G$ we are considering has $n - 1$ vertices, we are going to inductively assume that the property holds for $G$ such that $|E(G)| \geq 2(n-1) - 2$

  Now, because of the base case, consider a graph $G$ of $n$ vertices such that $n \geq 5$; moreover, suppose that $|E(G)| > 2n - 2$, and fix an edge $e \in E(G)$. Now, consider the graph $G - \{e\}$: clearly, this graph has still $n$ vertices but since we removed an edge it holds that $|E(G - \{e\})| > 2n - 2 - 1 = 2n - 3 \iff |E(G - \{e\})| \geq 2n - 2$, therefore we can apply the inductive hypothesis as we described, and the theorem holds.

  Therefore, we may assume that $|E(G)| = 2n - 2$. Fix an edge $e \in E(G)$, and consider the contracted graph $G/e$: this graph has $n - 1$ vertices, hence if $|E(G/e)| \geq 2(n-1) - 2$ by the inductive hypothesis the theorem holds trivially. Thus, we may assume that $|E(G/e)| < 2(n-1) - 2 = 2n - 4 \iff |E(G/e)| \leq 2n - 5$, and since we are assuming that $|E(G)| = 2n - 2$, this means that by contracting $e$ we removed at least 3 edges from $G$ — one of which is $e$ itself. Therefore, for each edge $e$ of $G$ we have the following structure
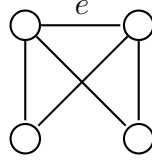
and in particular, the endpoints of $e$ must have at least two common neighbors. This implies that $e$ is contained in at least two distinct $K_3$'s. By applying the same argument on every edge $e$ of $G$, we get that $\delta \geq 3$ because the degree of the endpoints of $e$ must be at least 3. Moreover, if $\delta \geq 4$, by Corollary 1.1 we have that $|E(G)| \geq \frac{4n}{2} = 2n$ which is greater than $2n - 2$, contradicting our assumption that $|E(G)| = 2n - 2$. This implies that $\delta = 3$. Finally, fix a vertex $v$ of degree 3.

**Claim:** $G[\mathcal{N}(v) \cup \{v\}] = K_4$.

*Proof of the Claim.* By way of contradiction, suppose that there are two vertices $x, y \in \mathcal{N}(v)$ such that $x \nsim y$. Note that each edge of $G$ must be contained in at least two distinct 3-cliques, and in particular this must be true for $vx$. However, since we are assuming $xy \notin E(G)$ for the sake of contradiction, there must be at least one other vertex $z$ such that $v$, $x$ and $z$ form a $K_3$, which would violate the degree of $v$ $\lightning$. $\qquad\qquad\square$

- For $p = 5$, we have that $|E(G)| \geq (5 - 2)n - \binom{5-1}{2} + 1 = 3n - 6 + 1 = 3n - 5$. We proceed by induction on $n + m$ in the same fashion we did for the case of $p = 4$.

  For $n = p = 5$, we have that $|E(G)| \geq 3 \cdot 5 - 5 = 15 - 5 = 10$, and thus $G = K_5$. By applying the same reasoning of the case for $p = 4$, we can consider a graph $G$ such that $n \geq 6$, and fix an edge $e \in E(G)$; now, if we assume that $|E(G)| > 3n - 5$ then $|E(G - \{e\})| > 3n - 5 - 1 = 3n - 6 \iff |E(G - \{e\})| \geq 3n - 5$ therefore we can apply the inductive hypothesis and theorem holds.

  Thus, we may assume that $|E(G)| = 3n - 5$. Fix an edge $e \in E(G)$, and consider $G/e$: this graph has $n - 1$ vertices, and by the same reasoning as the case for $p = 4$ we may assume that $|E(G/e)| < 3(n-1) - 5 = 3n - 3 - 5 = 3n - 8 \iff |E(G/e)| \leq |E(G/e)| \leq 3n - 9$, and since we are assuming $|E(G)| = 3n - 5$, this means that by contracting $e$ we removed at least 4 edges from $G$ — one of which is $e$ itself. This implies that the endpoints of $e$ must have at least three distinct common neighbors, i.e. $e$ is contained in at least three distinct $K_3$'s.

  Thus, by applying the same argument on every edge $e$ of $G$, we get that $\delta \geq 4$. Moreover, suppose that $\delta \geq 6$: then by Corollary 1.1 we have that $|E(G)| \geq \frac{6n}{2} = 3n$ which is greater than $3n - 5$, contradicting our assumption. This implies that $\delta$ is either 4 or 5. If $\delta = 4$, then because every edge must be contained in at least three distinct 3-cliques, we can apply the same reasoning as the case for $p = 4$ for any vertex $v$ of degree 4, obtaining that $G[\mathcal{N}(v) \cup \{v\}] = K_5$. Otherwise, if $\delta = 5$, we have the following claim.

  **Claim:** If $\delta = 5$, for any $v \in V(G)$ such that $\deg(v) = 5$ it holds that $K_5$ is a minor of $G[\mathcal{N}(v) \cup \{v\}]$.

*Proof of the Claim.* Consider $G[\mathcal{N}(v)]$: by the same argument applied in the final claim of the case for $p = 4$, we know that each vertex $x$ in $G[\mathcal{N}(v)]$ must have $\deg_{G[\mathcal{N}(v)]}(x) \geq 3$, otherwise the edges connecting $v$ to $\mathcal{N}(v)$ would not be contained in at least three distinct 3-cliques. Now, if $G[\mathcal{N}(v)]$ is fully connected, we have that $G[\mathcal{N}(v)] = K_5$ hence the claim trivially holds, so we may assume the opposite. In fact, the degree bound we derived does not guarantee that $G[\mathcal{N}(v)]$ is fully connected; however, there cannot be more than 2 anti-edges, as shown in the figure below — the *dashed* edges are possible anti-edges of $G[\mathcal{N}(v)]$
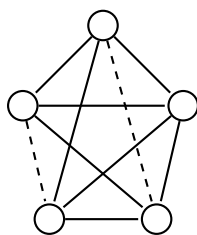


Hence, we have only two cases:

- if $G[\mathcal{N}(v)]$ has one anti-edge, consider an edge $e \in E(G[\mathcal{N}(v)])$ incident to the anti-edge; then $G[\mathcal{N}(v)]/e = K_4$ — we observe that the contraction will form one parallel edge to remove

- if $G[\mathcal{N}(v)]$ has two anti-edges, consider an edge $e$ incident to both the anti-edges; then $G[\mathcal{N}(v)]/e = K_4$ — we observe that the contraction does not form any parallel edges to remove

Therefore, in both cases we have that $G[\mathcal{N}(v) \cup \{v\}]/e = K_5$, where $e$ was chosen accordingly. $\qquad\square$

- The cases for $p = 6$ and $p = 7$ are analogous to the ones for $p = 4$ and $p = 5$.

$\square$

The cases for $2 \leq p \leq 5$ were initially established by Dirac [Dir64], and later independently by Gyori [Gyo82]. Mader [Mad68] extended the result to $p = 6$ and $p = 7$. For $p = 8$, Jorgensen [Jor94] showed that the theorem holds, except for an infinite family of counterexamples. Similarly, Song and Thomas [ST06] proved the result for $p = 9$, identifying two infinite families of exceptions. More recently, Zhu [Zhu21] demonstrated that the theorem also extends to $p = 10$, aside from a few infinite families. The proofs for $p = 9$ and $p = 10$ both involved computer assistance.

In a series of papers, **kostochaka** and Thomason [Tho01] independently proved that there exists a constant $c$ such that every graph with at least $cp\sqrt{\log p}\,n$ edges contains a $K_p$ minor. Moreover, thomason further showed that this bound is asymptotically tight: there exist Erdős–Rényi random graphs with only $c'p\sqrt{\log p}\,n$ edges and no $K_p$ minor, for some constant $c' < c$. Notably, in these examples, the number of vertices $n$ is also a function of $p$.

> **Theorem 4.5: Kostochka-Thomason theorem**
>
> There is a constant $c > 0$ such that any graph $G$ with $|E(G)| \geq cp\sqrt{\log p}n$ contains $K_p$ as minor.

Seymour and Thomas conjectured that the missing condition for Theorem 4.4 to hold for every $p \geq 2$ is the $(p-2)$-connectivity of the graph.

> **Conjecture 4.1: Seymour-Thomas conjecture**
>
> For all $p \in \mathbb{N}$ there is a constant $N_p > 0$ such that any $(p-2)$-connected graph $G$ with $n \geq N_p$ and $|E(G)| \geq (p-2)n - \binom{p-1}{2} + 1$ contains $K_p$ as minor.

The closest result for this conjecture was proven by Böhme, Kawarabayashi, Maharry, et al. [BKM+09].

> **Theorem 4.6**
>
> There is a constant $c > 0$ such that for all $p \in \mathbb{N}$ there is another constant $N_p > 0$ for which any $c(p+1)$-connected graph $G$ with $n \geq N_p$ contains $K_p$ as minor.

## 4.2 Vertex bounds

Up until now, we focused on finding conditions on the number of *edges* which guaranteed the existence of particular structures inside graphs. As mentioned before, we are now going to focus on bounds on the number of **vertices** that guarantee the existence of substructures in graphs.

### 4.2.1 Ramsey numbers

In 1917, Ramsey [Ram87] published a groundbreaking paper in which he proved the existence of the so-called **Ramsey number**.

> **Definition 4.7: Ramsey number**
>
> Given a value $t \in \mathbb{N}$, the **Ramsey number** $R(t)$ is the minimum number of vertices such that every graph with at least $R(t)$ vertices has $K_t$ as subgraph or contains an independent set of size $t$.

Trivially, $R(1) = 1$ since a single vertex is an independent set of size 1, and $R(2) = 2$ since a graph with two vertices can only be either $K_2$ or an independent set of size 2.

For the case when $t = 3$, clearly $R(3) \geq 3$ otherwise the graph cannot contain $K_3$ nor an independent set of size 3. Moreover, we can actually construct the following three graphs which lower-bound its value to be at least 6.
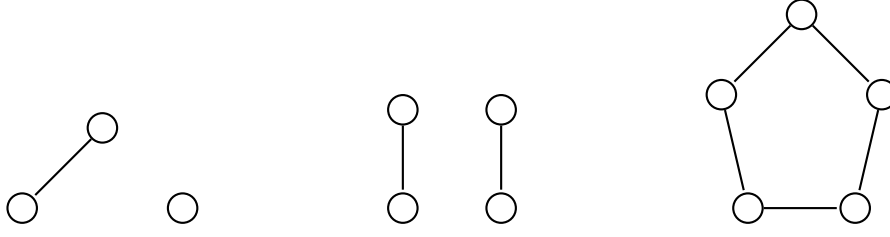
Figure 4.6: From left to right: the counterexamples with 3, 4 and 5 vertices that give the $R(3) > 5$ lower bound.

In fact, we can actually prove that this lower bound is tight.

**Theorem 4.7**

$R(3) = 6$

*Proof.* By way of contradiction, suppose that there is a graph $G$ with 6 vertices that does not contain $K_3$ as subgraph, nor any independent sets of size 3.

**Claim:** $\Delta(G) \leq 2$.

*Proof of the Claim.* By way of contradiction, suppose that there is a vertex $v$ of degree 3; however, if there was an edge with two endpoints in $\mathcal{N}(v)$ then there would be $K_3$ as subgraph of $G$. Therefore, the three vertices in $\mathcal{N}(v)$ cannot have edges between them, implying that $\mathcal{N}(v)$ forms an independent set of size 3 $\lightning$. $\square$

Fix a vertex $v$ with at least 3 non-neighbors — its existence is guaranteed by the previous claim, since $n = 6$ — and let the latter be $x$, $y$ and $z$. If two of the 3 non-neighbors of $v$ are not adjacent, say $x$ and $y$, then $\{v, x, y\}$ is an independent set of size 3. This implies that all of the non-neighbors of $v$ must be pairwise adjacent, meaning that $x$, $y$ and $z$ form a $K_3$ in $G$ $\lightning$. $\square$

What about $t = 4$? In 1965 Kalbfleisch [Kal65] proved that $R(4) > 17$ through the so-called *Kalbfleisch's graph*. We are going to show that this lower bound is actually tight, i.e. $R(4) = 18$. But first, we need to introduce the following definitions.

**Definition 4.8: Complementary graph**

Given a graph $G$, the **complement of** $G$ — denoted as $\overline{G}$ — is defined as follows:

- $V(\overline{G}) := V(G)$
- $E(\overline{G}) := \overline{E(G)} = V(G) \times V(G) - E(G)$

For instance, if we consider the following graph $G$
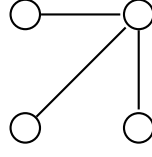
the following is $\overline{G}$, its complement
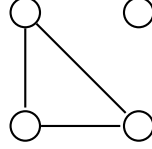


By definition, we observe that a graph $G$ contains an independent set of size $s$ if and only if $\overline{G}$ contains $K_s$ as subgraph.

We are going to use this definition in order to introduce a **generalized** version of the Ramsey number, in which the clique subgraph and the independent set can have different sizes.

### Definition 4.9: Generalized Ramsey number

Given two values $t, s \in \mathbb{N}$, the **generalized Ramsey number** $R(t, s)$ is the minimum number of vertices such that every graph $G$ with at least $R(t, s)$ vertices has $K_t$ as subgraph, or $\overline{G}$ has $K_s$ as subgraph.

By definition we observe that $R(t, t) = R(t)$. Moreover, we note that $R(t, s) = R(s, t)$ since $G = \overline{\overline{G}}$.

### Lemma 4.1

$R(4, 3) \leq 9$

*Proof.* By way of contradiction, let $G$ be a graph having 9 vertices that has no $K_4$ as subgraph nor any independent set of size 3.

By the same reasoning applied for the proof of Theorem 4.7, we know that $\Delta(G) \leq 5$, and that no vertex of $G$ can have more than 3 non-neighbors. Moreover, since $n = 9$, this implies that the degree of every vertex of $G$ is *exactly* 5, i.e. $G$ is 5-regular. However, we observe that $\sum_{x \in V(G)} \deg(x) = 5 \cdot 9 = 45$, which is impossible because 45 is odd and the sum of the degrees of $G$ must be even by the Handshaking lemma $\lightning$. $\qquad\square$

Thanks to the generalized version of the Ramsey number, we can prove the value of $R(4)$ as follows.

> **Corollary 4.3**
>
> $R(4) = 18$

*Proof.* By way of contradiction, let $G$ be a graph having 9 vertices that has no $K_4$ as subgraph nor any independent set of size 4, and fix a vertex $v \in V(G)$. Since $n = 18$, we observe that $v$ has either at least 9 neighbors or at least 9 non-neighbors. Thus

- if $|\mathcal{N}(v)| \geq 9$, then $\mathcal{N}(v)$ contains $K_3$ as subgraph or an independent set of size 4 by the previous lemma

- if $|V(G) - (\mathcal{N}(v) \cup \{v\})| \geq 9$, then $V(G) - (\mathcal{N}(v) \cup \{v\})$ contains $K_4$ as subgraph or an independent set of size 4, again by the previous lemma

meaning that $G$ has $K_4$ as subgraph or an independent set of size 4, contradicting the definition of $G$ ↯. $\qquad\square$

What about other values of $t$? Using a similar idea of this result, it can be proven that $R(5, 4) \leq 25$, which immediately provides an upper bound for $R(5) \leq 50$. However, we still don't know the exact value of $R(5)$: in 1989 Exoo [Exo89] proved that $R(5) \geq 43$, and in 2024 Angeltveit and McKay [AM24] proved that $R(5) \leq 46$.

What about the general case? At the beginning of this section, we mentioned that Ramsey proved the existence of $R(t)$ for any $t \geq 1$. The following is the theorem through which proved this result, which also provides an upper bound for any $R(t)$.

> **Theorem 4.8: Ramsey's theorem**
>
> For any $t \geq 1$, $R(t)$ exists. Moreover, if $t \neq 1$ then $R(t) \leq 2^{2t-3}$.

*Proof.* We already know the values for $R(1)$ and $R(2)$, therefore they exist. Hence, fix $t \geq 2$ and let $G$ be a graph with $2^{2t-3}$ vertices.

**Claim**: There are a sequence of sets $X_1, \ldots, X_{2t-3}$ and a sequence vertices $x_1, \ldots, x_{2t-3}$ such that for each $i \in [2t - 2]$ it holds that

1. $x_i \in X_i$ and $|X_i| \geq 2^{2t-2-i}$

2. $X_{i+1} \subseteq X_i - \{x_i\}$

3. $x_i$ is either adjacent to every vertex of $X_{i+1}$ or non-adjacent to all vertices of $X_{i+1}$

*Proof of the Claim.* We are going to construct the sequences inductively. First, let $X_1 := V(G)$, and fix $x_1 \in X_1$ arbitrarily. Now, let $X_2$ to be the larger between $\mathcal{N}(x_1)$ and $V(G) - (\mathcal{N}(x_1) \cup \{x_1\})$; then, fix $x_2 \in X_2$ arbitrarily. We observe that the first last two properties are satisfied by construction. For the first property, we observe that

$$\mathcal{N}(x_1) \cup (V(G) - (\mathcal{N}(x_1) \cup \{x_1\})) = X_1 - \{x_1\} = V(G) - \{x_1\}$$

meaning that the cardinality of their union is $|V(G) - \{x_1\}| = 2^{2t-3} - 1$, therefore

$$\begin{aligned}
|X_2| &= \max(|\mathcal{N}(x_1)|, |V(G) - (\mathcal{N}(x_1) \cup \{x_1\})|) \\
&\geq \frac{|\mathcal{N}(x_1) \cup (V(G) - (\mathcal{N}(x_1) \cup \{x_1\}))|}{2} \\
&= \frac{2^{2t-3} - 1}{2} \\
&\geq 2^{2t-4} \\
&= 2^{2t-2-2} \qquad (i = 2)
\end{aligned}$$

In general, suppose that we have defined $X_i$ and $x_i$ for some $i < 2t - 2$; then, let $X_{i+1}$ be the larger between $\mathcal{N}(x_i) \cap X_i$ and $X_i - (\mathcal{N}(x_i) \cup \{x_i\})$. Again, we observe that the last two properties of the claim are satisfied by construction of $X_{i+1}$. For the first property, we observe that

$$(\mathcal{N}(x_1) \cup X_i) \cup (X_i - (\mathcal{N}(x_i) \cup \{x_i\})) = X_i - \{x_i\}$$

meaning that the cardinality of their union is $|X_i - \{x_i\}| \geq 2^{2t-2-i} - 1$ by induction, therefore

$$\begin{aligned}
|X_{i+1}| &= \max(|\mathcal{N}(x_1) \cup X_i|, |X_i - (\mathcal{N}(x_i) \cup \{x_i\})|) \\
&\geq \frac{|(\mathcal{N}(x_1) \cup X_i) \cup (X_i - (\mathcal{N}(x_i) \cup \{x_i\}))|}{2} \\
&\geq \frac{2^{2t-2-i} - 1}{2} \\
&\geq 2^{2t-2-(i+1)}
\end{aligned}$$

TODO ⬚    <span style="color:orange">ricontrolla i conti che non funzionano</span>

Now, consider the sequence of sets and the sequence of vertices that we constructed.

**Claim:** For each $i, j \in [2t - 2]$ such that $i < j$ it holds that:

- $x_i \sim x_j \implies \forall j' \in [i+1, 2t-2] \quad x_i \sim x_{j'}$

- $x_i \nsim x_j \implies \forall j' \in [i+1, 2t-2] \quad x_i \nsim x_{j'}$

*Proof of the Claim.* Fix $i, j \in [2t - 2]$ such that $i < j$, and $j' \in [i + 1, 2t - 2]$. By construction of the sequence, since $j' \geq i + 1$ we have that $x_{j'} \in X_{j'} \subseteq X_{i+1}$. Moreover, by construction of $X_{i+1}$ we know that $x_i$ is either adjacent to all the vertices in $X_{i+1}$, or non-adjacent to all the vertices in $X_{i+1}$. Thus, if $x_i \sim x_j$ then $x_i \sim x_{j'}$, otherwise $x_i \nsim x_{j'}$. ⬚

In particular, this claim implies that if a vertex $x_i$ is adjacent (or non-adjacent) to a vertex $x_j$ that comes "after" $x_i$, then $x_i$ is actually adjacent (or non-adjacent) to *all* the vertices $x_{j'}$ that come "after" $x_i$.

Now, consider the set $\mathcal{U} := \mathcal{N}(x_{2t-2}) \cup \{x_1, \ldots, x_{2t-2}\}$. If $|\mathcal{U}| \geq t - 1$, by the second claim $\mathcal{U} \cup \{x_{2t-2}\}$ contains a $t$-clique; otherwise, if $|\mathcal{U}| \leq t - 2$, again by the second claim $V(G) - (\mathcal{N}(x_{2t-2}) \cup \{x_{2t-2}\})$ contains an independent set of size $2t - 2 - |\mathcal{U}| \geq 2t - 2 - (t - 2) = t$. ⬚

## 4.3 Exercises

> **Problem 4.1**
>
> Prove that $\mathrm{ex}(n, P_4) \leq n + 1$.

*Solution.* Let $G$ be a graph such that $|E(G)| = n+1$, and by way of contradiction suppose that $G$ does not contain $P_4$ as subgraph. By the number of edges, we know that $G$ cannot be a forest, implying that at least one of the components of $G$ contains a cycle. Moreover, we observe that

- no cycle can be longer than 3, otherwise that cycle would contain $P_4$ as subgraph trivially

- no component containing a cycle can have more than 3 vertices, otherwise it would contain $P_4$ as subgraph trivially — by connectivity of the components

Therefore, we get that every component of $G$ is either $K_3$ or a tree. Let $C_1, \ldots, C_\ell$ be the $K_3$-components of $G$, and let $T_1, \ldots, T_h$ be its tree components. Then, we have that

$$
\begin{aligned}
|E(G)| &= \sum_{i=1}^{\ell} |E(C_i)| + \sum_{i=1}^{h} |E(T_i)| \\
&= \sum_{i=1}^{\ell} |V(C_i)| + \sum_{i=1}^{h} |V(T_i)| - 1 \\
&= \sum_{i=1}^{\ell} |V(C_i)| + \sum_{i=1}^{h} |V(T_i)| - h \\
&= 3\ell + n - 3\ell - h \\
&= n - h
\end{aligned}
$$

However, this implies that

$$
n + 1 = |E(G)| = n - h \iff h = -1
$$

which is impossible because $h \geq 0$ ⨼. $\qquad \square$

> **Problem 4.2**
>
> Show that $\mathrm{ex}(n, K_{1,3}) = n + 1$.

*Solution.* First, we observe that any $C_n$ cycle graph on $n$ vertices has $n$ edges but does not contain $K_{1,3}$ as subgraph, therefore $\mathrm{ex}(n, K_{1,3}) \geq n + 1$.

Consider a graph of $n$ vertices and $n + 1$ edges — in particular, this implies that $\delta \geq 2$. By way of contradiction suppose that $\Delta(G) \leq 2$. If this is the case, we observe that

$$
\forall v \in V(G) \quad 2 \leq \delta \leq \deg(v) \leq \Delta \leq 2 \implies \deg(v) = 2
$$

However, by the Handshaking lemma we know that

$$\sum_{v \in V(G)} \deg(v) = 2\,|E(G)| = 2(n+1)$$

which means that the number of vertices of $G$ is given by $\frac{2(n+1)}{2} = n+1$ contradicting the fact that $G$ had $n$ vertices $\unicode{x21af}$.

In particular, if $\Delta(G) \geq 3$, it implies that $K_{1,3}$ must be a subgraph of $G$, concluding that $\text{ex}(n, K_{1,3}) \leq n+1$. $\qquad \square$

> ### Problem 4.3
>
> Find the value of $\text{ex}(n, K_{1,t})$ for all $t \geq 4$ and prove it.

*Solution.* We observe that if $n < t+1 \iff n \leq t$ it is impossible to have $K_{1,t}$ as subgraph since it requires at least $t+1$ vertices in the graph. Therefore, we may assume that $n \geq t+1$.

Note that to guarantee that $K_{1,t}$ is a subgraph of $G$, it suffices to guarantee that $G$ contains a vertex of degree at least $t$. Hence, the largest number of edges $G$ can have must be such that every vertex has degree $t-1$, i.e. the sum of the degrees of $G$ is $n(t-1)$. Hence, by the Handshaking lemma we get that

$$2\,|E(G)| = \sum_{v \in V(G)} \deg(v) = n(t-1) \iff |E(G)| = \frac{n}{2}(t-1)$$

We are now going to prove that is indeed $\text{ex}(n, K_{1,t}) = \frac{n}{2}(t-1) + 1$.

For the lower bound, consider the graph $K_{t-1,t-1}$. Clearly, each vertex has degree $t-1$ meaning that it cannot contain $K_{1,t}$ as subgraph, and by the Handshaking lemma the number of edges is given by

$$2\,|E(G)| = \sum_{v \in V(G)} \deg(v) = n(t-1) \iff |E(G)| = \frac{n}{2}(t-1)$$

**Claim:** For $t \geq 4$ it holds that $\text{ex}(n, K_{1,t}) \leq \frac{n}{2}(t-1) + 1$.

*Proof of the Claim.* Let $G$ be a graph of $n$ vertices and $\frac{n}{2}(t-1) + 1$ edges, and by way of contradiction suppose that $\Delta(G) \leq t-1$. By the Handshaking lemma, we have that

$$\sum_{v \in V(G)} \deg(v) = 2\,|E(G)| = 2\left[\frac{n}{2}(t-1) + 1\right] = n(t-1) + 2$$

meaning that the number of vertices of $G$ is given by

$$\frac{n(t-1) + 2}{t-1} \geq n + \frac{2}{t-1} > n$$

since $t \geq 4 \implies \frac{2}{t-1} \geq \frac{2}{3}$, contradicting the fact that $G$ had $n$ vertices $\unicode{x21af}$. $\qquad \square$

Together with the previous counterexample, we get that

$$n \geq t + 1 \implies \text{ex}(n, K_{1,t}) = \frac{n}{2}(t-1) + 1$$

<div>□</div>

### Definition 4.10

Given a graph $H$, let $R(H)$ be the minimum number of vertices such that for every graph $G$ of $R(H)$ vertices either $H \subseteq G$ or $H \subseteq \overline{G}$.

### Problem 4.4

Find the value of $R(P_4)$.

*Solution.* Trivially, the following graph shows that $R(P_4) \geq 5$ — the anti-edges are represented with dashed lines
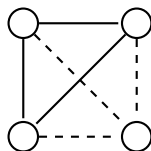


**Claim:** $R(P_4) \leq 5$.

*Proof of the Claim.* Let $G$ be a graph of 5 vertices; then $|E(G)| + |E(\overline{G})| = |E(K_5)| = \binom{5}{2} = 10$, which means that the largest between $G$ and $\overline{G}$ contains at least 5 edges — without loss of generality, suppose that $G$ is larger. However, a graph with 5 vertices and at least 5 edges must contain a cycle, hence $G$ contains a cycle. Pick the largest cycle in $G$: if the cycle has length 4 or 5, the theorem trivially holds, so we may assume that $G$ contains $C_3$ — let $x$ and $y$ be the two vertices not contained in $C_3$. Then, since $G$ must contain at least 5 edges on 5 vertices, and 3 edges form a $C_3$, there must be at least one edge connecting either $x$ or $y$ to $C_3$, meaning that $P_4$ is still contained in $G$. □

The claim, together with the previous graph, proves that $R(P_4) = 5$. □

### Problem 4.5

Find the value of $R(P_5)$.

*Solution.* The following graph shows that $R(P_5) \geq 6$ — observe that the graph consists of a $K_4$ with an isolated vertex.
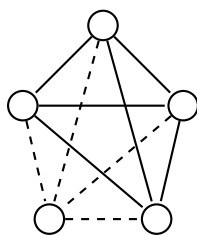
**Claim:** $R(P_5) \le 6$.

*Proof of the Claim.* Let $G$ be a graph of 6 vertices; then $|E(G)| + |E(\overline{G})| = |E(K_6)| = \binom{6}{2} = 15$, which means that the largest between $G$ and $\overline{G}$ contains at least 8 edges — without loss of generality, suppose that $G$ is larger. However, a graph with 6 vertices and at least 8 edges must contain a cycle, hence $G$ contains a cycle. Pick the largest cycle in $G$: if the cycle has length 5 or 6, the theorem trivially holds, so we may assume that our cycle is either a $C_3$ or a $C_4$.

- Suppose that $C_3 \subsetneq G$; we have two cases.

  - There are at least two vertices $x, y \in V(C_3)$ that have neighbors outside $C_3$. If there was a vertex $z \notin V(C_3)$ such that $x, y \sim z$, then $G$ contains $C_4$ contradicting the fact that $C_3$ was the largest cycle in $G$. Therefore, we may assume that there are two distinct vertices $u, v \notin V(C_3)$ such that $x \sim u$ and $y \sim v$, but $x \nsim v$ and $y \nsim u$. Moreover, $u \nsim v$ otherwise together with $C_3$ there would be $C_5$ inside $G$, again contradicting the maximality of the cycle we picked. Let $z$ be the third vertex outside $C_3$; if $x \sim z$ then $zx\ (x\ C\ y)\ yv$ is a $P_5$ in $G$, so we may assume that $x \nsim z$. Then $y \nsim u \nsim v \nsim x \nsim z$ is a $P_5$ in $\overline{G}$.

  - No vertex of $C_3$ has neighbors outside $C_3$; then $\overline{G}$ contains $K_{3,3}$ as subgraph, implying that $P_5 \subsetneq \overline{G}$.

- Suppose that $C_4 \subsetneq G$; if there is a vertex of $V(C_4)$ that has a neighbor outside $C_4$, there is a $P_5$ in $G$, so we may assume that no vertex of $V(C)$ has neighbors outside the cycle. However, this means that $\overline{G}$ contains $K_{2,4}$ as subgraph, implying that $P_5 \subsetneq \overline{G}$.

$\square$

The claim, together with the previous graph, proves that $R(P_5) = 6$. $\square$

> ### Problem 4.6
>
> For any $n$, find a lower bound for $R(P_n)$.

*Solution.* Trivially, the clique $K_{n-1}$ does not contain $P_n$, because its longest path is $P_{n-1}$.

**Claim:** Any complete bipartite graph that has $\lfloor \frac{n}{2} \rfloor - 1$ nodes in one of its partitions does not contain $P_n$.

*Proof of the Claim.* Any path in a complete bipartite graph must alternate between its two partitions, therefore the longest path will have length

$$2\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) + 1 \leq n - 2 + 1 = n - 1 < n$$

$\square$

Fix a number $n$, and consider the following graph $G := K_{n-1} \cup K_{\lfloor \frac{n}{2} \rfloor - 1}$. By the previous observation, $K_{n-1}$ does not contain $P_n$, and clearly $K_{\lfloor \frac{n}{2} \rfloor - 1}$ cannot contain $P_n$ either, since it contains even less vertices. Moreover, we observe that $\overline{G} = K_{n-1, \lfloor \frac{n}{2} \rfloor - 1}$ by construction of $G$, and by the claim we get that $\overline{G}$ does not contain $P_n$. Therefore, this graph concludes that

$$R(P_n) \geq |V(G)| + 1 = (n-1) + \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) + 1 = n + \left\lfloor \frac{n}{2} \right\rfloor - 1$$

$\square$

> ### Problem 4.7
>
> Find the value of $R(P_6)$.

*Solution.* By Problem 4.6, we know that

$$R(P_6) \geq 6 + \left\lfloor \frac{6}{2} \right\rfloor - 1 = 6 + 3 - 1 = 8$$

Let $G$ be a graph on 8 vertices, and by way of contradiction suppose that neither $G$ nor $\overline{G}$ contain $P_6$. Since $G$ has 8 vertices, then $|E(G)| + |E(\overline{G})| = |E(K_8)| = \binom{8}{2} = 28$, which means that the largest between $G$ and $\overline{G}$ contains at least 14 edges. However, a graph with 8 vertices and at least 14 edges must contain a cycle, hence the largest between $G$ and $\overline{G}$ contains a cycle. However, by the same argument, the smallest *may also* contain a cycle, hence let $C$ be the *largest* cycle in both $G$ and $\overline{G}$ — without loss of generality, suppose that $C$ is in $G$. In particular, if the cycle has length 6 or more, $C$ trivially contains $P_6$ contradicting our assumption, so we may assume that our cycle is either $C_3$, $C_4$ or $C_5$.

First, suppose that $C = C_3$, and let $\{c_i, c_j, c_k\}$ be the set of its vertices.

**Claim:** If $C = C_3$, there is no pair of distinct vertices $c_i, c_j \in V(C_3)$ such that for any $x_p, x_q \in V(G) - V(C_3)$ it holds that $c_i \sim x_p$ and $c_j \sim x_q$.

*Proof of the Claim.* Without loss of generality, suppose that $c_i \sim x_p$ and $c_j \sim x_q$ for the sake of contradiction. If $x_p = x_q$, then $x_p \sim c_i \sim c_k \sim c_j$ form $C_4$, contradicting the maximality of $C$ $\lightning$. Vice versa, if $x_p \neq x_q$, then for all $x_\ell \in V(G) - (V(C_3) \cup \{x_p, x_q\})$ it holds that $x_p, x_q \nsim x_\ell$, otherwise there would be $P_6$ in $G$, but this implies that there is $C_4$ in $\overline{G}$, again contradicting the maximality of $C$ $\lightning$. $\square$

Now, suppose that there is a vertex $c_i \in V(C_3)$ such that $c_i \sim x_p$ for some $x_p \in V(G) - V(C_3)$; then, by the previous claim, we know that for all $x_\ell \in V(G) - V(C_3)$ it holds that $c_j, c_k \nsim x_p$, i.e. $K_{2,5}$ is a subgraph of $\overline{G}[V(G) - \{c_i\}]$. This implies that for each

vertex $x_\ell \neq x_p$ outside $C_3$ it holds that $c_i, x_p \sim x_\ell$, otherwise the anti-edge from $x_\ell$ to $c_i$ or $x_p$ together with $K_{2,5}$ would form $P_6$ in $\overline{G}$. In particular, this implies that there are two vertices $x_{\ell_1}, x_{\ell_2} \neq x_p$ outside $C_3$ such that $c_i \sim x_{\ell_1} \sim x_p \sim x_{\ell_2}$ form $C_4$, contradicting the maximality of $G \not{\downarrow}$.

This implies that for all $c_i \in V(C_3)$ and $x_p \in V(G) - V(C_3)$ it holds that $c_i \not\sim x_p$, meaning that $\overline{G}$ contains $K_{3,5}$ as subgraph, which in turn contains $P_6$. This concludes that $C \neq C_3$.

Now, suppose that $C = C_4$.

**Claim:** If $C = C_4$, then for all $c_i \in V(C_4)$ and $x_\ell \in V(G) - V(C_4)$ it holds that $c_i \not\sim x_\ell$.

*Proof of the Claim.* By way of contradiction, suppose that there is a vertex $c_i \in V(C_4)$ such that $c_i \sim x_\ell$ for some $x_\ell \in V(G) - V(C_4)$; then for all $x_t \in V(G) - (V(C_4) \cup \{x_\ell\})$ it holds that $x_\ell \not\sim x_t$ otherwise there would be $P_6$ in $\overline{G}$. Moreover, let $c_j, c_k \in V(C_4)$ be the vertices such that $c_j, c_k \sim c_i$; then for all $x_t \in V(G) - (V(C_4) \cup \{x_\ell\})$ it holds that $c_j, c_k \not\sim x_t$ otherwise there would be $P_6$ in $\overline{G}$. However, this implies that the other 3 vertices $x_{t_1}, x_{t_2}, x_{t_3} \in V(G) - (V(C_4) \cup \{x_\ell\})$ are such that $x_{t_1} \not\sim x_\ell \not\sim x_{t_2} \not\sim c_k \not\sim x_{t_3} \not\sim c_j$ form $P_6$ in $\overline{G} \not{\downarrow}$. $\square$

In particular, the previous claim concludes that $\overline{G}$ contains $K_{4,4}$, which trivially contains $P_6$. Therefore, $C \neq C_3, C_4$.

Lastly, suppose that $C = C_5$. Then, if any of the vertices $c_i \in V(C_5)$ is adjacent to a vertex $x_\ell \in V(G) - V(C_5)$, then there is $P_6$ in $\overline{G}$, therefore for all $c_i \in V(C_5)$ and $x_\ell \in V(G) - V(C_5)$ it holds that $c_i \not\sim v_\ell$. In particular, this implies that $\overline{G}$ contains $K_{3,5}$, which clearly contains $P_6 \not{\downarrow}$. $\square$

# 5
# Graph decompositions

## 5.1 Blocks

> **Definition 5.1: Cut vertex**
>
> Given a graph $G$, a **cut vertex** of $G$ is a vertex $v \in V(G)$ such that $G[V(G) - \{v\}]$ has more connected components than $G$.
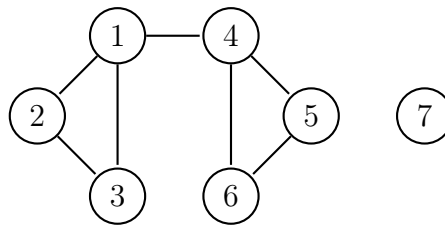
Figure 5.1: For instance, in this graph there are two cut vertices, namely 1 and 4.

Trivially, by definition we observe that no 2-connected graph can contain cut vertices.

> **Definition 5.2: Block**
>
> Given a graph $G$, a **block** is a *maximal* subgraph $H$ with no cut vertex, i.e. there is no $H' \subseteq G$ without cut vertices such that $H \subsetneq H'$.

For instance, in the previous example we have exactly four blocks, and the set of vertices that describe them are the following:

$$B_1 := \{1, 2, 3\}$$

$$B_2 := \{4, 5, 6\}$$
$$B_3 := \{1, 4\}$$
$$B_4 := \{7\}$$

We observe that any edge of a graph is itself a subgraph without cut vertices, which implies that *all* the edges of a graph will belong to a block. Furthermore, we observe that any cut vertex of a graph must belong to at least two different blocks, by definition.

Moreover, consider the following definition.

**Definition 5.3: Bridge**

Given a graph $G$, an edge $e \in E(G)$ is called **bridge** if $G - \{e\}$ has more components than $G$.

For instance, in the previous example the edge $\{1, 4\}$ is clearly a bridge.

Bridges and cut vertices are similar and linked concepts; in fact, if an edge $e$ is a bridge, clearly *at least one* of its endpoints must be a cut vertex. However, we observe that:

- the endpoints of a bridge may not be *both* cut vertices
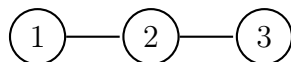


Figure 5.2: For instance, in this graph the edges $\{1, 2\}$ and $\{2, 3\}$ are clearly bridges, but the vertices 1 and 3 are not cut vertices.

- if an edge has a cut vertex as one of its endpoints, the edge *may not* be a bridge
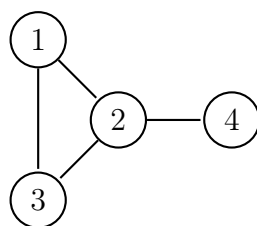


Figure 5.3: For instance, in this graph the vertex 2 is clearly a cut vertex, even if $\{1, 2\}$ and $\{2, 3\}$ are not bridges.

**Proposition 5.1**

A block of a graph can only be an isolated vertex, a bridge or a 2-connected subgraph.

*Proof.* If the block has only 1 vertex, clearly it must be an isolated vertex.

Consider a block $B$ consisting of 1 edge $uv$: if this edge were not a bridge, then removing it would not create more components than before, meaning that there must be at least another path beteween $u$ and $v$ in the graph that does not cross the edge $uv$. However, this implies that $uv$ is contained in a cycle, meaning that $B$ could have been extended with the cycle, contradicting the maximality of $B$ — we observe that a cycle cannot contain cut vertices.

Finally, consier a block $B$ with more than two vertices: if $B$ were not 2-connected, then by removing one vertex $v \in B$ we would get that $B - \{v\}$ is disconnected, meaning that $v$ was a cut vertex in the first place, contradicting the definition of $B$.  $\square$

### Lemma 5.1

Given a graph $G$, and two connected subgraphs $H, H' \subseteq G$ such that $V(H) \cap V(H') \neq \varnothing$, it holds that $H \cup H'$ is connected.

*Proof.* Let $z \in V(H) \cap V(H')$, which must exist by assumption, and two vertices $u \in V(H)$ and $v \in V(H')$:

- if $u, v \in V(H) \cap V(H')$, then there is a path $u \to v$ by connectivity of both $H$ and $H'$

- without loss of generality, if $u \in V(H)$ and $v \in V(H') - V(H)$, then there is a path $P$ of the form $u \to z$ by connectivity of $H$, and there is a path $Q$ of the form $z \to v$ by connectivity of $H'$, hence $P \cup Q$ is a path of the form $u \to v$

implying that $H \cup H'$ is still connected.  $\square$

### Proposition 5.2

Given a graph $G$, and two distinct blocks $H$ and $H'$ of $G$, it holds that $|V(H) \cap V(H')| \leq 1$.

*Proof.* By way of contradiction, suppose that $|V(H) \cap V(H')| \geq 2$. Let $z \in V(H \cup H')$; by Proposition 5.1 we get that both $H - \{z\}$ and $H' - \{z\}$ are still connected. Moreover, since $|V(H) \cap V(H')| \geq 2$, it holds that $|V(H)|, |V(H')| \geq 2$, meaning that

$$(H \cup H') - \{z\} = (H - \{z\}) \cup (H' - \{z\}) \neq \varnothing$$

which implies that $(H \cup H') - \{z\}$ is connected by the previous lemma.

Suppose that $|V(H \cup H')| = 2$; then we have only two possible cases:

- $H \cup H'$ consists of two isolated vertices, contradicting the fact that $|V(H) \cap V(H')| \geq 2$

- $H \cup H'$ consists of a single edge, meaning that the edge itself is one single block, hence $H = H'$, contradicting the fact that $H \neq H'$ by assumption

Hence, the only possibility is that $H \cup H'$ contains at least 3 vertices. Moreover, we observe that if $V(H) \cap V(H') \neq \varnothing$, then $H \cup H'$ is connected by Proposition 5.1.

However, we observe that if

- $H \cup H'$ is connected

- for any $z \in V(H \cup H')$ it holds that $(H \cup H') - \{z\}$ is connected

- $|V(H \cup H')| \geq 3$

it implies that $H \cup H'$ is 2-connected by definition. This means that $H \cup H'$ does not contain cut vertices, contradicting the maximality of the blocks $H, H' \subsetneq H \cup H' \, \nmid$.  $\square$

> **Proposition 5.3**
>
> Given a graph $G$, and two distinct blocks $H$ and $H'$ of $G$, if $V(H) \cap V(H') \neq \varnothing$ then any vertex in $V(H) \cap V(H')$ is a cut vertex of $G$.

*Proof.* Since $V(H) \cap V(H') \neq \varnothing$, we know that neither $H$ nor $H'$ are isolated vertices — if both of them were isolated vertices, they would have empty intersection, and if one of them were an isolated vertex it would have been part of the other block, meaning that $H$ and $H'$ actually were a single block. Moreover, if at least one between $H$ and $H'$ is a bridge, the theorem trivially holds. Hence, we may assume that $|V(H)|, |V(H')| \geq 3$ — and in particular, they are 2-connected by Proposition 5.1.

By the previous theorem, if $V(H) \cap V(H') \neq \varnothing$ it holds that $|V(H) \cap V(H')| = 1$, hence let $z$ be the vertex in the intersection of the two blocks, and by way of contradiction suppose that $z$ is not a cut vertex. Then, by definition this means that for any two fixed vertices $u \in V(H - \{z\})$ and $v \in V(H' - \{z\})$ there is a path $P$ that does not pass through $z$.

**Claim:** $H \cup H' \cup P$ is 2-connected.

*Proof of the Claim.* By Proposition 5.1, we know that both $H - \{z\}$ and $H' - \{z\}$ are connected, and since they both have a vertex in common with $P$, we get that

$$(H - \{z\}) \cup (H' - \{z\}) \cup P = (H \cup H' \cup P) - \{z\}$$

is still connected — since $P$ is a path.

Moreover, by the same proposition we note that $H \cup H'$ is connecetd. Now, let $p_1$ and $p_2$ be the endpoints of $P$ — without loss of generality, assume $p_1 \in V(H)$ and $p_2 \in V(H')$ — and fix a vertex $x \in V(H \cup H' \cup P)$; we have some cases to handle

- In the first case, we have that $x \in V(H \cup H') - \{z\}$. Fix two vertices $a, b \in V(H \cup H' \cup P) - \{x\}$

  - if $a, b \in V(P)$, they are trivially connected since $P$ is a path

  - if $a, b \in V(H) - \{z, x\}$, or $a, b \in V(H') - \{z, x\}$, then $a$ and $b$ are connected by 2-connectivity of $H$ and $H'$ — even if a path $a \to b$ passes through $x$, there must exist at least another path of the same form by Theorem 3.2 that does not pass through $x$

- without loss of generality, if $a \in V(H) - \{z, x\}$ and $b \in V(H') - \{z, x\}$, then by 2-connectivity of $H$ and $H'$ there are paths $a \to p_1$ and $p_2 \to b$ that do not pass through $x$, meaning that $(a \to p_1) \cup P \cup (p_2 \to b)$ is a path $a \to b$

  - without loss of generality, if $a \in V(H) - \{z, x\}$ and $b \in V(P) - \{p_1, p_2\}$, again by 2-connectivity of $H$ there is at least one path $a \to p_1$ that does not pass through $x$, meaning that $(a \to p_1) \cup (p_1 \ P \ b)$ is a path $a \to b$

  This proves that $(H \cup H' \cup P) - \{x\}$ is connected, for $x \in V(H \cup H') - \{z\}$.

- In the second case, we have that $x \in V(P) - V(H \cup H')$. Fix two vertices $a, b \in V(H \cup H' \cup P) - \{x\}$

  - if $a, b \in V(H \cup H') - V(P)$, then $a$ and $b$ are trivially connected by connectivity of $H \cup H'$

  - without loss of generality, if $a \in V(H) - V(P)$ and $b \in V(P) - \{x\}$ then

    * if by removing $x$ from $P$ we did not disconnect $a$ and $b$, there still is a path $a \to b$

    * otherwise, by connectivity of $H \cup H'$ there will be a path $a \to p_2$, hence $(a \to p_2) \cup (p_2 \ P \ b)$ is a path $a \to b$

  - similarly, if $a, b \in V(P) - \{x\}$, then

    * if by removing $x$ from $P$ we did not disconnect $a$ and $b$, there still is a path $a \to b$

    * otherwise, by connectivity of $H \cup H'$ there will be a path $p_1 \to p_2$, hence $(a \ P \ p_1) \cup (p_1 \to p_2) \cup (p_2 \ P \ b)$ is a path $a \to b$

  This proves that $(H \cup H' \cup P) - \{x\}$ is connected, for $x \in V(P) - V(H \cup H')$.

Finally, this means that no matter the choice of $x$, we have that $H \cup H' \cup P - \{x\}$ is still connected, i.e. $H \cup H' \cup P$ is 2-connected by definition. $\square$

In particular, since $H \cup H' \cup P$ is 2-connected by the claim, it cannot contain any cut vertices, implying that $H \cup H' \cup P$ contradicts the maximality of the blocks $H, H' \subsetneq H \cup H' \cup P \ \lightning$. $\square$

### 5.1.1  Block graph

> **Definition 5.4**
>
> Given a connected graph $G$, let $B_1, \ldots, B_k$ be its blocks, and let $z_1, \ldots, z_\ell$ be the set of its cut vertices. We define $\mathcal{B}(G)$ as the **block graph** of $G$ through the following sets of nodes and edges:
>
> - $V(\mathcal{B}(G)) = \{b_1, \ldots, b_k\} \cup \{z_1, \ldots, z_\ell\}$, where each $b_i$ is a vertex corresponding to a block $B_i$ of $G$
>
> - there is an edge $\{b_i, z_j\} \in E(\mathcal{B}(G))$ if and only if $z_j \in V(B_i)$

TODO _____ example

We observe that, by definition the only edges present in $\mathcal{B}(G)$ are between nodes representing blocks and nodes representing cut vertices of $G$. This means that $\{b_1, \ldots, b_k\}$ and $\{z_1, \ldots, z_\ell\}$ are independent sets, i.e. $\mathcal{B}(G)$ is bipartite.

**Theorem 5.1**

Given a connected graph $G$, then $\mathcal{B}(G)$ is a tree.

*Proof.* Since $G$ is connected, $\mathcal{B}(G)$ is trivially connected as well. By way of contradiction, suppose that it is not a tree, i.e. there is an *induced* cycle

$$C := b_{i_1} \ z_{j_1} \ b_{i_2} \ \ldots \ b_{i_{h-1}} \ z_{j_h} \ b_{i_h}$$

in $\mathcal{B}(G)$, where $z_{j_l} \in V(B_{i_l}) \cap V(B_{i_{l+1}})$ by definition. Moreover, since the block graph is bipartite by definition, we get that $|V(C)|$ must be even by Theorem 1.4. Additionally, since $C$ is induced, we have that for $i_l \notin \{j_l, j_l + 1\}$ it holds that $z_{j_l} \notin V(B_{i_l})$.

**Claim:** $\displaystyle\bigcup_{l=1}^{h} B_{i_l}$ has no cut vertex.

*Proof of the Claim.* Let $x \in V\left(\displaystyle\bigcup_{l=1}^{h} B_{i_l}\right)$, and without loss of generality suppose that $x \in V(B_{i_1})$. TODO _____ $\square$ da finire

In particular, this claim contradicts the maximality of the blocks $b_{i_1}, \ldots, b_{i_h}$ ↯. $\square$

**Proposition 5.4**

Given a connected graph $G$, every leaf of $\mathcal{B}(G)$ represents a block of $G$.

*Proof.* By way of contradiction, suppose that there is a leaf $v \in V(\mathcal{B}(G))$ that represents a cut vertex of $G$. Since $v$ is a leaf, it must have degree 1 in $\mathcal{B}(G)$, and by definition of the block graph we know that its only neighbor must be a vertex $b$ representing a block $B$ of $G$. Therefore, by definition of $\mathcal{B}(G)$ there we know that $v \in V(B)$, but if $v$ is a cut vertex of $G$ there must be at least another block $B'$ — represented by a vertex $b' \in V(\mathcal{B}(G))$ — in $G$ that contains $v$, meaning that $v \sim b, b'$, contradicting the fact that $v$ was a leaf of the block graph ↯. $\square$

## 5.2 Known decompositions

In this brief section, we are going to present characterizations of 2-connected and 3-connected graphs, which can be *decomposed* using specific types of structural decompositions.

### 5.2.1 2-connectivity

> **Definition 5.5: $H$-path**
>
> Given a graph $G$, and a subgraph $H \subseteq G$, an $H$-**path** $P$ is a non-trivial path with both ends in $V(H)$, no internal vertex in $V(H)$, and no edge in $E(H)$.

We observe that the last condition of the definition implies that *single edges* are not $H$-paths.

TODO _____ example

> **Definition 5.6: Ear decomposition**
>
> Given a graph $G$, an **ear decomposition** of $G$ is a sequence $C, P_1, \ldots, P_k$ such that
>
> - $C$ is a cycle
>
> - each $P_i$ is a $\left( \bigcup_{j=1}^{i-1} P_j \right)$-path
>
> - $C \cup P_1 \cup \ldots \cup P_k = G$
>
> The **size** of an ear decomposition is the number of its paths — in this case $k$.

TODO _____ example

The following theorem characterizes 2-connected graphs through ear decompositions.

> **Theorem 5.2**
>
> G is 2-connected if and only if it has an ear decomposition.

*Proof.*

*Direct implication.* If $|V(G)| = 3$, and $G$ is 2-connected, then $G = C_3$ and the theorem trivially holds, so we may assume that $|G| \geq 4$.

If $G$ is 2-connected, then $\delta \geq 2$ by Proposition 3.2, i.e. $G$ has a cycle $C$. Fix $H \subseteq G$ as large as possible — i.e. *maximal by subgraph containment* — such that $H$ has an ear decomposition.

By way of contradiction, suppose that $H \neq G$, meaning that there must be at least an edge $xy \in E(G) - E(H)$ — note that the vertex set of $H$ might still be $V(G)$. Moreover, since $|V(G)| \geq 4$ and $xy \in E(G) - E(H)$, then $|V(H)| \geq 2$. Then, since $G$ is 2-connected, by Corollary 3.1 there are at least two paths $P$ and $P'$ that connect $x$ to $V(H)$ such that $V(P) \cap V(P') = \{x\}$, and two paths $Q$ and $Q'$ that connect $y$ to $V(H)$ such that $V(Q) \cap V(Q') = \{y\}$. We have two cases.

- In the first case, all the paths intersect $V(P) \cap V(Q), V(P) \cap V(Q') \neq \varnothing$ and $V(P') \cap V(Q), V(P') \cap V(Q') \neq \varnothing$. Let $a \in V(P) \cap V(Q)$ and $b \in V(P') \cap V(Q')$; then $Q\ a\ x\ y\ b\ P'$ is an $H$-path.

- At least two paths do not intersect, and without loss of generality suppose that $V(P) \cap V(Q) = \varnothing$. Then $P\ x\ y\ Q$ is an $H$-path.

However, in both cases the $H$-path contradicts the maximality of $H\ \lightning$.

*Converse implication.* We procced by induction on the size of the ear decomposition.

For the base case, if $k = 0$ then $G = C$, which implies that $G$ is 2-connected. Now, assume inductively that if graph has an ear decomposition of size $k$ it is 2-connected, and consider a graph $G$ that has an ear decomposition of size $k + 1$.

By inductive hypothesis, we know that $H := C \cup P_1 \cup \ldots \cup P_k$ is 2-connected. Moreover, since $C, P_1, \ldots, P_{k+1}$ is an ear decomposition, we know that $P_{k+1}$ is an $H$-path. Fix $x \in V(G)$; we observe that

- $H$ is 2-connected, hence $H - \{x\}$ is still connected by definition

- every components of $P_{k+1} - \{x\}$ has a vertex in common with $H - \{x\}$

meaning that $(H - \{x\}) \cup (P_{k+1} - \{x\}) = (H \cup P_{k+1}) - \{x\}$ is connected. Lastly, since $C \subsetneq H$, then $|V(H)| \geq 3 \implies |V(H \cup P_{k+1})| \geq 3$, which concludes that $H \cup P_{k+1} = G$ is 2-connected.

$\square$

## 5.3    3-connectivity

The other type of graphs that we are going to discuss are 3-connected graphs. First, consider the following theorem — proved by Tutte [Tut61] in 1961.

> **Theorem 5.3**
>
> If a graph $G$ is 3-connected, and $n \geq 5$, then there exists an edge $e \in E(G)$ such that $G/e$ is still 3-connected.

*Proof.* By way of contradiction, suppose that there is no edge $e \in E(G)$ such that $G/e$ is still 3-connected. However, since $n \geq 5$, and $G/e$ is not 3-connected, it must admit a separation $(A, B)$ or order at most 2 by definition.

**Claim:** For any edge $e := xy \in E(G)$, there is a separation $(A_e, B_e)$ of order *exactly* 3 such that $x, y \in A_e \cap B_e$.

*Proof of the Claim.* Consider a separation $(A, B)$ of $G/e$ of *exactly* 2, and let $v_e \in V(G/e)$ be the vertex constructed after contracting $e$. If $v_e \in B - A$, then $(A, (B - \{v_e\}) \cup \{x, y\})$ would be a separation of $G$ of order 2, contradicting the 3-connectivity of $G$. This implies that $v_e \in A \cap B$, which in turn implies that $((A - v_e) \cup \{x, y\}, (B - v_e) \cup \{x, y\})$ is a

separation of order 3 of $G$. Therefore, for any edge $e := xy \in E(G)$ we can construct a separation as the statement of the claim requires. $\quad\square$

Thus, let $(A_e, B_e)$ be the separation that minimizes $|B_e|$, and let $z \in (A_e \cap B_e) - \{x, y\}$ be the third vertex of $A_e \cap B_e$; if $z$ has no neighbors in $B_e - A_e$, then by removing $e$ from $G$ we would disconnect the graph, contradicting the assumption of 3-connectivity. This means that there is a vertex $u \in B_e - A_e$ such that $z \sim u$. Hence, let $e' := zu$, and consider a non-trivial separation $(A_{e'}, B_{e'})$ of $G$ of order exactly 3 such that $z, u \in A_{e'} \cap B_{e'}$ — which exists by the previous claim.
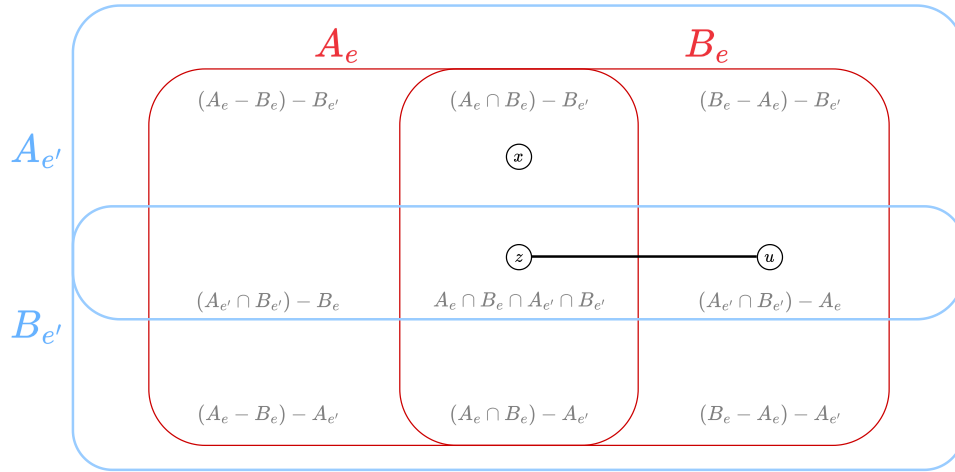


Figure 5.4: The two separations of $G$ and the various regions they induce.

**Claim:** $(A_e \cap B_e) - A_{e'} = \varnothing$.

*Proof of the Claim.* By definition of $(A_{e'}, B_{e'})$, we have that $z \in (A_e \cap B_e) \cap (A_{e'} \cap B_{e'})$. However, if both $x$ and $y$ are contained in $(A_e \cap B_e) \cap (A_{e'} \cap B_{e'})$, then $\{x, y, z, u\} \subseteq A_{e'} \cap B_{e'}$ contradicting the order of $(A_{e'}, B_{e'})$.

Then, without loss of generality suppose that $x \in A_{e'} - B_{e'}$; hence, $y \notin B_{e'} - A_{e'}$ otherwise the edge $xy$ would be an edge *across* $A_{e'} - B_{e'}$ and $B_{e'} - A_{e'}$, violating the fact that $(A_{e'}, B_{e'})$ is a separation. This implies that $(A_e \cap B_e) - A_{e'} = \varnothing$. $\quad\square$

**Claim:** $(A_e - B_e) - A_{e'} = \varnothing$.

*Proof of the Claim.* By way of contradiction, suppose that $(A_e - B_e) - A_{e'} \neq \varnothing$, and fix a vertex $w \in (A_e - B_e) - A_{e'}$. Since $(A_e, B_e)$ is a separation, we know that $w \nsim u$. Moreover, since $(A_{e'}, B_{e'})$ is a separation, $w$ can only be adjacent to vertices in $(A_{e'} \cap B_{e'}) - (B_e - A_e)$. Therefore, by removing $(A_{e'} \cap B_{e'}) - (B_e - A_e)$ from $G$ we disconnect $w$ from $u$ necessarily — since $(A_e \cap B_e) - A_{e'} = \varnothing$ by the previous claim. Moreover, since $(A_{e'} \cap B_{e'}) - (B_e - A_e)$ contains two vertices by choice on the order of $(A_{e'}, B_{e'})$, we get a contradiction on the 3-connectivity of $G \nlightning$. $\quad\square$

In particular, these two claims imply that

$$((A_e \cap B_e) - A_{e'}) \cup ((A_e - B_e) - A_{e'}) = A_e - A_{e'} = \varnothing$$

**Claim:** $(A_{e'} \cap B_{e'}) - B_e = \varnothing$.

*Proof of the Claim.* Consider the set $B_e - (A_e \cup A_{e'})$; since we picked $(A_{e'}, B_{e'})$ to be a non-trivial separation, we know that $B_{e'} - A_{e'} \neq \varnothing$. Therefore, the previous two claims imply that $B_e - (A_e \cup A_{e'}) \neq \varnothing$. Hence, let $v$ be a vertex in such set; since $(A_e, B_e)$ is a separation, we know that $v \not\sim x$ because $x \in (A_{e'} - B_{e'}) \cap (A_e \cap B_e)$.

Now, by way of contradiction suppose that $(A_{e'} \cap B_{e'}) - B_e \neq \varnothing$, and by choice of $(A_{e'}, B_{e'})$ we know that $A_{e'} \cap B_{e'}$ only contains 3 vertices. Therefore, let $w$ be the third vertex inside $(A_{e'} \cap B_{e'}) - B_e$; since $v \not\sim x$ by the previous observation, we know that $v$ can only be connected to $w$ through $z$ or $u$ — since $A_e - A_{e'} = \varnothing$ by the previous two claims.

This implies that by removing $\{z, u\}$ from $G$ we disconnect $v$ from $w$, contradicting the 3-connectivity of $G$ ↯. $\qquad\square$

Therefore, by these three claims we get that $B_{e'} \subseteq B_e$, but because $x \in (A_{e'} - B_{e'}) \cap (A_e \cap B_e)$ then $B_{e'} \subsetneq B_e$, which is a contradiction on the choice of the separation $(A_e, B_e)$ that minimized $|B_e|$ ↯. $\qquad\square$

From this theorem, we can immediately derive the direct implication of the following corollary, which gives the characterization of 3-connected graphs that we previously anticipated — the converse implication was also proved by Tutte. Unformally, we could say that every 3-connected graph can be constructed by repeatedly "un-contracting" edges.

> **Corollary 5.1**
>
> A graph $G$ such that $n \geq 5$ is 3-connected if and only if there exists a sequence $G_1, \ldots, G_k$ such that
>
> - for each $i \in [k-1]$ it holds that $G_i = G_{i+1}/e_{i+1}$ for some $e_{i+1} \in E(G_{i+1})$
> - $G_1 = K_4$ and $G_k = G$
> - for each $i \in [k]$, $G_i$ is 3-connected

# 6
# Planar graphs

### Definition 6.1: Straight line segment

Given two points $p, q \in \mathbb{R}^2$, the **straight line segment** (SLS) between $p$ and $q$ is the set

$$\{\alpha p + (1 - \alpha)q \mid \alpha \in [0, 1] \subset \mathbb{R}\}$$

### Definition 6.2: Polygon

A **polygon** is a subset of $\mathbb{R}^2$ formed by the unions of finitely many SLSs *homeomorphic* to the *unit circle* $S^1 := \{x \in \mathbb{R}^2 \mid ||x|| = 1\}$.

### Definition 6.3: Polygonal arc

A **polygonal arc** is a subset of $\mathbb{R}^2$ formed by the union of finitely many SLSs homeomorphic to the *unit interval* $[0, 1] \subset \mathbb{R}$.

Here, the term **homeomorphic** refers to the existence of an *homeomorphism* between the two sets, i.e. a continuous function between them whoch inverse is also continuous. In particular, for the case of polygonal arcs the images of 0 and 1 under such homeomorphism are the **endpoints** of the arc.

If the ends of a polygonal arc $P$ are $p, q \in \mathbb{R}^2$, we say that $P$ **links** $p$ and $q$, and $\mathring{P} := P - \{p, q\}$ is the **interior** of $P$.

### Definition 6.4: Open ball

Given a point $x \in \mathbb{R}^2$, and a radius $\varepsilon \in \mathbb{R}$, the **open ball** centered at $x$ having radius $\varepsilon$ is the following set of points

$$B_\varepsilon(x) := \{y \in \mathbb{R}^2 \mid ||x - y|| < \varepsilon\}$$

### Definition 6.5: Open and closed set

A set $X \subseteq \mathbb{R}^2$ is said to be

- **open** if for all $x \in X$ there is an $\varepsilon \in \mathbb{R}$ such that $B_\varepsilon(x) \subseteq X$

- **closed** if it is the complement of an open set in $\mathbb{R}^2$

We observe that, by definition, the union of two open sets is still open. Instead, the union of closed sets is guaranteed to be closed only if the sets are *finitely many*.

Let $O \subseteq \mathbb{R}^2$ be an open set; we observe that the property of "being linked by a polygonal arc that lies on $O$" induces an *equivalence relation* on $O$ — i.e. two points are related if they can be linked through a polygonal arc that lies inside $O$. Thus, we have the following definition.

### Definition 6.6: Region

Given $O \subseteq \mathbb{R}^2$, the equvialence classes of the equilvance relation induced by the property of "being linked by a polygonal arc that lies in $O$" are called **regions** of $O$.

Given an open set $O \subseteq \mathbb{R}^2$ and a closed set $X \subseteq \mathbb{R}^2$, we say that $X$ **separates** $O$ if $O - X$ has more regions than $O$.

### Definition 6.7: Frontier

Given a region $R$, the **frontier** of $R$ — written as $\mathrm{front}(R)$ — is the set of points $x \in \mathbb{R}^2$ such that every ball centered at $x$ intersects both $X$ and $\mathbb{R}^2 - X$.

Now that we discussed some preliminary definitions, we are ready to define **planar graphs**.

### Definition 6.8: Plane graph

A **plane graph** is a pair $G = (V, E)$ such that

- $V \subseteq \mathbb{R}^2$ is a set of points, called *vertices*
- $E$ is a set of polygonal arcs between points in $V$, called *edges*
- there are no pair of edges of $E$ that share both endpoints
- for each $e \in E$ it holds that $\mathring{e}$ is disjoint from $V \cup (E - e)$

In particular, the last condition implies that plane graphs do not admits edges that intersect.

We observe that, by definition, every plane graph $G$ is a closed set of points of $\mathbb{R}^2$ formed by the union of its vertices and its edges — implying that $\mathbb{R}^2 - G$ is open.

### Definition 6.9: Face

Given a plane graph $G$, a **face** of $G$ is a region of $\mathbb{R}^2 - G$. The set of faces of $G$ is denoted as $F(G)$.

We observe that there is exactly 1 unbounded face, which is called **outer face**, while all the other faces are called **inner faces**.

### Lemma 6.1: Handshaking lemma (faces)

Given a plane graph $G$, the sum of the number of edges seen by each face of $G$ is $2 |E(G)|$.

*Proof.* Trivially, by adding all the edges seen by each face of $G$, each edge is counted twice, therefore the sum is equal to $2 |E(G)|$. $\square$

### Definition 6.10: Planar graph

A graph $G$ is said to be **planar** if there exists a plane graph — called *drawing of $G$* — isomorphic to $G$.

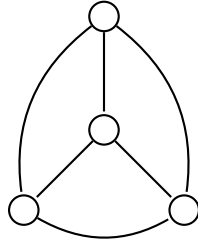In other words, a graph is said to be *planar* if it can be represented as a plane graph.

Figure 6.1: For instance, this is a *plane* representation of $K_4$, which implies that $K_4$ is *planar*.

# 6.1 Topological properties

**Theorem 6.1: Jordan curve theorem**

For every polygon $P \subseteq \mathbb{R}^2$, the set of points $\mathbb{R}^- P$ has exactly 2 regions, each of which has $P$ as its frontier.

TODO

**Lemma 6.2**

Let $P_1$, $P_2$ and $P_3$ be three polygonal arcs linking two points $x, y \in \mathbb{R}^2$ such that for each distinct $i, j \in [3]$ it holds that $\mathring{P}_i \cap \mathring{P}_j = \varnothing$. Then

1. $\mathbb{R}^2 - (P_1 \cup P_2 \cup P_3)$ has 3 regions with frontiers $P_1 \cup P_2$, $P_2 \cup P_3$ and $P_3 \cup P_1$

2. if $P$ is a polygonal arc between a point in $\mathring{P}_1$ and $\mathring{P}_3$, and $\mathring{P}$ lies in the region of $\mathbb{R}^2 - (P_1 \cup P_3)$ that contains $\mathring{P}_2$, then $\mathring{P} \cap \mathring{P}_2 \neq \varnothing$.

TODO

We observe that the intersection may contain more than one point, even an infinite amount.

**Lemma 6.3**

Let $X_1, X_2 \subseteq \mathbb{R}^2$ be two disjoint sets, each made up of a union of finitely many polygonal arcs and isolated points. Let $O$ be a region of $\mathbb{R}^2 - (X_1 \cup X_2)$; then, for any polygonal arc $P \subseteq O$ linking a point in $X_1$ and a point in $X_2$, it holds that $O - P$ is a region of $\mathbb{R}^2 - (X_1 \cup X_2 \cup P)$, and the number of total regions does not change.

TODO

> **Proposition 6.1**
>
> Let $G$ be a plane graph; then, given a face $f$ of $G$ and a subplane graph $H \subseteq G$, it holds that $H$ has a face $f'$ containing $f$. Moreover, if the frontier of $f$ lies in $H$, then $f = f'$.

*Proof.* Let $x, y \in \mathbb{R}^2$ be two points of $f$; then, by definition of face there is a polygonal arc linking $x$ and $y$ that does not intersect $G$. However, since $H \subseteq G$, they cannot intersect $H$ either, which in turn implies that $x$ and $y$ lie in a common face of $H$ by definition of face. Hence, there is a face $f'$ of $H$ that contains both $x$ and $y$.

TODO ☐

controlla simone come l'ha scritto

> **Theorem 6.2**
>
> A plane forest has exactly one face.

*Proof.* We proceed by induction on the number of edges.

> *Base case.* For the base case, if a plane forest $G$ is such that $|E(G)| = 0$ then the plane graph considered is a finites set $V(G)$ of isolated points; hence, $R - V(G)$ has only one region.
>
> *Inductive hypothesis.* Assume that a plane forest of $k$ edges has exactly one face.
>
> *Inductive step.* Consider a plane forest $G$ of $k+1$ edges; then, fix an edge $e \in E(G)$, and let $G' := G - e$. Clearly, $G'$ is still a plane forest but on $k$ edges, which implies that has exactly one face by inductive hypothesis — call this face $O$. Now, say that $e$ is a polygonal arc linking two vertices $x, y \in V(G)$; then, by definition of forest $x$ and $y$ must belong to different components of $G'$. Hence, by Lemma 6.3 we have that $G$ still has exactly one face, namely $O - \mathring{e}$.

☐

> **Lemma 6.4**
>
> Given a plane graph $G$, let $e \in E(G)$ be one of its edges; then, it holds that
>
> 1. given a face $f$ of $G$, either $e \subseteq \text{front}(f)$ or $\mathring{e} \cap \text{front}(f) = \varnothing$
>
> 2. if $e$ lies on a cycle of $G$, $e$ is in the frontier of exactly 2 faces of $G$
>
> 3. if $e$ does not lie on a cycle of $G$, $e$ is in the frontier of exactly 1 face

*Proof.* Fix an edge $e \in E(G)$, and a point $x_0 \in \mathring{e}$.

**Claim:** If $e$ lies on a cycle of $G$, $x_0$ lies in the frontier of exactly 2 faces of $G$, otherwise it lies in the frontier of exactly 1 face of $G$.

*Proof of the Claim.* By definition, $G$ is the union of finitely many SLSs and points; therefore, this implies that there exists a disc $D_0$ centered at $x_0$ ☐

TODO _____ ☐ [da finire]

TODO _____ [buco]

---

**Definition 6.11: Boundary**

Given a plane graph $G$, and a face $f$ of $G$, the **boundary** of $f$ — denoted as $\text{bound}(f)$ — is the plane subgraph of $G$ that forms $\text{front}(f)$.

---

**Lemma 6.5**

If $G$ is a plane graph that has two distinct faces $f_1$ and $f_2$ such that $\text{bound}(f_1) = \text{bound}(f_2)$, then $G$ is a cycle and $G = \text{bound}(f_1) = \text{bound}(f_2)$.

---

*Proof.* Let $H = \text{bound}(f_1) = \text{bound}(f_2) \subseteq G$; in particular, since $H \subseteq G$ by both statements of **??** $f_1$ and $f_2$ are also faces of $H$. In particular, this implies that $H$ has more than one face, meaning that by the contrapositive of Theorem 6.2 $H$ is not a plane forest, i.e. $H$ contains a cycle $C$. Let $F_1$ and $F_2$ be the faces of $C$; since $\text{bound}(f_1) = \text{bound}(f_2)$ it must be the case that $f_1$ and $f_2$ lie in distinct faces of $C$ — without loss of generality suppose that $f_1$ and $f_2$ lie in $F_1$ and $F_2$, respectively. Moreover, since $H$ is the boundary of both $f_1$ and $f_2$, this implies that no point of $H$ can lie inside $F_1$ or $F_2$. However, this can happen only if $H = C$, $f_1 = F_1$ and $f_2 = F_2$. Finally, by definition we have that $C \cup F_1 \cup F_2 = \mathbb{R}^2$, therefore all points of $G$ must lie on $G$, concluding that $G = C$. ☐

---

**Proposition 6.2**

Every face boundary of a 2-connected plane graph is a cycle.

---

*Proof.* Consider a 2-connected plane graph $G$; we proceed by induction on $n$.

*Base case.* If $G$ is 2-connected, then $n \geq 3$, therefore the base case is $n = 3$, for which we have that $G = K_3$ so the theorem trivially holds.

*Inductive hypothesis.* Assume the statement holds for 2-connected planar graphs on at most $n-1$ vertices.

*Inductive step.* Suppose $G$ has $n$ vertices; since $G$ is 2-connected, by Theorem 5.2 $G$ admits an ear decomposition $C, P_1, \ldots, P_k$. If $k = 0$ then $G = C$ and the theorem triviallyh holds, so we may assume that $k \neq 0$; hence, consider $H = C \cup P_1 \cup \ldots \cup P_{k-1}$. We observe that $H$ is the union of an ear decomposition, therefore by the same theorem $H$ is 2-connected. Moreover, $H$ cannot have more than $n-1$ vertices since $P_k$ is an $H$-path, therefore we can apply the inductive hypothesis and assume that each face boundary of $H$ is a cycle.

---

Now consider $P_k$; since each face boundary of $H$ is a cycle we know that there must be a face $f$ of $H$ that contains $\mathring{P}_k$ bounded by a cycle $C'$. Hence, by Lemma 6.2 this implies that $P_k$ divides $f$ into 2 faces $f'$ and $f''$ each bounded by a cycle — formed by $P_k$ and a subpath of $C'$.

$\square$

### Proposition 6.3

Let $G$ be a 3-connected plane graph, and let $C$ be a cycle subgraph of $G$; then $C$ is the boundary of a face of $G$ is and only if $C$ is induced and non-separating.

*Proof.*

*Direct implication.* Assume that a cycle $C$ of a 3-connected graph $G$ is the boundary of a face $f$ of $G$.

By way of contradiction, suppose that $C$ is not induced, which implies that there are two vertices $x, y \in V(C)$ such that $x \sim y$ in $G$. We observe that $\mathring{xy}$ must lie outside $f$, since we are assuming that $C = \text{bound}(f)$. Let $z_1, z_2 \in V(C)$ such that $x \to z_1 \to y \to z_2$ occur in this order while going around $C$. Let $P_1 := x \to z_1 \to y$, $P_2 := x \to z_2 \to y$; together with the edge $xy$, they form 3 pairwise disjoint polygonal arcs linking $x$ and $y$, therefore by Lemma 6.2 we know that they form 3 regions, and this must imply that there is no path connecting $z_1$ and $z_2$ inside $G - \{x, y\}$.

To prove that $C$ also non-separating, fix two vertices $x, y \in G - C$; since $G$ is 3-connected, by Theorem 3.2 there are 3 internally disjoint paths between $x$ and $y$ — say $P_1$, $P_2$ and $P_3$. Hence, by Lemma 6.2 we know that they induce 3 regions. Without loss of generality, suppose that $f$ lies inside the region bounded by $P_1 \cup P_2$; then $C \cap P_3 = \varnothing$ since $f$ is completely inside the region formed by the other two paths, meaning that $x$ and $y$ are connected through $P_3$ in $G - C$. Therefore, we conclude that $G - C$ is still connected, i.e. $C$ is non-separating.

*Converse implication.* Assume that $C$ is induced and non-separating, and let $f_1$ be the outer face and $f_2$ be the inner face of $C$ — considered as a subgraph of $G$. If both $f_1$ and $f_2$ contained a vertex of $G - C$, by removing the cycle we would disconnect such vertices (since $C$ is induced), contradicting the fact that $C$ is non-separating. This implies that at least one between $f_1$ and $f_2$ does not contain vertices of $G - C$; without loss of generality, suppose that $f_2$ does not contain vertices of $G - C$.

By way of contradiction, suppose that $C$ is not the boundary of $f_2$; this implies that there is a portion of $G - C$ that lies inside $f_2$. We observe that $f_2$ does not contain vertices of $G - C$, therefore the only way for a portion of the latter to lie in $f_2$ is through an edge of $G - E(C)$ — linking two vertices of $C$. However, the existence of such an edge would contradict the fact that $C$ is induced $\notrightarrow$.

$\square$

> **Theorem 6.3: Euler's formula**
>
> If $G$ is a connected plane graph on $\ell$ faces, then
>
> $$n - m + \ell = 2$$

*Proof.* We proceed by induction on $m$.

*Base case.* Since $G$ has to be connected, for the base case we have $m = n - 1$; in particular, this implies that $G$ is a tree, and by Theorem 6.2 we know that $\ell = 1$, and in fact

$$n - m + \ell = n - (n - 1) + 1 = 2$$

*Inductive hypothesis.* Assume that Euler's formula holds for connected plane graphs having $m - 1 \geq n$ edges.

*Inductive step.* Let $G$ be a connected plane graph on $m \geq n$ edges; in particular, this implies that $G$ has a cycle $C$. Fix an edge $e \in E(C)$; then, since $C$ is a cycle we know that $G - e$ is still connected, and since it has $m - 1 \geq n$ edges we can apply the inductive hypothesis, meaning that $n - (m - 1) + \ell' = 2$ — where $\ell'$ is the number of faces of $G - e$. Moreover, since $e$ lies in a cycle of $G$, by Lemma 6.4 we know that $e$ is in the frontier of exactly 2 faces of $G$. Let these two faces be $f_1$ and $f_2$ in $G$. This implies that $f_1 \cup f_2 \cup \mathring{e}$ is a face of $G - e$, and all the other faces are unchanged in $G - e$, meaning that if $g \neq f_1, f_2$ is a face of $G$, then $g$ is also a face of $G - e$. This impleis that $\ell' = \ell - 1$, therefore

$$n - (m - 1) + \ell' = n - m + 1 + \ell - 1 = 2 \iff n - m + \ell = 2$$

$\square$

## 6.2  Planarity conditions

> **Definition 6.12: Maximally plane graph**
>
> A plane graph $G$ is said to be **maximally plane** if no edge $e \notin E(G)$ can be added to $G$ such that $G \cup e$ is still plane.

> **Definition 6.13: Triangulation**
>
> A **triangulation** is a plane graph in which every face is a triangle.

We observe that even the outer face is bounded by a triangle.

Unsurprisingly, every maximally plane graph *must* be a triangulation, obtained by simply partition each face into a finite number of triangles.

**Proposition 6.4**

A plane graph $G$ such that $n \geq 3$ is maximally plane if and only $G$ is a triangulation.

*Proof.*

*Direct implication.* Let $G$ be maximally plane graph, and let $f$ be a face of $G$. We observe that $n \geq 3$ meaning that $G \neq K_2$; moreover, $\text{bound}(f)$ must be a complete graph, otherwise an edge could be added to $G$ violating the maximally plane condition of $G$. This implies that $\text{bound}(f) = K_t$. If $t = 3$ for each face $f$ the theorem trivially holds, so we may assume that there is a face such that $t \geq 4$ — and since $K_t \subseteq K_{t+1}$, it suffices to consider the case for $\text{bound}(f) = K_4$.

We observe that $K_4$ is 3-connected, therefore for any pair $x, y \in \text{bound}(f)$ by Theorem 3.2 there are 3 internally disjoint paths $P_1$, $P_2$ and $P_3$ connecting $x$ to $y$. Hence, by Lemma 6.2 $f$ lies inside one of the 3 regions induced $P_1 \cup P_2 \cup P_3$, implying that one of the three paths is not on the boundary of $f$, contradicting the definition of boundary.

*Converse implication.* Trivially, if every face is a triangle, there is no pair $x, y \in V(G)$ with $x \nsim y$ that lie common on face such that they can be connected, implying that $G$ is maximally plane by definition.

$\square$

**Theorem 6.4**

If $G$ is a plane graph such that $n \geq 3$, then $m \leq 3n - 6$. Moreover, if $G$ is a triangulation then $m = 3n - 6$.

*Proof.* By the previous proposition, if $G$ is maximally plane $G$ is a triangulation; moreover, every plane graph is the subgraph of a maximally plane graph defined on the same vertices. Therefore, it suffices to show that every triangulation has $3n - 6$ edges.

Let $G$ be a triangulation, and let $\ell$ be the number of faces of $G$; we observe that

- every face of $G$ sees 3 edges, by the Handshaking lemma for the faces we have that $2m \geq 3\ell$

- every face is a triangle, hence every edge of $G$ lies in 2 faces by Lemma 6.4.

This implies that $3\ell = 2m \iff \ell = \frac{2}{3}m$, and by Euler's formula we get that

$$n - m + \ell = n - m + \frac{2}{3}m = 2 \iff n - 2 = \frac{1}{3}m \iff m = 3n - 6$$

$\square$

In particular, the contrapositive of this theorem is enough to **certify** that a given graph

is *not* planar. For instance, consider $K_5$; then

$$|E(K_5)| = \binom{5}{2} = \frac{5 \cdot 4}{2} = 10 > 9 = 3 \cdot 5 - 6 = 3\,|V(K_5)| - 6$$

Is this theorem enough to prove that our original graph $K_{3,3}$ is not planar? Unfortunately, we have that

$$|E(K_{3,3})| = 3 \cdot 3 = 9 \leq 12 = 3 \cdot 6 - 6 = 3 \cdot |V(K_{3,3})| - 6$$

which is not enough to prove that $K_{3,3}$ is not planar. However, consider the following proposition.

---

**Proposition 6.5**

Given a 2-connected bipartite plane graph $G$, it holds that $m \leq 2n - 4$.

---

*Proof.* Let $\ell$ be the number of faces of $G$; we observe that

- $G$ is 2-connected and planar, hence by Theorem 5.2 is has an ear decomposition, therefore every edge must lie inside a cycles

- $G$ is bipartite, hence by Theorem 1.4 $G$ does not admit odd-length cycles, and in particular there is no $C_3$ inside $G$

Therefore, we get that every edge of $G$ must lie inside a cycle with at least 4 edges, therefore the sum on the number of edges of each face of $G$ is at least $4\ell$. Hence, by the Handshaking lemma on the faces we get that $2m \geq 4\ell \iff \ell \leq \frac{1}{2}m$. Lastly, by Euler's formula we conclude that

$$0 = n - m + \ell - 2 \leq n - m + \frac{1}{2}m - 2 \iff 1 \leq n - \frac{1}{2}m \iff m \leq 2m - 4$$

$\square$

Finally, the contrapositive of this last property is enough to conclude that $K_{3,3}$ is *not* planar, because

$$|E(K_{3,3})| = 3 \cdot 3 = 9 > 8 = 2 \cdot 6 - 4 = 2 \cdot |V(K_{3,3})| - 4$$

---

**Corollary 6.1**

$K_5$ and $K_{3,3}$ are not planar.

---

## 6.2.1 Kuratowski's theorem

Trivially, we observe that if $G$ has a subgraph $H \subseteq G$ such that $H$ is not planar, then $G$ will cannot be planar either. Moreover, now that we know that nor $K_5$ nor $K_{3,3}$, a trivial

conclusion is that any subdivision of these two graphs will not be planar. Therefore, from these two observation we get the following proposition.

**Lemma 6.6**

If $G$ contains $K_5$ or $K_{3,3}$ as topological minor, $G$ is not planar.

In a landmark paper published in 1930, Kuratowski [Kur30] proved that the converse is also true, thus providing a *complete* characterization to any planar graph. But before presenting the theorem, first we need to discuss some properties of minors and topological minors.

**Proposition 6.6**

A graph $G$ has $H$ as minor if and only if there exists a family of sets $X_v \subseteq V(G)$ for each $v \in V(H)$ such that

- $\forall u, v \in V(H) \quad u \neq v \implies X_u \cap X_v = \varnothing$

- for each $u \in V(H)$ it holds that $G[X_u]$ is connected

- for each $uv \in E(H)$ there is an edge $e_{uv} \in E(G)$ with one endpoint in $X_u$ and the other in $X_v$

*Proof.*

*Direct implication.* If $G$ has $H$ as minor, there is a sequence of contractions and vertex or edge deletions that yields $H$ starting from $G$. Let $H'$ be the following subgraph of $G$

- $V(H')$ contains the vertices of $G$ that were not removed to obtain $H$ from $G$, together with the endpoints of the edges that were contracted to obtain $H$ from $G$

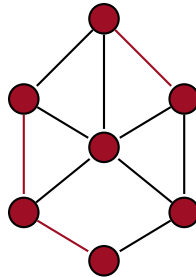- $E(G)$ contains the edges of $G$ that were contracted to obtain $H$ from $G$



Figure 6.2: For instance, this graph has $K_4$ as minor, and we are considering as $H'$ the subgraph colored in *red*.

We observe that, by construction, the number of connected components of $H'$ is

precisely $|V(H)|$; therefore, we can well-define $\{X_v \mid v \in V(H)\}$ where $X_v$ is a component of $H'$. Hence, given such family of subsets we observe that

- for any pair of distinct $u$ and $v$ in $V(H)$ it holds that $X_u \cap X_v = \varnothing$ by construction of $H'$

- the edges of $H$ obtained from $G$ are edges $e_{uv}$ connecting the connected components of $H'$ which correspond to $X_u$ and $X_v$, by construction of $H'$

*Converse implication.* For each $v \in V(H)$, fix a spanning tree $T_v$ on $G[X_v]$, and for each $uv \in E(H)$ fix an edge $e_{uv} \in E(G)$; now, to obtain $H$ we can simply

- remove all the edges *not* in $\bigcup\limits_{v \in V(H)} E(T_v) \cup \bigcup\limits_{uv \in E(H)} e_{uv}$ — i.e. the edges in each $G[X_v]$ outside the spanning tree $T_v$ for each $v \in V(H)$, and the edges that connect the various sets that are not the edges $e_{uv}$ that we fixed

- contract all the edges of $T_v$, for each $v \in V(H)$ — we observe that this contraction operation will leave one single vertex by connectivity of each tree $T_v$

$\square$

In Section 4.1.3 we stated that if $G$ has $H$ as topological minor, then $H$ is a minor of $G$. Moreover, we also provided an example to show that the converse does not always hold — i.e. it is not true in general that if $G$ has $H$ as minor, then $H$ is topological minor of $G$ as well. However, by adding an additional assumption on $G$ we can get the converse implication, as stated below.

> **Proposition 6.7**
>
> If $G$ contains $H$ as minor, and $\Delta(H) = 3$, then $G$ contains $H$ as topological minor.

*Proof.* Suppose that $G$ contains $H$ as minor, which implies that there exists a collection $\{X_v \mid v \in V(H)\}$ that satisfies Proposition 6.6. Now, for each pair of distinct $u$ and $v$ of $V(H)$ fix an edge $e_{uv} \in E(G)$; since $\Delta(H) = 3$, for any set $X_z$ there can be at most 3 edges of the form $e_{zv_i} \in E(G)$ for $1 \le i \le 3$. Now, if there are at most such 2 edges for all $X_z$, then $H$ is a topological minor of $G$, so we may assume there is at least one set $X_{\hat{z}}$ which has 3 such edges.

Fix a spanning tree $T_{\hat{z}}$ of $G[X_{\hat{z}}]$; since there are three edges $e_{\hat{z}v_1}, e_{\hat{z}v_2}, e_{\hat{z}v_3}$ and $T_{\hat{z}}$ is connected, there must be a vertex $r$ of degree 3 inside $T_{\hat{z}}$ such that there are 3 paths that start from $r$ and end in the endpoints of such edges that lie insize $X_{\hat{z}}$. Therefore, by fixing a spanning tree $T_u$ for each $G[X_u]$, we get that that $\bigcup\limits_{u \in V(H)} T_u \cup \bigcup\limits_{uv \in E(H)} e_{uv}$ is a subdivision of $H$ inside $G$, i.e. $H$ is a topological minor of $G$. $\square$

We can leverage this property to prove the following characterization about the special cases of $K_5$ and $K_{3,3}$.

## Lemma 6.7

A graph $G$ has $K_5$ or $K_{3,3}$ as topological minor if and only if it has $K_5$ or $K_{3,3}$ as minor.

*Proof.* Since being a topological minor implies being a minor, the direct implication of the statement is trivial. Moreover, since $\Delta(K_{3,3}) = 3$, if $G$ contains $K_{3,3}$ as minor, then it also contains $K_{3,3}$ as topological minor by the previous proposition. Therefore, to prove the statement it suffices to show that if $K_5$ is a minor of $G$, then $K_5$ or $K_{3,3}$ are topological minors of $G$.

Thus, suppose that $G$ contains $K_5$ as minor, i.e. there are sets $\{X_v \mid v \in V(K_5)\}$ that satisfy Proposition 6.6. TODO _____ $\square$

<span style="background:orange">da finire</span>

Finally, we are ready to discuss Kuratowski's theorem.

## Theorem 6.5: Kuratowski's theorem

A graph $G$ is planar if and only if $G$ does not contain $K_5$ nor $K_{3,3}$ as topological minors.

*Proof.* The observation discussed in Lemma 6.6 trivially proves the direct implication of the theorem. Moreover, the previous lemma implies that the statement is equivalent of proving that $G$ is planar if and only if $G$ does not contain $K_5$ nor $K_{3,3}$ as minor. We are going to prove the converse implication of this statement only for the case in which $G$ is 3-connected — the theorem can be extended to any graph $G$.

We proceed by induction on $n$. Since $G$ is 3-connected, the base case is for $n = 4$, which implies that $G = K_4$, which is indeed planar — as shown in Figure 6.1.

Assume that if a 3-connected graph $G$ on $n - 1$ vertices does not contain $K_5$ nor $K_{3,3}$ as minors, then $G$ is not planar. Now, consider a 3-connected graph $G$ on $n \geq 5$ vertices that does not contain $K_5$ nor $K_{3,3}$ as minors. Since $G$ is 3-connected, by Theorem 5.3 there exists an edge $xy \in E(G)$ such that $G/xy$ is still 3-connected; moreover, we know that $G/xy$ cannot contain $K_5$ nor $K_{3,3}$ as minors, otherwise they would be minors of $G$ as well since $G/xy$ is obtained from $G$ by contracting an edge, which would contradict the definition of $G$. Furthermore, $G/xy$ has $n - 1$ vertices, which implies that we can apply the inductive hypothesis on it, thus $G/xy$ is planar.

Let $v_{xy} \in V(G/xy)$ be the vertex obtained by contracting the edge $xy$, and consider $G/xy - \{v_{xy}\}$: since $G/xy$ is 3-connected, such graph is at least still 2-connected — and surely still planar — hence by Proposition 6.2 we have that every face of such graph is bounded by a cycle. This implies that there must exists a plane graph isomorphic to $G/xy - v_{xy}$ that has a face $f$ bounded by a cycle $C$ such that $\mathcal{N}(v_{xy})$ lies completely on $C$. Let $\mathcal{N}(v_{xy}) = \{v_1, \ldots, v_k\}$; then, by definition of contraction, each $v_i \in \mathcal{N}(v_{xy})$ is adjacent to at least one between $x$ and $y$. Now we are going to "un-contract" the edge $xy$ and consider how $G$ could be structured

**Claim 1:** $x$ and $y$ do not have 3 common neighbors.

*Proof of the Claim.* By way of contradiction suppose there are $v_1$, $v_2$ and $v_3$ such that $v_1, v_2, v_3 \sim x, y$; then, by contracting $C$ down to a triangle — which can always be done since $C$ is a cycle — we get that $G$ contains $K_5$ as minor, contradicting the definition of $G \, \natural$. $\qquad\square$

**Claim 2:** There cannot exist vertices $s$, $s'$, $t$ and $t'$ on $C$ such that $x \sim s, s'$ and $y \sim t, t'$ and $s \; t \; s' \; t'$ occur on $C$ on that cyclic order.

*Proof of the Claim.* If such vertices exist then $G$ contains $K_{3,3}$ as minor, contradicting its definition $\natural$



$\square$

TODO _____
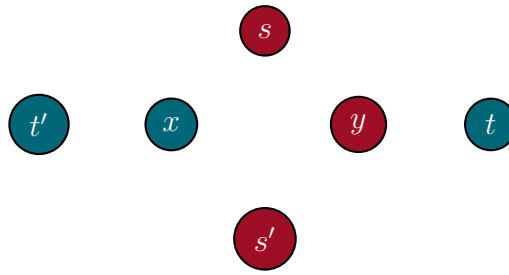
do edges of fig before

**Claim 3:** $x$ and $y$ have at least two neighbors on $C$.

*Proof of the Claim.* Since $G$ is 3-connected, then $\deg(x), \deg(y) \geq 3$ by Proposition 3.2, which implies that they must have at least two neighbors on $C$ — since all of their neighbors lie on $C$ and they are one the neighbor of the other. $\qquad\square$

**Claim 4:** If there exist two internally disjoint paths $P_1$ and $P_2$ on $C$ such that $\mathcal{N}(x) - \{y\} \subseteq V(P_1)$ and $\mathcal{N}(y) - \{x\} \subseteq P_2$, then $G$ is planar.

*Proof of the Claim.* Trivially, such paths imply that there is always a drawing of $G$ which allows to connect $x$ to $\mathcal{N}(x) - \{y\}$ and $y$ to $\mathcal{N}(y) - \{x\}$ without drawing overlapping edges — it may be necessary to "swap" the vertices $x$ and $y$ to obtain such drawing. $\qquad\square$

**Claim 5:** If $x$ or $y$ have exactly 2 neighbors on $C$, then $G$ is planar.

*Proof of the Claim.* Without loss of generality, suppose that $x$ has exactly two neighbors $v_1, v_2 \in V(C)$ — i.e. $\mathcal{N}(x) - \{y\} = \{v_1, v_2\} \subseteq V(C)$; we observe that $v_1$ and $v_2$ split $C$ into two internally disjoint paths $P_1$ and $P_2$ having $v_1$ and $v_2$ as endpoints such that $C = P_1 \cup P_2$.

Again, consider $G[C \cup \{x\}]$: since $x \sim v_1, v2$, such subplane graph has 3 faces — the outer face and two inner faces; without loss of generality, suppose that $y$ lies in the face formed by $P_2$ and $v_1 \to x \to v_2$. Now, if $\mathcal{N}(y) - \{x\} \subseteq V(P_2)$, then by Claim 4 $G$ is planar, so we may assume that there is at least one internal vertex of $P_1$ that is a neighbor of $y$. Then,

by Claim 3 we know that $y$ has at least another neighbor on $C$, and we know that such neighbor cannot be an internal vertex of $P_2$ otherwise we would contradict Claim 2. This implies that $\mathcal{N}(y) - \{x\} \subseteq P_1$, which means that $G$ is planar by Claim 4. $\qquad\square$

Therefore, by this claim we may assume that both $x$ and $y$ have at least 3 neighbors on $C$. Moreover, by Claim 1 $x$ and $y$ cannot share 3 neighbors, implying that there exists at least one vertex $v \in V(C)$ such that $x \nsim v$ and $y \sim v$ without loss of generality. Let $P_1$ be the shortest path of $C - \{v\}$ that contains all the neighbors of $x$ on $C$ — and in particular, its endpoints $v_i$ and $v_j$ are both neighbors of $x$. Moreover, since $y \sim v$, if $y$ had a neighbor $z$ internal to $P_1$ then $v$, $z$, $v_i$ and $v_j$ would contradict Claim 2.

Lastly, let $P_2$ be such that $C = P_1 \cup P_2$ and $P_1$ and $P_2$ are internally disjoint; the last observation implies that $\mathcal{N}(y) - \{x\}$ lies completely on $P_2$, therefore $G$ is planar by Claim 4.

$\qquad\square$

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