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# Mathematical Logic for Computer Science

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Lecture notes integrated with the book TODO

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# Information and Contacts

Personal notes and summaries collected as part of the *Mathematical Logic for Computer Science* course offered by the degree in Computer Science of the University of Rome "La Sapienza".

Further information and notes can be found at the following link:

<https://github.com/aflaag-notes>. Anyone can feel free to report inaccuracies, improvements or requests through the Issue system provided by GitHub itself or by contacting the author privately:

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The notes are constantly being updated, so please check if the changes have already been made in the most recent version.

## Suggested prerequisites:

TODO

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# 1

## Exercises

### 1.1 Homework 1

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#### Definition 1.1: Languages and propositions

A **propositional language** is a — possibly infinite — set  $\mathcal{L} = \{p_1, \dots, p_n\}$ , where each  $p_i$  is called **propositional variable**. Given a propositional language  $\mathcal{L}$ , the set of **propositions** over  $\mathcal{L}$ , denoted with  $\text{PROP}_{\mathcal{L}}$  is inductively defined as follows:

- each propositional variable in  $\mathcal{L}$  is a proposition, i.e.  $\mathcal{L} \subseteq \text{PROP}_{\mathcal{L}}$
- if  $A \in \text{PROP}_{\mathcal{L}}$ , then  $\neg A \in \text{PROP}_{\mathcal{L}}$
- if  $A, B \in \text{PROP}_{\mathcal{L}}$ , then  $(A \wedge B), (A \vee B), (A \rightarrow B), (A \leftrightarrow B) \in \text{PROP}_{\mathcal{L}}$

In other words, the set of propositions over a language is the set of *formulas* that can be constructed from the initial variables of  $\mathcal{L}$ , by using the Boolean connectives. Note that a propositional language may have an infinite number of propositional variables, even *uncountably infinite*. For instance, the following is a valid propositional language

$$\mathcal{L} := \{p_r \mid r \in \mathbb{R}\}$$

However, since propositions are inductively constructed starting from variables, each proposition of any language  $\mathcal{L}$  will still be defined over a *finite* number of propositional variables of  $\mathcal{L}$ .

Now that we provided a formal definition for languages and propositions, we are ready to discuss **assignments**.

**Definition 1.2: Assignment**

Given a propositional language  $\mathcal{L}$ , an **assignment**  $\alpha$  is a function  $\alpha : \mathcal{L} \rightarrow \{0, 1\}$  that assigns either 0 or 1 to all of  $\mathcal{L}$ 's propositional variable.

This definition can be inductively extended to **propositions** themselves: if  $A$  and  $B$  are two propositions, let  $\hat{\alpha} : \text{PROP}_{\mathcal{L}} \rightarrow \{0, 1\}$  be a function such that

$$A \in \mathcal{L} \implies \hat{\alpha}(A) = \alpha(A)$$

$$\hat{\alpha}(\neg A) = \begin{cases} 1 & \hat{\alpha}(A) = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{\alpha}(A \wedge B) = \begin{cases} 1 & \hat{\alpha}(A) = \hat{\alpha}(B) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{\alpha}(A \vee B) = \begin{cases} 1 & \hat{\alpha}(A) = 1 \vee \hat{\alpha}(B) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{\alpha}(A \rightarrow B) = \begin{cases} 0 & \hat{\alpha}(A) = 1 \wedge \hat{\alpha}(B) = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{\alpha}(A \leftrightarrow B) = \begin{cases} 1 & \hat{\alpha}(A) = \hat{\alpha}(B) \\ 0 & \text{otherwise} \end{cases}$$

Since it can be proven that the extension  $\hat{\alpha}$  of  $\alpha$  is unique, we will refer to  $\hat{\alpha}$  as  $\alpha$  directly. Two propositions  $A$  and  $B$  are said to be **equivalent** — written as  $A \equiv B$  — if and only if  $\forall \alpha \quad \alpha(A) = \alpha(B)$ .

**Definition 1.3: Satisfiability**

A proposition  $A$  is said to be **satisfiable** if there exists an assignment  $\alpha$  of its propositional variables such that  $\alpha(A) = 1$ . If there is no assignment that satisfies  $A$ ,  $A$  is said to be **unsatisfiable**, and if  $A$  is satisfied for any assignment  $\alpha$ , then  $A$  is said to be a **tautology**.

We will denote with SAT, UNSAT and TAUT respectively the sets of all satisfiable propositions, all unsatisfiable propositions and all tautologies.

Using symbols, we have that

- $A \in \text{SAT} \iff \exists \alpha \quad \alpha(A) = 1$
- $A \in \text{UNSAT} \iff \nexists \alpha \quad \alpha(A) = 1 \iff \forall \alpha \quad \alpha(A) = 0$

- $A \in \text{TAUT} \iff \forall \alpha \quad \alpha(A) = 1 \iff \nexists \alpha \quad \alpha(A) = 0$

The concept of satisfiability is strictly related to the concept of **logical consequence**, which is defined as follows.

**Definition 1.4: Logical consequence**

Given the propositions  $A_1, \dots, A_n, A$ , we say that  $A$  is a **logical consequence** of  $A_1, \dots, A_n$  if whenever  $A_1, \dots, A_n$  are true,  $A$  is also true. We will indicate this concept as follows:

$$A_1, \dots, A_n \models A$$

From its definition, the concept of logical consequence can be alternatively be expressed in terms of *unsatisfiability* and *tautology*.

**Theorem 1.1**

Given the formulas  $A_1, \dots, A_n, A$ , the following statements are equivalent:

- $A_1, \dots, A_n \models A$
- $(A_1 \wedge \dots \wedge A_n \rightarrow A) \in \text{TAUT}$
- $(A_1 \wedge \dots \wedge A_n \wedge A) \in \text{UNSAT}$

### 1.1.1 Theories

**Definition 1.5: Theory**

Given a language  $\mathcal{L}$ , a **theory**  $T$  over  $\mathcal{L}$  is a — finite or infinite — set of propositions defined on  $\mathcal{L}$ , i.e.  $T \subseteq \text{PROP}_{\mathcal{L}}$ .

As a natural extension of the *satisfiability* property previously discussed, a theory  $T$  will be said to be **satisfiable** — written as  $T \in \text{SAT}$  if and only if

$$\exists \alpha \quad \alpha(T) = 1$$

which is equivalent of saying that

$$\exists \alpha \quad \forall F \in T \quad \alpha(F) = 1$$

note that the assignment  $\alpha$  must be the same for all the propositions  $F$  of  $T$ .

Additionally, for infinite theories we can define another property.

**Definition 1.6: Finite satisfiability**

An infinite theory  $T$  is said to be **finitely satisfiable** if and only if

$$\forall T' \subset T \text{ finite} \quad T' \in \text{SAT}$$

We will denote with **FINSAT** the set of all finitely satisfiable theories.

However, the following theorem will prove that *satisfiability* and *finite satisfiability* are actually **equivalent**.

**Theorem 1.2: Compactness theorem (1st version)**

Given an infinite theory  $T$ , it holds that

$$T \in \text{SAT} \iff T \in \text{FINSAT}$$

*Proof.* In this proof we will assume that the propositions of the infinite theory  $T$  are *countably infinite*, however in its general form this theorem can be proved even without this assumption.

Since the direct implication of this statement is trivially true by definition, we just need to prove the converse implication.

**Claim:** Given a theory  $T \in \text{FINSAT}$ , and a proposition  $A$ , it must hold that  $T \cup \{A\} \in \text{FINSAT}$  or  $T \cup \{\neg A\} \in \text{FINSAT}$ .

*Proof of the Claim.* By way of contradiction, assume that  $T \cup \{A\}, T \cup \{\neg A\} \notin \text{FINSAT}$ .

By definition of finite satisfiability, if  $T \cup \{A\} \notin \text{FINSAT}$ , then there must exist a *finite* sub-theory  $T_0 \subset T \cup \{A\}$  such that  $T_0 \in \text{UNSAT}$ . Note that  $T \in \text{FINSAT}$ , therefore if  $T \cup \{A\} \notin \text{FINSAT}$  then it must be that  $A \in T_0$ . Let  $\widehat{T}_0$  be the theory such that  $T_0 := \widehat{T}_0 \cup \{A\}$ ; then

$$T_0 := \widehat{T}_0 \cup \{A\} \in \text{UNSAT} \iff \forall \alpha \quad \alpha(T_0) = 0$$

which implies that

$$\forall \alpha \quad \alpha(\widehat{T}_0) = 1 \implies \alpha(A) = 0$$

Analogously, we can apply the same reasoning for  $T \cup \{\neg A\}$ , and we get that there must exist a *finite* sub-theory  $T_1 \subset T \cup \{\neg A\}$  such that  $T_1 := \widehat{T}_1 \cup \{\neg A\} \in \text{UNSAT}$ , which implies that

$$\forall \alpha \quad \alpha(\widehat{T}_1) = 1 \implies \alpha(\neg A) = 0$$

Lastly, since  $\widehat{T}_0 \cup \widehat{T}_1 \subset T \in \text{FINSAT}$ , by finite satisfiability of  $T$  there must exist an assignment  $\alpha$  such that  $\alpha(\widehat{T}_0 \cup \widehat{T}_1) = 1$ , and therefore  $\alpha(\widehat{T}_0) = \alpha(\widehat{T}_1) = 1$ . However, for the previous observations this implies that  $\alpha(A) = \alpha(\neg A) = 0 \nmid$ .  $\square$

Since we are assuming that the propositions of  $T$  are *countably infinite*, and the number of variables in any proposition is finite by definition, we can fix an enumeration  $p_1, p_2, p_3, \dots$  on the — possibly infinite — propositional variables of  $T$ . Given this enumeration, define the following *chain* of sub-theories:

- $T_0 := T$
- $T_{i+1} := \begin{cases} T_i \cup \{p_i\} & T_i \cup \{p_i\} \in \text{FINSAT} \\ T_i \cup \{\neg p_i\} & T_i \cup \{\neg p_i\} \in \text{FINSAT} \end{cases}$

and note that, by definition, clearly

$$T =: T_0 \subseteq T_1 \subseteq T_2 \subseteq \dots$$

Moreover, let

$$T^* := \bigcup_{i \in \mathbb{N}} T_i$$

and note that since  $\forall i \quad T_i \in \text{FINSAT}$  by definition, then it must be that  $T^* \in \text{FINSAT}$  as well, as  $T^*$  is a chain defined *only* by inclusions of FINSAT theories.

Now, consider the following assignment:

$$\alpha^* : \{p_1, p_2, \dots\} \rightarrow \{0, 1\} : p_i \mapsto \begin{cases} 1 & p_i \in T^* \\ 0 & \neg p_i \in T^* \end{cases}$$

Note that this assignment is well defined, because by construction of  $T^*$  only one between  $p_i \in T^*$  and  $\neg p_i \in T^*$  can hold.

**Claim:**  $\alpha^*(T) = 1$ .

*Proof of the Claim.* Let  $A \in T$ , and let  $p_{i_1}, \dots, p_{i_k}$  be the propositional variables that appear in  $A$ . Then, for each  $j \in [k]$  let

$$p_{i_j}^* := \begin{cases} p_{i_j} & p_{i_j} \in T^* \\ \neg p_{i_j} & \neg p_{i_j} \in T^* \end{cases}$$

and consider the set  $\{A, p_{i_1}^*, \dots, p_{i_k}^*\}$ . Clearly, this is a finite subset of  $T^*$  — by definition of the various  $T_0, T_1, T_2, \dots$  that define  $T^*$  — hence  $T^* \in \text{FINSAT}$  implies that there must exist an assignment  $\beta_A$  that satisfies this set, i.e.

$$\beta_A(A) = \beta_A(p_{i_1}^*) = \dots = \beta_A(p_{i_k}^*) = 1$$

Note that, for each  $j \in [k]$ , it holds that

$$p_{i_j} \in T^* \implies p_{i_j}^* = p_{i_j} \wedge \alpha^*(p_{i_j}) = 1$$

and  $1 = \beta_A(p_{i_j}^*) = \beta_A(p_{i_j})$ ; analogously, it holds that

$$\neg p_{i_j} \in T^* \implies p_{i_j}^* = \neg p_{i_j} \wedge \alpha^*(p_{i_j}) = 0$$

and  $1 = \beta_A(p_{i_j}^*) = \beta_A(\neg p_{i_j}) = \neg \beta_A(p_{i_j}) \implies \beta_A(p_{i_j}) = 0$ . This proves that  $\alpha^* \equiv \beta_A$  for all of  $A$ 's variables, therefore it must also be true that  $\alpha^*(A) = \beta_A(A)$ .



Hence, this shows that for any proposition  $A \in T$  there is an assignment  $\beta_A$ , which satisfies  $A$ , that sets  $A$ 's variables as  $\alpha^*$  does. Therefore, since this will be true for any of the  $T$ 's propositions,  $\alpha^*$  satisfies  $T$ .  $\square$

This claim proves that there exists an assignment  $\alpha^*$  that satisfies  $T$ , hence  $T \in \text{SAT}$ , concluding the proof.  $\square$

The statement of this theorem is equivalent to the following one.

**Theorem 1.3: Compactness theorem (2nd version)**

Given an infinite theory  $T$ , and a proposition  $A$ , it holds that

$$T \models A \iff \exists T' \subset T \text{ finite } T' \models A$$

*Proof.*

*Direct implication.* We will prove the equivalence of the two formulations by assuming that the first version implies the second version of the theorem, and vice versa. First, notice that if there exist a  $T' \subset T$  finite sub-theory of  $T$  such that  $T' \models A$  for some proposition  $A$ , then it is trivially true that  $T \models A$  by definition of logical consequence. Hence, we just need to prove the direct implication of the second version of the theorem, assuming that the first version is true.

By way of contradiction, assume that  $T \models A$  and for each  $T' \subset T$  finite sub-theory of  $T$ ,  $T' \not\models A$ . By definition of logical consequence, this implies that

$$\forall T' \subset T \text{ finite } \exists \alpha \quad \alpha(T') = 1 \wedge \alpha(A) = 0$$

or, equivalently,  $\alpha(\neg A) = 1$ . This implies that

$$\forall T' \subset T \text{ finite } \exists \alpha \quad \alpha(T' \cup \{\neg A\}) = 1$$

which means that any finite subset of  $T \cup \{\neg A\}$  is SAT (recall that  $T' \subset T$ ). Hence, by definition  $T \cup \{\neg A\} \in \text{FINSAT}$ , and since we are assuming the first version of the theorem, we have that

$$T \cup \{\neg A\} \in \text{FINSAT} \iff T \cup \{\neg A\} \in \text{SAT}$$

which implies that  $T \not\models A$  by definition of logical consequence.

*Converse implication.* Consider a theory  $T$  and a proposition  $A$ ; by the second version of the theorem we know that  $T \models A$  if and only if there is a finite sub-theory  $T' \subset T$  such that  $T' \models A$ . Now, suppose that  $T \in \text{UNSAT}$ ; therefore, by definition we have that  $T \models 0$ , which implies that

$$T \models 0 \iff \exists T' \subset T \text{ finite } T' \models 0$$

but again,  $T' \models 0$  means that  $T' \in \text{UNSAT}$ . Hence, we proved that

$$T \in \text{UNSAT} \iff \exists T' \subset T \text{ finite } T' \in \text{UNSAT} \iff T \notin \text{FINSAT}$$

which is the contrapositive of the first version of the theorem.

□

The compactness theorem can be proven to be equivalent to a special case of [König's lemma](#) [Kön27], which states the following.

**Lemma 1.1: König's lemma (special case)**

Every infinite tree contains either a vertex of infinite degree, or an infinite path.

**Exercise 1.4** Let  $\mathcal{L} = \{E(x, y)\}$  be the language of graphs.

1. For each fixed  $n \in \mathbb{N}$ , write a sentence  $C_n$  such that for any graph  $\mathcal{G}$ ,  $\mathcal{G} \models C_n$  if and only if  $\mathcal{G}$  contains a cycle of length  $n$ .
2. Prove using Compactness that the property of being *a cycle* is not expressible by a theory in  $\mathcal{L}$  over the class of graphs.

*Solution.* Let  $\mathcal{L} = \{E(x, y)\}$  be the language of graphs.

1. The property “ $\mathcal{G}$  contains a cycle of length  $n$ ” can be written as follows

$$C_n := \exists x_1 \dots \exists x_n \left( \bigwedge_{\substack{1 \leq i, j \leq n \\ i \neq j}} \neg(x_i = x_j) \right) \wedge \left( \bigwedge_{1 \leq i \leq n-1} E(x_i, x_{i+1}) \wedge E(x_n, x_1) \right)$$

In fact, the first conjunction implies that  $x_1, \dots, x_n$  are *distinct*, and the second conjunction describes the existence of the  $n$ -long *cycle* itself.

2. Consider the property  $P_n :=$  “ $\mathcal{G}$  is *a cycle* of length  $n$ ”. This property can be expressed by *extending*  $C_n$  as follows:

$$\begin{aligned} V_n &:= \forall y \bigvee_{1 \leq j \leq n} (y = x_j) \\ E_n &:= \bigwedge_{1 \leq i \leq n-1} \bigwedge_{\substack{1 \leq j \leq n: \\ j \neq i+1}} \neg E(x_i, x_j) \wedge \bigwedge_{2 \leq j \leq n} \neg E(x_n, x_j) \\ C'_n &:= \exists x_1 \dots \exists x_n \quad C_n \wedge V_n \wedge E_n \end{aligned}$$

where we have that

- $V_n$  ensures that  $\mathcal{G}$  has *exactly*  $n$  vertices
- $E_n$  ensures that the only edges present in  $\mathcal{G}$  are the ones that describe the cycle graph of  $n$  vertices
- $C'_n$  describes our property  $P_n$

Now, consider the property  $P :=$  “ $\mathcal{G}$  is *a cycle*”, and in particular  $\neg P :=$  “ $\mathcal{G}$  is not *a cycle*”. We observe that we can build the following infinite theory

$$T^{\neg P} := \{\neg C'_n \mid n \in \mathbb{N}_{\geq 3}\}$$

for which it is easy to see that

$$\mathcal{G} \models \neg P \iff \neg P(\mathcal{G}) \text{ holds}$$

meaning that  $\neg P$  is expressible through  $T^{\neg P}$ .

**Claim:**  $T^{\neg P} \in \text{FINSAT}$ .

*Proof of the Claim.* Fix  $T_0 \subseteq_{fin} T^{\neg P}$ . We observe that  $T_0 := \{\neg C'_{i_1}, \dots, \neg C'_{i_k}\}$

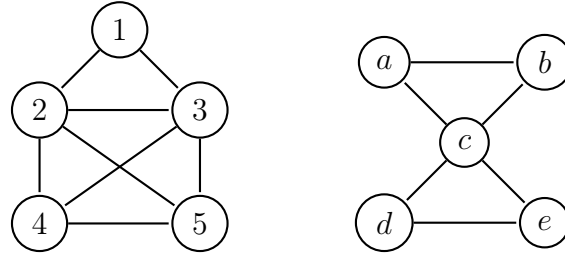
for some  $i_1, \dots, i_k \in \mathbb{N}$ . Now, if we consider  $i^* := \max_{j \in [k]} i_j$ , then the cycle graph that has  $i^* + 1$  vertices is clearly a structure that satisfies  $T_0$ .  $\square$

**Claim:**  $P$  is not expressible by a theory in  $\mathcal{L}$  over the class of graphs.

*Proof of the Claim.* By way of contradiction, suppose that  $P$  is expressible, i.e. there is a theory  $T^P$  for which  $P$  can be expressed. Then, consider the theory  $T := T^P \cup T^{\neg P}$ . By the previous claim, we have that  $T \in \text{FINSAT}$ , and by Compactness this is true if and only if  $T \in \text{SAT}$ . However, this is a contradiction, because a graph cannot be and not be a cycle at the same time.  $\square$

Finally, this last claim concludes the proof.  $\square$

**Exercise 2.1** Consider the following two structures  $\mathcal{G}_1$  and  $\mathcal{G}_2$  for the languages of graphs:



Write at least two sentences distinguishing the two structures. Discuss the EF-game played on these structures: for what  $k$  can the Duplicator win the  $k$ -rounds game? For what  $k$  can the Spoiler win?

*Solution.* Some properties that can distinguish these two structures are the following:

1. “ $\mathcal{G}$  contains a vertex of degree 3”, which is represented by the following sentence of rank 5

$$\exists x_1 \exists x_2 \exists x_3 \exists x_4 \left( \bigwedge_{\substack{1 \leq i, j \leq 4 \\ i \neq j}} \neg(x_i = x_j) \right) \wedge \left( \bigwedge_{2 \leq i \leq 4} E(x_1, x_i) \right) \wedge \left( \forall y \quad \neg E(x_1, y) \vee \bigvee_{2 \leq j \leq 4} (y = x_j) \right)$$

2. “ $\mathcal{G}$  contains edges as  $\mathcal{G}_1$ ”, which is represented by the following sentence of rank 5

$$\begin{aligned} \exists x_1 \exists x_2 \exists x_3 \exists x_4 \exists x_5 \quad & E(x_1, x_2) \wedge E(x_1, x_3) \wedge \\ & E(x_2, x_3) \wedge E(x_2, x_4) \wedge E(x_2, x_5) \wedge \\ & E(x_3, x_4) \wedge E(x_4, x_5) \wedge \\ & E(x_4, x_5) \end{aligned}$$

we observe that the edges of  $\mathcal{G}_2$  are not sufficient to distinguish the two sentences, because  $\mathcal{G}_2$  is a subgraph of  $\mathcal{G}_1$

3. “ $\mathcal{G}$  contains a cycle of length 5”, which is represented by  $C_5$  of the previous exercise, and has rank 5
4. “ $\mathcal{G}$  contains a cycle of length 4”, which is represented by  $C_4$  of the previous exercise, and has rank 4
5. “ $\mathcal{G}$  contains  $K_4$  as subgraph”, which is represented by the following sentence having rank 4

$$\exists x_1 \exists x_2 \exists x_3 \exists x_4 \left( \bigwedge_{\substack{1 \leq i, j \leq 4 \\ i \neq j}} \neg(x_i = x_j) \right) \wedge \left( \bigwedge_{\substack{1 \leq i, j \leq 4 \\ i \neq j}} E(x_i, x_j) \right)$$

These sentences *may seem* to suggest that the two structures are 3-equivalent, meaning that there is no sentence of rank 3 that can distinguish  $\mathcal{G}_1$  from  $\mathcal{G}_2$ . For now, let's focus on proving that they are *at least* 2-equivalent.

**Claim:** The Duplicator wins  $G_2(\mathcal{G}_1, \mathcal{G}_2)$ .

*Proof of the Claim.* Let  $s_i$  and  $d_i$  be the  $i$ -th nodes chosen by the Spoiler and the Duplicator, respectively. Then, we can define the following strategy for the Duplicator:

- if  $s_1 \in \{1, 4, 5\}$ , then the Duplicator chooses  $d_1 \in \{a, b, d, e\}$ , otherwise if  $s_1 \in \{2, 3\}$  then  $d_1 = c$
- similarly, if  $s_1 \in \{a, b, d, e\}$ , then the Duplicator chooses  $d_1 \in \{1, 4, 5\}$ , otherwise if  $s_1 = c$  then  $d_1 \in \{2, 3\}$

Then, no matter the choice of  $s_2$ , the Duplicator can always answer with a node  $d_2$  that preserves the partial isomorphism, in fact:

- if  $s_2 \sim s_1$ , it is guaranteed that there is a vertex  $d_2$  in the other structure such that  $d_2 \sim d_1$  because  $\delta(\mathcal{G}_1) = \delta(\mathcal{G}_2) = 2$  — and the same argument applies if  $s_2 \sim d_1$  for finding a vertex  $d_2 \sim s_1$
- if  $s_2 \not\sim s_1$ , the strategy that we provided for the Duplicator guarantees that there exists at least one vertex  $d_2$  in the other structure such that  $d_2 \not\sim d_1$  — and the same argument applies if  $s_2 \not\sim d_1$  for finding a vertex  $d_2 \not\sim s_1$

Thus, the Duplicator has a strategy to always win at least 2 rounds, therefore the Duplicator wins  $G_2(\mathcal{G}_1, \mathcal{G}_2)$  by Ehrenfeucht's theorem.  $\square$

Now that we proved that  $\mathcal{G}_1 \equiv_2 \mathcal{G}_2$ , is it true that they are also 3-equivalent? Unfortunately, the following claim proves that this is indeed false.

**Claim:** The Spoiler wins  $G_3(\mathcal{G}_1, \mathcal{G}_2)$ .

*Proof of the Claim.* The following is a strategy that guarantees the Spoiler to win in 3 rounds:

- let  $s_1 \in \{4, 5\}$

- by the previous claim, we know that the strategy for the Duplicator to win at least 2 rounds is to choose  $d_1 \in \{a, b, d, e\}$ , thus we may assume that  $d_1 \neq c$
- now, let  $s_2 = 1$
- to preserve the partial isomorphism, we observe that
  - if  $d_1 \in \{a, b\}$ , then  $d_2 \in \{d, e\}$
  - if  $d_1 \in \{d, e\}$ , then  $d_2 \in \{a, b\}$
- now, it suffices for the Spoiler to choose  $s_3$  in  $\mathcal{G}_2$  such that  $s_3 \sim d_2$  and  $s_3 \neq c$ : by construction of  $\mathcal{G}_2$ , we see that  $s_3 \approx d_1$ , but all the vertices in  $\{2, 3, 5\}$  are adjacent to  $s_1$ , which would violate the partial isomorphism

□

In fact, we can actually find a property that distinguishes  $\mathcal{G}_1$  from  $\mathcal{G}_2$  which can be written through a sentence of rank 3: “there are two vertices  $x_1$  and  $x_2$  of  $\mathcal{G}$  such that for each third vertex  $x_3$  there is a  $K_3$  as subgraph of  $\mathcal{G}$  such that  $V(K_3) = \{x_1, x_2, x_3\}$ ”

$$\exists x_1 \exists x_2 \forall x_3 \left( \bigwedge_{\substack{1 \leq i, j \leq 3 \\ i \neq j}} \neg(x_i = x_j) \right) \wedge E(x_1, x_2) \wedge E(x_2, x_3) \wedge E(x_3, x_1)$$

Let  $x_1, x_2$  and  $x_3$  be the three chosen vertices — and we may assume that  $x_1 \sim x_2$  otherwise the sentence is trivially unsatisfied. Then, we observe that

- in  $\mathcal{G}_1$  if  $\{x_1, x_2\} = \{2, 3\}$ , then for any other vertex  $x_3 \in \{1, 4, 5\}$  we can always find a  $K_3$  having  $x_1, x_2$  and  $x_3$  as its vertices
- in  $\mathcal{G}_2$  we have two cases
  - if  $\{x_1, x_2\} \subseteq \{a, b, c\}$ , the property is unsatisfied for  $x_3 \in \{d, e\}$
  - if  $\{x_1, x_2\} \subseteq \{c, d, e\}$ , the property is unsatisfied for  $x_3 \in \{a, b\}$

In conclusion, we have that  $\mathcal{G}_1 \equiv_2 \mathcal{G}_2$ , and that  $\mathcal{G}_1 \not\equiv_3 \mathcal{G}_2$ .

□

**Exercise 1.1** Let  $(W, R)$  be a *quasi-order*; that is, assume that  $R$  is transitive and reflexive. Define the binary relation  $\sim$  on  $W$  by putting  $s \sim t \iff R(s, t) \wedge R(t, s)$ .

(a) Show that  $\sim$  is an equivalence relation.

Let  $[s]$  denote the equivalence class of  $s$  under this relation, and define the following relation on the collection of equivalence classes:  $[s] \leq [t] \iff R(s, t)$ .

(b) Show that this relation is well-defined.

(c) Show that  $\leq$  is a partial order.

*Solution.* We prove the statements as follows.

(a) To prove that  $\sim$  is an equivalence relation, it suffices to show that  $\sim$  has the following properties:

- *reflexivity*:  $\forall s \in W \quad R(s, s)$  by reflexivity of  $R$ , therefore  $s \sim s$
- *symmetry*:  $\forall s, t \in W \quad s \sim t \iff R(s, t) \wedge R(t, s) \iff t \sim s$
- *transitivity*:  $\forall s, t, u \in W \quad \begin{cases} s \sim t \iff R(s, t) \wedge R(t, s) \\ t \sim u \iff R(t, u) \wedge R(u, t) \end{cases}$  and by transitivity of  $R$  we have that

$$- R(s, t) \wedge R(t, u) \implies R(s, u)$$

$$- R(u, t) \wedge R(t, s) \implies R(u, s)$$

$$\text{and by definition } R(s, u) \wedge R(u, s) \iff s \sim u$$

(b) To prove that  $\leq$  is well-defined, we need to show that

$$\forall s, t, s', t' \quad s \sim s' \wedge t \sim t' \implies ([s] \leq [t] \iff [s'] \leq [t'])$$

We observe that

- $s \sim s' \iff R(s, s') \wedge R(s', s)$
- $t \sim t' \iff R(t, t') \wedge R(t', t)$

therefore, we have that

- $[s] \leq [t] \iff R(s, t)$ , and by transitivity of  $R$  it holds that  $R(s', s) \wedge R(s, t) \implies R(s', t)$ ; therefore, by transitivity of  $R$  again we have that  $R(s', t) \wedge R(t, t') \implies R(s', t') \iff [s'] \leq [t']$
- $[s'] \leq [t'] \iff R(s', t')$ , and by transitivity of  $R$  it holds that  $R(s', t') \wedge R(t', t) \implies R(s', t)$ ; therefore, by transitivity of  $R$  again we have that  $R(s, s') \wedge R(s', t) \implies R(s, t) \iff [s] \leq [t]$

(c) To prove that  $\leq$  is a partial order, it suffices to show that  $\leq$  has the following properties:

- *reflexivity*:  $\forall s \in W \quad R(s, s)$  by reflexivity of  $R$ , and  $R(s, s) \iff [s] \leq [s]$

- *antisymmetry*:  $\forall s, t \in W \quad \left\{ \begin{array}{l} [s] \leq [t] \iff R(s, t) \\ [t] \leq [s] \iff R(t, s) \end{array} \right. \implies R(s, t) \wedge R(t, s) \iff s \sim t \iff [s] = [t]$
- *transitivity*:  $\forall s, t, u \in W \quad \left\{ \begin{array}{l} [s] \leq [t] \iff R(s, t) \\ [t] \leq [u] \iff R(t, u) \end{array} \right. \implies R(s, t) \wedge R(t, u) \implies R(s, u)$  by transitivity of  $R$ , and  $R(s, u) \iff [s] \leq [u]$

□

**Exercise 2.2** Let  $\mathcal{N} = (\mathbb{N}, S_1, S_2)$  and  $\mathcal{B} = (\mathbb{B}, R_1, R_2)$  be the following frames for a modal similarity type with two diamonds  $\Diamond_1, \Diamond_2$ . Here,  $\mathbb{N}$  is the set of natural numbers and  $\mathbb{B}$  is the set of strings of 0's and 1's, and the relations are defined by

$$\begin{aligned} S_1(m, n) &\iff n = m + 1 \\ S_2(m, n) &\iff m > n \\ R_1(s, t) &\iff t = s0 \vee t = s1 \\ R_2(s, t) &\iff t \sqsubset s \end{aligned}$$

where  $t \sqsubset s$  if and only if  $t$  is a *proper prefix* of  $s$  — i.e.  $t$  is a prefix of  $s$  such that  $t \neq s$  (thus  $t$  can be  $\varepsilon$ ). Which of the following formulas are valid on  $\mathcal{N}$  and  $\mathcal{B}$ , respectively?

- (a)  $(\Diamond_1 p \wedge \Diamond_2 q) \rightarrow \Diamond_1(p \wedge q)$
- (b)  $(\Diamond_2 p \wedge \Diamond_2 q) \rightarrow \Diamond_2(p \wedge q)$
- (c)  $(\Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r) \rightarrow (\Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r))$
- (d)  $p \rightarrow \Diamond_1 \Box_1 p$
- (e)  $p \rightarrow \Diamond_2 \Box_1 p$
- (f)  $p \rightarrow \Box_1 \Diamond_2 p$
- (g)  $p \rightarrow \Box_2 \Diamond_1 p$

*Solution.* First, consider the following extension to the  $\wedge$  operator on the inductive definition of satisfiability of formulas.

**Claim:** Given a model  $\mathfrak{M} = (W, R, V)$ , and a state  $w \in W$ , it holds that  $\mathfrak{M}, w \models \phi \wedge \psi \iff \mathfrak{M}, w \models \phi \wedge \mathfrak{M}, w \models \psi$ .

*Proof of the Claim.* By using De Morgan's law, we have that

$$\begin{aligned} \mathfrak{M}, w \models \phi \wedge \psi &= \neg(\neg\phi \vee \neg\psi) \iff \neg\mathfrak{M}, w \models \neg\phi \vee \neg\psi \\ &\iff \neg(\mathfrak{M}, w \models \neg\phi \vee \mathfrak{M}, w \models \neg\psi) \\ &\iff \neg(\neg\mathfrak{M}, w \models \phi \vee \neg\mathfrak{M}, w \models \psi) \\ &\iff \mathfrak{M}, w \models \phi \wedge \mathfrak{M}, w \models \psi \end{aligned}$$

□

For all the following propositions, we will assume that  $\mathfrak{M} = (\mathbb{N}, \mathcal{N}, V)$  and  $\mathfrak{M}' = (\mathbb{B}, \mathcal{B}, V)$  are two models.



(a)  $(\Diamond_1 p \wedge \Diamond_1 q) \rightarrow \Diamond_1(p \wedge q)$

- By the claim, for any  $m \in \mathbb{N}$  it holds that

$$\begin{aligned}
 \mathfrak{M}, m \models \Diamond_1 p \wedge \Diamond_1 q &\iff \mathfrak{M}, m \models \Diamond_1 p \wedge \mathfrak{M}, m \models \Diamond_1 q \\
 &\iff \begin{cases} \exists n_p \in \mathbb{N} & S_1(m, n_p) \wedge \mathfrak{M}, n_p \models p \\ \exists n_q \in \mathbb{N} & S_1(m, n_q) \wedge \mathfrak{M}, n_q \models q \end{cases} \\
 &\iff \begin{cases} \exists n_p \in \mathbb{N} & n_p = m + 1 \wedge n_p \in V(p) \\ \exists n_q \in \mathbb{N} & n_q = m + 1 \wedge n_q \in V(q) \end{cases} \\
 &\iff m + 1 \in V(p) \wedge m + 1 \in V(q) \\
 &\iff m + 1 \in V(p) \cap V(q)
 \end{aligned}$$

and again, by the claim we have that

$$\begin{aligned}
 \mathfrak{M}, m \models \Diamond_1(p \wedge q) &\iff \exists n \in \mathbb{N} \quad S_1(m, n) \wedge \mathfrak{M}, n \models p \wedge q \\
 &\iff \exists n \in \mathbb{N} \quad n = m + 1 \wedge (\mathfrak{M}, n \models p \wedge \mathfrak{M}, n \models q) \\
 &\iff \exists n \in \mathbb{N} \quad n = m + 1 \wedge (n \in V(p) \wedge n \in V(q)) \\
 &\iff \exists n \in \mathbb{N} \quad n = m + 1 \wedge n \in V(p) \cap V(q) \\
 &\iff m + 1 \in V(p) \cap V(q)
 \end{aligned}$$

from which we conclude that

$$\mathfrak{M}, m \models \Diamond_1 p \wedge \Diamond_1 q \iff m + 1 \in V(p) \cap V(q) \iff \mathfrak{M}, m \models \Diamond_1(p \wedge q)$$

implying that the formula is valid on  $\mathcal{N}$ .

- By the claim, for any  $s \in \mathbb{B}$  it holds that

$$\begin{aligned}
 \mathfrak{M}', s \models \Diamond_1 p \wedge \Diamond_1 q &\iff \mathfrak{M}', s \models \Diamond_1 p \wedge \mathfrak{M}', s \models \Diamond_1 q \\
 &\iff \begin{cases} \exists t_p \in \mathbb{B} & R_1(s, t_p) \wedge \mathfrak{M}', t_p \models p \\ \exists t_q \in \mathbb{B} & R_1(s, t_q) \wedge \mathfrak{M}', t_q \models q \end{cases} \\
 &\iff \begin{cases} \exists t_p \in \mathbb{B} & (t_p = s0 \vee t_p = s1) \wedge t_p \in V(p) \\ \exists t_q \in \mathbb{B} & (t_q = s0 \vee t_q = s1) \wedge t_q \in V(q) \end{cases} \\
 &\iff \begin{cases} s0 \in V(p) \vee s1 \in V(p) \\ s0 \in V(q) \vee s1 \in V(q) \end{cases} \\
 &\iff \{s0, s1\} \cap V(p) \neq \emptyset \wedge \{s0, s1\} \cap V(q) \neq \emptyset
 \end{aligned}$$

and again, by the claim we have that

$$\begin{aligned}
 \mathfrak{M}', s \models \Diamond_1(p \wedge q) &\iff \exists t \in \mathbb{B} \quad R_1(s, t) \wedge \mathfrak{M}', t \models p \wedge q \\
 &\iff \exists t \in \mathbb{B} \quad (t = s0 \vee t = s1) \wedge (\mathfrak{M}', t \models p \wedge \mathfrak{M}', t \models q) \\
 &\iff \exists t \in \mathbb{B} \quad (t = s0 \vee t = s1) \wedge (t \in V(p) \wedge t \in V(q)) \\
 &\iff \exists t \in \mathbb{B} \quad (t = s0 \vee t = s1) \wedge t \in V(p) \cap V(q) \\
 &\iff s0 \in V(p) \cap V(q) \vee s1 \in V(p) \cap V(q) \\
 &\iff \{s0, s1\} \cap V(p) \cap V(q) \neq \emptyset
 \end{aligned}$$

Now suppose  $V(p) = \{s0\}$  and  $V(q) = \{s1\}$ ; then we have that  $\{s0, s1\} \cap V(p) = \{s0\} \neq \emptyset \wedge \{s0, s1\} \cap V(q) = \{s1\} \neq \emptyset \iff \mathfrak{M}', s \models \Diamond_1 p \wedge \Diamond_1 q$  although  $\{s0, s1\} \cap V(p) \cap V(q) = \{s0, s1\} \cap \emptyset = \emptyset \iff \mathfrak{M}', s \not\models \Diamond_1(p \wedge q)$ , implying that the formula is not valid on  $\mathcal{B}$ .

(b)  $(\Diamond_2 p \wedge \Diamond_2 q) \rightarrow \Diamond_2(p \wedge q)$

- By definition, for any  $m \in \mathbb{N}$  it holds that

$$\begin{aligned} \mathfrak{M}, m \models \Diamond_2 p \wedge \Diamond_2 q &\iff \mathfrak{M}, m \models \Diamond_2 p \wedge \mathfrak{M}, m \models \Diamond_2 q \\ &\iff \begin{cases} \exists n_p \in \mathbb{N} & S_2(m, n_p) \wedge \mathfrak{M}, n_p \models p \\ \exists n_q \in \mathbb{N} & S_2(m, n_q) \wedge \mathfrak{M}, n_q \models q \end{cases} \\ &\iff \begin{cases} \exists n_p \in \mathbb{N} & m > n_p \wedge n_p \in V(p) \\ \exists n_q \in \mathbb{N} & m > n_q \wedge n_q \in V(q) \end{cases} \end{aligned}$$

and again, by the claim we have that

$$\begin{aligned} \mathfrak{M}, m \models \Diamond_2(p \wedge q) &\iff \exists n \in \mathbb{N} \quad S_2(m, n) \wedge \mathfrak{M}, n \models p \wedge q \\ &\iff \exists n \in \mathbb{N} \quad m > n \wedge (\mathfrak{M}, n \models p \wedge \mathfrak{M}, n \models q) \\ &\iff \exists n \in \mathbb{N} \quad m > n \wedge (n \in V(p) \wedge n \in V(q)) \\ &\iff \exists n \in \mathbb{N} \quad m > n \wedge n \in V(p) \cap V(q) \end{aligned}$$

Now take an  $n \geq 2$ , and consider  $n_p, n_q \in \mathbb{N}$  such that  $n_p \neq n_q \wedge n > n_p, n_q$ , and suppose that  $V(p) = \{n_p\}$  and  $V(q) = \{n_q\}$ ; then we have that  $\begin{cases} \exists n_p \in \mathbb{N} & n > n_p \wedge n_p \in V(p) \\ \exists n_q \in \mathbb{N} & n > n_q \wedge n_q \in V(q) \end{cases} \iff \mathfrak{M}, m \models \Diamond_2 p \wedge \Diamond_2 q$  although  $n_p \neq n_q \implies V(p) \cap V(q) = \emptyset \implies \nexists n \in \mathbb{N} \quad m > n \wedge n \in V(p) \cap V(q) \iff \mathfrak{M}, m \not\models \Diamond_2(p \wedge q)$ , implying that the formula is not valid on  $\mathcal{N}$ .

- By the claim, for any  $s \in \mathbb{B}$  it holds that

$$\begin{aligned} \mathfrak{M}', s \models \Diamond_2 p \wedge \Diamond_2 q &\iff \mathfrak{M}', s \models \Diamond_2 p \wedge \mathfrak{M}', s \models \Diamond_2 q \\ &\iff \begin{cases} \exists t_p \in \mathbb{B} & R_2(s, t_p) \wedge \mathfrak{M}', t_p \models p \\ \exists t_q \in \mathbb{B} & R_2(s, t_q) \wedge \mathfrak{M}', t_q \models q \end{cases} \\ &\iff \begin{cases} \exists t_p \in \mathbb{B} & t_p \sqsubset s \wedge t_p \in V(p) \\ \exists t_q \in \mathbb{B} & t_q \sqsubset s \wedge t_q \in V(q) \end{cases} \end{aligned}$$

and again, by the claim we have that

$$\begin{aligned} \mathfrak{M}', s \models \Diamond_2(p \wedge q) &\iff \exists t \in \mathbb{B} \quad t \sqsubset s \wedge (\mathfrak{M}', t \models p \wedge \mathfrak{M}', t \models q) \\ &\iff \exists t \in \mathbb{B} \quad t \sqsubset s \wedge (t \in V(p) \wedge t \in V(q)) \\ &\iff \exists t \in \mathbb{B} \quad t \sqsubset s \wedge t \in V(p) \cap V(q) \end{aligned}$$

Now take  $s = 000$ , consider  $t_p = 0$  and  $t_q = 0$ , and suppose that  $V(p) = \{t_p\} = \{0\}$  and  $V(q) = \{t_q\} = \{00\}$ ; we observe that  $t_p = 0 \sqsubset 000 = s$  and  $t_q = 00 \sqsubset 000 = s$ , therefore  $\begin{cases} \exists t_p \in \mathbb{B} & t_p \sqsubset s \wedge t_p \in V(p) \\ \exists t_q \in \mathbb{B} & t_q \sqsubset s \wedge t_q \in V(q) \end{cases} \iff \mathfrak{M}', s \models \Diamond_2 p \wedge \Diamond_2 q$

although  $V(p) \cap V(q) = \{t_p\} \cap \{t_q\} = \{0\} \cap \{00\} = \emptyset \implies \nexists t \in \mathbb{B} \quad t \sqsubset s \wedge t \in V(p) \cap V(q) \iff \mathfrak{M}', s \not\models \Diamond_2(p \wedge q)$ , implying that the formula is not valid on  $\mathcal{B}$ .

$$(c) (\Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r) \rightarrow (\Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r))$$

- By the claim, for any  $m \in \mathbb{N}$  it holds that

$$\begin{aligned} \mathfrak{M}, m \models \Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r &\iff \mathfrak{M}, m \models \Diamond_1 p \wedge \mathfrak{M}, m \models \Diamond_1 q \wedge \mathfrak{M}, m \models \Diamond_1 r \\ &\iff \begin{cases} \exists n_p \in \mathbb{N} & S_1(m, n_p) \wedge \mathfrak{M}, n_p \models \Diamond_1 p \\ \exists n_q \in \mathbb{N} & S_1(m, n_q) \wedge \mathfrak{M}, n_q \models \Diamond_1 q \\ \exists n_r \in \mathbb{N} & S_1(m, n_r) \wedge \mathfrak{M}, n_r \models \Diamond_1 r \end{cases} \\ &\iff \begin{cases} \exists n_p \in \mathbb{N} & n_p = m + 1 \wedge n_p \in V(p) \\ \exists n_q \in \mathbb{N} & n_q = m + 1 \wedge n_q \in V(q) \\ \exists n_r \in \mathbb{N} & n_r = m + 1 \wedge n_r \in V(r) \end{cases} \\ &\iff \begin{cases} m + 1 \in V(p) \\ m + 1 \in V(q) \\ m + 1 \in V(r) \end{cases} \\ &\iff m + 1 \in V(p) \cap V(q) \cap V(r) \end{aligned}$$

and again, by the claim we have that

$$\begin{aligned} \mathfrak{M}, m \models \Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r) &\iff (\mathfrak{M}, m \models \Diamond_1(p \wedge q)) \\ &\quad \vee (\mathfrak{M}, m \models \Diamond_1(p \wedge r)) \\ &\quad \vee (\mathfrak{M}, m \models \Diamond_1(q \wedge r)) \\ &\iff (\exists n_1 \in \mathbb{N} \quad S_1(m, n_1) \wedge \mathfrak{M}, n_1 \models p \wedge q) \\ &\quad \vee (\exists n_2 \in \mathbb{N} \quad S_1(m, n_2) \wedge \mathfrak{M}, n_2 \models p \wedge r) \\ &\quad \vee (\exists n_3 \in \mathbb{N} \quad S_1(m, n_3) \wedge \mathfrak{M}, n_3 \models q \wedge r) \\ &\iff (\exists n_1 \in \mathbb{N} \quad n_1 = m + 1 \wedge n_1 \in V(p) \cap V(q)) \\ &\quad \vee (\exists n_2 \in \mathbb{N} \quad n_2 = m + 1 \wedge n_2 \in V(p) \cap V(r)) \\ &\quad \vee (\exists n_3 \in \mathbb{N} \quad n_3 = m + 1 \wedge n_3 \in V(q) \cap V(r)) \\ &\iff (m + 1 \in V(p) \cap V(q)) \\ &\quad \vee (m + 1 \in V(p) \cap V(r)) \\ &\quad \vee (m + 1 \in V(q) \cap V(r)) \end{aligned}$$

Hence, we see that

$$\begin{aligned} \mathfrak{M}, m \models \Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r &\iff m + 1 \in V(p) \cap V(q) \cap V(r) \\ &\implies \begin{cases} m + 1 \in V(p) \cap V(q) \\ m + 1 \in V(p) \cap V(r) \\ m + 1 \in V(q) \cap V(r) \end{cases} \\ &\iff \mathfrak{M}, m \models \Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r) \end{aligned}$$

implying that the formula is valid in  $\mathcal{N}$ .

- By the claim, for any  $s \in \mathbb{B}$  it holds that

$$\begin{aligned}
\mathfrak{M}', s \models \Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r &\iff \mathfrak{M}', s \models \Diamond_1 p \wedge \mathfrak{M}', s \models \Diamond_1 q \wedge \mathfrak{M}', s \models \Diamond_1 r \\
&\iff \begin{cases} \exists t_p \in \mathbb{B} & R_1(s, t_p) \wedge \mathfrak{M}', t_p \models p \\ \exists t_q \in \mathbb{B} & R_1(s, t_q) \wedge \mathfrak{M}', t_q \models q \\ \exists t_r \in \mathbb{B} & R_1(s, t_r) \wedge \mathfrak{M}', t_r \models r \end{cases} \\
&\iff \begin{cases} \exists t_p \in \mathbb{B} & (t_p = s0 \vee t_p = s1) \wedge t_p \in V(p) \\ \exists t_q \in \mathbb{B} & (t_q = s0 \vee t_q = s1) \wedge t_q \in V(q) \\ \exists t_r \in \mathbb{B} & (t_r = s0 \vee t_r = s1) \wedge t_r \in V(r) \end{cases} \\
&\iff \begin{cases} \{s0, s1\} \cap V(p) \neq \emptyset \\ \{s0, s1\} \cap V(q) \neq \emptyset \\ \{s0, s1\} \cap V(r) \neq \emptyset \end{cases}
\end{aligned}$$

and again, by the claim we have that

$$\begin{aligned}
\mathfrak{M}', s \models \Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r) &\iff (\mathfrak{M}', s \models \Diamond_1(p \wedge q)) \\
&\vee (\mathfrak{M}', s \models \Diamond_1(p \wedge r)) \\
&\vee (\mathfrak{M}', s \models \Diamond_1(q \wedge r)) \\
&\iff (\exists t_1 \in \mathbb{B} \quad R_1(s, t_1) \wedge \mathfrak{M}', t_1 \models p \wedge q) \\
&\vee (\exists t_2 \in \mathbb{B} \quad R_1(s, t_2) \wedge \mathfrak{M}', t_2 \models p \wedge r) \\
&\vee (\exists t_3 \in \mathbb{B} \quad R_1(s, t_3) \wedge \mathfrak{M}', t_3 \models q \wedge r) \\
&\iff (\exists t_1 \in \mathbb{B} \quad (t_1 = s0 \vee t_1 = s1) \wedge t_1 \in V(p) \cap V(q)) \\
&\vee (\exists t_2 \in \mathbb{B} \quad (t_2 = s0 \vee t_2 = s1) \wedge t_2 \in V(p) \cap V(r)) \\
&\vee (\exists t_3 \in \mathbb{B} \quad (t_3 = s0 \vee t_3 = s1) \wedge t_3 \in V(q) \cap V(r)) \\
&\iff \{s0, s1\} \cap V(p) \cap V(q) \neq \emptyset \\
&\vee \{s0, s1\} \cap V(p) \cap V(r) \neq \emptyset \\
&\vee \{s0, s1\} \cap V(q) \cap V(r) \neq \emptyset
\end{aligned}$$

Now suppose that  $\mathfrak{M}', s \models \Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r$ , which happens if and only if

$$\begin{cases} \{s0, s1\} \cap V(p) \neq \emptyset \\ \{s0, s1\} \cap V(q) \neq \emptyset \\ \{s0, s1\} \cap V(r) \neq \emptyset \end{cases} \text{ as proved previously; by the pigeonhole principle,}$$

since there are 2 strings in  $\{s0, s1\}$  and we have 3 sets  $V(p)$ ,  $V(q)$  and  $V(r)$ , there must be at least one string  $x \in \{s0, s1\}$  such that  $x \in V(a) \cap V(b)$ , where  $a, b \in \{p, q, r\}$  distinct. Without loss of generality, suppose that  $x = s0$  and  $a = p$  and  $b = q$ ; then we have that

$$\begin{aligned}
x = s0 \in V(p) \cap V(q) &\implies x \in \{s0, s1\} \cap V(p) \cap V(q) \\
&\implies \{s0, s1\} \cap V(p) \cap V(q) \neq \emptyset \\
&\implies \mathfrak{M}', s \models \Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r)
\end{aligned}$$

implying that the formula is valid on  $\mathcal{B}$ .

(d)  $p \rightarrow \Diamond_1 \Box_2 p$

- By definition, for any  $m \in \mathbb{N}$  it holds that

$$\begin{aligned}
\mathfrak{M}, m \models \Diamond_1 \Box_2 p &\iff \exists n \in \mathbb{N} \quad S_1(m, n) \wedge \mathfrak{M}, n \models \Box_2 p \\
&\iff \exists n \in \mathbb{N} \quad n = m + 1 \wedge (\forall k \in \mathbb{N} \quad S_2(n, k) \implies \mathfrak{M}, k \models p) \\
&\iff \exists n \in \mathbb{N} \quad n = m + 1 \wedge (\forall k \in \mathbb{N} \quad n > k \implies k \in V(p)) \\
&\iff \forall k \in \mathbb{N} \quad m + 1 > k \implies k \in V(p) \\
&\iff V(p) = \{k \in \mathbb{N} \mid m + 1 > k\}
\end{aligned}$$

Now take  $m = 1 \in \mathbb{N}$ , and suppose  $V(p) = \{m\} = \{1\}$ ; then  $m \in V(p) \iff \mathfrak{M}, m \models p$ , however for instance  $k = 0 \in \mathbb{N}$  is such that  $m + 1 = 1 + 1 = 2 > 0 = k$  although  $k = 0 \notin V(p)$ , therefore  $\exists k \in \mathbb{N} \quad m + 1 > k \wedge k \notin V(p) \implies V(p) \neq \{k \in \mathbb{N} \mid m + 1 > k\} \iff \mathfrak{M}, m \not\models \Diamond_1 \Box_2 p$  which implies that the formula is not valid on  $\mathcal{N}$ .

- By definition, for any  $s \in \mathbb{B}$  it holds that

$$\begin{aligned}
\mathfrak{M}', s \models \Diamond_1 \Box_2 p &\iff \exists t \in \mathbb{B} \quad R_1(s, t) \wedge \mathfrak{M}', t \models \Box_2 p \\
&\iff \exists t \in \mathbb{B} \quad (t = s0 \vee t = s1) \wedge (\forall u \in \mathbb{B} \quad R_2(t, u) \implies \mathfrak{M}', u \models p) \\
&\iff \exists t \in \mathbb{B} \quad (t = s0 \vee t = s1) \wedge (\forall u \in \mathbb{B} \quad u \sqsubset t \implies u \in V(p))
\end{aligned}$$

Now take  $s = 00 \in \mathbb{B}$ , and suppose  $V(p) = \{s\} = \{00\}$ ; then  $s \in V(p) \iff \mathfrak{M}', s \models p$ , however if  $t = s0$  or  $t = s1$ , there still is  $u = 0$  such that  $u = 0 \sqsubset 00 = t$  although  $u = 0 \notin V(p)$ , therefore  $(t = s0 \vee t = s1) \implies (\exists u \in \mathbb{B} \quad u \sqsubset t \wedge u \notin V(p)) \iff \mathfrak{M}', s \not\models \Diamond_1 \Box_2 p$  which implies that the formula is not valid on  $\mathcal{B}$ .

(e)  $p \rightarrow \Diamond_2 \Box_1 p$

- By definition, for any  $m \in \mathbb{N}$  it holds that

$$\begin{aligned}
\mathfrak{M}, m \models \Diamond_2 \Box_1 p &\iff \exists n \in \mathbb{N} \quad S_2(m, n) \wedge \mathfrak{M}, n \models \Box_1 p \\
&\iff \exists n \in \mathbb{N} \quad m > n \wedge (\forall k \in \mathbb{N} \quad S_1(n, k) \implies \mathfrak{M}, k \models p) \\
&\iff \exists n \in \mathbb{N} \quad m > n \wedge (\forall k \in \mathbb{N} \quad k = n + 1 \implies k \in V(p)) \\
&\iff \exists n \in \mathbb{N} \quad m > n \wedge n + 1 \in V(p)
\end{aligned}$$

Now take  $m = 0 \in \mathbb{N}$ , and suppose  $V(p) = \{m\} = \{0\}$ ; then  $m \in V(p) \iff \mathfrak{M}, m \models p$ , however there is no  $n \in \mathbb{N}$  such that  $m = 0 > n$ , therefore  $\nexists n \in \mathbb{N} \quad m > n \wedge n + 1 \in V(p) \iff \mathfrak{M}, m \not\models \Diamond_2 \Box_1 p$  which implies that the formula is not valid on  $\mathcal{N}$ .

- By definition, for any  $s \in \mathbb{B}$  it holds that

$$\begin{aligned}
\mathfrak{M}', s \models \Diamond_2 \Box_1 p &\iff \exists t \in \mathbb{B} \quad R_2(s, t) \wedge \mathfrak{M}', t \models \Box_1 p \\
&\iff \exists t \in \mathbb{B} \quad t \sqsubset s \wedge (\forall u \in \mathbb{B} \quad R_1(t, u) \implies \mathfrak{M}', u \models p) \\
&\iff \exists t \in \mathbb{B} \quad t \sqsubset s \wedge (\forall u \in \mathbb{B} \quad (u = t0 \vee u = t1) \implies u \in V(p)) \\
&\iff \exists t \in \mathbb{B} \quad t \sqsubset s \wedge (t0 \in V(p) \vee t1 \in V(p))
\end{aligned}$$

Now take  $s = \varepsilon \in \mathbb{B}$ , and suppose  $V(p) = \{s\} = \{\varepsilon\}$ ; then  $s \in V(p) \iff \mathfrak{M}', s \models p$ , however there is no  $t \in \mathbb{B}$  such that  $t \sqsubset s$ , therefore  $\nexists t \in \mathbb{B} \quad t \sqsubset s \wedge (t0 \in V(p) \vee t1 \in V(p)) \iff \mathfrak{M}', s \not\models \Diamond_2 \Box_1 p$  which implies that the formula is not valid on  $\mathcal{B}$ .

(f)  $p \rightarrow \Box_1 \Diamond_2 p$ 

- By definition, for any  $m \in \mathbb{N}$  it holds that

$$\begin{aligned}
\mathfrak{M}, m \models \Box_1 \Diamond_2 p &\iff \forall n \in \mathbb{N} \quad S_1(m, n) \implies \mathfrak{M}, n \models \Diamond_2 p \\
&\iff \forall n \in \mathbb{N} \quad n = m + 1 \implies (\exists k \in \mathbb{N} \quad S_2(n, k) \wedge \mathfrak{M}, k \models p) \\
&\iff \forall n \in \mathbb{N} \quad n = m + 1 \implies (\exists k \in \mathbb{N} \quad n > k \wedge k \in V(p)) \\
&\iff \exists k \in \mathbb{N} \quad m + 1 > k \wedge k \in V(p)
\end{aligned}$$

Now suppose  $m \in V(p) \iff \mathfrak{M}, m \models p$ ; we observe that for every  $m \in \mathbb{N}$  it holds that  $m + 1 > m$ , therefore  $m + 1 > m \wedge m \in V(p) \iff \exists k \in \mathbb{N} \quad m + 1 > k \wedge k \in V(p) \iff \mathfrak{M}, m \models \Box_1 \Diamond_2 p$ , which implies that the formula is valid on  $\mathcal{N}$ .

- By definition, for any  $s \in \mathbb{B}$  it holds that

$$\begin{aligned}
\mathfrak{M}', s \models \Box_1 \Diamond_2 p &\iff \forall t \in \mathbb{B} \quad R_1(s, t) \implies \mathfrak{M}', t \models \Diamond_2 p \\
&\iff \forall t \in \mathbb{B} \quad (t = s0 \vee t = s1) \implies (\exists u \in \mathbb{B} \quad R_2(t, u) \wedge \mathfrak{M}', u \models p) \\
&\iff \forall t \in \mathbb{B} \quad (t = s0 \vee t = s1) \implies (\exists u \in \mathbb{B} \quad u \sqsubset t \wedge u \in V(p))
\end{aligned}$$

Now suppose  $s \in V(p) \iff \mathfrak{M}', s \models p$ ; we observe that for every  $s \in \mathbb{B}$  it holds that  $s \sqsubset s0, s1$ , therefore  $\exists u \in \mathbb{B} \quad (u \sqsubset s0 \vee u \sqsubset s1) \wedge u \in V(p) \iff \mathfrak{M}', s \models \Box_1 \Diamond_2 p$ , which implies that the formula is valid on  $\mathcal{B}$ .

(g)  $p \rightarrow \Box_2 \Diamond_1 p$ 

- By definition, for any  $m \in \mathbb{N}$  it holds that

$$\begin{aligned}
\mathfrak{M}, m \models \Box_2 \Diamond_1 p &\iff \forall n \in \mathbb{N} \quad S_2(m, n) \implies \mathfrak{M}, n \models \Diamond_1 p \\
&\iff \forall n \in \mathbb{N} \quad m > n \implies (\exists k \in \mathbb{N} \quad S_1(n, k) \wedge \mathfrak{M}, k \models p) \\
&\iff \forall n \in \mathbb{N} \quad m > n \implies (\exists k \in \mathbb{N} \quad k = n + 1 \wedge k \in V(p)) \\
&\iff \forall n \in \mathbb{N} \quad m > n \implies n + 1 \in V(p)
\end{aligned}$$

Now take  $m = 2 \in \mathbb{N}$ , and suppose  $V(p) = \{m\} = \{2\}$ ; then  $m \in V(p) \iff \mathfrak{M}, m \models p$ , however for instance  $n = 0 \in \mathbb{N}$  is such that  $m = 2 > 0 = n$  and  $n + 1 = 0 + 1 = 1 \notin V(p)$ , therefore  $\exists n \in \mathbb{N} \quad m > n \wedge n + 1 \notin V(p) \iff \mathfrak{M}, m \not\models \Box_2 \Diamond_1 p$ , which implies that the formula is not valid on  $\mathcal{N}$ .

- By definition, for any  $s \in \mathbb{B}$  it holds that

$$\begin{aligned}
\mathfrak{M}', s \models \Box_2 \Diamond_1 p &\iff \forall t \in \mathbb{B} \quad R_2(s, t) \implies \mathfrak{M}', t \models \Diamond_1 p \\
&\iff \forall s \in \mathbb{B} \quad t \sqsubset s \implies (\exists u \in \mathbb{B} \quad R_1(t, u) \wedge \mathfrak{M}', u \models p) \\
&\iff \forall s \in \mathbb{B} \quad t \sqsubset s \implies (\exists u \in \mathbb{B} \quad (u = t0 \vee u = t1) \implies u \in V(p))
\end{aligned}$$

Now take  $s = 000 \in \mathbb{B}$ , and suppose  $V(p) = \{s\} = \{000\}$ ; then  $s \in V(p) \iff \mathfrak{M}', s \models p$ , however for instance  $t = 0 \in \mathbb{B}$  is such that  $t = 0 \sqsubset 000 = s$  although there is no  $u = t0 = 00$  or  $u = t1 = 01$  such that  $u \in V(p)$ , therefore  $\exists t \in \mathbb{B} \quad t \sqsubset s \wedge (\nexists u \in \mathbb{B} \quad (u = t0 \vee u = t1) \wedge u \in V(p)) \iff \mathfrak{M}', s \not\models \Box_2 \Diamond_1 p$  which implies that the formula is not valid on  $\mathcal{B}$ .

□

**Exercise 3.2** Consider the basic modal language, and the tuple  $\mathfrak{f} = (\mathbb{N}, <, A)$  where  $A$  is the collection of finite and co-finite subsets of  $\mathbb{N}$ . Show that  $\mathfrak{f}$  is a general frame.

*Solution.* First, consider the following two claims.

**Claim:** If  $X \subseteq \mathbb{N}$  is finite, and  $Y \subseteq \mathbb{N}$  is co-finite, then  $X \cup Y$  is co-finite.

*Proof of the Claim.* Since  $X$  is finite,  $\mathbb{N} - X$  is finite, and since  $Y$  is co-finite,  $\mathbb{N} - Y$  is finite; this implies that

$$\mathbb{N} - (X \cup Y) = (\mathbb{N} - X) \cap (\mathbb{N} - Y)$$

is the intersection of a co-finite and a finite set. In particular, we observe that

- such intersection will be a subset of  $\mathbb{N} - Y$  by definition of intersection
- $\mathbb{N} - Y$  is finite
- a subset of a finite set is always finite

concluding that such intersection must be finite as well. Lastly, by definition we have that  $\mathbb{N} - (X \cup Y)$  is finite if and only if  $X \cup Y$  is co-finite. □

**Claim:** If  $X, Y \subseteq \mathbb{N}$  are co-finite, then  $X \cup Y$  is co-finite.

*Proof of the Claim.* By repeating the same argument of the previous claim, we have that  $\mathbb{N} - (X \cup Y) = (\mathbb{N} - X) \cap (\mathbb{N} - Y)$  except that in this case both  $\mathbb{N} - X$  and  $\mathbb{N} - Y$  are finite, which implies that their intersection must be finite, hence  $\mathbb{N} - (X \cup Y)$  is co-finite by definition. □

To prove that  $\mathfrak{f}$  is a general frame, it suffices to prove that the set  $A$  is closed under the following

- *union:* fix two sets  $X, Y \in A$ ; then, by definition of  $A$ , we have that
  - if both  $X$  and  $Y$  are finite, then  $X \cup Y$  is finite, hence  $X \cup Y \in A$
  - without loss of generality, if  $X$  is finite and  $Y$  is co-finite, by the first claim  $X \cup Y$  is co-finite, therefore  $X \cup Y \in A$
  - if both  $X$  and  $Y$  are co-finite, then  $X \cup Y$  is co-finite by the second claim, therefore  $X \cup Y \in A$
- *relative complement:* fix a set  $X \in A$ ; then, by definition of  $A$  we trivially have that
  - if  $X$  is finite, then  $\mathbb{N} - X$  is co-finite, hence  $\mathbb{N} - X \in A$
  - if  $X$  is co-finite, then  $\mathbb{N} - X$  is finite, hence  $\mathbb{N} - X \in A$
- *modal operations:* fix two sets  $X, Y \in A$ ; by definition, we have that

$$\begin{aligned} m_{<}(X, Y) &= \{n \in \mathbb{N} \mid \exists x \in X, y \in Y \quad R_{<}(n, x, y)\} \\ &= \{n \in \mathbb{N} \mid \exists x \in X, y \in Y \quad n < x \wedge n < y\} \end{aligned}$$

Therefore, we have that

- if both  $X$  and  $Y$  are finite, then

$$\begin{aligned} m_{<}(X, Y) &= \{n \in \mathbb{N} \mid n < \max(X), \max(Y)\} \\ &= \{n \in \mathbb{N} \mid n < \min(\max(X), \max(Y))\} \end{aligned}$$

which implies that  $m_{<}(X, Y)$  is an “initial segment of  $\mathbb{N}$ ”, thus it is finite, hence  $m_{<}(X, Y) \in A$

- without loss of generality, if  $X$  is finite and  $Y$  is co-finite, then  $\max(Y)$  is not defined, thus

$$m_{<}(X, Y) = \{n \in \mathbb{N} \mid n < \max(X)\}$$

which implies that  $m_{<}(X, Y)$  is still finite, hence  $m_{<}(X, Y) \in A$

- if both  $X$  and  $Y$  are co-finite, then neither  $\max(X)$  nor  $\max(Y)$  are defined, therefore

$$m_{<}(X, Y) = \mathbb{N}$$

which implies that  $\mathbb{N} - m_{<}(X, Y) = \mathbb{N} - \mathbb{N} = \emptyset$  meaning that  $m_{<}(X, Y)$  is co-finite, hence  $m_{<}(X, Y) \in A$

□

### Exercise 5.1 TODO

scrivimi

*Solution.* TODO

scrivimi

First consider the following claim.

**Claim:** If  $p \rightarrow q$  can be  $K$ -proved, and  $q \rightarrow r$  can be  $K$ -proved, then  $p \rightarrow r$  can be  $K$ -proved in 4 steps.

*Proof of the Claim.* Consider a  $K$ -proof in which  $p \rightarrow q$  is proved at step  $i$ , and  $q \rightarrow r$  is proved at step  $j$  — without loss of generality suppose that  $i < j$ ; then we have that

$$\begin{array}{ll} \dots & \\ i. \vdash p \rightarrow q & \\ \dots & \\ j. \vdash q \rightarrow r & \\ j+1. \vdash (a \rightarrow b) \rightarrow ((b \rightarrow c) \rightarrow (a \rightarrow c)) & (T) \\ j+2. \vdash (p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r)) & (S(j+1)) \\ j+3. \vdash (q \rightarrow r) \rightarrow (p \rightarrow r) & (MP(i, j+2)) \\ j+4. \vdash p \rightarrow r & (MP(j, j+3)) \end{array}$$

□



- 
1.  $\vdash \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$  (K)
  2.  $\vdash (\neg a \vee \neg b) \rightarrow (a \rightarrow \neg b)$  (T)
  3.  $\vdash (\neg p \vee \neg q) \rightarrow (p \rightarrow \neg q)$  (S(2))
  4.  $\vdash \Box((\neg p \vee \neg q) \rightarrow (p \rightarrow \neg q))$  (G(3))
  5.  $\vdash \Box((\neg p \vee \neg q) \rightarrow (p \rightarrow \neg q)) \rightarrow (\Box(\neg p \vee \neg q) \rightarrow \Box(p \rightarrow \neg q))$  (S(1))
  6.  $\vdash \Box(\neg p \vee \neg q) \rightarrow \Box(p \rightarrow \neg q)$  (MP(4))
  7.  $\vdash \Box(p \rightarrow \neg q) \rightarrow (\Box p \rightarrow \Box \neg q)$  (S(1))
  - ...
  11.  $\vdash \Box(\neg p \vee \neg q) \rightarrow (\Box p \rightarrow \Box \neg q)$  (C(6, 7))
  12.  $\vdash (a \rightarrow b) \rightarrow (\neg a \vee b)$  (T)
  13.  $\vdash (\Box p \rightarrow \Box \neg q) \rightarrow (\neg \Box p \vee \Box \neg q)$  (S(12))
  - ...
  17.  $\vdash \Box(\neg p \vee \neg q) \rightarrow (\neg \Box p \vee \Box \neg q)$  (C(11, 13))
  18.  $\vdash (\neg a \vee b) \rightarrow \neg(a \wedge \neg b)$  (T)
  19.  $\vdash (\neg \Box p \vee \Box \neg q) \rightarrow \neg(\Box p \wedge \neg \Box \neg q)$  (S(18))
  - ...
  23.  $\vdash \Box(\neg p \vee \neg q) \rightarrow \neg(\Box p \wedge \neg \Box \neg q)$  (C(17, 19))
  24.  $\vdash (a \rightarrow \neg b) \rightarrow (b \rightarrow \neg a)$  (T)
  25.  $\vdash (\Box(\neg p \vee \neg q) \rightarrow \neg(\Box p \wedge \neg \Box \neg q)) \rightarrow ((\Box p \wedge \neg \Box \neg q) \rightarrow \neg \Box(\neg p \vee \neg q))$  (S(24))
  26.  $\vdash (\Box p \wedge \neg \Box \neg q) \rightarrow \neg \Box(\neg p \vee \neg q)$  (MP(25))
  27.  $\vdash \neg(a \wedge b) \rightarrow (\neg a \vee \neg b)$  (T)
  28.  $\vdash \neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$  (S(27))
  29.  $\vdash \Box((p \wedge q) \rightarrow (\neg p \vee \neg q))$  (G(28))
  30.  $\vdash \Box((p \wedge q) \rightarrow (\neg p \vee \neg q)) \rightarrow (\Box(\neg(p \wedge q)) \rightarrow \Box(\neg p \vee \neg q))$  (S(1))
  31.  $\vdash \Box \neg(p \wedge q) \rightarrow \Box(\neg p \vee \neg q)$  (MP(29))
  32.  $\vdash (a \rightarrow b) \rightarrow (\neg b \rightarrow \neg a)$  (T)
  33.  $\vdash (\Box \neg(p \wedge q) \rightarrow \Box(\neg p \vee \neg q)) \rightarrow (\neg \Box(\neg p \vee \neg q) \rightarrow \neg \Box \neg(p \wedge q))$  (S(32))
  34.  $\vdash \neg \Box(\neg p \vee \neg q) \rightarrow \neg \Box \neg(p \wedge q)$  (MP(33))
  - ...
  39.  $\vdash (\Box p \wedge \neg \Box \neg q) \rightarrow \neg \Box \neg(p \wedge q)$  (C(26, 34))
  40.  $\Diamond p \leftrightarrow \neg \Box \neg p$  (K)

□