Marp

Markdown Presentation Ecosystem

https://marp.app/

Syntax - Formulas

Given Φ_0 the set of *atomic formulas*, for any $\phi, \psi \in \Phi_0$

- $\phi \in \mathrm{Form}(\Phi_0)$
- $\neg\phi \in \mathrm{Form}(\Phi_0)$
- $\phi \vee \psi \in \mathrm{Form}(\Phi_0)$
- $[\alpha]\phi \in \mathrm{Form}(\Phi_0)$

where $\alpha \in \mathrm{Prog}(\Pi_0)$

Syntax - Programs

Given Π_0 the set of *atomic programs*, for any $\alpha, \beta \in \Pi_0$

- $\alpha \in \mathrm{Prog}(\Pi_0)$
- $(\alpha; \beta) \in \mathrm{Prog}(\Pi_0)$
- $(\alpha \cup \beta) \in \mathrm{Prog}(\Pi_0)$
- $\alpha^* \in \mathrm{Prog}(\Pi_0)$
- $\phi? \in \mathrm{Prog}(\Pi_0)$

where $\phi \in \mathrm{Form}(\Phi_0)$

Syntax - Relations

$$(x, y) \in R(\pi) \iff x \xrightarrow{\pi} y$$

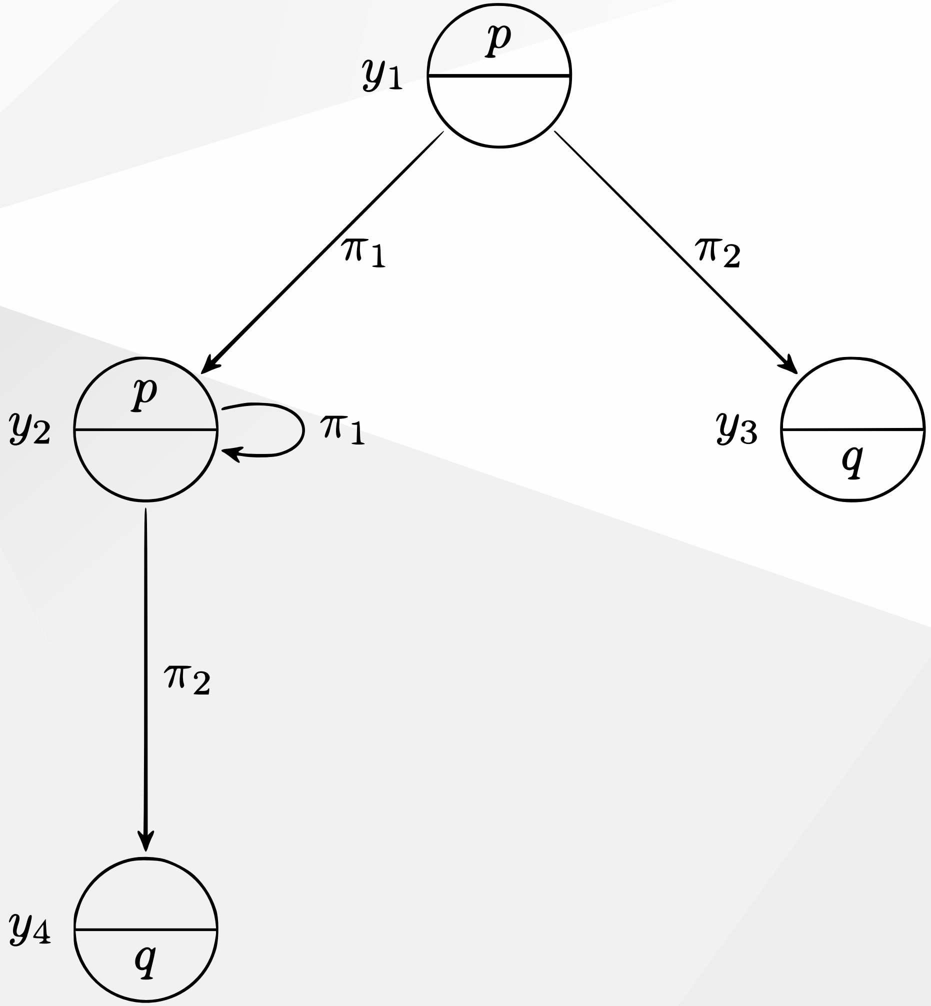
- $(x, y) \in R(\alpha; \beta) \iff \exists z \in W \quad (x, z) \in R(\alpha) \wedge (z, y) \in R(\beta)$
- $(x, y) \in R(\alpha \cup \beta) \iff (x, y) \in R(\alpha) \vee (x, y) \in R(\beta)$
- $(x, y) \in R(\alpha^*) \iff \exists z_0, \ldots, z_n \in W \quad \begin{cases} z_0 = x \\ z_n = y \\ (z_{k-1}, z_k) \in R(\alpha) \end{cases}$
- $(x, y) \in R(\phi?) \iff x = y \wedge y \in V(\phi)$

Syntax - Valuations

$$x \in V(p) \iff p \text{ is true at } x$$

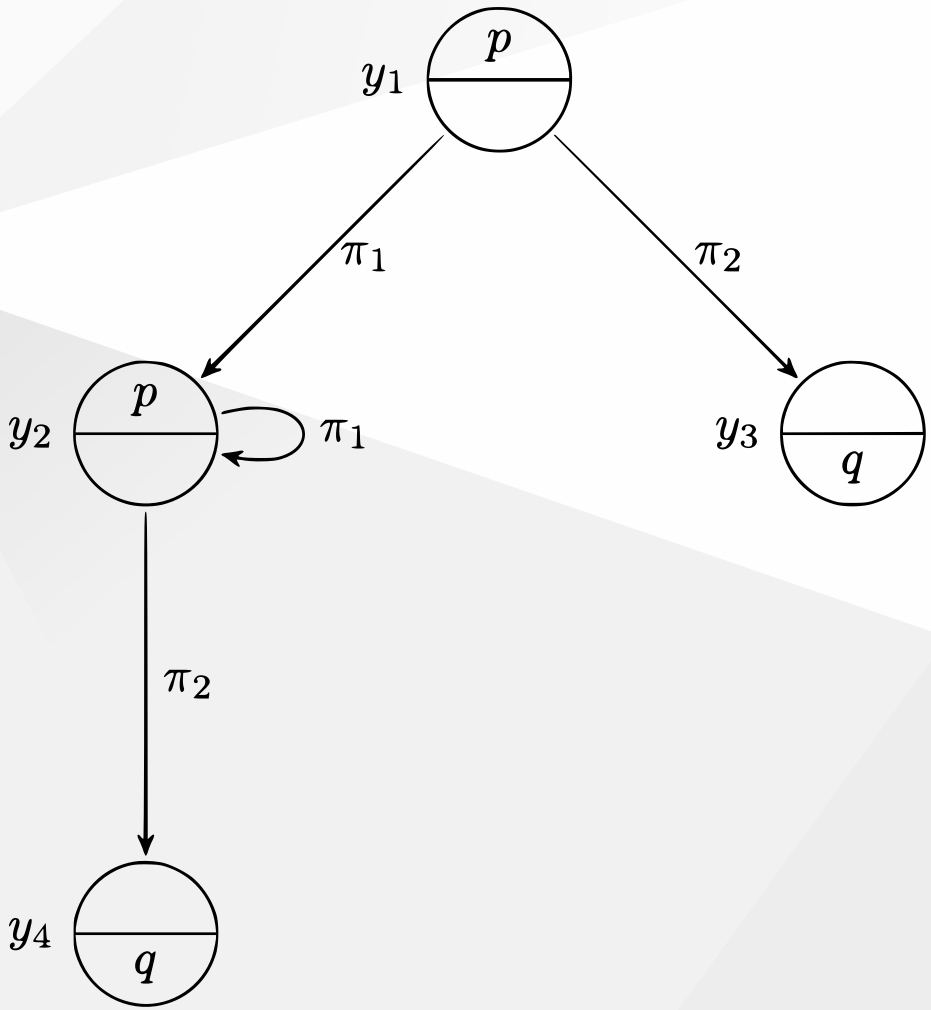
- $V(\bot) = \varnothing$
- $V(\top) = W$
- $V(\neg\phi) = W - V(\phi)$
- $V(\phi \vee \psi) = V(\phi) \cup V(\psi)$
- $V([\alpha]\phi) = \{x \mid \forall y \in W \quad (x, y) \in R(\alpha) \implies y \in V(\phi)\}$

LTS



- $W = \{y_1, y_2, y_3, y_4\}$
- $R(\pi_1) = \{(y_1, y_2), (y_2, y_2)\}$
- $R(\pi_2) = \{(y_1, y_3), (y_2, y_4)\}$
- $V(p) = \{y_1, y_2\}$
- $V(q) = \{y_3, y_4\}$

LTS



- $\mathfrak{M}, y_1 \models \langle \pi_1^*; \pi_2 \rangle q$
- $\mathfrak{M}, y_2 \models [\pi_1^*] p$
- $\mathfrak{M}, y_1 \models [\pi_1 \cup \pi_2](p \vee q)$
- $\mathfrak{M}, y_3 \models [\pi_1 \cup \pi_2] \bot$

Axiomatization - Validity

We write $\mathfrak{M}, w \models \phi$ if and only if $w \in V(\phi)$

ϕ is *valid* in $\mathfrak{M}$, written as $\mathfrak{M} \models \phi$, if and only if

$$\mathfrak{M} \models \phi \iff \forall w \in W \quad \mathfrak{M}, w \models \phi$$

ϕ is **valid**, written as $\models \phi$, if and only if

$$\models \phi \iff \forall \mathfrak{M} \quad \mathfrak{M} \models \phi$$

Axiomatization - Goal

The goal is to define a **decudibility predicate** $\vdash$ such that $\vdash$-deductions are both *sound* and *complete* in terms of validity, i.e. for any ϕ it holds that

$$\vdash \phi \iff \models \phi$$

Axiomatization - K and N axioms

$$\text{(K)} \qquad [\alpha](\phi \to \psi) \to ([\alpha]\phi \to [\alpha]\psi)$$

$$\text{(N)} \qquad \frac{\phi}{[\pi]\phi}$$

A modal logic is **normal** if it obeys (K) and (N)

Axiomatization - PDL axioms

PDL is the *least normal* modal logic containing every instance of

$$
\begin{array}{ll}
(\mathrm{A1}) & [\alpha;\beta]\phi \leftrightarrow [\alpha][\beta]\phi \\
(\mathrm{A2}) & [\alpha\cup\beta]\phi \leftrightarrow [\alpha]\phi \wedge [\beta]\phi \\
(\mathrm{A3}) & [\alpha^*]\phi \leftrightarrow \phi \wedge [\alpha][\alpha^*]\phi \\
(\mathrm{A4}) & [\phi?]\psi \leftrightarrow (\phi \rightarrow \psi)
\end{array}
$$

and closed under the *loop invariance* rule of inference

$$
(\mathrm{I}) \qquad \frac{\phi \rightarrow [\alpha]\phi}{\phi \rightarrow [\alpha^*]\phi}
$$

Axiomatization - $\vdash$-deducibility

A formula ϕ is $\vdash$-*deducible* from $\Sigma \subseteq \mathrm{Form}(\Phi_0)$ if there exists a sequence $\phi_0, \ldots, \phi_n$ such that $\phi_n = \phi$, and for all $i \in [n]$

- ϕ_i is an instance of an axiom schema
- ϕ_i is an instance of a formula of Σ
- ϕ_i comes from earlier formulas of the sequence by inference

Are $\vdash$-deductions sound and complete?

Completeness - Segerberg's axioms

In 1977 Segerberg proposed to replace

$$(\mathrm{I}) \qquad \frac{\phi \to [\alpha]\phi}{\phi \to [\alpha^*]\phi}$$

with the following fifth axiom

$$(\mathrm{A5}) \qquad \phi \land [\alpha^*](\phi \to [\alpha]\phi) \to [\alpha^*]\phi$$

in order to prove that such axiomatization was sound and complete

Completeness - Segerberg's axioms

Indeed, it is easy to prove that (I) can be replaced with $(\mathrm{A5})$

1. $\vdash \phi \to [\alpha]\phi$	(premise)
2. $\vdash [\alpha^*](\phi \to [\alpha]\phi)$	(from 1 using (N) with $\pi = \alpha^*$)
3. $\vdash \phi \land [\alpha^*](\phi \to [\alpha]\phi) \to [\alpha^*]\phi$	(A5)
4. $\vdash [\alpha^*](\phi \to [\alpha]\phi) \to (\phi \to [\alpha^*]\phi)$	(from 3 through prop. reasoning)
5. $\vdash \phi \to [\alpha^*]\phi$	(from 2 and 4 using *Modus Ponens*)

Completeness

To prove that $\vdash$ is sound w.r.t. $\models$, i.e. that

$$\vdash \phi \implies \models \phi$$

a proof by induction on the length of ϕ's deduction in $\vdash$ suffices.

So, what about completeness? It requires to prove that

$$\models \phi \implies \vdash \phi$$

Completeness

Segerberg's work was the first attempt to prove the completeness of $\vdash$, however in 1978 he found a flaw in his argument.

Then in the same year Parikh published what is now considered the first proof of the completeness of $\vdash$.

Completeness - Goldblatt

Since then, different alternative proof theories of PDL have also been sought after. For example, in 1992 Goldblatt proposed the $\vdash'$-deducibility predicate, which is based on the same first four axiom schemas along with this *infinitary* rule of inference

$$(\mathrm{I}') \qquad \frac{\{[\beta][\alpha^n]\phi \mid n \in \mathbb{N}\}}{[\beta][\alpha^*]\phi}$$

Goldblatt was able to prove that $\vdash'$ is both sound and complete.

Complexity - PDL satisfiability

ϕ is *satisfiable* in $\mathfrak{M}$ if there is a world $w \in W$ such that $\mathfrak{M}, w \models \phi$

ϕ is **satisfiable** if there is a model $\mathfrak{M}$ such that ϕ is satisfiable in $\mathfrak{M}$

$$\text{PDL-SAT} := \{\langle\phi\rangle \mid \phi \text{ is a satisfiable PDL formula}\}$$

Complexity - Unsatisfiable formulas

ϕ is *unsatisfiable* if and only if $\neg\phi$ is *valid*

Therefore, we can use the recursive definition of valid PDL formulas and build a procedure P that enumerates all the $\vdash$-deducible formulas

Hence, given enough time if $\neg\phi$ is $\vdash$-deducible P will eventually find it and determine that ϕ is *unsatisfiable*

This proves that $\mathrm{PDL\text{-}SAT} \in$ coREC

Complexity - Satisfiable formulas

However, if ϕ is satisfiable P never terminates.

Nonetheless, we can leverage the **finite model property** of PDL

$$\forall \phi \in \mathrm{Form}(\Phi_0) \quad \langle \phi \rangle \in \mathrm{PDL\text{-}SAT} \implies \exists \mathfrak{M}_{fin} \text{ finite} \quad \phi \text{ satisfiable in } \mathfrak{M}_{fin}$$

Therefore, there is a procedure P' that enumerates all the finite models $\mathfrak{M}_{fin}$ and checks for each model if ϕ is satisfiable in $\mathfrak{M}_{fin}$.

Thus, P and P' can be run in parallel to decide $\mathrm{PDL\text{-}SAT}$. However, this is *very* inefficient, can we do any better?

Complexity

Kozen and Parikh proved that PDL has also the **small model property**

$$\forall \phi \in \text{Form}(\Phi_0) \quad \langle \phi \rangle \in \text{PDL-SAT} \implies \exists \mathfrak{M}_{fin} \text{ finite} \quad \begin{cases} |\mathfrak{M}_{fin}| < \exp(|\phi|) \\ \phi \text{ satisfiable in } \mathfrak{M}_{fin} \end{cases}$$

This property implies that we can stop P' as soon as all the "small" models have been exhausted, to conclude that ϕ is not satisfiable.

This concludes that $\text{PDL-SAT} \in$ NEXP. In 1980 Pratt was able to prove that $\text{PDL-SAT} \in$ EXP-complete.