"Sapienza" University of Rome
Faculty of Information Engineering,
Informatics and Statistics
Department of Computer Science

Graph Theory

Lecture notes integrated with the book TODO

Author
Alessio Bandiera

March 6, 2025

Contents

Information and Contacts

Personal notes and summaries collected as part of the *Graph Theory* course offered by the degree in Computer Science of the University of Rome "La Sapienza".

Further information and notes can be found at the following link:
https://github.com/aflaag-notes. Anyone can feel free to report inaccuracies, improvements or requests through the Issue system provided by GitHub itself or by contacting the author privately:

- Email: **alessio.bandiera02@gmail.com**
- LinkedIn: **Alessio Bandiera**

The notes are constantly being updated, so please check if the changes have already been made in the most recent version.

Suggested prerequisites:

- Progettazione degli Algoritmi

Licence:

1

Basics of Graph Theory

placeholder

write a small introduction on graph theory

1.1 Introduction

Definition 1.1: Graph

A **graph** is a pair $G = (V, E)$, where V is the — finite — set of **vertices** of the graph, and E is the set of **edges**.

For now, will assume to be working with **simple** and **undirected** graphs, i.e. graphs in which the set of edges is defined as follows

$$E \subseteq [V]^2 = \{\{x, y\} \mid x, y \in V \land x \neq v\}$$

where the notation $\{x, y\}$ will be used to indicate an edge between two nodes $x, y \in V$, and will be replaced with $xy = yx$ directly — the *set* notation for edges is used to highlight that edges have no direction. We will indicate with n and m the cardinality of $|V|$ and $|E|$, respectively.

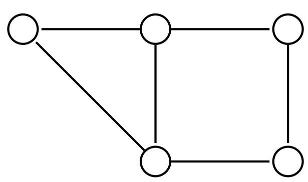


Figure 1.1: A simple graph.

Note that, in this definition, we are assuming that each edge has exactly 2 *distinct* end-points — i.e. the graphs do not admit **loops** — and there cannot exist two edges with the same endpoints. In fact, if we drop these assumption we obtain what is called a **multigraph**.

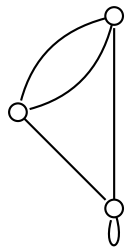


Figure 1.2: A multigraph.

Definition 1.2: Subgraph

Given a graph $G = (V, E)$, a **subgraph** $G' = (V', E')$ of G is a graph such that $V' \subseteq V$ and $E' \subseteq E$.

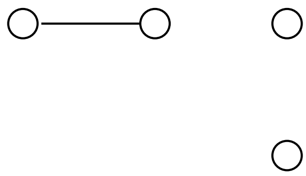


Figure 1.3: This is a subgraph of the graph shown in Figure 1.1.

Definition 1.3: Induced subgraph

Given a graph $G = (V, E)$, a subgraph $G' = (V', E')$ of G is **induced** if every edge of G with both ends in V is an edge of V'.

This definition is *stricter* than the previous one: in fact, the last graph is *not* an example of an induced subgraph, but the following is:

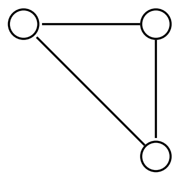
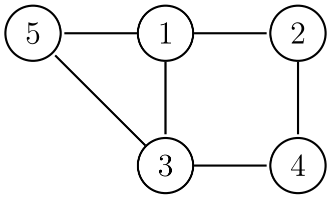


Figure 1.4: This is an *induced* subgraph of the graph shown in Figure 1.1.

Note that every induced subgraph of a graph is **unique** by definition, and we indicate each induced subgraph as follows: suppose that the graph in Figure 1.1 had the following *labeling* on the vertices



then, the induced subgraph in Figure 1.4 would have been referred to as $G[\{1, 2, 5\}]$.

Intuitively, two vertices $x, y \in V$ are said to be **adjacent**, if there is an edge $xy \in E$, and we write $x \sim y$. If there is no such edge, we write $x \nsim y$ for non-adjacency. The **neighborhood** of a vertex $x \in V$ is the set of vertices that are adjacent to x, and it will be indicated as follows

$$\mathcal{N}(x) := \{y \in V \mid x \sim y\}$$

The **degree** of a vertex $x \in V$, denoted with $\deg(x)$, is exactly $|\mathcal{N}(x)|$. We will use the following notation for the **minimum** and **maximum** degree of a graph, respectively

$$\delta := \min_{x \in V} \deg(x) \qquad \Delta := \max_{x \in V} \deg(x)$$

Lemma 1.1: Handshaking lemma

Given a graph $G = (V, E)$, it holds that

$$\sum_{x \in V} \deg(x) = 2\,|E|$$

Proof. Trivially, the sum of the degrees counts every edge in E exactly twice, once for each of the 2 endpoints. □

Definition 1.4: k-regular graph

A graph G is said to be **k-regular** if every vertex of G has degree k.

Note that in a k-regular graph it holds that

$$\sum_{x \in V} \deg(x) = k \cdot n$$

Proposition 1.1

There are no k-regular graphs with k odd and an odd number of vertices.

Proof. By way of contradiction, suppose that there exists a k-regular graph $G = (V, E)$ such that both k and n are odd; however, by the handshaking lemma we would get that

$$2\,|E| = \sum_{x \in V} \deg(x) = k \cdot n$$

but the product of two odd numbers, namely k and n, is still an odd number, while $2\,|E|$ must be even $\lightning$. $\square$

1.1.1 Important structures

Definition 1.5: Path

A **path** is a *graph* with vertex set $x_0, \ldots, x_n$ and edge set $e_1, \ldots, e_n$ such that $e_i = x_{i-1}x_i$. We say that a path P of vertices $x_0, \ldots, x_n$ **links** together x_0 and x_n, and the **length** of P is the number of edges between x_0 and x_n, i.e. $|\{e_1, \ldots, e_n\}|$, namely n in this case.

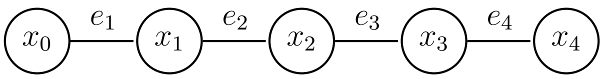


Figure 1.5: A path graph of length 4 that links x_0 and x_4.

Definition 1.6: Walk

A **walk** is a *sequence*

placeholder

da finire

Note that there is a subtle difference between the definitions of **path** and **walk**: the definition of a path implies that this is always a *graph* on its own, while a walk is defined as a *sequence*. Nonetheless, we will treat *paths* as if they where *sequences* as well. This assumption holds for the following structures that will be discussed as well.

placeholder

da finire