Q.3

3.a

3.b

Q.5

5.a

We model this as a geometric random variable (R.V.) with a success probability $p = \frac{1}{k}$. The value of the face is of no consequence to us as the probability of seeing that face in a fair die is still the same. Using the fact that expectation of a geometric R.V. with success probability p is $\frac{1}{p}$. Hence the expected number of trials here is $\frac{1}{\frac{1}{k}} = k$

5.b

We model this as a geometric random variable (R.V.) with a success probability p. At the very start, picking any die results in a new die, and hence $p = \frac{k}{k} = 1$. As we pick out, say, i dice, the probability of now seeing a new die is $\frac{k-i}{k}$. The expectation of a geometric R.V. with success probability p is $\frac{1}{p}$.

Putting the pieces together at the very start with success probability is $\frac{k}{k}$ our expected trials are $\frac{1}{\frac{k}{k}} = \frac{k}{k}$. In the next attempt the number of expected trials to see a new face is $\frac{1}{\frac{k-1}{k}} = \frac{k}{k-1}$ and so on until we only have one die remaining where the number of expected trials to see a new dice is $\frac{1}{\frac{1}{k}} = \frac{k}{1}$

By linearity of expectations, the total expected number of trials to see all faces once is -

$$\sum_{i=0}^{k-1} \frac{k}{k-i}$$

This can be written as

$$k * \sum_{i=0}^{k-1} \frac{1}{k-i}$$

Now in the limiting case of the die having a large number of face i.e. k is very large, the expression

$$k * \sum_{i=0}^{k-1} \frac{1}{k-i} = k * \log(k)$$

5.c

Modelling the problem as a Geometric R.V. we can build several cases and then average over the number of expected rolls.

Note: The expectation of a geometric R.V. with success probability p is $\frac{1}{p}$.

At the very start the success probability of our R.V. is 1 and hence the expected value is $\frac{1}{1} = 1$. At this point any of the 3 faces could've been chosen. Let's break into cases:

1. Face 1 was picked: Remaining probability mass for success is $1 - \frac{1}{4} = \frac{3}{4}$. Hence now the number of expected trials to see a new face is $\frac{1}{\frac{3}{4}} = \frac{4}{3}$. Again we now have 2 more cases of which face was picked.
   - Face 2 was picked: Remaining probability mass for success is $\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$. Hence now the number of expected trials to see a new face is $\frac{1}{\frac{1}{4}} = 4$. At this point the number of expected rolls is $1 + \frac{4}{3} + 4 = \frac{19}{3}$
   - Face 3 was picked: Remaining probability mass for success is $\frac{3}{4} - \frac{1}{4} = \frac{1}{2}$. Hence now the number of expected trials to see a new face is $\frac{1}{\frac{1}{2}} = 2$. At this point the number of expected rolls is $1 + \frac{4}{3} + 2 = \frac{13}{3}$
2. Face 2 was picked: Remaining probability mass for success is $1 - \frac{1}{2} = \frac{1}{2}$. Hence now the number of expected trials to see a new face is $\frac{1}{\frac{1}{2}} = 2$. Again we now have 2 more cases of which face was picked.
   - Face 1 was picked: Remaining probability mass for success is $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. Hence now the number of expected trials to see a new face is $\frac{1}{\frac{1}{4}} = 4$. At this point the number of expected rolls is $1 + 2 + 4 = 7$
   - Face 3 was picked: Remaining probability mass for success is $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. Hence now the number of expected trials to see a new face is $\frac{1}{\frac{1}{4}} = 4$. At this point the number of expected rolls is $1 + 2 + 4 = 7$
3. Face 3 was picked: Remaining probability mass for success is $1 - \frac{1}{4} = \frac{3}{4}$. Hence now the number of expected trials to see a new face is $\frac{1}{\frac{3}{4}} = \frac{4}{3}$. Again we now have 2 more cases of which face was picked.
   - Face 1 was picked: Remaining probability mass for success is $\frac{3}{4} - \frac{1}{4} = \frac{1}{2}$. Hence now the number of expected trials to see a new face is $\frac{1}{\frac{1}{2}} = 2$. At this point the number of expected rolls is $1 + \frac{4}{3} + 2 = \frac{13}{3}$
   - Face 2 was picked: Remaining probability mass for success is $\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$. Hence now the number of expected trials to see a new face is $\frac{1}{\frac{1}{4}} = 4$. At this point the number of expected rolls is $1 + \frac{4}{3} + 4 = \frac{19}{3}$

Hence average number of expected steps can be taken as the probability weighted average of the expected number of trials. However the weight here would be the same as irrespective of the order of rolling the face, the net product of the probabilities is the same. Hence we can take a simple average -

$$ExpectedNumberofTrials = \frac{\frac{19}{3} + \frac{13}{3} + 7 + 7 + \frac{13}{3} + \frac{19}{3}}{6} = 5.89$$