Marp

Markdown Presentation Ecosystem

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Syntax - Formulas

Given Φ_0 the set of *atomic formulas*, for any $\phi, \psi \in \Phi_0$

- $\phi \in \mathrm{Form}(\Phi_0)$
- $\neg\phi \in \mathrm{Form}(\Phi_0)$
- $\phi \vee \psi \in \mathrm{Form}(\Phi_0)$
- $[\alpha]\phi \in \mathrm{Form}(\Phi_0)$

where $\alpha \in \mathrm{Prog}(\Pi_0)$

Syntax - Programs

Given Π_0 the set of *atomic programs*, for any $\alpha, \beta \in \Pi_0$

- $\alpha \in \mathrm{Prog}(\Pi_0)$
- $(\alpha; \beta) \in \mathrm{Prog}(\Pi_0)$
- $(\alpha \cup \beta) \in \mathrm{Prog}(\Pi_0)$
- $\alpha^* \in \mathrm{Prog}(\Pi_0)$
- $\phi? \in \mathrm{Prog}(\Pi_0)$

where $\phi \in \mathrm{Form}(\Phi_0)$

Syntax - Relations

$$(x, y) \in R(\pi) \iff x \xrightarrow{\pi} y$$

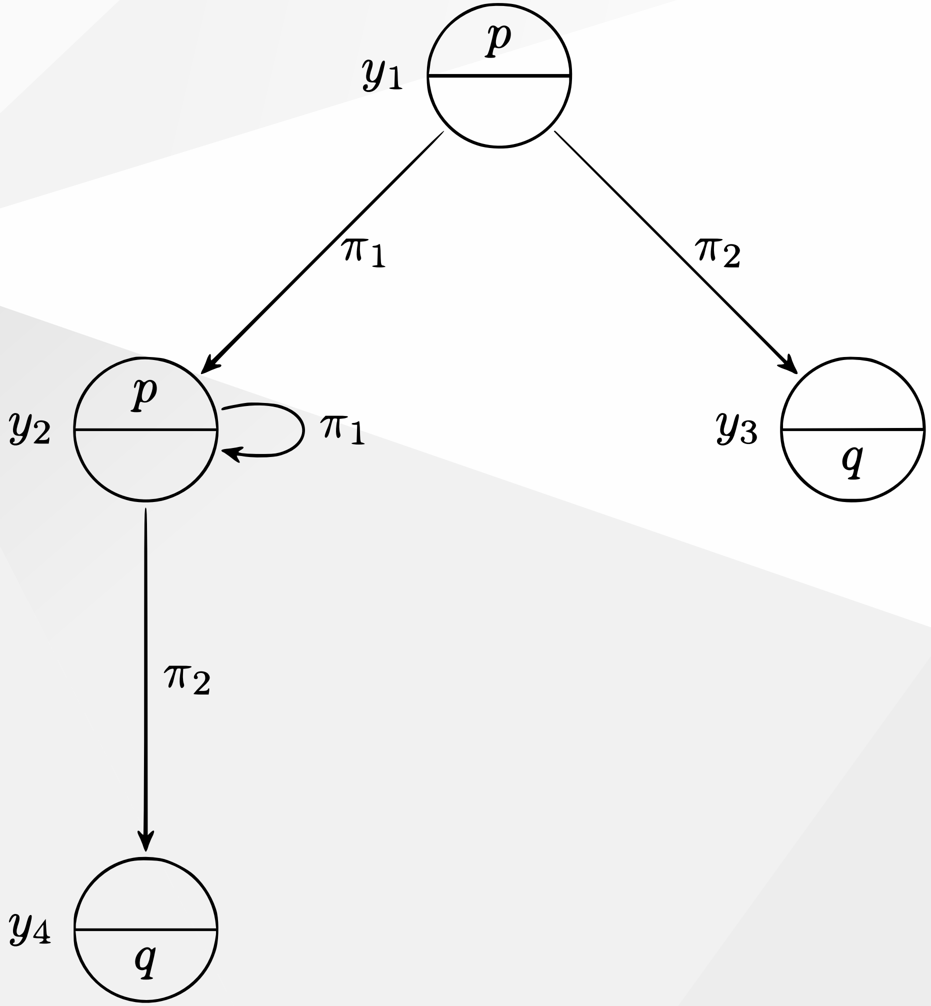
- $(x, y) \in R(\alpha; \beta) \iff \exists z \in W \quad (x, z) \in R(\alpha) \wedge (z, y) \in R(\beta)$
- $(x, y) \in R(\alpha \cup \beta) \iff (x, y) \in R(\alpha) \vee (x, y) \in R(\beta)$
- $(x, y) \in R(\alpha^*) \iff \exists z_0, \ldots, z_n \in W \quad \begin{cases} z_0 = x \\ z_n = y \\ (z_{k-1}, z_k) \in R(\alpha) \end{cases}$
- $(x, y) \in R(\phi?) \iff x = y \wedge y \in V(\phi)$

Syntax - Valuations

$$x \in V(p) \iff p \text{ is true at } x$$

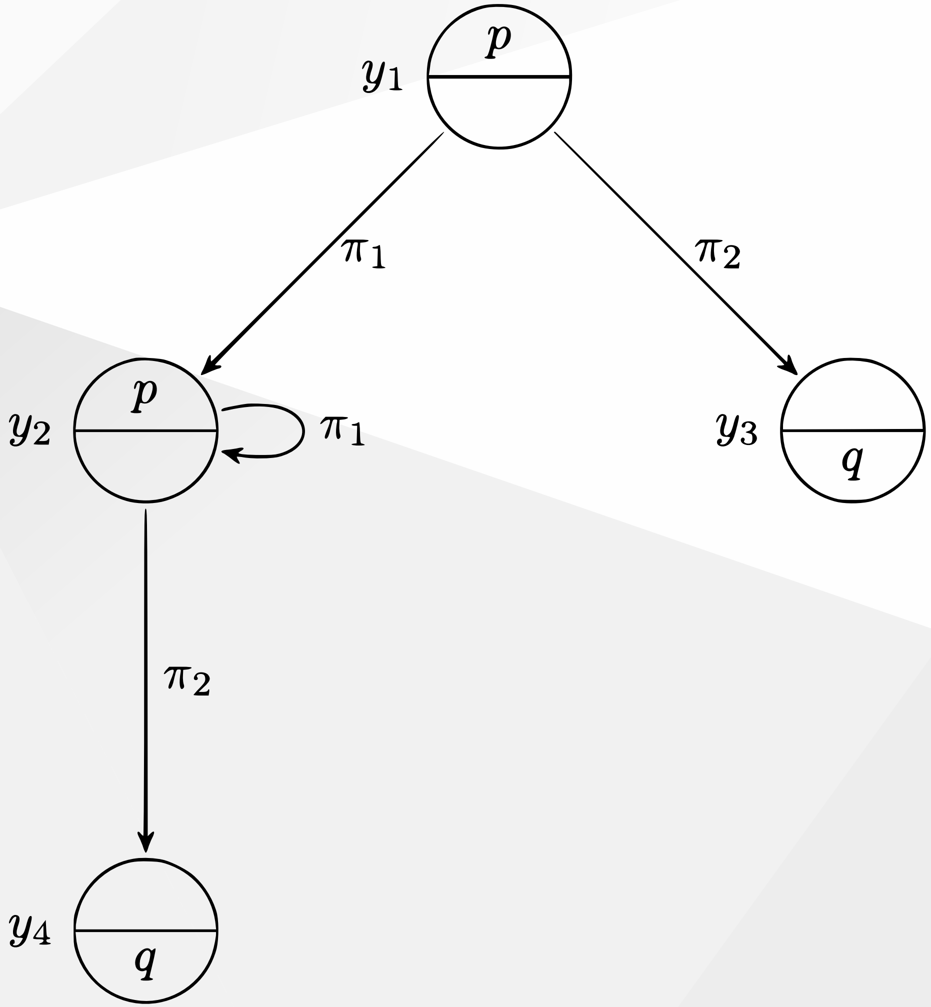
- $V(\bot) = \varnothing$
- $V(\top) = W$
- $V(\neg\phi) = W - V(\phi)$
- $V(\phi \vee \psi) = V(\phi) \cup V(\psi)$
- $V([\alpha]\phi) = \{x \mid \forall y \in W \quad (x, y) \in R(\alpha) \implies y \in V(\phi)\}$

LTS



- $W = \{y_1, y_2, y_3, y_4\}$
- $R(\pi_1) = \{(y_1, y_2), (y_2, y_2)\}$
- $R(\pi_2) = \{(y_1, y_3), (y_2, y_4)\}$
- $V(p) = \{y_1, y_2\}$
- $V(q) = \{y_3, y_4\}$

LTS



- $\mathfrak{M}, y_1 \models \langle \pi_1^*; \pi_2 \rangle q$
- $\mathfrak{M}, y_2 \models [\pi_1^*] p$
- $\mathfrak{M}, y_1 \models [\pi_1 \cup \pi_2](p \vee q)$
- $\mathfrak{M}, y_3 \models [\pi_1 \cup \pi_2] \bot$

Axiomatization - Validity

We write $\mathfrak{M}, w \models \phi$ if and only if $w \in V(\phi)$

ϕ is *valid* in $\mathfrak{M}$, written as $\mathfrak{M} \models \phi$, if and only if

$$\mathfrak{M} \models \phi \iff \forall w \in W \quad \mathfrak{M}, w \models \phi$$

ϕ is **valid**, written as $\models \phi$, if and only if

$$\models \phi \iff \forall \mathfrak{M} \quad \mathfrak{M} \models \phi$$

Axiomatization - Goal

The goal is to define a **decudibility predicate** $\vdash$ such that $\vdash$-deductions are both *sound* and *complete* in terms of validity, i.e. for any ϕ it holds that

$$\vdash \phi \iff \models \phi$$

Axiomatization - K and N axioms

$$(\mathrm{K}) \qquad [\alpha](\phi \to \psi) \to ([\alpha]\phi \to [\alpha]\psi)$$

$$(\mathrm{N}) \qquad \frac{\phi}{[\pi]\phi}$$

A modal logic is **normal** if it obeys (K) and (N)

Axiomatization - PDL axioms

PDL is the *least normal* modal logic containing every instance of

$$
\begin{array}{ll}
(\mathrm{A1}) & [\alpha;\beta]\phi \leftrightarrow [\alpha][\beta]\phi \\
(\mathrm{A2}) & [\alpha \cup \beta]\phi \leftrightarrow [\alpha]\phi \wedge [\beta]\phi \\
(\mathrm{A3}) & [\alpha^*]\phi \leftrightarrow \phi \wedge [\alpha][\alpha^*]\phi \\
(\mathrm{A4}) & [\phi?]\psi \leftrightarrow (\phi \rightarrow \psi)
\end{array}
$$

and closed under the *loop invariance* rule of inference

$$
(\mathrm{I}) \qquad \frac{\phi \rightarrow [\alpha]\phi}{\phi \rightarrow [\alpha^*]\phi}
$$

Axiomatization - $\vdash$-deducibility

A formula ϕ is $\vdash$-*deducible* from $\Sigma \subseteq \mathrm{Form}(\Phi_0)$ if there exists a sequence $\phi_0, \ldots, \phi_n$ such that $\phi_n = \phi$, and for all $i \in [n]$

- ϕ_i is an instance of an axiom schema
- ϕ_i is an instance of a formula of Σ
- ϕ_i comes from earlier formulas of the sequence by inference

Are $\vdash$-deductions sound and complete?

Completeness - Segerberg's axioms

In 1977 Segerberg proposed to replace

$$\text{(I)} \qquad \frac{\phi \to [\alpha]\phi}{\phi \to [\alpha^*]\phi}$$

with the following fifth axiom

$$\text{(A5)} \qquad \phi \wedge [\alpha^*](\phi \to [\alpha]\phi) \to [\alpha^*]\phi$$

in order to prove that such axiomatization was sound and complete

Completeness - Segerberg's axioms

Indeed, it is easy to prove that (I) can be replaced with $(\mathrm{A5})$

1. $\vdash \phi \to [\alpha]\phi$	(premise)
2. $\vdash [\alpha^*](\phi \to [\alpha]\phi)$	(from 1 using (N) with $\pi = \alpha^*$)
3. $\vdash \phi \wedge [\alpha^*](\phi \to [\alpha]\phi) \to [\alpha^*]\phi$	(A5)
4. $\vdash [\alpha^*](\phi \to [\alpha]\phi) \to (\phi \to [\alpha^*]\phi)$	(from 3 through prop. reasoning)
5. $\vdash \phi \to [\alpha^*]\phi$	(from 2 and 4 using *Modus Ponens*)

Completeness

To prove that $\vdash$ is sound w.r.t. $\models$, i.e. that

$$\vdash \phi \implies \models \phi$$

a proof by induction on the length of ϕ's deduction in $\vdash$ suffices.

So, what about completeness? It requires to prove that

$$\models \phi \implies \vdash \phi$$

Completeness

Segerberg's work was the first attempt to prove the completeness of $\vdash$, however in 1978 he found a flaw in his argument.

Then in the same year Parikh published what is now considered the first proof of the completeness of $\vdash$.

Completeness - Goldblatt

Since then, different alternative proof theories of PDL have also been sought after. For example, in 1992 Goldblatt proposed the $\vdash'$-deducibility predicate, which is based on the same first four axiom schemas along with this *infinitary* rule of inference

$$(\mathrm{I}') \qquad \frac{\{[\beta][\alpha^n]\phi \mid n \in \mathbb{N}\}}{[\beta][\alpha^*]\phi}$$

Goldblatt was able to prove that $\vdash'$ is both sound and complete.

Complexity - PDL satisfiability

ϕ is *satisfiable* in $\mathfrak{M}$ if there is a world $w \in W$ such that $\mathfrak{M}, w \models \phi$

ϕ is **satisfiable** if there is a model $\mathfrak{M}$ such that ϕ is satisfiable in $\mathfrak{M}$

$$\text{PDL-SAT} := \{\langle\phi\rangle \mid \phi \text{ is a satisfiable PDL formula}\}$$

Complexity - Unsatisfiable formulas

ϕ is *unsatisfiable* if and only if $\neg\phi$ is *valid*

Therefore, we can use the recursive definition of valid PDL formulas and build a procedure P that enumerates all the $\vdash$-deducible formulas

Hence, given enough time if $\neg\phi$ is $\vdash$-deducible P will eventually find it and determine that ϕ is *unsatisfiable*

This proves that $\mathrm{PDL\text{-}SAT} \in$ coREC

Complexity - Satisfiable formulas

However, if ϕ is satisfiable P never terminates.

Nonetheless, we can leverage the **finite model property** of PDL

$$\forall \phi \in \mathrm{Form}(\Phi_0) \quad \langle \phi \rangle \in \mathrm{PDL\text{-}SAT} \implies \exists \mathfrak{M}_{fin} \text{ finite} \quad \phi \text{ satisfiable in } \mathfrak{M}_{fin}$$

Therefore, there is a procedure P' that enumerates all the finite models $\mathfrak{M}_{fin}$ and checks for each model if ϕ is satisfiable in $\mathfrak{M}_{fin}$.

Thus, P and P' can be run in parallel to decide $\mathrm{PDL\text{-}SAT}$. However, this is *very* inefficient, can we do any better?

Complexity

Kozen and Parikh proved that PDL has also the **small model property**

$$\forall \phi \in \mathrm{Form}(\Phi_0) \quad \langle \phi \rangle \in \mathrm{PDL\text{-}SAT} \implies \exists \mathfrak{M}_{fin} \text{ finite } \quad \begin{cases} |\mathfrak{M}_{fin}| < \exp(|\phi|) \\ \phi \text{ satisfiable in } \mathfrak{M}_{fin} \end{cases}$$

This property implies that we can stop P' as soon as all the "small" models have been exhausted, to conclude that ϕ is not satisfiable.

This concludes that $\mathrm{PDL\text{-}SAT} \in$ NEXP. In 1980 Pratt was able to prove that $\mathrm{PDL\text{-}SAT} \in$ EXP-$\mathrm{complete}$.

Variants

Over the years multiple versions of PDL have been studied

We will discuss the following

- Test-free PDL
- CPDL
- IPDL

Variants - Test-free PDL

The "$?$" operator seems *different* with respect to the other programs, can we remove this operator from PDL?

Let PDL_0 be the test-free version of PDL. In 1981 Berman and Paterson proved that this PDL formula

$$\langle(\phi?;\pi)^*;\neg\phi?;\pi;\phi?\rangle\top$$

has no PDL_0 equivalent formula. This formula can be rewritten as

$$\langle\text{while }\phi\text{ do }\pi\rangle\langle\pi\rangle\phi$$

Variants - CPDL

CPDL is a variant which adds the **converse** operator to PDL programs

$$(x, y) \in R(\alpha^{-1}) \iff (y, x) \in R(\alpha)$$

To get a sound a complete system, two additional axioms are needed

$$\begin{aligned} &(\mathrm{A6}) \qquad \phi \to [\alpha] \langle \alpha^{-1} \rangle \phi \\ &(\mathrm{A7}) \qquad \phi \to [\alpha^{-1}] \langle \alpha \rangle \phi \end{aligned}$$

As for PDL, CPDL has the **small model property**, hence it holds that $\mathrm{CPDL\text{-}SAT} \in$ EXP-complete

Variants - CPDL

What about the expressive power? Consider these two models:

$\mathfrak{M} = (M, R, V)$	$\mathfrak{M}' = (W', R', V')$
$W = \{x, y\}$	$W' = \{y'\}$
$R(\pi) = \{(x, y)\}$	$R'(\pi) = \varnothing$
$V(x) = V(y) = \varnothing$	$V'(y') = \varnothing$

Variants - CPDL

$\mathfrak{M} = (M, R, V)$	$\mathfrak{M}' = (W', R', V')$
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From the perspective of PDL y and y' are *indistinguishable*, in fact

$$\mathfrak{M}, y \models \phi \iff \mathfrak{M}', y' \models \phi$$

Variants - CPDL

$\mathfrak{M} = (M, R, V)$	$\mathfrak{M}' = (W', R', V')$
$W = \{x, y\}$	$W' = \{y'\}$
$R(\pi) = \{(x, y)\}$	$R'(\pi) = \varnothing$
$V(x) = V(y) = \varnothing$	$V'(y') = \varnothing$

However CPDL *can* distinguish y and y' because

$$\mathfrak{M}, y \models \langle \pi^{-1} \rangle \top \qquad \mathfrak{M}', y' \not\models \langle \pi^{-1} \rangle \top$$

meaning that CPDL has **more expressive power** than PDL