SAPIENZA
UNIVERSITÀ DI ROMA

"Sapienza" University of Rome
Faculty of Information Engineering,
Informatics and Statistics
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Mathematical Logic for Computer Science

Lecture notes integrated with the book TODO

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Contents

Information and Contacts

Personal notes and summaries collected as part of the *Mathematical Logic for Computer Science* course offered by the degree in Computer Science of the University of Rome "La Sapienza".

Further information and notes can be found at the following link:
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Suggested prerequisites:

TODO

Licence:

1
Homeworks

1.1 Homework 1

Exercise 1.4 Let $\mathcal{L} = \{E(x, y)\}$ be the language of graphs.

1. For each fixed $n \in \mathbb{N}$, write a sentence C_n such that for any graph $\mathcal{G}$, $\mathcal{G} \models C_n$ if and only if $\mathcal{G}$ contains a cycle of length n.
2. Prove using Compactness that the property of being *a cycle* is not expressible by a theory in $\mathcal{L}$ over the class of graphs.

Solution. Let $\mathcal{L} = \{E(x, y)\}$ be the language of graphs.

1. The property "$\mathcal{G}$ contains a cycle of length n" can be written as follows

$$C_n := \exists x_1 \dots \exists x_n \quad \left(\bigwedge_{\substack{1 \le i,j \le n \\ i \ne j}} \neg(x_i = x_j) \right) \wedge \left(\bigwedge_{1 \le i \le n-1} E(x_i, x_{i+1}) \wedge E(x_n, x_1) \right)$$

In fact, the first conjunction implies that $x_1, \dots, x_n$ are *distinct*, and the second conjunction describes the existence of the n-long *cycle* itself.

2. Consider the property $P_n :=$ "$\mathcal{G}$ is *a cycle* of length n". This property can be expressed by *extending* C_n as follows:

$$\begin{aligned} V_n &:= \forall y \bigvee_{1 \le j \le n} (y = x_j) \\ E_n &:= \bigwedge_{1 \le i \le n-1} \bigwedge_{\substack{1 \le j \le n: \\ j \ne i+1}} \neg E(x_i, x_j) \wedge \bigwedge_{2 \le j \le n} \neg E(x_n, x_j) \\ C_n' &:= \exists x_1 \dots \exists x_n \quad C_n \wedge V_n \wedge E_n \end{aligned}$$

where we have that

- V_n ensures that $\mathcal{G}$ has *exactly* n vertices
- E_n ensures that the only edges present in $\mathcal{G}$ are the ones that describe the cycle graph of n vertices
- C_n' describes our property P_n

Now, consider the property $P :=$ "$\mathcal{G}$ is *a cycle*", and in particular $\neg P :=$ "$\mathcal{G}$ is not *a cycle*". We observe that we can build the following infinte theory

$$T^{\neg P} := \{\neg C_n' \mid n \in \mathbb{N}_{\ge 3}\}$$

for which it is easy to see that

$$\mathcal{G} \models \neg P \iff \neg P(\mathcal{G}) \text{ holds}$$

meaning that $\neg P$ is expressible through $T^{\neg P}$.

Claim: $T^{\neg P} \in \mathsf{FINSAT}$.

Proof of the Claim. Fix $T_0 \underset{fin}{\subseteq} T^{\neg P}$. We observe that $T_0 := \{\neg C_{i_1}', \dots, \neg C_{i_k}'\}$

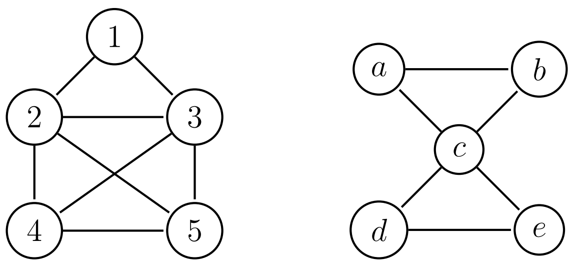
for some $i_1, \ldots, i_k \in \mathbb{N}$. Now, if we consider $i^* := \max_{j \in [k]} i_j$, then the cycle graph that has $i^* + 1$ vertices is clearly a structure that satisfies T_0. $\square$

Claim: P is not expressible by a theory in $\mathcal{L}$ over the class of graphs.

Proof of the Claim. By way of contradiction, suppose that P is expressible, i.e. there is a theory T^P for which P can be expressed. Then, consider the theory $T := T^P \cup T^{\neg P}$. By the previous claim, we have that $T \in \mathsf{FINSAT}$, and by Compactness this is true if and only if $T \in \mathsf{SAT}$. However, this is a contradiction, because a graph cannot be and not be a cycle at the same time. $\square$

Finally, this last claim concludes the proof. $\square$

Exercise 2.1 Consider the following two structures $\mathcal{G}_1$ and $\mathcal{G}_2$ for the languages of graphs:



Write at least two sentences distinguishing the two structures. Discuss the EF-game played on these structures: for what k can the Duplicator win the k-rounds game? For what k can the Spoiler win?

Solution. Some properties that can distinguish these two structures are the following:

1. "$\mathcal{G}$ contains a vertex of degree 3", which is represented by the following sentence of rank 5

$$\exists x_1 \exists x_2 \exists x_3 \exists x_4 \quad \left(\bigwedge_{\substack{1 \le i,j \le 4 \\ i \ne j}} \neg(x_i = x_j) \right) \wedge \left(\bigwedge_{2 \le i \le 4} E(x_1, x_i) \right) \wedge \left(\forall y \quad \neg E(x_1, y) \vee \bigvee_{2 \le j \le 4} (y = x_j) \right)$$

2. "$\mathcal{G}$ contains edges as $\mathcal{G}_1$", which is represented by the following sentence of rank 5

$$\begin{aligned} \exists x_1 \exists x_2 \exists x_3 \exists x_4 \exists x_5 \quad & E(x_1, x_2) \wedge E(x_1, x_3) \wedge \\ & E(x_2, x_3) \wedge E(x_2, x_4) \wedge E(x_2, x_5) \wedge \\ & E(x_3, x_4) \wedge E(x_4, x_5) \wedge \\ & E(x_4, x_5) \end{aligned}$$

we observe that the edges of $\mathcal{G}_2$ are not sufficient to distinguish the two sentences, because $\mathcal{G}_2$ is a subgraph of $\mathcal{G}_1$

3. "$\mathcal{G}$ contains a cycle of length 5", which is represented by C_5 of the previous exercise, and has rank 5
4. "$\mathcal{G}$ contains a cycle of length 4", which is represented by C_4 of the previous exercise, and has rank 4
5. "$\mathcal{G}$ contains K_4 as subgraph", which is represented by the following sentence having rank 4

$$\exists x_1 \exists x_2 \exists x_3 \exists x_4 \quad \left(\bigwedge_{\substack{1 \leq i,j \leq 4 \\ i \neq j}} \neg(x_i = x_j) \right) \land \left(\bigwedge_{\substack{1 \leq i,j \leq 4 \\ i \neq j}} E(x_i, x_j) \right)$$

These sentences *may seem* to suggest that the two structures are 3-equivalent, meaning that there is no sentence of rank 3 that can distinguish $\mathcal{G}_1$ from $\mathcal{G}_2$. For now, let's focus on proving that they are *at least* 2-equivalent.

Claim: The Duplicator wins $G_2(\mathcal{G}_1, \mathcal{G}_2)$.

Proof of the Claim. Let s_i and d_i be the i-th nodes chosen by the Spoiler and the Duplicator, respectively. Then, we can define the following strategy for the Duplicator:

- if $s_1 \in \{1, 4, 5\}$, then the Duplicator chooses $d_1 \in \{a, b, d, e\}$, otherwise if $s_1 \in \{2, 3\}$ then $d_1 = c$
- similarly, if $s_1 \in \{a, b, d, e\}$, then the Duplicator chooses $d_1 \in \{1, 4, 5\}$, otherwise if $s_1 = c$ then $d_1 \in \{2, 3\}$

Then, no matter the choice of s_2, the Duplicator can always answer with a node d_2 that preserves the partial isomorphism, in fact:

- if $s_2 \sim s_1$, it is guaranteed that there is a vertex d_2 in the other structure such that $d_2 \sim d_1$ because $\delta(\mathcal{G}_1) = \delta(\mathcal{G}_2) = 2$ — and the same argument applies if $s_2 \sim d_1$ for finding a vertex $d_2 \sim s_1$
- if $s_2 \nsim s_1$, the strategy that we provided for the Duplicator guarantees that there exists at least one vertex d_2 in the other structure such that $d_2 \nsim d_1$ — and the same argument applies if $s_2 \nsim d_1$ for finding a vertex $d_2 \nsim s_1$

Thus, the Duplicator has a strategy to always win at least 2 rounds, therefore the Duplicator wins $G_2(\mathcal{G}_1, \mathcal{G}_2)$ by Ehrenfeucht's theorem. □

Now that we proved that $\mathcal{G}_1 \equiv_2 \mathcal{G}_2$, is it true that they are also 3-equivalent? Unfortunately, the following claim proves that this is indeed false.

Claim: The Spoiler wins $G_3(\mathcal{G}_1, \mathcal{G}_2)$.

Proof of the Claim. The following is a strategy that guarantees the Spoiler to win in 3 rounds:

- let $s_1 \in \{4, 5\}$

- by the previous claim, we know that the strategy for the Duplicator to win at least 2 rounds is to choose $d_1 \in \{a, b, d, e\}$, thus we may assume that $d_1 \neq c$
- now, let $s_2 = 1$
- to preserve the partial isomorphism, we observe that
  - if $d_1 \in \{a, b\}$, then $d_2 \in \{d, e\}$
  - if $d_1 \in \{d, e\}$, then $d_2 \in \{a, b\}$
- now, it suffices for the Spoiler to choose s_3 in $\mathcal{G}_2$ such that $s_3 \sim d_2$ and $s_3 \neq c$: by construction of $\mathcal{G}_2$, we see that $s_3 \nsim d_1$, but all the vertices in $\{2, 3, 5\}$ are adjacent to s_1, which would violate the partial isomorphism

□

In fact, we can actually find a property that distinguishes $\mathcal{G}_1$ from $\mathcal{G}_2$ which can be written through a sentence of rank 3: "there are two vertices x_1 and x_2 of $\mathcal{G}$ such that for each third vertex x_3 there is a K_3 as subgraph of $\mathcal{G}$ such that $V(K_3) = \{x_1, x_2, x_3\}$"

$$\exists x_1 \exists x_2 \forall x_3 \quad \left(\bigwedge_{\substack{1 \le i,j \le 3 \\ i \neq j}} \neg(x_i = x_j) \right) \land E(x_1, x_2) \land E(x_2, x_3) \land E(x_3, x_1)$$

Let x_1, x_2 and x_3 be the three choosen vertices — and we may assume that $x_1 \sim x_2$ otherwise the sentence is trivially unsatisfied. Then, we observe that

- in $\mathcal{G}_1$ if $\{x_1, x_2\} = \{2, 3\}$, then for any other vertex $x_3 \in \{1, 4, 5\}$ we can always find a K_3 having x_1, x_2 and x_3 as its vertices
- in $\mathcal{G}_2$ we have two cases
  - if $\{x_1, x_2\} \subseteq \{a, b, c\}$, the property is unsatisfied for $x_3 \in \{d, e\}$
  - if $\{x_1, x_2\} \subseteq \{c, d, e\}$, the property is unsatisfied for $x_3 \in \{a, b\}$

In conclusion, we have that $\mathcal{G}_1 \equiv_2 \mathcal{G}_2$, and that $\mathcal{G}_1 \not\equiv_3 \mathcal{G}_2$. □

1.2 Homework 2

Exercise 1.1 Let (W, R) be a *quasi-order*; that is, assume that R is transitive and reflexive. Define the binary relation $\sim$ on W by putting $s \sim t \iff R(s,t) \wedge R(t,s)$.

(a) Show that $\sim$ is an equivalence relation.

Let $[s]$ denote the equivalence class of s under this relation, and define the following relation on the collection of equivalence classes: $[s] \leq [t] \iff R(s,t)$.

(b) Show that this relation is well-defined.

(c) Show that $\leq$ is a partial order.

Solution. We prove the statements as follows.

(a) To prove that $\sim$ is an equivalence relation, it suffices to show that $\sim$ has the following properties:

- *reflexivity*: $\forall s \in W \quad R(s,s)$ by reflexivity or R, therefore $s \sim s$
- *symmetry*: $\forall s,t \in W \quad s \sim t \iff R(s,t) \wedge R(t,s) \iff t \sim s$
- *transitivity*: $\forall s,t,u \in W \quad \begin{cases} s \sim t \iff R(s,t) \wedge R(t,s) \\ t \sim u \iff R(t,u) \wedge R(u,t) \end{cases}$ and by transitivity of R we have that
  - $R(s,t) \wedge R(t,u) \implies R(s,u)$
  - $R(u,t) \wedge R(t,s) \implies R(u,s)$

  and by definition $R(s,u) \wedge R(u,s) \iff s \sim u$

(b) To prove that $\leq$ is well-defined, we need to show that

$$\forall s,t,s',t' \quad s \sim s' \wedge t \sim t' \implies ([s] \leq [t] \iff [s'] \leq [t'])$$

We observe that

- $s \sim s' \iff R(s,s') \wedge R(s',s)$
- $t \sim t' \iff R(t,t') \wedge R(t',t)$

therefore, we have that

- $[s] \leq [t] \iff R(s,t)$, and by transitivity of R it holds that $R(s',s) \wedge R(s,t) \implies R(s',t)$; therefore, by transitivity of R again we have that $R(s',t) \wedge R(t,t') \implies R(s',t') \iff [s'] \leq [t']$
- $[s'] \leq [t'] \iff R(s',t')$, and by transitivity of R it holds that $R(s',t') \wedge R(t',t) \implies R(s',t)$; therefore, by transitivity of R again we have that $R(s,s') \wedge R(s',t) \implies R(s,t) \iff [s] \leq [t]$

(c) To prove that $\leq$ is a partial order, it suffices to show that $\leq$ has the following properties:

- *reflexivity*: $\forall s \in W \quad R(s,s)$ by reflexivity of R, and $R(s,s) \iff [s] \leq [s]$

- *antisymmetry*: $\forall s,t \in W$ $\begin{cases} [s] \leq [t] \iff R(s,t) \\ [t] \leq [s] \iff R(t,s) \end{cases} \implies R(s,t) \wedge R(t,s) \iff s \sim t \iff [s] = [t]$
- *transitivity*: $\forall s,t,u \in W$ $\begin{cases} [s] \leq [t] \iff R(s,t) \\ [t] \leq [u] \iff R(t,u) \end{cases} \implies R(s,t) \wedge R(t,u) \implies R(s,u)$ by transitivity of R, and $R(s,u) \iff [s] \leq [u]$

$\square$

Exercise 2.2 Let $\mathcal{N} = (\mathbb{N}, S_1, S_2)$ and $\mathcal{B} = (\mathbb{B}, R_1, R_2)$ be the following frames for a modal similarity type with two diamonds $\Diamond_1$, $\Diamond_2$. Here, $\mathbb{N}$ is the set of natural numbers and $\mathbb{B}$ is the set of strings of 0's and 1's, and the relations are defined by

$$\begin{aligned} S_1(m,n) &\iff n = m+1 \\ S_2(m,n) &\iff m > n \\ R_1(s,t) &\iff t = s0 \vee t = s1 \\ R_2(s,t) &\iff t \sqsubset s \end{aligned}$$

where $t \sqsubset s$ if and only if t is a *proper prefix* of s — i.e. t is a prefix of s such that $t \neq s$ (thus t can be ε). Which of the following formulas are valid on $\mathcal{N}$ and $\mathcal{B}$, respectively?

(a) $(\Diamond_1 p \wedge \Diamond_2 q) \to \Diamond_1(p \wedge q)$

(b) $(\Diamond_2 p \wedge \Diamond_2 q) \to \Diamond_2(p \wedge q)$

(c) $(\Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r) \to (\Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r))$

(d) $p \to \Diamond_1 \Box_1 p$

(e) $p \to \Diamond_2 \Box_1 p$

(f) $p \to \Box_1 \Diamond_2 p$

(g) $p \to \Box_2 \Diamond_1 p$

Solution. First, consider the following extension to the $\wedge$ operator on the inductive definition of satisfiability of formulas.

Claim: Given a model $\mathfrak{M} = (W, R, V)$, and a state $w \in W$, it holds that $\mathfrak{M}, w \models \phi \wedge \psi \iff \mathfrak{M}, w \models \phi \wedge \mathfrak{M}, w \models \psi$.

Proof of the Claim. By using De Morgan's law, we have that

$$\begin{aligned} \mathfrak{M}, w \models \phi \wedge \psi = \neg(\neg\phi \vee \neg\psi) &\iff \neg\mathfrak{M}, w \models \neg\phi \vee \neg\psi \\ &\iff \neg(\mathfrak{M}, w \models \neg\phi \vee \mathfrak{M}, w \models \neg\psi) \\ &\iff \neg(\neg\mathfrak{M}, w \models \phi \vee \neg\mathfrak{M}, w \models \psi) \\ &\iff \mathfrak{M}, w \models \phi \wedge \mathfrak{M}, w \models \psi \end{aligned}$$

$\square$

For all the following propositions, we will assume that $\mathfrak{M} = (\mathbb{N}, \mathcal{N}, V)$ and $\mathfrak{M}' = (\mathbb{B}, \mathcal{B}, V)$ are two models.

(a) $(\Diamond_1 p \wedge \Diamond_1 q) \to \Diamond_1(p \wedge q)$

- By the claim, for any $m \in \mathbb{N}$ it holds that

$$
\begin{aligned}
\mathfrak{M}, m \models \Diamond_1 p \wedge \Diamond_1 q &\iff \mathfrak{M}, m \models \Diamond_1 p \wedge \mathfrak{M}, m \models \Diamond_1 q \\
&\iff \begin{cases} \exists n_p \in \mathbb{N} & S_1(m, n_p) \wedge \mathfrak{M}, n_p \models p \\ \exists n_q \in \mathbb{N} & S_1(m, n_q) \wedge \mathfrak{M}, n_q \models q \end{cases} \\
&\iff \begin{cases} \exists n_p \in \mathbb{N} & n_p = m+1 \wedge n_p \in V(p) \\ \exists n_q \in \mathbb{N} & n_q = m+1 \wedge n_q \in V(p) \end{cases} \\
&\iff m+1 \in V(p) \wedge m+1 \in V(q) \\
&\iff m+1 \in V(p) \cap V(q)
\end{aligned}
$$

  and again, by the claim we have that

$$
\begin{aligned}
\mathfrak{M}, m \models \Diamond_1(p \wedge q) &\iff \exists n \in \mathbb{N} \quad S_1(m, n) \wedge \mathfrak{M}, n \models p \wedge q \\
&\iff \exists n \in \mathbb{N} \quad n = m+1 \wedge (\mathfrak{M}, n \models p \wedge \mathfrak{M}, n \models q) \\
&\iff \exists n \in \mathbb{N} \quad n = m+1 \wedge (n \in V(p) \wedge n \in V(q)) \\
&\iff \exists n \in \mathbb{N} \quad n = m+1 \wedge n \in V(p) \cap V(q) \\
&\iff m+1 \in V(p) \cap V(q)
\end{aligned}
$$

  from which we conclude that

$$
\mathfrak{M}, m \models \Diamond_1 p \wedge \Diamond_1 q \iff m+1 \in V(p) \cap V(q) \iff \mathfrak{M}, m \models \Diamond_1(p \wedge q)
$$

  implying that the formula is valid on $\mathcal{N}$.

- By the claim, for any $s \in \mathbb{B}$ it holds that

$$
\begin{aligned}
\mathfrak{M}', s \models \Diamond_1 p \wedge \Diamond_1 q &\iff \mathfrak{M}', s \models \Diamond_1 p \wedge \mathfrak{M}', s \models \Diamond_1 q \\
&\iff \begin{cases} \exists t_p \in \mathbb{B} & R_1(s, t_p) \wedge \mathfrak{M}', t_p \models p \\ \exists t_q \in \mathbb{B} & R_1(s, t_q) \wedge \mathfrak{M}', t_q \models q \end{cases} \\
&\iff \begin{cases} \exists t_p \in \mathbb{B} & (t_p = s0 \vee t_p = s1) \wedge t_p \in V(p) \\ \exists t_q \in \mathbb{B} & (t_q = s0 \vee t_q = s1) \wedge t_q \in V(q) \end{cases} \\
&\iff \begin{cases} s0 \in V(p) \vee s1 \in V(p) \\ s0 \in V(q) \vee s1 \in V(q) \end{cases} \\
&\iff \{s0, s1\} \cap V(p) \neq \varnothing \wedge \{s0, s1\} \cap V(q) \neq \varnothing
\end{aligned}
$$

  and again, by the claim we have that

$$
\begin{aligned}
\mathfrak{M}', s \models \Diamond_1(p \wedge q) \models &\iff \exists t \in \mathbb{B} \quad R_1(s, t) \wedge \mathfrak{M}', t \models p \wedge q \\
&\iff \exists t \in \mathbb{B} \quad (t = s0 \vee t = s1) \wedge (\mathfrak{M}', t \models p \wedge \mathfrak{M}', t \models q) \\
&\iff \exists t \in \mathbb{B} \quad (t = s0 \vee t = s1) \wedge (t \in V(p) \wedge t \in V(q)) \\
&\iff \exists t \in \mathbb{B} \quad (t = s0 \vee t = s1) \wedge t \in V(p) \cap V(q) \\
&\iff s0 \in V(p) \cap V(q) \vee s1 \in V(p) \cap V(q) \\
&\iff \{s0, s1\} \cap V(p) \cap V(q) \neq \varnothing
\end{aligned}
$$

Now suppose $V(p) = \{s0\}$ and $V(q) = \{s1\}$; then we have that $\{s0, s1\} \cap V(p) = \{s0\} \neq \varnothing \wedge \{s0, s1\} \cap V(q) = \{s1\} \neq \varnothing \iff \mathfrak{M}', s \models \Diamond_1 p \wedge \Diamond_1 q$ although $\{s0, s1\} \cap V(p) \cap V(q) = \{s0, s1\} \cap \varnothing = \varnothing \iff \mathfrak{M}', s \not\models \Diamond_1(p \wedge q)$, implying that the formula is not valid on $\mathcal{B}$.

(b) $(\Diamond_2 p \wedge \Diamond_2 q) \to \Diamond_2(p \wedge q)$

- By definition, for any $m \in \mathbb{N}$ it holds that

$$\begin{aligned}
\mathfrak{M}, m \models \Diamond_2 p \wedge \Diamond_2 q &\iff \mathfrak{M}, m \models \Diamond_2 p \wedge \mathfrak{M}, m \models \Diamond_2 q \\
&\iff \begin{cases} \exists n_p \in \mathbb{N} \quad S_2(m, n_p) \wedge \mathfrak{M}, n_p \models p \\ \exists n_q \in \mathbb{N} \quad S_2(m, n_q) \wedge \mathfrak{M}, n_q \models q \end{cases} \\
&\iff \begin{cases} \exists n_p \in \mathbb{N} \quad m > n_p \wedge n_p \in V(p) \\ \exists n_q \in \mathbb{N} \quad m > n_q \wedge n_q \in V(q) \end{cases}
\end{aligned}$$

and again, by the claim we have that

$$\begin{aligned}
\mathfrak{M}, m \models \Diamond_2(p \wedge q) &\iff \exists n \in \mathbb{N} \quad S_2(m, n) \wedge \mathfrak{M}, n \models p \wedge q \\
&\iff \exists n \in \mathbb{N} \quad m > n \wedge (\mathfrak{M}, n \models P \wedge \mathfrak{M}, n \models q) \\
&\iff \exists n \in \mathbb{N} \quad m > n \wedge (n \in V(p) \wedge n \in V(q)) \\
&\iff \exists n \in \mathbb{N} \quad m > n \wedge n \in V(p) \cap n \in V(q)
\end{aligned}$$

Now take an $n \geq 2$, and consider $n_p, n_q \in \mathbb{N}$ such that $n_p \neq n_q \wedge n > n_p, n_q$, and suppose that $V(p) = \{n_p\}$ and $V(q) = \{n_q\}$; then we have that $\begin{cases} \exists n_p \in \mathbb{N} \quad n > n_p \wedge n_p \in V(p) \\ \exists n_q \in \mathbb{N} \quad n > n_q \wedge n_q \in V(q) \end{cases} \iff \mathfrak{M}, m \models \Diamond_2 p \wedge \Diamond_2 q$ although $n_p \neq n_q \implies V(p) \cap V(q) = \varnothing \implies \nexists n \in \mathbb{N} \quad m > n \wedge n \in V(p) \cap V(q) \iff \mathfrak{M}, m \not\models \Diamond_2(p \wedge q)$, implying that the formula is not valid on $\mathcal{N}$.

- By the claim, for any $s \in \mathbb{B}$ it holds that

$$\begin{aligned}
\mathfrak{M}', s \models \Diamond_2 p \wedge \Diamond_2 q &\iff \mathfrak{M}', s \models \Diamond_2 p \wedge \mathfrak{M}', s \models \Diamond_2 q \\
&\iff \begin{cases} \exists t_p \in \mathbb{B} \quad R_2(s, t_p) \wedge \mathfrak{M}', t_p \models p \\ \exists t_q \in \mathbb{B} \quad R_2(s, t_q) \wedge \mathfrak{M}', t_q \models q \end{cases} \\
&\iff \begin{cases} \exists t_p \in \mathbb{B} \quad t_p \sqsubset s \wedge t_p \in V(p) \\ \exists t_q \in \mathbb{B} \quad t_q \sqsubset s \wedge t_q \in V(p) \end{cases}
\end{aligned}$$

and again, by the claim we have that

$$\begin{aligned}
\mathfrak{M}' s \models \Diamond_2(p \wedge q) &\iff \exists t \in \mathbb{B} \quad t \sqsubset s \wedge (\mathfrak{M}', t \models p \wedge \mathfrak{M}', t \models q) \\
&\iff \exists t \in \mathbb{B} \quad t \sqsubset s \wedge (t \in V(p) \wedge t \in V(q)) \\
&\iff \exists t \in \mathbb{B} \quad t \sqsubset s \wedge t \in V(p) \cap V(q)
\end{aligned}$$

Now take $s = 000$, consider $t_p = 0$ and $t_q = 0$, and suppose that $V(p) = \{t_p\} = \{0\}$ and $V(q) = \{t_q\} = \{00\}$; we observe that $t_p = 0 \sqsubset 000 = s$ and $t_q = 00 \sqsubset 000 = s$, therefore $\begin{cases} \exists t_p \in \mathbb{B} \quad t_p \sqsubset s \wedge t_p \in V(p) \\ \exists t_q \in \mathbb{B} \quad t_q \sqsubset s \wedge t_q \in V(q) \end{cases} \iff \mathfrak{M}', s \models \Diamond_2 p \wedge \Diamond_2 q$

although $V(p) \cap V(q) = \{t_p\} \cap \{t_q\} = \{0\} \cap \{00\} = \varnothing \implies \nexists t \in \mathbb{B} \quad t \sqsubset s \wedge t \in V(p) \cap V(q) \iff \mathfrak{M}', s \not\models \Diamond_2(p \wedge q)$, implying that the formula is not valid on $\mathcal{B}$.

(c) $(\Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r) \to (\Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r))$

- By the claim, for any $m \in \mathbb{N}$ it holds that

$$\begin{aligned}
\mathfrak{M}, m \models \Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r &\iff \mathfrak{M}, m \models \Diamond_1 p \wedge \mathfrak{M}, m \models \Diamond_1 q \wedge \mathfrak{M}, m \models \Diamond_1 r \\
&\iff \begin{cases} \exists n_p \in \mathbb{N} \quad S_1(m, n_p) \wedge \mathfrak{M}, n_p \models \Diamond_1 p \\ \exists n_q \in \mathbb{N} \quad S_1(m, n_q) \wedge \mathfrak{M}, n_q \models \Diamond_1 q \\ \exists n_r \in \mathbb{N} \quad S_1(m, n_r) \wedge \mathfrak{M}, n_r \models \Diamond_1 r \end{cases} \\
&\iff \begin{cases} \exists n_p \in \mathbb{N} \quad n_p = m+1 \wedge n_p \in V(p) \\ \exists n_q \in \mathbb{N} \quad n_q = m+1 \wedge n_q \in V(q) \\ \exists n_r \in \mathbb{N} \quad n_r = m+1 \wedge n_r \in V(r) \end{cases} \\
&\iff \begin{cases} m+1 \in V(p) \\ m+1 \in V(q) \\ m+1 \in V(r) \end{cases} \\
&\iff m+1 \in V(p) \cap V(q) \cap V(r)
\end{aligned}$$

and again, by the claim we have that

$$\begin{aligned}
\mathfrak{M}, m \models \Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r) &\iff (\mathfrak{M}, m \models \Diamond_1(p \wedge q)) \\
&\qquad \vee (\mathfrak{M}, m \models \Diamond_1(p \wedge r)) \\
&\qquad \vee (\mathfrak{M}, m \models \Diamond_1(q \wedge r)) \\
&\iff (\exists n_1 \in \mathbb{N} \quad S_1(m, n_1) \wedge \mathfrak{M}, n_1 \models p \wedge q) \\
&\qquad \vee (\exists n_2 \in \mathbb{N} \quad S_1(m, n_2) \wedge \mathfrak{M}, n_2 \models p \wedge r) \\
&\qquad \vee (\exists n_3 \in \mathbb{N} \quad S_1(m, n_3) \wedge \mathfrak{M}, n_3 \models q \wedge r) \\
&\iff (\exists n_1 \in \mathbb{N} \quad n_1 = m+1 \wedge n_1 \in V(p) \cap V(q)) \\
&\qquad \vee (\exists n_2 \in \mathbb{N} \quad n_2 = m+1 \wedge n_2 \in V(p) \cap V(r)) \\
&\qquad \vee (\exists n_3 \in \mathbb{N} \quad n_3 = m+1 \wedge n_3 \in V(q) \cap V(r)) \\
&\iff (m+1 \in V(p) \cap V(q)) \\
&\qquad \vee (m+1 \in V(p) \cap V(r)) \\
&\qquad \vee (m+1 \in V(q) \cap V(r))
\end{aligned}$$

Hence, we see that

$$\begin{aligned}
\mathfrak{M}, m \models \Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r &\iff m+1 \in V(p) \cap V(q) \cap V(r) \\
&\implies \begin{cases} m+1 \in V(p) \cap V(q) \\ m+1 \in V(p) \cap V(r) \\ m+1 \in V(q) \cap V(r) \end{cases} \\
&\iff \mathfrak{M}, m \models \Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r)
\end{aligned}$$

implying that the formula is valid in $\mathcal{N}$.

- By the claim, for any $s \in \mathbb{B}$ it holds that

$$\begin{aligned}
\mathfrak{M}', s \models \Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r &\iff \mathfrak{M}', s \models \Diamond_1 p \wedge \mathfrak{M}', s \models \Diamond_1 q \wedge \mathfrak{M}', s \models \Diamond_1 r \\
&\iff \begin{cases} \exists t_p \in \mathbb{B} \quad R_1(s, t_p) \wedge \mathfrak{M}', t_p \models p \\ \exists t_q \in \mathbb{B} \quad R_1(s, t_q) \wedge \mathfrak{M}', t_q \models q \\ \exists t_r \in \mathbb{B} \quad R_1(s, t_r) \wedge \mathfrak{M}', t_r \models r \end{cases} \\
&\iff \begin{cases} \exists t_p \in \mathbb{B} \quad (t_p = s0 \vee t_p = s1) \wedge t_p \in V(p) \\ \exists t_p \in \mathbb{B} \quad (t_q = s0 \vee t_q = s1) \wedge t_q \in V(q) \\ \exists t_r \in \mathbb{B} \quad (t_r = s0 \vee t_r = s1) \wedge t_r \in V(r) \end{cases} \\
&\iff \begin{cases} \{s0, s1\} \cap V(p) \neq \varnothing \\ \{s0, s1\} \cap V(q) \neq \varnothing \\ \{s0, s1\} \cap V(r) \neq \varnothing \end{cases}
\end{aligned}$$

and again, by the claim we have that

$$\begin{aligned}
\mathfrak{M}', s \models \Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r) &\iff (\mathfrak{M}', s \models \Diamond_1(p \wedge q)) \\
&\qquad \vee (\mathfrak{M}', s \models \Diamond_1(p \wedge r)) \\
&\qquad \vee (\mathfrak{M}', s \models \Diamond_1(q \wedge r)) \\
&\iff (\exists t_1 \in \mathbb{B} \quad R_1(s, t_1) \wedge \mathfrak{M}', t_1 \models p \wedge q) \\
&\qquad \vee (\exists t_2 \in \mathbb{B} \quad R_1(s, t_2) \wedge \mathfrak{M}', t_2 \models p \wedge r) \\
&\qquad \vee (\exists t_3 \in \mathbb{B} \quad R_1(s, t_3) \wedge \mathfrak{M}', t_3 \models q \wedge r) \\
&\iff (\exists t_1 \in \mathbb{B} \quad (t_1 = s0 \vee t_1 = s1) \wedge t_1 \in V(p) \cap V(q)) \\
&\qquad \vee (\exists t_2 \in \mathbb{B} \quad (t_2 = s0 \vee t_2 = s1) \wedge t_2 \in V(p) \cap V(r)) \\
&\qquad \vee (\exists t_3 \in \mathbb{B} \quad (t_3 = s0 \vee t_3 = s1) \wedge t_3 \in V(q) \cap V(r)) \\
&\iff \{s0, s1\} \cap V(p) \cap V(q) \neq \varnothing \\
&\qquad \vee \{s0, s1\} \cap V(p) \cap V(q) \neq \varnothing \\
&\qquad \vee \{s0, s1\} \cap V(q) \cap V(r) \neq \varnothing
\end{aligned}$$

Now suppose that $\mathfrak{M}', s \models \Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r$, which happens if and only if $\begin{cases} \{s0, s1\} \cap V(p) \neq \varnothing \\ \{s0, s1\} \cap V(q) \neq \varnothing \\ \{s0, s1\} \cap V(r) \neq \varnothing \end{cases}$ as proved previously; by the pigeonhole principle, since there are 2 strings in $\{s0, s1\}$ and we have 3 sets $V(p)$, $V(q)$ and $V(r)$, there must be at least one string $x \in \{s0, s1\}$ such that $x \in V(a) \cap V(b)$, where $a, b \in \{p, q, r\}$ distinct. Without loss of generality, suppose that $x = s0$ and $a = p$ and $b = q$; then we have that

$$\begin{aligned}
x = s0 \in V(p) \cap V(q) &\implies x \in \{s0, s1\} \cap V(p) \cap V(q) \\
&\implies \{s0, s1\} \cap V(p) \cap V(q) \neq \varnothing \\
&\implies \mathfrak{M}', s \models \Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r)
\end{aligned}$$

implying that the formula is valid on $\mathcal{B}$.

(d) $p \to \Diamond_1 \Box_2 p$

- By definition, for any $m \in \mathbb{N}$ it holds that

$$\begin{aligned}\mathfrak{M}, m \models \Diamond_1 \Box_2 p &\iff \exists n \in \mathbb{N} \quad S_1(m,n) \land \mathfrak{M}, n \models \Box_2 p \\ &\iff \exists n \in \mathbb{N} \quad n = m+1 \land (\forall k \in \mathbb{N} \quad S_2(n,k) \implies \mathfrak{M}, k \models p) \\ &\iff \exists n \in \mathbb{N} \quad n = m+1 \land (\forall k \in \mathbb{N} \quad n > k \implies k \in V(p)) \\ &\iff \forall k \in \mathbb{N} \quad m+1 > k \implies k \in V(p) \\ &\iff V(p) = \{k \in \mathbb{N} \mid m+1 > k\}\end{aligned}$$

Now take $m = 1 \in \mathbb{N}$, and suppose $V(p) = \{m\} = \{1\}$; then $m \in V(p) \iff \mathfrak{M}, m \models p$, however for instance $k = 0 \in \mathbb{N}$ is such that $m + 1 = 1 + 1 = 2 > 0 = k$ although $k = 0 \notin V(p)$, therefore $\exists k \in \mathbb{N} \quad m+1 > k \land k \notin V(p) \implies V(p) \neq \{k \in \mathbb{N} \mid m+1 > k\} \iff \mathfrak{M}, m \not\models \Diamond_1 \Box_2 p$ which implies that the formula is not valid on $\mathcal{N}$.

- By definition, for any $s \in \mathbb{B}$ it holds that

$$\begin{aligned}\mathfrak{M}', s \models \Diamond_1 \Box_2 p &\iff \exists t \in \mathbb{B} \quad R_1(s,t) \land \mathfrak{M}', t \models \Box_2 p \\ &\iff \exists t \in \mathbb{B} \quad (t = s0 \lor t = s1) \land (\forall u \in \mathbb{B} \quad R_2(t,u) \implies \mathfrak{M}', u \models p) \\ &\iff \exists t \in \mathbb{B} \quad (t = s0 \lor t = s1) \land (\forall u \in \mathbb{B} \quad u \sqsubset t \implies u \in V(p))\end{aligned}$$

Now take $s = 00 \in \mathbb{B}$, and suppose $V(p) = \{s\} = \{00\}$; then $s \in V(p) \iff \mathfrak{M}', s \models p$, however if $t = s0$ or $t = s1$, there still is $u = 0$ such that $u = 0 \sqsubset 00 = t$ although $u = 0 \notin V(p)$, therefore $(t = s0 \lor t = s1) \implies (\exists u \in \mathbb{B} \quad u \sqsubset t \land u \notin V(p)) \iff \mathfrak{M}', s \not\models \Diamond_1 \Box_2 p$ which implies that the formula is not valid on $\mathcal{B}$.

(e) $p \to \Diamond_2 \Box_1 p$

- By definition, for any $m \in \mathbb{N}$ it holds that

$$\begin{aligned}\mathfrak{M}, m \models \Diamond_2 \Box_1 p &\iff \exists n \in \mathbb{N} \quad S_2(m,n) \land \mathfrak{M}, n \models \Box_1 p \\ &\iff \exists n \in \mathbb{N} \quad m > n \land (\forall k \in \mathbb{N} \quad S_1(n,k) \implies \mathfrak{M}, k \models p) \\ &\iff \exists n \in \mathbb{N} \quad m > n \land (\forall k \in \mathbb{N} \quad k = n+1 \implies k \in V(p)) \\ &\iff \exists n \in \mathbb{N} \quad m > n \land n+1 \in V(p)\end{aligned}$$

Now take $m = 0 \in \mathbb{N}$, and suppose $V(p) = \{m\} = \{0\}$; then $m \in V(p) \iff \mathfrak{M}, m \models p$, however there is no $n \in \mathbb{N}$ such that $m = 0 > n$, therefore $\nexists n \in \mathbb{N} \quad m > n \land n+1 \in V(p) \iff \mathfrak{M}, m \not\models \Diamond_2 \Box_1 p$ which implies that the formula is not valid on $\mathcal{N}$.

- By definition, for any $s \in \mathbb{B}$ it holds that

$$\begin{aligned}\mathfrak{M}', s \models \Diamond_2 \Box_1 p &\iff \exists t \in \mathbb{B} \quad R_2(s,t) \land \mathfrak{M}', t \models \Box_1 p \\ &\iff \exists t \in \mathbb{B} \quad t \sqsubset s \land (\forall u \in \mathbb{B} \quad R_1(t,u) \implies \mathfrak{M}', u \models p) \\ &\iff \exists t \in \mathbb{B} \quad t \sqsubset s \land (\forall u \in \mathbb{B} \quad (u = t0 \lor u = t1) \implies u \in V(p)) \\ &\iff \exists t \in \mathbb{B} \quad t \sqsubset s \land (t0 \in V(p) \lor t1 \in V(p))\end{aligned}$$

Now take $s = \varepsilon \in \mathbb{B}$, and suppose $V(p) = \{s\} = \{\varepsilon\}$; then $s \in V(p) \iff \mathfrak{M}', s \models p$, however there is no $t \in \mathbb{B}$ such that $t \sqsubset s$, therefore $\nexists t \in \mathbb{B} \quad t \sqsubset s \land (t0 \in V(p) \lor t1 \in V(p)) \iff \mathfrak{M}', s \not\models \Diamond_2 \Box_1 p$ which implies that the formula is not valid on $\mathcal{B}$.

(f) $p \to \Box_1 \Diamond_2 p$

- By definition, for any $m \in \mathbb{N}$ it holds that

$$\begin{aligned}
\mathfrak{M}, m \models \Box_1\Diamond_2 p &\iff \forall n \in \mathbb{N} \quad S_1(m,n) \implies \mathfrak{M}, n \models \Diamond_2 p \\
&\iff \forall n \in \mathbb{N} \quad n = m+1 \implies (\exists k \in \mathbb{N} \quad S_2(n,k) \land \mathfrak{M}, k \models p) \\
&\iff \forall n \in \mathbb{N} \quad n = m+1 \implies (\exists k \in \mathbb{N} \quad n > k \land k \in V(p)) \\
&\iff \exists k \in \mathbb{N} \quad m+1 > k \land k \in V(p)
\end{aligned}$$

Now suppose $m \in V(p) \iff \mathfrak{M}, m \models p$; we observe that for every $m \in \mathbb{N}$ it holds that $m+1 > m$, therefore $m+1 > m \land m \in V(p) \iff \exists k \in \mathbb{N} \quad m+1 > k \land k \in V(p) \iff \mathfrak{M}, m \models \Box_1\Diamond_2 p$, which implies that the formula is valid on $\mathcal{N}$.

- By definition, for any $s \in \mathbb{B}$ it holds that

$$\begin{aligned}
\mathfrak{M}', s \models \Box_1\Diamond_2 p &\iff \forall t \in \mathbb{B} \quad R_1(s,t) \implies \mathfrak{M}', t \models \Diamond_2 p \\
&\iff \forall t \in \mathbb{B} \quad (t = s0 \lor t = s1) \implies (\exists u \in \mathbb{B} \quad R_2(t,u) \land \mathfrak{M}', u \models p) \\
&\iff \forall t \in \mathbb{B} \quad (t = s0 \lor t = s1) \implies (\exists u \in \mathbb{B} \quad u \sqsubset t \land u \in V(p))
\end{aligned}$$

Now suppose $s \in V(p) \iff \mathfrak{M}', s \models p$; we observe that for every $s \in \mathbb{B}$ it holds that $s \sqsubset s0, s1$, therefore $\exists u \in \mathbb{B} \quad (u \sqsubset s0 \lor u \sqsubset s1) \land u \in V(p) \iff \mathfrak{M}', s \models \Box_1\Diamond_2 p$, which implies that the formula is valid on $\mathcal{B}$.

(g) $p \to \Box_2 \Diamond_1 p$

- By definition, for any $m \in \mathbb{N}$ it holds that

$$\begin{aligned}
\mathfrak{M}, m \models \Box_2\Diamond_1 p &\iff \forall n \in \mathbb{N} \quad S_2(m,n) \implies \mathfrak{M}, n \models \Diamond_2 p \\
&\iff \forall n \in \mathbb{N} \quad m > n \implies (\exists k \in \mathbb{N} \quad S_1(n,k) \land \mathfrak{M}, k \models p) \\
&\iff \forall n \in \mathbb{N} \quad m > n \implies (\exists k \in \mathbb{N} \quad k = n+1 \land k \in V(p)) \\
&\iff \forall n \in \mathbb{N} \quad m > n \implies n+1 \in V(p)
\end{aligned}$$

Now take $m = 2 \in \mathbb{N}$, and suppose $V(p) = \{m\} = \{2\}$; then $m \in V(p) \iff \mathfrak{M}, m \models p$, however for instance $n = 0 \in \mathbb{N}$ is such that $m = 2 > 0 = n$ and $n+1 = 0+1 = 1 \notin V(p)$, therefore $\exists n \in \mathbb{N} \quad m > n \land n+1 \notin V(p) \iff \mathfrak{M}, m \not\models \Box_2\Diamond_1 p$, which implies that the formula is not valid on $\mathcal{N}$.

- By definition, for any $s \in \mathbb{B}$ it holds that

$$\begin{aligned}
\mathfrak{M}', s \models \Box_2\Diamond_1 p &\iff \forall t \in \mathbb{B} \quad R_2(s,t) \implies \mathfrak{M}', t \models \Diamond_2 p \\
&\iff \forall s \in \mathbb{B} \quad t \sqsubset s \implies (\exists u \in \mathbb{B} \quad R_1(t,u) \land \mathfrak{M}', u \models p) \\
&\iff \forall s \in \mathbb{B} \quad t \sqsubset s \implies (\exists u \in \mathbb{B} \quad (u = t0 \lor u = t1) \implies u \in V(p))
\end{aligned}$$

Now take $s = 000 \in \mathbb{B}$, and suppose $V(p) = \{s\} = \{000\}$; then $s \in V(p) \iff \mathfrak{M}', s \models p$, however for instance $t = 0 \in \mathbb{B}$ is such that $t = 0 \sqsubset 000 = s$ although there is no $u = t0 = 00$ or $u = t1 = 01$ such that $u \in V(p)$, therefore $\exists t \in \mathbb{B} \quad t \sqsubset u \land (\nexists u \in \quad (u = t0 \lor u = t1) \land u \in V(p)) \iff \mathfrak{M}', s \not\models \Box_2\Diamond_1 p$ which implies that the formula is not valid on $\mathcal{B}$.

□

Exercise 3.2 Consider the basic modal language, and the tuple $\mathfrak{f} = (\mathbb{N}, <, A)$ where A is the collection of finite and co-finite subsets of $\mathbb{N}$. Show that $\mathfrak{f}$ is a general frame.

Solution. First, consider the follwing two claims.

Claim 1: If $X \subseteq \mathbb{N}$ is finite, and $Y \subseteq \mathbb{N}$ is co-finite, then $X \cup Y$ is co-finite.

Proof of the Claim. Since X is finite, $\mathbb{N} - X$ is finite, and since Y is co-finite, $\mathbb{N} - Y$ is finite; this implies that

$$\mathbb{N} - (X \cup Y) = (\mathbb{N} - X) \cap (\mathbb{N} - Y)$$

is the intersection of a co-finite and a finite set. In particular, we observe that

- such intersection will be a subset of $\mathbb{N} - Y$ by definition of intersection
- $\mathbb{N} - Y$ is finite
- a subset of a finite set is always finite

concluding that such intersection must be finite as well. Lastly, by definition we have that $\mathbb{N} - (X \cup Y)$ is finite if and only if $X \cup Y$ is co-finite. □

Claim 2: If $X, Y \subseteq \mathbb{N}$ are co-finite, then $X \cup Y$ is co-finite.

Proof of the Claim. By repeating the same argument of the previous claim, we have that $\mathbb{N} - (X \cup Y) = (\mathbb{N} - X) \cap (\mathbb{N} - Y)$ except that in this case both $\mathbb{N} - X$ and $\mathbb{N} - Y$ are finite, which implies that their intersection must be finite, hence $\mathbb{N} - (X \cup Y)$ is co-finite by definition. □

To prove that $\mathfrak{f}$ is a general frame, it suffices to prove that the set A is closed under the following

- *union*: fix two sets $X, Y \in A$; then, by definition of A, we have that
  - if both X and Y are finite, then $X \cup Y$ is finite, hence $X \cup Y \in A$
  - without loss of generality, if X is finite and Y is co-finite, by Claim 1 $X \cup Y$ is co-finite, therefore $X \cup Y \in A$
  - if both X and Y are finite, then $X \cup Y$ is co-finite by Claim 2, therefore $X \cup Y \in A$
- *relative complement*: fix a set $X \in A$; then, by definition of A we trivially have that
  - if X is finite, then $\mathbb{N} - X$ is co-finite, hence $\mathbb{N} - X \in A$
  - if X is co-finite, then $\mathbb{N} - X$ is finite, hence $\mathbb{N} - X \in A$
- *modal operations*: assume that "$<$" is the relation referring to a unary modal operator $\langle < \rangle$, and fix a set $X \in A$; by definition, we have that

$$m_{\langle < \rangle}(X) = \{n \in \mathbb{N} \mid \exists x \in X \quad n < x\}$$

therefore, we have that

- if X is finite, then

$$m_{\langle<\rangle}(X) = \{n \in \mathbb{N} \mid n < \max(X)\}$$

therefore $m_{\langle<\rangle}(X)$ is an "initial segment of $\mathbb{N}$", implying that is finite, hence $m_{\langle<\rangle}(X) \in A$

- if X is co-finite, then $\mathbb{N} - X$ is finite, implying that X is infinite; this implies that $\max(X)$ is not defined, therefore

$$m_{\langle<\rangle}(X) = \mathbb{N} \implies \mathbb{N} - m_{\langle<\rangle}(X) = \mathbb{N} - \mathbb{N} = \varnothing$$

and since $\varnothing$ is finite, we conclude that $m_{\langle<\rangle}(X)$ is co-finite, thus $m_{\langle<\rangle}(X) \in A$

$\square$

Exercise 4.3 TODO . scrivimi

Solution. TODO scrivimi

Claim: TODO scrivimi

Proof of the Claim. Assume that $\Sigma \models^g_{\mathsf{F}} \phi$; fix a frame $\mathcal{F} \in \mathsf{F}$ defined over a world W, a model $\mathfrak{M}$ over $\mathcal{F}$, and a world $w \in W$. Suppose that $\mathfrak{M}, w \models \Pi$; then, by the claim of Exercise 2.2 we know that

$$\begin{aligned} & \forall \sigma \in \Sigma, n \in \mathbb{N} \quad \mathfrak{M}, w \models \Box^n \sigma \\ \iff & \forall \sigma \in \Sigma, n \in \mathbb{N}, x_1, \ldots, x_n \in W \quad w \xrightarrow{R} x_1 \xrightarrow{R} \ldots \xrightarrow{R} x_n \implies \mathfrak{M}, x_n \models \sigma \end{aligned}$$

Now, consider the following restriction of W

$$W_w := \{v \in W_w \mid \exists n \in \mathbb{N}, x_1, \ldots, x_{n-1} \in W \quad w \xrightarrow{R} x_1 \xrightarrow{R} \ldots \xrightarrow{R} x_{n-1} \xrightarrow{R} v\}$$

where $v \in W_w$ if and onlhy if v can be R-reached from w through a sequence of R-accessible elements. Moreover, consider the following restriction of R

$$R_w := (W_w \times W_w) \cap R$$

in which we consider the tuples of R that connect elements of W_w. Then, since F is the class of all frames, we know that $\mathcal{F}_w = (W_w, R_w) \in \mathsf{F}$. Lastly, consider a model $\mathfrak{M}_w$ over $\mathcal{F}_w$ such that $\mathfrak{M}_w = (W_w, R_w, V_w)$ where

$$V_w = W_w \cap V$$

Consider some $x_1, \ldots, x_n \in W$ such that $w \xrightarrow{R} x_1 \xrightarrow{R} \ldots \xrightarrow{R} x_n$; by definition of W_w, this implies that all $x_1, \ldots, x_n$ are R-reachable, which implies that $x_1, \ldots, x_n \in W_w$; this means that

$$\begin{aligned} & \forall \sigma \in \Sigma, n \in \mathbb{N}, x_1, \ldots, x_n \in W \quad w \xrightarrow{R} x_1 \xrightarrow{R} \ldots \xrightarrow{R} x_n \implies \mathfrak{M}, x_n \models \sigma \\ \implies & \forall \sigma \in \Sigma, n \in \mathbb{N}, x_1 \ldots, x_n \in W_W \quad w \xrightarrow{R} x_1 \xrightarrow{R} \ldots \xrightarrow{R} x_n \implies \mathfrak{M}, x_n \models \sigma \end{aligned}$$

Moreover, since $x_1, \ldots, x_n$ are elements of W_w, by definition of R_w it holds that

$$\begin{aligned}
&\forall \sigma \in \Sigma, n \in \mathbb{N}, x_1 \ldots, x_n \in W_W \quad w \xrightarrow{R} x_1 \xrightarrow{R} \ldots \xrightarrow{R} x_n \implies \mathfrak{M}, x_n \models \sigma \\
\implies &\forall \sigma \in \Sigma, n \in \mathbb{N}, x_1 \ldots, x_n \in W_W \quad w \xrightarrow{R_w} x_1 \xrightarrow{R_w} \ldots \xrightarrow{R_w} x_n \implies \mathfrak{M}, x_n \models \sigma
\end{aligned}$$

Furthermore, since $W_w \subseteq W$ and $R_w \subseteq R$, by definition of $\mathfrak{M}_w$ we get that

$$\begin{aligned}
&\forall \sigma \in \Sigma, n \in \mathbb{N}, x_1 \ldots, x_n \in W_W \quad w \xrightarrow{R_w} x_1 \xrightarrow{R_w} \ldots \xrightarrow{R_w} x_n \implies \mathfrak{M}, x_n \models \sigma \\
\implies &\forall \sigma \in \Sigma, n \in \mathbb{N}, x_1 \ldots, x_n \in W_W \quad w \xrightarrow{R_w} x_1 \xrightarrow{R_w} \ldots \xrightarrow{R_w} x_n \implies \mathfrak{M}_w, x_n \models \sigma
\end{aligned}$$

Now fix $v \in W_w$; by definition there are $y_1, \ldots, y_k \in W$ such that $w \xrightarrow{R_w} y_1 \xrightarrow{R_w} \ldots \xrightarrow{R_w} y_k \xrightarrow{R} v$, therefore by the previous observation we get that

$$\begin{aligned}
&\forall \sigma \in \Sigma, v \in W_w \quad \mathfrak{M}_w, v \models \sigma \\
\implies &\forall v \in W_w, \sigma \in \Sigma \quad \mathfrak{M}_w, v \models \sigma \\
\implies &\forall v \in W_w \quad \mathfrak{M}_w, v \models \Sigma \\
\implies &\forall v \in W_w \quad \mathfrak{M}_w, v \models \phi && (\Sigma \models^g_{\mathsf{F}} \phi) \\
\iff &\forall v \in W_w \quad w \in V_w(\phi) \\
\implies &W_w \subseteq V_w(\phi) \subseteq V(\phi)
\end{aligned}$$

Lastly, we observe that $w \in W_w$, and since $W_w \subseteq V(\phi)$ we have that $w \in V(\phi)$, which happens if and only if $\mathfrak{M}, w \models \phi$. $\square$

Claim: TODO scrivimi

Proof of the Claim. Assume that $\Pi \models_{\mathsf{F}} \phi$; fix a frame $\mathcal{F} \in \mathsf{F}$ defined over a world W, and a model $\mathfrak{M}$ over $\mathcal{F}$. Suppose that $\forall w \in W \quad \mathfrak{M}, w \models \Sigma$; then, by the claim of Exercise 2.2 we know that

$$\forall w \in W \quad \mathfrak{M}, w \models \Sigma \iff \forall w \in W, \sigma \in \Sigma \quad \mathfrak{M}, w \models \sigma$$

We observe that this implies that

$$\forall n \in \mathbb{N}, w, x_1, \ldots, x_n \in W, \sigma \in \Sigma \quad R(w, x_1) \wedge R(x_1, x_2) \wedge \ldots \wedge R(x_{n-1}, x_n) \implies \mathfrak{M}, x_n \models \sigma$$

therefore

$$\forall n \in \mathbb{N}, w \in W, \sigma \in \Sigma \quad \mathfrak{M}, w \models \Box^n \sigma$$

Finally, this concludes that

$$\forall w \in W \quad \mathfrak{M}, w \models \Pi \implies \forall w \in W \quad \mathfrak{M}, w \models \phi$$

This proves that for any frame $\mathcal{F} \in \mathsf{F}$ defined over a world W, and any model $\mathfrak{M}$ over $\mathcal{F}$ it holds that

$$\forall w \in W \quad \mathfrak{M}, w \models \Sigma \implies \forall w \in W \quad \mathfrak{M}, w \models \phi$$

which implies that $\Sigma \models^g_{\mathsf{F}} \phi$ by definition. $\square$

$\square$

Exercise 5.1 Give K-proofs of $(\Box p \wedge \Diamond q) \to \Diamond(p \wedge q)$ and $\Diamond(p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q)$.

Solution. In the first section of the solution, we are going to prove some useful derivations that will be extensively used in the actual K-proof of the two propositions. The right side of each line will be one of the following:

- (K): the K axiom
- (D): the Dual axiom
- (T): a propositional Tautology
- (MP(i, j)): the Modus Ponens rule applied on lines i and j
- (S(i)): the Substitution rule applied on line i
- (G(i)): the Generalization rule applied on line i
- ($\mathrm{C}_k(i_1, \dots i_n)$): the k-th Claim applied on lines $i_1, \dots, i_n$ — $k \in [7]$ and n depends on the number of lines the Claim refers to

Claim 1: If $p \to q$ can be K-proved, and $q \to r$ can be K-proved, then $p \to r$ can be K-proved in 4 steps.

Proof of the Claim. Consider a K-proof in which $p \to q$ is proved at step i, and $q \to r$ is proved at step j — without loss of generality suppose that $i < j$; then we have that

$$
\begin{array}{rll}
 & \dots & \\
i. & \vdash p \to q & \\
 & \dots & \\
j. & \vdash q \to r & \\
j+1. & \vdash (a \to b) \to ((b \to c) \to (a \to c)) & \text{(T)} \\
j+2. & \vdash (p \to q) \to ((q \to r) \to (p \to r)) & (\mathrm{S}(j+1)) \\
j+3. & \vdash (q \to r) \to (p \to r) & (\mathrm{MP}(i,\, j+2)) \\
j+4. & \vdash p \to r & (\mathrm{MP}(j,\, j+3))
\end{array}
$$

$\Box$

Claim 2: If $p \to q$ can be K-proved, then $\Box p \to \Box q$ can be K-proved in 4 steps.

Proof of the Claim. Consider a K-proof in which $p \to q$ is proved at step i; then, we have that

$$
\begin{array}{rll}
 & \dots & \\
i. & \vdash p \to q & \\
i+1. & \vdash \Box(p \to q) & (\mathrm{G}(i)) \\
i+2. & \vdash \Box(a \to b) \to (\Box a \to \Box b) & \text{(K)} \\
i+3. & \vdash \Box(p \to q) \to (\Box p \to \Box q) & (\mathrm{S}(i+2)) \\
i+4. & \vdash \Box p \to \Box q & (\mathrm{MP}(i+1,\, i+3))
\end{array}
$$

$\Box$

Claim 3: If $p \to q$ can be K-proved, and $p \to r$ can be K-proved, then $p \to q \wedge r$ can be K-proved in 4 steps.

Proof of the Claim. Consider a K-proof in which $p \to q$ is proved at step i, and $p \to r$ is proved at step j — without loss of generality suppose $i < j$; then, we have that

$$\begin{array}{rll} & \dots & \\ i. & \vdash p \to q & \\ & \dots & \\ j. & \vdash p \to r & \\ j+1. & \vdash (a \to b) \to ((a \to c) \to (a \to b \wedge c)) & \text{(T)} \\ j+2. & \vdash (p \to q) \to ((p \to r) \to (p \to q \wedge r)) & (\mathrm{S}(j+1)) \\ j+3. & \vdash (p \to r) \to (p \to q \wedge r) & (\mathrm{MP}(i,\, j+2)) \\ j+4. & \vdash p \to q \wedge r & (\mathrm{MP}(j,\, j+3)) \end{array}$$

$\square$

Claim 4: If $p \to q$ can be K-proved, then $\neg q \to \neg p$ can be K-proved in 3 steps. Moreover, if $p \to \neg q$ can be K-proved, then $q \to \neg p$ can be K-proved in 3 steps.

Proof of the Claim. Consider a K-proof in which $p \to q$ is proved at step i; then, we have that

$$\begin{array}{rll} & \dots & \\ i. & \vdash p \to q & \\ i+1. & \vdash (a \to b) \to (\neg b \to \neg a) & \text{(T)} \\ i+2. & \vdash (p \to q) \to (\neg q \to \neg p) & (\mathrm{S}(i+1)) \\ i+3. & \vdash \neg q \to \neg p & (\mathrm{MP}(i,\, i+2)) \end{array}$$

The same K-proof can be used to prove the rest of the claim by using the propositional tautology $(a \to \neg b) \to (b \to \neg a)$. $\square$

Claim 5: If $p \leftrightarrow q$ can be K-proved, then $p \to q$ and $q \to p$ can be K-proved in 3 steps.

Proof of the Claim. Consider a K-proof in which $p \leftrightarrow q$ is proved at step i; then, we have that

$$\begin{array}{rll} & \dots & \\ i. & \vdash p \leftrightarrow q & \\ i+1. & \vdash (a \leftrightarrow b) \to (a \to b) & \text{(T)} \\ i+2. & \vdash (p \leftrightarrow q) \to (p \to q) & (\mathrm{S}(i+1)) \\ i+3. & \vdash p \to q & (\mathrm{MP}(i,\, i+2)) \end{array}$$

The case for $q \to p$ can be proved analogously by using the propositional tautology $(a \leftrightarrow b) \to (b \to a)$. $\square$

Claim 6: If $p \to (q \to r)$ can be K-proved, then $p \wedge q \to r$ can be K-proved in 3 steps.

Proof of the Claim. Consider a K-proof in which $p \to (q \to r)$ is proved at step i; then, we have that

$$\begin{array}{rll} & \ldots & \\ i. & \vdash p \to (q \to r) & \\ i+1. & \vdash (a \to (b \to c)) \to (a \wedge b \to c) & \text{(T)} \\ i+2. & \vdash (p \to (q \to r)) \to (p \wedge q \to r) & (\mathrm{S}(i+1)) \\ i+3. & \vdash p \wedge q \to r & (\mathrm{MP}(i,\, i+2)) \end{array}$$

$\square$

Claim 7: If $p \to q$ can be K-proved, and $q \to p$ can be K-proved, then $p \leftrightarrow q$ can be proved in 4 steps.

Proof of the Claim. Consider a K-proof in which $p \to q$ is proved at step i, and $q \to p$ can be proved at step j — without loss of generality suppose $i < j$; then, we have that

$$\begin{array}{rll} & \ldots & \\ i. & \vdash p \to q & \\ & \ldots & \\ j. & \vdash q \to p & \\ j+1. & \vdash (a \to b) \to ((b \to a) \to (a \leftrightarrow b)) & \text{(T)} \\ j+2. & \vdash (p \to q) \to ((q \to p) \to (p \leftrightarrow q)) & (\mathrm{S}(j+1)) \\ j+3. & \vdash (q \to p) \to (p \leftrightarrow q) & (\mathrm{MP}(i,\, j+2)) \\ j+4. & \vdash p \leftrightarrow q & (\mathrm{MP}(j,\, j+3)) \end{array}$$

$\square$

Now that we proved some preliminary claims, we can prove the two given propositions.

Claim 8: $(\Box p \wedge \Diamond q) \to \Diamond(p \wedge q)$ is K-provable.

Proof of the Claim.

$$
\begin{array}{rll}
1. & \vdash \Box(p \to q) \to (\Box p \to \Box q) & \text{(K)} \\
2. & \vdash (\neg a \lor \neg b) \to (a \to \neg b) & \text{(T)} \\
3. & \vdash (\neg p \lor \neg q) \to (p \to \neg q) & \text{(S(2))} \\
 & \dots & \\
7. & \vdash \Box(\neg p \lor \neg q) \to \Box(p \to \neg q) & (\text{C}_2(3)) \\
8. & \vdash \Box(p \to \neg q) \to (\Box p \to \Box\neg q) & \text{(S(1))} \\
 & \dots & \\
12. & \vdash \Box(\neg p \lor \neg q) \to (\Box p \to \Box\neg q) & (\text{C}_1(7,8)) \\
13. & \vdash (a \to b) \to (\neg a \lor b) & \text{(T)} \\
14. & \vdash (\Box p \to \Box\neg q) \to (\neg\Box p \lor \Box\neg q) & \text{(S(13))} \\
 & \dots & \\
18. & \vdash \Box(\neg p \lor \neg q) \to (\neg\Box p \lor \Box\neg q) & (\text{C}_1(12,14)) \\
19. & \vdash (\neg a \lor b) \to \neg(a \land \neg b) & \text{(T)} \\
20. & \vdash (\neg\Box p \lor \Box\neg q) \to \neg(\Box p \land \neg\Box\neg q) & \text{(S(19))} \\
 & \dots & \\
24. & \vdash \Box(\neg p \lor \neg q) \to \neg(\Box p \land \neg\Box\neg q) & (\text{C}_1(18,20)) \\
 & \dots & \\
27. & \vdash (\Box p \land \neg\Box\neg q) \to \neg\Box(\neg p \lor \neg q) & (\text{C}_4(24)) \\
28. & \vdash \neg(a \land b) \to (\neg a \lor \neg b) & \text{(T)} \\
29. & \vdash \neg(p \land q) \to (\neg p \lor \neg q) & \text{(S(27))} \\
 & \dots & \\
33. & \vdash \Box\neg(p \land q) \to \Box(\neg p \lor \neg q) & (\text{C}_2(28)) \\
 & \dots & \\
36. & \vdash \neg\Box(\neg p \lor \neg q) \to \neg\Box\neg(p \land q) & (\text{C}_4(32)) \\
 & \dots & \\
40. & \vdash (\Box p \land \neg\Box\neg q) \to \neg\Box\neg(p \land q) & (\text{C}_1(27,36)) \\
41. & \vdash \Diamond a \leftrightarrow \neg\Box\neg a & \text{(D)} \\
42. & \vdash \Diamond(p \land q) \leftrightarrow \neg\Box\neg(p \land q) & \text{(S(41))} \\
 & \dots & \\
45. & \vdash \neg\Box\neg(p \land q) \to \Diamond(p \land q) & (\text{C}_5(42)) \\
 & \dots & \\
49. & \vdash (\Box p \land \neg\Box\neg q) \to \Diamond(p \land q) & (\text{C}_1(40,45))
\end{array}
$$

50. $\vdash \Diamond q \leftrightarrow \neg\Box\neg q$	(S(41))
...	
53. $\vdash \Diamond q \rightarrow \neg\Box\neg q$	(C_5(50))
54. $\vdash (b \rightarrow c) \rightarrow (a \wedge b \rightarrow a \wedge c)$	(T)
55. $\vdash (\Diamond q \rightarrow \neg\Box\neg q) \rightarrow (\Box p \wedge \Diamond q \rightarrow \Box p \wedge \neg\Box\neg q)$	(S(54))
56. $\vdash \Box p \wedge \Diamond q \rightarrow \Box p \wedge \neg\Box\neg q$	(MP(53, 55))
...	
60. $\vdash (\Box p \wedge \Diamond q) \rightarrow \Diamond(p \wedge q)$	($C_1(56, 49)$)

$\Box$

This claim concludes the K-proof of the first proposition. To K-prove the second proposition, we are going to split the K-proof into 3 claims.

Claim 9: $(\Diamond p \vee \Diamond q) \rightarrow \Diamond(p \vee q)$ is K-provable.

Proof of the Claim.

1.	$\vdash \neg p \wedge \neg q \to \neg p$	(T)
	$\ldots$	
5.	$\vdash \Box(\neg p \wedge \neg q) \to \Box\neg p$	($\mathrm{C}_2(1)$)
6.	$\vdash \neg p \wedge \neg q \to \neg q$	(T)
	$\ldots$	
10.	$\vdash \Box(\neg p \wedge \neg q) \to \Box\neg q$	($\mathrm{C}_2(2)$)
	$\ldots$	
14.	$\vdash \Box(\neg p \wedge \neg q) \to \Box\neg p \wedge \Box\neg q$	($\mathrm{C}_3(10)$)
	$\ldots$	
17.	$\vdash \neg(\Box\neg p \wedge \Box\neg q) \to \neg\Box(\neg p \wedge \neg q)$	($\mathrm{C}_4(14)$)
18.	$\vdash \neg a \vee \neg b \to \neg(a \wedge b)$	(T)
19.	$\vdash \neg\Box\neg p \vee \neg\Box\neg q \to \neg(\Box\neg p \wedge \Box\neg q)$	(S(18))
	$\ldots$	
23.	$\vdash \neg\Box\neg p \vee \neg\Box\neg q \to \neg\Box(\neg p \wedge \neg q)$	($\mathrm{C}_1(22, 17)$)
24.	$\vdash \neg(a \vee b) \to \neg a \wedge \neg b$	(T)
25.	$\vdash \neg(p \vee q) \to \neg p \wedge \neg q$	(S(24))
	$\ldots$	
29.	$\vdash \Box\neg(p \vee q) \to \Box(\neg p \wedge \neg q)$	($\mathrm{C}_2(25)$)
	$\ldots$	
32.	$\vdash \neg\Box(\neg p \wedge \neg q) \to \neg\Box\neg(p \vee q)$	($\mathrm{C}_4(29)$)
	$\ldots$	
36.	$\vdash \neg\Box\neg p \vee \neg\Box\neg q \to \neg\Box\neg(p \vee q)$	($\mathrm{C}_1(23, 32)$)
37.	$\vdash \Diamond a \leftrightarrow \neg\Box\neg a$	(D)
	$\ldots$	
40.	$\vdash \neg\Box\neg a \to \Diamond a$	($\mathrm{C}_5(37)$)
41.	$\vdash \neg\Box\neg(p \vee q) \to \Diamond(p \vee q)$	(S(40))
	$\ldots$	
45.	$\vdash \neg\Box\neg p \vee \neg\Box\neg q \to \Diamond(p \vee q)$	($\mathrm{C}_1(36, 41)$)
	$\ldots$	
49.	$\vdash \Diamond a \to \neg\Box\neg a$	($\mathrm{C}_5(37)$)
50.	$\vdash \Diamond p \to \neg\Box\neg p$	(S(49))
51.	$\vdash \Diamond q \to \neg\Box\neg q$	(S(49))
52.	$\vdash (a \to c) \to ((b \to d) \to (a \vee b \to c \vee d))$	(T)
53.	$\vdash (\Diamond p \to \neg\Box\neg p) \to ((\Diamond q \to \neg\Box\neg q) \to (\Diamond p \vee \Diamond q \to \neg\Box\neg p \vee \neg\Box\neg q))$	(S(52))
54.	$\vdash (\Diamond q \to \neg\Box\neg q) \to (\Diamond p \vee \Diamond q \to \neg\Box\neg p \vee \neg\Box\neg q)$	(MP(50, 53))
55.	$\vdash \Diamond p \vee \Diamond q \to \neg\Box\neg p \vee \neg\Box\neg q$	(MP(51, 54))
	$\ldots$	
59.	$\vdash \Diamond p \vee \Diamond q \to \Diamond(p \vee q)$	($\mathrm{C}_1(55, 48)$)

$\square$

Claim 10: $\Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$ is K-provable.

Proof of the Claim.

1. $\vdash \neg p \to (\neg q \to \neg p \wedge \neg q)$	(T)
$\ldots$	
5. $\vdash \square\neg p \to \square(\neg q \to \neg p \vee \neg q)$	($C_2(1)$)
6. $\vdash \square(a \to b) \to (\square a \to \square b)$	(K)
7. $\vdash \square(\neg q \to \neg p \wedge \neg q) \to (\square\neg q \to \square(\neg p \wedge \neg q))$	(S(6))
$\ldots$	
11. $\vdash \square\neg p \to (\square\neg q \to \square(\neg p \wedge \neg q))$	($C_1(5,7)$)
$\ldots$	
14. $\vdash \square\neg p \wedge \square\neg q \to \square(\neg p \wedge \neg q)$	($C_6(11)$)
$\ldots$	
17. $\vdash \neg\square(\neg p \wedge \neg q) \to \neg(\square\neg p \wedge \square\neg q)$	($C_4(14)$)
18. $\vdash \neg(a \wedge b) \to \neg a \vee \neg b$	(T)
19. $\vdash \neg(\square\neg p \wedge \square\neg q) \to \neg\square\neg p \vee \neg\square\neg q$	(S(18))
$\ldots$	
23. $\vdash \neg\square(\neg p \wedge \neg q) \to \neg\square\neg p \vee \neg\square\neg q$	($C_1(17,19)$)
24. $\vdash (\neg a \wedge \neg b) \to \neg(a \vee b)$	(T)
25. $\vdash (\neg p \wedge \neg q) \to \neg(p \vee q)$	(S(24))
$\ldots$	
29. $\vdash \square(\neg p \wedge \neg q) \to \square\neg(p \vee q)$	($C_2(25)$)
$\ldots$	
32. $\vdash \neg\square\neg(p \vee q) \to \neg\square(\neg p \wedge \neg q)$	($C_4(29)$)
$\ldots$	
36. $\vdash \neg\square\neg(p \vee q) \to \neg\square\neg p \vee \neg\square\neg q$	($C_1(32,23)$)
37. $\vdash \Diamond a \leftrightarrow \neg\square\neg a$	(D)
$\ldots$	

$\dots$

41. $\vdash \neg\Box\neg a \to \Diamond a$ $\quad$ $(\mathrm{C}_5(37))$
42. $\vdash \neg\Box\neg p \to \Diamond p$ $\quad$ $(\mathrm{S}(41))$
43. $\vdash \neg\Box\neg q \to \Diamond q$ $\quad$ $(\mathrm{S}(41))$
44. $\vdash (a \to c) \to ((b \to d) \to (a \vee b \to c \vee d))$ $\quad$ (T)
45. $\vdash (\neg\Box\neg p \to \Diamond p) \to ((\neg\Box\neg q \to \Diamond q) \to (\neg\Box\neg p \vee \neg\Box\neg q \to \Diamond p \vee \Diamond q))$ $\quad$ $(\mathrm{S}(44))$
46. $\vdash (\neg\Box\neg q \to \Diamond q) \to (\neg\Box\neg p \vee \neg\Box\neg q \to \Diamond p \vee \Diamond q)$ $\quad$ $(\mathrm{MP}(41,\ 45))$
47. $\vdash \neg\Box\neg p \vee \neg\Box\neg q \to \Diamond p \vee \Diamond q$ $\quad$ $(\mathrm{MP}(42,\ 46))$

$\dots$

51. $\vdash \Diamond a \to \neg\Box\neg a$ $\quad$ $(\mathrm{C}_5(37))$
52. $\vdash \Diamond(p \vee q) \to \neg\Box\neg(p \vee q)$ $\quad$ $(\mathrm{S}(51))$

$\dots$

56. $\vdash \Diamond(p \vee q) \to (\neg\Box\neg p \vee \neg\Box\neg q)$ $\quad$ $(\mathrm{C}_1(52, 36))$

$\dots$

60. $\vdash \Diamond(p \vee q) \to \Diamond p \vee \Diamond q$ $\quad$ $(\mathrm{C}_1(56, 47))$

$\Box$

Claim 11: $\Diamond(p \vee q) \leftrightarrow \Diamond(p \vee \Diamond q)$ is K-provable.

Proof of the Claim.

$$\begin{array}{rll}
 & \dots & \\
59. & \vdash \Diamond p \vee \Diamond q \to \Diamond(p \vee q) & (\mathrm{C}_9) \\
 & \dots & \\
119. & \vdash \Diamond(p \vee q) \to \Diamond p \vee \Diamond q & (\mathrm{C}_{10}) \\
 & \dots & \\
123. & \vdash \Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q & (\mathrm{C}_7(2, 1))
\end{array}$$

$\Box$

This last claim concludes that the second proposition is K-provable as well. $\Box$