→ System of linear equations is a collection of linear equations involving the same variables ($x_1, x_2, \ldots x_n$).
A sol^n to the system is a list ($s_1, s_2, \ldots s_n$) that satisfies each eq^n.

→ N-Tuple or Row vector:
The list of numbers ($s_1, s_2 \ldots s_n$).

→ Two systems are **equivalent** if they have the same sol^n set.

. Classification of linear systems:

→ Consistent: If it has 1 or infinitely many solutions.

→ Inconsistent: If it has no solution.

. MATRICES:

→ We can find if a system is consistent / inconsistent, has a unique sol^n without solving the system by setting up the eq^n using a matrix and analysing it's properties.

. $m \times 1$: column vector $1 \times n$: row vector

P.T.O.

→ Augmented Matrix:

For a system of eq^n;

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 \cdots\cdots a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 \cdots\cdots a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 \cdots\cdots a_{mn}x_n &= b_m
\end{aligned}$$

We define an Augmented matrix.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \overbrace{b_1}^{\text{Augmented column.}} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

→ The system of eq^n can be represented $Ax = b$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

• Elementary Row Operations:

i) Replacement, i.e; $R_i \leftrightarrow R_i + kR_j$. ii) Interchange iii) Scaling.

• Row Echelon Form:

→ A matrix is in Row echelon form if
• leading non-zero entries of any row are to the right of the pivots in the row above it.

→ leading non-zero entries of a row in a Matrix that's in row echelon form are called Pivots.

→ If an augmented matrix has a pivot in the augmented column, it is inconsistent. (when reduced to row echelon form).

→ If every other column other than the augmented column has a pivot then the system has a unique sol^n

• Why Elementary Row Operations make Sense?

→ Applying elementary row operations on a matrix is the same as pre-multiplying the matrix by an elementary matrix.

(differing from elementary matrix by 1 elementary row operation).

i) Replacement:

ii) Interchange:

iii) Scaling:

$R_i \longrightarrow R_i + cR_j$ $\qquad$ $R_i \leftrightarrow R_j$ $\qquad$ $R_i \rightarrow kR_i$

$$E = \overbrace{\begin{bmatrix} 1 & & & \\ & 1 & & c_{(i,j)} \\ & & \ddots & \\ & & & 1 \end{bmatrix}}^{m} \qquad E = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & 1_{(i,j)} & \\ & & 1_{(j,i)} & \ddots \; 0 & \\ & & & & 1 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & & \\ & \ddots \; k_{(i,i)} & \\ & & \ddots \; 1 \end{bmatrix}$$

$(j > i)$.

$$E^{-1} = \begin{bmatrix} 1 & & & \\ & 1 & & -c_{(i,j)} \\ & & \ddots & \\ & & & 1 \end{bmatrix} \qquad E^{-1} = E \qquad E^{-1} = \begin{bmatrix} 1 & & \\ & \ddots \; \frac{1}{k_{(i,i)}} & \\ & & \ddots \; 1 \end{bmatrix}$$

→ The inverse of a matrix is $\boxed{A^{-1} = \dfrac{adj(A)}{|A|}}$ CTM

where adj.(A) is the transpose of matrix obtained by replacing every element with it's co-factor.

- → All elementary matrices are invertible.
- → For triangular matrices, det. is product of diagonal elements.

• Reduced Row Echelon Form:

If a matrix in row Echelon form satisfies the following conditions,
i) Pivot = 1,
ii) Pivot column should have 1 as it's only non-zero entry,
then it is said to be in RREF form.

→ Each matrix is equivalent to one & only one RREF.

→ Since RREF is unique, pivot columns are fixed for a matrix, and are same for any echelon form.

→ The variables corresponding to pivot columns of RREF : basic variables
other : free variables

• The existence of free variables ⇒ infinitely many solⁿ.
The solⁿ is expressed in parametric form, with free variable as the parameter.

• VECTORS:

→ Abstract definition of a vector is described later, currently, they will be assumed as n-tuples or ordered list of numbers.

- n-tuples are $n \times 1$ column vectors / $1 \times n$ row vectors.
- Set of all n-tuples is $\mathbb{R}^n$.

→ Multiplication of 2 vectors is not always possible.

→ $\mathbb{R}^n$ is not closed under dot product, closed under cross product.

beyond scope

→ $$y = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots \; c_p \vec{v}_p$$

weights

→ $Ax = b$, can be written using vectors,

$$A = [\vec{a}_1 \;\; \vec{a}_2 \;\; \cdots \;\; \vec{a}_n] \quad , \quad x = \begin{bmatrix} x_1 \\ \vdots \\ \end{bmatrix}$$

$$Ax = x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots \; x_n \vec{a}_n .$$

→ $$A = [\vec{a}_1 \;\; \vec{a}_2 \;\; \cdots \;\; \vec{a}_n] \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$Ax = b$$

$$\downarrow$$

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \end{bmatrix} + \cdots \; x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

• The above eqⁿ has the same solution set as given by the augmented matrix. $[\vec{a}_1 \; \vec{a}_2 \; --- \; \vec{b}]$

We can also say that $Ax = b$ has a solⁿ only when $\vec{b}$ can be expressed as a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3 --- \vec{a}_n$.

• Homogeneous linear System:

A homogeneous system of linear eqⁿ is of the form $Ax = 0$.

It always has a trivial solⁿ; existence of free variable $\Rightarrow \infty$ solⁿ.

• Parametric Vector Form:

If the solⁿ set is described explicitly in terms of linear combination of vectors, with free variables or constants as weights.

e.g.; $2x_1 + 3x_2 - 4x_3 + x_4 = 0$
$x_2 - 3x_3 + 2x_4 = 0.$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -5/2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 5/2 \\ -2 \\ 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} \frac{b_1 - 3b_2}{2} \\ b_2 \\ 0 \\ 0 \end{bmatrix}$$

→ To represent in parametric vector form, express basic variables in terms of free variables after reducing to row echelon form.

- Write them in the form as shown in e.g. (on previous page).

- **Imp. Theorem:**

If 'p' satisfies $Ax = b$, then the general solⁿ of $Ax = b$ is

$$\boxed{w = p + V_h}$$ CTM

V_h → Solⁿ of $Ax = 0$.

Proof: $Ap = b$ → Suppose 'w' is a solⁿ,

$Aw = b$.

$A(w - p) = 0$. $\Rightarrow$ $w - p = V_h$ → where V_h is any solⁿ of $Ax = 0$.

$\therefore w = p + V_h$.

Conversely, if V_h is any solⁿ of $Ax = 0$.

$AV_h = 0$

$A(w - p) = 0$.

$Aw - Ap = 0$.

$Aw = b$.

→ AB can also be thought of as $[Ab_1 \; Ab_2 \; --- \; Ab_p]$.

or as $\begin{bmatrix} a_1 B \\ \vdots \end{bmatrix}$

• Invertible Matrices:

→ Inverse of a matrix is unique. $\left[\begin{array}{l} \because AB = AC = I \\ \therefore B = C \end{array}\right]$

→ $(A^{-1})^T = (A^T)^{-1}$

→ $(AB)^{-1} = B^{-1}A^{-1}$.

• Theorem: An $n \times n$ matrix is invertible iff A is row reducible to I. The sequence of row operations that reduces $A \to I$ also reduces $I \to A^{-1}$. $[\because (E_q \cdots E_1)A = I, \quad (E_q \cdots E_1) = A^{-1}]$

Proof: i) A is invertible if RREF is invertible.

ii) RREF is invertible only when RREF = I.

i) $RREF = E_q E_{q-1} --- E_1 A$.

$A = (E_1^{-1} --- E_q^{-1})\, RREF$.

(Product of invertible matrices is an invertible matrix.)

ii). Assume $Rf \neq I$. & Rf is invertible

Then there is a free variable

$\Rightarrow (Rf)x = 0$ has non-trivial sol^n.

(here b)

$(Rf)b = 0$.

$b = 0$

. Invertibility of Elementary Matrices:

Proof: A matrix is invertible if $EF = FE = I$

ρ_{ij} denotes $n \times n$ matrix with $(i,j)^{th} = 1$ and rest all 0.

If E is elementary matrix that corresponds to $R_i \to R_i + cR_j$

$E = I + c\rho_{ij}$

$F = I - c\rho_{ij}$

$EF = I - c^2 \rho_{ij}^2 = FE. \qquad (i \neq j).$

$$(\rho_{ij}^2)_{kp} = \sum_{m=1}^{n} (\rho_{ij})_{km} \cdot (\rho_{ij})_{mp}$$

$$= 0 \qquad (\text{when } i \neq j).$$

→ Similarly for $R_i \leftrightarrow R_j$ and $R_i \to cR_i$

. Invertible Matrix Theorem:

If A is an $n \times n$ matrix, then following are all true or all false.

i) A is an invertible matrix.

ii) A is row equivalent to identity matrix.

iii) A has n pivot positions.

iv) The eq^n $Ax=0$ has only trivial sol^n.

v) The eq^n $Ax=b$ has atleast one sol^n.

vi) $\exists$ $C_{n\times n}$ s.t. $AC=I$ and $D_{n\times n}$ s.t. $DA=I$.

vii) A^T is a invertible matrix.

. **Linear Span:**

If $v_1, v_2, --- v_p \in \mathbb{R}^n$, then set of linear combinations is denoted by Span$\{v_1, v_2, --- v_n\}$ and is called the subset of $\mathbb{R}^n$ generated by $v_1, v_2, --- v_n$.

. **Linear Independence:**

A set of vectors $v_1, v_2, ---- v_p$ is linearly independent if

$$x_1 v_1 + x_2 v_2 + --- x_p v_p = 0$$

has only trivial sol^n. If $\exists$ $c_1, c_2, --- c_p$ s.t.

$$c_1 v_1 + c_2 v_2 + --- c_p v_p = 0.$$

then $V_1, V_2 \ldots\ldots V_p$ are called linearly dependent.

- **Scalar Multiples:**

A set of vectors $\{V_1, V_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent iff neither of the vectors is a multiple of the other.

Proof: i) let $V_1 = cV_2$

$\Rightarrow V_1 - cV_2 = 0.$

ii) a) If linearly independent;

By contradiction, if vectors are multiples of each other, then they're dependent.

b) If vectors are not multiples, then they're not dependent.

→ An indexed set of vectors is linearly dependent if one of the vectors can be expressed as a linear combination.

Proof: → If $V_j = c_1V_1 + \text{---} \; c_pV_p.$

then $c_1V_1 + \text{--} (-V_j) + c_pV_p = 0 \Rightarrow$ dependent.

→ If dependent, $c_1V_1 + c_2V_2 + \text{---} \; c_pV_p = 0$

$\therefore V_j = \dfrac{-c_1V_1 + \text{--}}{c_j}$

→ If a set contains more vectors than there are entries in each vector, then the set is linearly dependent.
[$\because$ $Ax=0$ has free variables, $\therefore$ non-trivial sol^n].

→ If any vector is a zero vector in a set, then the set is linearly dependent.
[$\because$ $v_j=0$ $\therefore$ $c_j=1$ and rest $=0$ is a sol^n].

. LU Factorisation:

If A is an $m \times n$ matrix, it can be expressed as;

$$A = LU.$$

where, L: lower triangular square matrix. $(m \times m)$
U: echelon form. $(m \times n)$.

Assumption: It is assumed that A can be reduced to echelon form by scaling or row replacements of the type
$R_i \longleftrightarrow R_i + cR_j$ $(i>j)$. [$\because$ lower triangular matrix].

$$E_p E_{p-1} \text{----} E_1 A = U$$

$$A = E_1^{-1} E_2^{-1} \cdots E_p^{-1} U$$
$$A = LU.$$

(inverse of lower triangular matrix is lower $\triangle$)

(product of lower $\triangle$ is lower $\triangle$)

→ Find L s.t. $(E_p \cdots E_1) L = I$.

• VECTOR SPACES:

→ A vector space is a non-empty set of V of vectors, on which 2 operations; addition and multiplication by a scalar are defined, subject to axioms.

i) $u+v$ is in V. , cu in V.

ii) $\exists$ 0 s.t. $0+u = u$. , $1.u = u$

iii) $u+v = v+u$

iv) $(u+v)+w = u+(v+w)$

v) $c(u+v) = cu+cv$.

$(c+d)u = cu + du$.

$(cd)u = c(du)$

vi) $\exists$ $-u$ s.t. $u+(-u) = 0$.

• **Subspace:** A subspace of a vector space is a subset H that:

i) is closed under addition.

ii) is closed under scalar multiplication.

eg. lines through origin $\subset \mathbb{R}^n$.

• **Span:** let S be a subset of a vector space V. The set of all elements of V that can be expressed as a linear combination of elements of S is called span S.

→ Span S is a subspace of V.

Proof:

i) S is finite.

$S = \{V_1, V_2 \ldots\ldots V_n\}$

let $v, w \in$ Span S.

$\Rightarrow \exists\ c_1 \ldots c_n,\ d_1, \ldots d_n$

s.t. $V = c_1 V_1 + \ldots\ c_n V_n$

$W = d_1 V_1 + \ldots\ d_n V_n$

$V + W \in$ Span S.

$kV, kW \in$ Span S.

ii) S is infinite.

let V & $W \in$ Span S.

$\Rightarrow \exists\ V_1, V_2 \ldots V_n \in S$

& $W_1, W_2 \ldots W_m \in S$.

$V = c_1 V_1 + \ldots\ c_n V_n$

$W = d_1 W_1 + \ldots\ d_m W_m$

$S_1 = \{V_1, V_2 \ldots V_n\}$

$S_2 = \{W_1, W_2, \ldots W_m\}$.

Span $S_i \subset$ Span $(S_1 \cup S_2)$ $(i = 1 \& 2)$

$V, W \in$ Span $(S_1 \cup S_2)$.

Span $(S_1 \cup S_2)$ is subspace of V.

$V + W \in$ Span $(S_1 \cup S_2)$.

$V + W \in$ Span (S).

$\rightarrow \quad v \in Span(S).$

$v = c_1 v_1 + \cdots\ c_n v_n.$

$kv = k(c_1 v_1 + \cdots\ k_n v_n)$

$\rightarrow$ 0 in a vector space is unique.

Proof: let $0, z$; $\quad 0+z=0, \quad z+0=z.$

$\therefore \quad 0=z.$

$\rightarrow$ $-u$ for u is unique.

Proof: let $-u, z$; $\quad u+(-u)=0, \quad u+z=0.$

$-u=z.$

Alt. $\rightarrow$ $-u = -u+(u+z)$

$-u=z.$

$\rightarrow$ $0u=0, \ -u=(-1)u, \ c0=0.$

Proof: $v = 0u.$
$= (0+0)u.$
$= 0u+0u$
$v = v+v.$

$0 = v+(-v)$
$= v+v-v$
$0 = v.$
$\therefore 0u = v = 0.$

$u+(-1)u$
$(1+(-1))u$
$0(u)=0.$

$w = c0.$
$c(0) = c(0+0)$
$w = w+w$
$\therefore w=0.$

→ If V is a vector space ; $\{V_1, V_2, \ldots V_p\}$ is a linearly independent system if

$c_1V_1 + \cdots c_pV_p = 0$ has only trivial solⁿ.

• Linear Transformation:

let V, W be vector spaces. A func. $T: V \to W$ is said to be a linear transformation if:

i) $T(V_1+V_2) = T(V_1) + T(V_2)$. $\quad (V_1, V_2 \in V)$

ii) $T(cv) = cT(v)$. $\quad (v \in V)$.

e.g. $V = \mathbb{R}^n$, $W = \mathbb{R}^m$

$A = m \times n$.

$T: V \to W$; $\quad T(v) = Av$.

• Null Space:

• The null space of a $m \times n$ matrix, written as $\text{Nul } A$ is set of all solⁿ to $Ax = 0$

• $\text{Nul } A$ is a subspace of $\mathbb{R}^n$.

Proof: i) $x, y \in \text{Nul } A$.

$Ax = 0$, $Ay = 0$.

$A(x+y) = 0$.

ii) $Ax = 0$

$A(kx) = 0$.

- **Column Space:**
  - The column space of $m \times n$ matrix A, $(\text{Col } A)$ is set of all linear combinations of columns $[a_1, a_2 \ldots a_n]$ of A.

$$\text{Col } A = \text{Span}\{a_1, a_2, \ldots a_n\}$$

  - Column A is a subspace of $\mathbb{R}^n$.

Proof: Span of set is a subspace.

- **Row Space:**
  - The Row space of $m \times n$ matrix A called Row A, is set of all linear combinations of rows of A

$$\text{Row } A = \text{Col } A^T$$

  - Nul A, Row A, Col A are fundamental subspaces assosiated with A.

- **Kernel T:**

  - V, W: vector space. $T: V \to W$.

$$\ker T = \{x \in V : T(x) = 0\}.$$

  - $\ker T$ is a subspace of V.

Proof: $x, y \in V$.

$T(x) + T(y) = 0$.

ii) $T(kx) = kT(x) = 0$.

- **Range T:**

- V, W: vector spaces T: linear transformation.

$$\text{Range } T = \{ x \in W : x = T(v),\ v \in V \}.$$

- Range T (R(T)) is a subspace of W.

Proof: x, y and v_1, v_2.

$x = T(v_1)$

$y = T(v_2)$.

$x + y = T(v_1 + v_2)$.

$x = T(v_1)$

$kx = T(kv_1)$.

- **Basis:**

- If S is an infinite subset of V, S is linearly independent if every finite subset is linearly independent.

- $\mathcal{B} \subset V$ is basis of V if (V: Vector space)

i) $\mathcal{B}$ is linearly independent.

ii) $\mathcal{B}$ spans V.

→ Pivot columns of A form a basis for Col A.

Proof: . The columns $e_1, e_2 \ldots e_n$ of an $n \times n$ identity matrix form a basis for $\mathbb{R}^n$.

. Any subset of linearly independent set is linearly independent.

. $A' = EA$

l: no. of non-zero rows of A'.

Pivot columns of A' are $e_1, \ldots, e_l$

Since all columns have 0's from $l+1$ to m.

$$\text{Col } A' = \text{span}\{e_1, \ldots e_l\}.$$

$E^{-1}e_1, \ldots E^{-1}e_l$ are pivot columns of A.

let $b \in \text{Col} A \Rightarrow \exists \; c_1, \ldots c_n$

$$b = c_1 a_1 \ldots c_n a_n.$$

$$Eb = c_1 E a_1 + \ldots c_n E a_n$$

$$= c_1 e_1 + \ldots c_n e_l.$$

$$\therefore Eb \in \text{col}(A').$$

$$b = c_1 E^{-1} e_1 + \ldots c_n E^{-1} e_l.$$

$b \in$ Span of pivot colums. of A'.

→ If $\{v_1, v_2, \dots v_n\}$ be a linearly independent set in V.
If $w \notin \text{Span}\{v_1, \dots v_n\}$, then the set $\{v_1, \dots v_n, w\}$ is a linearly independent set.

Proof: If $c_1 v_1 + \dots c_n v_n + c_{n+1} w = 0$

$c_{n+1} \neq 0$.

$$\therefore \quad w = \frac{-c_1}{c_{n+1}} v_1 + \dots + \frac{-c_n}{c_{n+1}} v_n$$

→ Contradiction

- **Spanning Set Theorem:**

→ $S = \{v_1, \dots v_p\} \subset V$.
$H = \text{Span } S$.

- If one of the vectors, say v_k, is a linear combination of the remaining vectors in S, the set formed by removing v_k still spans H.
- If $H \neq \{0\}$, some subset of S is basis of H.

Proof: i) $v_k = d_1 v_1 + \dots d_{k-1} v_{k-1} + d_{k+1} v_{k+1} + \dots d_p v_p$.

$$w = c_1 v_1 + \dots c_k v_k + \dots c_p v_p$$
$$= c_1 v_1 + \dots c_k (d_1 v_1 + \dots d_p v_p) + c_p v_p.$$

ii) let B is linearly independent subset of S having maximum cardinality.

If possible, $w \in S \notin \text{Span } B$.

$\Rightarrow B \cup \{w\}$ is l.i., but B is of max. cardinality

$\therefore \quad S \subset \text{Span } B.$

$H = \text{Span } S \subset \text{Span } B.$

$\therefore \quad B$ is basis of H.

=X=

→ $T: V \to W \quad , \; T(0) = 0.$

Proof: $T(0) + T(0) = T(0+0)$

$$T(0) + \cancel{T(0)} = \cancel{T(0)}$$

$$T(0) = 0.$$

→ If $\{v_1, \ldots, v_p\}$ is a linearly dependent subset of V; $T: V \to W$ then $\{T(v_1), \ldots T(v_p)\}$ is linearly dependent subset of W.

Proof: $c_1 v_1 + \cdots + c_p v_p = 0. \qquad$ (l.d.).

$$c_1 T(v_1) + \cdots \quad c_p T(v_p) = T(0).$$

- This may not necessarily hold true for linear independence.

[images may not be linearly independent].

only when T is 1-1 function.

- $T: V \to W$ is 1-1 iff $\ker T = 0$.

<u>Proof:</u> If T is 1-1, $v \in V$ s.t. $T(v) = 0$.

$\Rightarrow T(0) = 0$

$v = 0$.

If $\ker(T) = 0$. $\Rightarrow$ only $T = 0$

$\Rightarrow T(v) = T(w)$ $v, w \in V$

$T(v - w) = 0$

$v - w = 0$.

$v = w$

- If $T: V \to W$ is 1-1, then:
  If $\{V_1, \ldots V_p\}$ is linearly independent then $\{T(V_1), \ldots, T(V_p)\}$ is linearly independent.

<u>Proof:</u> $T(C_1 V_1 + \ldots C_n V_n) = 0$.

$\ker T = \{0\}$

$\Rightarrow C_1 V_1 + \ldots C_n V_n = 0$.

$\Rightarrow C_1, C_2, \ldots C_n = 0$ $\because V_1, V_2 \ldots V_n$ are l.i.

• DETERMINANTS:

$\to$ For $n \times n$ matrix $A = [a_{ij}]_{n \times n}$,

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \qquad i = \text{const.}$$

→ If A is triangular matrix, $\det(A) = \prod_{i=1}^{n} a_{ii}$

(prove by induction).