7 FORMULARIUM

7.1 GASLAWS, WATER VAPOUR AND THERMODYNAMICS (APPENDIX B)

(T in Kelvin, unless otherwise stated)

Equation of state	$p = \rho \frac{R^*}{M} T = \rho R T$
Hydrostatic equilibrium	$dp = -g\rho dz$ $g = 9.8\ \mathrm{m\ s^{-2}}$
Thermodynamics	$dQ = c_v dT + p d\alpha$ $dQ = c_p dT - \frac{dp}{\rho}$ $c_p - c_v = R$
Dry air	$M_\mathrm{d} = 29\ \mathrm{kg\ kmol^{-1}}$
Dry-adiabatic	$\frac{dT}{T} = \frac{R}{c_p}\frac{dp}{\rho}$ $\theta = T\left(\frac{p_o}{p}\right)^{\frac{R}{c_p}}$
Relative humidity RH	$RH = \frac{e}{e_\mathrm{sat}}$
Specific humidity q	$q = \frac{\rho_v}{\rho} \approx \frac{R_d}{R_v}\frac{e}{p}$ $q = \frac{w}{1+w}$
Mixing ratio w	$w = \frac{\rho_v}{\rho_d}$
Dew point T_d	$e = e_\mathrm{sat}(T_\mathrm{d})$
Virtual temperature T_v	$T_v = \frac{T}{1-\left(1-\frac{R_d}{R_v}\right)\frac{e}{p}} \approx T(1+0.61q)$
Wet bulb temperature T_w and psychrometer-equation	$e = e_\mathrm{sat}(T_w) - \frac{R_v}{R_d}\frac{c_p}{L_v} p (T - T_w) = e_\mathrm{sat}(T_w) - \gamma (T - T_w)$ with $\gamma = \frac{R_v}{R_d}\frac{c_p}{L_v} p$

Thermodynamic properties of dry and moist air

	Dependent on				
Constant	***T***	***q***	***p***	**Value**	**Unit**
R^*				8314	$\mathrm{J\ kmol^{-1}K^{-1}}$
R_v				462	$\mathrm{J\ kg^{-1}K^{-1}}$
R_d				287	$\mathrm{J\ kg^{-1}K^{-1}}$
R		●		$287(1+0.61q)$	$\mathrm{J\ kg^{-1}K^{-1}}$
c_pd				1004	$\mathrm{J\ kg^{-1}K^{-1}}$
c_vd				717	$\mathrm{J\ kg^{-1}K^{-1}}$
c_p		●		$1004(1+0.84q)$	$\mathrm{J\ kg^{-1}K^{-1}}$
c_v		●		$717(1+0.37q)$	$\mathrm{J\ kg^{-1}K^{-1}}$
L_v	●			$2501000(1-0.00095(T-273.15))$	$\mathrm{J\ kg^{-1}}$
γ	●	●	●	$65.5\frac{1+0.84q}{1-0.00095(T-273.15)}\frac{p}{101300}$	$\mathrm{Pa\ K^{-1}}$
e_sat (over water)	●			$611.2\, exp\left[\frac{17.62(T-273.15)}{-30.03+T}\right]$	Pa
s (over water)	●			$e_\mathrm{sat}(T)\frac{4284}{(-30.03+T)^2}$ (with e_sat over water)	$\mathrm{Pa\ K^{-1}}$
e_sat (over ice)	●			$611.2\, exp\left[\frac{22.46(T-273.15)}{-0.53+T}\right]$	Pa

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	s (over ice)	•			$e_{\mathrm{sat}}(T)\,\frac{6123}{(-0.53+T)^2}$ (with e_{sat} over ice)	Pa K^{-1}

7.2 Radiation (partly from Appendix A) and soil heat flux

Planck curve	$M_{b\lambda} = \frac{C_1}{\lambda^5\left[exp\left(\frac{C_2}{\lambda T}\right)-1\right]}$ W m^{-2} µm^{-1} C_1 = 3.74 10^8 W m^{-2} µm^4, C_2 = 1.439 10^4 µm K
Stefan-Boltzmann	$M_\mathrm{b} = \sigma T^4$ W m^{-2} with σ = 5.67 10^{-8} W m^{-2}K^{-4}
Wien	$\lambda_\mathrm{m} T$ = 2897.8 µm K
Lambert-Beer	$dI_\lambda = -k_\lambda \rho I_\lambda ds$
Radiation at top of atmosphere	$K_0^\downarrow = \overline{I_0}\left(\frac{\overline{d}_{\mathrm{Sun}}}{d_{\mathrm{Sun}}}\right)^2 cos(\theta_z)$ $\overline{I_0}$ = 1365 W m^{-2} $\cos(\theta_z) = \sin(\delta)\sin(\phi)+\cos(\delta)\cos(\phi)\cos(\omega)$ $\overline{K_0^\downarrow}^{24} = \frac{\overline{I_0}}{\pi}\left(\frac{\overline{d}_{\mathrm{Sun}}}{d_{\mathrm{Sun}}}\right)^2 [\omega_S\sin(\delta)\sin(\phi)+\cos(\delta)\cos(\phi)\sin(\omega_s)]$ (θ_z: zenit angle, δ: declination angle, ϕ: latitude, ω: hour angle, ω_s : daylength (radians))
Global radiation (empirical estimate daily sum)	$\overline{K^\downarrow} = \overline{K_o^\downarrow}\left(0.2+0.55\frac{n}{N_d}\right)$ (N_d:daylength (hours) $= 2\frac{24}{2\pi}\omega_s$)
Extinction of radiation	$I_\lambda = I_{\lambda 0}\,exp\left[-m_r\int_0^\infty k_{\lambda,i}q_i\rho dz\right] = I_{\lambda 0}\tau_{\lambda\theta,i}$
Global radiation as sum of direct and diffuse	$K^\downarrow = S + D = I\cos(\theta_z) + D$
Optical thickness	$\delta_{\lambda,i} \equiv \int_0^\infty k_{\lambda,i}q_i\rho dz$
Optical mass (vertical, relative)	$m_a = \int_\infty^0 \rho ds$ $m_v = \int_\infty^0 \rho dz$ $m_a = m_v m_r$ $m_r \approx [\cos(\theta_z)]^{-1}$
Transmissivities	$\tau_{b,\theta} = \frac{I}{I_0}$ $\tau_b = \frac{K^\downarrow}{K_0^\downarrow}$
Models for broadband transmissivities	$\tau_{b,\theta} = \left[(\tau_{bv,cda})^{m_r}\right]^{T_L} = \exp(\delta_{b,cda} m_r T_L)$ $\frac{K^\downarrow}{K_0^\downarrow} = \tau_b = 0.868\,e^{-0.0387\,m_r T_{L,2}}$ (with T_L the Linke turbidity and $\delta_{\mathrm{b,cda}}$ the broadband optical thickness of a clean, dry atmosphere).
Brunt	$\varepsilon_{\mathrm{a,clear}} = c_1 + c_2\sqrt{e_a}$ with e_a in mbar and c_1 = 0.52 and c_2 = 0.065 hPa$^{-0.5}$
Incoming longwave radiation (empirical estimate)	$L^\downarrow = \varepsilon_a \sigma T_a^4$ W m^{-2}, with $\varepsilon_a = f_{\mathrm{cloud}} + (1-f_{\mathrm{cloud}})\varepsilon_{\mathrm{a,clear}}$
Upwelling longwave radiation	$L^\uparrow = {L_e}^\uparrow + (1-\varepsilon_s)L^\downarrow$ with ${L_e}^\uparrow \approx \varepsilon_s \sigma T_s^4$

Soil heat flux	$G = -\lambda_s \frac{\partial T}{\partial z_d}$ $\frac{\partial G}{\partial z_d} = -\rho_s c_s \frac{\partial T}{\partial t}$ $\frac{\partial T}{\partial t} = \kappa_s \frac{\partial^2 T}{\partial {z_d}^2}$
Sine-wave forcing of T at surface	$T(z_d, t) = \overline{T} + A(0) e^{-z_d/D} \sin\left(\omega\, t - \frac{z_d}{D}\right)$ Damping depth (penetration depth): $D = \sqrt{\frac{2\lambda}{\omega \rho_s c_s}} = \sqrt{\frac{2\kappa_s}{\omega}}$ Amplitude at depth z_d: $A(z_d) = A(0) e^{-z_d/D}$ Flux density at depth z_d and time t: $G(z_d, t) = A(0) e^{-z_d/D} \sqrt{\omega \rho_s c_s \lambda_s} \sin\left[\omega\, t - \frac{z_d}{D} + \frac{\pi}{4}\right]$
Frost penetration depth	$z_f = a\sqrt{-I_n}$

7.3 TURBULENT FLUXES

flux densities	$E = \overline{\rho}\ \ \overline{w'q'}$ $H = \overline{\rho} c_p \overline{w'T'}$ $\tau = -\overline{\rho}\ \ \overline{u'w'}$
K-theory	$E = -\overline{\rho} K_e \frac{\partial \overline{q}}{\partial z}$ $H = -\overline{\rho} c_p K_h \frac{\partial \overline{\theta}}{\partial z}$ $\tau = \overline{\rho} K_m \frac{\partial \overline{u}}{\partial z}$
Turbulent scales	$q_* = -\frac{E}{\overline{\rho} u_*}$ $\theta_* = -\frac{H}{\overline{\rho} c_p u_*}$ $u_* = \sqrt{\frac{\tau}{\overline{\rho}}}$ $\theta_{v*} = -\frac{H_v}{\overline{\rho} c_p u_*}$
Obukhov length	$L = \frac{\overline{\theta}_v}{\kappa g} \frac{u_*^2}{\theta_{v*}} = -\frac{\overline{\rho} c_p \overline{\theta}_v u_*^3}{\kappa g H_v}$
Aerodynamische resisistance	$r_{\mathrm{ax}} = \int_{z_1}^{z_2} \frac{dz}{K_x}$ with *x* is *h*, *e* or *m*; $H = -\overline{\rho} c_p \frac{\overline{\theta}(z_2) - \overline{\theta}(z_1)}{r_{\mathrm{ah}}}$ $E = -\overline{\rho} \frac{\overline{q}(z_2) - \overline{q}(z_1)}{r_{\mathrm{ae}}}$ for neutral conditions: $r_{\mathrm{ae}} = r_{\mathrm{ah}} = r_{\mathrm{am}} = \frac{ln\left[\frac{z_2}{z_1}\right]}{\kappa u_*}$ and $r_{\mathrm{ah}} = \frac{ln\left[\frac{z_{\theta 2}}{z_{\theta 1}}\right] ln\left[\frac{z_{u2}}{z_{u1}}\right]}{\kappa^2 [\overline{u}(z_2) - \overline{u}(z_1)]}$ (κ=0.4, von Karman constant)

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TKE and TKE equation	$\bar{e} = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$ $\frac{d\bar{e}}{dt} = -\overline{u'w'}\frac{\partial\bar{u}}{\partial z} + g\frac{\overline{w'\theta_v'}}{\overline{\theta_v}} - \varepsilon + \cdots$
Richardson numbers	$Ri_f = \frac{g}{\overline{\theta_v}}\frac{\overline{w'\theta_v'}}{\overline{u'w'}\frac{\partial\bar{u}}{\partial z}}$ $Ri_g = \frac{g}{\overline{\theta_v}}\frac{\frac{\partial\overline{\theta_v}}{\partial z}}{\left[\frac{\partial\bar{u}}{\partial z}\right]^2}$ $Ri_f = \frac{K_h}{K_m}Ri_g$
Buoyancy flux	Turbulent flux (kinematic): $\overline{w'\theta_v'} = \overline{w'\theta'}(1 + 0.61\bar{q}) + 0.61\bar{\theta}\,\overline{w'q'}$ Surface flux (energy): $H_v = H(1 + 0.61\bar{q}) + 0.61c_p\bar{\theta}E$
Flux-gradient relationships	$\frac{\partial\bar{\theta}}{\partial z}\frac{\kappa z}{\theta_*} = \phi_h\left(\frac{z}{L}\right)$ $\frac{\partial\bar{q}}{\partial z}\frac{\kappa z}{q_*} = \phi_e\left(\frac{z}{L}\right)$ $\frac{\partial\bar{u}}{\partial z}\frac{\kappa z}{u_*} = \phi_m\left(\frac{z}{L}\right)$ $Ri_g = \frac{\phi_h}{\phi_m^2}\frac{z}{L}$ One can use: $\phi_h = \phi_e = \phi_m^2 = \left(1 - 16\frac{z}{L}\right)^{-1/2}$ $(L < 0)$, $\phi_h = \phi_e = \phi_m = \left(1 + 5\frac{z}{L}\right)$ $(L > 0)$, Then: $\frac{z}{L} = Ri_g$ (when L<0), $\frac{z}{L} = \frac{Ri_g}{1-5Ri_g}$ (when L>0) Neutral: $\frac{\partial\bar{\theta}}{\partial z}\frac{\kappa z}{\theta_*} = 1$, $\frac{\partial\bar{q}}{\partial z}\frac{\kappa z}{q_*} = 1$, $\frac{\partial\bar{u}}{\partial z}\frac{\kappa z}{u_*} = 1$
Integrated flux-gradient relationships	$\bar{u}(z_{u2}) - \bar{u}(z_{u1}) = \frac{u_*}{\kappa}\left[ln\left(\frac{z_{u2}}{z_{u1}}\right) - \Psi_m\left(\frac{z_{u2}}{L}\right) + \Psi_m\left(\frac{z_{u1}}{L}\right)\right]$ $\bar{\theta}(z_{\theta 2}) - \bar{\theta}(z_{\theta 1}) = \frac{\theta_*}{\kappa}\left[ln\left(\frac{z_{\theta 2}}{z_{\theta 1}}\right) - \Psi_h\left(\frac{z_{\theta 2}}{L}\right) + \Psi_h\left(\frac{z_{\theta 1}}{L}\right)\right]$ $\left.\begin{aligned}\Psi_m\left(\frac{z}{L}\right) &= 2\ln\left(\frac{1+x}{2}\right) + \ln\left(\frac{1+x^2}{2}\right) - 2\arctan(x) + \frac{\pi}{2}, \text{ with } x = \left(1 - 16\frac{z}{L}\right)^{\frac{1}{4}} \\ \Psi_h\left(\frac{z}{L}\right) &= 2\ln\left(\frac{1+x^2}{2}\right), \text{ with } x = \left(1 - 16\frac{z}{L}\right)^{\frac{1}{4}}\end{aligned}\right\} \text{for } \frac{z}{L} < 0$ $\left.\Psi_m\left(\frac{z}{L}\right) = \Psi_h\left(\frac{z}{L}\right) = -5\frac{z}{L}\right\} \text{for } \frac{z}{L} > 0$ Neutral: $\bar{u}(z_{u2}) - \bar{u}(z_{u1}) = \frac{u_*}{\kappa}ln\left(\frac{z_{u2}}{z_{u1}}\right)$ $\bar{\theta}(z_{\theta 2}) - \bar{\theta}(z_{\theta 1}) = \frac{\theta_*}{\kappa}ln\left(\frac{z_{\theta 2}}{z_{\theta 1}}\right)$
General equations for H and u_*	$H = \frac{-\bar{\rho}c_p\kappa^2\Delta\bar{\theta}\Delta\bar{u}}{\left[\ln\left(\frac{z_{\theta 2}}{z_{\theta 1}}\right) - \Psi_h\left(\frac{z_{\theta 2}}{L}\right) + \Psi_h\left(\frac{z_{\theta 1}}{L}\right)\right]\left[ln\left(\frac{z_{u2}}{z_{u1}}\right) - \Psi_m\left(\frac{z_{u2}}{L}\right) + \Psi_m\left(\frac{z_{u1}}{L}\right)\right]}$ $u_* = \frac{\kappa\Delta\bar{u}}{\left[\ln\left(\frac{z_{u2}}{z_{u1}}\right) - \Psi_m\left(\frac{z_{u2}}{L}\right) + \Psi_m\left(\frac{z_{u1}}{L}\right)\right]}$

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Analytical solutions (L<0)	$H=\frac{-\bar{\rho}c_p\kappa^2\Delta\bar{\theta}\Delta\bar{u}}{\left[ln\left(\frac{z_2}{z_1}\right)\right]^2}(1-16Ri_{b*})^{3/4}$ $u_*=\frac{\kappa\Delta\bar{u}}{\left[ln\left(\frac{z_2}{z_1}\right)\right]}(1-16Ri_{b*})^{1/4}$ with $Ri_{b*}=\sqrt{z_1z_2}\left[ln\left(\frac{z_2}{z_1}\right)\right]\frac{g}{\bar{\theta}_v}\frac{\Delta\bar{\theta}_v}{(\Delta\bar{u})^2}$, and $L=\frac{\bar{\theta}_v}{g}\frac{(\Delta\bar{u})^2}{\Delta\bar{\theta}_v\,ln\left(\frac{z_2}{z_1}\right)}$
Analytical solutions (L>0)	$H=\frac{-\bar{\rho}c_p\kappa^2\Delta\bar{\theta}\Delta\bar{u}}{\left[ln\left(\frac{z_2}{z_1}\right)\right]^2}(1-5Ri_b)^2$ $u_*=\frac{\kappa\Delta\bar{u}}{\left[ln\left(\frac{z_2}{z_1}\right)\right]}(1-5Ri_b)$ (for $Ri_b<0.2$) With $Ri_b=(z_2-z_1)\frac{g}{\bar{\theta}_v}\frac{\Delta\bar{\theta}_v}{(\Delta\bar{u})^2}$, and $L=\frac{(z_2-z_1)}{ln\left(\frac{z_2}{z_1}\right)}\frac{(1-5Ri_b)}{Ri_b}$

7.4 SOIL WATER FLOW

Warrilow model	$\frac{d\theta}{dt}=\frac{1}{D_r}[P-E(\theta)-D(\theta)]$ $E(\theta)=\beta_w(\theta)E_p\quad\text{with}\quad\beta_w=\begin{bmatrix}1 & \text{for} & \theta_c<\theta<\theta_s\\ \frac{\theta-\theta_w}{\theta_c-\theta_w} & \text{for} & \theta_w<\theta<\theta_c\\ 0 & \text{for} & \theta<\theta_w\end{bmatrix}$ $D=k(\theta)=k_s\left(\frac{\theta-\theta_w}{\theta_s-\theta_w}\right)^n$
Vapour pressure	$\ln\left(\frac{e}{e_{\text{sat}}}\right)=7.5x10^{-7}\ \text{cm}^{-1}(h+\pi)$
Effective conductivity	$k_{\text{eff}}=\frac{\sum_{j=1}^{N}L_j}{\sum_{j=1}^{N}\frac{L_j}{k_j}}$
Continuity eq.	$\frac{\partial\theta}{\partial t}=-\frac{\partial q}{\partial z}-S$
Richards' equation	$\frac{\partial\theta}{\partial t}=C(h)\frac{\partial h}{\partial t}=\frac{\partial\left[k(h)\left(\frac{\partial h}{\partial z}+1\right)\right]}{\partial z}-S(z)$
Soil hydraulic functions	$\theta(h)=\theta_r+\frac{\theta_s-\theta_r}{(1+\|\alpha h\|^n)^{\frac{n-1}{n}}}$ $k(\theta)=k_sS_e^{\lambda}\left[1-\left(1-S_e^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}\right]^2$ with $S_e=\frac{\theta-\theta_r}{\theta_s-\theta_r}$
Infiltration	Horton: $I=I_f+(I_0-I_f)e^{-\beta t};\qquad I_{\text{cum}}=I_ft+\frac{I_0-I_f}{\beta}\left[1-e^{-\beta t}\right]$ Green Ampt horizontal: $I_{\text{cum}}=\left[-2k_th_f(\theta_t-\theta_i)\right]^{½}t^{½}=St^{½}$ Green Ampt vertical: $t=\frac{\theta_t-\theta_i}{k_t}\left[s_f+h_f\ln\left(\frac{s_f-h_f}{-h_f}\right)\right]$
Capillary rise	$z_c=\frac{2\sigma\cos\varphi}{\rho gr}$ $Z=\int_0^Z dz=-\int_0^h\frac{k(h)}{q+k(h)}dh$

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7.5 SOLUTE TRANSPORT

Solute flux	$J = J_{dif} + J_{con} + J_{dis} = qC_l - \theta(D_{dif} + D_{dis})\dfrac{\partial C_l}{\partial z}$
Mass balance	$\dfrac{\partial C_T}{\partial t} = -\dfrac{\partial J}{\partial z} - S_s$
General transport equation	$\dfrac{\partial}{\partial t}(\rho_b C_a + \theta C_l) = \dfrac{\partial}{\partial z}\left(\theta D_e \dfrac{\partial C_l}{\partial z}\right) - \dfrac{\partial}{\partial z}(qC_l) - S_s$
Pulse breakthrough	$C(z,t) = \dfrac{-zC_0}{2\sqrt{\pi D_e t^3}} \exp\left(-\dfrac{(z-vt)^2}{4D_e t}\right)$
Retardation factor	$R = 1 + \rho_b S_d/\theta$
First order decay	$M(t) = M_0 \mathrm{e}^{-\mu t}$
Salinization root zones	$C_l(z) = \dfrac{C_0}{1 + (L_f - 1)\dfrac{\lvert z \rvert}{D_r}}$ $\dfrac{\bar{C}}{C_0} = \dfrac{1}{1 - L_f} \ln\left(\dfrac{1}{L_f}\right)$
Residence time in groundwater	$T_{res} = \dfrac{\phi H}{R} \ln\left(\dfrac{L}{2x}\right)$ $C_{out} = C_{in} + (C_{orig} - C_{in})e^{\frac{-Rt}{\phi H}}$

7.6 PLANT TRANSPORT PROCESSES

Root water uptake (macroscopic)	$S_p(z) = \dfrac{L_{\mathrm{root}}(z)}{\int_{-D_r}^{0} L_{\mathrm{root}}(z)\partial z} T_p$ $S(z) = \alpha_{\mathrm{rw}}(z)\ \alpha_{\mathrm{rs}}(z)\ S_p(z)$ with $T_a = \int_{-D_r}^{0} S(z)\partial z$
Flow in vessels	$v = -\dfrac{r^2}{8\eta}\dfrac{\partial H_p}{\partial x}$
Fluxes at stomate level	Transpiration: $E = -\rho \dfrac{q_e - q_i}{r_s}$ Photosynthesis: $A_n = -F_c = \rho \dfrac{q_{\mathrm{ce}} - q_{\mathrm{ci}}}{r_{\mathrm{s,c}}} = \rho \dfrac{q_{\mathrm{ce}} - q_{\mathrm{ci}}}{1.6 r_s} = \dfrac{g_s}{1.6}\rho(q_{\mathrm{ce}} - q_{\mathrm{ci}})$
Crop yield	$DM_a = 10\mu \dfrac{\boldsymbol{T}_a}{D_a}$
Leaf temperature	$T_{\mathrm{leaf}} = T_a + \dfrac{r_b}{\rho c_p}(Q *_{\mathrm{leaf}} - L_v E_{\mathrm{leaf}})$

7.7 COMBINATION METHODS

Energy balance method	$\beta \equiv \dfrac{H}{L_v E} = \dfrac{c_p}{L_v}\dfrac{\Delta\bar{\theta}}{\Delta\bar{q}} = \gamma \dfrac{\Delta\bar{\theta}}{\Delta\bar{e}}$ $H = \beta \dfrac{Q * - G}{1 + \beta}$ $L_v E = \dfrac{Q * - G}{1 + \beta}$

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Penman or Penman-Monteith	$L_vE = \dfrac{s(Q^* - G) + \dfrac{\rho c_p}{r_a}\left[e_{\text{sat}}(\overline{T}_a) - \overline{e}_a\right]}{s + \gamma}$ $L_vE = \dfrac{s(Q^* - G) + \dfrac{\rho c_p}{r_a}\left[e_{\text{sat}}(\overline{T}_a) - \overline{e}_a\right]}{s + \gamma\left(1 + \dfrac{r_c}{r_a}\right)}$
Priestley-Taylor	$L_vE = \alpha_{\text{PT}}\dfrac{s}{s + \gamma}(Q^* - G)$ with $\alpha_{PT} \approx 1.26$
Makkink	$L_vE = 0.65\dfrac{s}{s + \gamma}K^{\downarrow}$