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Mathematical Logic for Computer Science

Lecture notes integrated with the book TODO

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Contents

Information and Contacts

Personal notes and summaries collected as part of the *Mathematical Logic for Computer Science* course offered by the degree in Computer Science of the University of Rome "La Sapienza".

Further information and notes can be found at the following link:
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Suggested prerequisites:

TODO

Licence:

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1.1 Introduction

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Definition 1.1: Languages and propositions

A **propositional language** is a — possibly infinite — set $\mathcal{L} = \{p_1, \ldots, p_n\}$, where each p_i is called **propositional variable**. Given a propositional language $\mathcal{L}$, the set of **propositions** over $\mathcal{L}$, denoted with $\mathsf{PROP}_{\mathcal{L}}$ is inductively defined as follows:

- each propositional variable in $\mathcal{L}$ is a proposition, i.e. $\mathcal{L} \subseteq \mathsf{PROP}_{\mathcal{L}}$
- if $A \in \mathsf{PROP}_{\mathcal{L}}$, then $\neg A \in \mathsf{PROP}_{\mathcal{L}}$
- if $A, B \in \mathsf{PROP}_{\mathcal{L}}$, then $(A \wedge B), (A \vee B), (A \to B), (A \leftrightarrow B) \in \mathsf{PROP}_{\mathcal{L}}$

In other words, the set of propositions over a language is the set of *formulas* that can be constructed from the initial variables of $\mathcal{L}$, by using the Boolean connectives. Note that a propositional language may have an infinite number of propositional variables, even *uncountably infinite*. For instance, the following is a valid propositional language

$$\mathcal{L} := \{p_r \mid r \in \mathbb{R}\}$$

However, since propositions are inductively constructed starting from variables, each proposition of any language $\mathcal{L}$ will still be defined over a *finite* number of propositional variables of $\mathcal{L}$.

Now that we provided a formal definition for languages and propositions, we are ready to discuss **assignments**.

Definition 1.2: Assignment

Given a propositional language $\mathcal{L}$, an **assignment** α is a function $\alpha : \mathcal{L} \to \{0, 1\}$ that assigns either 0 or 1 to all of $\mathcal{L}$'s propositional variable.

This definition can be inductively extended to **propositions** themselves: if A and B are two propositions, let $\widehat{\alpha} : \mathsf{PROP}_{\mathcal{L}} \to \{0, 1\}$ be a function such that

$$A \in \mathcal{L} \implies \widehat{\alpha}(A) = \alpha(A)$$

$$\widehat{\alpha}(\neg A) = \begin{cases} 1 & \widehat{\alpha}(A) = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\widehat{\alpha}(A \wedge B) = \begin{cases} 1 & \widehat{\alpha}(A) = \widehat{\alpha}(B) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\widehat{\alpha}(A \vee B) = \begin{cases} 1 & \widehat{\alpha}(A) = 1 \vee \widehat{\alpha}(B) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\widehat{\alpha}(A \to B) = \begin{cases} 0 & \widehat{\alpha}(A) = 1 \wedge \widehat{\alpha}(B) = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\widehat{\alpha}(A \leftrightarrow B) = \begin{cases} 1 & \widehat{\alpha}(A) = \widehat{\alpha}(B) \\ 0 & \text{otherwise} \end{cases}$$

Since it can be proven that the extension $\widehat{\alpha}$ of α is unique, we will refer to $\widehat{\alpha}$ as α directly. Two propositions A and B are said to be **equivalent** — written as $A \equiv B$ — if and only if $\forall \alpha \quad \alpha(A) = \alpha(B)$.

Definition 1.3: Satisfiability

A proposition A is said to be **satisfiable** if there exists an assignment α of its propositional variables such that $\alpha(A) = 1$. If there is no assignment that satisfies A, A is said to be **unsatisfiable**, and if A is satisfied for any assignment α, then A is said to be a **tautology**.

We will denote with SAT, UNSAT and TAUT respectively the sets of all satisfiable propositions, all unsatisfiable propositions and all tautologies.

Using symbols, we have that

- $A \in \mathsf{SAT} \iff \exists \alpha \quad \alpha(A) = 1$
- $A \in \mathsf{UNSAT} \iff \nexists \alpha \quad \alpha(A) = 1 \iff \forall \alpha \quad \alpha(A) = 0$

- $A \in \mathsf{TAUT} \iff \forall\alpha \quad \alpha(A) = 1 \iff \nexists\alpha \quad \alpha(A) = 0$

The concept of satisfiability is strictly related to the concept of **logical consequence**, which is defined as follows.

Definition 1.4: Logical consequence

Given the propositions $A_1, \ldots, A_n, A$, we say that A is a **logical consequence** of $A_1, \ldots, A_n$ if whenever $A_1, \ldots, A_n$ are true, A is also true. We will indicate this concept as follows:

$$A_1, \ldots, A_n \models A$$

From its definition, the concept of logical consequence can be alternatively be expressed in terms of *unsatisfiability* and *tautology*.

Theorem 1.1

Given the formulas $A_1, \ldots, A_n, A$, the following statements are equivalent:

- $A_1, \ldots, A_n \models A$
- $(A_1 \wedge \ldots \wedge A_n \rightarrow A) \in \mathsf{TAUT}$
- $(A_1 \wedge \ldots \wedge A_n \wedge A) \in \mathsf{UNSAT}$

1.1.1 Theories

Definition 1.5: Theory

A **theory** is a — possibly infinite — set of propositions (or hypothesis).

As a natural extension of the *satisfiability* property previously discussed, a theory T will be said to be **satisfiable** — written as $T \in \mathsf{SAT}$ if and only if

$$\exists\alpha \quad \alpha(T) = 1$$

which is equivalent of saying that

$$\exists\alpha \quad \forall F \in T \quad \alpha(F) = 1$$

note that the assignment α must be the same for all the propositions F of T.

Additionally, for infinite theories we can define another property.

Definition 1.6: Finite satisfiability

An infinite theory T is said to be **finitely satisfiable** if and only if

$$\forall T' \subset T \text{ finite} \quad T' \in \mathsf{SAT}$$

We will denote with FINSAT the set of all finitely satisfiable theories.

However, the following theorem will prove that *satisfiability* and *finite satisfiability* are actually **equivalent**.

Theorem 1.2: Compactness theorem

Given an infinite theory T, it holds that

$$T \in \mathsf{SAT} \iff T \in \mathsf{FINSAT}$$

Proof. In this proof we will assume that the propositions of the infinite theory T are *countably infinite*, however in its general form this theorem can be proved even without this assumption.

Since the direct implication of this statement is trivially true by definition, we just need to prove the converse implication.

Claim: Given a theory $T \in \mathsf{FINSAT}$, and a proposition A, it must hold that $T \cup \{A\} \in \mathsf{FINSAT}$ or $T \cup \{\neg A\} \in \mathsf{FINSAT}$.

Proof of the Claim. By way of contradiction, assume that $T \cup \{A\}, T \cup \{\neg A\} \notin \mathsf{FINSAT}$.

By definition of finite satisfiability, if $T \cup \{A\} \notin \mathsf{FINSAT}$, then there must exist a *finite* sub-theory $T_0 \subset T \cup \{A\}$ such that $T_0 \in \mathsf{UNSAT}$. Note that $T \in \mathsf{FINSAT}$, therefore if $T \cup \{A\} \notin \mathsf{FINSAT}$ then it must be that $A \in T_0$. Let $\widehat{T_0}$ be the theory such that $T_0 := \widehat{T_0} \cup \{A\}$; then

$$T_0 := \widehat{T_0} \cup \{A\} \in \mathsf{UNSAT} \iff \forall \alpha \quad \alpha(T_0) = 0$$

which implies that

$$\forall \alpha \quad \alpha(\widehat{T_0}) = 1 \implies \alpha(A) = 0$$

Analogously, we can apply the same reasoning for $T \cup \{\neg A\}$, and we get that there must exist a *finite* sub-theory $T_1 \subset T \cup \{\neg A\}$ such that $T_1 := \widehat{T_1} \cup \{\neg A\} \in \mathsf{UNSAT}$, which implies that

$$\forall \alpha \quad \alpha(\widehat{T_1}) = 1 \implies \alpha(\neg A) = 0$$

Lastly, since $\widehat{T_0} \cup \widehat{T_1} \subset T \in \mathsf{FINSAT}$, by finite satisfiability of T there must exist an assignment α such that $\alpha(\widehat{T_0} \cup \widehat{T_1}) = 1$, and therefore $\alpha(\widehat{T_0}) = \alpha(\widehat{T_1}) = 1$. However, for the previous observations this implies that $\alpha(A) = \alpha(\neg A) = 0 \; \lightning$. $\square$

Since we are assuming that the propositions of T are *countably infinite*, and the number of variables in any proposition is finite by definition, we can fix an enumeration $p_1, p_2, p_3, \ldots$ on the — possibly infinite — propositional variables of T. Given this enumeration, define the following *chain* of sub-theories:

- $T_0 := T$
- $T_{i+1} := \begin{cases} T_i \cup \{p_i\} & T_i \cup \{p_i\} \in \mathsf{FINSAT} \\ T_i \cup \{\neg p_i\} & T_i \cup \{\neg p_i\} \in \mathsf{FINSAT} \end{cases}$

and note that, by definition, clearly

$$T =: T_0 \subseteq T_1 \subseteq T_2 \subseteq \ldots$$

Moreover, let

$$T^* := \bigcup_{i \in \mathbb{N}} T_i$$

and note that since $\forall i \quad T_i \in \mathsf{FINSAT}$ by definition, then it must be that $T^* \in \mathsf{FINSAT}$ as well, as T^* is a chain defined *only* by inclusions of FINSAT theories.

Now, consider the following assignment:

$$\alpha^* : \{p_1, p_2, \ldots\} \to \{0, 1\} : p_i \mapsto \begin{cases} 1 & p_i \in T^* \\ 0 & \neg p_i \in T^* \end{cases}$$

Note that this assignment is well defined, because by construction of T^* only one between $p_i \in T^*$ and $\neg p_i \in T^*$ can hold.

Claim: $\alpha^*(T) = 1$.

Proof of the Claim. Let $A \in T$, and let $p_{i_1}, \ldots, p_{i_k}$ the be the propositional variables that appear in A. Then, for each $j \in [k]$ let

$$p^*_{i_j} := \begin{cases} p_{i_j} & p_{i_j} \in T^* \\ \neg p_{i_j} & \neg p_{i_j} \in T^* \end{cases}$$

and consider the set $\left\{A, p^*_{i_1}, \ldots, p^*_{i_k}\right\}$. Clearly, this is a finite subset of T^* , hence $T^* \in$ FINSAT implies that there must exist an assignment β_A that satisfies this set, i.e. non sono d'accordo

$$\beta_A(A) = \beta_A\left(p^*_{i_1}\right) = \ldots = \beta_A\left(p^*_{i_k}\right) = 1$$

Note that, for each $j \in [k]$, it holds that

$$p_{i_j} \in T^* \implies p^*_{i_j} = p_{i_j} \wedge \alpha^*\left(p_{i_j}\right) = 1$$

and $1 = \beta_A(p^*_{i_j}) = \beta_A\left(p_{i_j}\right)$; analogously, it holds that

$$p_{i_j} \notin T^* \implies p^*_{i_j} = \neg p_{i_j} \wedge \alpha^*\left(p_{i_j}\right) = 0$$

and $1 = \beta_A(p^*_{i_j}) = \beta_A\left(\neg p_{i_j}\right) = \neg \beta_A\left(p_{i_j}\right) \implies \beta_A\left(p_{i_j}\right) = 0$. This proves that $\alpha^* \equiv \beta_A$ for all of A's variables, therefore it must also be true that $\alpha^*(A) = \beta_A(A)$. $\square$

not enough for whole T? missing last part

This claim proves that there exists an assignment α^* that satisfies T, hence $T \in \mathsf{SAT}$, concluding the proof. $\square$

The statement of this theorem is equivalent to the following one.

Corollary 1.1

Given an infinite theory T, and a proposition A, it holds that

$$T \models A \iff \exists T' \subset T \text{ finite} \quad T' \models A$$

Proof. placeholder □

prove their equivalence

The compactness theorem can be proven to be equivalent to a special case of Kőnig's lemma [1], which states the following.

Lemma 1.1: Kőnig's lemma (special case)

Every infinite tree contains either a vertex of infinite degree, or an infinite path.

Bibliography

[1] Dénes König. "Über eine Schlussweise aus dem Endlichen ins Unendliche". In: *Acta Sci. Math.(Szeged)* 3.2-3 (1927), pp. 121–130.