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Models of Computation

Lecture notes integrated with the book TODO

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Contents

Information and Contacts	1
1 TODO	2
1.1 λ -calculus	2
1.1.1 β -reduction	4
1.1.2 Data structures	7
1.1.3 Recursion	11
1.1.4 Recursive functions	13
1.1.5 Combinatory Logic	16
1.2 Exercises	18

Information and Contacts

Personal notes and summaries collected as part of the *Models of Computation* course offered by the degree in Computer Science of the University of Rome "La Sapienza".

Further information and notes can be found at the following link:

<https://github.com/aflaag-notes>. Anyone can feel free to report inaccuracies, improvements or requests through the Issue system provided by GitHub itself or by contacting the author privately:

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The notes are constantly being updated, so please check if the changes have already been made in the most recent version.

Suggested prerequisites:

- Linguaggi di Programmazione
- Tecniche di Programmazione Funzionale ed Imperativa

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1

TODO

1.1 λ -calculus

In this first section, examples will be omitted from this notes, refer to the notes of the “[Linguaggi di Programmazione](#)” course for further details.

Definition 1: λ -calculus

Let Var be the set of all possible variables; thus, the **set Λ of all possible λ -terms** is defined by the following rules:

$$[var] \frac{x \in \text{Var}}{x \in \Lambda}$$

$$[appl] \frac{M \in \Lambda \quad N \in \Lambda}{MN \in \Lambda}$$

$$[abs] \frac{x \in \text{Var} \quad M \in \Lambda}{\lambda x.M \in \Lambda}$$

The terms of the form $\lambda x.M$ are called **λ -abstractions**, and MN is the function application of M to N . Note that function application *associates to the left*, therefore

$$MNL = (MN)L \neq M(NL)$$

Lambda calculus can be alternatively defined with the [Backus Normal Form](#) (BNF), as follows:

$$M, N ::= x \mid \lambda x.M \mid MN$$

Although *all functions in lambda calculus are unary*, the following definition can expand

this concept.

Definition 2: Currying

Currying (named after [Haskell Curry](#)) is defined as follows:

$$\lambda x_1.(\dots(\lambda x_n.y)) \equiv \lambda x_1 \dots x_n.y$$

Additionally, the following notation

$$\lambda \vec{x}.f \vec{x}$$

will denote **vectors** in place of

$$\lambda x_1 \dots x_n.f x_1 \dots x_n$$

Definition 3: Boundness

A variable is said to be **bound** if it is declared in a λ -abstraction, otherwise it is said to be **free**.

A term that has no free variables is said to be **closed** or **combinator**.

Example 1.1 (Boundness). Consider the following term:

$$\lambda x.xy$$

In this example, x is *bound*, and y is *free*.

Definition 4: Notable combinators

The following are some of the **notable combinators**:

$$\begin{aligned} I &\equiv \lambda x.x \\ K &\equiv \lambda xy.x \\ O &\equiv \lambda xy.y \\ S &\equiv \lambda xyz.xz(yz) \\ B &\equiv \lambda fgx.f(gx) \\ C &\equiv \lambda abc.acb \\ W &\equiv \lambda xy.xyy \end{aligned}$$

In particular

- I stands for “Identity”, in fact it is the identity function
- K stands for “Konstant”, meaning *constant* in german, and in fact K yields a constant function

- S stands for “Satz”, which translates to *substitution* in german, and plays a central role in [Section 1.1.5](#)
- B stands for “Belegung”, meaningn *binding* in german, in fact B is the composition operator
- C stands for “Commutator”, which is precisely what the C combinator performs
- W is the duplicating combinator, and it is called W because the letter W resambles the shape of two V’s placed side by side

Definition 5: Free variables

Given a λ -term, the function

$$\text{free} : \Lambda \rightarrow \mathcal{P}(\text{Var})$$

returns the **set of free variables in M** , and it is defined recursively as follows:

$$\begin{cases} \text{free}(x) := \{x\} \\ \text{free}(MN) := \text{free}(M) \cup \text{free}(N) \\ \text{free}(\lambda x.M) := \text{free}(M) - \{x\} \end{cases}$$

1.1.1 β -reduction

Definition 6: Substitution

The **substitution** operation is recursively defined by the following rules:

$$x[N/x] = N$$

$$y[N/x] = y$$

$$(PQ)[N/x] = P[N/x] Q[N/x]$$

$$(\lambda y.P)[N/x] = \lambda y.(P[N/x]) \text{ if } y \neq x$$

$$(\lambda x.P)[N/x] = \lambda x.P$$

where $M[N/x]$ means that *each instance of x in M is replaced with N* . Note that **only free variables may be substituted**.

Lemma 1: Substitution lemma

Let $M, N, L \in \Lambda$; if $x \neq y$ and $x \notin \text{free}(L)$, then

$$M[M/x][L/y] \equiv M[L/y][N[L/y]/x]$$

Proof. By induction on the structure of M , the details are omitted. \square

Definition 7: Inference rules

The following are the **inference rules** for the lambda calculus:

$$(\alpha) \lambda x.M \equiv (\lambda y.M)[y/x]$$

$$(\beta) (\lambda x.M)N \xrightarrow{\beta} M[N/x]$$

$$(\mu) \frac{M \xrightarrow{\beta} M'}{NM \xrightarrow{\beta} NM'}$$

$$(\nu) \frac{M \xrightarrow{\beta} M'}{MN \xrightarrow{\beta} M'N}$$

$$(\xi) \frac{M \xrightarrow{\beta} M'}{\lambda x.M \xrightarrow{\beta} \lambda x.M'}$$

Note that the β -rule is effectively *one step of the computation* described by the λ -term.

If $M \equiv N$ is provable in the λ -calculus, it will be written as

$$\lambda \vdash M \equiv N$$

Additionally, if a term N can be derived from M through β -reductions

$$M \xrightarrow{\beta} \dots \xrightarrow{\beta} N$$

it will be written as $M \rightsquigarrow N$.

Definition 8: Normal form

If a term can be β -reduced, it is called **β -redex**, or simply **redex** (*reducible expression*), and the reduced term is called **β -reduct**, or simply **reduct**.

If a term has no redexes, it is said to be in **normal form**.

Observation 1: Variable capture

Consider the following λ -term:

$$(\lambda x t. t x)(\lambda t. y) \xrightarrow{\beta} \lambda t. t(\lambda t. y)$$

Note that the two t s are *different*. In fact, underlining the λ -abstractions to which they are bounded to can help clarifying their distinction:

$$(\lambda x \underline{t}. \underline{t} x)(\lambda t. y) \xrightarrow{\beta} \lambda \underline{t}. \underline{t}(\lambda t. y)$$

Now, consider the following λ -term, similar to the previous one:

$$(\lambda x y. y x)(\lambda t. y) \xrightarrow{\beta} \lambda y. y(\lambda t. y)$$

This β -reduction created a problem, because now the two y s *are the same*, even though they were not originally. In fact, the previous term can be relabeled as follows:

$$(\lambda x \underline{y}. \underline{y} x)(\lambda t. y) \xrightarrow{\beta} \lambda \underline{y}. \underline{y}(\lambda t. \underline{y})$$

This happened because

$$\text{free}(\lambda t. y) = \{y\} - \{t\} = \{y\}$$

therefore y was **captured** by the y that was already present in the leftmost λ -abstraction. This phenomena is called **variable capturing**, and constitutes a problem when reducing β -redexes. In particular, to reduce this second λ -abstraction, it is necessary to apply a substitution, by using the α rule (refer to [Definition 7](#)):

$$\lambda x y. y x = \lambda x (\lambda y. y x) = \lambda x. ((\lambda y. y x)[u/y]) = \lambda x. (\lambda u. u x) = \lambda x u. u x$$

which means that the β -reduction can now be performed without any issue:

$$(\lambda x u. u x)(\lambda t. y) \xrightarrow{\beta} \lambda u. u(\lambda t. y)$$

where y is still free. Note that it would not have been *safe* to rename the other (free) y , because in general *renaming free variables can create capturing problems as well*. For example, y could have not been substituted with t , as it would otherwise be captured by the t in the λ -term $\lambda t. y$, as follows:

$$(\lambda t. y)[t/y] = \lambda t. t$$

Fortunately, variable capturing can be solved by employing the following *variable naming convention*.

Definition 9: Variable naming convention

To avoid variable capturing problems, it is sufficient to follow this **variable naming convention**: *bound and free variables must have different names between them.*

From now on, it will be assumed that any β -reduction is performed by renaming opportunely the **bound** variables, such that in each step of the computation the naming convention is followed.

1.1.2 Data structures**Definition 10: Tuples**

A **tuple** of the form

$$(M_1, \dots, M_k)$$

can be represented in λ -calculus as follows:

$$\lambda x.xM_1 \dots M_k$$

In λ -calculus, tuples will be represented as follows

$$[M_1, \dots, M_k]$$

To access the elements of a tuple, *projectors* are used, which are defined below.

Definition 11: Projector

A **projector** has the following form

$$\lambda x.x\pi_j^k$$

where

$$\pi_j^k \equiv \lambda x_1 \dots x_k.x_j$$

Example 1.2 (Projectors). Given a tuple $\lambda x.xM_1 \dots M_k$, its j -th element can be accessed as follows:

$$\lambda x.x\pi_j^k(\lambda x.xM_1 \dots M_k) \xrightarrow{\beta} (\lambda x.xM_1 \dots M_k)\pi_j^k \xrightarrow{\beta} \pi_j^k M_1 \dots M_k \xrightarrow{\beta} M_j$$

Definition 12: Booleans

Booleans can be defined in λ -calculus as follows:

$$T \equiv \lambda xy.x$$

$$F \equiv \lambda xy.y$$

Definition 13: Conditionals

Conditionals can be defined in λ -calculus as follows:

$$\text{ite} = \lambda xyz. xyz$$

Note that *ite* stands for “*if-then-else*”, and in fact, the term behaves exactly like a condition when used in conjunction with the λ -booleans.

Observation 2: Conditionals

The term *ite* correctly behaves as a *conditional* when used with T and F. In fact, when used in an term such as

$$\text{ite } C \ A \ B$$

if *C* is a λ -boolean, the term will be β -reduced to *A* if $C \equiv T$, and it will be evaluated to *B* if $C \equiv F$. Indeed

$$\begin{aligned} \text{ite } T \ A \ B &\equiv (\lambda xyz. xyz) \ T \ A \ B \\ &\xrightarrow{\beta} T \ A \ B \\ &\equiv (\lambda xy. x) \ A \ B \\ &\xrightarrow{\beta} A \end{aligned}$$

and

$$\begin{aligned} \text{ite } F \ A \ B &\equiv (\lambda xyz. xyz) \ F \ A \ B \\ &\xrightarrow{\beta} F \ A \ B \\ &\equiv (\lambda xy. y) \ A \ B \\ &\xrightarrow{\beta} B \end{aligned}$$

Additionally, it is possible to define the logical **not** operator, as follows:

$$\text{not} \equiv \lambda t. tFT$$

which clearly negates the boolean given as input

$$\begin{aligned} \text{not } T &\equiv (\lambda t. tFT) \ T \xrightarrow{\beta} F \\ \text{not } F &\equiv (\lambda t. tFT) \ F \xrightarrow{\beta} T \end{aligned}$$

Definition 14: Church numerals

The **Church numerals** are defined by a mapping between natural numbers \mathbb{N} and λ -abstractions:

$$\varphi : \mathbb{N} \rightarrow \Lambda : n \mapsto \lambda xy. \underbrace{x(\dots(xy))}_{n \text{ times}}$$

The Church numeral corresponding to $n \in \mathbb{N}$ will be represented as $\underline{n} \in \Lambda$.

Example 1.3 (Church numerals). For example, the number $\underline{0}$ is represented as

$$\varphi(0) = \underline{0} = \lambda xy.y \equiv F \equiv O$$

number $\underline{1}$ as

$$\varphi(1) = \underline{1} = \lambda xy.xy \equiv T \equiv K$$

and number $\underline{2}$ as

$$\varphi(2) = \underline{2} = \lambda xy.x(xy)$$

and so on.

Observation 3: Church numerals are iterators

Note that Church numerals are **iterators**, in the sense that \underline{n} replicates any input function f for n times

$$\begin{aligned} \underline{n} f \chi &\equiv (\lambda xy.x(\underbrace{\dots(xy)}_{n \text{ times}})) f \chi \\ &\xrightarrow{\beta} \underbrace{f(\dots(f \chi))}_{n \text{ times}} \end{aligned}$$

The following are some important λ -functions for Church numerals:

- **successor function:**

$$\underline{s} \equiv \lambda xyz.xy(yz)$$

which given a Church numeral \underline{n} , it returns $\underline{n+1}$, which is easy to show

$$\begin{aligned} \underline{s} \underline{n} &\equiv (\lambda abc.ab(bc)) (\lambda xy.x(\underbrace{\dots(xy)}_{n \text{ times}})) \\ &\xrightarrow{\beta} \lambda bc.(\lambda xy.x(\underbrace{\dots(xy)}_{n \text{ times}}) b (bc)) \\ &\xrightarrow{\beta} \lambda bc.(\lambda y.b(\underbrace{\dots(by)}_{n \text{ times}}) (bc)) \\ &\xrightarrow{\beta} \lambda bc.\underbrace{b(\dots(bc))}_{n+1 \text{ times}} \end{aligned}$$

- **is-zero function:**

$$\underline{z} \equiv \lambda f.f(\lambda t.F)T$$

which given a Church numeral \underline{n} , it returns T if and only if \underline{n} is $\underline{0}$, and it can be proven as follows

$$\begin{aligned} \underline{z} \underline{0} &\equiv (\lambda f.f(\lambda t.F)T) (\lambda xy.y) \\ &\xrightarrow{\beta} (\lambda xy.y)(\lambda t.F)T \\ &\xrightarrow{\beta} T \end{aligned}$$

and

$$\begin{aligned}
 \underline{z} \ \underline{n} &\equiv (\lambda f.f(\lambda t.F)T) (\lambda xy.\underbrace{x(\dots(xy))}_{n \text{ times}}) \\
 &\xrightarrow{\beta} (\lambda xy.\underbrace{x(\dots(xy))}_{n \text{ times}})(\lambda t.F)T \\
 &\xrightarrow{\beta} (\lambda t.F)(\underbrace{\dots((\lambda t.F)T)}_{n \text{ times}}) \\
 &\xrightarrow{\beta} F
 \end{aligned}$$

- **addition function:**

$$\text{add} \equiv \lambda ab.a \ \underline{s} \ b$$

a proof of this function is omitted, but it can be intuitively explained by using [Observation 3](#), which suggests that if \underline{a} and \underline{b} are two Church numerals, then $\underline{a} \ \underline{s} \ \underline{b}$ is the repeated application of \underline{s} exactly a times to \underline{b}

- **predecessor function:**

$$P \equiv \lambda x.x(\lambda abz.zb(\underline{s} \ b))$$

which is a term such that

$$\begin{aligned}
 P \ [\underline{m}, \underline{n}] &\equiv (\lambda x.x(\lambda abz.zb(\underline{s} \ b))) (\lambda u.u \ \underline{m} \ \underline{n}) \\
 &\xrightarrow{\beta} (\lambda u.u \ \underline{m} \ \underline{n})(\lambda abz.zb(\underline{s} \ b)) \\
 &\xrightarrow{\beta} (\lambda abz.zb(\underline{s} \ b)) \ \underline{m} \ \underline{n} \\
 &\xrightarrow{\beta} \lambda z.z \ \underline{n} \ (\underline{s} \ \underline{n}) \\
 &\equiv [\underline{n}, \underline{n} + 1]
 \end{aligned}$$

therefore

$$\begin{aligned}
 \underline{n} \ P \ [\underline{0}, \underline{0}] &\xrightarrow{\beta} \underbrace{P(\dots(P \ [\underline{0}, \underline{0}]))}_{n \text{ times}} \\
 &\xrightarrow{\beta} \underbrace{P(\dots(P \ [\underline{0}, \underline{1}]))}_{n-1 \text{ times}} \\
 &\xrightarrow{\beta} \underbrace{P(\dots(P \ [\underline{1}, \underline{2}]))}_{n-2 \text{ times}} \\
 &\xrightarrow{\beta} \dots \\
 &\xrightarrow{\beta} [\underline{n} - 1, \underline{n}]
 \end{aligned}$$

which means that the following term

$$\underline{p} \equiv \lambda x.(x \ P \ [\underline{0}, \underline{0}])T$$

is such that

$$\underline{p} \ \underline{n} \rightsquigarrow \underline{n} - 1$$

Church numerals are not the only way to define a system of numerals in λ -calculus. In fact, [Hank Barendregt](#) proposed the following system in 1976.

Definition 15: Barendregt numerals

The Barendregt numerals are defined inductively as follows:

$$\begin{aligned} \ulcorner 0 \urcorner &\equiv I \\ \ulcorner n + 1 \urcorner &\equiv [F, \ulcorner n \urcorner] \end{aligned}$$

Therefore, we have that

$$\ulcorner n \urcorner \equiv \underbrace{[F, \dots, [F, I]]}_{n \text{ times}}$$

From this definition, the following λ -function for Barendregt numerals can be defined:

- **successor function:**

$$s \equiv \lambda x.x[F, x]$$

- **predecessor function:**

$$f \equiv \lambda x.x F$$

which consumes the outermost tuple

- **is-zero function:**

$$z \equiv \lambda x.x T$$

because it returns F , the first element of any tuple, for any Barendregt numeral, except when the input is not a tuple but I itself, which is precisely $\ulcorner 0 \urcorner$

1.1.3 Recursion

Definition 16: Fixed point

Given a function $f : X \rightarrow Y$, an element $x \in X$ is said to be a **fixed point of f** if and only if $f(x) = x$.

Example 1.4 (Fixed points). Given a function $f(x) = x^2 - 3x + 4$, $x = 2$ is a *fixed point* of f , because

$$f(x) = 2^2 - 3 \cdot 2 + 4 = 4 - 6 + 4 = 2 = x$$

and thus $f(x) = x$.

Example 1.5 (Functions are fixed points). Consider the following function

$$F(g) := h(x) = \begin{cases} 1 & x = 0 \\ x \cdot g(x - 1) & x > 0 \end{cases}$$

that takes a function as input, and returns a function h ; for instance, plugging in the following function

$$\text{succ} : x \rightarrow x + 1$$

we get that F returns the following function

$$F(\text{succ}) \equiv h(x) = \begin{cases} 1 & x = 0 \\ x \cdot \text{succ}(x - 1) = x \cdot x = x^2 & x > 0 \end{cases}$$

which is the function that returns 1 if $x = 0$, and x^2 otherwise.

It's easy to check that the *fixed point* of F is the following function:

$$\text{fact}(x) := \begin{cases} 1 & x = 0 \\ x \cdot \text{fact}(x - 1) & x > 0 \end{cases}$$

which computes the factorial of x , because

$$F(\text{fact}) \equiv h(x) = \begin{cases} 1 & x = 0 \\ x \cdot \text{fact}(x - 1) & x > 0 \end{cases} \equiv \text{fact}$$

Definition 17: Kleene's combinator

The **fixed point operator**, **Y combinator** or **Kleene's combinator** (named after [Stephen Kleene](#)) is defined as follows:

$$Y \equiv \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$$

The Y combinator can be alternatively defined as follows:

$$Y \equiv (\lambda xy.y(xy))(\lambda xy.y(xy))$$

Proposition 1: Fixed point operator

Given a function, Kleene's combinator returns its fixed point.

Proof. If the Kleene's combinator can return the fixed point of a given function h , it means that Yh is h 's fixed point. Therefore, the statement that has to be proved is that

$$h(Yh) \equiv Yh$$

This can be proved for both formulations of the Y combinator, as follows:

$$\begin{aligned} Yh &\xrightarrow{\beta} (\lambda f.(\lambda x.f(xx))(\lambda x.f(xx)))h \\ &\xrightarrow{\beta} (\lambda x.h(xx))(\lambda x.h(xx)) \\ &\xrightarrow{\beta} h(\lambda x.h(xx)\lambda x.h(xx)) \\ &\xrightarrow{\beta} h(Yh) \end{aligned}$$

and for the alternative formulation

$$\begin{aligned}
Yh &\xrightarrow{\beta} ((\lambda xy. y(xy))(\lambda xy'. y'(xy'))h) \\
&\xrightarrow{\beta} (\lambda y. y((\lambda xy'. y'(xy'))(\lambda xy''. y''(xy''))y))h \\
&\xrightarrow{\beta} h((\lambda xy'. y'(xy'))(\lambda xy''. y''(xy''))h) \\
&\xrightarrow{\beta} h(Yh)
\end{aligned}$$

□

Note that the Y combinator can be used to perform *recursion* inside λ -calculus, because of the following property:

$$\begin{aligned}
h(Yh) &= Yh \\
h(h(Yh)) &= h(Yh) = Yh \\
&\vdots \\
h(\dots(h(Yh))) &= Yh
\end{aligned}$$

1.1.4 Recursive functions

Definition 18: Numeric function

A **numeric function** is a map $f : \mathbb{N}^p \rightarrow \mathbb{N}$ for some p .

Definition 19: λ -definable function

A function is said to be **λ -definable** if there exists a closed term F such that

$$F \ \underline{n_1} \dots \underline{n_p} \equiv \underline{f(n_1, \dots, n_p)}$$

If that is the case, f is said to be **λ -defined** by F .

Definition 20: Initial functions

The following are the so called **initial functions**:

$$\begin{aligned}
U_r^i(x_1, \dots, x_r) &= x_i, \quad 1 \leq i \leq r \\
\text{succ}(n) &= n + 1 \\
\text{zero}(n) &= 0
\end{aligned}$$

In particular, the first equations are called **projection functions**, the second is the **successor function** and the last is called **constant function**.

Definition 21: Composition

Given an m -ary function $h(x_1, \dots, x_m)$ and k m -ary functions

$$g_1(x_1, \dots, x_k), \dots, g_m(x_1, \dots, x_k)$$

, the **composition operator** is defined as follows:

$$f := h \circ (g_1, \dots, g_m)$$

where

$$f(\vec{x}) = h(g_1(\vec{x}), \dots, g_m(\vec{x}))$$

Lemma 2: Composition

The λ -definable functions are closed under *composition*.

Proof. Let h, g_1, \dots, g_m be defined by the λ -terms H, G_1, \dots, G_m respectively. Then

$$F \equiv \lambda \vec{x}. H(G_1 \vec{x}) \dots (G_m \vec{x})$$

λ -defines $h \circ (g_1, \dots, g_m)$. □

Definition 22: Primitive recursion

Given a k -ary function $g(x_1, \dots, x_k)$ and a $(k+2)$ -ary function $h(y, z, x_1, \dots, x_k)$, the **primitive recursion operator** is a $(k+1)$ -ary function ρ is defined as follows:

$$f := \rho(g, h)$$

where

$$\begin{aligned} f(0, \vec{n}) &= g(\vec{x}) \\ f(\text{succ}(n), \vec{x}) &= h(y, f(y, \vec{x}), \vec{x}) \end{aligned}$$

Lemma 3: Primitive recursion

The λ -definable functions are closed under *primitive recursion*.

Proof. Let f be a function such that

$$\begin{aligned} f(0) &= g \\ f(k+1) &= h(f(k), k) \end{aligned}$$

which is a weaker version of the *primitive recursion operator*, but the proof for general \vec{n} is similar (details are omitted).

Consider the following term

$$T \equiv \lambda p. [\underline{s}(p \ T), H(p \ F)(p \ T)]$$

where H λ -defines h , and note how it computes over the following input

$$\begin{aligned} T \ [\underline{k}, \underline{f(x)}] &\rightsquigarrow [\underline{s} \ \underline{k}, H \ \underline{f(x)} \ \underline{k}] \\ &\xrightarrow{\beta} [\underline{k+1}, H \ \underline{f(x)} \ \underline{k}] \\ &\equiv [\underline{k+1}, \underline{f(k+1)}] \end{aligned}$$

(the last step follows from f 's definition) therefore, T describes the following function

$$t : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N} : (k, f(k)) \mapsto (k+1, f(k+1))$$

Hence, by induction, it follows that

$$[\underline{k}, \underline{f(k+1)}] \equiv T^k \ [\underline{0}, \underline{f(0)}]$$

and because Church numerals are iterators (as shown in [Observation 3](#)), it follows that

$$\underline{f(k)} \equiv \underline{k} \ T \ [\underline{0}, \underline{f(0)}] \ F$$

Finally, f can be λ -defined by the following term

$$F \equiv \lambda k. k \ T \ [\underline{0}, \underline{G}] \ F$$

where G λ -defines g . □

Definition 23: Minimalization

Given a $(k+1)$ -ary function $f(y, x_1, \dots, x_k)$, the **minimalization operator** is a k -ary function $\mu(f)$ defined as follows:

$$\mu(f)(\vec{x}) = z \iff \forall i \in [0, z-1] \quad f(i, \vec{x}) > 0 \wedge f(z, \vec{x}) = 0$$

In other words, that *minimalization operator* seeks the smallest argument that causes the function to return zero. If there is no such argument, or if an argument is encountered for which f is not defined, then the search never terminates, and $\mu(f)$ is not defined for \vec{x} .

Lemma 4: Minimalization

The λ -definable functions are closed under minimalization

Proof. TODO □

Definition 24: Recursive functions

The class \mathcal{R} is the smallest class of *numeric functions* that contains *all initial functions*, and is closed under *composition*, *primitive recursion* and *minimalization*.

Theorem 1: Recursive functions

All recursive functions are λ -definable, therefore

$$\mathcal{R} \subseteq \Lambda$$

Proof. The following terms λ -define *initial functions*:

$$\begin{aligned} U_p^i &\equiv \pi_j^k \\ \text{succ} &\equiv \underline{s} \\ \text{zero} &\equiv \underline{0} \end{aligned}$$

Thus, the theorem follows directly from [Lemma 2](#), [Lemma 3](#) and [Lemma 4](#). \square

1.1.5 Combinatory Logic**Definition 25: Combinatory Logic**

Combinatory Logic is a simplified model of computation, related to λ -calculus, defined as follows:

$$\begin{aligned} [\text{var}] \quad & \frac{x \in \text{Var}}{x \in \text{CL}} \\ [\text{appl}] \quad & \frac{U \in \Lambda \quad V \in \Lambda}{UV \in \text{CL}} \\ [\text{const}] \quad & \frac{x \in \text{Const}}{x \in \text{CL}} \end{aligned}$$

where the set Const is defined by the following **constants**:

$$\text{Const} := \{S, K, I, B, C\}$$

Association rules are the same as the ones for λ -calculus. Note that, in combinatory logic, there are **no abstractions**, therefore *all variables are free*. To compute with CL, **reductions** are defined as follows:

$$\begin{aligned} Sxyz &\triangleright xy(yz) \\ Kxy &\triangleright x \\ Ix &\triangleright x \\ Bxyz &\triangleright x(yz) \\ Cxyz &\triangleright xzy \end{aligned}$$

Theorem 2: Equivalence of λ -calculus and CL

Every λ -term can be written as a CL term, and viceversa.

Proof. Given a CL term $U \in \text{CL}$:

- if $U \in \text{CL}$ because $U \in \text{Var}$, then $U \in \Lambda$ by definition of Λ
- if $U \in \text{CL}$ because $U \equiv (MN)$ where $M, N \in \text{CL}$, then $(MN) \in \Lambda$ because applications are defined in λ -calculus as well, and $M, N \in \Lambda$ inductively
- finally, if $U \in \text{CL}$ because $U \in \text{Const}$, then simply convert the CL constant into a λ -combinator (refer to [Definition 4](#))

This proves that $\text{CL} \subseteq \Lambda$. To show the other inclusion, a similar reasoning can be applied, except for λ -abstractions, which do not exist in CL. Nevertheless, the following recursive algorithm returns a CL term equivalent to any given λ -abstraction. Let the *swap* operation be defined as follows:

$$\text{swap} : \Lambda_{\text{abstr}} \rightarrow \text{CL} : \lambda x.P \mapsto [x]P$$

then, the conversion algorithm defines additional rules to the *swap* operation as follows:

$$\begin{aligned} [x]Ux &= U \\ [x]x &= \text{I} \\ [x]U &= \text{KU} \\ [x]UW &= \text{BU}([x]W) \\ [x]VU &= \text{C}([x]V)U \\ [x]VW &= \text{S}([x]V)([x]W) \end{aligned}$$

where $U, V, W \in \text{CL}$ are three CL terms such that $x \notin \text{free}(U)$ and $x \in \text{free}(V), \text{free}(W)$. □

An intuition of the correctness of the algorithm can be provided as follows (bounded variables are renamed differently from [Definition 4](#), to match the definitions in the algorithm):

- the first rule is a special case
- $\text{swap}(\text{I}) = \text{swap}(\lambda x.x) = [x]x$
- $\text{swap}(\text{KU}) = \text{swap}(\lambda x.U) = [x]U$
- $\text{swap}(\text{BU}(\lambda a.W)) = \text{swap}(\lambda x.U(\lambda a.W)x) = [x]UW$
- $\text{swap}(\text{C}(\lambda a.V)U) = \text{swap}(\lambda x.(\lambda a.V)xU) = [x]VU$
- $\text{swap}(\text{S}(\lambda a.V)(\lambda b.W)) = \text{swap}(\lambda x.(\lambda a.V)x(\lambda b.W)x) = [x]VW$

Example 1.6 (Conversion Λ to CL). Consider the following λ -term

$$\lambda xy.ytx$$

it can be converted into a CL term as follows

$$\begin{aligned} \text{swap}(\lambda xy.ytx) &= \text{swap}([x](\lambda y.ytx)) \\ &= [x]([y](yt)x) \\ &= [x](C([y]yt)x) \\ &= [x](C(C([y]y)t)x) \\ &= [x]C(CIt)x \\ &= C(CIt) \end{aligned}$$

and in fact

$$\begin{aligned} C(CIt)xy &\triangleright CIt yx \\ &\triangleright Iyt \\ &\triangleright ytx \end{aligned}$$

Because CL and Λ are equivalent, Church numerals can be defined in CL as well:

- $\text{swap}(\underline{0}) = \text{swap}(\lambda xy.y) = [x][y]y = [x]I = KI$ and in fact $KIxy \triangleright Iy \triangleright y$
- $\text{swap}(\underline{1}) = \text{swap}(\lambda xy.xy) = [x][y]xy = [x]x = I$ and in fact $Ixy \triangleright xy$
- $\text{swap}(\underline{2}) = \text{swap}(\lambda xy.x(xy)) = [x][y]x(xy) = [x]Bx([y]xy) = [x]Bxx = S([x]Bx)([x]x) = SBI$ and in fact $SBIxy \triangleright Bx(Ix)y \triangleright Bxxy \triangleright x(xy)$
- in general, we have that

$$\text{swap}(\underline{n}) = \underbrace{SB(\dots(SBI))}_{n-1 \text{ times}}$$

1.2 Exercises

Problem 1: Solve for X

Find X such that

$$Xx = \lambda t.t(Xx)$$

Solution. The term is

$$X \equiv (\lambda fbt.t(fb))X \implies X \equiv Y(\lambda fbt.t(fb))$$

because

$$\begin{aligned} Xx &\xrightarrow{\beta} (\lambda fbt.t(fb))Xx \\ &\xrightarrow{\beta} (\lambda bt.t(Xb))x \\ &\xrightarrow{\beta} \lambda t.t(Xx) \end{aligned}$$

Problem 2: Solve for H

Find H such that

$$H(\lambda x_1 x_2 x_3. P) = \lambda x_3 x_2 x_1. P$$

Solution. The term is

$$H \equiv \lambda f x_3 x_2 x_1. f x_1 x_2 x_3$$

because

$$\begin{aligned} H(\lambda x_1 x_2 x_3. P) &\xrightarrow{\beta} (\lambda f x_3 x_2 x_1. f x_1 x_2 x_3)(\lambda x_1 x_2 x_3. P) \\ &\xrightarrow{\beta} \lambda x_3 x_2 x_1. (\lambda x_1 x_2 x_3. P) x_1 x_2 x_3 \\ &\xrightarrow{\beta} \lambda x_3 x_2 x_1. P \end{aligned}$$

Problem 3: Solve for X

Find X such that

$$Xxyz = Xz(uv)$$

Solution. The term is

$$X \equiv (\lambda tabc. tc(uv))X \implies X \equiv Y(\lambda tabc. tc(uv))$$

because

$$\begin{aligned} (\lambda tabc. tc(uv))Xxyz &\xrightarrow{\beta} (\lambda abc. Xc(uv))xyz \\ &\xrightarrow{\beta} Xz(uv) \end{aligned}$$

Problem 4: Solve for Δ

Find Δ such that

$$\begin{cases} \Delta S = y_1 \\ \Delta K = y_2 \\ \Delta I = y_3 \end{cases}$$

Solution. Assume that

$$\Delta \equiv \lambda x. x P_1 P_2 P_3$$

for some λ -terms P_1 , P_2 and P_3 ; then

$$\begin{cases} \Delta S \xrightarrow{\beta} S P_1 P_2 P_3 \xrightarrow{\beta} P_1 P_3 (P_2 P_3) \\ \Delta K \xrightarrow{\beta} K P_1 P_2 P_3 \xrightarrow{\beta} P_1 P_3 \\ \Delta I \xrightarrow{\beta} I P_1 P_2 P_3 \xrightarrow{\beta} P_1 P_2 P_3 \end{cases}$$

However, this cannot be a correct assumption, because if

$$\Delta K \xrightarrow{\beta} P_1 P_3 = y_2$$

then

$$\Delta S \xrightarrow{\beta} P_1 P_3 (P_2 P_3) = y_2 (P_2 P_3) \neq y_1$$

which means that ΔS cannot be evaluated to y_1 . This issue can be solved by assuming that

$$\Delta \equiv \lambda x.x P_1 P_2 P_3 P_4$$

for some other term λ -term P_4 , in fact

$$\begin{cases} \Delta S \xrightarrow{\beta} S P_1 P_2 P_3 P_4 \xrightarrow{\beta} P_1 P_3 (P_2 P_3) P_4 \\ \Delta K \xrightarrow{\beta} K P_1 P_2 P_3 P_4 \xrightarrow{\beta} P_1 P_3 P_4 \\ \Delta I \xrightarrow{\beta} I P_1 P_2 P_3 P_4 \xrightarrow{\beta} P_1 P_2 P_3 P_4 \end{cases}$$

and if $P_1 = \lambda xy.y$ then

$$\Delta K \xrightarrow{\beta} P_1 P_3 P_4 \xrightarrow{\beta} (\lambda xy.y) P_3 P_4 \xrightarrow{\beta} P_4$$

which means that P_4 must be y_2 . Moreover

$$\Delta I \xrightarrow{\beta} P_1 P_2 P_3 P_4 \equiv (\lambda xy.y) P_2 P_3 y_2 \xrightarrow{\beta} P_3 y_2 = y_3 \iff P_3 = \lambda t.y_3$$

and finally

$$\Delta S \xrightarrow{\beta} P_1 P_3 (P_2 P_3) P_4 \equiv (\lambda xy.y) (\lambda t.y_3) (P_2 (\lambda t.y_3)) y_2 \iff P_2 = \lambda ab.y_1$$

Problem 5: Solve for Δ

Find Δ such that

$$\begin{cases} \Delta(\lambda xyz.x(xyI)x) = y_1 \\ \Delta(\lambda xyz.x(xyz)x) = y_2 \end{cases}$$

Solution. Let $\text{flip} \equiv \lambda xy.yx$; Δ can be defined as follows:

$$\Delta \equiv \lambda x.x \text{ flip } T \text{ not } (\lambda x.xy_1y_2)$$

in fact

$$\begin{aligned} (\lambda x.x \text{ flip } T \text{ not } (\lambda x.xy_1y_2))(\lambda xyz.x(xyI)x) &\rightsquigarrow \text{flip}(\text{flip } T \text{ I})\text{flip } (\lambda x.xy_1y_2) \\ &\xrightarrow{\beta} \text{flip}(\text{I } T)\text{flip } (\lambda x.xy_1y_2) \\ &\xrightarrow{\beta} \text{flip } T \text{ flip } (\lambda x.xy_1y_2) \\ &\xrightarrow{\beta} \text{flip } T (\lambda x.xy_1y_2) \\ &\xrightarrow{\beta} (\lambda x.xy_1y_2) T \\ &\xrightarrow{\beta} y_1 \end{aligned}$$

and

$$\begin{aligned} (\lambda x.x \text{ flip } T \text{ not } (\lambda x.xy_1y_2))(\lambda xyz.x(xyz)x) &\rightsquigarrow \text{flip}(\text{flip } T \text{ not})\text{flip } (\lambda x.xy_1y_2) \\ &\xrightarrow{\beta} \text{flip}(\text{not } T)\text{flip } (\lambda x.xy_1y_2) \\ &\xrightarrow{\beta} \text{flip}(\text{not } T) (\lambda x.xy_1y_2) \\ &\xrightarrow{\beta} \text{flip } F (\lambda x.xy_1y_2) \\ &\xrightarrow{\beta} (\lambda x.xy_1y_2) F \\ &\xrightarrow{\beta} y_2 \end{aligned}$$