



SAPIENZA
UNIVERSITÀ DI ROMA

“SAPIENZA” UNIVERSITY OF ROME
FACULTY OF INFORMATION ENGINEERING,
INFORMATICS AND STATISTICS
DEPARTMENT OF COMPUTER SCIENCE

Mathematical Logic for Computer Science

Lecture notes integrated with the book TODO

Author
Alessio Bandiera

May 20, 2025

Contents

Information and Contacts	1
1 Homeworks	2
1.1 Homework 1	2
1.2 Homework 2	6

Information and Contacts

Personal notes and summaries collected as part of the *Mathematical Logic for Computer Science* course offered by the degree in Computer Science of the University of Rome "La Sapienza".

Further information and notes can be found at the following link:

<https://github.com/aflaag-notes>. Anyone can feel free to report inaccuracies, improvements or requests through the Issue system provided by GitHub itself or by contacting the author privately:

- Email: alessio.bandiera02@gmail.com
- LinkedIn: [Alessio Bandiera](#)

The notes are constantly being updated, so please check if the changes have already been made in the most recent version.

Suggested prerequisites:

TODO

Licence:

These documents are distributed under the [GNU Free Documentation License](#), a form of copyleft intended for use on a manual, textbook or other documents. Material licensed under the current version of the license can be used for any purpose, as long as the use meets certain conditions:

- All previous authors of the work must be **attributed**.
- All changes to the work must be **logged**.
- All derivative works must be **licensed under the same license**.
- The full text of the license, unmodified invariant sections as defined by the author if any, and any other added warranty disclaimers (such as a general disclaimer alerting readers that the document may not be accurate for example) and copyright notices from previous versions must be maintained.
- Technical measures such as DRM may not be used to control or obstruct distribution or editing of the document.

1

Homeworks

1.1 Homework 1

Exercise 1.4 Let $\mathcal{L} = \{E(x, y)\}$ be the language of graphs.

1. For each fixed $n \in \mathbb{N}$, write a sentence C_n such that for any graph \mathcal{G} , $\mathcal{G} \models C_n$ if and only if \mathcal{G} contains a cycle of length n .
2. Prove using Compactness that the property of being *a cycle* is not expressible by a theory in \mathcal{L} over the class of graphs.

Solution. Let $\mathcal{L} = \{E(x, y)\}$ be the language of graphs.

1. The property “ \mathcal{G} contains a cycle of length n ” can be written as follows

$$C_n := \exists x_1 \dots \exists x_n \left(\bigwedge_{\substack{1 \leq i, j \leq n \\ i \neq j}} \neg(x_i = x_j) \right) \wedge \left(\bigwedge_{1 \leq i \leq n-1} E(x_i, x_{i+1}) \wedge E(x_n, x_1) \right)$$

In fact, the first conjunction implies that x_1, \dots, x_n are *distinct*, and the second conjunction describes the existence of the n -long *cycle* itself.

2. Consider the property $P_n :=$ “ \mathcal{G} is *a cycle* of length n ”. This property can be expressed by *extending* C_n as follows:

$$\begin{aligned} V_n &:= \forall y \bigvee_{1 \leq j \leq n} (y = x_j) \\ E_n &:= \bigwedge_{1 \leq i \leq n-1} \bigwedge_{\substack{1 \leq j \leq n: \\ j \neq i+1}} \neg E(x_i, x_j) \wedge \bigwedge_{2 \leq j \leq n} \neg E(x_n, x_j) \\ C'_n &:= \exists x_1 \dots \exists x_n \quad C_n \wedge V_n \wedge E_n \end{aligned}$$

where we have that

- V_n ensures that \mathcal{G} has *exactly* n vertices
- E_n ensures that the only edges present in \mathcal{G} are the ones that describe the cycle graph of n vertices
- C'_n describes our property P_n

Now, consider the property $P :=$ “ \mathcal{G} is *a cycle*”, and in particular $\neg P :=$ “ \mathcal{G} is not *a cycle*”. We observe that we can build the following infinite theory

$$T^{\neg P} := \{\neg C'_n \mid n \in \mathbb{N}_{\geq 3}\}$$

for which it is easy to see that

$$\mathcal{G} \models \neg P \iff \neg P(\mathcal{G}) \text{ holds}$$

meaning that $\neg P$ is expressible through $T^{\neg P}$.

Claim: $T^{\neg P} \in \text{FINSAT}$.

Proof of the Claim. Fix $T_0 \subseteq_{fin} T^{\neg P}$. We observe that $T_0 := \{\neg C'_{i_1}, \dots, \neg C'_{i_k}\}$

for some $i_1, \dots, i_k \in \mathbb{N}$. Now, if we consider $i^* := \max_{j \in [k]} i_j$, then the cycle graph that has $i^* + 1$ vertices is clearly a structure that satisfies T_0 . \square

Claim: P is not expressible by a theory in \mathcal{L} over the class of graphs.

Proof of the Claim. By way of contradiction, suppose that P is expressible, i.e. there is a theory T^P for which P can be expressed. Then, consider the theory $T := T^P \cup T^{\neg P}$. By the previous claim, we have that $T \in \text{FINSAT}$, and by Compactness this is true if and only if $T \in \text{SAT}$. However, this is a contradiction, because a graph cannot be and not be a cycle at the same time. \square

Finally, this last claim concludes the proof. \square

Exercise 2.1 Consider the following two structures \mathcal{G}_1 and \mathcal{G}_2 for the languages of graphs:



Write at least two sentences distinguishing the two structures. Discuss the EF-game played on these structures: for what k can the Duplicator win the k -rounds game? For what k can the Spoiler win?

Solution. Some properties that can distinguish these two structures are the following:

1. “ \mathcal{G} contains a vertex of degree 3”, which is represented by the following sentence of rank 5

$$\exists x_1 \exists x_2 \exists x_3 \exists x_4 \left(\bigwedge_{\substack{1 \leq i, j \leq 4 \\ i \neq j}} \neg(x_i = x_j) \right) \wedge \left(\bigwedge_{2 \leq i \leq 4} E(x_1, x_i) \right) \wedge \left(\forall y \quad \neg E(x_1, y) \vee \bigvee_{2 \leq j \leq 4} (y = x_j) \right)$$

2. “ \mathcal{G} contains edges as \mathcal{G}_1 ”, which is represented by the following sentence of rank 5

$$\begin{aligned} \exists x_1 \exists x_2 \exists x_3 \exists x_4 \exists x_5 \quad & E(x_1, x_2) \wedge E(x_1, x_3) \wedge \\ & E(x_2, x_3) \wedge E(x_2, x_4) \wedge E(x_2, x_5) \wedge \\ & E(x_3, x_4) \wedge E(x_4, x_5) \wedge \\ & E(x_4, x_5) \end{aligned}$$

we observe that the edges of \mathcal{G}_2 are not sufficient to distinguish the two sentences, because \mathcal{G}_2 is a subgraph of \mathcal{G}_1

3. “ \mathcal{G} contains a cycle of length 5”, which is represented by C_5 of the previous exercise, and has rank 5
4. “ \mathcal{G} contains a cycle of length 4”, which is represented by C_4 of the previous exercise, and has rank 4
5. “ \mathcal{G} contains K_4 as subgraph”, which is represented by the following sentence having rank 4

$$\exists x_1 \exists x_2 \exists x_3 \exists x_4 \left(\bigwedge_{\substack{1 \leq i, j \leq 4 \\ i \neq j}} \neg(x_i = x_j) \right) \wedge \left(\bigwedge_{\substack{1 \leq i, j \leq 4 \\ i \neq j}} E(x_i, x_j) \right)$$

These sentences *may seem* to suggest that the two structures are 3-equivalent, meaning that there is no sentence of rank 3 that can distinguish \mathcal{G}_1 from \mathcal{G}_2 . For now, let's focus on proving that they are *at least* 2-equivalent.

Claim: The Duplicator wins $G_2(\mathcal{G}_1, \mathcal{G}_2)$.

Proof of the Claim. Let s_i and d_i be the i -th nodes chosen by the Spoiler and the Duplicator, respectively. Then, we can define the following strategy for the Duplicator:

- if $s_1 \in \{1, 4, 5\}$, then the Duplicator chooses $d_1 \in \{a, b, d, e\}$, otherwise if $s_1 \in \{2, 3\}$ then $d_1 = c$
- similarly, if $s_1 \in \{a, b, d, e\}$, then the Duplicator chooses $d_1 \in \{1, 4, 5\}$, otherwise if $s_1 = c$ then $d_1 \in \{2, 3\}$

Then, no matter the choice of s_2 , the Duplicator can always answer with a node d_2 that preserves the partial isomorphism, in fact:

- if $s_2 \sim s_1$, it is guaranteed that there is a vertex d_2 in the other structure such that $d_2 \sim d_1$ because $\delta(\mathcal{G}_1) = \delta(\mathcal{G}_2) = 2$ — and the same argument applies if $s_2 \sim d_1$ for finding a vertex $d_2 \sim s_1$
- if $s_2 \not\sim s_1$, the strategy that we provided for the Duplicator guarantees that there exists at least one vertex d_2 in the other structure such that $d_2 \not\sim d_1$ — and the same argument applies if $s_2 \not\sim d_1$ for finding a vertex $d_2 \not\sim s_1$

Thus, the Duplicator has a strategy to always win at least 2 rounds, therefore the Duplicator wins $G_2(\mathcal{G}_1, \mathcal{G}_2)$ by Ehrenfeucht's theorem. \square

Now that we proved that $\mathcal{G}_1 \equiv_2 \mathcal{G}_2$, is it true that they are also 3-equivalent? Unfortunately, the following claim proves that this is indeed false.

Claim: The Spoiler wins $G_3(\mathcal{G}_1, \mathcal{G}_2)$.

Proof of the Claim. The following is a strategy that guarantees the Spoiler to win in 3 rounds:

- let $s_1 \in \{4, 5\}$

- by the previous claim, we know that the strategy for the Duplicator to win at least 2 rounds is to choose $d_1 \in \{a, b, d, e\}$, thus we may assume that $d_1 \neq c$
- now, let $s_2 = 1$
- to preserve the partial isomorphism, we observe that
 - if $d_1 \in \{a, b\}$, then $d_2 \in \{d, e\}$
 - if $d_1 \in \{d, e\}$, then $d_2 \in \{a, b\}$
- now, it suffices for the Spoiler to choose s_3 in \mathcal{G}_2 such that $s_3 \sim d_2$ and $s_3 \neq c$: by construction of \mathcal{G}_2 , we see that $s_3 \approx d_1$, but all the vertices in $\{2, 3, 5\}$ are adjacent to s_1 , which would violate the partial isomorphism

□

In fact, we can actually find a property that distinguishes \mathcal{G}_1 from \mathcal{G}_2 which can be written through a sentence of rank 3: “there are two vertices x_1 and x_2 of \mathcal{G} such that for each third vertex x_3 there is a K_3 as subgraph of \mathcal{G} such that $V(K_3) = \{x_1, x_2, x_3\}$ ”

$$\exists x_1 \exists x_2 \forall x_3 \left(\bigwedge_{\substack{1 \leq i, j \leq 3 \\ i \neq j}} \neg(x_i = x_j) \right) \wedge E(x_1, x_2) \wedge E(x_2, x_3) \wedge E(x_3, x_1)$$

Let x_1, x_2 and x_3 be the three chosen vertices — and we may assume that $x_1 \sim x_2$ otherwise the sentence is trivially unsatisfied. Then, we observe that

- in \mathcal{G}_1 if $\{x_1, x_2\} = \{2, 3\}$, then for any other vertex $x_3 \in \{1, 4, 5\}$ we can always find a K_3 having x_1, x_2 and x_3 as its vertices
- in \mathcal{G}_2 we have two cases
 - if $\{x_1, x_2\} \subseteq \{a, b, c\}$, the property is unsatisfied for $x_3 \in \{d, e\}$
 - if $\{x_1, x_2\} \subseteq \{c, d, e\}$, the property is unsatisfied for $x_3 \in \{a, b\}$

In conclusion, we have that $\mathcal{G}_1 \equiv_2 \mathcal{G}_2$, and that $\mathcal{G}_1 \not\equiv_3 \mathcal{G}_2$.

□

1.2 Homework 2

Exercise 1.1 Let (W, R) be a *quasi-order*; that is, assume that R is transitive and reflexive. Define the binary relation \sim on W by putting $s \sim t \iff R(s, t) \wedge R(t, s)$.

(a) Show that \sim is an equivalence relation.

Let $[s]$ denote the equivalence class of s under this relation, and define the following relation on the collection of equivalence classes: $[s] \leq [t] \iff R(s, t)$.

(b) Show that this relation is well-defined.

(c) Show that \leq is a partial order.

Solution. We prove the statements as follows.

(a) To prove that \sim is an equivalence relation, it suffices to show that \sim has the following properties:

- *reflexivity*: $\forall s \in W \quad R(s, s)$ by reflexivity of R , therefore $s \sim s$
- *symmetry*: $\forall s, t \in W \quad s \sim t \iff R(s, t) \wedge R(t, s) \iff t \sim s$
- *transitivity*: $\forall s, t, u \in W \quad \begin{cases} s \sim t \iff R(s, t) \wedge R(t, s) \\ t \sim u \iff R(t, u) \wedge R(u, t) \end{cases}$ and by transitivity of R we have that

$$- R(s, t) \wedge R(t, u) \implies R(s, u)$$

$$- R(u, t) \wedge R(t, s) \implies R(u, s)$$

$$\text{and by definition } R(s, u) \wedge R(u, s) \iff s \sim u$$

(b) To prove that \leq is well-defined, we need to show that

$$\forall s, t, s', t' \quad s \sim s' \wedge t \sim t' \implies ([s] \leq [t] \iff [s'] \leq [t'])$$

We observe that

- $s \sim s' \iff R(s, s') \wedge R(s', s)$
- $t \sim t' \iff R(t, t') \wedge R(t', t)$

therefore, we have that

- $[s] \leq [t] \iff R(s, t)$, and by transitivity of R it holds that $R(s', s) \wedge R(s, t) \implies R(s', t)$; therefore, by transitivity of R again we have that $R(s', t) \wedge R(t, t') \implies R(s', t') \iff [s'] \leq [t']$
- $[s'] \leq [t'] \iff R(s', t')$, and by transitivity of R it holds that $R(s', t') \wedge R(t', t) \implies R(s', t)$; therefore, by transitivity of R again we have that $R(s, s') \wedge R(s', t) \implies R(s, t) \iff [s] \leq [t]$

(c) To prove that \leq is a partial order, it suffices to show that \leq has the following properties:

- *reflexivity*: $\forall s \in W \quad R(s, s)$ by reflexivity of R , and $R(s, s) \iff [s] \leq [s]$

- *antisymmetry*: $\forall s, t \in W \quad \begin{cases} [s] \leq [t] \iff R(s, t) \\ [t] \leq [s] \iff R(t, s) \end{cases} \implies R(s, t) \wedge R(t, s) \iff s \sim t \iff [s] = [t]$
- *transitivity*: $\forall s, t, u \in W \quad \begin{cases} [s] \leq [t] \iff R(s, t) \\ [t] \leq [u] \iff R(t, u) \end{cases} \implies R(s, t) \wedge R(t, u) \implies R(s, u)$ by transitivity of R , and $R(s, u) \iff [s] \leq [u]$

□

Exercise 2.2 Let $\mathcal{N} = (\mathbb{N}, S_1, S_2)$ and $\mathcal{B} = (\mathbb{B}, R_1, R_2)$ be the following frames for a modal similarity type with two diamonds \Diamond_1, \Diamond_2 . Here, \mathbb{N} is the set of natural numbers and \mathbb{B} is the set of strings of 0's and 1's, and the relations are defined by

$$\begin{aligned} S_1(m, n) &\iff n = m + 1 \\ S_2(m, n) &\iff m > n \\ R_1(s, t) &\iff t = s0 \vee t = s1 \\ R_2(s, t) &\iff t \sqsubset s \end{aligned}$$

where $t \sqsubset s$ if and only if t is a *proper prefix* of s — i.e. t is a prefix of s such that $t \neq s$ (thus t can be ε). Which of the following formulas are valid on \mathcal{N} and \mathcal{B} , respectively?

- (a) $(\Diamond_1 p \wedge \Diamond_2 q) \rightarrow \Diamond_1(p \wedge q)$
- (b) $(\Diamond_2 p \wedge \Diamond_2 q) \rightarrow \Diamond_2(p \wedge q)$
- (c) $(\Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r) \rightarrow (\Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r))$
- (d) $p \rightarrow \Diamond_1 \Box_1 p$
- (e) $p \rightarrow \Diamond_2 \Box_1 p$
- (f) $p \rightarrow \Box_1 \Diamond_2 p$
- (g) $p \rightarrow \Box_2 \Diamond_1 p$

Solution. First, consider the following extension to the \wedge operator on the inductive definition of satisfiability of formulas.

Claim: Given a model $\mathfrak{M} = (W, R, V)$, and a state $w \in W$, it holds that $\mathfrak{M}, w \models \phi \wedge \psi \iff \mathfrak{M}, w \models \phi \wedge \mathfrak{M}, w \models \psi$.

Proof of the Claim. By using De Morgan's law, we have that

$$\begin{aligned} \mathfrak{M}, w \models \phi \wedge \psi &= \neg(\neg\phi \vee \neg\psi) \iff \neg\mathfrak{M}, w \models \neg\phi \vee \neg\psi \\ &\iff \neg(\mathfrak{M}, w \models \neg\phi \vee \mathfrak{M}, w \models \neg\psi) \\ &\iff \neg(\neg\mathfrak{M}, w \models \phi \vee \neg\mathfrak{M}, w \models \psi) \\ &\iff \mathfrak{M}, w \models \phi \wedge \mathfrak{M}, w \models \psi \end{aligned}$$

□

For all the following propositions, we will assume that $\mathfrak{M} = (\mathbb{N}, \mathcal{N}, V)$ and $\mathfrak{M}' = (\mathbb{B}, \mathcal{B}, V)$ are two models.

(a) $(\Diamond_1 p \wedge \Diamond_1 q) \rightarrow \Diamond_1(p \wedge q)$

- By the claim, for any $m \in \mathbb{N}$ it holds that

$$\begin{aligned}
 \mathfrak{M}, m \models \Diamond_1 p \wedge \Diamond_1 q &\iff \mathfrak{M}, m \models \Diamond_1 p \wedge \mathfrak{M}, m \models \Diamond_1 q \\
 &\iff \begin{cases} \exists n_p \in \mathbb{N} & S_1(m, n_p) \wedge \mathfrak{M}, n_p \models p \\ \exists n_q \in \mathbb{N} & S_1(m, n_q) \wedge \mathfrak{M}, n_q \models q \end{cases} \\
 &\iff \begin{cases} \exists n_p \in \mathbb{N} & n_p = m + 1 \wedge n_p \in V(p) \\ \exists n_q \in \mathbb{N} & n_q = m + 1 \wedge n_q \in V(q) \end{cases} \\
 &\iff m + 1 \in V(p) \wedge m + 1 \in V(q) \\
 &\iff m + 1 \in V(p) \cap V(q)
 \end{aligned}$$

and again, by the claim we have that

$$\begin{aligned}
 \mathfrak{M}, m \models \Diamond_1(p \wedge q) &\iff \exists n \in \mathbb{N} \quad S_1(m, n) \wedge \mathfrak{M}, n \models p \wedge q \\
 &\iff \exists n \in \mathbb{N} \quad n = m + 1 \wedge (\mathfrak{M}, n \models p \wedge \mathfrak{M}, n \models q) \\
 &\iff \exists n \in \mathbb{N} \quad n = m + 1 \wedge (n \in V(p) \wedge n \in V(q)) \\
 &\iff \exists n \in \mathbb{N} \quad n = m + 1 \wedge n \in V(p) \cap V(q) \\
 &\iff m + 1 \in V(p) \cap V(q)
 \end{aligned}$$

from which we conclude that

$$\mathfrak{M}, m \models \Diamond_1 p \wedge \Diamond_1 q \iff m + 1 \in V(p) \cap V(q) \iff \mathfrak{M}, m \models \Diamond_1(p \wedge q)$$

implying that the formula is valid on \mathcal{N} .

- By the claim, for any $s \in \mathbb{B}$ it holds that

$$\begin{aligned}
 \mathfrak{M}', s \models \Diamond_1 p \wedge \Diamond_1 q &\iff \mathfrak{M}', s \models \Diamond_1 p \wedge \mathfrak{M}', s \models \Diamond_1 q \\
 &\iff \begin{cases} \exists t_p \in \mathbb{B} & R_1(s, t_p) \wedge \mathfrak{M}', t_p \models p \\ \exists t_q \in \mathbb{B} & R_1(s, t_q) \wedge \mathfrak{M}', t_q \models q \end{cases} \\
 &\iff \begin{cases} \exists t_p \in \mathbb{B} & (t_p = s0 \vee t_p = s1) \wedge t_p \in V(p) \\ \exists t_q \in \mathbb{B} & (t_q = s0 \vee t_q = s1) \wedge t_q \in V(q) \end{cases} \\
 &\iff \begin{cases} s0 \in V(p) \vee s1 \in V(p) \\ s0 \in V(q) \vee s1 \in V(q) \end{cases} \\
 &\iff \{s0, s1\} \cap V(p) \neq \emptyset \wedge \{s0, s1\} \cap V(q) \neq \emptyset
 \end{aligned}$$

and again, by the claim we have that

$$\begin{aligned}
 \mathfrak{M}', s \models \Diamond_1(p \wedge q) &\iff \exists t \in \mathbb{B} \quad R_1(s, t) \wedge \mathfrak{M}', t \models p \wedge q \\
 &\iff \exists t \in \mathbb{B} \quad (t = s0 \vee t = s1) \wedge (\mathfrak{M}', t \models p \wedge \mathfrak{M}', t \models q) \\
 &\iff \exists t \in \mathbb{B} \quad (t = s0 \vee t = s1) \wedge (t \in V(p) \wedge t \in V(q)) \\
 &\iff \exists t \in \mathbb{B} \quad (t = s0 \vee t = s1) \wedge t \in V(p) \cap V(q) \\
 &\iff s0 \in V(p) \cap V(q) \vee s1 \in V(p) \cap V(q) \\
 &\iff \{s0, s1\} \cap V(p) \cap V(q) \neq \emptyset
 \end{aligned}$$

Now suppose $V(p) = \{s0\}$ and $V(q) = \{s1\}$; then we have that $\{s0, s1\} \cap V(p) = \{s0\} \neq \emptyset \wedge \{s0, s1\} \cap V(q) = \{s1\} \neq \emptyset \iff \mathfrak{M}', s \models \Diamond_1 p \wedge \Diamond_1 q$ although $\{s0, s1\} \cap V(p) \cap V(q) = \{s0, s1\} \cap \emptyset = \emptyset \iff \mathfrak{M}', s \not\models \Diamond_1(p \wedge q)$, implying that the formula is not valid on \mathcal{B} .

(b) $(\Diamond_2 p \wedge \Diamond_2 q) \rightarrow \Diamond_2(p \wedge q)$

- By definition, for any $m \in \mathbb{N}$ it holds that

$$\begin{aligned} \mathfrak{M}, m \models \Diamond_2 p \wedge \Diamond_2 q &\iff \mathfrak{M}, m \models \Diamond_2 p \wedge \mathfrak{M}, m \models \Diamond_2 q \\ &\iff \begin{cases} \exists n_p \in \mathbb{N} & S_2(m, n_p) \wedge \mathfrak{M}, n_p \models p \\ \exists n_q \in \mathbb{N} & S_2(m, n_q) \wedge \mathfrak{M}, n_q \models q \end{cases} \\ &\iff \begin{cases} \exists n_p \in \mathbb{N} & m > n_p \wedge n_p \in V(p) \\ \exists n_q \in \mathbb{N} & m > n_q \wedge n_q \in V(q) \end{cases} \end{aligned}$$

and again, by the claim we have that

$$\begin{aligned} \mathfrak{M}, m \models \Diamond_2(p \wedge q) &\iff \exists n \in \mathbb{N} \quad S_2(m, n) \wedge \mathfrak{M}, n \models p \wedge q \\ &\iff \exists n \in \mathbb{N} \quad m > n \wedge (\mathfrak{M}, n \models p \wedge \mathfrak{M}, n \models q) \\ &\iff \exists n \in \mathbb{N} \quad m > n \wedge (n \in V(p) \wedge n \in V(q)) \\ &\iff \exists n \in \mathbb{N} \quad m > n \wedge n \in V(p) \cap V(q) \end{aligned}$$

Now take an $n \geq 2$, and consider $n_p, n_q \in \mathbb{N}$ such that $n_p \neq n_q \wedge n > n_p, n_q$, and suppose that $V(p) = \{n_p\}$ and $V(q) = \{n_q\}$; then we have that $\begin{cases} \exists n_p \in \mathbb{N} & n > n_p \wedge n_p \in V(p) \\ \exists n_q \in \mathbb{N} & n > n_q \wedge n_q \in V(q) \end{cases} \iff \mathfrak{M}, m \models \Diamond_2 p \wedge \Diamond_2 q$ although $n_p \neq n_q \implies V(p) \cap V(q) = \emptyset \implies \nexists n \in \mathbb{N} \quad m > n \wedge n \in V(p) \cap V(q) \iff \mathfrak{M}, m \not\models \Diamond_2(p \wedge q)$, implying that the formula is not valid on \mathcal{N} .

- By the claim, for any $s \in \mathbb{B}$ it holds that

$$\begin{aligned} \mathfrak{M}', s \models \Diamond_2 p \wedge \Diamond_2 q &\iff \mathfrak{M}', s \models \Diamond_2 p \wedge \mathfrak{M}', s \models \Diamond_2 q \\ &\iff \begin{cases} \exists t_p \in \mathbb{B} & R_2(s, t_p) \wedge \mathfrak{M}', t_p \models p \\ \exists t_q \in \mathbb{B} & R_2(s, t_q) \wedge \mathfrak{M}', t_q \models q \end{cases} \\ &\iff \begin{cases} \exists t_p \in \mathbb{B} & t_p \sqsubset s \wedge t_p \in V(p) \\ \exists t_q \in \mathbb{B} & t_q \sqsubset s \wedge t_q \in V(q) \end{cases} \end{aligned}$$

and again, by the claim we have that

$$\begin{aligned} \mathfrak{M}', s \models \Diamond_2(p \wedge q) &\iff \exists t \in \mathbb{B} \quad t \sqsubset s \wedge (\mathfrak{M}', t \models p \wedge \mathfrak{M}', t \models q) \\ &\iff \exists t \in \mathbb{B} \quad t \sqsubset s \wedge (t \in V(p) \wedge t \in V(q)) \\ &\iff \exists t \in \mathbb{B} \quad t \sqsubset s \wedge t \in V(p) \cap V(q) \end{aligned}$$

Now take $s = 000$, consider $t_p = 0$ and $t_q = 0$, and suppose that $V(p) = \{t_p\} = \{0\}$ and $V(q) = \{t_q\} = \{00\}$; we observe that $t_p = 0 \sqsubset 000 = s$ and $t_q = 00 \sqsubset 000 = s$, therefore $\begin{cases} \exists t_p \in \mathbb{B} & t_p \sqsubset s \wedge t_p \in V(p) \\ \exists t_q \in \mathbb{B} & t_q \sqsubset s \wedge t_q \in V(q) \end{cases} \iff \mathfrak{M}', s \models \Diamond_2 p \wedge \Diamond_2 q$

although $V(p) \cap V(q) = \{t_p\} \cap \{t_q\} = \{0\} \cap \{00\} = \emptyset \implies \nexists t \in \mathbb{B} \quad t \sqsubset s \wedge t \in V(p) \cap V(q) \iff \mathfrak{M}', s \not\models \Diamond_2(p \wedge q)$, implying that the formula is not valid on \mathcal{B} .

$$(c) (\Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r) \rightarrow (\Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r))$$

- By the claim, for any $m \in \mathbb{N}$ it holds that

$$\begin{aligned} \mathfrak{M}, m \models \Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r &\iff \mathfrak{M}, m \models \Diamond_1 p \wedge \mathfrak{M}, m \models \Diamond_1 q \wedge \mathfrak{M}, m \models \Diamond_1 r \\ &\iff \begin{cases} \exists n_p \in \mathbb{N} & S_1(m, n_p) \wedge \mathfrak{M}, n_p \models \Diamond_1 p \\ \exists n_q \in \mathbb{N} & S_1(m, n_q) \wedge \mathfrak{M}, n_q \models \Diamond_1 q \\ \exists n_r \in \mathbb{N} & S_1(m, n_r) \wedge \mathfrak{M}, n_r \models \Diamond_1 r \end{cases} \\ &\iff \begin{cases} \exists n_p \in \mathbb{N} & n_p = m + 1 \wedge n_p \in V(p) \\ \exists n_q \in \mathbb{N} & n_q = m + 1 \wedge n_q \in V(q) \\ \exists n_r \in \mathbb{N} & n_r = m + 1 \wedge n_r \in V(r) \end{cases} \\ &\iff \begin{cases} m + 1 \in V(p) \\ m + 1 \in V(q) \\ m + 1 \in V(r) \end{cases} \\ &\iff m + 1 \in V(p) \cap V(q) \cap V(r) \end{aligned}$$

and again, by the claim we have that

$$\begin{aligned} \mathfrak{M}, m \models \Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r) &\iff (\mathfrak{M}, m \models \Diamond_1(p \wedge q)) \\ &\quad \vee (\mathfrak{M}, m \models \Diamond_1(p \wedge r)) \\ &\quad \vee (\mathfrak{M}, m \models \Diamond_1(q \wedge r)) \\ &\iff (\exists n_1 \in \mathbb{N} \quad S_1(m, n_1) \wedge \mathfrak{M}, n_1 \models p \wedge q) \\ &\quad \vee (\exists n_2 \in \mathbb{N} \quad S_1(m, n_2) \wedge \mathfrak{M}, n_2 \models p \wedge r) \\ &\quad \vee (\exists n_3 \in \mathbb{N} \quad S_1(m, n_3) \wedge \mathfrak{M}, n_3 \models q \wedge r) \\ &\iff (\exists n_1 \in \mathbb{N} \quad n_1 = m + 1 \wedge n_1 \in V(p) \cap V(q)) \\ &\quad \vee (\exists n_2 \in \mathbb{N} \quad n_2 = m + 1 \wedge n_2 \in V(p) \cap V(r)) \\ &\quad \vee (\exists n_3 \in \mathbb{N} \quad n_3 = m + 1 \wedge n_3 \in V(q) \cap V(r)) \\ &\iff (m + 1 \in V(p) \cap V(q)) \\ &\quad \vee (m + 1 \in V(p) \cap V(r)) \\ &\quad \vee (m + 1 \in V(q) \cap V(r)) \end{aligned}$$

Hence, we see that

$$\begin{aligned} \mathfrak{M}, m \models \Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r &\iff m + 1 \in V(p) \cap V(q) \cap V(r) \\ &\implies \begin{cases} m + 1 \in V(p) \cap V(q) \\ m + 1 \in V(p) \cap V(r) \\ m + 1 \in V(q) \cap V(r) \end{cases} \\ &\iff \mathfrak{M}, m \models \Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r) \end{aligned}$$

implying that the formula is valid in \mathcal{N} .

- By the claim, for any $s \in \mathbb{B}$ it holds that

$$\begin{aligned}
\mathfrak{M}', s \models \Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r &\iff \mathfrak{M}', s \models \Diamond_1 p \wedge \mathfrak{M}', s \models \Diamond_1 q \wedge \mathfrak{M}', s \models \Diamond_1 r \\
&\iff \begin{cases} \exists t_p \in \mathbb{B} & R_1(s, t_p) \wedge \mathfrak{M}', t_p \models p \\ \exists t_q \in \mathbb{B} & R_1(s, t_q) \wedge \mathfrak{M}', t_q \models q \\ \exists t_r \in \mathbb{B} & R_1(s, t_r) \wedge \mathfrak{M}', t_r \models r \end{cases} \\
&\iff \begin{cases} \exists t_p \in \mathbb{B} & (t_p = s0 \vee t_p = s1) \wedge t_p \in V(p) \\ \exists t_q \in \mathbb{B} & (t_q = s0 \vee t_q = s1) \wedge t_q \in V(q) \\ \exists t_r \in \mathbb{B} & (t_r = s0 \vee t_r = s1) \wedge t_r \in V(r) \end{cases} \\
&\iff \begin{cases} \{s0, s1\} \cap V(p) \neq \emptyset \\ \{s0, s1\} \cap V(q) \neq \emptyset \\ \{s0, s1\} \cap V(r) \neq \emptyset \end{cases}
\end{aligned}$$

and again, by the claim we have that

$$\begin{aligned}
\mathfrak{M}', s \models \Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r) &\iff (\mathfrak{M}', s \models \Diamond_1(p \wedge q)) \\
&\vee (\mathfrak{M}', s \models \Diamond_1(p \wedge r)) \\
&\vee (\mathfrak{M}', s \models \Diamond_1(q \wedge r)) \\
&\iff (\exists t_1 \in \mathbb{B} \quad R_1(s, t_1) \wedge \mathfrak{M}', t_1 \models p \wedge q) \\
&\vee (\exists t_2 \in \mathbb{B} \quad R_1(s, t_2) \wedge \mathfrak{M}', t_2 \models p \wedge r) \\
&\vee (\exists t_3 \in \mathbb{B} \quad R_1(s, t_3) \wedge \mathfrak{M}', t_3 \models q \wedge r) \\
&\iff (\exists t_1 \in \mathbb{B} \quad (t_1 = s0 \vee t_1 = s1) \wedge t_1 \in V(p) \cap V(q)) \\
&\vee (\exists t_2 \in \mathbb{B} \quad (t_2 = s0 \vee t_2 = s1) \wedge t_2 \in V(p) \cap V(r)) \\
&\vee (\exists t_3 \in \mathbb{B} \quad (t_3 = s0 \vee t_3 = s1) \wedge t_3 \in V(q) \cap V(r)) \\
&\iff \{s0, s1\} \cap V(p) \cap V(q) \neq \emptyset \\
&\vee \{s0, s1\} \cap V(p) \cap V(r) \neq \emptyset \\
&\vee \{s0, s1\} \cap V(q) \cap V(r) \neq \emptyset
\end{aligned}$$

Now suppose that $\mathfrak{M}', s \models \Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r$, which happens if and only if

$$\begin{cases} \{s0, s1\} \cap V(p) \neq \emptyset \\ \{s0, s1\} \cap V(q) \neq \emptyset \\ \{s0, s1\} \cap V(r) \neq \emptyset \end{cases} \text{ as proved previously; by the pigeonhole principle,}$$

since there are 2 strings in $\{s0, s1\}$ and we have 3 sets $V(p)$, $V(q)$ and $V(r)$, there must be at least one string $x \in \{s0, s1\}$ such that $x \in V(a) \cap V(b)$, where $a, b \in \{p, q, r\}$ distinct. Without loss of generality, suppose that $x = s0$ and $a = p$ and $b = q$; then we have that

$$\begin{aligned}
x = s0 \in V(p) \cap V(q) &\implies x \in \{s0, s1\} \cap V(p) \cap V(q) \\
&\implies \{s0, s1\} \cap V(p) \cap V(q) \neq \emptyset \\
&\implies \mathfrak{M}', s \models \Diamond_1(p \wedge q) \vee \Diamond_1(p \wedge r) \vee \Diamond_1(q \wedge r)
\end{aligned}$$

implying that the formula is valid on \mathcal{B} .

(d) $p \rightarrow \Diamond_1 \Box_2 p$

- By definition, for any $m \in \mathbb{N}$ it holds that

$$\begin{aligned}
\mathfrak{M}, m \models \Diamond_1 \Box_2 p &\iff \exists n \in \mathbb{N} \quad S_1(m, n) \wedge \mathfrak{M}, n \models \Box_2 p \\
&\iff \exists n \in \mathbb{N} \quad n = m + 1 \wedge (\forall k \in \mathbb{N} \quad S_2(n, k) \implies \mathfrak{M}, k \models p) \\
&\iff \exists n \in \mathbb{N} \quad n = m + 1 \wedge (\forall k \in \mathbb{N} \quad n > k \implies k \in V(p)) \\
&\iff \forall k \in \mathbb{N} \quad m + 1 > k \implies k \in V(p) \\
&\iff V(p) = \{k \in \mathbb{N} \mid m + 1 > k\}
\end{aligned}$$

Now take $m = 1 \in \mathbb{N}$, and suppose $V(p) = \{m\} = \{1\}$; then $m \in V(p) \iff \mathfrak{M}, m \models p$, however for instance $k = 0 \in \mathbb{N}$ is such that $m + 1 = 1 + 1 = 2 > 0 = k$ although $k = 0 \notin V(p)$, therefore $\exists k \in \mathbb{N} \quad m + 1 > k \wedge k \notin V(p) \implies V(p) \neq \{k \in \mathbb{N} \mid m + 1 > k\} \iff \mathfrak{M}, m \not\models \Diamond_1 \Box_2 p$ which implies that the formula is not valid on \mathcal{N} .

- By definition, for any $s \in \mathbb{B}$ it holds that

$$\begin{aligned}
\mathfrak{M}', s \models \Diamond_1 \Box_2 p &\iff \exists t \in \mathbb{B} \quad R_1(s, t) \wedge \mathfrak{M}', t \models \Box_2 p \\
&\iff \exists t \in \mathbb{B} \quad (t = s0 \vee t = s1) \wedge (\forall u \in \mathbb{B} \quad R_2(t, u) \implies \mathfrak{M}', u \models p) \\
&\iff \exists t \in \mathbb{B} \quad (t = s0 \vee t = s1) \wedge (\forall u \in \mathbb{B} \quad u \sqsubset t \implies u \in V(p))
\end{aligned}$$

Now take $s = 00 \in \mathbb{B}$, and suppose $V(p) = \{s\} = \{00\}$; then $s \in V(p) \iff \mathfrak{M}', s \models p$, however if $t = s0$ or $t = s1$, there still is $u = 0$ such that $u = 0 \sqsubset 00 = t$ although $u = 0 \notin V(p)$, therefore $(t = s0 \vee t = s1) \implies (\exists u \in \mathbb{B} \quad u \sqsubset t \wedge u \notin V(p)) \iff \mathfrak{M}', s \not\models \Diamond_1 \Box_2 p$ which implies that the formula is not valid on \mathcal{B} .

(e) $p \rightarrow \Diamond_2 \Box_1 p$

- By definition, for any $m \in \mathbb{N}$ it holds that

$$\begin{aligned}
\mathfrak{M}, m \models \Diamond_2 \Box_1 p &\iff \exists n \in \mathbb{N} \quad S_2(m, n) \wedge \mathfrak{M}, n \models \Box_1 p \\
&\iff \exists n \in \mathbb{N} \quad m > n \wedge (\forall k \in \mathbb{N} \quad S_1(n, k) \implies \mathfrak{M}, k \models p) \\
&\iff \exists n \in \mathbb{N} \quad m > n \wedge (\forall k \in \mathbb{N} \quad k = n + 1 \implies k \in V(p)) \\
&\iff \exists n \in \mathbb{N} \quad m > n \wedge n + 1 \in V(p)
\end{aligned}$$

Now take $m = 0 \in \mathbb{N}$, and suppose $V(p) = \{m\} = \{0\}$; then $m \in V(p) \iff \mathfrak{M}, m \models p$, however there is no $n \in \mathbb{N}$ such that $m = 0 > n$, therefore $\nexists n \in \mathbb{N} \quad m > n \wedge n + 1 \in V(p) \iff \mathfrak{M}, m \not\models \Diamond_2 \Box_1 p$ which implies that the formula is not valid on \mathcal{N} .

- By definition, for any $s \in \mathbb{B}$ it holds that

$$\begin{aligned}
\mathfrak{M}', s \models \Diamond_2 \Box_1 p &\iff \exists t \in \mathbb{B} \quad R_2(s, t) \wedge \mathfrak{M}', t \models \Box_1 p \\
&\iff \exists t \in \mathbb{B} \quad t \sqsubset s \wedge (\forall u \in \mathbb{B} \quad R_1(t, u) \implies \mathfrak{M}', u \models p) \\
&\iff \exists t \in \mathbb{B} \quad t \sqsubset s \wedge (\forall u \in \mathbb{B} \quad (u = t0 \vee u = t1) \implies u \in V(p)) \\
&\iff \exists t \in \mathbb{B} \quad t \sqsubset s \wedge (t0 \in V(p) \vee t1 \in V(p))
\end{aligned}$$

Now take $s = \varepsilon \in \mathbb{B}$, and suppose $V(p) = \{s\} = \{\varepsilon\}$; then $s \in V(p) \iff \mathfrak{M}', s \models p$, however there is no $t \in \mathbb{B}$ such that $t \sqsubset s$, therefore $\nexists t \in \mathbb{B} \quad t \sqsubset s \wedge (t0 \in V(p) \vee t1 \in V(p)) \iff \mathfrak{M}', s \not\models \Diamond_2 \Box_1 p$ which implies that the formula is not valid on \mathcal{B} .

(f) $p \rightarrow \Box_1 \Diamond_2 p$

- By definition, for any $m \in \mathbb{N}$ it holds that

$$\begin{aligned}
\mathfrak{M}, m \models \Box_1 \Diamond_2 p &\iff \forall n \in \mathbb{N} \quad S_1(m, n) \implies \mathfrak{M}, n \models \Diamond_2 p \\
&\iff \forall n \in \mathbb{N} \quad n = m + 1 \implies (\exists k \in \mathbb{N} \quad S_2(n, k) \wedge \mathfrak{M}, k \models p) \\
&\iff \forall n \in \mathbb{N} \quad n = m + 1 \implies (\exists k \in \mathbb{N} \quad n > k \wedge k \in V(p)) \\
&\iff \exists k \in \mathbb{N} \quad m + 1 > k \wedge k \in V(p)
\end{aligned}$$

Now suppose $m \in V(p) \iff \mathfrak{M}, m \models p$; we observe that for every $m \in \mathbb{N}$ it holds that $m + 1 > m$, therefore $m + 1 > m \wedge m \in V(p) \iff \exists k \in \mathbb{N} \quad m + 1 > k \wedge k \in V(p) \iff \mathfrak{M}, m \models \Box_1 \Diamond_2 p$, which implies that the formula is valid on \mathcal{N} .

- By definition, for any $s \in \mathbb{B}$ it holds that

$$\begin{aligned}
\mathfrak{M}', s \models \Box_1 \Diamond_2 p &\iff \forall t \in \mathbb{B} \quad R_1(s, t) \implies \mathfrak{M}', t \models \Diamond_2 p \\
&\iff \forall t \in \mathbb{B} \quad (t = s0 \vee t = s1) \implies (\exists u \in \mathbb{B} \quad R_2(t, u) \wedge \mathfrak{M}', u \models p) \\
&\iff \forall t \in \mathbb{B} \quad (t = s0 \vee t = s1) \implies (\exists u \in \mathbb{B} \quad u \sqsubset t \wedge u \in V(p))
\end{aligned}$$

Now suppose $s \in V(p) \iff \mathfrak{M}', s \models p$; we observe that for every $s \in \mathbb{B}$ it holds that $s \sqsubset s0, s1$, therefore $\exists u \in \mathbb{B} \quad (u \sqsubset s0 \vee u \sqsubset s1) \wedge u \in V(p) \iff \mathfrak{M}', s \models \Box_1 \Diamond_2 p$, which implies that the formula is valid on \mathcal{B} .

(g) $p \rightarrow \Box_2 \Diamond_1 p$

- By definition, for any $m \in \mathbb{N}$ it holds that

$$\begin{aligned}
\mathfrak{M}, m \models \Box_2 \Diamond_1 p &\iff \forall n \in \mathbb{N} \quad S_2(m, n) \implies \mathfrak{M}, n \models \Diamond_1 p \\
&\iff \forall n \in \mathbb{N} \quad m > n \implies (\exists k \in \mathbb{N} \quad S_1(n, k) \wedge \mathfrak{M}, k \models p) \\
&\iff \forall n \in \mathbb{N} \quad m > n \implies (\exists k \in \mathbb{N} \quad k = n + 1 \wedge k \in V(p)) \\
&\iff \forall n \in \mathbb{N} \quad m > n \implies n + 1 \in V(p)
\end{aligned}$$

Now take $m = 2 \in \mathbb{N}$, and suppose $V(p) = \{m\} = \{2\}$; then $m \in V(p) \iff \mathfrak{M}, m \models p$, however for instance $n = 0 \in \mathbb{N}$ is such that $m = 2 > 0 = n$ and $n + 1 = 0 + 1 = 1 \notin V(p)$, therefore $\exists n \in \mathbb{N} \quad m > n \wedge n + 1 \notin V(p) \iff \mathfrak{M}, m \not\models \Box_2 \Diamond_1 p$, which implies that the formula is not valid on \mathcal{N} .

- By definition, for any $s \in \mathbb{B}$ it holds that

$$\begin{aligned}
\mathfrak{M}', s \models \Box_2 \Diamond_1 p &\iff \forall t \in \mathbb{B} \quad R_2(s, t) \implies \mathfrak{M}', t \models \Diamond_1 p \\
&\iff \forall s \in \mathbb{B} \quad t \sqsubset s \implies (\exists u \in \mathbb{B} \quad R_1(t, u) \wedge \mathfrak{M}', u \models p) \\
&\iff \forall s \in \mathbb{B} \quad t \sqsubset s \implies (\exists u \in \mathbb{B} \quad (u = t0 \vee u = t1) \implies u \in V(p))
\end{aligned}$$

Now take $s = 000 \in \mathbb{B}$, and suppose $V(p) = \{s\} = \{000\}$; then $s \in V(p) \iff \mathfrak{M}', s \models p$, however for instance $t = 0 \in \mathbb{B}$ is such that $t = 0 \sqsubset 000 = s$ although there is no $u = t0 = 00$ or $u = t1 = 01$ such that $u \in V(p)$, therefore $\exists t \in \mathbb{B} \quad t \sqsubset s \wedge (\nexists u \in \mathbb{B} \quad (u = t0 \vee u = t1) \wedge u \in V(p)) \iff \mathfrak{M}', s \not\models \Box_2 \Diamond_1 p$ which implies that the formula is not valid on \mathcal{B} .

□

Exercise 3.2 Consider the basic modal language, and the tuple $\mathfrak{f} = (\mathbb{N}, <, A)$ where A is the collection of finite and co-finite subsets of \mathbb{N} . Show that \mathfrak{f} is a general frame.

Solution. First, consider the following two claims.

Claim 1: If $X \subseteq \mathbb{N}$ is finite, and $Y \subseteq \mathbb{N}$ is co-finite, then $X \cup Y$ is co-finite.

Proof of the Claim. Since X is finite, $\mathbb{N} - X$ is finite, and since Y is co-finite, $\mathbb{N} - Y$ is finite; this implies that

$$\mathbb{N} - (X \cup Y) = (\mathbb{N} - X) \cap (\mathbb{N} - Y)$$

is the intersection of a co-finite and a finite set. In particular, we observe that

- such intersection will be a subset of $\mathbb{N} - Y$ by definition of intersection
- $\mathbb{N} - Y$ is finite
- a subset of a finite set is always finite

concluding that such intersection must be finite as well. Lastly, by definition we have that $\mathbb{N} - (X \cup Y)$ is finite if and only if $X \cup Y$ is co-finite. □

Claim 2: If $X, Y \subseteq \mathbb{N}$ are co-finite, then $X \cup Y$ is co-finite.

Proof of the Claim. By repeating the same argument of the previous claim, we have that $\mathbb{N} - (X \cup Y) = (\mathbb{N} - X) \cap (\mathbb{N} - Y)$ except that in this case both $\mathbb{N} - X$ and $\mathbb{N} - Y$ are finite, which implies that their intersection must be finite, hence $\mathbb{N} - (X \cup Y)$ is co-finite by definition. □

To prove that \mathfrak{f} is a general frame, it suffices to prove that the set A is closed under the following

- *union:* fix two sets $X, Y \in A$; then, by definition of A , we have that
 - if both X and Y are finite, then $X \cup Y$ is finite, hence $X \cup Y \in A$
 - without loss of generality, if X is finite and Y is co-finite, by Claim 1 $X \cup Y$ is co-finite, therefore $X \cup Y \in A$
 - if both X and Y are finite, then $X \cup Y$ is co-finite by Claim 2, therefore $X \cup Y \in A$
- *relative complement:* fix a set $X \in A$; then, by definition of A we trivially have that
 - if X is finite, then $\mathbb{N} - X$ is co-finite, hence $\mathbb{N} - X \in A$
 - if X is co-finite, then $\mathbb{N} - X$ is finite, hence $\mathbb{N} - X \in A$
- *modal operations:* assume that “ $<$ ” is the relation referring to a unary modal operator $\langle < \rangle$, and fix a set $X \in A$; by definition, we have that

$$m_{\langle < \rangle}(X) = \{n \in \mathbb{N} \mid \exists x \in X \quad n < x\}$$

therefore, we have that

- if X is finite, then

$$m_{\langle \cdot \rangle}(X) = \{n \in \mathbb{N} \mid n < \max(X)\}$$

therefore $m_{\langle \cdot \rangle}(X)$ is an “initial segment of \mathbb{N} ”, implying that it is finite, hence $m_{\langle \cdot \rangle}(X) \in A$

- if X is co-finite, then $\mathbb{N} - X$ is finite, implying that X is infinite; this implies that $\max(X)$ is not defined, therefore

$$m_{\langle \cdot \rangle}(X) = \mathbb{N} \implies \mathbb{N} - m_{\langle \cdot \rangle}(X) = \mathbb{N} - \mathbb{N} = \emptyset$$

and since \emptyset is finite, we conclude that $m_{\langle \cdot \rangle}(X)$ is co-finite, thus $m_{\langle \cdot \rangle}(X) \in A$

□

Exercise 5.1 Give K -proofs of $(\Box p \wedge \Diamond q) \rightarrow \Diamond(p \wedge q)$ and $\Diamond(p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q)$.

Solution. In the first section of the solution, we are going to prove some useful derivations that will be extensively used in the actual K -proof of the two propositions. The right side of each line will be one of the following:

- (K): the K axiom
- (D): the Dual axiom
- (T): a propositional Tautology
- (MP(i, j)): the Modus Ponens rule applied on lines i and j
- (S(i)): the Substitution rule applied on line i
- (G(i)): the Generalization rule applied on line i
- (C $_k$ (i_1, \dots, i_n)): the k -th Claim applied on lines i_1, \dots, i_n — $k \in [7]$ and n depends on the number of lines the Claim refers to

Claim 1: If $p \rightarrow q$ can be K -proved, and $q \rightarrow r$ can be K -proved, then $p \rightarrow r$ can be K -proved in 4 steps.

Proof of the Claim. Consider a K -proof in which $p \rightarrow q$ is proved at step i , and $q \rightarrow r$ is proved at step j — without loss of generality suppose that $i < j$; then we have that

$$\begin{array}{ll}
 \dots & \\
 i. \vdash p \rightarrow q & \\
 \dots & \\
 j. \vdash q \rightarrow r & \\
 j+1. \vdash (a \rightarrow b) \rightarrow ((b \rightarrow c) \rightarrow (a \rightarrow c)) & \text{(T)} \\
 j+2. \vdash (p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r)) & \text{(S}(j+1)) \\
 j+3. \vdash (q \rightarrow r) \rightarrow (p \rightarrow r) & \text{(MP}(i, j+2)) \\
 j+4. \vdash p \rightarrow r & \text{(MP}(j, j+3))
 \end{array}$$

□

Claim 2: If $p \rightarrow q$ can be K -proved, then $\Box p \rightarrow \Box q$ can be K -proved in 4 steps.

Proof of the Claim. Consider a K -proof in which $p \rightarrow q$ is proved at step i ; then, we have that

$$\begin{array}{ll}
 \dots & \\
 i. \vdash p \rightarrow q & \\
 i + 1. \vdash \Box(p \rightarrow q) & (G(i)) \\
 i + 2. \vdash \Box(a \rightarrow b) \rightarrow (\Box a \rightarrow \Box b) & (K) \\
 i + 3. \vdash \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q) & (S(i + 2)) \\
 i + 4. \vdash \Box p \rightarrow \Box q & (MP(i + 1, i + 3))
 \end{array}$$

□

Claim 3: If $p \rightarrow q$ can be K -proved, and $p \rightarrow r$ can be K -proved, then $p \rightarrow q \wedge r$ can be K -proved in 4 steps.

Proof of the Claim. Consider a K -proof in which $p \rightarrow q$ is proved at step i , and $p \rightarrow r$ is proved at step j — without loss of generality suppose $i < j$; then, we have that

$$\begin{array}{ll}
 \dots & \\
 i. \vdash p \rightarrow q & \\
 \dots & \\
 j. \vdash p \rightarrow r & \\
 j + 1. \vdash (a \rightarrow b) \rightarrow ((a \rightarrow c) \rightarrow (a \rightarrow b \wedge c)) & (T) \\
 j + 2. \vdash (p \rightarrow q) \rightarrow ((p \rightarrow r) \rightarrow (p \rightarrow q \wedge r)) & (S(j + 1)) \\
 j + 3. \vdash (p \rightarrow r) \rightarrow (p \rightarrow q \wedge r) & (MP(i, j + 2)) \\
 j + 4. \vdash p \rightarrow q \wedge r & (MP(j, j + 3))
 \end{array}$$

□

Claim 4: If $p \rightarrow q$ can be K -proved, then $\neg q \rightarrow \neg p$ can be K -proved in 3 steps. Moreover, if $p \rightarrow \neg q$ can be K -proved, then $q \rightarrow \neg p$ can be K -proved in 3 steps.

Proof of the Claim. Consider a K -proof in which $p \rightarrow q$ is proved at step i ; then, we have that

$$\begin{array}{ll}
 \dots & \\
 i. \vdash p \rightarrow q & \\
 i + 1. \vdash (a \rightarrow b) \rightarrow (\neg b \rightarrow \neg a) & (T) \\
 i + 2. \vdash (p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p) & (S(i + 1)) \\
 i + 3. \vdash \neg q \rightarrow \neg p & (MP(i, i + 2))
 \end{array}$$

The same K -proof can be used to prove the rest of the claim by using the propositional tautology $(a \rightarrow \neg b) \rightarrow (b \rightarrow \neg a)$. □

Claim 5: If $p \leftrightarrow q$ can be K -proved, then $p \rightarrow q$ and $q \rightarrow p$ can be K -proved in 3 steps.

Proof of the Claim. Consider a K -proof in which $p \leftrightarrow q$ is proved at step i ; then, we have that

$$\begin{array}{ll}
 \dots & \\
 i. \vdash p \leftrightarrow q & \\
 i + 1. \vdash (a \leftrightarrow b) \rightarrow (a \rightarrow b) & (T) \\
 i + 2. \vdash (p \leftrightarrow q) \rightarrow (p \rightarrow q) & (S(i + 1)) \\
 i + 3. \vdash p \rightarrow q & (MP(i, i + 2))
 \end{array}$$

The case for $q \rightarrow p$ can be proved analogously by using the propositional tautology $(a \leftrightarrow b) \rightarrow (b \rightarrow a)$. \square

Claim 6: If $p \rightarrow (q \rightarrow r)$ can be K -proved, then $p \wedge q \rightarrow r$ can be K -proved in 3 steps.

Proof of the Claim. Consider a K -proof in which $p \rightarrow (q \rightarrow r)$ is proved at step i ; then, we have that

$$\begin{array}{ll}
 \dots & \\
 i. \vdash p \rightarrow (q \rightarrow r) & \\
 i + 1. \vdash (a \rightarrow (b \rightarrow c)) \rightarrow (a \wedge b \rightarrow c) & (T) \\
 i + 2. \vdash (p \rightarrow (q \rightarrow r)) \rightarrow (p \wedge q \rightarrow r) & (S(i + 1)) \\
 i + 3. \vdash p \wedge q \rightarrow r & (MP(i, i + 2))
 \end{array}$$

\square

Claim 7: If $p \rightarrow q$ can be K -proved, and $q \rightarrow p$ can be K -proved, then $p \leftrightarrow q$ can be proved in 4 steps.

Proof of the Claim. Consider a K -proof in which $p \rightarrow q$ is proved at step i , and $q \rightarrow p$ can be proved at step j — without loss of generality suppose $i < j$; then, we have that

$$\begin{array}{ll}
 \dots & \\
 i. \vdash p \rightarrow q & \\
 \dots & \\
 j. \vdash q \rightarrow p & \\
 j + 1. \vdash (a \rightarrow b) \rightarrow ((b \rightarrow a) \rightarrow (a \leftrightarrow b)) & (T) \\
 j + 2. \vdash (p \rightarrow q) \rightarrow ((q \rightarrow p) \rightarrow (p \leftrightarrow q)) & (S(j + 1)) \\
 j + 3. \vdash (q \rightarrow p) \rightarrow (p \leftrightarrow q) & (MP(i, j + 2)) \\
 j + 4. \vdash p \leftrightarrow q & (MP(j, j + 3))
 \end{array}$$

\square

Now that we proved some preliminary claims, we can prove the two given propositions.

Claim 8: $(\Box p \wedge \Diamond q) \rightarrow \Diamond(p \wedge q)$ is K -provable.

Proof of the Claim.

1. $\vdash \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$ (K)
2. $\vdash (\neg a \vee \neg b) \rightarrow (a \rightarrow \neg b)$ (T)
3. $\vdash (\neg p \vee \neg q) \rightarrow (p \rightarrow \neg q)$ (S(2))
- ...
7. $\vdash \Box(\neg p \vee \neg q) \rightarrow \Box(p \rightarrow \neg q)$ ($C_2(3)$)
8. $\vdash \Box(p \rightarrow \neg q) \rightarrow (\Box p \rightarrow \Box \neg q)$ (S(1))
- ...
12. $\vdash \Box(\neg p \vee \neg q) \rightarrow (\Box p \rightarrow \Box \neg q)$ ($C_1(7, 8)$)
13. $\vdash (a \rightarrow b) \rightarrow (\neg a \vee b)$ (T)
14. $\vdash (\Box p \rightarrow \Box \neg q) \rightarrow (\neg \Box p \vee \Box \neg q)$ (S(13))
- ...
18. $\vdash \Box(\neg p \vee \neg q) \rightarrow (\neg \Box p \vee \Box \neg q)$ ($C_1(12, 14)$)
19. $\vdash (\neg a \vee b) \rightarrow \neg(a \wedge \neg b)$ (T)
20. $\vdash (\neg \Box p \vee \Box \neg q) \rightarrow \neg(\Box p \wedge \neg \Box \neg q)$ (S(19))
- ...
24. $\vdash \Box(\neg p \vee \neg q) \rightarrow \neg(\Box p \wedge \neg \Box \neg q)$ ($C_1(18, 20)$)
- ...
27. $\vdash (\Box p \wedge \neg \Box \neg q) \rightarrow \neg \Box(\neg p \vee \neg q)$ ($C_4(24)$)
28. $\vdash \neg(a \wedge b) \rightarrow (\neg a \vee \neg b)$ (T)
29. $\vdash \neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$ (S(27))
- ...
33. $\vdash \Box \neg(p \wedge q) \rightarrow \Box(\neg p \vee \neg q)$ ($C_2(28)$)
- ...
36. $\vdash \neg \Box(\neg p \vee \neg q) \rightarrow \neg \Box \neg(p \wedge q)$ ($C_4(32)$)
- ...
40. $\vdash (\Box p \wedge \neg \Box \neg q) \rightarrow \neg \Box \neg(p \wedge q)$ ($C_1(27, 36)$)
41. $\vdash \Diamond a \leftrightarrow \neg \Box \neg a$ (D)
42. $\vdash \Diamond(p \wedge q) \leftrightarrow \neg \Box \neg(p \wedge q)$ (S(41))
- ...
45. $\vdash \neg \Box \neg(p \wedge q) \rightarrow \Diamond(p \wedge q)$ ($C_5(42)$)
- ...
49. $\vdash (\Box p \wedge \neg \Box \neg q) \rightarrow \Diamond(p \wedge q)$ ($C_1(40, 45)$)

50. $\vdash \Diamond q \leftrightarrow \neg \Box \neg q$ (S(41))
...
53. $\vdash \Diamond q \rightarrow \neg \Box \neg q$ ($C_5(50)$)
54. $\vdash (b \rightarrow c) \rightarrow (a \wedge b \rightarrow a \wedge c)$ (T)
55. $\vdash (\Diamond q \rightarrow \neg \Box \neg q) \rightarrow (\Box p \wedge \Diamond q \rightarrow \Box p \wedge \neg \Box \neg q)$ (S(54))
56. $\vdash \Box p \wedge \Diamond q \rightarrow \Box p \wedge \neg \Box \neg q$ (MP(53, 55))
...
60. $\vdash (\Box p \wedge \Diamond q) \rightarrow \Diamond(p \wedge q)$ ($C_1(56, 49)$)

□

This claim concludes the K -proof of the first proposition. To K -prove the second proposition, we are going to split the K -proof into 3 claims.

Claim 9: $(\Diamond p \vee \Diamond q) \rightarrow \Diamond(p \vee q)$ is K -provable.

Proof of the Claim.

1. $\vdash \neg p \wedge \neg q \rightarrow \neg p$ (T)
- ...
5. $\vdash \Box(\neg p \wedge \neg q) \rightarrow \Box \neg p$ ($C_2(1)$)
6. $\vdash \neg p \wedge \neg q \rightarrow \neg q$ (T)
- ...
10. $\vdash \Box(\neg p \wedge \neg q) \rightarrow \Box \neg q$ ($C_2(2)$)
- ...
14. $\vdash \Box(\neg p \wedge \neg q) \rightarrow \Box \neg p \wedge \Box \neg q$ ($C_3(10)$)
- ...
17. $\vdash \neg(\Box \neg p \wedge \Box \neg q) \rightarrow \neg \Box(\neg p \wedge \neg q)$ ($C_4(14)$)
18. $\vdash \neg a \vee \neg b \rightarrow \neg(a \wedge b)$ (T)
19. $\vdash \neg \Box \neg p \vee \neg \Box \neg q \rightarrow \neg(\Box \neg p \wedge \Box \neg q)$ (S(18))
- ...
23. $\vdash \neg \Box \neg p \vee \neg \Box \neg q \rightarrow \neg \Box(\neg p \wedge \neg q)$ ($C_1(22, 17)$)
24. $\vdash \neg(a \vee b) \rightarrow \neg a \wedge \neg b$ (T)
25. $\vdash \neg(p \vee q) \rightarrow \neg p \wedge \neg q$ (S(24))
- ...
29. $\vdash \Box \neg(p \vee q) \rightarrow \Box(\neg p \wedge \neg q)$ ($C_2(25)$)
- ...
32. $\vdash \neg \Box(\neg p \wedge \neg q) \rightarrow \neg \Box \neg(p \vee q)$ ($C_4(29)$)
- ...
36. $\vdash \neg \Box \neg p \vee \neg \Box \neg q \rightarrow \neg \Box \neg(p \vee q)$ ($C_1(23, 32)$)
37. $\vdash \Diamond a \leftrightarrow \neg \Box \neg a$ (D)
- ...
40. $\vdash \neg \Box \neg a \rightarrow \Diamond a$ ($C_5(37)$)
41. $\vdash \neg \Box \neg(p \vee q) \rightarrow \Diamond(p \vee q)$ (S(40))
- ...
45. $\vdash \neg \Box \neg p \vee \neg \Box \neg q \rightarrow \Diamond(p \vee q)$ ($C_1(36, 41)$)
- ...
49. $\vdash \Diamond a \rightarrow \neg \Box \neg a$ ($C_5(37)$)
50. $\vdash \Diamond p \rightarrow \neg \Box \neg p$ (S(49))
51. $\vdash \Diamond q \rightarrow \neg \Box \neg q$ (S(49))
52. $\vdash (a \rightarrow c) \rightarrow ((b \rightarrow d) \rightarrow (a \vee b \rightarrow c \vee d))$ (T)
53. $\vdash (\Diamond p \rightarrow \neg \Box \neg p) \rightarrow ((\Diamond q \rightarrow \neg \Box \neg q) \rightarrow (\Diamond p \vee \Diamond q \rightarrow \neg \Box \neg p \vee \neg \Box \neg q))$ (S(52))
54. $\vdash (\Diamond q \rightarrow \neg \Box \neg q) \rightarrow (\Diamond p \vee \Diamond q \rightarrow \neg \Box \neg p \vee \neg \Box \neg q)$ (MP(50, 53))
55. $\vdash \Diamond p \vee \Diamond q \rightarrow \neg \Box \neg p \vee \neg \Box \neg q$ (MP(51, 54))
- ...
59. $\vdash \Diamond p \vee \Diamond q \rightarrow \Diamond(p \vee q)$ ($C_1(55, 48)$)

□

Claim 10: $\Diamond(p \vee q) \rightarrow (\Diamond p \vee \Diamond q)$ is K -provable.

Proof of the Claim.

1. $\vdash \neg p \rightarrow (\neg q \rightarrow \neg p \wedge \neg q)$ (T)
- ...
5. $\vdash \Box \neg p \rightarrow \Box(\neg q \rightarrow \neg p \vee \neg q)$ ($C_2(1)$)
6. $\vdash \Box(a \rightarrow b) \rightarrow (\Box a \rightarrow \Box b)$ (K)
7. $\vdash \Box(\neg q \rightarrow \neg p \wedge \neg q) \rightarrow (\Box \neg q \rightarrow \Box(\neg p \wedge \neg q))$ (S(6))
- ...
11. $\vdash \Box \neg p \rightarrow (\Box \neg q \rightarrow \Box(\neg p \wedge \neg q))$ ($C_1(5, 7)$)
- ...
14. $\vdash \Box \neg p \wedge \Box \neg q \rightarrow \Box(\neg p \wedge \neg q)$ ($C_6(11)$)
- ...
17. $\vdash \neg \Box(\neg p \wedge \neg q) \rightarrow \neg(\Box \neg p \wedge \Box \neg q)$ ($C_4(14)$)
18. $\vdash \neg(a \wedge b) \rightarrow \neg a \vee \neg b$ (T)
19. $\vdash \neg(\Box \neg p \wedge \Box \neg q) \rightarrow \neg \Box \neg p \vee \neg \Box \neg q$ (S(18))
- ...
23. $\vdash \neg \Box(\neg p \wedge \neg q) \rightarrow \neg \Box \neg p \vee \neg \Box \neg q$ ($C_1(17, 19)$)
24. $\vdash (\neg a \wedge \neg b) \rightarrow \neg(a \vee b)$ (T)
25. $\vdash (\neg p \wedge \neg q) \rightarrow \neg(p \vee q)$ (S(24))
- ...
29. $\vdash \Box(\neg p \wedge \neg q) \rightarrow \Box \neg(p \vee q)$ ($C_2(25)$)
- ...
32. $\vdash \neg \Box \neg(p \vee q) \rightarrow \neg \Box(\neg p \wedge \neg q)$ ($C_4(29)$)
- ...
36. $\vdash \neg \Box \neg(p \vee q) \rightarrow \neg \Box \neg p \vee \neg \Box \neg q$ ($C_1(32, 23)$)
37. $\vdash \Diamond a \leftrightarrow \neg \Box \neg a$ (D)
- ...

- ...
41. $\vdash \neg \Box \neg a \rightarrow \Diamond a$ (C₅(37))
42. $\vdash \neg \Box \neg p \rightarrow \Diamond p$ (S(41))
43. $\vdash \neg \Box \neg q \rightarrow \Diamond q$ (S(41))
44. $\vdash (a \rightarrow c) \rightarrow ((b \rightarrow d) \rightarrow (a \vee b \rightarrow c \vee d))$ (T)
45. $\vdash (\neg \Box \neg p \rightarrow \Diamond p) \rightarrow ((\neg \Box \neg q \rightarrow \Diamond q) \rightarrow (\neg \Box \neg p \vee \neg \Box \neg q \rightarrow \Diamond p \vee \Diamond q))$ (S(44))
46. $\vdash (\neg \Box \neg q \rightarrow \Diamond q) \rightarrow (\neg \Box \neg p \vee \neg \Box \neg q \rightarrow \Diamond p \vee \Diamond q)$ (MP(41, 45))
47. $\vdash \neg \Box \neg p \vee \neg \Box \neg q \rightarrow \Diamond p \vee \Diamond q$ (MP(42, 46))
- ...
51. $\vdash \Diamond a \rightarrow \neg \Box \neg a$ (C₅(37))
52. $\vdash \Diamond(p \vee q) \rightarrow \neg \Box \neg(p \vee q)$ (S(51))
- ...
56. $\vdash \Diamond(p \vee q) \rightarrow (\neg \Box \neg p \vee \neg \Box \neg q)$ (C₁(52, 36))
- ...
60. $\vdash \Diamond(p \vee q) \rightarrow \Diamond p \vee \Diamond q$ (C₁(56, 47))

□

Claim 11: $\Diamond(p \vee q) \leftrightarrow \Diamond(p \vee \Diamond q)$ is K -provable.

Proof of the Claim.

- ...
59. $\vdash \Diamond p \vee \Diamond q \rightarrow \Diamond(p \vee q)$ (C₉)
- ...
119. $\vdash \Diamond(p \vee q) \rightarrow \Diamond p \vee \Diamond q$ (C₁₀)
- ...
123. $\vdash \Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$ (C₇(2, 1))

□

This last claim concludes that the second proposition is K -provable as well.

□