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# Mathematical Logic for Computer Science

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Lecture notes integrated with the book TODO

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# Information and Contacts

Personal notes and summaries collected as part of the *Mathematical Logic for Computer Science* course offered by the degree in Computer Science of the University of Rome "La Sapienza".

Further information and notes can be found at the following link:

<https://github.com/aflaag-notes>. Anyone can feel free to report inaccuracies, improvements or requests through the Issue system provided by GitHub itself or by contacting the author privately:

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## Suggested prerequisites:

TODO

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# 1

## Homeworks

### 1.1 Homework 1

**Exercise 1.4** Let  $\mathcal{L} = \{E(x, y)\}$  be the language of graphs.

1. For each fixed  $n \in \mathbb{N}$ , write a sentence  $C_n$  such that for any graph  $\mathcal{G}$ ,  $\mathcal{G} \models C_n$  if and only if  $\mathcal{G}$  contains a cycle of length  $n$ .
2. Prove using Compactness that the property of being *a cycle* is not expressible by a theory in  $\mathcal{L}$  over the class of graphs.

*Solution.* Let  $\mathcal{L} = \{E(x, y)\}$  be the language of graphs.

1. The property “ $\mathcal{G}$  contains a cycle of length  $n$ ” can be written as follows

$$C_n := \exists x_1 \dots \exists x_n \left( \bigwedge_{\substack{1 \leq i, j \leq n \\ i \neq j}} \neg(x_i = x_j) \right) \wedge \left( \bigwedge_{1 \leq i \leq n-1} E(x_i, x_{i+1}) \wedge E(x_n, x_1) \right)$$

In fact, the first conjunction implies that  $x_1, \dots, x_n$  are *distinct*, and the second conjunction describes the existence of the  $n$ -long *cycle* itself.

2. Consider the property  $P_n :=$  “ $\mathcal{G}$  is *a cycle* of length  $n$ ”. This property can be expressed by *extending*  $C_n$  as follows:

$$\begin{aligned} V_n &:= \forall y \bigvee_{1 \leq j \leq n} (y = x_j) \\ E_n &:= \bigwedge_{1 \leq i \leq n-1} \bigwedge_{\substack{1 \leq j \leq n: \\ j \neq i+1}} \neg E(x_i, x_j) \wedge \bigwedge_{2 \leq j \leq n} \neg E(x_n, x_j) \\ C'_n &:= \exists x_1 \dots \exists x_n \quad C_n \wedge V_n \wedge E_n \end{aligned}$$

where we have that

- $V_n$  ensures that  $\mathcal{G}$  has *exactly*  $n$  vertices
- $E_n$  ensures that the only edges present in  $\mathcal{G}$  are the ones that describe the cycle graph of  $n$  vertices
- $C'_n$  describes our property  $P_n$

Now, consider the property  $P :=$  “ $\mathcal{G}$  is *a cycle*”, and in particular  $\neg P :=$  “ $\mathcal{G}$  is not *a cycle*”. We observe that we can build the following infinite theory

$$T^{\neg P} := \{\neg C'_n \mid n \in \mathbb{N}_{\geq 3}\}$$

for which it is easy to see that

$$\mathcal{G} \models \neg P \iff \neg P(\mathcal{G}) \text{ holds}$$

meaning that  $\neg P$  is expressible through  $T^{\neg P}$ .

**Claim:**  $T^{\neg P} \in \text{FINSAT}$ .

*Proof of the Claim.* Fix  $T_0 \subseteq_{fin} T^{\neg P}$ . We observe that  $T_0 := \{\neg C'_{i_1}, \dots, \neg C'_{i_k}\}$

for some  $i_1, \dots, i_k \in \mathbb{N}$ . Now, if we consider  $i^* := \max_{j \in [k]} i_j$ , then the cycle graph that has  $i^* + 1$  vertices is clearly a structure that satisfies  $T_0$ .  $\square$

**Claim:**  $P$  is not expressible by a theory in  $\mathcal{L}$  over the class of graphs.

*Proof of the Claim.* By way of contradiction, suppose that  $P$  is expressible, i.e. there is a theory  $T^P$  for which  $P$  can be expressed. Then, consider the theory  $T := T^P \cup T^{\neg P}$ . By the previous claim, we have that  $T \in \text{FINSAT}$ , and by Compactness this is true if and only if  $T \in \text{SAT}$ . However, this is a contradiction, because a graph cannot be and not be a cycle at the same time.  $\square$

Finally, this last claim concludes the proof.  $\square$

**Exercise 2.1** Consider the following two structures  $\mathcal{G}_1$  and  $\mathcal{G}_2$  for the languages of graphs:



Write at least two sentences distinguishing the two structures. Discuss the EF-game played on these structures: for what  $k$  can the Duplicator win the  $k$ -rounds game? For what  $k$  can the Spoiler win?

*Solution.* Some properties that can distinguish these two structures are the following:

1. “ $\mathcal{G}$  contains a vertex of degree 3”, which is represented by the following sentence of rank 5

$$\exists x_1 \exists x_2 \exists x_3 \exists x_4 \left( \bigwedge_{\substack{1 \leq i, j \leq 4 \\ i \neq j}} \neg(x_i = x_j) \right) \wedge \left( \bigwedge_{2 \leq i \leq 4} E(x_1, x_i) \right) \wedge \left( \forall y \neg E(x_1, y) \vee \bigvee_{2 \leq j \leq 4} (y = x_j) \right)$$

2. “ $\mathcal{G}$  contains edges as  $\mathcal{G}_1$ ”, which is represented by the following sentence of rank 5

$$\begin{aligned} \exists x_1 \exists x_2 \exists x_3 \exists x_4 \exists x_5 \quad & E(x_1, x_2) \wedge E(x_1, x_3) \wedge \\ & E(x_2, x_3) \wedge E(x_2, x_4) \wedge E(x_2, x_5) \wedge \\ & E(x_3, x_4) \wedge E(x_4, x_5) \wedge \\ & E(x_4, x_5) \end{aligned}$$

we observe that the edges of  $\mathcal{G}_2$  are not sufficient to distinguish the two sentences, because  $\mathcal{G}_2$  is a subgraph of  $\mathcal{G}_1$

3. “ $\mathcal{G}$  contains a cycle of length 5”, which is represented by  $C_5$  of the previous exercise, and has rank 5
4. “ $\mathcal{G}$  contains a cycle of length 4”, which is represented by  $C_4$  of the previous exercise, and has rank 4
5. “ $\mathcal{G}$  contains  $K_4$  as subgraph”, which is represented by the following sentence having rank 4

$$\exists x_1 \exists x_2 \exists x_3 \exists x_4 \left( \bigwedge_{\substack{1 \leq i, j \leq 4 \\ i \neq j}} \neg(x_i = x_j) \right) \wedge \left( \bigwedge_{\substack{1 \leq i, j \leq 4 \\ i \neq j}} E(x_i, x_j) \right)$$

These sentences *may seem* to suggest that the two structures are 3-equivalent, meaning that there is no sentence of rank 3 that can distinguish  $\mathcal{G}_1$  from  $\mathcal{G}_2$ . For now, let's focus on proving that they are *at least* 2-equivalent.

**Claim:** The Duplicator wins  $G_2(\mathcal{G}_1, \mathcal{G}_2)$ .

*Proof of the Claim.* Let  $s_i$  and  $d_i$  be the  $i$ -th nodes chosen by the Spoiler and the Duplicator, respectively. Then, we can define the following strategy for the Duplicator:

- if  $s_1 \in \{1, 4, 5\}$ , then the Duplicator chooses  $d_1 \in \{a, b, d, e\}$ , otherwise if  $s_1 \in \{2, 3\}$  then  $d_1 = c$
- similarly, if  $s_1 \in \{a, b, d, e\}$ , then the Duplicator chooses  $d_1 \in \{1, 4, 5\}$ , otherwise if  $s_1 = c$  then  $d_1 \in \{2, 3\}$

Then, no matter the choice of  $s_2$ , the Duplicator can always answer with a node  $d_2$  that preserves the partial isomorphism, in fact:

- if  $s_2 \sim s_1$ , it is guaranteed that there is a vertex  $d_2$  in the other structure such that  $d_2 \sim d_1$  because  $\delta(\mathcal{G}_1) = \delta(\mathcal{G}_2) = 2$  — and the same argument applies if  $s_2 \sim d_1$  for finding a vertex  $d_2 \sim s_1$
- if  $s_2 \not\sim s_1$ , the strategy that we provided for the Duplicator guarantees that there exists at least one vertex  $d_2$  in the other structure such that  $d_2 \not\sim d_1$  — and the same argument applies if  $s_2 \not\sim d_1$  for finding a vertex  $d_2 \not\sim s_1$

Thus, the Duplicator has a strategy to always win at least 2 rounds, therefore the Duplicator wins  $G_2(\mathcal{G}_1, \mathcal{G}_2)$  by Ehrenfeucht's theorem.  $\square$

Now that we proved that  $\mathcal{G}_1 \equiv_2 \mathcal{G}_2$ , is it true that they are also 3-equivalent? Unfortunately, the following claim proves that this is indeed false.

**Claim:** The Spoiler wins  $G_3(\mathcal{G}_1, \mathcal{G}_2)$ .

*Proof of the Claim.* The following is a strategy that guarantees the Spoiler to win in 3 rounds:

- let  $s_1 \in \{4, 5\}$

- by the previous claim, we know that the strategy for the Duplicator to win at least 2 rounds is to choose  $d_1 \in \{a, b, d, e\}$ , thus we may assume that  $d_1 \neq c$
- now, let  $s_2 = 1$
- to preserve the partial isomorphism, we observe that
  - if  $d_1 \in \{a, b\}$ , then  $d_2 \in \{d, e\}$
  - if  $d_1 \in \{d, e\}$ , then  $d_2 \in \{a, b\}$
- now, it suffices for the Spoiler to choose  $s_3$  in  $\mathcal{G}_2$  such that  $s_3 \sim d_2$  and  $s_3 \neq c$ : by construction of  $\mathcal{G}_2$ , we see that  $s_3 \approx d_1$ , but all the vertices in  $\{2, 3, 5\}$  are adjacent to  $s_1$ , which would violate the partial isomorphism

□

In fact, we can actually find a property that distinguishes  $\mathcal{G}_1$  from  $\mathcal{G}_2$  which can be written through a sentence of rank 3: “there are two vertices  $x_1$  and  $x_2$  of  $\mathcal{G}$  such that for each third vertex  $x_3$  there is a  $K_3$  as subgraph of  $\mathcal{G}$  such that  $V(K_3) = \{x_1, x_2, x_3\}$ ”

$$\exists x_1 \exists x_2 \forall x_3 \left( \bigwedge_{\substack{1 \leq i, j \leq 3 \\ i \neq j}} \neg(x_i = x_j) \right) \wedge E(x_1, x_2) \wedge E(x_2, x_3) \wedge E(x_3, x_1)$$

Let  $x_1, x_2$  and  $x_3$  be the three chosen vertices — and we may assume that  $x_1 \sim x_2$  otherwise the sentence is trivially unsatisfied. Then, we observe that

- in  $\mathcal{G}_1$  if  $\{x_1, x_2\} = \{2, 3\}$ , then for any other vertex  $x_3 \in \{1, 4, 5\}$  we can always find a  $K_3$  having  $x_1, x_2$  and  $x_3$  as its vertices
- in  $\mathcal{G}_2$  we have two cases
  - if  $\{x_1, x_2\} \subseteq \{a, b, c\}$ , the property is unsatisfied for  $x_3 \in \{d, e\}$
  - if  $\{x_1, x_2\} \subseteq \{c, d, e\}$ , the property is unsatisfied for  $x_3 \in \{a, b\}$

In conclusion, we have that  $\mathcal{G}_1 \equiv_2 \mathcal{G}_2$ , and that  $\mathcal{G}_1 \not\equiv_3 \mathcal{G}_2$ .

□

## 1.2 Homework 2



**Exercise 1.1** Let  $(W, R)$  be a *quasi-order*; that is, assume that  $R$  is transitive and reflexive. Define the binary relation  $\sim$  on  $W$  by putting  $s \sim t \iff R(s, t) \wedge R(t, s)$ .

(a) Show that  $\sim$  is an equivalence relation.

Let  $[s]$  denote the equivalence class of  $s$  under this relation, and define the following relation on the collection of equivalence classes:  $[s] \leq [t] \iff R(s, t)$ .

(b) Show that this relation is well-defined.

(c) Show that  $\leq$  is a partial order.

*Solution.* We prove the statements as follows.

(a) To prove that  $\sim$  is an equivalence relation, it suffices to show that  $\sim$  has the following properties:

- *reflexivity*:  $\forall s \in W \quad R(s, s)$  by reflexivity of  $R$ , therefore  $s \sim s$
- *symmetry*:  $\forall s, t \in W \quad s \sim t \iff R(s, t) \wedge R(t, s) \iff t \sim s$
- *transitivity*:  $\forall s, t, u \in W \quad \begin{cases} s \sim t \iff R(s, t) \wedge R(t, s) \\ t \sim u \iff R(t, u) \wedge R(u, t) \end{cases}$  and by transitivity of  $R$  we have that

$$- R(s, t) \wedge R(t, u) \implies R(s, u)$$

$$- R(u, t) \wedge R(t, s) \implies R(u, s)$$

$$\text{and by definition } R(s, u) \wedge R(u, s) \iff s \sim u$$

(b) To prove that  $\leq$  is well-defined, we need to show that

$$\forall s, t, s', t' \quad s \sim s' \wedge t \sim t' \implies ([s] \leq [t] \iff [s'] \leq [t'])$$

We observe that

- $s \sim s' \iff R(s, s') \wedge R(s', s)$
- $t \sim t' \iff R(t, t') \wedge R(t', t)$

therefore, we have that

- $[s] \leq [t] \iff R(s, t)$ , and by transitivity of  $R$  it holds that  $R(s', s) \wedge R(s, t) \implies R(s', t)$ ; therefore, by transitivity of  $R$  again we have that  $R(s', t) \wedge R(t, t') \implies R(s', t') \iff [s'] \leq [t']$
- $[s'] \leq [t'] \iff R(s', t')$ , and by transitivity of  $R$  it holds that  $R(s', t') \wedge R(t', t) \implies R(s', t)$ ; therefore, by transitivity of  $R$  again we have that  $R(s, s') \wedge R(s', t) \implies R(s, t) \iff [s] \leq [t]$

(c) To prove that  $\leq$  is a partial order, it suffices to show that  $\leq$  has the following properties:

- *reflexivity*:  $\forall s \in W \quad R(s, s)$  by reflexivity of  $R$ , and  $R(s, s) \iff [s] \leq [s]$

- *antisymmetry*:  $\forall s, t \in W \quad \left\{ \begin{array}{l} [s] \leq [t] \iff R(s, t) \\ [t] \leq [s] \iff R(t, s) \end{array} \right. \implies R(s, t) \wedge R(t, s) \iff s \sim t \iff [s] = [t]$
- *transitivity*:  $\forall s, t, u \in W \quad \left\{ \begin{array}{l} [s] \leq [t] \iff R(s, t) \\ [t] \leq [u] \iff R(t, u) \end{array} \right. \implies R(s, t) \wedge R(t, u) \implies R(s, u)$  by transitivity of  $R$ , and  $R(s, u) \iff [s] \leq [u]$

□

**Exercise 2.2** Let  $\mathcal{N} = (\mathbb{N}, S_1, S_2)$  and  $\mathcal{B} = (\mathbb{B}, R_1, R_2)$  be the following frames for a modal similarity type with two diamonds  $\Diamond_1, \Diamond_2$ . Here,  $\mathbb{N}$  is the set of natural numbers and  $\mathbb{B}$  is the set of strings of 0's and 1's, and the relations are defined by

$$\begin{aligned} S_1(m, n) &\iff n = m + 1 \\ S_2(m, n) &\iff m > n \\ R_1(s, t) &\iff t = s0 \vee t = s1 \\ R_2(s, t) &\iff t \sqsubset s \end{aligned}$$

where  $t \sqsubset s$  if and only if  $t$  is a *proper prefix* of  $s$  — i.e.  $t$  is a prefix of  $s$  such that  $t \neq s$  (thus  $t$  can be  $\varepsilon$ ). Which of the following formulas are valid on  $\mathcal{N}$  and  $\mathcal{B}$ , respectively?

- (a)  $(\Diamond_1 p \wedge \Diamond_2 q) \rightarrow \Diamond_1(p \wedge q)$
- (b)  $(\Diamond_2 p \wedge \Diamond_2 q) \rightarrow \Diamond_2(p \wedge q)$
- (c)  $(\Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r) \rightarrow (\Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r))$
- (d)  $p \rightarrow \Diamond_1 \Box_1 p$
- (e)  $p \rightarrow \Diamond_2 \Box_1 p$
- (f)  $p \rightarrow \Box_1 \Diamond_2 p$
- (g)  $p \rightarrow \Box_2 \Diamond_1 p$

*Solution.* First, consider the following extension to the  $\wedge$  operator on the inductive definition of satisfiability of formulas.

**Claim:** Given a model  $\mathfrak{M} = (W, R, V)$ , and a state  $w \in W$ , it holds that  $\mathfrak{M}, w \models \phi \wedge \psi \iff \mathfrak{M}, w \models \phi \wedge \mathfrak{M}, w \models \psi$ .

*Proof of the Claim.* By using De Morgan's law, we have that

$$\begin{aligned} \mathfrak{M}, w \models \phi \wedge \psi &= \neg(\neg\phi \vee \neg\psi) \iff \neg\mathfrak{M}, w \models \neg\phi \vee \neg\psi \\ &\iff \neg(\mathfrak{M}, w \models \neg\phi \vee \mathfrak{M}, w \models \neg\psi) \\ &\iff \neg(\neg\mathfrak{M}, w \models \phi \vee \neg\mathfrak{M}, w \models \psi) \\ &\iff \mathfrak{M}, w \models \phi \wedge \mathfrak{M}, w \models \psi \end{aligned}$$

□

For all the following propositions, we will assume that  $\mathfrak{M} = (\mathbb{N}, \mathcal{N}, V)$  and  $\mathfrak{M}' = (\mathbb{B}, \mathcal{B}, V)$  are two models.

(a)  $(\Diamond_1 p \wedge \Diamond_1 q) \rightarrow \Diamond_1(p \wedge q)$

- By the claim, for any  $m \in \mathbb{N}$  it holds that

$$\begin{aligned}
 \mathfrak{M}, m \models \Diamond_1 p \wedge \Diamond_1 q &\iff \mathfrak{M}, m \models \Diamond_1 p \wedge \mathfrak{M}, m \models \Diamond_1 q \\
 &\iff \begin{cases} \exists n_p \in \mathbb{N} & S_1(m, n_p) \wedge \mathfrak{M}, n_p \models p \\ \exists n_q \in \mathbb{N} & S_1(m, n_q) \wedge \mathfrak{M}, n_q \models q \end{cases} \\
 &\iff \begin{cases} \exists n_p \in \mathbb{N} & n_p = m + 1 \wedge n_p \in V(p) \\ \exists n_q \in \mathbb{N} & n_q = m + 1 \wedge n_q \in V(q) \end{cases} \\
 &\iff m + 1 \in V(p) \wedge m + 1 \in V(q) \\
 &\iff m + 1 \in V(p) \cap V(q)
 \end{aligned}$$

and again, by the claim we have that

$$\begin{aligned}
 \mathfrak{M}, m \models \Diamond_1(p \wedge q) &\iff \exists n \in \mathbb{N} \quad S_1(m, n) \wedge \mathfrak{M}, n \models p \wedge q \\
 &\iff \exists n \in \mathbb{N} \quad n = m + 1 \wedge (\mathfrak{M}, n \models p \wedge \mathfrak{M}, n \models q) \\
 &\iff \exists n \in \mathbb{N} \quad n = m + 1 \wedge (n \in V(p) \wedge n \in V(q)) \\
 &\iff \exists n \in \mathbb{N} \quad n = m + 1 \wedge n \in V(p) \cap V(q) \\
 &\iff m + 1 \in V(p) \cap V(q)
 \end{aligned}$$

from which we conclude that

$$\mathfrak{M}, m \models \Diamond_1 p \wedge \Diamond_1 q \iff m + 1 \in V(p) \cap V(q) \iff \mathfrak{M}, m \models \Diamond_1(p \wedge q)$$

implying that the formula is valid on  $\mathcal{N}$ .

- By the claim, for any  $s \in \mathbb{B}$  it holds that

$$\begin{aligned}
 \mathfrak{M}', s \models \Diamond_1 p \wedge \Diamond_1 q &\iff \mathfrak{M}', s \models \Diamond_1 p \wedge \mathfrak{M}', s \models \Diamond_1 q \\
 &\iff \begin{cases} \exists t_p \in \mathbb{B} & R_1(s, t_p) \wedge \mathfrak{M}', t_p \models p \\ \exists t_q \in \mathbb{B} & R_1(s, t_q) \wedge \mathfrak{M}', t_q \models q \end{cases} \\
 &\iff \begin{cases} \exists t_p \in \mathbb{B} & (t_p = s0 \vee t_p = s1) \wedge t_p \in V(p) \\ \exists t_q \in \mathbb{B} & (t_q = s0 \vee t_q = s1) \wedge t_q \in V(q) \end{cases} \\
 &\iff \begin{cases} s0 \in V(p) \vee s1 \in V(p) \\ s0 \in V(q) \vee s1 \in V(q) \end{cases} \\
 &\iff \{s0, s1\} \cap V(p) \neq \emptyset \wedge \{s0, s1\} \cap V(q) \neq \emptyset
 \end{aligned}$$

and again, by the claim we have that

$$\begin{aligned}
 \mathfrak{M}', s \models \Diamond_1(p \wedge q) &\iff \exists t \in \mathbb{B} \quad R_1(s, t) \wedge \mathfrak{M}', t \models p \wedge q \\
 &\iff \exists t \in \mathbb{B} \quad (t = s0 \vee t = s1) \wedge (\mathfrak{M}', t \models p \wedge \mathfrak{M}', t \models q) \\
 &\iff \exists t \in \mathbb{B} \quad (t = s0 \vee t = s1) \wedge (t \in V(p) \wedge t \in V(q)) \\
 &\iff \exists t \in \mathbb{B} \quad (t = s0 \vee t = s1) \wedge t \in V(p) \cap V(q) \\
 &\iff s0 \in V(p) \cap V(q) \vee s1 \in V(p) \cap V(q) \\
 &\iff \{s0, s1\} \cap V(p) \cap V(q) \neq \emptyset
 \end{aligned}$$

Now suppose  $V(p) = \{s0\}$  and  $V(q) = \{s1\}$ ; then we have that  $\{s0, s1\} \cap V(p) = \{s0\} \neq \emptyset \wedge \{s0, s1\} \cap V(q) = \{s1\} \neq \emptyset \iff \mathfrak{M}', s \models \Diamond_1 p \wedge \Diamond_1 q$  although  $\{s0, s1\} \cap V(p) \cap V(q) = \{s0, s1\} \cap \emptyset = \emptyset \iff \mathfrak{M}', s \not\models \Diamond_1(p \wedge q)$ , implying that the formula is not valid on  $\mathcal{B}$ .

(b)  $(\Diamond_2 p \wedge \Diamond_2 q) \rightarrow \Diamond_2(p \wedge q)$

- By definition, for any  $m \in \mathbb{N}$  it holds that

$$\begin{aligned} \mathfrak{M}, m \models \Diamond_2 p \wedge \Diamond_2 q &\iff \mathfrak{M}, m \models \Diamond_2 p \wedge \mathfrak{M}, m \models \Diamond_2 q \\ &\iff \begin{cases} \exists n_p \in \mathbb{N} & S_2(m, n_p) \wedge \mathfrak{M}, n_p \models p \\ \exists n_q \in \mathbb{N} & S_2(m, n_q) \wedge \mathfrak{M}, n_q \models q \end{cases} \\ &\iff \begin{cases} \exists n_p \in \mathbb{N} & m > n_p \wedge n_p \in V(p) \\ \exists n_q \in \mathbb{N} & m > n_q \wedge n_q \in V(q) \end{cases} \end{aligned}$$

and again, by the claim we have that

$$\begin{aligned} \mathfrak{M}, m \models \Diamond_2(p \wedge q) &\iff \exists n \in \mathbb{N} \quad S_2(m, n) \wedge \mathfrak{M}, n \models p \wedge q \\ &\iff \exists n \in \mathbb{N} \quad m > n \wedge (\mathfrak{M}, n \models P \wedge \mathfrak{M}, n \models q) \\ &\iff \exists n \in \mathbb{N} \quad m > n \wedge (n \in V(p) \wedge n \in V(q)) \\ &\iff \exists n \in \mathbb{N} \quad m > n \wedge n \in V(p) \cap V(q) \end{aligned}$$

Now take an  $n \geq 2$ , and consider  $n_p, n_q \in \mathbb{N}$  such that  $n_p \neq n_q \wedge n > n_p, n_q$ , and suppose that  $V(p) = \{n_p\}$  and  $V(q) = \{n_q\}$ ; then we have that  $\begin{cases} \exists n_p \in \mathbb{N} & n > n_p \wedge n_p \in V(p) \\ \exists n_q \in \mathbb{N} & n > n_q \wedge n_q \in V(q) \end{cases} \iff \mathfrak{M}, m \models \Diamond_2 p \wedge \Diamond_2 q$  although  $n_p \neq n_q \implies V(p) \cap V(q) = \emptyset \implies \nexists n \in \mathbb{N} \quad m > n \wedge n \in V(p) \cap V(q) \iff \mathfrak{M}, m \not\models \Diamond_2(p \wedge q)$ , implying that the formula is not valid on  $\mathcal{N}$ .

- By the claim, for any  $s \in \mathbb{B}$  it holds that

$$\begin{aligned} \mathfrak{M}', s \models \Diamond_2 p \wedge \Diamond_2 q &\iff \mathfrak{M}', s \models \Diamond_2 p \wedge \mathfrak{M}', s \models \Diamond_2 q \\ &\iff \begin{cases} \exists t_p \in \mathbb{B} & R_2(s, t_p) \wedge \mathfrak{M}', t_p \models p \\ \exists t_q \in \mathbb{B} & R_2(s, t_q) \wedge \mathfrak{M}', t_q \models q \end{cases} \\ &\iff \begin{cases} \exists t_p \in \mathbb{B} & t_p \sqsubset s \wedge t_p \in V(p) \\ \exists t_q \in \mathbb{B} & t_q \sqsubset s \wedge t_q \in V(q) \end{cases} \end{aligned}$$

and again, by the claim we have that

$$\begin{aligned} \mathfrak{M}', s \models \Diamond_2(p \wedge q) &\iff \exists t \in \mathbb{B} \quad t \sqsubset s \wedge (\mathfrak{M}', t \models p \wedge \mathfrak{M}', t \models q) \\ &\iff \exists t \in \mathbb{B} \quad t \sqsubset s \wedge (t \in V(p) \wedge t \in V(q)) \\ &\iff \exists t \in \mathbb{B} \quad t \sqsubset s \wedge t \in V(p) \cap V(q) \end{aligned}$$

Now take  $s = 000$ , consider  $t_p = 0$  and  $t_q = 0$ , and suppose that  $V(p) = \{t_p\} = \{0\}$  and  $V(q) = \{t_q\} = \{00\}$ ; we observe that  $t_p = 0 \sqsubset 000 = s$  and  $t_q = 00 \sqsubset 000 = s$ , therefore  $\begin{cases} \exists t_p \in \mathbb{B} & t_p \sqsubset s \wedge t_p \in V(p) \\ \exists t_q \in \mathbb{B} & t_q \sqsubset s \wedge t_q \in V(q) \end{cases} \iff \mathfrak{M}', s \models \Diamond_2 p \wedge \Diamond_2 q$

although  $V(p) \cap V(q) = \{t_p\} \cap \{t_q\} = \{0\} \cap \{00\} = \emptyset \implies \nexists t \in \mathbb{B} \quad t \sqsubset s \wedge t \in V(p) \cap V(q) \iff \mathfrak{M}', s \not\models \Diamond_2(p \wedge q)$ , implying that the formula is not valid on  $\mathcal{B}$ .

$$(c) (\Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r) \rightarrow (\Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r))$$

- By the claim, for any  $m \in \mathbb{N}$  it holds that

$$\begin{aligned} \mathfrak{M}, m \models \Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r &\iff \mathfrak{M}, m \models \Diamond_1 p \wedge \mathfrak{M}, m \models \Diamond_1 q \wedge \mathfrak{M}, m \models \Diamond_1 r \\ &\iff \begin{cases} \exists n_p \in \mathbb{N} & S_1(m, n_p) \wedge \mathfrak{M}, n_p \models \Diamond_1 p \\ \exists n_q \in \mathbb{N} & S_1(m, n_q) \wedge \mathfrak{M}, n_q \models \Diamond_1 q \\ \exists n_r \in \mathbb{N} & S_1(m, n_r) \wedge \mathfrak{M}, n_r \models \Diamond_1 r \end{cases} \\ &\iff \begin{cases} \exists n_p \in \mathbb{N} & n_p = m + 1 \wedge n_p \in V(p) \\ \exists n_q \in \mathbb{N} & n_q = m + 1 \wedge n_q \in V(q) \\ \exists n_r \in \mathbb{N} & n_r = m + 1 \wedge n_r \in V(r) \end{cases} \\ &\iff \begin{cases} m + 1 \in V(p) \\ m + 1 \in V(q) \\ m + 1 \in V(r) \end{cases} \\ &\iff m + 1 \in V(p) \cap V(q) \cap V(r) \end{aligned}$$

and again, by the claim we have that

$$\begin{aligned} \mathfrak{M}, m \models \Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r) &\iff (\mathfrak{M}, m \models \Diamond_1(p \wedge q)) \\ &\quad \vee (\mathfrak{M}, m \models \Diamond_1(p \wedge r)) \\ &\quad \vee (\mathfrak{M}, m \models \Diamond_1(q \wedge r)) \\ &\iff (\exists n_1 \in \mathbb{N} \quad S_1(m, n_1) \wedge \mathfrak{M}, n_1 \models p \wedge q) \\ &\quad \vee (\exists n_2 \in \mathbb{N} \quad S_1(m, n_2) \wedge \mathfrak{M}, n_2 \models p \wedge r) \\ &\quad \vee (\exists n_3 \in \mathbb{N} \quad S_1(m, n_3) \wedge \mathfrak{M}, n_3 \models q \wedge r) \\ &\iff (\exists n_1 \in \mathbb{N} \quad n_1 = m + 1 \wedge n_1 \in V(p) \cap V(q)) \\ &\quad \vee (\exists n_2 \in \mathbb{N} \quad n_2 = m + 1 \wedge n_2 \in V(p) \cap V(r)) \\ &\quad \vee (\exists n_3 \in \mathbb{N} \quad n_3 = m + 1 \wedge n_3 \in V(q) \cap V(r)) \\ &\iff (m + 1 \in V(p) \cap V(q)) \\ &\quad \vee (m + 1 \in V(p) \cap V(r)) \\ &\quad \vee (m + 1 \in V(q) \cap V(r)) \end{aligned}$$

Hence, we see that

$$\begin{aligned} \mathfrak{M}, m \models \Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r &\iff m + 1 \in V(p) \cap V(q) \cap V(r) \\ &\implies \begin{cases} m + 1 \in V(p) \cap V(q) \\ m + 1 \in V(p) \cap V(r) \\ m + 1 \in V(q) \cap V(r) \end{cases} \\ &\iff \mathfrak{M}, m \models \Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r) \end{aligned}$$

implying that the formula is valid in  $\mathcal{N}$ .

- By the claim, for any  $s \in \mathbb{B}$  it holds that

$$\begin{aligned}
\mathfrak{M}', s \models \Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r &\iff \mathfrak{M}', s \models \Diamond_1 p \wedge \mathfrak{M}', s \models \Diamond_1 q \wedge \mathfrak{M}', s \models \Diamond_1 r \\
&\iff \begin{cases} \exists t_p \in \mathbb{B} & R_1(s, t_p) \wedge \mathfrak{M}', t_p \models p \\ \exists t_q \in \mathbb{B} & R_1(s, t_q) \wedge \mathfrak{M}', t_q \models q \\ \exists t_r \in \mathbb{B} & R_1(s, t_r) \wedge \mathfrak{M}', t_r \models r \end{cases} \\
&\iff \begin{cases} \exists t_p \in \mathbb{B} & (t_p = s0 \vee t_p = s1) \wedge t_p \in V(p) \\ \exists t_q \in \mathbb{B} & (t_q = s0 \vee t_q = s1) \wedge t_q \in V(q) \\ \exists t_r \in \mathbb{B} & (t_r = s0 \vee t_r = s1) \wedge t_r \in V(r) \end{cases} \\
&\iff \begin{cases} \{s0, s1\} \cap V(p) \neq \emptyset \\ \{s0, s1\} \cap V(q) \neq \emptyset \\ \{s0, s1\} \cap V(r) \neq \emptyset \end{cases}
\end{aligned}$$

and again, by the claim we have that

$$\begin{aligned}
\mathfrak{M}', s \models \Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r) &\iff (\mathfrak{M}', s \models \Diamond_1(p \wedge q)) \\
&\vee (\mathfrak{M}', s \models \Diamond_1(p \wedge r)) \\
&\vee (\mathfrak{M}', s \models \Diamond_1(q \wedge r)) \\
&\iff (\exists t_1 \in \mathbb{B} \quad R_1(s, t_1) \wedge \mathfrak{M}', t_1 \models p \wedge q) \\
&\vee (\exists t_2 \in \mathbb{B} \quad R_1(s, t_2) \wedge \mathfrak{M}', t_2 \models p \wedge r) \\
&\vee (\exists t_3 \in \mathbb{B} \quad R_1(s, t_3) \wedge \mathfrak{M}', t_3 \models q \wedge r) \\
&\iff (\exists t_1 \in \mathbb{B} \quad (t_1 = s0 \vee t_1 = s1) \wedge t_1 \in V(p) \cap V(q)) \\
&\vee (\exists t_2 \in \mathbb{B} \quad (t_2 = s0 \vee t_2 = s1) \wedge t_2 \in V(p) \cap V(r)) \\
&\vee (\exists t_3 \in \mathbb{B} \quad (t_3 = s0 \vee t_3 = s1) \wedge t_3 \in V(q) \cap V(r)) \\
&\iff \{s0, s1\} \cap V(p) \cap V(q) \neq \emptyset \\
&\vee \{s0, s1\} \cap V(p) \cap V(r) \neq \emptyset \\
&\vee \{s0, s1\} \cap V(q) \cap V(r) \neq \emptyset
\end{aligned}$$

Now suppose that  $\mathfrak{M}', s \models \Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r$ , which happens if and only if

$$\begin{cases} \{s0, s1\} \cap V(p) \neq \emptyset \\ \{s0, s1\} \cap V(q) \neq \emptyset \\ \{s0, s1\} \cap V(r) \neq \emptyset \end{cases} \text{ as proved previously; by the pigeonhole principle,}$$

since there are 2 strings in  $\{s0, s1\}$  and we have 3 sets  $V(p)$ ,  $V(q)$  and  $V(r)$ , there must be at least one string  $x \in \{s0, s1\}$  such that  $x \in V(a) \cap V(b)$ , where  $a, b \in \{p, q, r\}$  distinct. Without loss of generality, suppose that  $x = s0$  and  $a = p$  and  $b = q$ ; then we have that

$$\begin{aligned}
x = s0 \in V(p) \cap V(q) &\implies x \in \{s0, s1\} \cap V(p) \cap V(q) \\
&\implies \{s0, s1\} \cap V(p) \cap V(q) \neq \emptyset \\
&\implies \mathfrak{M}', s \models \Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r)
\end{aligned}$$

implying that the formula is valid on  $\mathcal{B}$ .

(d)  $p \rightarrow \Diamond_1 \Box_2 p$

- By definition, for any  $m \in \mathbb{N}$  it holds that

$$\begin{aligned}
\mathfrak{M}, m \models \Diamond_1 \Box_2 p &\iff \exists n \in \mathbb{N} \quad S_1(m, n) \wedge \mathfrak{M}, n \models \Box_2 p \\
&\iff \exists n \in \mathbb{N} \quad n = m + 1 \wedge (\forall k \in \mathbb{N} \quad S_2(n, k) \implies \mathfrak{M}, k \models p) \\
&\iff \exists n \in \mathbb{N} \quad n = m + 1 \wedge (\forall k \in \mathbb{N} \quad n > k \implies k \in V(p)) \\
&\iff \forall k \in \mathbb{N} \quad m + 1 > k \implies k \in V(p) \\
&\iff V(p) = \{k \in \mathbb{N} \mid m + 1 > k\}
\end{aligned}$$

Now take  $m = 1 \in \mathbb{N}$ , and suppose  $V(p) = \{m\} = \{1\}$ ; then  $m \in V(p) \iff \mathfrak{M}, m \models p$ , however for instance  $k = 0 \in \mathbb{N}$  is such that  $m + 1 = 1 + 1 = 2 > 0 = k$  although  $k = 0 \notin V(p)$ , therefore  $\exists k \in \mathbb{N} \quad m + 1 > k \wedge k \notin V(p) \implies V(p) \neq \{k \in \mathbb{N} \mid m + 1 > k\} \iff \mathfrak{M}, m \not\models \Diamond_1 \Box_2 p$  which implies that the formula is not valid on  $\mathcal{N}$ .

- By definition, for any  $s \in \mathbb{B}$  it holds that

$$\begin{aligned}
\mathfrak{M}', s \models \Diamond_1 \Box_2 p &\iff \exists t \in \mathbb{B} \quad R_1(s, t) \wedge \mathfrak{M}', t \models \Box_2 p \\
&\iff \exists t \in \mathbb{B} \quad (t = s0 \vee t = s1) \wedge (\forall u \in \mathbb{B} \quad R_2(t, u) \implies \mathfrak{M}', u \models p) \\
&\iff \exists t \in \mathbb{B} \quad (t = s0 \vee t = s1) \wedge (\forall u \in \mathbb{B} \quad u \sqsubset t \implies u \in V(p))
\end{aligned}$$

Now take  $s = 00 \in \mathbb{B}$ , and suppose  $V(p) = \{s\} = \{00\}$ ; then  $s \in V(p) \iff \mathfrak{M}', s \models p$ , however if  $t = s0$  or  $t = s1$ , there still is  $u = 0$  such that  $u = 0 \sqsubset 00 = t$  although  $u = 0 \notin V(p)$ , therefore  $(t = s0 \vee t = s1) \implies (\exists u \in \mathbb{B} \quad u \sqsubset t \wedge u \notin V(p)) \iff \mathfrak{M}', s \not\models \Diamond_1 \Box_2 p$  which implies that the formula is not valid on  $\mathcal{B}$ .

(e)  $p \rightarrow \Diamond_2 \Box_1 p$

- By definition, for any  $m \in \mathbb{N}$  it holds that

$$\begin{aligned}
\mathfrak{M}, m \models \Diamond_2 \Box_1 p &\iff \exists n \in \mathbb{N} \quad S_2(m, n) \wedge \mathfrak{M}, n \models \Box_1 p \\
&\iff \exists n \in \mathbb{N} \quad m > n \wedge (\forall k \in \mathbb{N} \quad S_1(n, k) \implies \mathfrak{M}, k \models p) \\
&\iff \exists n \in \mathbb{N} \quad m > n \wedge (\forall k \in \mathbb{N} \quad k = n + 1 \implies k \in V(p)) \\
&\iff \exists n \in \mathbb{N} \quad m > n \wedge n + 1 \in V(p)
\end{aligned}$$

Now take  $m = 0 \in \mathbb{N}$ , and suppose  $V(p) = \{m\} = \{0\}$ ; then  $m \in V(p) \iff \mathfrak{M}, m \models p$ , however there is no  $n \in \mathbb{N}$  such that  $m = 0 > n$ , therefore  $\nexists n \in \mathbb{N} \quad m > n \wedge n + 1 \in V(p) \iff \mathfrak{M}, m \not\models \Diamond_2 \Box_1 p$  which implies that the formula is not valid on  $\mathcal{N}$ .

- By definition, for any  $s \in \mathbb{B}$  it holds that

$$\begin{aligned}
\mathfrak{M}', s \models \Diamond_2 \Box_1 p &\iff \exists t \in \mathbb{B} \quad R_2(s, t) \wedge \mathfrak{M}', t \models \Box_1 p \\
&\iff \exists t \in \mathbb{B} \quad t \sqsubset s \wedge (\forall u \in \mathbb{B} \quad R_1(t, u) \implies \mathfrak{M}', u \models p) \\
&\iff \exists t \in \mathbb{B} \quad t \sqsubset s \wedge (\forall u \in \mathbb{B} \quad (u = t0 \vee u = t1) \implies u \in V(p)) \\
&\iff \exists t \in \mathbb{B} \quad t \sqsubset s \wedge (t0 \in V(p) \vee t1 \in V(p))
\end{aligned}$$

Now take  $s = \varepsilon \in \mathbb{B}$ , and suppose  $V(p) = \{s\} = \{\varepsilon\}$ ; then  $s \in V(p) \iff \mathfrak{M}', s \models p$ , however there is no  $t \in \mathbb{B}$  such that  $t \sqsubset s$ , therefore  $\nexists t \in \mathbb{B} \quad t \sqsubset s \wedge (t0 \in V(p) \vee t1 \in V(p)) \iff \mathfrak{M}', s \not\models \Diamond_2 \Box_1 p$  which implies that the formula is not valid on  $\mathcal{B}$ .

(f)  $p \rightarrow \Box_1 \Diamond_2 p$

- By definition, for any  $m \in \mathbb{N}$  it holds that

$$\begin{aligned} \mathfrak{M}, m \models \Box_1 \Diamond_2 p &\iff \forall n \in \mathbb{N} \quad S_1(m, n) \implies \mathfrak{M}, n \models \Diamond_2 p \\ &\iff \forall n \in \mathbb{N} \quad n = m + 1 \implies (\exists k \in \mathbb{N} \quad S_2(n, k) \wedge \mathfrak{M}, k \models p) \\ &\iff \forall n \in \mathbb{N} \quad n = m + 1 \implies (\exists k \in \mathbb{N} \quad n > k \wedge k \in V(p)) \\ &\iff \exists k \in \mathbb{N} \quad m + 1 > k \wedge k \in V(p) \end{aligned}$$

Now suppose  $m \in V(p) \iff \mathfrak{M}, m \models p$ ; we observe that for every  $m \in \mathbb{N}$  it holds that  $m + 1 > m$ , therefore  $m + 1 > m \wedge m \in V(p) \iff \exists k \in \mathbb{N} \quad m + 1 > k \wedge k \in V(p) \iff \mathfrak{M}, m \models \Box_1 \Diamond_2 p$ , which implies that the formula is valid on  $\mathcal{N}$ .

- By definition, for any  $s \in \mathbb{B}$  it holds that

$$\begin{aligned} \mathfrak{M}', s \models \Box_1 \Diamond_2 p &\iff \forall t \in \mathbb{B} \quad R_1(s, t) \implies \mathfrak{M}', t \models \Diamond_2 p \\ &\iff \forall t \in \mathbb{B} \quad (t = s0 \vee t = s1) \implies (\exists u \in \mathbb{B} \quad R_2(t, u) \wedge \mathfrak{M}', u \models p) \\ &\iff \forall t \in \mathbb{B} \quad (t = s0 \vee t = s1) \implies (\exists u \in \mathbb{B} \quad u \sqsubset t \wedge u \in V(p)) \end{aligned}$$

Now suppose  $s \in V(p) \iff \mathfrak{M}', s \models p$ ; we observe that for every  $s \in \mathbb{B}$  it holds that  $s \sqsubset s0, s1$ , therefore  $\exists u \in \mathbb{B} \quad (u \sqsubset s0 \vee u \sqsubset s1) \wedge u \in V(p) \iff \mathfrak{M}', s \models \Box_1 \Diamond_2 p$ , which implies that the formula is valid on  $\mathcal{B}$ .

(g)  $p \rightarrow \Box_2 \Diamond_1 p$

- By definition, for any  $m \in \mathbb{N}$  it holds that

$$\begin{aligned} \mathfrak{M}, m \models \Box_2 \Diamond_1 p &\iff \forall n \in \mathbb{N} \quad S_2(m, n) \implies \mathfrak{M}, n \models \Diamond_1 p \\ &\iff \forall n \in \mathbb{N} \quad m > n \implies (\exists k \in \mathbb{N} \quad S_1(n, k) \wedge \mathfrak{M}, k \models p) \\ &\iff \forall n \in \mathbb{N} \quad m > n \implies (\exists k \in \mathbb{N} \quad k = n + 1 \wedge k \in V(p)) \\ &\iff \forall n \in \mathbb{N} \quad m > n \implies n + 1 \in V(p) \end{aligned}$$

Now take  $m = 2 \in \mathbb{N}$ , and suppose  $V(p) = \{m\} = \{2\}$ ; then  $m \in V(p) \iff \mathfrak{M}, m \models p$ , however for instance  $n = 0 \in \mathbb{N}$  is such that  $m = 2 > 0 = n$  and  $n + 1 = 0 + 1 = 1 \notin V(p)$ , therefore  $\exists n \in \mathbb{N} \quad m > n \wedge n + 1 \notin V(p) \iff \mathfrak{M}, m \not\models \Box_2 \Diamond_1 p$ , which implies that the formula is not valid on  $\mathcal{N}$ .

- By definition, for any  $s \in \mathbb{B}$  it holds that

$$\begin{aligned} \mathfrak{M}', s \models \Box_2 \Diamond_1 p &\iff \forall t \in \mathbb{B} \quad R_2(s, t) \implies \mathfrak{M}', t \models \Diamond_1 p \\ &\iff \forall s \in \mathbb{B} \quad t \sqsubset s \implies (\exists u \in \mathbb{B} \quad R_1(t, u) \wedge \mathfrak{M}', u \models p) \\ &\iff \forall s \in \mathbb{B} \quad t \sqsubset s \implies (\exists u \in \mathbb{B} \quad (u = t0 \vee u = t1) \implies u \in V(p)) \end{aligned}$$

Now take  $s = 000 \in \mathbb{B}$ , and suppose  $V(p) = \{s\} = \{000\}$ ; then  $s \in V(p) \iff \mathfrak{M}', s \models p$ , however for instance  $t = 0 \in \mathbb{B}$  is such that  $t = 0 \sqsubset 000 = s$  although there is no  $u = t0 = 00$  or  $u = t1 = 01$  such that  $u \in V(p)$ , therefore  $\exists t \in \mathbb{B} \quad t \sqsubset s \wedge (\nexists u \in \mathbb{B} \quad (u = t0 \vee u = t1) \wedge u \in V(p)) \iff \mathfrak{M}', s \not\models \Box_2 \Diamond_1 p$  which implies that the formula is not valid on  $\mathcal{B}$ .



□

**Exercise 3.2** Consider the basic modal language, and the tuple  $\mathfrak{f} = (\mathbb{N}, <, A)$  where  $A$  is the collection of finite and co-finite subsets of  $\mathbb{N}$ . Show that  $\mathfrak{f}$  is a general frame.

*Solution.* First, consider the following two claims.

**Claim 1:** If  $X \subseteq \mathbb{N}$  is finite, and  $Y \subseteq \mathbb{N}$  is co-finite, then  $X \cup Y$  is co-finite.

*Proof of the Claim.* Since  $X$  is finite,  $\mathbb{N} - X$  is finite, and since  $Y$  is co-finite,  $\mathbb{N} - Y$  is finite; this implies that

$$\mathbb{N} - (X \cup Y) = (\mathbb{N} - X) \cap (\mathbb{N} - Y)$$

is the intersection of a co-finite and a finite set. In particular, we observe that

- such intersection will be a subset of  $\mathbb{N} - Y$  by definition of intersection
- $\mathbb{N} - Y$  is finite
- a subset of a finite set is always finite

concluding that such intersection must be finite as well. Lastly, by definition we have that  $\mathbb{N} - (X \cup Y)$  is finite if and only if  $X \cup Y$  is co-finite. □

**Claim 2:** If  $X, Y \subseteq \mathbb{N}$  are co-finite, then  $X \cup Y$  is co-finite.

*Proof of the Claim.* By repeating the same argument of the previous claim, we have that  $\mathbb{N} - (X \cup Y) = (\mathbb{N} - X) \cap (\mathbb{N} - Y)$  except that in this case both  $\mathbb{N} - X$  and  $\mathbb{N} - Y$  are finite, which implies that their intersection must be finite, hence  $\mathbb{N} - (X \cup Y)$  is co-finite by definition. □

To prove that  $\mathfrak{f}$  is a general frame, it suffices to prove that the set  $A$  is closed under the following

- *union:* fix two sets  $X, Y \in A$ ; then, by definition of  $A$ , we have that
  - if both  $X$  and  $Y$  are finite, then  $X \cup Y$  is finite, hence  $X \cup Y \in A$
  - without loss of generality, if  $X$  is finite and  $Y$  is co-finite, by Claim 1  $X \cup Y$  is co-finite, therefore  $X \cup Y \in A$
  - if both  $X$  and  $Y$  are finite, then  $X \cup Y$  is co-finite by Claim 2, therefore  $X \cup Y \in A$
- *relative complement:* fix a set  $X \in A$ ; then, by definition of  $A$  we trivially have that
  - if  $X$  is finite, then  $\mathbb{N} - X$  is co-finite, hence  $\mathbb{N} - X \in A$
  - if  $X$  is co-finite, then  $\mathbb{N} - X$  is finite, hence  $\mathbb{N} - X \in A$
- *modal operations:* assume that “ $<$ ” is the relation referring to a unary modal operator  $\langle < \rangle$ , and fix a set  $X \in A$ ; by definition, we have that

$$m_{\langle < \rangle}(X) = \{n \in \mathbb{N} \mid \exists x \in X \quad n < x\}$$

therefore, we have that

- if  $X$  is finite, then

$$m_{(<)}(X) = \{n \in \mathbb{N} \mid n < \max(X)\}$$

therefore  $m_{(<)}(X)$  is an “initial segment of  $\mathbb{N}$ ”, implying that is finite, hence  $m_{(<)}(X) \in A$

- if  $X$  is co-finite, then  $\mathbb{N} - X$  is finite, implying that  $X$  is infinite; this implies that  $\max(X)$  is not defined, therefore

$$m_{(<)}(X) = \mathbb{N} \implies \mathbb{N} - m_{(<)}(X) = \mathbb{N} - \mathbb{N} = \emptyset$$

and since  $\emptyset$  is finite, we conclude that  $m_{(<)}(X)$  is co-finite, thus  $m_{(<)}(X) \in A$

□

**Exercise 4.3** TODO .

scrivimi

*Solution.* TODO

scrivimi

**Claim:**

*Proof of the Claim.* Assume that  $\Pi \models_{\mathcal{F}} \phi$ ; fix a frame  $\mathcal{F} \in \mathbf{F}$  defined over a world  $W$ , and a model  $\mathfrak{M}$  over  $\mathcal{F}$ . Suppose that  $\forall w \in W \quad \mathfrak{M}, w \models \Sigma$ ; then, by the claim of Exercise 2.2 we know that

$$\forall w \in W \quad \mathfrak{M}, w \models \Sigma \iff \forall w \in W, \sigma \in \Sigma \quad \mathfrak{M}, w \models \sigma$$

We observe that this implies that

$$\forall n \in \mathbb{N}, w, x_1, \dots, x_n \in W, \sigma \in \Sigma \quad R(w, x_1) \wedge R(x_1, x_2) \wedge \dots \wedge R(x_{n-1}, x_n) \implies \mathfrak{M}, x_n \models \sigma$$

therefore

$$\forall n \in \mathbb{N}, w \in W, \sigma \in \Sigma \quad \mathfrak{M}, w \models \Box^n \sigma$$

Finally, this concludes that

$$\forall w \in W \quad \mathfrak{M}, w \models \Pi \implies \forall w \in W \quad \mathfrak{M}, w \models \phi$$

This proves that for any frame  $\mathcal{F} \in \mathbf{F}$  defined over a world  $W$ , and any model  $\mathfrak{M}$  over  $\mathcal{F}$  it holds that

$$\forall w \in W \quad \mathfrak{M}, w \models \Sigma \implies \forall w \in W \quad \mathfrak{M}, w \models \phi$$

which implies that  $\Sigma \models_{\mathcal{F}}^g \phi$  by definition.

□

□

**Exercise 5.1** Give  $K$ -proofs of  $(\Box p \wedge \Diamond q) \rightarrow \Diamond(p \wedge q)$  and  $\Diamond(p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q)$ .

*Solution.* In the first section of the solution, we are going to prove some useful derivations that will be extensively used in the actual  $K$ -proof of the two propositions. The right side of each line will be one of the following:

- (K): the K axiom
- (D): the Dual axiom
- (T): a propositional Tautology
- (MP( $i, j$ )): the Modus Ponens rule applied on lines  $i$  and  $j$
- (S( $i$ )): the Substitution rule applied on line  $i$
- (G( $i$ )): the Generalization rule applied on line  $i$
- (C $_k$ ( $i_1, \dots, i_n$ )): the  $k$ -th Claim applied on lines  $i_1, \dots, i_n$  —  $k \in [7]$  and  $n$  depends on the number of lines the Claim refers to

**Claim 1:** If  $p \rightarrow q$  can be  $K$ -proved, and  $q \rightarrow r$  can be  $K$ -proved, then  $p \rightarrow r$  can be  $K$ -proved in 4 steps.

*Proof of the Claim.* Consider a  $K$ -proof in which  $p \rightarrow q$  is proved at step  $i$ , and  $q \rightarrow r$  is proved at step  $j$  — without loss of generality suppose that  $i < j$ ; then we have that

$$\begin{array}{ll}
 \dots & \\
 i. \vdash p \rightarrow q & \\
 \dots & \\
 j. \vdash q \rightarrow r & \\
 j+1. \vdash (a \rightarrow b) \rightarrow ((b \rightarrow c) \rightarrow (a \rightarrow c)) & \text{(T)} \\
 j+2. \vdash (p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r)) & \text{(S}(j+1)) \\
 j+3. \vdash (q \rightarrow r) \rightarrow (p \rightarrow r) & \text{(MP}(i, j+2)) \\
 j+4. \vdash p \rightarrow r & \text{(MP}(j, j+3))
 \end{array}$$

□

**Claim 2:** If  $p \rightarrow q$  can be  $K$ -proved, then  $\Box p \rightarrow \Box q$  can be  $K$ -proved in 4 steps.

*Proof of the Claim.* Consider a  $K$ -proof in which  $p \rightarrow q$  is proved at step  $i$ ; then, we have that

$$\begin{array}{ll}
 \dots & \\
 i. \vdash p \rightarrow q & \\
 i+1. \vdash \Box(p \rightarrow q) & \text{(G}(i)) \\
 i+2. \vdash \Box(a \rightarrow b) \rightarrow (\Box a \rightarrow \Box b) & \text{(K)} \\
 i+3. \vdash \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q) & \text{(S}(i+2)) \\
 i+4. \vdash \Box p \rightarrow \Box q & \text{(MP}(i+1, i+3))
 \end{array}$$

□

**Claim 3:** If  $p \rightarrow q$  can be  $K$ -proved, and  $p \rightarrow r$  can be  $K$ -proved, then  $p \rightarrow q \wedge r$  can be  $K$ -proved in 4 steps.

*Proof of the Claim.* Consider a  $K$ -proof in which  $p \rightarrow q$  is proved at step  $i$ , and  $p \rightarrow r$  is proved at step  $j$  — without loss of generality suppose  $i < j$ ; then, we have that

$$\begin{array}{ll}
 \dots & \\
 i. \vdash p \rightarrow q & \\
 \dots & \\
 j. \vdash p \rightarrow r & \\
 j+1. \vdash (a \rightarrow b) \rightarrow ((a \rightarrow c) \rightarrow (a \rightarrow b \wedge c)) & (T) \\
 j+2. \vdash (p \rightarrow q) \rightarrow ((p \rightarrow r) \rightarrow (p \rightarrow q \wedge r)) & (S(j+1)) \\
 j+3. \vdash (p \rightarrow r) \rightarrow (p \rightarrow q \wedge r) & (MP(i, j+2)) \\
 j+4. \vdash p \rightarrow q \wedge r & (MP(j, j+3))
 \end{array}$$

□

**Claim 4:** If  $p \rightarrow q$  can be  $K$ -proved, then  $\neg q \rightarrow \neg p$  can be  $K$ -proved in 3 steps. Moreover, if  $p \rightarrow \neg q$  can be  $K$ -proved, then  $q \rightarrow \neg p$  can be  $K$ -proved in 3 steps.

*Proof of the Claim.* Consider a  $K$ -proof in which  $p \rightarrow q$  is proved at step  $i$ ; then, we have that

$$\begin{array}{ll}
 \dots & \\
 i. \vdash p \rightarrow q & \\
 i+1. \vdash (a \rightarrow b) \rightarrow (\neg b \rightarrow \neg a) & (T) \\
 i+2. \vdash (p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p) & (S(i+1)) \\
 i+3. \vdash \neg q \rightarrow \neg p & (MP(i, i+2))
 \end{array}$$

The same  $K$ -proof can be used to prove the rest of the claim by using the propositional tautology  $(a \rightarrow \neg b) \rightarrow (b \rightarrow \neg a)$ . □

**Claim 5:** If  $p \leftrightarrow q$  can be  $K$ -proved, then  $p \rightarrow q$  and  $q \rightarrow p$  can be  $K$ -proved in 3 steps.

*Proof of the Claim.* Consider a  $K$ -proof in which  $p \leftrightarrow q$  is proved at step  $i$ ; then, we have that

$$\begin{array}{ll}
 \dots & \\
 i. \vdash p \leftrightarrow q & \\
 i+1. \vdash (a \leftrightarrow b) \rightarrow (a \rightarrow b) & (T) \\
 i+2. \vdash (p \leftrightarrow q) \rightarrow (p \rightarrow q) & (S(i+1)) \\
 i+3. \vdash p \rightarrow q & (MP(i, i+2))
 \end{array}$$

The case for  $q \rightarrow p$  can be proved analogously by using the propositional tautology  $(a \leftrightarrow b) \rightarrow (b \rightarrow a)$ . □

**Claim 6:** If  $p \rightarrow (q \rightarrow r)$  can be  $K$ -proved, then  $p \wedge q \rightarrow r$  can be  $K$ -proved in 3 steps.

*Proof of the Claim.* Consider a  $K$ -proof in which  $p \rightarrow (q \rightarrow r)$  is proved at step  $i$ ;

then, we have that

$$\begin{array}{ll}
 \dots & \\
 i. \vdash p \rightarrow (q \rightarrow r) & \\
 i+1. \vdash (a \rightarrow (b \rightarrow c)) \rightarrow (a \wedge b \rightarrow c) & (T) \\
 i+2. \vdash (p \rightarrow (q \rightarrow r)) \rightarrow (p \wedge q \rightarrow r) & (S(i+1)) \\
 i+3. \vdash p \wedge q \rightarrow r & (MP(i, i+2))
 \end{array}$$

□

**Claim 7:** If  $p \rightarrow q$  can be  $K$ -proved, and  $q \rightarrow p$  can be  $K$ -proved, then  $p \leftrightarrow q$  can be proved in 4 steps.

*Proof of the Claim.* Consider a  $K$ -proof in which  $p \rightarrow q$  is proved at step  $i$ , and  $q \rightarrow p$  can be proved at step  $j$  — without loss of generality suppose  $i < j$ ; then, we have that

$$\begin{array}{ll}
 \dots & \\
 i. \vdash p \rightarrow q & \\
 \dots & \\
 j. \vdash q \rightarrow p & \\
 j+1. \vdash (a \rightarrow b) \rightarrow ((b \rightarrow a) \rightarrow (a \leftrightarrow b)) & (T) \\
 j+2. \vdash (p \rightarrow q) \rightarrow ((q \rightarrow p) \rightarrow (p \leftrightarrow q)) & (S(j+1)) \\
 j+3. \vdash (q \rightarrow p) \rightarrow (p \leftrightarrow q) & (MP(i, j+2)) \\
 j+4. \vdash p \leftrightarrow q & (MP(j, j+3))
 \end{array}$$

□

Now that we proved some preliminary claims, we can prove the two given propositions.

**Claim 8:**  $(\Box p \wedge \Diamond q) \rightarrow \Diamond(p \wedge q)$  is  $K$ -provable.

*Proof of the Claim.*

1.  $\vdash \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$  (K)
2.  $\vdash (\neg a \vee \neg b) \rightarrow (a \rightarrow \neg b)$  (T)
3.  $\vdash (\neg p \vee \neg q) \rightarrow (p \rightarrow \neg q)$  (S(2))
- ...
7.  $\vdash \Box(\neg p \vee \neg q) \rightarrow \Box(p \rightarrow \neg q)$  ( $C_2(3)$ )
8.  $\vdash \Box(p \rightarrow \neg q) \rightarrow (\Box p \rightarrow \Box \neg q)$  (S(1))
- ...
12.  $\vdash \Box(\neg p \vee \neg q) \rightarrow (\Box p \rightarrow \Box \neg q)$  ( $C_1(7, 8)$ )
13.  $\vdash (a \rightarrow b) \rightarrow (\neg a \vee b)$  (T)
14.  $\vdash (\Box p \rightarrow \Box \neg q) \rightarrow (\neg \Box p \vee \Box \neg q)$  (S(13))
- ...
18.  $\vdash \Box(\neg p \vee \neg q) \rightarrow (\neg \Box p \vee \Box \neg q)$  ( $C_1(12, 14)$ )
19.  $\vdash (\neg a \vee b) \rightarrow \neg(a \wedge \neg b)$  (T)
20.  $\vdash (\neg \Box p \vee \Box \neg q) \rightarrow \neg(\Box p \wedge \neg \Box \neg q)$  (S(19))
- ...
24.  $\vdash \Box(\neg p \vee \neg q) \rightarrow \neg(\Box p \wedge \neg \Box \neg q)$  ( $C_1(18, 20)$ )
- ...
27.  $\vdash (\Box p \wedge \neg \Box \neg q) \rightarrow \neg \Box(\neg p \vee \neg q)$  ( $C_4(24)$ )
28.  $\vdash \neg(a \wedge b) \rightarrow (\neg a \vee \neg b)$  (T)
29.  $\vdash \neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$  (S(27))
- ...
33.  $\vdash \Box \neg(p \wedge q) \rightarrow \Box(\neg p \vee \neg q)$  ( $C_2(28)$ )
- ...
36.  $\vdash \neg \Box(\neg p \vee \neg q) \rightarrow \neg \Box \neg(p \wedge q)$  ( $C_4(32)$ )
- ...
40.  $\vdash (\Box p \wedge \neg \Box \neg q) \rightarrow \neg \Box \neg(p \wedge q)$  ( $C_1(27, 36)$ )
41.  $\vdash \Diamond a \leftrightarrow \neg \Box \neg a$  (D)
42.  $\vdash \Diamond(p \wedge q) \leftrightarrow \neg \Box \neg(p \wedge q)$  (S(41))
- ...
45.  $\vdash \neg \Box \neg(p \wedge q) \rightarrow \Diamond(p \wedge q)$  ( $C_5(42)$ )
- ...
49.  $\vdash (\Box p \wedge \neg \Box \neg q) \rightarrow \Diamond(p \wedge q)$  ( $C_1(40, 45)$ )

50.  $\vdash \Diamond q \leftrightarrow \neg \Box \neg q$  (S(41))  
...  
53.  $\vdash \Diamond q \rightarrow \neg \Box \neg q$  ( $C_5(50)$ )  
54.  $\vdash (b \rightarrow c) \rightarrow (a \wedge b \rightarrow a \wedge c)$  (T)  
55.  $\vdash (\Diamond q \rightarrow \neg \Box \neg q) \rightarrow (\Box p \wedge \Diamond q \rightarrow \Box p \wedge \neg \Box \neg q)$  (S(54))  
56.  $\vdash \Box p \wedge \Diamond q \rightarrow \Box p \wedge \neg \Box \neg q$  (MP(53, 55))  
...  
60.  $\vdash (\Box p \wedge \Diamond q) \rightarrow \Diamond(p \wedge q)$  ( $C_1(56, 49)$ )

□

This claim concludes the  $K$ -proof of the first proposition. To  $K$ -prove the second proposition, we are going to split the  $K$ -proof into 3 claims.

**Claim 9:**  $(\Diamond p \vee \Diamond q) \rightarrow \Diamond(p \vee q)$  is  $K$ -provable.

*Proof of the Claim.*

1.  $\vdash \neg p \wedge \neg q \rightarrow \neg p$  (T)
- ...
5.  $\vdash \Box(\neg p \wedge \neg q) \rightarrow \Box \neg p$  ( $C_2(1)$ )
6.  $\vdash \neg p \wedge \neg q \rightarrow \neg q$  (T)
- ...
10.  $\vdash \Box(\neg p \wedge \neg q) \rightarrow \Box \neg q$  ( $C_2(2)$ )
- ...
14.  $\vdash \Box(\neg p \wedge \neg q) \rightarrow \Box \neg p \wedge \Box \neg q$  ( $C_3(10)$ )
- ...
17.  $\vdash \neg(\Box \neg p \wedge \Box \neg q) \rightarrow \neg \Box(\neg p \wedge \neg q)$  ( $C_4(14)$ )
18.  $\vdash \neg a \vee \neg b \rightarrow \neg(a \wedge b)$  (T)
19.  $\vdash \neg \Box \neg p \vee \neg \Box \neg q \rightarrow \neg(\Box \neg p \wedge \Box \neg q)$  (S(18))
- ...
23.  $\vdash \neg \Box \neg p \vee \neg \Box \neg q \rightarrow \neg \Box(\neg p \wedge \neg q)$  ( $C_1(22, 17)$ )
24.  $\vdash \neg(a \vee b) \rightarrow \neg a \wedge \neg b$  (T)
25.  $\vdash \neg(p \vee q) \rightarrow \neg p \wedge \neg q$  (S(24))
- ...
29.  $\vdash \Box \neg(p \vee q) \rightarrow \Box(\neg p \wedge \neg q)$  ( $C_2(25)$ )
- ...
32.  $\vdash \neg \Box(\neg p \wedge \neg q) \rightarrow \neg \Box \neg(p \vee q)$  ( $C_4(29)$ )
- ...
36.  $\vdash \neg \Box \neg p \vee \neg \Box \neg q \rightarrow \neg \Box \neg(p \vee q)$  ( $C_1(23, 32)$ )
37.  $\vdash \Diamond a \leftrightarrow \neg \Box \neg a$  (D)
- ...
40.  $\vdash \neg \Box \neg a \rightarrow \Diamond a$  ( $C_5(37)$ )
41.  $\vdash \neg \Box \neg(p \vee q) \rightarrow \Diamond(p \vee q)$  (S(40))
- ...
45.  $\vdash \neg \Box \neg p \vee \neg \Box \neg q \rightarrow \Diamond(p \vee q)$  ( $C_1(36, 41)$ )
- ...
49.  $\vdash \Diamond a \rightarrow \neg \Box \neg a$  ( $C_5(37)$ )
50.  $\vdash \Diamond p \rightarrow \neg \Box \neg p$  (S(49))
51.  $\vdash \Diamond q \rightarrow \neg \Box \neg q$  (S(49))
52.  $\vdash (a \rightarrow c) \rightarrow ((b \rightarrow d) \rightarrow (a \vee b \rightarrow c \vee d))$  (T)
53.  $\vdash (\Diamond p \rightarrow \neg \Box \neg p) \rightarrow ((\Diamond q \rightarrow \neg \Box \neg q) \rightarrow (\Diamond p \vee \Diamond q \rightarrow \neg \Box \neg p \vee \neg \Box \neg q))$  (S(52))
54.  $\vdash (\Diamond q \rightarrow \neg \Box \neg q) \rightarrow (\Diamond p \vee \Diamond q \rightarrow \neg \Box \neg p \vee \neg \Box \neg q)$  (MP(50, 53))
55.  $\vdash \Diamond p \vee \Diamond q \rightarrow \neg \Box \neg p \vee \neg \Box \neg q$  (MP(51, 54))
- ...
59.  $\vdash \Diamond p \vee \Diamond q \rightarrow \Diamond(p \vee q)$  ( $C_1(55, 48)$ )



□

**Claim 10:**  $\Diamond(p \vee q) \rightarrow (\Diamond p \vee \Diamond q)$  is  $K$ -provable.

*Proof of the Claim.*

1.  $\vdash \neg p \rightarrow (\neg q \rightarrow \neg p \wedge \neg q)$  (T)
- ...
5.  $\vdash \Box \neg p \rightarrow \Box(\neg q \rightarrow \neg p \vee \neg q)$  ( $C_2(1)$ )
6.  $\vdash \Box(a \rightarrow b) \rightarrow (\Box a \rightarrow \Box b)$  (K)
7.  $\vdash \Box(\neg q \rightarrow \neg p \wedge \neg q) \rightarrow (\Box \neg q \rightarrow \Box(\neg p \wedge \neg q))$  (S(6))
- ...
11.  $\vdash \Box \neg p \rightarrow (\Box \neg q \rightarrow \Box(\neg p \wedge \neg q))$  ( $C_1(5, 7)$ )
- ...
14.  $\vdash \Box \neg p \wedge \Box \neg q \rightarrow \Box(\neg p \wedge \neg q)$  ( $C_6(11)$ )
- ...
17.  $\vdash \neg \Box(\neg p \wedge \neg q) \rightarrow \neg(\Box \neg p \wedge \Box \neg q)$  ( $C_4(14)$ )
18.  $\vdash \neg(a \wedge b) \rightarrow \neg a \vee \neg b$  (T)
19.  $\vdash \neg(\Box \neg p \wedge \Box \neg q) \rightarrow \neg \Box \neg p \vee \neg \Box \neg q$  (S(18))
- ...
23.  $\vdash \neg \Box(\neg p \wedge \neg q) \rightarrow \neg \Box \neg p \vee \neg \Box \neg q$  ( $C_1(17, 19)$ )
24.  $\vdash (\neg a \wedge \neg b) \rightarrow \neg(a \vee b)$  (T)
25.  $\vdash (\neg p \wedge \neg q) \rightarrow \neg(p \vee q)$  (S(24))
- ...
29.  $\vdash \Box(\neg p \wedge \neg q) \rightarrow \Box \neg(p \vee q)$  ( $C_2(25)$ )
- ...
32.  $\vdash \neg \Box \neg(p \vee q) \rightarrow \neg \Box(\neg p \wedge \neg q)$  ( $C_4(29)$ )
- ...
36.  $\vdash \neg \Box \neg(p \vee q) \rightarrow \neg \Box \neg p \vee \neg \Box \neg q$  ( $C_1(32, 23)$ )
37.  $\vdash \Diamond a \leftrightarrow \neg \Box \neg a$  (D)
- ...

- ...
41.  $\vdash \neg \Box \neg a \rightarrow \Diamond a$  (C<sub>5</sub>(37))
42.  $\vdash \neg \Box \neg p \rightarrow \Diamond p$  (S(41))
43.  $\vdash \neg \Box \neg q \rightarrow \Diamond q$  (S(41))
44.  $\vdash (a \rightarrow c) \rightarrow ((b \rightarrow d) \rightarrow (a \vee b \rightarrow c \vee d))$  (T)
45.  $\vdash (\neg \Box \neg p \rightarrow \Diamond p) \rightarrow ((\neg \Box \neg q \rightarrow \Diamond q) \rightarrow (\neg \Box \neg p \vee \neg \Box \neg q \rightarrow \Diamond p \vee \Diamond q))$  (S(44))
46.  $\vdash (\neg \Box \neg q \rightarrow \Diamond q) \rightarrow (\neg \Box \neg p \vee \neg \Box \neg q \rightarrow \Diamond p \vee \Diamond q)$  (MP(41, 45))
47.  $\vdash \neg \Box \neg p \vee \neg \Box \neg q \rightarrow \Diamond p \vee \Diamond q$  (MP(42, 46))
- ...
51.  $\vdash \Diamond a \rightarrow \neg \Box \neg a$  (C<sub>5</sub>(37))
52.  $\vdash \Diamond(p \vee q) \rightarrow \neg \Box \neg(p \vee q)$  (S(51))
- ...
56.  $\vdash \Diamond(p \vee q) \rightarrow (\neg \Box \neg p \vee \neg \Box \neg q)$  (C<sub>1</sub>(52, 36))
- ...
60.  $\vdash \Diamond(p \vee q) \rightarrow \Diamond p \vee \Diamond q$  (C<sub>1</sub>(56, 47))

□

**Claim 11:**  $\Diamond(p \vee q) \leftrightarrow \Diamond(p \vee \Diamond q)$  is  $K$ -provable.

*Proof of the Claim.*

- ...
59.  $\vdash \Diamond p \vee \Diamond q \rightarrow \Diamond(p \vee q)$  (C<sub>9</sub>)
- ...
119.  $\vdash \Diamond(p \vee q) \rightarrow \Diamond p \vee \Diamond q$  (C<sub>10</sub>)
- ...
123.  $\vdash \Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$  (C<sub>7</sub>(2, 1))

□

This last claim concludes that the second proposition is  $K$ -provable as well.

□