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Models of Computation

Lecture notes integrated with the book TODO

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Information and Contacts

Personal notes and summaries collected as part of the *Models of Computation* course offered by the degree in Computer Science of the University of Rome "La Sapienza".

Further information and notes can be found at the following link:

<https://github.com/aflaag-notes>. Anyone can feel free to report inaccuracies, improvements or requests through the Issue system provided by GitHub itself or by contacting the author privately:

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The notes are constantly being updated, so please check if the changes have already been made in the most recent version.

Suggested prerequisites:

- Linguaggi di Programmazione
- Tecniche di Programmazione Funzionale ed Imperativa

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1.1 TODO

1.1.1 TODO

In this first section, examples will be omitted from this notes, refer to the notes of the “[Linguaggi di Programmazione](#)” course for further details.

Definition 1.1: Lambda calculus

Let Var be the set of all possible variables; thus, the **set Λ of all possible λ -terms** is defined by the following rules:

$$[var] \frac{x \in \text{Var}}{x \in \Lambda}$$

$$[appl] \frac{M \in \Lambda \quad N \in \Lambda}{MN \in \Lambda}$$

$$[abs] \frac{x \in \text{Var} \quad M \in \Lambda}{\lambda x.M \in \Lambda}$$

The terms of the form $\lambda x.M$ are called **λ -abstractions**, and MN is the function application of M to N . Note that function application *associates to the left*, therefore

$$MNL = (MN)L \neq M(NL)$$

Lambda calculus can be alternatively defined with the **Backus Normal Form** (BNF), as follows:

$$M, N ::= x \mid \lambda x.M \mid MN$$

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Although *all functions in lambda calculus are unary*, the following definition can expand this concept.

Definition 1.2: Currying

Currying (named after [Haskell Curry](#)) is defined as follows:

$$\lambda x_1.(\dots(\lambda x_n.y)) \equiv \lambda x_1 \dots x_n.y$$

Definition 1.3: Boundness

A variable is said to be **bound** if it is declared in a λ -abstraction, otherwise it is said to be **free**.

A term that has no free variables is said to be **closed** or **combinator**.

Example 1.1 (Boundness). Consider the following term:

$$\lambda x.xy$$

In this example, x is *bound*, and y is *free*.

Definition 1.4: Notable combinators

The following are some of the **notable combinators**:

$$\begin{aligned} I &\equiv \lambda x.x \\ K &\equiv \lambda xy.x \\ O &\equiv \lambda xy.y \\ S &\equiv \lambda xyz.xz(yz) \\ B &\equiv \lambda fgx.f(gx) \\ C &\equiv \lambda abc.acb \\ W &\equiv \lambda xy.xyy \end{aligned}$$

Definition 1.5: Free variables

Given a λ -term, the function

$$\text{free} : \Lambda \rightarrow \mathcal{P}(\text{Var})$$

returns the **set of free variables in** M , and it is defined recursively as follows:

$$\begin{cases} \text{free}(x) := \{x\} \\ \text{free}(MN) := \text{free}(M) \cup \text{free}(N) \\ \text{free}(\lambda x.M) := \text{free}(M) - \{x\} \end{cases}$$

Definition 1.6: Substitution

The **substitution** operation is recursively defined by the following rules:

$$x[N/x] = N$$

$$y[N/x] = y$$

$$(PQ)[N/x] = P[N/x] Q[N/x]$$

$$(\lambda t.P)[N/x] = \lambda t.(P[N/x])$$

where $M[N/x]$ means that *each instance of x in M is replaced with N* . Note that **only free variables may be substituted**.

Definition 1.7: Inference rules

The following are the **inference rules** for the lambda calculus:

$$(\alpha) \lambda x.M \equiv (\lambda y.M)[y/x]$$

$$(\beta) (\lambda x.M)N \xrightarrow{\beta} M[N/x]$$

$$(\mu) \frac{M \xrightarrow{\beta} M'}{NM \xrightarrow{\beta} NM'}$$

$$(\nu) \frac{M \xrightarrow{\beta} M'}{MN \xrightarrow{\beta} M'N}$$

$$(\xi) \frac{M \xrightarrow{\beta} M'}{\lambda x.M \xrightarrow{\beta} \lambda x.M'}$$

Note that the β -rule is effectively *one step of the computation* described by the λ -term.

Definition 1.8: Normal form

If a term can be β -reduced, it is called **β -redex**, or simply **redex** (*reducible expression*), and the reduced term is called **β -reduct**, or simply **reduct**.

If a term has no redexes, it is said to be in **normal form**.

Observation 1.1: Variable capture

Consider the following λ -term:

$$(\lambda x t. t x)(\lambda t. y) \xrightarrow{\beta} \lambda t. t(\lambda t. y)$$

Note that the two t s are *different*. In fact, underlining the λ -abstractions to which they are bounded to can help clarifying their distinction:

$$(\lambda x \underline{t}. \underline{t} x)(\lambda t. y) \xrightarrow{\beta} \lambda \underline{t}. \underline{t}(\lambda t. y)$$

Now, consider the following λ -term, similar to the previous one:

$$(\lambda x y. y x)(\lambda t. y) \xrightarrow{\beta} \lambda y. y(\lambda t. y)$$

This β -reduction created a problem, because now the two y s *are the same*, even though they were not originally. In fact, the previous term can be relabeled as follows:

$$(\lambda x \underline{y}. \underline{y} x)(\lambda t. y) \xrightarrow{\beta} \lambda \underline{y}. \underline{y}(\lambda t. \underline{y})$$

This happened because

$$\text{free}(\lambda t. y) = \{y\} - \{t\} = \{y\}$$

therefore y was **captured** by the y that was already present in the leftmost λ -abstraction. This phenomena is called **variable capturing**, and constitutes a problem when reducing β -redexes. In particular, to reduce this second λ -abstraction, it is necessary to apply a substitution, by using the α rule (refer to [Definition 1.7](#)):

$$\lambda x y. y x = \lambda x (\lambda y. y x) = \lambda x. ((\lambda y. y x)[u/y]) = \lambda x. (\lambda u. u x) = \lambda x u. u x$$

which means that the β -reduction can now be performed without any issue:

$$(\lambda x u. u x)(\lambda t. y) \xrightarrow{\beta} \lambda u. u(\lambda t. y)$$

where y is still free. Note that it would not have been *safe* to rename the other (free) y , because in general *renaming free variables can create capturing problems as well*. For example, y could have not been substituted with t , as it would otherwise be captured by the t in the λ -term $\lambda t. y$, as follows:

$$(\lambda t. y)[t/y] = \lambda t. t$$

Fortunately, variable capturing can be solved by employing the following *variable naming convention*.

Definition 1.9: Variable naming convention

To avoid variable capturing problems, it is sufficient to follow this **variable naming convention**: *bound and free variables must have different names between them.*

From now on, it will be assumed that any β -reduction is performed by renaming opportunely the **bound** variables, such that in each step of the computation the naming convention is followed.

Definition 1.10: Tuples

A **tuple** of the form

$$(M_1, \dots, M_k)$$

can be represented in λ -calculus as follows:

$$\lambda x.xM_1 \dots M_k$$

To access the elements of a tuple, *projectors* are used, which are defined below

Definition 1.11: Projector

A **projector** has the following form

$$\lambda x.x\pi_j^k$$

where

$$\pi_j^k \equiv \lambda x_1 \dots x_k.x_j$$

Example 1.2 (Projectors). Given a tuple $\lambda x.xM_1 \dots M_k$, its j -th element can be accessed as follows:

$$\lambda x.x\pi_j^k(\lambda x.xM_1 \dots M_k) \xrightarrow{\beta} (\lambda x.xM_1 \dots M_k)\pi_j^k \xrightarrow{\beta} \pi_j^k M_1 \dots M_k \xrightarrow{\beta} M_j$$

Definition 1.12: Booleans

Booleans can be defined in λ -calculus as follows:

$$\mathbf{T} \equiv \lambda xy.x$$

$$\mathbf{F} \equiv \lambda xy.y$$

Definition 1.13: Conditionals

Conditionals can be defined in λ -calculus as follows:

$$\mathbf{ite} = \lambda xyz.xyz$$

Note that *ite* stands for “*if-then-else*”, and in fact, the term behaves exactly like a condition when used in conjunction with the λ -booleans.

Observation 1.2: Conditionals

The term *ite* correctly behaves as a *conditional* when used with T and F. In fact, when used in an term such as

$$\text{ite } C \ A \ B$$

if C is a λ -boolean, the term will be β -reduced to A if $C \equiv T$, and it will be evaluated to B if $C \equiv F$. Indeed

$$\begin{aligned} \text{ite } T \ A \ B &\equiv (\lambda xyz. xyz) \ T \ A \ B \\ &\xrightarrow{\beta} T \ A \ B \\ &\equiv (\lambda xy. x) \ A \ B \\ &\xrightarrow{\beta} A \end{aligned}$$

and

$$\begin{aligned} \text{ite } F \ A \ B &\equiv (\lambda xyz. xyz) \ F \ A \ B \\ &\xrightarrow{\beta} F \ A \ B \\ &\equiv (\lambda xy. y) \ A \ B \\ &\xrightarrow{\beta} B \end{aligned}$$

Definition 1.14: Church numerals

The **Church numerals** are defined by a mapping between natural numbers \mathbb{N} and λ -abstractions:

$$\varphi : \mathbb{N} \rightarrow \Lambda : n \mapsto \lambda xy. x(\underbrace{\dots (xy)}_{n \text{ times}})$$

The Church numeral corresponding to $n \in \mathbb{N}$ will be represented as $\underline{n} \in \Lambda$.

Example 1.3 (Church numerals). For example, the number $\underline{0}$ is represented as

$$\varphi(0) = \underline{0} = \lambda xy. y \equiv F \equiv O$$

number $\underline{1}$ as

$$\varphi(1) = \underline{1} = \lambda xy. xy \equiv T \equiv K$$

and number $\underline{2}$ as

$$\varphi(2) = \underline{2} = \lambda xy. x(xy)$$

and so on.

Observation 1.3: Church numerals are iterators

Note that Church numerals are **iterators**, in the sense that \underline{n} replicates any input function f for n times

$$\begin{aligned}\underline{n} f \chi &\equiv (\lambda xy. \underbrace{x(\dots(xy))}_{n \text{ times}}) f \chi \\ &\xrightarrow{\beta} \underbrace{f(\dots(f \chi))}_{n \text{ times}}\end{aligned}$$

The following are some important λ -functions for Church numerals:

- **successor function:**

$$\underline{s} \equiv \lambda xyz. xy(yz)$$

which given a Church numeral \underline{n} , it returns $\underline{n+1}$, which is easy to show

$$\begin{aligned}\underline{s} \underline{n} &\equiv (\lambda abc. ab(bc)) (\lambda xy. \underbrace{x(\dots(xy))}_{n \text{ times}}) \\ &\xrightarrow{\beta} \lambda bc. (\lambda xy. \underbrace{x(\dots(xy))}_{n \text{ times}}) b (bc) \\ &\xrightarrow{\beta} \lambda bc. (\lambda y. \underbrace{b(\dots(by))}_{n \text{ times}}) (bc) \\ &\xrightarrow{\beta} \lambda bc. \underbrace{b(\dots(b c))}_{n+1 \text{ times}}\end{aligned}$$

- **is-zero function:**

$$\underline{z} \equiv \lambda f. f(\lambda t. F) T$$

which given a Church numeral \underline{n} , it returns T if and only if \underline{n} is $\underline{0}$, and it can be proven as follows

$$\begin{aligned}\underline{z} \underline{0} &\equiv (\lambda f. f(\lambda t. F) T) (\lambda xy. y) \\ &\xrightarrow{\beta} (\lambda xy. y)(\lambda t. F) T \\ &\xrightarrow{\beta} T\end{aligned}$$

and

$$\begin{aligned}\underline{z} \underline{n} &\equiv (\lambda f. f(\lambda t. F) T) (\lambda xy. \underbrace{x(\dots(xy))}_{n \text{ times}}) \\ &\xrightarrow{\beta} (\lambda xy. \underbrace{x(\dots(xy))}_{n \text{ times}})(\lambda t. F) T \\ &\xrightarrow{\beta} \underbrace{(\lambda t. F)(\dots((\lambda t. F) T))}_{n \text{ times}} \\ &\xrightarrow{\beta} F\end{aligned}$$

- **addition function:**

$$\text{add} \equiv \lambda ab. a \ \underline{s} \ b$$

a proof of this function is omitted, but it can be intuitively explained by using [Observation 1.3](#), which suggests that if \underline{a} and \underline{b} are two Church numerals, then $\underline{a} \ \underline{s} \ \underline{b}$ is the repeated application of \underline{s} exactly a times to \underline{b}

Definition 1.15: Fixed point

Given a function $f : X \rightarrow Y$, an element $x \in X$ is said to be a **fixed point of f** if and only if $f(x) = x$.

Example 1.4 (Fixed points). Given a function $f(x) = x^2 - 3x + 4$, $x = 2$ is a *fixed point* of f , because

$$f(x) = 2^2 - 3 \cdot 2 + 4 = 4 - 6 + 4 = 2 = x$$

and thus $f(x) = x$.

Example 1.5 (Functions are fixed points). Consider the following function

$$F(g) \equiv h(x) = \begin{cases} 1 & x = 0 \\ x \cdot g(x-1) & x > 0 \end{cases}$$

that takes a function as input, and returns a function h ; for instance, plugging in the following function

$$\text{succ} : x \rightarrow x + 1$$

we get that F returns the following function

$$F(\text{succ}) \equiv h(x) = \begin{cases} 1 & x = 0 \\ x \cdot \text{succ}(x-1) = x \cdot x = x^2 & x > 0 \end{cases}$$

which is the function that returns 1 if $x = 0$, and x^2 otherwise.

It's easy to check that the *fixed point* of F is the following function:

$$\text{fact}(x) := \begin{cases} 1 & x = 0 \\ x \cdot \text{fact}(x-1) & x > 0 \end{cases}$$

which computes the factorial of x , because

$$F(\text{fact}) \equiv h(x) = \begin{cases} 1 & x = 0 \\ x \cdot \text{fact}(x-1) & x > 0 \end{cases} \equiv \text{fact}$$

Definition 1.16: Kleene's combinator

The **fixed point operator**, **Y combinator** or **Kleene's combinator** (named after [Stephen Kleene](#)) is defined as follows:

$$Y \equiv \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$$

The Y combinator can be alternatively defined as follows:

$$Y \equiv (\lambda xy.y(xxy))(\lambda xy.y(xxy))$$

Proposition 1.1: Fixed point operator

Given a function, Kleene's combinator returns its fixed point.

Proof. If the Kleene's combinator can return the fixed point of a given function h , it means that Yh is h 's fixed point. Therefore, the statement that has to be proved is that

$$h(Yh) \equiv Yh$$

This can be proved for both formulations of the Y combinator, as follows:

$$\begin{aligned} Yh &\xrightarrow{\beta} (\lambda f.(\lambda x.f(xx))(\lambda x.f(xx)))h \\ &\xrightarrow{\beta} (\lambda x.h(xx))(\lambda x.h(xx)) \\ &\xrightarrow{\beta} h(\lambda x.h(xx)\lambda x.h(xx)) \\ &\xrightarrow{\beta} h(Yh) \end{aligned}$$

and for the alternative formulation

$$\begin{aligned} Yh &\xrightarrow{\beta} ((\lambda xy.y(xxy))(\lambda xy'.y'(xxy'))h \\ &\xrightarrow{\beta} (\lambda y.y((\lambda xy'.y'(xxy'))(\lambda xy''.y''(xxy''))y))h \\ &\xrightarrow{\beta} h((\lambda xy'.y'(xxy'))(\lambda xy''.y''(xxy''))h) \\ &\xrightarrow{\beta} h(Yh) \end{aligned}$$

□

Note that the Y combinator can be used to perform *recursion* inside λ -calculus, because of the following property:

$$\begin{aligned} h(Yh) &= Yh \\ h(h(Yh)) &= h(Yh) = Yh \\ &\vdots \\ h(\dots(h(Yh))) &= Yh \end{aligned}$$

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1.2 Exercises

Problem 1.1: Solve for X

Find X such that

$$Xx = \lambda t.t(Xx)$$

Solution. The term is

$$X \equiv (\lambda fbt.t(fb))X$$

because

$$\begin{aligned} Xx &\xrightarrow{\beta} (\lambda fbt.t(fb))Xx \\ &\xrightarrow{\beta} (\lambda bt.t(Xb))x \\ &\xrightarrow{\beta} \lambda t.t(Xx) \end{aligned}$$

Problem 1.2: Solve for H

Find H such that

$$H(\lambda x_1x_2x_3.P) = \lambda x_3x_2x_1.P$$

Solution. The term is

$$H \equiv \lambda f x_3 x_2 x_1. f x_1 x_2 x_3$$

because

$$\begin{aligned} H(\lambda x_1x_2x_3.P) &\xrightarrow{\beta} (\lambda f x_3 x_2 x_1. f x_1 x_2 x_3)(\lambda x_1x_2x_3.P) \\ &\xrightarrow{\beta} \lambda x_3 x_2 x_1. (\lambda x_1x_2x_3.P) x_1 x_2 x_3 \\ &\xrightarrow{\beta} \lambda x_3 x_2 x_1. P \end{aligned}$$

Problem 1.3: Solve for X

Find X such that

$$Xxyz = Xz(uv)$$

Solution. The term is

$$X \equiv (\lambda tabc.tc(uv))X$$

because

$$\begin{aligned} (\lambda tabc.tc(uv))Xxyz &\xrightarrow{\beta} (\lambda abc.Xc(uv))xyz \\ &\xrightarrow{\beta} Xz(uv) \end{aligned}$$

Problem 1.4: Solve for Δ

Find Δ such that

$$\begin{cases} \Delta S = y_1 \\ \Delta K = y_2 \\ \Delta I = y_3 \end{cases}$$

Solution. Assume that

$$\Delta \equiv \lambda x.x P_1 P_2 P_3$$

for some λ -terms P_1 , P_2 and P_3 ; then

$$\begin{cases} \Delta S \xrightarrow{\beta} S P_1 P_2 P_3 \xrightarrow{\beta} P_1 P_3 (P_2 P_3) \\ \Delta K \xrightarrow{\beta} K P_1 P_2 P_3 \xrightarrow{\beta} P_1 P_3 \\ \Delta I \xrightarrow{\beta} I P_1 P_2 P_3 \xrightarrow{\beta} P_1 P_2 P_3 \end{cases}$$

However, this cannot be a correct assumption, because if

$$\Delta K \xrightarrow{\beta} P_1 P_3 = y_2$$

then

$$\Delta S \xrightarrow{\beta} P_1 P_3 (P_2 P_3) = y_2 (P_2 P_3) \neq y_1$$

which means that ΔS cannot be evaluated to y_1 . This issue can be solved by assuming that

$$\Delta \equiv \lambda x.x P_1 P_2 P_3 P_4$$

for some other term λ -term P_4 , in fact

$$\begin{cases} \Delta S \xrightarrow{\beta} S P_1 P_2 P_3 P_4 \xrightarrow{\beta} P_1 P_3 (P_2 P_3) P_4 \\ \Delta K \xrightarrow{\beta} K P_1 P_2 P_3 P_4 \xrightarrow{\beta} P_1 P_3 P_4 \\ \Delta I \xrightarrow{\beta} I P_1 P_2 P_3 P_4 \xrightarrow{\beta} P_1 P_2 P_3 P_4 \end{cases}$$

and if $P_1 = \lambda xy.y$ then

$$\Delta K \xrightarrow{\beta} P_1 P_3 P_4 \xrightarrow{\beta} (\lambda xy.y) P_3 P_4 \xrightarrow{\beta} P_4$$

which means that P_4 must be y_2 . Moreover

$$\Delta I \xrightarrow{\beta} P_1 P_2 P_3 P_4 \equiv (\lambda xy.y) P_2 P_3 y_2 \xrightarrow{\beta} P_3 y_2 = y_3 \iff P_3 = \lambda t.y_3$$

and finally

$$\Delta S \xrightarrow{\beta} P_1 P_3 (P_2 P_3) P_4 \equiv (\lambda xy.y) (\lambda t.y_3) (P_2 (\lambda t.y_3)) y_2 \iff P_2 = \lambda ab.y_1$$