DSc Assignment 1

Mohammad Aflah Khan, 2020082

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Q.3

3.a

Ω is the sample space P is the power set such that it has the size 2^{Ω} and $P(A) = \frac{|A|}{|\Omega|}$ where P is the probability measure function

For a probability measure $P : F \to [0, 1]$ to be a valid probability measure, it needs to follow 2 criteria -

1. Measure of Ω is 1
2. P is $\sigma - additive$

Proving Condition 1 -

$P(A) = \frac{|A|}{|\Omega|}$ Now since $A = \Omega$, hence $P(\Omega) = 1$

Proving Condition 2 -

Hence we need to prove - $P(\bigcup_{i=1}^{\infty} A_i) = \Sigma_{i=1}^{\infty} P(A_i)$

Proof:

$P(\bigcup_{i=1}^{\infty} A_i)$ is simply the number of elements in the union set

Now since all A_i are pairwise disjoint, i.e. their intersection is ϕ

Also $P(\bigcup_{i=1}^{\infty} A_i) = \frac{|\bigcup_{i=1}^{\infty} A_i|}{|\Omega|}$ by our probability measure

Since $\bigcup_{i=1}^{\infty} A_i = \Sigma_{i=1}^{\infty} P(A_i)$, when we divide both sides by $|\Omega|$ and apply the definition of our probability measure -

We get $P(\bigcup_{i=1}^{\infty} A_i) = \Sigma_{i=1}^{\infty} P(A_i)$

Hence proved

3.b

For upper-bound:

Given -

$P(\bigcup_{1 \leq i \leq n} A_i) = \Sigma_{1 \leq i \leq n} P(A_i) - \Sigma_{1 \leq i \leq j \leq n} P(A_i \cap A_j) + \Sigma_{1 \leq i \leq j \leq k \leq n} P(A_i \cap A_j \cap A_k) - \Sigma_{1 \leq i \leq j \leq k \leq l \leq n} P(A_i \cap A_j \cap A_k \cap A_l) + ...$

This can be rewritten as

$P(\bigcup_{1 \leq i \leq n} A_i) = \Sigma_{1 \leq i \leq n} P(A_i) - \Sigma_{1 \leq i \leq j \leq n} P(A_i \cap A_j) + \Sigma_{1 \leq i \leq j \leq k \leq n} P(A_i \cap A_j \cap A_k) - C$

where,

$C = \Sigma_{1 \le i \le j \le k \le l \le n} P(A_i \cap A_j \cap A_k \cap A_l) + ...$

Now if $C \ge 0$ then

$P(\bigcup_{1 \le i \le n} A_i) \le \Sigma_{1 \le i \le n} P(A_i) - \Sigma_{1 \le i \le j \le n} P(A_i \cap A_j) + \Sigma_{1 \le i \le j \le k \le n} P(A_i \cap A_j \cap A_k)$

Hence if we prove $C \ge 0$ it is sufficient to show that the bound holds

Since all probabilities individually are greater than or equal to 0 and C has terms with alternating signs. Now we might have even or odd number of terms in C, if we show that $C \ge 0$ for even number of terms, it will hold for odd number of terms too as odd entries are always added and are always non negative. Now, for even number of terms we can show that each pair of terms starting from intersection taken k at a time is greater than or atleast equal to intersection $k+1$ at a time where k starts from 4. This is because introduction of a new term in intersection can only even keep the intersection the same or reduce it. Hence such pair of terms is individually greater than or equal to 0 and hence as a net $C \ge 0$ for both even and odd number of terms.

Therefore the upperbound holds

For lower-bound:

Given -

$P(\bigcup_{1 \le i \le n} A_i) = \Sigma_{1 \le i \le n} P(A_i) - \Sigma_{1 \le i \le j \le n} P(A_i \cap A_j) + \Sigma_{1 \le i \le j \le k \le n} P(A_i \cap A_j \cap A_k) - \Sigma_{1 \le i \le j \le k \le l \le n} P(A_i \cap A_j \cap A_k \cap A_l) + ...$

Which can be written as

$P(\bigcup_{1 \le i \le n} A_i) = \Sigma_{1 \le i \le n} P(A_i) - \Sigma_{1 \le i \le j \le n} P(A_i \cap A_j) + \Sigma_{1 \le i \le j \le k \le n} P(A_i \cap A_j \cap A_k) - \Sigma_{1 \le i \le j \le k \le l \le n} P(A_i \cap A_j \cap A_k \cap A_l) + C$

If we can show $C \ge 0$ then

$P(\bigcup_{1 \le i \le n} A_i) \ge \Sigma_{1 \le i \le n} P(A_i) - \Sigma_{1 \le i \le j \le n} P(A_i \cap A_j) + \Sigma_{1 \le i \le j \le k \le n} P(A_i \cap A_j \cap A_k) - \Sigma_{1 \le i \le j \le k \le l \le n} P(A_i \cap A_j \cap A_k \cap A_l)$

Since all probabilities individually are greater than or equal to 0 and C has terms with alternating signs. Now we might have even or odd number of terms in C, if we show that $C \ge 0$ for even number of terms, it will hold for odd number of terms too as odd entries are always added and are always non negative. Now, for even number of terms we can show that each pair of terms starting from intersection taken k at a time is greater than or atleast equal to intersection $k+1$ at a time where k starts from 5. This is because introduction of a new term in intersection can only even keep the intersection the same or reduce it. Hence such pair of terms is individually greater than or equal to 0 and hence as a net $C \ge 0$ for both even and odd number of terms.

Note: This is the exact same argument as for the upper-bound.

Hence Proved

Q.5

5.a

We model this as a geometric random variable (R.V.) with a success probability $p = \frac{1}{k}$. The value of the face is of no consequence to us as the probability of seeing that face in a fair die is still the same. Using the fact that expectation of a geometric R.V. with success probability p is $\frac{1}{p}$. Hence the expected number of trials here is $\frac{1}{\frac{1}{k}} = k$

5.b

We model this as a geometric random variable (R.V.) with a success probability p. At the very start, picking any die results in a new die, and hence $p = \frac{k}{k} = 1$. As we pick out, say, i dice, the probability of now seeing a new die is $\frac{k-i}{k}$. The expectation of a geometric R.V. with success probability p is $\frac{1}{p}$.

Putting the pieces together at the very start with success probability is $\frac{k}{k}$ our expected trials are $\frac{1}{\frac{k}{k}} = \frac{k}{k}$. In the next attempt the number of expected trials to see a new face is $\frac{1}{\frac{k-1}{k}} = \frac{k}{k-1}$ and so on until we only have one die remaining where the number of expected trials to see a new dice is $\frac{1}{\frac{1}{k}} = \frac{k}{1}$

By linearity of expectations, the total expected number of trials to see all faces once is -

$$\sum_{i=0}^{k-1} \frac{k}{k-i}$$

This can be written as

$$k * \sum_{i=0}^{k-1} \frac{1}{k-i}$$

Now in the limiting case of the die having a large number of faces i.e. k is very large, the expression

$$k * \sum_{i=0}^{k-1} \frac{1}{k-i} = k * \log(k)$$

5.c

Approach 1:

Using the closed form solution derived here in Section 3 which is derived using the Maximum-Minimums identity we get -

$E[X_(p_1, .., p_N)] = \sum_i \frac{1}{p_i} - \sum_{i \leq j} \frac{1}{p_i + p_j} + ... + (-1)^{N+1} \frac{1}{p_1 + ... + p_N}$

Plugging in the values given, we get -

$$ExpectedNumberofTrials = 6.33$$

Approach - 2:

Modelling the problem as a Geometric R.V. we can build several cases and then average over the number of expected rolls.

Note: The expectation of a geometric R.V. with success probability p is $\frac{1}{p}$.

At the very start the success probability of our R.V. is 1 and hence the expected value is $\frac{1}{1} = 1$. At this point any of the 3 faces could've been chosen. Let's break into cases:

1. Face 1 was picked: Remaining probability mass for success is $1 - \frac{1}{4} = \frac{3}{4}$. Hence now the number of expected trials to see a new face is $\frac{1}{\frac{3}{4}} = \frac{4}{3}$. Again we now have 2 more cases of which face was picked.
   - Face 2 was picked: Remaining probability mass for success is $\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$. Hence now the number of expected trials to see a new face is $\frac{1}{\frac{1}{4}} = 4$. At this point the number of expected rolls is $1 + \frac{4}{3} + 4 = \frac{19}{3}$
   - Face 3 was picked: Remaining probability mass for success is $\frac{3}{4} - \frac{1}{4} = \frac{1}{2}$. Hence now the number of expected trials to see a new face is $\frac{1}{\frac{1}{2}} = 2$. At this point the number of expected rolls is $1 + \frac{4}{3} + 2 = \frac{13}{3}$
2. Face 2 was picked: Remaining probability mass for success is $1 - \frac{1}{2} = \frac{1}{2}$. Hence now the number of expected trials to see a new face is $\frac{1}{\frac{1}{2}} = 2$. Again we now have 2 more cases of which face was picked.
   - Face 1 was picked: Remaining probability mass for success is $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. Hence now the number of expected trials to see a new face is $\frac{1}{\frac{1}{4}} = 4$. At this point the number of expected rolls is $1 + 2 + 4 = 7$
   - Face 3 was picked: Remaining probability mass for success is $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. Hence now the number of expected trials to see a new face is $\frac{1}{\frac{1}{4}} = 4$. At this point the number of expected rolls is $1 + 2 + 4 = 7$
3. Face 3 was picked: Remaining probability mass for success is $1 - \frac{1}{4} = \frac{3}{4}$. Hence now the number of expected trials to see a new face is $\frac{1}{\frac{3}{4}} = \frac{4}{3}$. Again we now have 2 more cases of which face was picked.
   - Face 1 was picked: Remaining probability mass for success is $\frac{3}{4} - \frac{1}{4} = \frac{1}{2}$. Hence now the number of expected trials to see a new face is $\frac{1}{\frac{1}{2}} = 2$. At this point the number of expected rolls is $1 + \frac{4}{3} + 2 = \frac{13}{3}$
   - Face 2 was picked: Remaining probability mass for success is $\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$. Hence now the number of expected trials to see a new face is $\frac{1}{\frac{1}{4}} = 4$. At this point the number of expected rolls is $1 + \frac{4}{3} + 4 = \frac{19}{3}$

Hence average number of expected steps can be taken as the probability weighted average of the expected number of trials. However the weight here would be the same as irrespective of the order of rolling the face, the net product of the probabilities is the same. Hence we can take a simple average -

$$ExpectedNumberofTrials = \frac{\frac{19}{3} + \frac{13}{3} + 7 + 7 + \frac{13}{3} + \frac{19}{3}}{6} = 5.89$$