AMATH271: Dynamics of a Double Pendulum

Paul Coones and Aidan Minto

December 7th 2020

.

Abstract

In this paper, we will analyze both the numerical solutions to the fully non-linearized double pendulum, and the analytical solutions to the linearized double pendulum. In particular, we discuss the dependence of the systems behaviour on its initial conditions and its long-term behaviour by presenting the results of various numerical simulations. In accordance to theory, our results suggest the double pendulum is both non-periodic and highly sensitive to initial conditions.

1 Introduction and Background Theory

The double pendulum consists of two point masses connected by rigid rods, with one of the two masses connected to a pivot which is free to rotate, as demonstrated in figure 1. It turns out that the addition of a second point mass gives rise to erratic and complex behaviour not observed in the typical single pendulum [1] [2]. Due to the complexity of the system, it was not until the 20th-century the double pendulum could be studied in depth, as the solutions to its equations of motion can only be solved numerically [2]. Even though, the first studies of the behaviour of the double pendulum began much earlier in the 18th-century [2]. The double pendulum is also well-known as one of the simplest examples of a nonlinear system which exhibits chaotic behaviour. In particular, this notion of chaos is characterized by non-periodic behaviour, and the sensitivity of the system to its initial conditions [1].

In section 2, we begin by making use of Lagrangian formulation of classical mechanics in order to fully derive the equations of motion governing the double pendulum. Subsequently in section 3, we linearize the previously derived equations making use of a variety of assumptions. Finally, in section 4, we numerically analyze the fully non-linearized system. More specifically, we present the results of various numerical simulations for varying initial conditions, in an attempt to demonstrate chaotic behaviour.

2 Deriving the Equations of Motion

To begin, with reference to figure 1 we are able to deduce the xy-coordinates of each bob of the pendulum in terms of the chosen generalized coordinates ϕ_1 and ϕ_2:

$$\begin{aligned} x_1 &= \ell_1 \sin\phi_1, \\ y_1 &= -\ell_1 \cos\phi_1, \\ x_2 &= x_1 + \ell_2 \sin\phi_2 = \ell_1 \sin\phi_1 + \ell_2 \sin\phi_2, \\ y_2 &= y_1 - \ell_2 \cos\phi_2 = -\ell_1 \cos\phi_1 - \ell_2 \cos\phi_2. \end{aligned}$$

From this, we have:

$$\dot{x}_1 = \ell_1 \dot{\phi}_1 \cos\phi_1, \tag{1}$$

$$\dot{y}_1 = -\ell_1 \dot{\phi}_1 \sin\phi_1, \tag{2}$$

$$\dot{x}_2 = \ell_1 \cos\phi_1 \dot{\phi}_1 + \ell_2 \cos\phi_2 \dot{\phi}_2, \tag{3}$$

$$\dot{y}_2 = \ell_1 \sin\phi_1 \dot{\phi}_1 + \ell_2 \sin\phi_2 \dot{\phi}_2. \tag{4}$$

Now, we notice that as the coordinate ϕ_1 increases from 0, mass 1 with a mass denoted by m_1 rises to a height of $y_1 = \ell_1 \cos\phi_1$ from its initial height of ℓ_1 (at $\phi_1 = 0$) and thus gains gravitational potential energy according to:

$$U_1 = m_1 g \ell_1 \left(1 - \cos\phi_1\right).$$

Likewise, as ϕ_2 increases from 0 alongside the increase of ϕ_1, mass 2 with a mass denoted by m_2 rises to a height $\ell_2 \left(1 - cos\phi_2\right)$ from its initial height of ℓ_2 relative to its support. However, we must also account for the rise of the support (or equivalently mass 1), which we know has risen by $\ell_1 \left(1 - cos\phi_1\right)$ from our prior work. Combining these two facts, we conclude m_2 gains gravitational potential according to:

$$U_2 = m_2 g \left(\ell_1 \left(1 - \cos\phi_1\right) + \ell_2 \left(1 - \cos\phi_2\right)\right).$$

Therefore, the potential of the system is given by:

$$U = U_1 + U_2 = m_1 g \ell_1 \left(1 - \cos\phi_1\right) + m_2 g \left(\ell_1 \left(1 - \cos\phi_1\right) + \ell_2 \left(1 - \cos\phi_2\right)\right).$$

Equivalently, this may be written as:

$$U = \left(m_1 + m_2\right) g \ell_1 \left(1 - \cos\phi_1\right) + m_2 g \ell_2 \left(1 - \cos\phi_2\right). \tag{5}$$

Now, to find the kinetic energy of our system in terms of our generalized coordinates ϕ_1 and ϕ_2, we make use of equations (1) through (4) given that by definition:

$$T_1 := \frac{1}{2} m_1 \left|\left|\dot{\vec{x}}_{m1}\right|\right|^2,$$

where $\dot{\vec{x}}_{m1}$ is the velocity of mass 1. Implying that:

$$T_1 = \frac{1}{2} m_1 \left(\dot{x}_1^2 + \dot{y}_1^2\right).$$

So, inserting the results of equations (1) and (2), we have:

$$\begin{aligned} T_1 &= \frac{1}{2} m_1 \left(\left(\ell_1 \dot{\phi}_1 \cos\phi_1\right)^2 + \left(-\ell_1 \dot{\phi}_1 \sin\phi_1\right)^2\right), \\ &= \frac{1}{2} m_1 \left(\ell_1^2 \dot{\phi}_1^2 \cos^2\phi_1 + \ell_1^2 \dot{\phi}_1^2 \sin^2\phi_1\right). \end{aligned}$$

Which simplifies as:

$$T_1 = \frac{1}{2} m_1 \ell_1^2 \dot{\phi}_1^2. \tag{6}$$

Likewise, for the kinetic energy of mass 2 we have by definition:

$$T_2 := \frac{1}{2} m_2 \left|\left|\dot{\vec{x}}_{m2}\right|\right|^2$$

where $\dot{\vec{x}}_{m2}$ is the velocity of mass 2. As such:

$$T_2 = \frac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{y}_2^2\right),$$

So by inserting the results of equations (3) and (4), we obtain:

$$\begin{aligned} T_2 &= \frac{1}{2} m_2 \left(\left(\ell_1 \cos\phi_1 \dot{\phi}_1 + \ell_2 \cos\phi_2 \dot{\phi}_2\right)^2 + \left(\ell_1 \sin\phi_1 \dot{\phi}_1 + \ell_2 \sin\phi_2 \dot{\phi}_2\right)^2\right), \\ &= \frac{1}{2} m_2 \left(\ell_1^2 \dot{\phi}_1^2 \cos^2\phi_1 + \ell_2^2 \dot{\phi}_2^2 \cos^2\phi_2 + 2\ell_1 \ell_2 \dot{\phi}_1 \dot{\phi}_2 \cos\phi_1 \cos\phi_2 + \ell_1^2 \dot{\phi}_1^2 \sin^2\phi_1 + \ell_2^2 \dot{\phi}_2^2 \sin^2\phi_2 + 2\ell_1 \ell_2 \dot{\phi}_1 \dot{\phi}_2 \sin\phi_1 \sin\phi_2, \right), \\ &= \frac{1}{2} m_2 \left(\ell_1^2 \dot{\phi}_1^2 \left(\cos^2\phi_1 + \sin^2\phi_1\right) + \ell_2^2 \dot{\phi}_2^2 \left(\cos^2\phi_2 + \sin^2\phi_2\right) + 2\ell_1 \ell_2 \dot{\phi}_1 \dot{\phi}_2 \left(\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2\right)\right), \\ &= \frac{1}{2} m_2 \left(\ell_1^2 \dot{\phi}_1^2 + \ell_2^2 \dot{\phi}_2^2 + 2\ell_1 \ell_2 \dot{\phi}_1 \dot{\phi}_2 \cos\left(\phi_1 - \phi_2\right)\right). \end{aligned}$$

From the above result, we conclude the kinetic energy of mass 2 with respect to our generalized coordinates is given by:

$$T_2 = \frac{1}{2} m_2 \left(\ell_1^2 \dot{\phi}_1^2 + \ell_2^2 \dot{\phi}_2^2 + 2\ell_1 \ell_2 \dot{\phi}_1 \dot{\phi}_2 \cos\left(\phi_1 - \phi_2\right) \right). \tag{7}$$

Therefore, the total kinetic energy of the system (as per equations (6) and (7)):

$$T = T_1 + T_2 = \frac{1}{2} m_1 \ell_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 \left(\ell_1^2 \dot{\phi}_1^2 + \ell_2^2 \dot{\phi}_2^2 + 2\ell_1 \ell_2 \dot{\phi}_1 \dot{\phi}_2 \cos\left(\phi_1 - \phi_2\right) \right). \tag{8}$$

$$\tag{9}$$

Which simplifies as

$$T = \frac{1}{2} \left(m_1 + m_2\right) \ell_1^2 \dot{\phi}_1^2 + m_2 \ell_1 \ell_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) + \frac{1}{2} m_2 \ell_2^2 \dot{\phi}_2^2. \tag{10}$$

With the potential energy and kinetic energy of our system is terms of our generalized coordinates given by equations (5) and (10) respectively, we find the Lagrangian of our system defined by:

$$\mathcal{L} := T - U,$$

is given by:

$$\mathcal{L} = \frac{1}{2} \left(m_1 + m_2\right) \ell_1^2 \dot{\phi}_1^2 + m_2 \ell_1 \ell_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) + \frac{1}{2} m_2 \ell_2^2 \dot{\phi}_2^2 - m_1 g \ell_1 \left(1 - \cos\phi_1\right) + m_2 g \left(\ell_1 \left(1 - \cos\phi_1\right) + \ell_2 \left(1 - \cos\phi_2\right)\right). \tag{11}$$

As such, we can determine our Euler-Lagrange equation for ϕ_1 given by:

$$\frac{\partial \mathcal{L}}{\partial \phi_1} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} \right). \tag{12}$$

To do so, we find:

$$\frac{\partial \mathcal{L}}{\partial \phi_1} = -\left(m_1 + m_2\right) g \ell_1 \sin\phi_1 - m_2 \ell_1 \ell_2 \dot{\phi}_1 \dot{\phi}_2 \sin(\phi_1 - \phi_2),$$

and:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} \right) &= \frac{\mathrm{d}}{\mathrm{d}t} \left((m_1 + m_2) \ell_1^2 \dot{\phi}_1^2 + m_2 \ell_1 \ell_2 \dot{\phi}_2 \cos(\phi_1 - \phi_2) \right), \\ &= (m_1 + m_2) \ell_1^2 \ddot{\phi}_1^2 + m_2 \ell_1 \ell_2 \left(\ddot{\phi}_2 \cos(\phi_1 - \phi_2) - \dot{\phi}_2 \sin(\phi_1 - \phi_2)(\dot{\phi}_1 - \dot{\phi}_2) \right). \end{aligned}$$

Therefore, by inserting the above results into equation (12), we have:

$$\begin{aligned} -\left(m_1 + m_2\right) g \ell_1 \sin\phi_1 - m_2 \ell_1 \ell_2 \dot{\phi}_1 \dot{\phi}_2 \sin(\phi_1 - \phi_2) = (m_1 + m_2) \ell_1^2 \ddot{\phi}_1^2 + \\ m_2 \ell_1 \ell_2 \left(\ddot{\phi}_2 \cos(\phi_1 - \phi_2) - \dot{\phi}_2 \sin(\phi_1 - \phi_2)(\dot{\phi}_1 - \dot{\phi}_2) \right). \end{aligned}$$

Moving all terms to the right-hand side and cancelling terms where appropriate, we have our simplified Euler-Lagrange equation for ϕ_1 given by:

$$0 = (m_1 + m_2) \ell_1^2 \ddot{\phi}_1 + m_2 \ell_1 \ell_2 \left(\ddot{\phi}_2 \cos(\phi_1 - \phi_2) + \dot{\phi}_2^2 \sin(\phi_1 - \phi_2) \right) + (m_1 + m_2) g \ell_1 \sin\phi_1. \tag{13}$$

Now, we proceed to do the same for ϕ_2, given its Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \phi_2} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_2} \right). \tag{14}$$

Notably, we have:

$$\frac{\partial \mathcal{L}}{\partial \phi_2} = -m_2 g \ell_2 \sin\phi_2 - m_2 \ell_1 \ell_2 \dot{\phi}_1 \dot{\phi}_2 \sin(\phi_1 - \phi_2),$$

and:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_2}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(m_2 \ell_1 \dot{\phi}_2 + m_2 \ell_1 \ell_2 \dot{\phi}_1 \cos(\phi_1 - \phi_2)\right),\\ &= m_2 \ell_1 \ddot{\phi}_2 + 2\ell_1 \ell_2 \left(\ddot{\phi}_1 \cos(\phi_1 - \phi_2) - \dot{\phi}_1 \sin(\phi_1 - \phi_2)(\dot{\phi}_1 - \dot{\phi}_2)\right).\end{aligned}$$

Inserting the above results into equation (14), we have:

$$-m_2 g \ell_2 \sin\phi_2 - m_2 \ell_1 \ell_2 \dot{\phi}_1 \dot{\phi}_2 \sin(\phi_1 - \phi_2) = m_2 \ell_1 \ddot{\phi}_2 + 2\ell_1 \ell_2 \left(\ddot{\phi}_1 \cos(\phi_1 - \phi_2) - \dot{\phi}_1 \sin(\phi_1 - \phi_2)(\dot{\phi}_1 - \dot{\phi}_2)\right),$$

which simplifies to give us our simplified Euler-Lagrange equation for ϕ_2:

$$0 = m_2 \ell_2^2 \ddot{\phi}_2 + m_2 \ell_1 \ell_2 \left(\ddot{\phi}_1 \cos(\phi_1 - \phi_2) - \dot{\phi}_1^2 \sin(\phi_1 - \phi_2)\right) + m_2 g \ell_2 \sin\phi_2. \tag{15}$$

In all, we conclude from equations (13) and (15) that our system obeys the two equations of motion:

$$\begin{aligned}0 &= (m_1 + m_2)\ell_1^2 \ddot{\phi}_1 + m_2 \ell_1 \ell_2 \left(\ddot{\phi}_2 \cos(\phi_1 - \phi_2) + \dot{\phi}_2^2 \sin(\phi_1 - \phi_2)\right) + (m_1 + m_2) g \ell_1 \sin\phi_1\\ 0 &= m_2 \ell_2^2 \ddot{\phi}_2 + m_2 \ell_1 \ell_2 \left(\ddot{\phi}_1 \cos(\phi_1 - \phi_2) - \dot{\phi}_1^2 \sin(\phi_1 - \phi_2)\right) + m_2 g \ell_2 \sin\phi_2\end{aligned}$$

* Note that throughout this derivation, we have partially followed along and/or made reference to Taylor's derivation for the equations of motion of the double pendulum [1].

3 Analysis

In the following section, we will linearize our equations of motion provided by by equations (13) and (15), so that we may subsequently obtain exact solutions to the motion with the appropriate assumptions. To begin, we make use of the small-angle approximation $\sin(x) \approx x$:

$$\begin{aligned}0 &= (m_1 + m_2)\ell_1^2 \ddot{\phi}_1 + m_2 \ell_1 \ell_2 (\ddot{\phi}_2 \cos(\phi_1 - \phi_2) + \phi_2^2 (\phi_1 - \phi_2)) + (m_1 + m_2) g \ell_1 \phi_1,\\ 0 &= m_2 \ell_2^2 \ddot{\phi}_2 + m_2 \ell_1 \ell_2 (\ddot{\phi}_1 \cos(\phi_1 - \phi_2) - \phi_1^2 (\phi_1 - \phi_2)) + m_2 g \ell_2 \phi_2.\end{aligned}$$

Moreover, applying the similar small-angle approximation $\cos(x) \approx 1$, we have:

$$\begin{aligned}0 &= (m_1 + m_2)\ell_1^2 \ddot{\phi}_1 + m_2 \ell_1 \ell_2 (\ddot{\phi}_2 + \phi_2^2 (\phi_1 - \phi_2)) + (m_1 + m_2) g \ell_1 \phi_1,\\ 0 &= m_2 \ell_2^2 \ddot{\phi}_2 + m_2 \ell_1 \ell_2 (\ddot{\phi}_1 - \phi_1^2 (\phi_1 - \phi_2)) + m_2 g \ell_2 \phi_2.\end{aligned}$$

Now, given that ϕ_1 and ϕ_2 must be small in order for our small-angle approximations to hold, it follows that the doubly small quantities ϕ_1^2 and ϕ_2^2 may be approximated as zero. Thus, the linearized equations of motion for our system are given by:

$$\begin{aligned}0 &= (m_1 + m_2)\ell_1^2 \ddot{\phi}_1 + m_2 \ell_1 \ell_2 \ddot{\phi}_2 + (m_1 + m_2) g \ell_1 \phi_1,\\ 0 &= m_2 \ell_2^2 \ddot{\phi}_2 + m_2 \ell_1 \ell_2 \ddot{\phi}_1 + m_2 g \ell_2 \phi_2.\end{aligned}$$

Moving all second order derivatives to the left-hand side:

$$\begin{aligned}(m_1 + m_2)\ell_1^2 \ddot{\phi}_1 + m_2 \ell_1 \ell_2 \ddot{\phi}_2 &= -(m_1 + m_2) g \ell_1 \phi_1,\\ m_2 \ell_1 \ell_2 \ddot{\phi}_1 + m_2 \ell_2^2 \ddot{\phi}_2 &= -m_2 g \ell_2 \phi_2.\end{aligned}$$

Now, if we let $\boldsymbol{\phi} = \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}$ and $\ddot{\boldsymbol{\phi}} = \begin{bmatrix}\ddot{\phi}_1\\ \ddot{\phi}_2\end{bmatrix}$, we combine the above equations into a single matrix equation:

$$\begin{bmatrix}(m_1+m_2)\ell_1^2 & m_2\ell_1\ell_2\\ m_2\ell_1\ell_2 & m_2\ell_2^2\end{bmatrix}\ddot{\boldsymbol{\phi}} = \begin{bmatrix}-(m_1+m_2)g\ell_1 & 0\\ 0 & -m_2g\ell_2\end{bmatrix}\boldsymbol{\phi}$$

Making the assumption that the pendulum bobs have equal masses with $m_1 = m_2 = m$, and equal rod lengths with $\ell_1 = \ell_2 = \ell$. It makes sense to express this vector differential equation in terms of $\omega_0 = \sqrt{\frac{g}{\ell}}$ since ω_0 is the natural frequency of a simple pendulum with length ℓ and gravitational field g. The vector differential equation simplifies:

$$\begin{bmatrix}2m\ell^2 & m\ell^2\\ m\ell^2 & m\ell^2\end{bmatrix}\ddot{\boldsymbol{\phi}} = \begin{bmatrix}-2mg\ell & 0\\ -mg\ell & \end{bmatrix}\boldsymbol{\phi}$$

$$m\ell^2\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\ddot{\boldsymbol{\phi}} = m\ell^2\begin{bmatrix}-2\frac{g}{\ell} & 0\\ 0 & -\frac{g}{\ell}\end{bmatrix}\boldsymbol{\phi}$$

Canceling and expressing in terms of ω_0,

$$m\ell^2\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\ddot{\boldsymbol{\phi}} = m\ell^2\begin{bmatrix}-2\omega_0^2 & 0\\ 0 & -\omega_0^2\end{bmatrix}\boldsymbol{\phi} \tag{16}$$

$$\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\ddot{\boldsymbol{\phi}} = \begin{bmatrix}-2\omega_0^2 & 0\\ 0 & -\omega_0^2\end{bmatrix}\boldsymbol{\phi} \tag{17}$$

It makes sense that there might be solutions to this equation such that both ϕ_1 and ϕ_2 oscillate with the same angular frequency ω. If

$$\phi_1 = c_1\cos(\omega t - \delta_1)$$
$$\phi_2 = c_2\cos(\omega t - \delta_2)$$

are solutions to the differential equation, then the corresponding ϕ_1 and ϕ_2 with sines instead of cosines also solve the equation. Then the sums

$$\phi_1(t) = c_1\cos(\omega t - \delta_1) + c_1 i\sin(\omega t - \delta_1) = c_1 e^{i(\omega t - \delta_1)}$$
$$\phi_2(t) = c_2\cos(\omega t - \delta_2) + c_2 i\sin(\omega t - \delta_2) = c_2 e^{i(\omega t - \delta_2)}$$

are also solutions to the equation. These complex solutions can be simplified into a form:

$$\boldsymbol{\phi_c} = \begin{bmatrix}c_1e^{-i\delta_1}e^{i\omega t}\\ c_2e^{-i\delta_2}e^{i\omega t}\end{bmatrix} = e^{i\omega t}\begin{bmatrix}d_1\\ d_2\end{bmatrix}$$

The actual motion of the double pendulum will be equal to the real part:

$$\boldsymbol{\phi} = \mathbb{R}e\boldsymbol{\phi_c}$$

Substituting these solutions $\boldsymbol{\phi_c}$ into equation (17),

$$\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\left(-\omega^2\begin{bmatrix}d_1\\ d_2\end{bmatrix}\right) = \begin{bmatrix}-2\omega_0^2 & 0\\ 0 & -\omega_0^2\end{bmatrix}\begin{bmatrix}d_1\\ d_2\end{bmatrix}$$

$$\begin{bmatrix}-2\omega^2 & -\omega^2\\ -\omega^2 & -\omega^2\end{bmatrix}\begin{bmatrix}d_1\\ d_2\end{bmatrix} = \begin{bmatrix}-2\omega_0^2 & 0\\ 0 & -\omega_0^2\end{bmatrix}\begin{bmatrix}d_1\\ d_2\end{bmatrix}$$

$$\mathbf{0} = \begin{bmatrix}2\omega^2 - 2\omega_0^2 & \omega^2\\ \omega^2 & \omega^2 - \omega_0^2\end{bmatrix}\begin{bmatrix}d_1\\ d_2\end{bmatrix}$$

If $\begin{bmatrix}d_1\\ d_2\end{bmatrix} \neq \mathbf{0}$, then the solutions must satisfy:

$$\det\begin{bmatrix}2\omega^2 - 2\omega_0^2 & \omega^2\\ \omega^2 & \omega^2 - \omega_0^2\end{bmatrix} = 0.$$

Solving for ω,

$$\begin{aligned}
2(\omega^2-\omega_0^2)^2-\omega^4&=0\\
2(\omega^4-2\omega^2\omega_0^2+\omega_0^4)-\omega^4&=0\\
2\omega^4-4\omega^2\omega_0^2+2\omega_0^4-\omega^4&=0\\
\omega^4-4\omega_0^2\omega^2+2\omega_0^4&=0
\end{aligned}$$

Completing the square,

$$\begin{aligned}
(\omega^4-4\omega_0^2\omega^2+4\omega_0^4)-2\omega_0^4&=0\\
(\omega^2-2\omega_0^2)^2&=2\omega_0^4\\
\omega^2-2\omega_0^2&=\pm\sqrt{2}\omega_0^2\\
\omega^2&=(2\pm\sqrt{2})\omega_0^2.
\end{aligned}$$

Then the normal modes have frequencies $\omega \in \{\omega_0\sqrt{2+\sqrt{2}}, \omega_0\sqrt{2-\sqrt{2}}\} \approx \{1.848\omega_0, 0.765\omega_0\}$. We now proceed to find the eigenvectors $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ for the normal mode $\omega^2 = (2+\sqrt{2})\omega_0^2$.

$$\begin{aligned}
\mathbf{0} &= \begin{bmatrix} 2(2+\sqrt{2})\omega_0^2-2\omega_0^2 & (2+\sqrt{2})\omega_0^2 \\ (2+\sqrt{2})\omega_0^2 & (2+\sqrt{2})\omega_0^2-\omega_0^2 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}\\
\mathbf{0} &= \begin{bmatrix} (2\sqrt{2}+2)\omega_0^2 & (2+\sqrt{2})\omega_0^2 \\ (2+\sqrt{2})\omega_0^2 & (\sqrt{2}+1)\omega_0^2 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}\\
\mathbf{0} &= \begin{bmatrix} 2\omega_0^2 & \sqrt{2}\omega_0^2 \\ \sqrt{2}\omega_0^2 & \omega_0^2 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}
\end{aligned}$$

For this normal mode, $d_2 = -\sqrt{2}d_1$. Let $d_1 = f_a e^{-i\delta_a}$ to allow for a nonzero phase shift. Then $\boldsymbol{\phi_c} = e^{i\omega t}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = f_a e^{i(\omega t-\delta_a)}\begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}$. The real part is $\boldsymbol{\phi} = \mathbb{R}e\boldsymbol{\phi_c} = f_a\begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}\cos\Big(\sqrt{2+\sqrt{2}}\omega_0 t-\delta_a\Big)$. We now find the eigenvectors for $\omega^2 = (2-\sqrt{2})\omega_0^2$:

$$\begin{aligned}
\mathbf{0} &= \begin{bmatrix} 2(2-\sqrt{2})\omega_0^2-2\omega_0^2 & (2-\sqrt{2})\omega_0^2 \\ (2-\sqrt{2})\omega_0^2 & (2-\sqrt{2})\omega_0^2-\omega_0^2 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}\\
\mathbf{0} &= \begin{bmatrix} (-2\sqrt{2}+2)\omega_0^2 & (2-\sqrt{2})\omega_0^2 \\ (2-\sqrt{2})\omega_0^2 & (-\sqrt{2}+1)\omega_0^2 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}\\
\mathbf{0} &= \begin{bmatrix} -2\omega_0^2 & \sqrt{2}\omega_0^2 \\ \sqrt{2}\omega_0^2 & -\omega_0^2 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}
\end{aligned}$$

Then $d_2 = d_1\sqrt{2}$. Using the same procedure as for the first normal mode, let $d_1 = f_b e^{-i\delta_b}$. $\boldsymbol{\phi_c} = e^{i\omega t}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = f_b e^{i(\omega t-\delta_b)}\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$. The real part is $\boldsymbol{\phi} = \mathbb{R}e\boldsymbol{\phi_c} = f_b\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}\cos\Big(\sqrt{2-\sqrt{2}}\omega_0 t-\delta_b\Big)$. Because our vector differential equation is linear and homogeneous, all linear combinations of solutions are also solutions. The general equation of motion is:

$$\boldsymbol{\phi} = f_a\begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}\cos\Big(\sqrt{2+\sqrt{2}}\omega_0 t-\delta_a\Big) + f_b\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}\cos\Big(\sqrt{2-\sqrt{2}}\omega_0 t-\delta_b\Big).$$

The first term in $\boldsymbol{\phi}$ describes motion in which both angles oscillate perfectly sinusoidally, but the amplitude of oscillation of the lower angle is $-\sqrt{2}$ that of the upper angle so the angles are precisely out of phase. The second term in $\boldsymbol{\phi}$ describes motion in which both angles oscillate perfectly sinusoidally, but the amplitude of oscillation of the lower angle is $\sqrt{2}$ of the upper angle, but in the same direction. Assuming no dissipation of energy, the motion of the second normal mode reminds us of the swinging motion of a continuous non-rigid object, like that of a swinging spaghetti.

Each normal mode is a periodic function. A condition for their sum to be periodic is that their frequencies have a least common integer multiple. We wish to find integers g and h such that $g\sqrt{2+\sqrt{2}}\omega_0 = h\sqrt{2-\sqrt{2}}\omega_0$. Then g/h would be rational.

$$\frac{g}{h} = \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}$$
$$\frac{g}{h} = \frac{2-\sqrt{2}}{\sqrt{2+\sqrt{2}}\sqrt{2-\sqrt{2}}}$$
$$\frac{g}{h} = \frac{2-\sqrt{2}}{\sqrt{4-2}} = \frac{2-\sqrt{2}}{\sqrt{2}} = \sqrt{2}-1.$$

The frequency of the second normal mode is $\sqrt{2}-1$ times that of the first normal mode. The fact that this factor $\sqrt{2}-1$ is irrational means that the sum of oscillations $\boldsymbol{\phi}$ is not a periodic function for conditions in which the masses and lengths are equal.

* Note that throughout this derivation, we have partially followed along and/or made reference to Taylor's derivation for the equations of motion of the double pendulum [1].

4 Nonlinear Solutions

Using Maple to numerically solve the nonlinear system of differential equations provided by the two equations of motion (13) and (15), we will now investigate the motion further through graphical and numerical analysis. More specifically, we will determine the effects of varying the chosen parameters of the scenario, $\phi_1(0)$ and $\phi_2(0)$, the initial angles of each mass of the pendulum with respect to the vertical, while keeping all others potential parameters constant. Where we note these varying parameters were chosen specifically as they coincide exactly with the general coordinates of the system at-hand, and represent the initial conditions of the system which we expect to be highly sensitive. With that said, we will assume the values m_1 = 4kg, m_2 = 1kg, ℓ_1 = 5m, ℓ_2 = 3m, and g = $9.81ms^{-1}$ alongside the initial conditions $\dot{\phi}_1(0) = 0.1ms^{-1}$ and $\dot{\phi}_2 = 0.2ms^1$. Moreover, we assert that these values are held constant throughout the following discussions.

Now, we consider the scenario with the specified parameters $\phi_1(0) = -\frac{\pi}{7}$ and $\phi_2(0) = \frac{\pi}{3}$, implying the masses are initially unaligned from one another. As such, we obtain figures 2 and 3, which demonstrate the time-progression of the generalized coordinates ϕ_1 and ϕ_2 respectively, over the interval from t = 0s to t = 35s. From these figures alongside work in Maple, we are able to conclude that both ϕ_1 and ϕ_2 are periodic with what appears to be the same period $T \approx 15.5s$. However, the oscillations of ϕ_1 and ϕ_2 are not in phase, as seen clearly by the nonlinear relationship when graphing ϕ_1 against ϕ_2 as in figure 4. Finally, we in figure 5 graph the motion of the double pendulum in the xy-plane over the same time interval from t = 0s to t = 35s, which reveals the time-progression of each mass. From this graph, we notice the lack of overlapping lines (of mass 2) implies directly that the system is non-periodic over the given time interval, as the system never appears to return to the same state. As such, the motion as a whole is non-periodic even though both ϕ_1 and ϕ_2 appear to exhibit periodic behaviour themselves.

If we now consider the case in which the masses are initially aligned, say with $\phi_1(0) = \frac{\pi}{5}$ and $\phi_2(0) = \frac{\pi}{5}$, we obtain figures 6 through 9. Of all these figures however, we in particular note figures 6 and 7, as we are able to conclude alongside work in Maple that ϕ_1 and ϕ_2 oscillate in this case with a period $\tau \approx 20.16s$. Interestingly, this period differs from our prior case where we had found a period $T \approx 15.5s$. From this, we conclude that the periods in which ϕ_1 and ϕ_2 oscillate in time are dependent on their initial states (that is our parameters) $\phi_1(0)$ and $\phi_2(0)$, given that they are the only values which have been changed from the unaligned case. Furthermore, we note figure 9, as when compared to the prior case's figure 5, it is evident that the behaviour of the double pendulum as a whole has drastically changed. In turn, this allows to suspect that the double pendulum is highly dependent on its initial conditions, which of course includes the initial alignment of the masses with respect to the vertical given by $\phi_1(0)$ and $\phi_2(0)$.

To further expand on this claim of the system's sensitivity to its initial conditions, we will return to the case with $\phi_1(0) = -\frac{\pi}{7}$ and $\phi_2(0) = \frac{\pi}{3}$. However, we now graph the motion of the double pendulum over the longer time interval from t = 0s to t = 100s, yielding figure 10. With this figure in mind, we now perturb $\phi_1(0)$ and $\phi_2(0)$ both by 0.05 radians (or equivalently 2.86 degrees). As such, we have $\phi_1(0) = -\frac{\pi}{7} + 0.05$ and $\phi_2(0) = \frac{\pi}{3} + 0.05$. With these new perturbed parameters, we obtain figure 11 for the motion of the double pendulum over the 100 second time interval. Evidently, the differences between figures 10 and 11 are quite large, and thus appear to verify our claim of the system's sensitivity to its initial conditions. In particular, as time-progresses, the two figures appear to diverge at an increasing rate. Moreover, we observe in both figures the lack of overlapping lines of mass 2, implying that in both cases, the system as a whole exhibits non-periodic in the long-term as expected.

5 Conclusions

In summary, we have discussed how to derive, linearize, and approximately solve the governing equations of motion of the one of the simplest nonlinear chaotic systems, the double pendulum. We have also presented a variety of simulations which appear to demonstrate chaos, as per the non-periodic and highly sensitive behaviour of the double pendulum to its initial conditions. In the approximation for small upper and lower angles, there are special cases in which the pendulum moves in predictable, periodic ways. In particular, in the case where the amplitude of the lower angle is exactly $\pm\sqrt{2}$ times that of the upper angle, both angles oscillate in simple harmonic motion. However, when we have small angles that are not aligned in these $\pm\sqrt{2}$ ratios, the pendulums oscillate in a predictable, but non-periodic way. For the nonlinear solution, we simulated the motion of the double pendulum with masses initially aligned and determined that it moves very similarly to a single pendulum. We examined the result of a change in the initial angle of just 0.05 radians, and saw that the path of motion of the double pendulum is drastically different in just the first 100 seconds. This indicates that the nonlinear double pendulum's behaviour cannot be reliably predicted from approximate initial conditions.

6 Figures

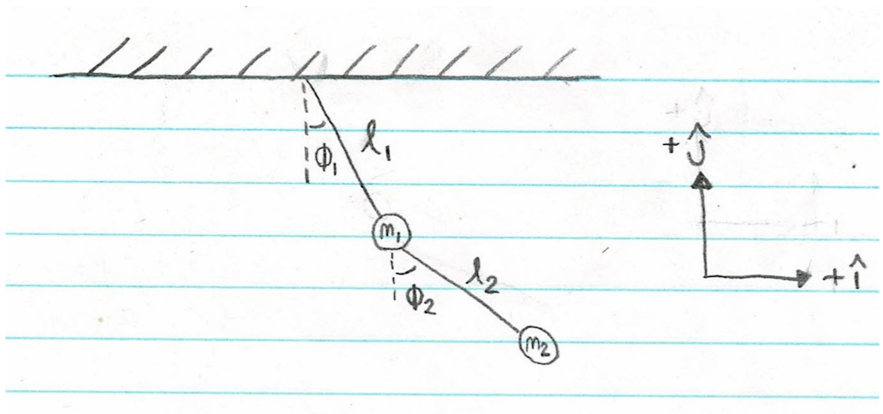

Figure 1: This figure simply illustrates the double pendulum scenario.

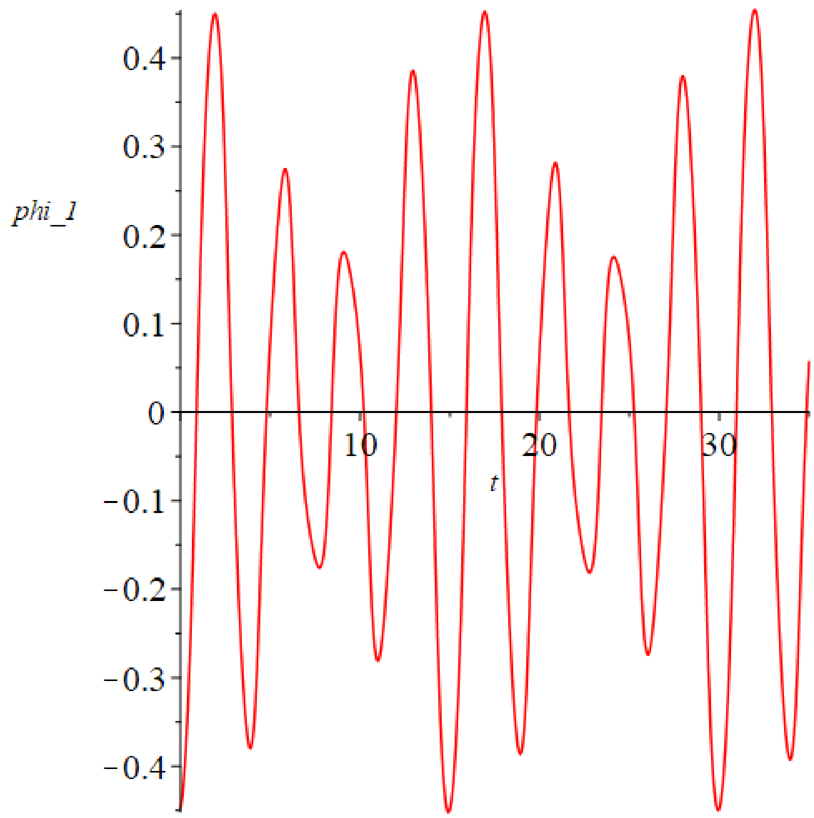

Figure 2: This figure demonstrates the time progression of the generalized coordinate ϕ_1 of the double pendulum (defined in figure 1) over the time interval from t = 0s to t = 35s, and with the specified parameters: $m_1 = 4$ kg, $m_2 = 1$kg, $\ell_1 = 5m$, $\ell_2 = 3m$, $\phi_1 = -\frac{\pi}{7}$, $\phi_2 = \frac{\pi}{3}$, $\dot{\phi_1} = 0.1s^{-1}$, and $\dot{\phi_2} = 0.2s^{-1}$. More specifically, we have the 4 kilogram mass initially aligned at an angle of $-\frac{\pi}{7}$ with respect to the vertical, supporting the 1 kilogram mass 2 which is initially aligned at the angle $\frac{\pi}{3}$ with respect to the vertical. Furthermore, mass 1 has an initial angular velocity of $0.1s^{-1}$, while mass 2 has an initial angular velocity of $0.2s^{-1}$.

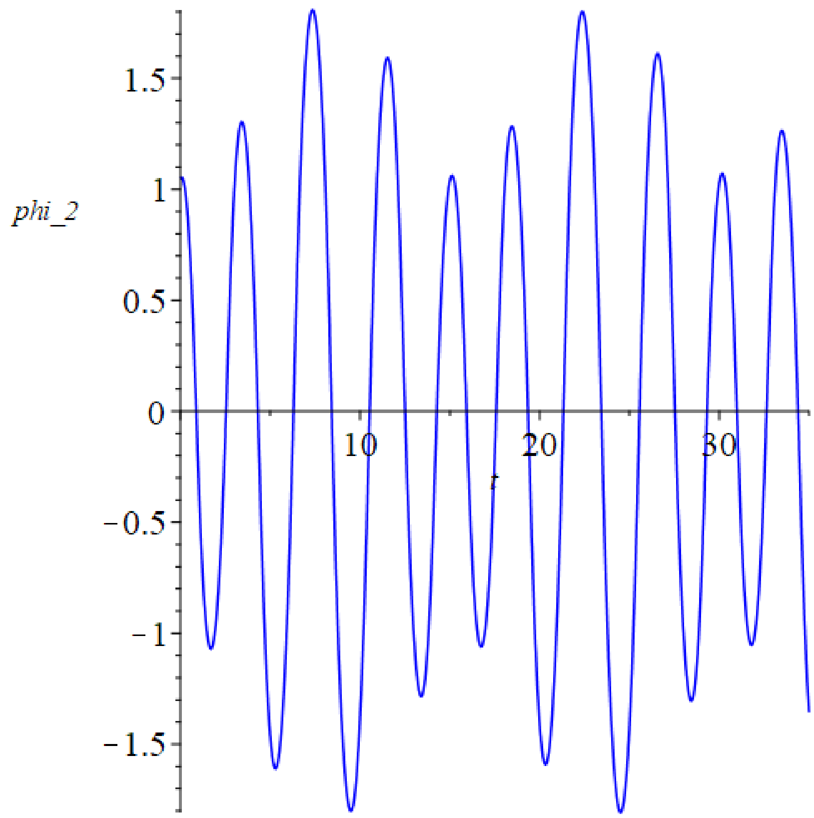

Figure 3: This figure demonstrates the time progression of the generalized coordinate ϕ_2 of the double pendulum (defined in figure 1) over the time interval from t = 0s to t = 35s, and with the specified parameters: $m_1 = 4$ kg, $m_2 = 1$kg, $\ell_1 = 5m$, $\ell_2 = 3m$, $\phi_1 = -\frac{\pi}{7}$, $\phi_2 = \frac{\pi}{3}$, $\dot{\phi_1} = 0.1s^{-1}$, and $\dot{\phi_2} = 0.2s^{-1}$. More specifically, we have the 4 kilogram mass initially aligned at an angle of $-\frac{\pi}{7}$ with respect to the vertical, supporting the 1 kilogram mass 2 which is initially aligned at the angle $\frac{\pi}{3}$ with respect to the vertical. Furthermore, mass 1 has an initial angular velocity of $0.1s^{-1}$, while mass 2 has an initial angular velocity of $0.2s^{-1}$.

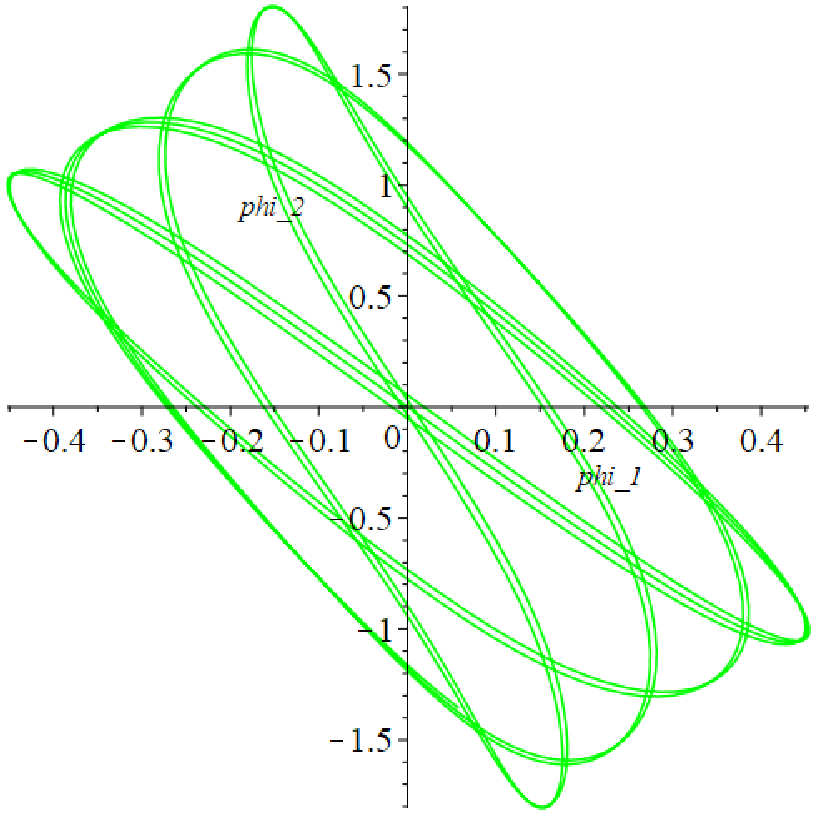

Figure 4: This figure demonstrates the relationship between the generalized coordinates ϕ_1 and ϕ_2 of the double pendulum (defined in figure 1) over the time interval from t = 0s to t = 35s, and with the specified parameters: $m_1 = 4$ kg, $m_2 = 1$kg, $\ell_1 = 5m$, $\ell_2 = 3m$, $\phi_1 = -\frac{\pi}{7}$, $\phi_2 = \frac{\pi}{3}$, $\dot{\phi_1} = 0.1s^{-1}$, and $\dot{\phi_2} = 0.2s^{-1}$. More specifically, we have the 4 kilogram mass initially aligned at an angle of $-\frac{\pi}{7}$ with respect to the vertical, supporting the 1 kilogram mass 2 which is initially aligned at the angle $\frac{\pi}{3}$ with respect to the vertical. Furthermore, mass 1 has an initial angular velocity of $0.1s^{-1}$, while mass 2 has an initial angular velocity of $0.2s^{-1}$.

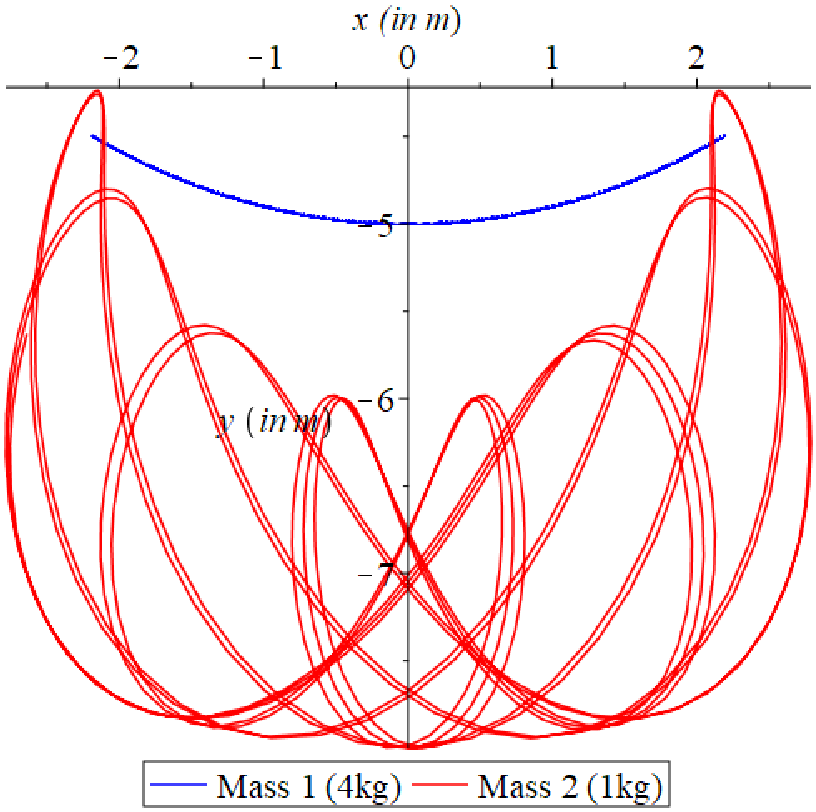

Figure 5: This figure demonstrates the motion of each bob of a double pendulum over the time interval from t = 0s to t = 35s, and with the specified parameters: $m_1 = 4$ kg, $m_2 = 1$kg, $\ell_1 = 5m$, $\ell_2 = 3m$, $\phi_1 = -\frac{\pi}{7}$, $\phi_2 = \frac{\pi}{3}$, $\dot{\phi_1} = 0.1s^{-1}$, and $\dot{\phi_2} = 0.2s^{-1}$. More specifically, we have the 4 kilogram mass initially aligned at an angle of $-\frac{\pi}{7}$ with respect to the vertical, supporting the 1 kilogram mass 2 which is initially aligned at the angle $\frac{\pi}{3}$ with respect to the vertical. Furthermore, mass 1 has an initial angular velocity of $0.1s^{-1}$, while mass 2 has an initial angular velocity of $0.2s^{-1}$.

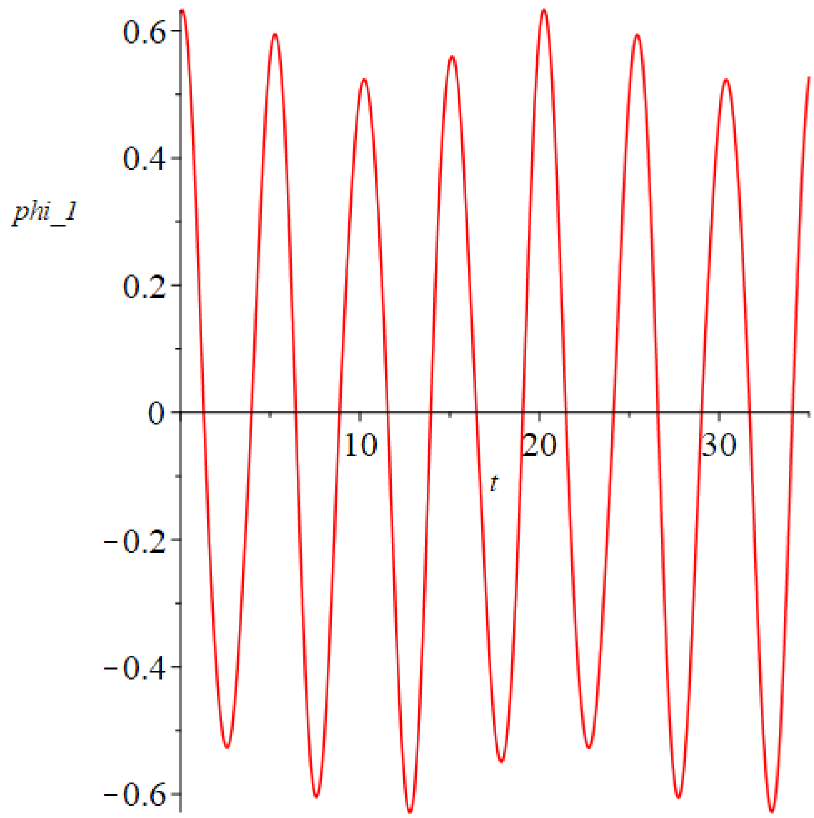

Figure 6: This figure demonstrates the time progression of the generalized coordinate ϕ_1 of the double pendulum (defined in figure 1) over the time interval from t = 0s to t = 35s, and with the specified parameters: $m_1 = 4$ kg, $m_2 = 1$kg, $\ell_1 = 5m$, $\ell_2 = 4m$, $\phi_1 = \frac{\pi}{5}$, $\phi_2 = \frac{\pi}{5}$, $\dot{\phi}_1 = 0.1s^{-1}$, and $\dot{\phi}_2 = 0.2s^{-1}$. More specifically, we have both the 4 kilogram mass 1 and the 1 kilogram mass 2 initially aligned with one another at angles of $\frac{\pi}{5}$ with respect to the vertical. Furthermore, while mass 1 has an initial angular velocity of $0.1s^{-1}$, mass 2 has an initial angular velocity of $0.2s^{-1}$.

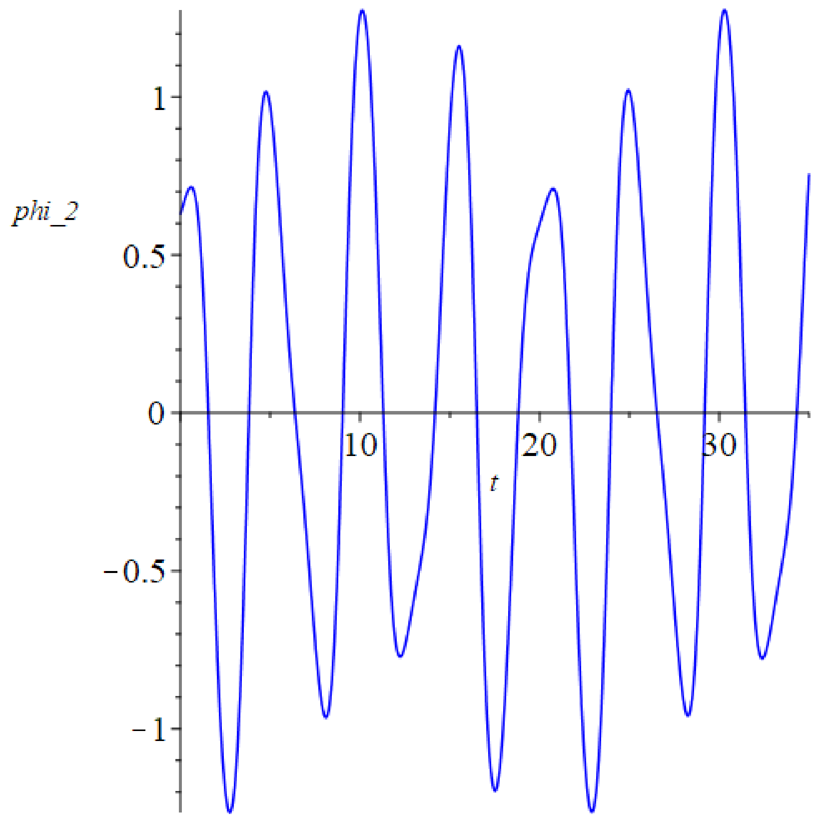

Figure 7: This figure demonstrates the time progression of the generalized coordinate ϕ_2 of the double pendulum (defined in figure 1) over the time interval from t = 0s to t = 35s, and with the specified parameters: $m_1 = 4$ kg, $m_2 = 1$kg, $\ell_1 = 5m$, $\ell_2 = 4m$, $\phi_1 = \frac{\pi}{5}$, $\phi_2 = \frac{\pi}{5}$, $\dot{\phi_1} = 0.1s^{-1}$, and $\dot{\phi_2} = 0.2s^{-1}$. More specifically, we have both the 4 kilogram mass 1 and the 1 kilogram mass 2 initially aligned with one another at angles of $\frac{\pi}{5}$ with respect to the vertical. Furthermore, while mass 1 has an initial angular velocity of $0.1s^{-1}$, mass 2 has an initial angular velocity of $0.2s^{-1}$.

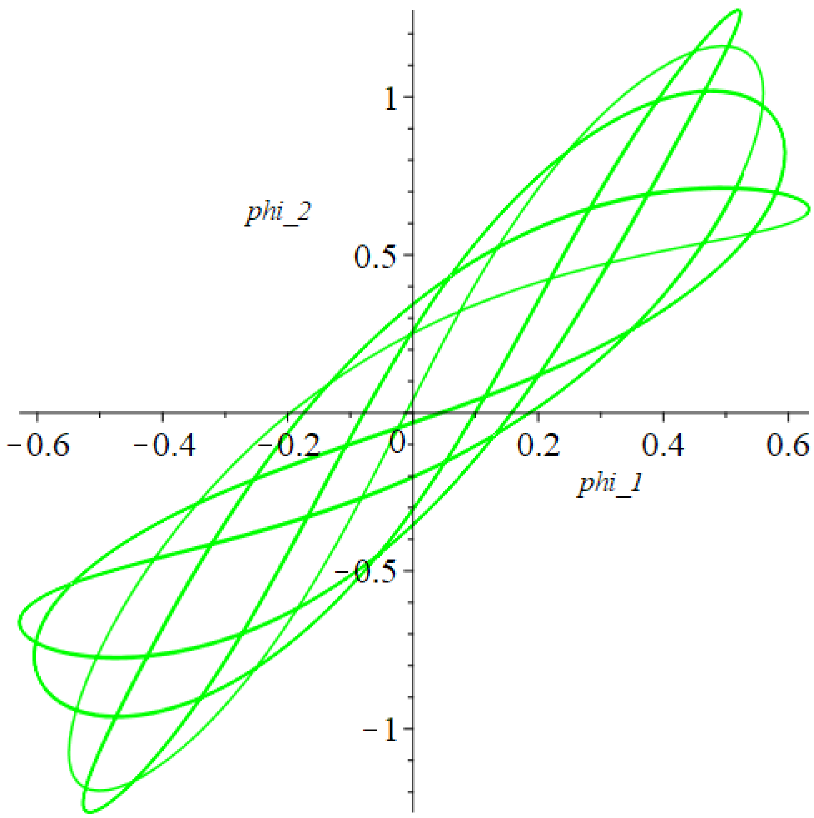

Figure 8: This figure demonstrates the relationship between the generalized coordinates ϕ_1 and ϕ_2 of the double pendulum (defined in figure 1) over the time interval from t = 0s to t = 35s, and with the specified parameters: $m_1 = 4$ kg, $m_2 = 1$kg, $\ell_1 = 5m$, $\ell_2 = 4m$, $\phi_1 = \frac{\pi}{5}$, $\phi_2 = \frac{\pi}{5}$, $\dot{\phi_1} = 0.1s^{-1}$, and $\dot{\phi_2} = 0.2s^{-1}$. More specifically, we have both the 4 kilogram mass 1 and the 1 kilogram mass 2 initially aligned with one another at angles of $\frac{\pi}{5}$ with respect to the vertical. Furthermore, while mass 1 has an initial angular velocity of $0.1s^{-1}$, mass 2 has an initial angular velocity of $0.2s^{-1}$.

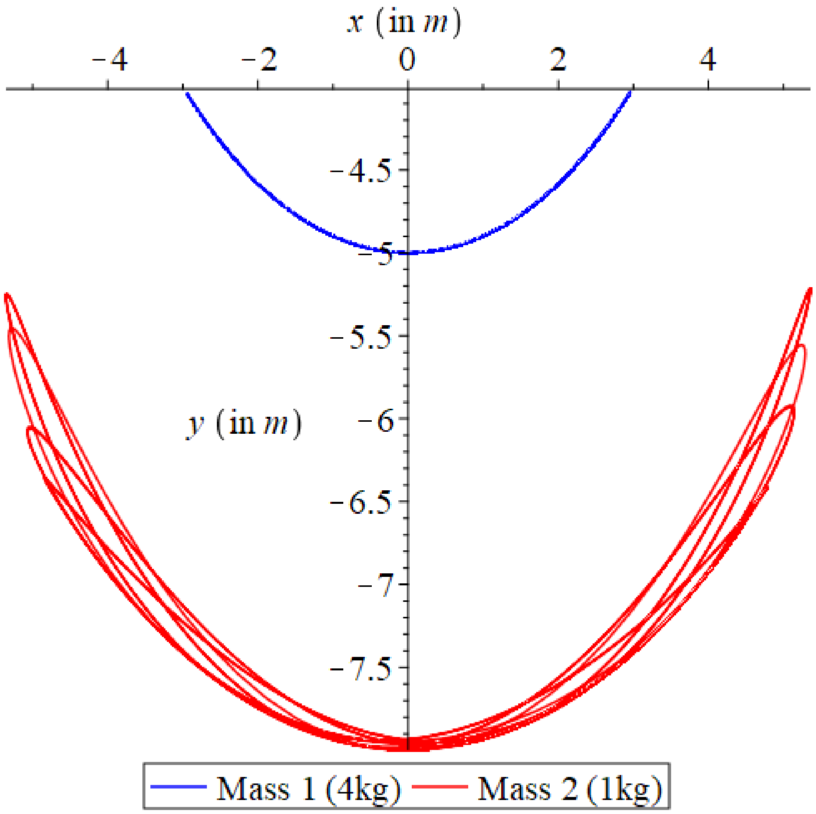

Figure 9: This figure demonstrates the motion of each bob of a double pendulum over the time interval from t = 0s to t = 35s, and with the specified parameters: $m_1 = 4$ kg, $m_2 = 1$kg, $\ell_1 = 5m$, $\ell_2 = 4m$, $\phi_1 = \frac{\pi}{5}$, $\phi_2 = \frac{\pi}{5}$, $\dot{\phi_1} = 0.1s^{-1}$, and $\dot{\phi_2} = 0.2s^{-1}$. More specifically, we have both the 4 kilogram mass 1 and the 1 kilogram mass 2 initially aligned with one another at angles of $\frac{\pi}{5}$ with respect to the vertical. Furthermore, while mass 1 has an initial angular velocity of $0.1s^{-1}$, mass 2 has an initial angular velocity of $0.2s^{-1}$.

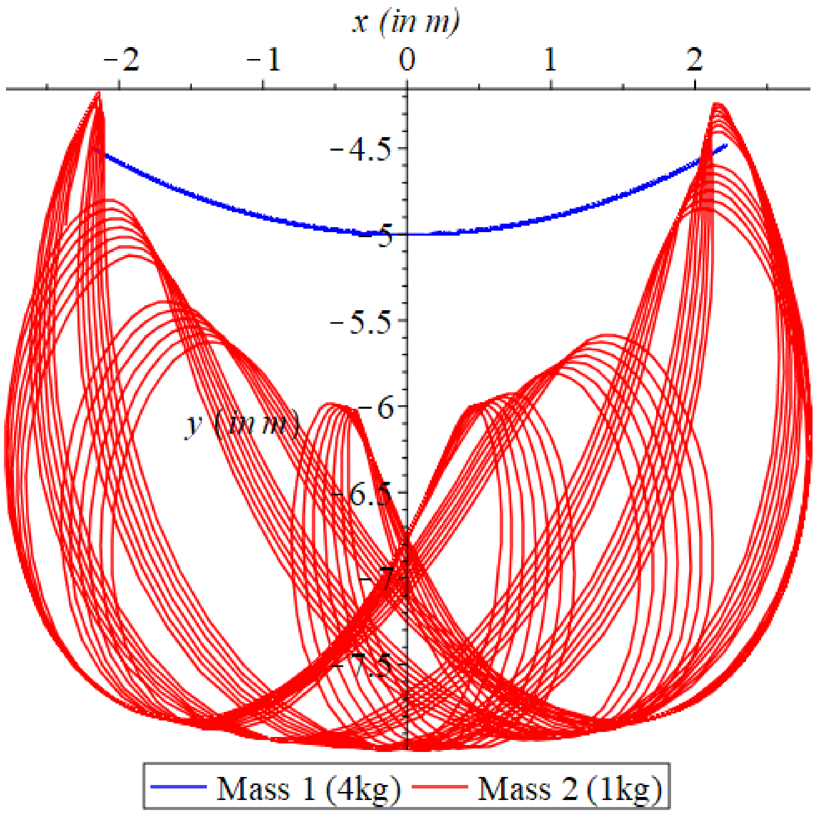

Figure 10: This figure demonstrates the motion of each bob of a double pendulum over the time interval from t = 0s to t = 100ss, and with the specified parameters: $m_1 = 4$ kg, $m_2 =$ 1kg, $\ell_1 = 5m$, $\ell_2 = 3m$, $\phi_1 = -\frac{\pi}{7}$, $\phi_2 = \frac{\pi}{3}$, $\dot{\phi}_1 = 0.1s^{-1}$, and $\dot{\phi}_2 = 0.2s^{-1}$. More specifically, we have the 4 kilogram mass initially aligned at an angle of $-\frac{\pi}{7}$ with respect to the vertical, supporting the 1 kilogram mass 2 which is initially aligned at the angle $\frac{\pi}{3}$ with respect to the vertical. Furthermore, mass 1 has an initial angular velocity of $0.1s^{-1}$, while mass 2 has an initial angular velocity of $0.2s^{-1}$.

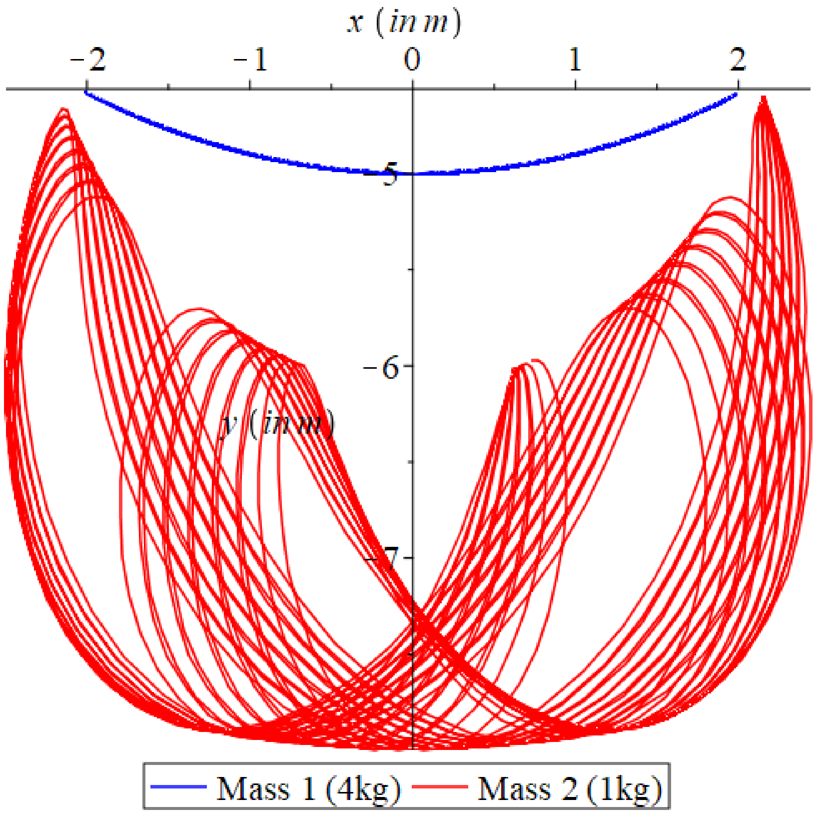

Figure 11: This figure demonstrates the motion of each bob of a double pendulum over the time interval from t = 0s to t = 100s, and with the specified parameters: $m_1 = 4$ kg, $m_2 = 1$kg, $\ell_1 = 5m$, $\ell_2 = 3m$, $\phi_1 = -\frac{\pi}{7} + 0.05$, $\phi_2 = \frac{\pi}{3} + 0.05$, $\dot{\phi_1} = 0.1s^{-1}$, and $\dot{\phi_2} = 0.2s^{-1}$. More specifically, we have the 4 kilogram mass initially aligned at an angle of $-\frac{\pi}{7} + 0.05$ with respect to the vertical, supporting the 1 kilogram mass 2 which is initially aligned at the angle $\frac{\pi}{3} + 0.05$ with respect to the vertical. Furthermore, mass 1 has an initial angular velocity of $0.1s^{-1}$, while mass 2 has an initial angular velocity of $0.2s^{-1}$.

References

[1] Taylor, J. R. *Classical Mechanics*. University Science Books, 2005.

[2] Chen, Joe. *Chaos from Simplicity: An Introduction to the Double Pendulum*. University of Canterbury, 2008.