SAPIENZA
UNIVERSITÀ DI ROMA

"SAPIENZA" UNIVERSITY OF ROME
FACULTY OF INFORMATION ENGINEERING,
INFORMATICS AND STATISTICS
DEPARTMENT OF COMPUTER SCIENCE

Mathematical Logic for Computer Science

Lecture notes integrated with the book TODO

Author
Alessio Bandiera

March 3, 2025

Contents

Information and Contacts

Personal notes and summaries collected as part of the *Mathematical Logic for Computer Science* course offered by the degree in Computer Science of the University of Rome "La Sapienza".

Further information and notes can be found at the following link:
https://github.com/aflaag-notes. Anyone can feel free to report inaccuracies, improvements or requests through the Issue system provided by GitHub itself or by contacting the author privately:

- Email: **alessio.bandiera02@gmail.com**
- LinkedIn: **Alessio Bandiera**

The notes are constantly being updated, so please check if the changes have already been made in the most recent version.

Suggested prerequisites:

TODO

Licence:

1

TODO

1.1 TODO

placeholder

missing introduction

Definition 1.1: Theory

A **theory** is a — possibly infinite — set of propositions (or hypothesis).

As a natural extension of the *satisfiability* property discussed above, a theory T will be said to be **satisfiable** — written as $T \in \mathsf{SAT}$ if and only if

$$\exists \alpha \quad \alpha(T) = 1$$

which is equivalent of saying that

$$\exists \alpha \quad \forall F \in T \quad \alpha(F) = 1$$

(note that the assignment α must be the same for all the propositions F of T).

Additionally, for infinite theories we can define another property.

Definition 1.2: Finite satisfiability

An infinite theory T is said to be **finitely satisfiable** if and only if

$$\forall T' \subset T \text{ finite} \quad T' \in \mathsf{SAT}$$

We will denote with FINSAT the set of all finitely satisfiable theories.

However, the following theorem will prove that *satisfiability* and *finite satisfiability* are actually **equivalent**.

Theorem 1.1: Compactness theorem

Given an infinite theory T, it holds that

$$T \in \mathsf{SAT} \iff T \in \mathsf{FINSAT}$$

Proof. In this proof we will assume that the propositions of the infinite theory T are *countably infinite*, however in its general form this theorem can be proved even without this assumption.

Since the direct implication of this statement is trivially true by definition, we just need to prove the converse implication.

Claim: Given a theory $T \in \mathsf{FINSAT}$, and a proposition A, it must hold that $T \cup \{A\} \in \mathsf{FINSAT}$ or $T \cup \{\neg A\} \in \mathsf{FINSAT}$.

Proof of the Claim. By way of contradiction, assume that $T \cup \{A\}, T \cup \{\neg A\} \notin \mathsf{FINSAT}$.

By definition of finite satisfiability, if $T \cup \{A\} \notin \mathsf{FINSAT}$, then there must exist a *finite* sub-theory $T_0 \subset T \cup \{A\}$ such that $T_0 \in \mathsf{UNSAT}$. Note that $T \in \mathsf{FINSAT}$, therefore if $T \cup \{A\} \notin \mathsf{FINSAT}$ then it must be that $A \in T_0$. Let $\widehat{T_0}$ be the theory such that $T_0 := \widehat{T_0} \cup \{A\}$; then

$$T_0 := \widehat{T_0} \cup \{A\} \in \mathsf{UNSAT} \iff \forall \alpha \quad \alpha(T_0) = 0$$

which implies that

$$\forall \alpha \quad \alpha(\widehat{T_0}) = 1 \implies \alpha(A) = 0$$

Analogously, we can apply the same reasoning for $T \cup \{\neg A\}$, and we get that there must exist a *finite* sub-theory $T_1 \subset T \cup \{\neg A\}$ such that $T_1 := \widehat{T_1} \cup \{\neg A\} \in \mathsf{UNSAT}$, which implies that

$$\forall \alpha \quad \alpha(\widehat{T_1}) = 1 \implies \alpha(\neg A) = 0$$

Lastly, since $\widehat{T_0} \cup \widehat{T_1} \subset T \in \mathsf{FINSAT}$, by finite satisfiability of T there must exist an assignment α such that $\alpha(\widehat{T_0} \cup \widehat{T_1}) = 1$, and therefore $\alpha(\widehat{T_0}) = \alpha(\widehat{T_1}) = 1$. However, for previous observations this implies that $\alpha(A) = \alpha(\neg A) = 0$ $\lightning$. □

TODO tododa finire □

The statement of this theorem is equivalent of the following.

Corollary 1.1

Given an infinite theory T, and a proposition A, it holds that

$$T \models A \iff \exists T' \subset T \text{ finite} \quad T' \models A$$

Proof. placeholder □

prove their equiva- lence