Chapter 9

Numerical Integration

Core Topics	**Complementary Topics**
Rectangle and midpoint methods (*9.2*).	Estimation of error (*9.8*).
Trapezoidal method (*9.3*).	Richardson's extrapolation (*9.9*).
Simpson's methods (*9.4*).	Romberg integration (*9.10*).
Gauss quadrature (*9.5*).	Improper integrals (*9.11*).
Evaluation of multiple integrals (*9.6*).	
Use of MATLAB built-in functions for integration (9.7).	

9.1 BACKGROUND

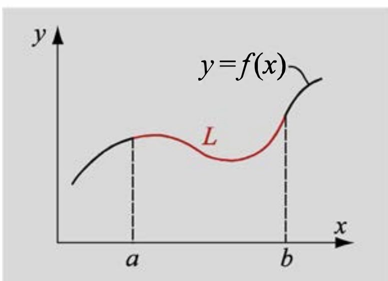


Figure 9-1: Length of a curve.

Integration is frequently encountered when solving problems and calculating quantities in engineering and science. Integration and integrals are also used when solving differential equations. One of the simplest examples for the application of integration is the calculation of the length of a curve (Fig. 9-1). When a curve in the x-y plane is given by the equation $y = f(x)$, the length L of the curve between the points $x = a$ and $x = b$ is given by:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

In engineering there are circumstances that involve experimental or test data, where a physical quantity that has to be determined may be expressed as an integral of other quantities that are measured. For example, the total rate of heat flow through a cross section of width W and height $(b - a)$ is related to the local heat flux via an integral (see Fig. 9-2):

$$\dot{Q} = \int_{y=a}^{y=b} \dot{q}'' W\, dy$$

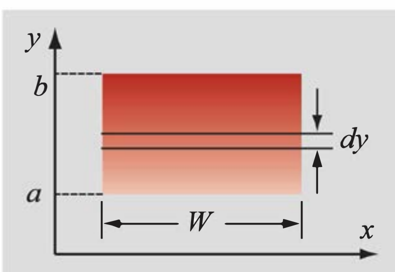


Figure 9-2: Heat flux through a rectangular cross section.

where $\dot{q}''$ is the heat flux and $\dot{Q}$ is the heat flow rate. Experimental measurements may yield discrete values for the heat flux along the surface as a function of y, but the quantity to be determined may be the total heat flow rate. In this instance, the integrand may be specified as a known set of values for each value of y.

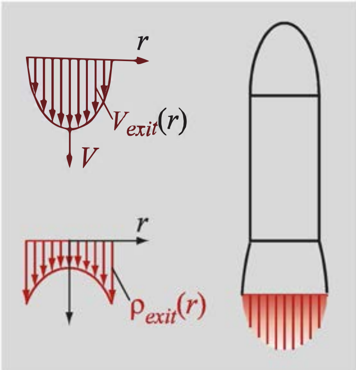


Figure 9-3: Exhaust of a rocket engine.

As yet another illustration, consider the exhaust of a rocket engine generating thrust. As shown in Fig. 9-3, the velocity and density of the flow exiting the rocket nozzle are not uniform over the cross-sectional area. For a circular cross section, both will vary with the radial coordinate r. The resulting expression for the magnitude of the thrust can be obtained from conservation of momentum at steady state:

$$T = \int_0^R 2\pi\rho(r)V_{exit}^2(r)rdr$$

where T is the thrust, $\rho(r)$ is the mass density of the fluid, $V_{exit}(r)$ is the velocity profile at the exit plane of the engine, r is the radial coordinate, and R is the radius of the rocket nozzle at the exit plane. Computational fluid dynamics calculations can yield $\rho(r)$ and $V_{exit}(r)$, but the thrust (a quantity that can be measured in experiments or tests) must be obtained by integration.

The general form of a definite integral (also called an antiderivative) is:

$$I(f) = \int_a^b f(x)dx \tag{9.1}$$

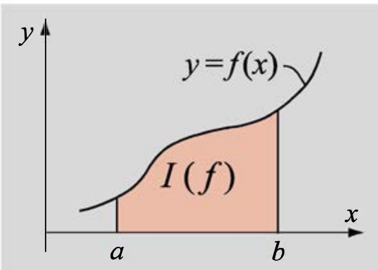


Figure 9-4: Definite integral of *f*(*x*) between *a* and *b*.

where $f(x)$, called the integrand, is a function of the independent variable x, and a and b are the limits of the integration. The value of the integral $I(f)$ is a number when a and b are numbers. Graphically, as shown in Fig. 9-4, the value of the integral corresponds to the shaded area under the curve of $f(x)$ between a and b.

The need for numerical integration

The integrand can be an analytical function or a set of discrete points (tabulated data). When the integrand is a mathematical expression for which the antiderivative can be found easily, the value of the definite integral can be determined analytically. Numerical integration is needed when analytical integration is difficult or not possible, and when the integrand is given as a set of discrete points.

9.1.1 Overview of Approaches in Numerical Integration

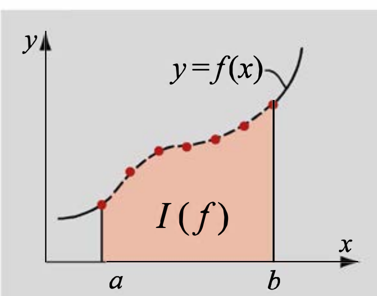


Figure 9-5: Finite number of points are used in numerical integration.

Numerical evaluation of a single integral deals with estimating the number $I(f)$ that is the integral of a function $f(x)$ over an interval from a to b. If the integrand $f(x)$ is an analytical function, the numerical integration is done by using a finite number of points at which the integrand is evaluated (Fig. 9-5). One strategy is to use only the end-points of the interval, $(a, f(a))$ and $(b, f(b))$. This, however, might not give an accurate enough result, especially if the interval is wide and/or the integrand varies significantly within the interval. Higher accuracy can be achieved by using a composite method where the interval $[a, b]$ is divided into smaller subintervals. The integral over each subinterval is calculated, and the results are added together to give the value of the whole integral. If the integrand $f(x)$ is given as a set of discrete points (tabulated data), the numerical integration is done by using these points.

In all cases the numerical integration is carried out by using a set of discrete points for the integrand. When the integrand is an analytical function, the location of the points within the interval $[a, b]$ can be defined by the user or is defined by the integration method. When the integrand is a given set of tabulated points (like data measured in an experiment), the location of the points is fixed and cannot be changed.

Various methods have been developed for carrying out numerical integration. In each of these methods, a formula is derived for calculating an approximate value of the integral from discrete values of the integrand. The methods can be divided into groups called open methods and closed methods.

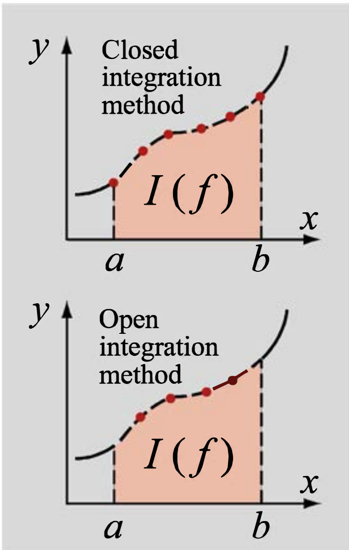


Figure 9-6: Closed and open integration methods.

Closed and open methods

In closed integration methods, the endpoints of the interval (and the integrand) are used in the formula that estimates the value of the integral. In open integration methods, the interval of integration extends beyond the range of the endpoints that are actually used for calculating the value of the integral (Fig. 9-6). The trapezoidal (Section 9.3) and Simpson's (Section 9.4) methods are closed methods, whereas the midpoint method (Section 9.2) and Gauss quadrature (Section 9.5) are open methods.

There are various methods for calculating the value of an integral from the set of discrete points of the integrand. Most commonly, it is done by using Newton–Cotes integration formulas.

Newton–Cotes integration formulas

In numerical integration methods that use Newton–Cotes integration formulas, the value of the integrand between the discrete points is estimated using a function that can be easily integrated. The value of the integral is then obtained by integration. When the original integrand is an analytical function, the Newton–Cotes formula replaces it with a simpler function. When the original integrand is in the form of data points, the Newton–Cotes formula interpolates the integrand between the given points. Most commonly, as with the trapezoidal method (Section 9.3) and Simpson's methods (Section 9.4), the Newton–Cotes integration formulas are polynomials of different degrees.

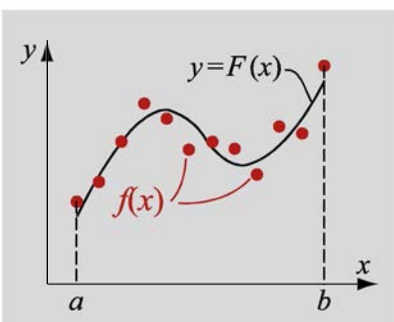


Figure 9-7: Integrating $f(x)$ using an integrable $F(x)$ function.

A different option for integration, once the integrand $f(x)$ is specified as discrete points, is to curve-fit the points with a function $F(x)$ that best fits the points. In other words, as shown in Fig. 9-7, $f(x) \approx F(x)$, where $F(x)$ is a polynomial or a simple function whose antiderivative can be found easily. Then, the integral

$$I(f) = \int_a^b f(x)dx \approx \int_a^b F(x)dx$$

is evaluated by direct analytical methods from calculus. This procedure requires numerical methods for finding $F(x)$ (Chapters 6 and 8), but may not require a numerical method to evaluate the integral if $F(x)$ is an integrable function.

9.2 RECTANGLE AND MIDPOINT METHODS

Rectangle method

The simplest approximation for $\int_a^b f(x)dx$ is to take $f(x)$ over the interval $x \in [a, b]$ as a constant equal to the value of $f(x)$ at either one of the endpoints (Fig. 9-8).

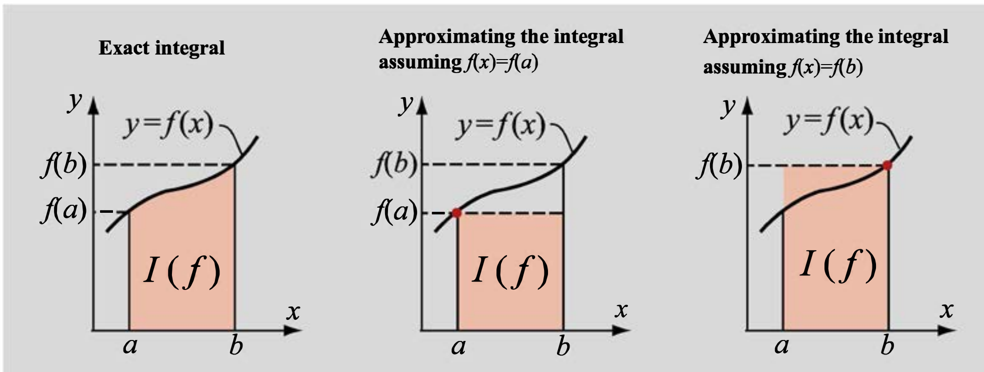


Figure 9-8: Integration using the rectangle method.

The integral can then be calculated in one of two ways:

$$I(f) = \int_a^b f(a)dx = f(a)(b-a) \quad \text{or} \quad I(f) = \int_a^b f(b)dx = f(b)(b-a) \tag{9.2}$$

As Fig. 9-8 shows, the actual integral is approximated by an area of a rectangle. Obviously for the monotonically increasing function shown, the value of the integral is underestimated when $f(x)$ is assumed to be equal to $f(a)$, and overestimated when $f(x)$ is assumed to be equal to $f(b)$. Moreover, the error can be large. When the integrand is an analytical function, the error can be significantly reduced by using the composite rectangle method.

Composite rectangle method

In the composite rectangle method the domain $[a, b]$ is divided into N subintervals. The integral in each subinterval is calculated with the rectangle method, and the value of the whole integral is obtained by adding the values of the integrals in the subintervals. This is shown in Fig. 9-9 where the interval $[a, b]$ is divided into N subintervals by defining the points $x_1, x_2, \ldots, x_{N+1}$. The first point is $x_1 = a$ and the last point is $x_{N+1} = b$ (it takes $N+1$ points to define N intervals). Figure 9-9 shows subintervals with the same width, but in general, the subintervals can have arbitrary width. In this way smaller intervals can be used in regions where the value of the integrand changes rapidly (large slopes) and larger intervals can be used when the integrand changes more gradually.

In Fig. 9-9, the integrand in each subinterval is assumed to have the value of the integrand at the beginning of the subinterval. By using Eq. (9.2) for each subinterval, the integral over the whole domain can be written as the sum of the integrals in the subintervals:

$$I(f) = \int_a^b f(x)dx \approx \overbrace{f(x_1)(x_2-x_1)}^{I_1} + \overbrace{f(x_2)(x_3-x_2)}^{I_2} + \ldots + \overbrace{f(x_i)(x_{i+1}-x_i)}^{I_i}$$

$$+ \ldots + \underbrace{f(x_N)(x_{N+1}-x_N)}_{I_N} = \sum_{i=1}^{N}[f(x_i)(x_{i+1}-x_i)] \quad (9.3)$$

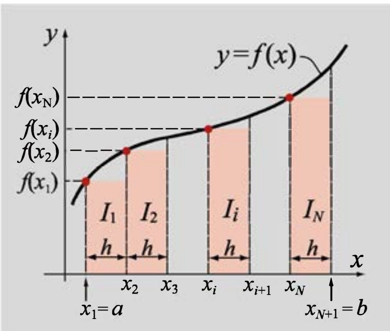


Figure 9-9: The composite rectangle method.

When the subintervals have the same width, h, Eq. (9.3) can be simplified to:

$$\boxed{I(f) = \int_a^b f(x)dx \approx h\sum_{i=1}^{N} f(x_i)} \quad (9.4)$$

Equation (9.4) is the formula for the composite rectangle method for the case where the subintervals have identical width h.

Midpoint method

An improvement over the naive rectangle method is the midpoint method. Instead of approximating the integrand by the values of the function at $x = a$ or at $x = b$, the value of the integrand at the middle of the interval, that is, $f\left(\frac{(a+b)}{2}\right)$, is used. Substituting into Eq. (9.1) yields:

$$I(f) = \int_a^b f(x)dx \approx \int_a^b f\left(\frac{a+b}{2}\right)dx = f\left(\frac{a+b}{2}\right)(b-a) \quad (9.5)$$

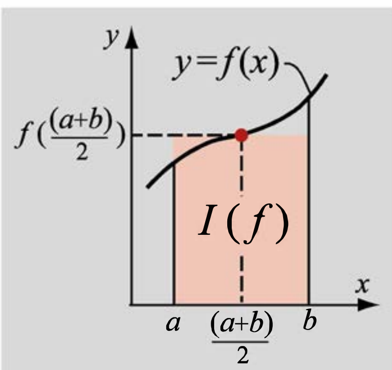


Figure 9-10: The midpoint method.

This method is depicted graphically in Fig. 9-10. As can be seen, the value of the integral is still approximated as the area of a rectangle, but with an important difference—the area is that of an ***equivalent*** rectangle. This turns out to be more accurate than the rectangle method because for a monotonic function as shown in the figure, the regions of the area under the curve that are ignored may be approximately offset by those regions above the curve that are included. However, this is not true for all cases, so that this method may still not be accurate enough. As in the rectangle method, the accuracy can be increased using a composite midpoint method.

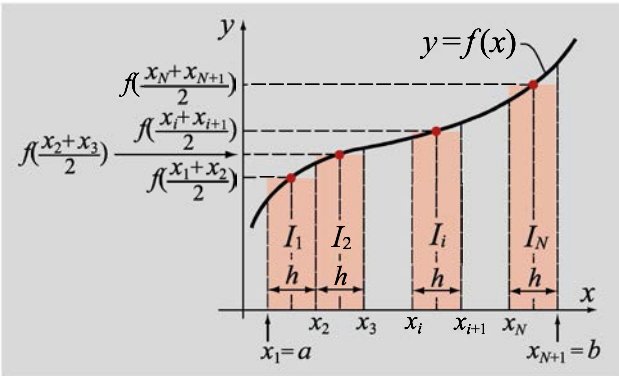


Figure 9-11: The composite midpoint method.

Composite midpoint method

In the composite midpoint method, the domain $[a, b]$ is divided into N subintervals. The integral in each subinterval is calculated with the midpoint method, and the value of the whole integral is obtained by adding the values of the integrals in the subintervals. This is shown in Fig. 9-11 where the interval $[a, b]$ is divided into N subintervals by defining the points $x_1, x_2, \ldots, x_{N+1}$. The first point is $x_1 = a$ and the last point is $x_{N+1} = b$ (it takes $N + 1$ points to define N intervals). Figure 9-11 shows subintervals with the same width, but in general, the subintervals can have arbitrary width.

By using Eq. (9.5) for each subinterval, the integral over the whole domain can be written as the sum of the integrals in the subintervals:

$$I(f) = \int_a^b f(x)dx \approx \overbrace{f\left(\frac{x_1+x_2}{2}\right)(x_2-x_1)}^{I_1} + \overbrace{f\left(\frac{x_2+x_3}{2}\right)(x_3-x_2)}^{I_2} + \ldots$$

$$+ \overbrace{f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)}^{I_i} + \ldots + \overbrace{f\left(\frac{x_N+x_{N+1}}{2}\right)(x_{N+1}-x_N)}^{I_N}$$

$$= \sum_{i=1}^{N}\left[f\left(\frac{x_i+x_{i+1}}{2}\right)(x_{i+1}-x_i)\right] \tag{9.6}$$

When the subintervals have the same width, h, Eq. (9.6) can be simplified to:

$$\boxed{I(f) = \int_a^b f(x)dx \approx h\sum_{i=1}^{N} f\left(\frac{x_i+x_{i+1}}{2}\right)} \tag{9.7}$$

Equation (9.7) is the formula for the composite midpoint method for the case where the subintervals have identical width h.

9.3 TRAPEZOIDAL METHOD

A refinement over the simple rectangle and midpoint methods is to use a linear function to approximate the integrand over the interval of integration (Fig. 9-12). Newton's form of interpolating polynomials with two points $x = a$ and $x = b$, yields:

$$f(x) \approx f(a) + (x-a)f[a, b] = f(a) + (x-a)\frac{[f(b)-f(a)]}{b-a} \tag{9.8}$$

Substituting Eq. (9.8) into Eq. (9.1) and integrating analytically gives:

$$I(f) \approx \int_a^b \left(f(a) + (x-a)\frac{[f(b)-f(a)]}{b-a} \right) dx$$

$$= f(a)(b-a) + \frac{1}{2}[f(b)-f(a)](b-a) \tag{9.9}$$

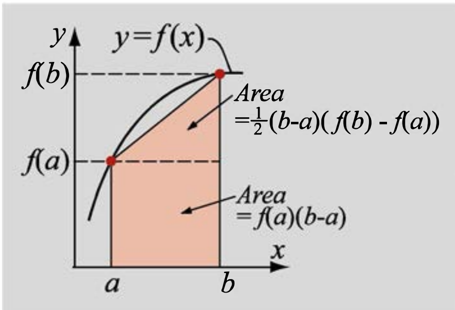


Figure 9-12: The trapezoidal method.

Simplifying the result gives an approximate formula popularly known as the trapezoidal rule or trapezoidal method:

$$\boxed{I(f) \approx \frac{[f(a)+f(b)]}{2}(b-a)} \tag{9.10}$$

Examining the result before the simplification, that is, the right-hand side of Eq. (9.9), shows that the first term, $f(a)(b-a)$, represents the area of a rectangle of height $f(a)$ and length $(b-a)$. The second term, $\frac{1}{2}[f(b)-f(a)](b-a)$, is the area of the triangle whose base is $(b-a)$ and whose height is $[f(b)-f(a)]$. These are shown in Fig. 9-12 and serve to reinforce the notion that in this method the area under the curve $f(x)$ is approximated by the area of the trapezoid (rectangle + triangle). As shown in Fig. 9-12, this is more accurate than using a rectangle to approximate the shape of the region under $f(x)$.

As with the rectangle and midpoint methods, the trapezoidal method can be easily extended to yield any desired level of accuracy by subdividing the interval $[a, b]$ into subintervals.

9.3.1 Composite Trapezoidal Method

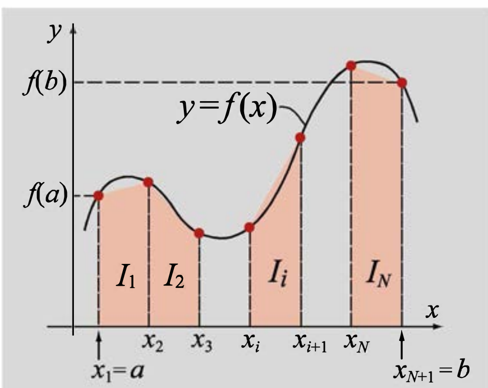


Figure 9-13: The composite trapezoidal method.

The integral over the interval $[a, b]$ can be evaluated more accurately by dividing the interval into subintervals, evaluating the integral for each subintervals (with the trapezoidal method), and adding the results. As shown in Fig. 9-13, the interval $[a, b]$ is divided into N subintervals by defining the points $x_1, x_2, \ldots, x_{N+1}$ where the first point is $x_1 = a$ and the last point is $x_{N+1} = b$ (it takes $N+1$ points to define N intervals).

The integral over the whole interval can be written as the sum of the integrals in the subintervals:

$$I(f) = \int_a^b f(x)dx = \overbrace{\int_{x_1=a}^{x_2} f(x)dx}^{I_1} + \overbrace{\int_{x_2}^{x_3} f(x)dx}^{I_2} + \ldots + \overbrace{\int_{x_i}^{x_{i+1}} f(x)dx}^{I_i}$$

$$+ \ldots + \underbrace{\int_{x_N}^{x_{N+1}} f(x)dx}_{I_N} = \sum_{i=1}^{N} \int_{x_i}^{x_{i+1}} f(x)dx \tag{9.11}$$

Applying the trapezoidal method to each subinterval $[x_i, x_{i+1}]$ yields:

$$I_i(f) = \int_{x_i}^{x_{i+1}} f(x)dx \approx \frac{[f(x_i) + f(x_{i+1})]}{2}(x_{i+1} - x_i)$$

Substituting the trapezoidal approximation in the right side of Eq. (9.11) gives:

$$I(f) = \int_a^b f(x)dx \approx \frac{1}{2}\sum_{i=1}^{N}[f(x_i) + f(x_{i+1})](x_{i+1} - x_i) \qquad (9.12)$$

Equation 9.12 is the general formula for the composite trapezoidal method. Note that the subintervals $[x_i, x_{i+1}]$ need not be identical (i.e., equally spaced) at all. In other words, each of the subintervals can be of different width. If, however, the subintervals are all the same width, that is, if

$$(x_2 - x_1) = (x_3 - x_2) = \ldots = (x_{i+1} - x_i) = \ldots = (x_N - x_{N-1}) = h$$

then Eq. (9.12) can be simplified to:

$$I(f) \approx \frac{h}{2}\sum_{i=1}^{N}[f(x_{i+1}) + f(x_i)]$$

This can be further reduced to a formula that lends itself to programming by expanding the summation:

$$I(f) \approx \frac{h}{2}[f(a) + 2f(x_2) + 2f(x_3) + \ldots + 2f(x_N) + f(b)]$$

or,

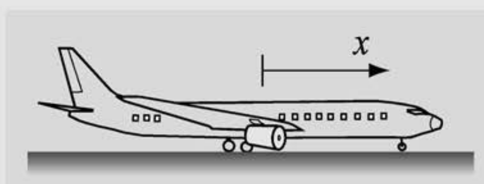
$$I(f) \approx \frac{h}{2}[f(a) + f(b)] + h\sum_{i=2}^{N} f(x_i) \qquad (9.13)$$

Equation (9.13) is the formula for the composite trapezoidal method for the case where the subintervals have identical width h.

Example 9-1 shows how the composite trapezoidal method is programmed in MATLAB and then used for solving a problem.

Example 9-1: Distance traveled by a decelerating airplane.

A Boeing 737-200 airplane of mass $m = 97000$ kg lands at a speed of 93 m/s (about 181 knots) and applies its thrust reversers at $t = 0$. The force F that is applied to the airplane, as it decelerates, is given by $F = -5v^2 - 570000$, where v is the airplane's velocity. Using Newton's second law of motion and flow dynamics, the relationship between the velocity and the position x of the airplane can be written as:



$$mv\frac{dv}{dx} = -5v^2 - 570000$$

where x is the distance measured from the location of the jet at $t = 0$.

Determine how far the airplane travels before its speed is reduced to 40 m/s (about 78 knots) by using the composite trapezoidal method to evaluate the integral resulting from the governing differential equation.

SOLUTION

Even though the governing equation is an ODE, it can be expressed as an integral in this case. This is done by separating the variables such that the speed v appears on one side of the equation and x appears on the other.

$$\frac{97000 v dv}{(-5v^2 - 570000)} = dx$$

Next, both sides are integrated. For x the limits of integration are from 0 to an arbitrary location x, and for v the limits are from 93 m/s to 40 m/s.

$$\int_0^x dx = -\int_{93}^{40} \frac{97000v}{(5v^2 + 570000)} dv = \int_{40}^{93} \frac{97000v}{(5v^2 + 570000)} dv \tag{9.14}$$

The objective of this example is to show how the definite integral on the right-hand side of the equation can be determined numerically using the composite trapezoidal method. In this problem, however, the integration can also be carried out analytically. For comparison, the integration is done both ways.

Analytical Integration

The integration can be carried out analytically by using substitution. By substituting $z = 5u^2 + 570000$, the integration can be performed to obtain the value $x = 574.1494$ m.

Numerical Integration

To carry out the numerical integration, the following user-defined function, named `trapezoidal`, is created.

Program 9-1: Function file, integration trapezoidal method.

```
function I = trapezoidal(Fun,a,b,N)
% trapezoidal numerically integrate using the composite trapezoidal method.
% Input Variables:
% Fun  Name for the function to be integrated.
% (Fun is assumed to be written with element-by-element calculations.)
% a  Lower limit of integration.
% b  Upper limit of integration.
% N  Number of subintervals.
% Output Variable:
% I  Value of the integral.

h = (b-a)/N;                    Calculate the width h of the subintervals.
x = a:h:b;                      Create a vector x with the coordinates of the subintervals.
F = Fun(x);                     Create a vector F with the values of the integrand at each point x.

I=h*(F(1)+F(N+1))/2+h*sum(F(2:N));   Calculate the value of the integral according to Eq. (9.13).
```

The function `trapezoidal` is used next in the Command Window to determine the value of the integral in Eq. (9.14). To examine the effect of the number of subintervals on the result, the function is used three times using $N = 10, 100$, and 1000. The display in the Command Window is:

```
>> format long g
>> Vel = @ (v) 97000*v./(5*v.^2+570000);
>> distance = trapezoidal(Vel,40,93,10)
distance =
        574.085485133712
>> distance = trapezoidal(Vel,40,93,100)
distance =
        574.148773931409
>> distance = trapezoidal(Vel,40,93,1000)
distance =
        574.149406775129
```

Define an anonymous function for the integrand. Note element-by-element calculations.

As expected, the results show that the integral is evaluated more accurately as the number of subintervals is increased. When $N = 1000$, the answer is the same as that calculated analytically to four decimal places.

Example 9-1 reveals two key points:

- It is important to check results from numerical computations (performed either by hand or by computer), against known analytical solutions. In the event an analytical solution is not available, it is necessary to check the answer by another numerical method and to compare the two results.
- In most problems involving numerical integration, it is possible to improve on the accuracy of an answer by taking more subintervals, that is, by reducing the size of the subinterval.

9.4 SIMPSON'S METHODS

The trapezoidal method described in the last section relies on approximating the integrand by a straight line. A better approximation can possibly be obtained by approximating the integrand with a nonlinear function that can be easily integrated. One class of such methods, called Simpson's rules or Simpson's methods, uses quadratic (Simpson's 1/3 method) and cubic (Simpson's 3/8 method) polynomials to approximate the integrand.

9.4.1 Simpson's 1/3 Method

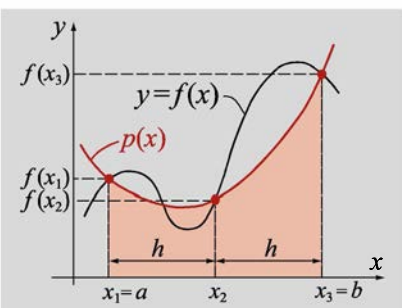


Figure 9-14: Simpson's 1/3 Method.

In this method, a quadratic (second-order) polynomial is used to approximate the integrand (Fig. 9-14). The coefficients of a quadratic polynomial can be determined from three points. For an integral over the domain $[a, b]$, the three points used are the two endpoints $x_1 = a$,

$x_3 = b$, and the midpoint $x_2 = (a+b)/2$. The polynomial can be written in the form:

$$p(x) = \alpha + \beta(x - x_1) + \gamma(x - x_1)(x - x_2) \tag{9.15}$$

where α, β, and γ are unknown constants evaluated from the condition that the polynomial passes through the points, $p(x_1) = f(x_1)$, $p(x_2) = f(x_2)$, and $p(x_3) = f(x_3)$. These conditions yield:

$$\alpha = f(x_1),\quad \beta = [f(x_2) - f(x_1)]/(x_2 - x_1)\text{ , and } \gamma = \frac{f(x_3) - 2f(x_2) + f(x_1)}{2(h)^2}$$

where $h = (b-a)/2$. Substituting the constants back in Eq. (9.15) and integrating $p(x)$ over the interval $[a, b]$ gives:

$$\begin{aligned} I = \int_{x_1}^{x_3} f(x)dx \approx \int_{x_1}^{x_3} p(x)dx &= \frac{h}{3}[f(x_1) + 4f(x_2) + f(x_3)] \\ &= \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \end{aligned} \tag{9.16}$$

The value of the integral is shown in Fig. 9-14 as the shaded area between the curve of $p(x)$ and the x axis. The name 1/3 in the method comes from the fact that there is a factor of 1/3 multiplying the expression in the brackets in Eq. (9.16).

As with the rectangular and trapezoidal methods, a more accurate evaluation of the integral can be done with a composite Simpson's 1/3 method. The whole interval is divided into small subintervals. Simpson's 1/3 method is used to calculate the value of the integral in each subinterval, and the values are added together.

Composite Simpson's 1/3 method

In the composite Simpson's 1/3 method (Fig. 9-15) the whole interval $[a, b]$ is divided into N subintervals. In general, the subintervals can have arbitrary width. The derivation here, however, is limited to the case where the subintervals have equal width h, where $h = (b-a)/N$. Since three points are needed for defining a quadratic polynomial, the Simpson's 1/3 method is applied to two adjacent subintervals at a time (the first two, the third and fourth together, and so on). Consequently, the whole interval has to be divided into an ***even number*** of subintervals.

The integral over the whole interval can be written as the sum of the integrals of couples of adjacent subintervals.

$$\begin{aligned} I(f) = \int_a^b f(x)dx &= \overbrace{\int_{x_1=a}^{x_3} f(x)dx}^{I_2} + \overbrace{\int_{x_3}^{x_5} f(x)dx}^{I_4} + \ldots + \overbrace{\int_{x_{i-1}}^{x_{i+1}} f(x)dx}^{I_i} + \ldots \\ &+ \underbrace{\int_{x_{N-1}}^{x_{N+1}=b} f(x)dx}_{I_N} = \sum_{i=2,4,6}^{N} \int_{x_{i-1}}^{x_{i+1}} f(x)dx \end{aligned} \tag{9.17}$$

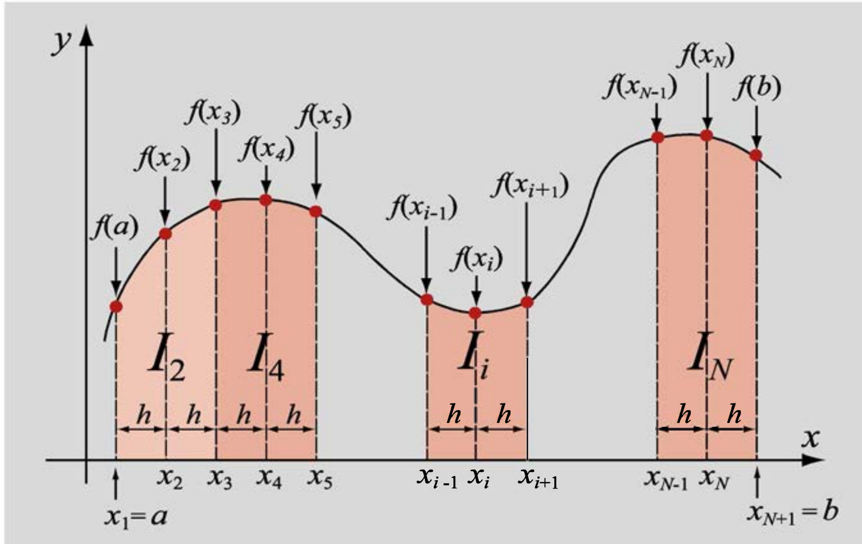


Figure 9-15: Composite Simpson's 1/3 method.

By using Eq. (9.16), the integral over two adjacent intervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ can be written in terms of the Simpson's 1/3 method by:

$$I_i(f) = \int_{x_{i-1}}^{x_{i+1}} f(x)dx \approx \frac{h}{3}[f(x_{i-1}) + 4f(x_i) + f(x_{i+1})] \tag{9.18}$$

where $h = x_{i+1} - x_i = x_i - x_{i-1}$. Substituting Eq. (9.18) in Eq. (9.17) for each of the integrals gives:

$$I(f) \approx \frac{h}{3}[f(a) + 4f(x_2) + f(x_3) + f(x_3) + 4f(x_4) + f(x_5) + f(x_5) + 4f(x_6) + f(x_7) + \ldots + f(x_{N-1}) + 4f(x_N) + f(b)]$$

By collecting similar terms, the right side of the last equation can be simplified to give the general equation for the composite Simpson's 1/3 method for equally spaced subintervals:

$$I(f) \approx \frac{h}{3}\left[f(a) + 4\sum_{i=2,4,6}^{N} f(x_i) + 2\sum_{j=3,5,7}^{N-1} f(x_j) + f(b)\right] \tag{9.19}$$

where $h = (b-a)/N$.

Equation (9.19) is the composite ***Simpson's 1/3 formula*** for numerical integration. It is important to point out that Eq. (9.19) can be used only if two conditions are satisfied:

- The subintervals must be ***equally spaced***.
- The ***number of subintervals*** within $[a, b]$ ***must be an even number***.

Equation (9.19) is a weighted addition of the value of the function at the points that define the subintervals. The weight is 4 at all the points x_i with an even index. These are the middle points of each set of two adjacent subintervals (see Eq. (9.18)). The weight is 2 at all the points x_i with an odd index (except the first and last points). These points are at the interface between adjacent pairs of subintervals. Each point is used once as the right endpoint of a pair of subinterval and once as the left endpoint of the next pair of subintervals. The endpoints are used only once. Figure 9-16 illustrates the weighted addition according

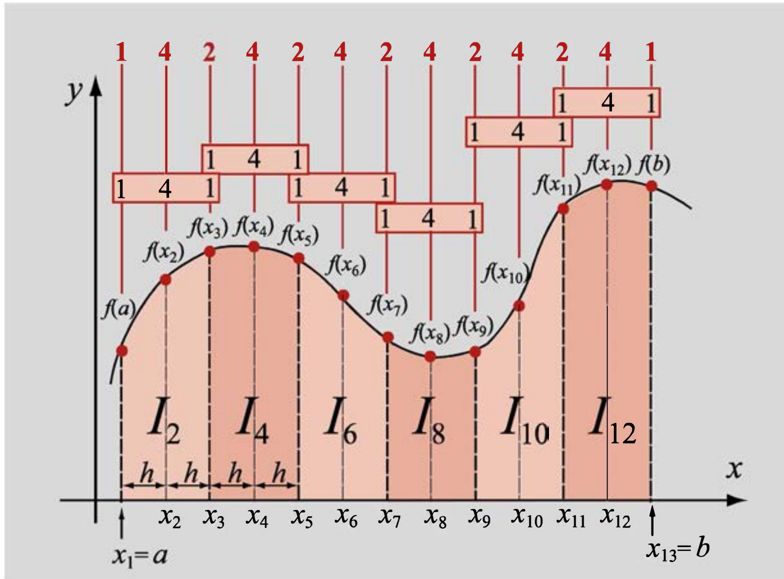


Figure 9-16: Weighted addition with the composite Simpson's 1/3 method.

to Eq. (9.19) for a domain $[a, b]$ that is divided into 12 subintervals. Applying Eq. (9.19) to this illustration gives:

$$I(f) \approx \frac{h}{3}\{ f(a) + 4[f(x_2) + f(x_4) + f(x_6) + f(x_8) + f(x_{10}) + f(x_{12})] \\ + 2[f(x_3) + f(x_5) + f(x_7) + f(x_9) + f(x_{11})] + f(b)\}$$

9.4.2 Simpson's 3/8 Method

In this method a cubic (third-order) polynomial is used to approximate the integrand (Fig. 9-17). A third-order polynomial can be determined from four points. For an integral over the domain $[a, b]$, the four points used are the two endpoints $x_1 = a$ and $x_4 = b$, and two points x_2 and x_3 that divide the interval into three equal sections. The polynomial can be written in the form:

$$p(x) = c_3x^3 + c_2x^2 + c_1x + c_0$$

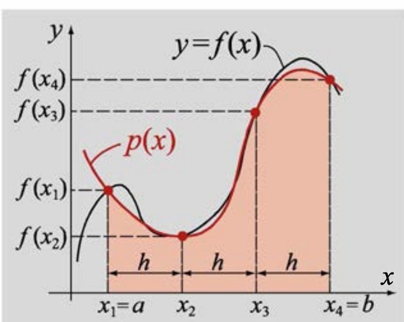


Figure 9-17: Simpson's 3/8 method.

where c_3, c_2, c_1, and c_0 are constants evaluated from the conditions that the polynomial passes through the points, $p(x_1) = f(x_1)$, $p(x_2) = f(x_2)$, $p(x_3) = f(x_3)$, and $p(x_4) = f(x_4)$. Once the constants are determined, the polynomial can be easily integrated to give:

$$I = \int_a^b f(x)dx \approx \int_a^b p(x)dx = \frac{3}{8}h[f(a) + 3f(x_2) + 3f(x_3) + f(b)] \quad (9.20)$$

The value of the integral is shown in Fig. 9-17 as the shaded area between the curve of $p(x)$ and the x axis. The name 3/8 method comes from the 3/8 factor in the expression in Eq. (9.20). Notice that Eq. (9.20) is a weighted addition of the values of $f(x)$ at the two endpoints $x_1 = a$ and $x_4 = b$, and the two points x_2 and x_3 that divide the interval into three equal sections.

As with the other methods, a more accurate evaluation of the integral can be done by using a composite Simpson's 3/8 method.

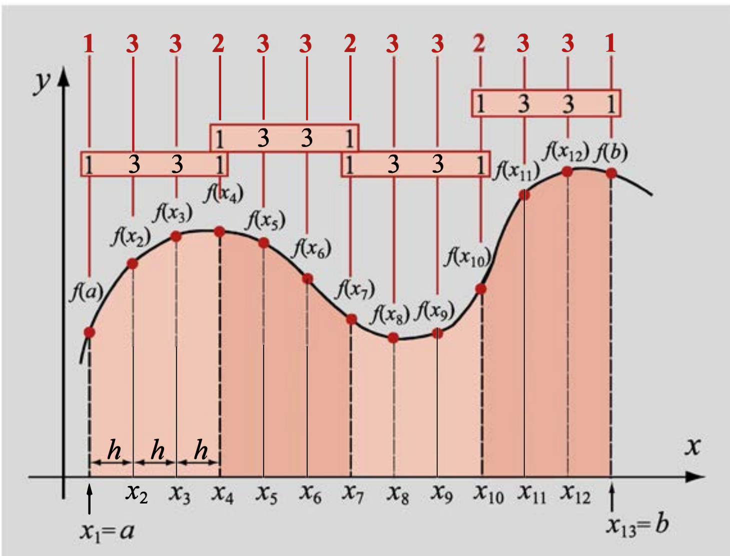


Figure 9-18: Weighted addition with the composite Simpson's 3/8 method.

Composite Simpson's 3/8 Method

In the composite Simpson's 3/8 method, the whole interval $[a, b]$ is divided into N subintervals. In general, the subintervals can have arbitrary width. The derivation here, however, is limited to the case where the subintervals have an equal width h, where $h = (b-a)/N$. Since four points are needed for constructing a cubic polynomial, the Simpson's 3/8 method is applied to three adjacent subintervals at a time (the first three, the fourth, fifth, and sixth intervals together, and so on). Consequently, the whole interval has to be divided into a number of subintervals that is divisible by 3.

The integration in each group of three adjacent subintervals is evaluated by using Eq. (9.20). The integral over the whole domain is obtained by adding the integrals in the subinterval groups. The process is illustrated in Fig. 9-18 where the whole domain $[a, b]$ is divided into 12 subintervals that are grouped in four groups of three subintervals. Using Eq. (9.20) for each group and adding the four equations gives:

$$I(f) \approx \frac{3h}{8}\{ f(a) + 3[f(x_2) + f(x_3) + f(x_5) + f(x_6) + f(x_8) + f(x_9) + f(x_{11}) + f(x_{12})] + 2[f(x_4) + f(x_7) + f(x_{10})] + f(b)\} \quad (9.21)$$

For the general case when the domain $[a, b]$ is divided into N subintervals (where N is at least 6 and divisible by 3), Eq. (9.21) can be generalized to:

$$I(f) \approx \frac{3h}{8}\left[f(a) + 3\sum_{i=2,5,8}^{N-1}[f(x_i) + f(x_{i+1})] + 2\sum_{j=4,7,10}^{N-2} f(x_j) + f(b)\right] \quad (9.22)$$

Equation (9.22) is ***Simpson's 3/8 method*** for numerical integration. Simpson's 3/8 method can be used if the following two conditions are met:

- The subintervals are ***equally spaced.***
- The ***number of subintervals*** within $[a, b]$ ***must be divisible by 3***.

Since Simpson's 1/3 method is only valid for an even number of subintervals and Simpson's 3/8 method is only valid for a number of subintervals that is divisible by 3, a combination of both can be used for integration when there are any odd number of intervals. This is done by using Simpson's 3/8 method for the first three subintervals ($[a, x_2]$, $[x_2, x_3]$, and $[x_3, x_4]$) or for the last three subintervals ($[x_{N-2}, x_{N-1}]$, $[x_{N-1}, x_N]$, and $[x_N, x_b]$), and using Simpson's 1/3 method for the remaining even number of subintervals. Such a combined strategy works because the order of the numerical error is the same for both methods (see Section 9.8).

9.5 GAUSS QUADRATURE

Background

In all the integration methods that have been presented so far, the integral of $f(x)$ over the interval $[a, b]$ was evaluated by approximating $f(x)$ with a polynomial that could be easily integrated. Depending on the integration method, the approximating polynomial and $f(x)$ have the same value at one (rectangular and midpoint methods), two (trapezoidal method), or more points (Simpson's methods) within the interval. The integral is evaluated from the value of $f(x)$ at the common points with the approximating polynomial. When two or more points are used, the value of the integral is calculated from weighted addition of the values of $f(x)$ at the different points. The location of the common points is predetermined in each of the integration methods. All the methods have been illustrated so far, using points that are equally spaced. The various methods are summarized in the following table.

Integration Method	Values of the function used in evaluating the integral
Rectangle Equation (9.2)	$f(a)$ or $f(b)$ (Either one of the endpoints.)
Midpoint Equation (9.5)	$f((a+b)/2)$ (The middle point.)
Trapezoidal Equation (9.9)	$f(a)$ and $f(b)$ (Both endpoints.)

Integration Method	Values of the function used in evaluating the integral
Simpson's 1/3 Equation (9.16)	$f(a)$, $f(b)$, and $f((a+b)/2)$ (Both endpoints and the middle point.)
Simpson's 3/8 Equation (9.20)	$f(a)$, $f(b)$, $f\left(a+\frac{1}{3}(a+b)\right)$, and $f\left(a+\frac{2}{3}(a+b)\right)$ (Both endpoints and two points that divide the interval into three equal-width subintervals.)

In Gauss quadrature, the integral is also evaluated by using weighted addition of the values of $f(x)$ at different points (called Gauss points) within the interval $[a, b]$. The Gauss points, however, are not equally spaced and do not include the endpoints. The location of the points and the corresponding weights of $f(x)$ are determined in such a way as to minimize the error.

General form of Gauss quadrature

The general form of Gauss quadrature is:

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} C_i f(x_i) \tag{9.23}$$

where the coefficients C_i are the weights and the x_i are points (Gauss points) within the interval $[a, b]$. For example, for $n = 2$ and $n = 3$ Eq. (9.23) has the form:

$$\int_a^b f(x)dx \approx C_1 f(x_1) + C_2 f(x_2)\,, \quad \int_a^b f(x)dx \approx C_1 f(x_1) + C_2 f(x_2) + C_3 f(x_3)$$

The value of the coefficients C_i and the location of the points x_i depend on the values of n, a, and b, and are determined such that the right side of Eq. (9.23) is exactly equal to the left side for specified functions $f(x)$.

Gauss quadrature integration of $\int_{-1}^{1} f(x)dx$

For the domain $[-1, 1]$ the form of Gauss quadrature is:

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} C_i f(x_i) \tag{9.24}$$

The coefficients C_i and the location of the Gauss points x_i are determined by enforcing Eq. (9.24) to be exact for the cases when $f(x) = 1, x, x^2, x^3, \ldots$. The number of cases that have to be considered depends on the value of n. For example, when $n = 2$:

$$\int_{-1}^{1} f(x)dx \approx C_1 f(x_1) + C_2 f(x_2) \tag{9.25}$$

The four constants C_1, C_2, x_1, and x_2 are determined by enforcing Eq. (9.25) to be exact when applied to the following four cases:

Case 1: $f(x) = 1$ $\qquad \int_{-1}^{1}(1)dx = 2 = C_1 + C_2$

Case 2: $f(x) = x$ $\qquad \int_{-1}^{1}xdx = 0 = C_1x_1 + C_2x_2$

Case 3: $f(x) = x^2$ $\qquad \int_{-1}^{1}x^2dx = \frac{2}{3} = C_1x_1^2 + C_2x_2^2$

Case 4: $f(x) = x^3$ $\qquad \int_{-1}^{1}x^3dx = 0 = C_1x_1^3 + C_2x_2^3$

The four cases provide a set of four equations for the four unknowns. The equations are nonlinear, which means that multiple solutions can exist. One particular solution can be obtained by imposing an additional requirement. Here the requirement is that the points x_1, and x_2 should be symmetrically located about $x = 0$ ($x_1 = -x_2$). From the second equation, this requirement implies that $C_1 = C_2$. With these requirements, solving the equations gives:

$$C_1 = 1, \ C_2 = 1, \ x_1 = -\frac{1}{\sqrt{3}} = -0.57735027, \ x_2 = \frac{1}{\sqrt{3}} = 0.57735027$$

Substituting the constants back in Eq. (9.25) gives (for $n = 2$):

$$\int_{-1}^{1} f(x)dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \tag{9.26}$$

The right-hand side of Eq. (9.26) gives the exact value for the integral on the left hand side of the equation when $f(x) = 1$, $f(x) = x$, $f(x) = x^2$, or $f(x) = x^3$. This is illustrated in Fig. 9-19 for the case where $f(x) = x^2$. In this case:

$$\int_{-1}^{1}x^2dx = \frac{2}{3} = 1\left(-\frac{1}{\sqrt{3}}\right)^2 + 1\left(\frac{1}{\sqrt{3}}\right)^2 \tag{9.27}$$

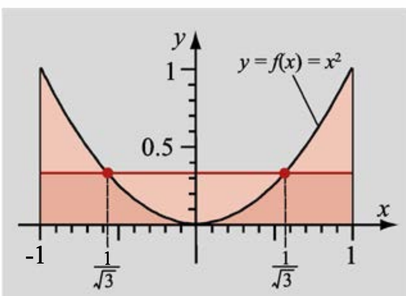


Figure 9-19: Gauss quadrature integration of $f(x) = x^2$.

The value of the integral $\int_{-1}^{1}x^2dx$ is the area under the curve $f(x) = x^2$. The right-hand side of Eq. (9.27) is the area under the colored horizontal line. The two areas are identical since the area between $f(x)$ and the colored line for $|x| > 1/(\sqrt{3})$ is the same as the area that is between the colored line and $f(x)$ for $|x| < 1/(\sqrt{3})$ (the light shaded areas that are above and below the colored line have the same area).

When $f(x)$ is a function that is different from $f(x) = 1$, $f(x) = x$, $f(x) = x^2$, or $f(x) = x^3$, or any linear combination of these, Gauss quadrature gives an approximate value for the integral. For example, if $f(x) = \cos(x)$, the exact value of the integral (the left-hand side of Eq. (9.26) is:

$$\int_{-1}^{1} \cos(x)dx = \sin(x)\Big|_{-1}^{1} = \sin(1) - \sin(-1) = 1.68294197$$

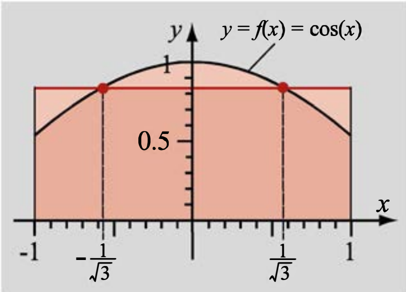


Figure 9-20: Gauss quadrature integration of $f(x) = \cos(x)$.

The approximate value of the integral according to Gauss quadrature (the right-hand side of Eq. (9.26)) is:

$$\cos\left(\frac{-1}{\sqrt{3}}\right) + \cos\left(\frac{1}{\sqrt{3}}\right) = 1.67582366$$

These results show that Gauss quadrature gives a very good approximation (error of a 4.2%) for the integral, but not the exact value. The last integration is illustrated in Fig. 9-20, where the exact integration is the area under the curve $f(x) = \cos(x)$ and the approximate value of the integral according to Gauss quadrature is the area under the red line. In this case the two areas are not exactly identical. The light shaded area under the red line is a little bit (4.2%) smaller than the light shaded area that is above the red line.

The accuracy of Gauss quadrature can be increased by using a higher value for n in Eq. (9.24). For $n = 3$ the equation has the form:

$$\int_{-1}^{1} f(x)dx \approx C_1 f(x_1) + C_2 f(x_2) + C_3 f(x_3) \tag{9.28}$$

In this case there are six constants: C_1, C_2, C_3, x_1, x_2, and x_3. The constants are determined by enforcing Eq. (9.28) to be exact when $f(x) = 1$, $f(x) = x$, $f(x) = x^2$, $f(x) = x^3$, $f(x) = x^4$, and $f(x) = x^5$. This gives a set of six equations with six unknowns. (The process of finding the unknowns is the same as was done when the value of n was 2.) The constants that are determined are:

$C_1 = 0.5555556$, $C_2 = 0.8888889$, $C_3 = 0.5555556$

$x_1 = -0.77459667$, $x_2 = 0$, $x_3 = 0.77459667$

The Gauss quadrature equation for $n = 3$ is then:

$$\int_{-1}^{1} f(x)dx \approx 0.5555556 f(-0.77459667) + 0.8888889 f(0) + 0.5555556 f(0.77459667) \tag{9.29}$$

As an example, the integral when $f(x) = \cos(x)$ is estimated again by using Eq. (9.29):

$$\int_{-1}^{1} \cos(x)dx \approx 0.5555556\cos(-0.77459667) + 0.8888889\cos(0) + 0.5555556\cos(0.77459667) = 1.68285982$$

This value is almost identical to the exact value that was calculated earlier.

The accuracy of Gauss quadrature can be increased even more by using higher values for n. The general equation for estimating the value of an integral is:

$$\int_{-1}^{1} f(x)dx \approx C_1 f(x_1) + C_2 f(x_2) + C_3 f(x_3) + \ldots + C_n f(x_n) \quad (9.30)$$

Table 9-1 lists the values of the coefficients C_i and the location of the Gauss points x_i for $n = 2, 3, 4, 5, and\ 6$.

Table 9-1: Weight coefficients and Gauss points coordinates.

n (Number of points)	Coefficients C_i (weights)	Gauss points x_i
2	$C_1 = 1$ $C_2 = 1$	$x_1 = -0.57735027$ $x_2 = 0.57735027$
3	$C_1 = 0.5555556$ $C_2 = 0.8888889$ $C_3 = 0.5555556$	$x_1 = -0.77459667$ $x_2 = 0$ $x_3 = 0.77459667$
4	$C_1 = 0.3478548$ $C_2 = 0.6521452$ $C_3 = 0.6521452$ $C_4 = 0.3478548$	$x_1 = -0.86113631$ $x_2 = -0.33998104$ $x_3 = 0.33998104$ $x_4 = 0.86113631$
5	$C_1 = 0.2369269$ $C_2 = 0.4786287$ $C_3 = 0.5688889$ $C_4 = 0.4786287$ $C_5 = 0.2369269$	$x_1 = -0.90617985$ $x_2 = -0.53846931$ $x_3 = 0$ $x_4 = 0.53846931$ $x_5 = 0.90617985$
6	$C_1 = 0.1713245$ $C_2 = 0.3607616$ $C_3 = 0.4679139$ $C_4 = 0.4679139$ $C_5 = 0.3607616$ $C_6 = 0.1713245$	$x_1 = -0.93246951$ $x_2 = -0.66120938$ $x_3 = -0.23861919$ $x_4 = 0.23861919$ $x_5 = 0.66120938$ $x_6 = 0.93246951$

Gauss quadrature integration of $\int_a^b f(x)dx$

The weight coefficients and the coordinates of the Gauss points given in Table 9-1 are valid only when the interval of the integration is $[-1, 1]$. In general, however, the interval can have any domain $[a, b]$. Gauss quadrature with the coefficients and Gauss points determined for the $[-1, 1]$ interval can still be used for a general domain. This is done by using a transformation. The integral $\int_a^b f(x)dx$ is transformed into an integral in the form $\int_{-1}^{1} f(t)dt$. This is done by changing variables:

$$x = \frac{1}{2}[t(b-a)+a+b] \quad \text{and} \quad dx = \frac{1}{2}(b-a)dt \tag{9.31}$$

The integration then has the form:

$$\int_a^b f(x)dx = \int_{-1}^{1} f\left(\frac{(b-a)t+a+b}{2}\right)\frac{(b-a)}{2}dt$$

Example 9-2 shows how to use the transformation.

Example 9-2: Evaluation of a single definite integral using fourth-order Gauss quadrature.

Evaluate $\int_0^3 e^{-x^2}dx$ using four-point Gauss quadrature.

SOLUTION

Step 1: Since the limits of integration are $[0, 3]$, the integral has to be transformed to the form $\int_{-1}^{1} f(t)dt$. In the present problem $a = 0$ and $b = 3$. Substituting these values in Eq. (9.31) gives:

$$x = \frac{1}{2}[t(b-a)+a+b] = \frac{1}{2}[t(3-0)+0+3] = \frac{3}{2}(t+1) \quad \text{and} \quad dx = \frac{1}{2}(b-a)dt = \frac{1}{2}(3-0)dt = \frac{3}{2}dt$$

Substituting these values in the integral gives:

$$I = \int_0^3 e^{-x^2}dx = \int_{-1}^{1} f(t)dt = \int_{-1}^{1} \frac{3}{2}e^{-\left[\frac{3}{2}(t+1)\right]^2}dt$$

Step 2: Use four-point Gauss quadrature to evaluate the integral. From Eq. (9.30), and using Table 9-1:

$$I = \int_{-1}^{1} f(t)dt \approx C_1 f(t_1) + C_2 f(t_2) + C_3 f(t_3) + C_4 f(t_4) = 0.3478548 \cdot f(-0.86113631)$$
$$+ 0.6521452 \cdot f(-0.33998104) + 0.6521452 \cdot f(0.33998104) + 0.3478548 \cdot f(0.86113631)$$

Evaluating $f(t) = \frac{3}{2}e^{-\left[\frac{3}{2}(t+1)\right]^2}$ gives:

$$I = 0.3478548\frac{3}{2}e^{-\left[\frac{3}{2}((-0.86113631)+1)\right]^2} + 0.6521452\frac{3}{2}e^{-\left[\frac{3}{2}((-0.33998104)+1)\right]^2}$$
$$+ 0.6521452\frac{3}{2}e^{-\left[\frac{3}{2}(0.33998104+1)\right]^2} + 0.3478548\frac{3}{2}e^{-\left[\frac{3}{2}(0.86113631+1)\right]^2} = 0.8841359$$

The exact value of the integral (when carried out analytically) is 0.8862073. The error is only about 1%.

9.6 EVALUATION OF MULTIPLE INTEGRALS

Double and triple integrals often arise in two-dimensional and three-dimensional problems. A two-dimensional (double integral) has the form:

$$I = \int_A f(x, y)\, dA = \int_a^b \left[\int_{y=g(x)}^{y=p(x)} f(x, y) dy\right] dx \tag{9.32}$$

The integrand $f(x, y)$ is a function of the independent variables x and y. The limits of integration of the inner integral may be a function of x, as in Eq. (9.32), or may be constants, (When they are constants, the integration is over a rectangular region.) Figure 9-21 schematically shows the surface of the function $f(x, y)$ and the projection of the surface on the x-y plane. In this illustration, the domain $[a, b]$ in the x direction is

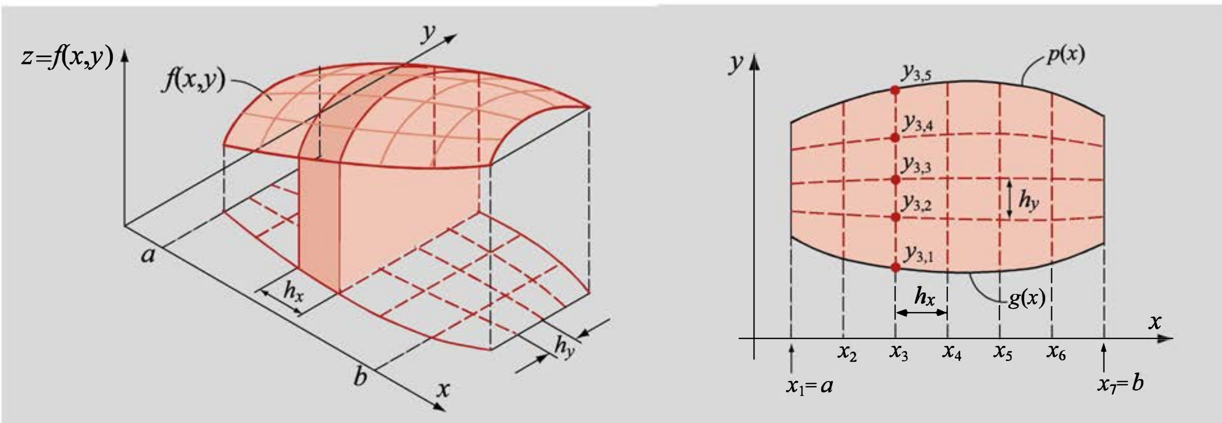


Figure 9-21: Function and domain for double integration.

divided into six equally spaced subintervals. In the y direction the domain $[g(x), p(x)]$ is a function of x, and at every x the y direction is divided into four equally spaced subintervals.

The double integration in Eq. (9.32) can be separated into two parts. The inner integral can be written as:

$$G(x) = \int_{y=g(x)}^{y=p(x)} f(x, y) dy \tag{9.33}$$

and the outer integral can be written with $G(x)$ as its integrand:

$$I = \int_a^b G(x)\, dx \tag{9.34}$$

The outer integral is evaluated by using one of the numerical methods described in the previous sections. For example, if Simpson's 1/3 method is used, then the outer integral is evaluated by:

$$I(G) \approx \frac{h_x}{3}\{G(a) + 4[G(x_2) + G(x_4) + G(x_6)] + 2[G(x_3) + G(x_5)] + f(b)\} \tag{9.35}$$

where $h_x = \frac{b-a}{6}$. Each of the G terms in Eq. (9.35) is an inner integral that has to be integrated according to Eq. (9.33) using the appropriate value of x. In general, the integral $G(x_i)$ can be written as:

$$G(x_i) = \int_{y=g(x_i)}^{y=p(x_i)} f(x_i, y)dy$$

and then integrated numerically. For example, using Simpson's 1/3 method to integrate $G(x_3)$ gives:

$$G(x_3) = \int_{y=g(x_3)}^{y=p(x_3)} f(x_3, y)dy$$

$$= \frac{h_y}{3}\{f(x_3, y_{3,1}) + 4[f(x_3, y_{3,2}) + f(x_3, y_{3,4})] + 2[f(x_3, y_{3,3})] + f(x_3, y_{3,5})\}$$

where $h_y = \dfrac{p(x_3) - g(x_3)}{4}$.

In general, the domain of integration can be divided into any number of subintervals, and the integration can be done with any numerical method.

9.7 USE OF MATLAB BUILT-IN FUNCTIONS FOR INTEGRATION

MATLAB has several built-in functions for carrying out integration. The following describes how to use MATLAB's functions `quad`, `quadl`, and `trapz` for evaluating single integrals, and the function `dblquad` for evaluating double integrals. The `quad`, `quadl`, and `dblquad` commands are used to integrate functions, while the `trapz` function is used to integrate tabulated data.

The `quad` *command*

The form of the `quad` command is:

```
I = quad(function,a,b)
```

The value of the integral. The function to be integrated. The integration limits.

- The function can be entered as a string expression, or as a function handle.
- The function $f(x)$ must be written for an argument x that is a vector (use element-by-element operations), such that it calculates the value of the function for each element of x.
- The user has to make sure that the function does not have a vertical asymptote (singularity) between a and b.
- `quad` calculates the integral with an absolute error that is smaller than 1.0×10^{-6}. This number can be changed by adding an optional `tol` argument to the command:

```
q = quad(function,a,b,tol)
```

`tol` is a number that defines the maximum error. With larger `tol` the integral is calculated less accurately but more quickly.

The quad command uses an adaptive Simpson's method of integration. Adaptive methods are integration schemes that selectively refine the domain of integration, depending on the behavior of the integrand. If the integrand varies sharply in the neighborhood of a point within the domain of integration, then the subintervals in this vicinity are divided into smaller subintervals.

The `quadl` command:

The form of the `quadl` (the last letter is a lower case L) command is exactly the same as the `quad` command:

```
I = quadl(function,a,b)
```

The value of the integral. | The function to be integrated. | The integration limits.

All the comments listed above for the `quad` command are valid for the `quadl` command. The difference between the two commands is in the numerical method used for calculating the integration. The `quadl` uses the adaptive Lobatto method.

The `trapz` command

The built-in function `trapz` can be used for integrating a function that is given as discrete data points. It uses the trapezoidal method of numerical integration. The form of the command is:

```
q = trapz(x,y)
```

where `x` and `y` are vectors with the x and y coordinates of the points, respectively. The two vectors must be of the same length.

The `dblquad` command

The built-in function `dblquad` can be used to evaluate a double integral. The format of the command is:

```
I = dblquad(function, xmin, xmax, ymin, ymax)
```

The value of the integral. | The function to be integrated. | The integration limits.

- The function can be entered as a string, or as a function handle.
- The function $f(x, y)$ must be written for an argument x that is a vector (use element-by-element operations) and for an argument y that is a scalar.

- The limits of integration are constants.

In the format shown above, the integration is done using the `quad` function and the default tolerance, which is 1.0×10^{-6}. The tolerance can be changed by adding an optional `tol` argument to the command, and the method of integration can be changed to quadl by adding it as an argument:

```
q = dblquad(function,xmin,xmax,ymin,ymax,tol,quadl)
```

9.8 ESTIMATION OF ERROR IN NUMERICAL INTEGRATION

The error is the difference between the value of the numerically calculated integral and the exact value of the integral. When the integrand is a set of tabulated data points, an exact value does not really exist and an error cannot be calculated or even estimated. When the integrand is a function, the error can be calculated if the exact value of the integral can be determined analytically. However, if the value of the integral can be calculated analytically, there is no real need to calculate the value of the integral numerically. A common situation is that the integrand is a mathematical expression, and the integral is evaluated numerically because an exact result obtained by analytical integration is difficult or impossible. In this case, the error can be estimated in some of the numerical integration methods. As an illustration, an estimation of the error in the rectangle method is presented in some detail.

In the rectangle method, the integral of $f(x)$ over the interval $x \in [a, b]$ is calculated by assuming that $f(x) = f(a)$ within the interval:

$$I(f) = \int_a^b f(x)dx \approx \int_a^b f(a)dx = f(a)(b-a)$$

The error E is then:

$$E = \int_a^b f(x)dx - f(a)(b-a) \tag{9.36}$$

An estimate of the error can be obtained by writing the one-term Taylor series expansion with a remainder (see Chapter 2) of $f(x)$ near the point $x = a$:

$$f(x) = f(a) + f'(\xi)(x-a) \tag{9.37}$$

where ξ is a point between a and b. Integrating both sides of Eq. (9.37) gives:

$$\int_a^b f(x)dx = \int_a^b [f(a) + f'(\xi)(x-a)]dx = f(a)(b-a) + \frac{1}{2}f'(\xi)(b-a)^2 \tag{9.38}$$

The error according to Eq. (9.36) can be determined using Eq. (9.38):

$$E = \int_a^b f(x)dx - f(a)(b-a) = \frac{1}{2}f'(\xi)(b-a)^2 \tag{9.39}$$

Equation (9.39) shows that the error depends on $(b-a)$ and the values of the first derivative of $f(x)$ within the interval $[a, b]$. Obviously, the error can be large if the domain is large and/or the value of the derivatives is large. The error, however, can be reduced significantly if the composite rectangle method is used. The domain is then divided into subintervals of width h where $h = x_{i+1} - x_i$. Equation (9.39) can be used to estimate the error for a subinterval:

$$E = \int_{x_i}^{x_{i+1}} f(x)dx - f(x_i)h = \frac{1}{2} f'(\xi_i) h^2 \tag{9.40}$$

where ξ_i is a point between x_i and x_{i+1}. Now, the magnitude of the error can be controlled by the size of h. When h is very small (much smaller than 1), the error in the subinterval becomes very small. For the whole interval $[a, b]$, an estimate of the error is obtained by adding the errors from all the subintervals. For the case where h is the same for all subintervals:

$$E = \frac{1}{2} h^2 \sum_{i=1}^{N} f'(\xi_i) \tag{9.41}$$

If an average value of the derivative $\overline{f'}$ in the interval $[a, b]$ can be estimated by:

$$\overline{f'} \approx \frac{\sum_{i=1}^{N} f'(\xi_i)}{N} \tag{9.42}$$

then Eq. (9.41) can be simplified by using Eq. (9.42) and recalling that $h = (b-a)/N$:

$$E = \frac{(b-a)}{2} h \overline{f'} = O(h) \tag{9.43}$$

This equation is an estimate of the error for the composite rectangle method. The error is proportional to h since $\frac{(b-a)}{2}\overline{f'}$ is a constant. It is written as $O(h)$, which means of the order of h.

The error in the composite midpoint, composite trapezoidal, and composite Simpson's methods can be estimated in a similar way. The details are beyond the scope of this book, and the results are as follows.

Composite midpoint method: $$E = \frac{(b-a)}{24} \overline{f''} h^2 = O(h^2) \tag{9.44}$$

Composite trapezoidal method: $$E = -\frac{(b-a)}{12} \overline{f''} h^2 = O(h^2) \tag{9.45}$$

Composite Simpson's 1/3 method: $$E = -\frac{(b-a)}{180} \overline{f^{IV}} h^4 = O(h^4) \tag{9.46}$$

Composite Simpson's 3/8 method: $$E = -\frac{(b-a)}{80} \overline{f^{IV}} h^4 = O(h^4) \tag{9.47}$$

Note that if the average value of the derivatives in Eqs. (9.43)–(9.47) can be bounded, then bounds can be found for the errors. Unfortunately, such bounds are difficult to find so that the exact magnitude of the error is difficult to calculate in practice.

9.9 RICHARDSON'S EXTRAPOLATION

Richardson's extrapolation is a method for obtaining a more accurate estimate of the value of an integral from two less accurate estimates. For example, two estimates calculated with an error $O(h^2)$ can be used for calculating an estimate with an error $O(h^4)$. This section starts by deriving Richardson's extrapolation formula for this case by considering two initial estimates that are calculated with the composite trapezoidal method (error $O(h^2)$). Next, Richardson's extrapolation formula for obtaining an estimate with an error $O(h^6)$ from two estimates with an error $O(h^4)$ is derived. Finally, a general Richardson's extrapolation formula is presented. This formula uses known estimates of an integral with an error of order h^n, to calculate a new estimate that has an increase of 1 (and possibly 2) in the order of accuracy (i.e., with error $O(h^{n+1})$ or $O(h^{n+2})$).

Richardson's extrapolation from two estimates with an error $O(h^2)$

When an integral $I(f)_h$ is numerically evaluated with a method whose truncation error can be written in terms of even powers of *h*, starting with h^2, then the true (unknown) value of the integral $I(f)$ can be expressed as the sum of $I(f)_h$ and the error:

$$I(f) = I(f)_h + Ch^2 + Dh^4 + \dots \tag{9.48}$$

where *C*, *D*, ..., are constants. For example, if the composite trapezoidal method is used for calculating $I(f)_h$ (with an error given by Eq. (9.45)), then $I(f)$ can be expressed by:

$$I(f) = I(f)_h - \frac{(b-a)}{12}\overline{f''}h^2 \tag{9.49}$$

Two estimated values of an integral $I(f)_{h_1}$ and $I(f)_{h_2}$ can be calculated by using a different number of subintervals (in one estimate $h = h_1$ and in the other $h = h_2$). Substituting each of the estimates in Eq. (9.48) gives:

$$I(f) = I(f)_{h_1} + Ch_1^2 \tag{9.50}$$

and

$$I(f) = I(f)_{h_2} + Ch_2^2 \tag{9.51}$$

If it is assumed that C is the same (the average value of the second derivative $\overline{f''}$ is independent of the value of h), then Eqs. (9.50) and (9.51) can be solved for $I(f)$ in terms of $I(f)_{h_1}$ and $I(f)_{h_2}$:

$$I(f) = \frac{I(f)_{h_1} - \left(\frac{h_1}{h_2}\right)^2 I(f)_{h_2}}{1 - \left(\frac{h_1}{h_2}\right)^2} \tag{9.52}$$

Equation (9.52) gives a new estimate for $I(f)$, which has an error $O(h^4)$, from the values of $I(f)_h$ and $I(f)_{h_2}$, each of which have an error $O(h^2)$. The proof that Eq. (9.52) has an error $O(h^4)$ is beyond the scope of this book.[1]

A special case is when $h_2 = \frac{1}{2}h_1$. The two estimates of the value of the integral used for the extrapolation are such that the second estimate has double the number of subintervals compared with the first estimate. In this case Eq. (9.52) reduces to:

$$I(f) = \frac{4I(f)_{h_2} - I(f)_{h_1}}{3} \tag{9.53}$$

Richardson's extrapolation from two estimates with an error O(h^4)

When an integral $I(f)_h$ is numerically evaluated with a method whose truncation error can be written in terms of even powers of h, starting with h^4, then the true (unknown) value of the integral $I(f)$ can be expressed as the sum of $I(f)_h$ and the error:

$$I(f) = I(f)_h + Ch^4 + Dh^6 + \ldots \tag{9.54}$$

where C, D, ..., are constants.

Two estimated values of an integral $I(f)_{h_1}$ and $I(f)_{h_2}$ can be calculated with the same method (which has an error $O(h^4)$) by using a different number of subintervals (in one estimate $h = h_1$ and in the other $h = h_2$) Substituting each of the estimates in Eq. (9.54) gives:

$$I(f) = I(f)_{h_1} + Ch_1^4 \tag{9.55}$$

and

$$I(f) = I(f)_{h_2} + Ch_2^4 \tag{9.56}$$

If it is assumed that C is the same (the average value of the fourth deriv-

1. The interested reader is referred to: P. J. Davis, and P. Rabinowitz, *Numerical Integration*, Blaisdell Publishing Company, Waltham, Massachusetts, 1967, pp. 52–55, 166; L. F. Richardson and J. A. Gaunt, Phil. Trans. Roy. Soc. London A, Vol. 226, pp. 299-361, 1927.

ative $\overline{f^{IV}}$ is independent of the value of h), then Eqs. (9.55) and (9.56) can be solved for $I(f)$ in terms of $I(f)_{h_1}$ and $I(f)_{h_2}$:

$$I(f) = \frac{I(f)_{h_1} - \left(\frac{h_1}{h_2}\right)^4 I(f)_{h_2}}{1 - \left(\frac{h_1}{h_2}\right)^4} \tag{9.57}$$

Equation (9.57) gives an estimate for $I(f)$ with an error $O(h^6)$ from the values of $I(f)_{h_1}$ and $I(f)_{h_2}$, each of which were calculated with an error $O(h^4)$. (The proof is, again, beyond the scope of this book. See footnote on the previous page.)

A special case is when $h_2 = \frac{1}{2}h_1$. The two estimates of the value of the integral used for the extrapolation are such that in the second estimate the number of subintervals is doubled compared with the first estimate. In this case Eq. (9.57) reduces to:

$$I(f) = \frac{16I(f)_{h_2} - I(f)_{h_1}}{15} \tag{9.58}$$

Richardson's general extrapolation formula

A general extrapolation formula can be derived for the case when the two initial estimates of the value of the integral have the same estimated error of order h^p and are obtained such that in one the number of subintervals is twice the number of subintervals of the other.

If I_n is an estimate of the value of the integral that is obtained by using n subintervals and I_{2n} is an estimate of the value of the integral that is obtained by using $2n$ subintervals, where in both the estimated error is of order h^p, then a new estimate for the value of the integral can be calculated by:

$$I = \frac{2^p I_{2n} - I_n}{2^p - 1} \tag{9.59}$$

In general, the new estimate of the integral has an estimated error of order $h^{(p+1)}$. The error is of order $h^{(p+2)}$ when the truncation error can be written in terms of even powers of h. Substituting $p = 2$ and $p = 4$ in Eq. (9.59) gives Eqs. (9.53) and (9.58), respectively. In the same way, the extrapolation equation for using two estimates with an error $O(h^6)$ to obtain a new estimate with an error $O(h^8)$ is obtained by substituting $p = 6$ in Eq. (9.59):

$$I = \frac{2^6 I_{2n} - I_n}{2^6 - 1} = \frac{64}{63}I_{2n} - \frac{1}{63}I_n \tag{9.60}$$

9.10 ROMBERG INTEGRATION

Romberg integration is a scheme for improving the accuracy of the estimate of the value of an integral by successive application of Richardson's extrapolation formula (see Section 9.9). The scheme uses a series of initial estimates of the integral calculated with the composite trapezoidal method by using different numbers of subintervals. The Romberg integration scheme, illustrated in Fig. 9-22, follows these steps:

Step 1: The value of the integral is calculated with the composite trapezoidal method several times. In the first time, the number of subintervals is n, and in each calculation that follows the number of subintervals is doubled. The values obtained are listed in the first (left) column in Fig. 9-22. In the first row $I_{1,1}$ is calculated with the composite trapezoidal method using n subintervals. In the second row $I_{2,1}$ is calculated using $2n$ subintervals, $I_{3,1}$ using $4n$ subintervals, and so on. The error in the calculations of the integrals in the first column is $O(h^2)$.

Step 2: Richardson's extrapolation formula, Eq. (9.53), is used for obtaining improved estimates for the value of the integral from the values listed in the first (left) column in Fig. 9-22. This is the first level of the Romberg integration. The first two values $I_{1,1}$ and $I_{2,1}$ give the estimate $I_{1,2}$:

$$I_{1,2} = \frac{4I_{2,1} - I_{1,1}}{3} \tag{9.61}$$

The second and third values ($I_{2,1}$ and $I_{3,1}$) give the estimate $I_{2,2}$:

$$I_{2,2} = \frac{4I_{3,1} - I_{2,1}}{3} \tag{9.62}$$

and so on. The new improved estimates are listed in the second column in Fig. 9-22. According to Richardson's extrapolation formula they have an error of $O(h^4)$.

Step 3: Richardson's extrapolation formula, Eq. (9.58), is used for obtaining improved estimates for the value of the integral from the values listed in the second column in Fig. 9-22. This is the second level of the Romberg integration. The first two values $I_{1,2}$ and $I_{2,2}$ give the estimate $I_{1,3}$:

$$I_{1,3} = \frac{16I_{2,2} - I_{1,2}}{15} \tag{9.63}$$

The second and third values ($I_{2,2}$ and $I_{3,2}$) give the estimate $I_{2,3}$:

$$I_{2,3} = \frac{16I_{3,2} - I_{2,2}}{15} \tag{9.64}$$

and so on. The new improved estimates are listed in the third column in

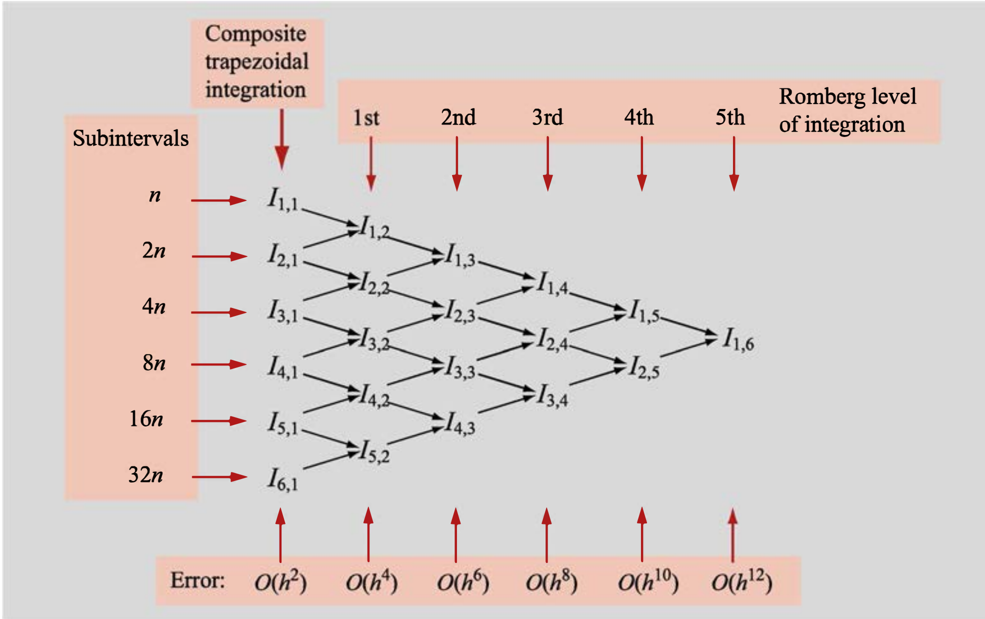


Figure 9-22: Romberg integration method.

Fig. 9-22. According to Richardson's extrapolation formula they have an error of $O(h^6)$.

Step 4 and beyond: The process of calculating improved estimates for the value of the integral can continue where each new column is a higher level of Romberg integration.

The equation for calculating the extrapolated values in each level from the values in the previous level can be written in a general form:

$$I_{i,j} = \frac{4^{j-1} I_{i+1,j-1} - I_{i,j-1}}{4^{j-1} - 1} \tag{9.65}$$

The values of the first column $I_{1,1}$ through $I_{k,1}$ are calculated by using the composite trapezoidal method. Then, the extrapolated values in the rest of the columns are calculated by using Eq. (9.65) for $j = 2, 3, \ldots, k$ and in each column $i = 1, 2, \ldots, (k - j + 1)$, where k is the number of elements in the first column. The highest level of Romberg integration that can be calculated is $k - 1$. The process can continue until there is only one term in the last column (highest level of Romberg integration), or the process can be stopped when the differences between the improved estimated values of the integral are smaller than a predetermined tolerance. Example 9-3 shows an application of Romberg integration.

Example 9-3: Romberg integration with comparison to the composite trapezoidal method.

Evaluate $\int_0^1 \frac{1}{(1+x)}dx$ using three levels of Romberg integration. Use an initial step size of $h = 1$ (one subinterval). Compare your result with the exact answer. What number of subintervals would be required if you were to use the composite trapezoidal method to obtain the same level of accuracy?

SOLUTION

Exact answer: The exact answer to this problem can be obtained analytically. The answer is: $\ln(2) = 0.69314718$.

Romberg Integration

To carry out the numerical integration, a user-defined function, which is listed below, named `Romberg` is created.

Program 9-2: Function file. Romberg integration.

```
function IR = Romberg(Fun,a,b,Ni,Levels)
% Romberg numerically integrate using the Romberg integration method.
% Input Variables:
% Fun  Name for the function to be integrated.
% (Fun is assumed to be written with element-by-element calculations.)
% a  Lower limit of integration.
% b  Upper limit of integration.
% Ni Initial number of subintervals.
% Levels Number of levels of Romberg integration.
% Output Variable:
% IR  A matrix with the estimated values of the integral.

% Creating the first column with the composite trapezoidal method:
for i=1:Levels + 1
  Nsubinter=Ni*2^(i - 1);
  IR(i,1)=trapezoidal(Fun,a,b,Nsubinter);
end
```

Create the first column of Fig. 9-22 by using the user-defined function **`trapezoidal`** (listed in Section 9.3).

```
% Calculating the extrapolated values using Eq. (9.65):
for j=2:Levels + 1
  for i=1:(Levels-j+2)
    IR(i,j)=(4^(j-1)*IR(i+1,j-1)-IR(i,j-1))/(4^(j-1)-1);
  end
end
```

Calculate the extrapolated values, level after level, using Eq. (9.65).

The function Romberg is next used in the Command Window to determine the value of the integral $\int_0^1 \frac{1}{(1+x)}dx$. The initial number of subintervals (for the first estimate with the composite trapezoidal method) is 1.

```
>> format long
>> integ = @ (x) 1./(1+x);
```

Define an anonymous function for the integrand. Note element-by-element calculations.

```
>> IntVal = Romberg(integ,0,1,1,3)
IntVal =
  0.75000000000000   0.69444444444444   0.69317460317460   0.69314747764483
  0.70833333333333   0.69325396825397   0.69314790148123                  0
  0.69702380952381   0.69315453065453                  0                  0
  0.69412185037185                  0                  0                  0
```

Estimates from composite trapezoidal integration.

Romberg level 1.

Romberg level 2.

Romberg level 3.

The results show that with the composite trapezoidal method (first column) the most accurate value that is obtained (using eight subintervals) is accurate to two decimal places. The first-level Romberg integration (second column) increases the accuracy to four decimal places. The second-level Romberg integration (third column) increases the accuracy to six decimal places. The result from the third-level Romberg integration (fourth column) is also accurate to six decimal places, but the value is closer to the exact answer.

The number of calculations that was executed is 10 (4 in the composite trapezoidal method and 6 in the Romberg integration procedure). To obtain an estimate for the integral with an accuracy of six decimal places by applying only the composite trapezoidal method, the method has to be applied with 276 subintervals. This is shown below where `trapezoidal` is used in the Command Window.

```
>> ITrap = trapezoidal(integ,0,1,277)
ITrap =
   0.69314799511374
```

9.11 IMPROPER INTEGRALS

In all integrals $\int_a^b f(x)dx$ that have been considered so far in this chapter, the limits of integration a and b are finite, and the integrand $f(x)$ is finite and continuous in the domain of integration. There are, however, situations in science and engineering where one, or both integration limits, are infinite, and cases where the integrand is not continuous within the range of integration. For example, in statistics the integral $\int_{-\infty}^{b} \frac{1}{\sqrt{2\pi}} e^{\left(\frac{-x^2}{2}\right)} dx$ is used to calculate the cumulative probability that a quantity will have a value of b or smaller.

9.11.1 Integrals with Singularities

An integral $\int_a^b f(x)dx$ has a singularity when there is a point c within the domain, $a \le c \le b$, where the value of the integrand $f(c)$ is not defined ($|f(x)| \to \infty$ as $x \to c$). If the singularity is not at one of the endpoints, the integral can always be written as a sum of two integrals. One over

$[a, c]$ and one over $[c, b]$. Mathematically, integrals that have a singularity at one of the endpoints might or might not have a finite value. For example, the function $1/(\sqrt{x})$ has a singular point at $x = 0$, but the integral of this function over $[0, 2]$ has a value of 2, $\int_0^1 \frac{1}{\sqrt{x}} dx = 2$. On the other hand, the integral $\int_0^1 \frac{1}{x} dx$ does not have a finite value.

Numerically, there are several ways of integrating an integral that has a finite value when the integrand has a singularity at one of the endpoints. One possibility is to use an open integration method where the endpoints are not used for determining the integral. Two such methods presented in this chapter are the composite midpoint method (Section 9.2) and Gauss quadrature (Section 9.5). Another possibility is to use a numerical method that uses the value of the integrand at the endpoint, but instead of using the endpoint itself, for example, $x = a$, the integration starts at a point that is very close to the end point $x = a + \varepsilon$ where $\varepsilon \ll |a|$.

In some cases it is also possible to eliminate a singularity analytically. This can be done by using a change of variable or transformation. Subsequently, the transformed integral can be integrated numerically.

9.11.2 Integrals with Unbounded Limits

Integrals with one or two unbounded limits can have one of the following forms:

$$I = \int_{-\infty}^{b} f(x)dx, \quad I = \int_{a}^{\infty} f(x)dx, \quad I = \int_{-\infty}^{\infty} f(x)dx \tag{9.66}$$

In general, integrals with unbounded limits might have a finite value (converge) or might not have a finite value (diverge). When the integral has a finite value, it is possible to carry out the integration numerically. Typically, the integrand of such an integral has a finite value over a small range of the domain of integration and a value close to zero everywhere else. The numerical integration can then be done by replacing the unbounded limit (or limits) with a finite limit (or limits) where the value of the integrand is close to zero. Then, the numerical integration can be carried out with any of the methods described in this chapter. The integration is done successively, where in each the absolute value of the limit is increased. The calculations stop when the value of the integral does not change much with successive integrations.

In some cases it is also possible to use a change of variable to transform the integral such that the transformed integral will have bounded limits. Subsequently, the transformed integral can be integrated numerically.

9.12 PROBLEMS

Problems to be solved by hand

Solve the following problems by hand. When needed, use a calculator, or write a MATLAB script file to carry out the calculations. If using MATLAB, do not use built-in functions for integration.

9.1 The function $f(x)$ is given in the following tabulated form. Compute $\int_0^{1.8} f(x)dx$ with $h = 0.3$ and with $h = 0.4$.

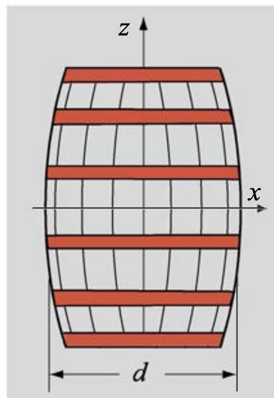
(*a*) Use the composite rectangle method.
(*b*) Use the composite trapezoidal method.
(*c*) Use the composite Simpson's 3/8 method.

x	0	0.3	0.6	0.9	1.2	1.5	1.8
$f(x)$	0.5	0.6	0.8	1.3	2	3.2	4.8

9.2 To estimate the surface area and volume of a wine barrel, the diameter of the barrel is measured at different points along the barrel. The surface area, S, and volume, V, can be determined by:

$$S = 2\pi \int_0^L r dz \quad \text{and} \quad V = \pi \int_0^L r^2 dz$$

Use the data given in the table to determine the volume and surface area of the barrel.



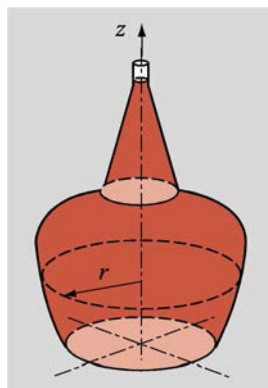
z (in.)	–18	–12	–6	0	6	12	18
d (in.)	0	2.6	3.2	4.8	5.6	6	6.2

(*a*) Use the composite trapezoidal method.
(*b*) Use the composite Simpson's 1/3 method.
(*c*) Use the composite Simpson's 3/8 method.

9.3 To estimate the surface area and volume of a wine bottle, the radius of the bottle is measured at different heights. The surface area, S, and volume, V, can be determined by:

$$S = 2\pi \int_0^L r dz \quad \text{and} \quad V = \pi \int_0^L r^2 dz$$

Use the data given below to determine the volume and surface area of the vase:

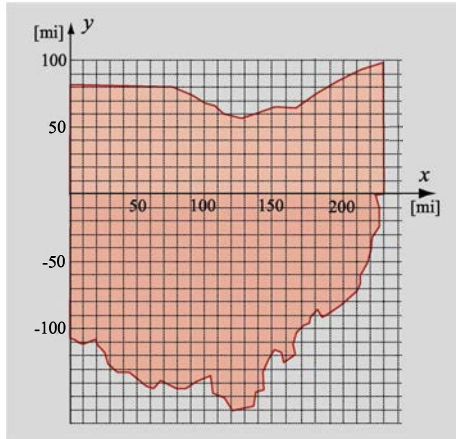


z (cm)	0	2	4	6	8	10	12	14	16	18
r (cm)	10	11	11.9	12.4	13	13.5	13.8	14.1	13.6	12.1
z (cm)	20	22	24	26	28	30	32	34	36	
r (cm)	8.9	4.7	4.1	3.5	3.0	2.4	1.9	1.2	1.0	

(*a*) Use the composite rectangle method.
(*b*) Use the composite trapezoidal method.
(*c*) Use the composite Simpson's 3/8 method.

9.4 An approximate map of the state of Ohio is shown in the figure. For determining the area of the state, the map is divided into two parts (one above and one below the x-axis). Determine the area of the state by numerically integrating the two areas. For each part, make a list of the coordinate y of the border as a function of x. Start with $x = 0$ and use increments of 10 mi, such that the last point is $x = 230$ mi.

Once the tabulated data is available, determine the integrals once with the composite trapezoidal method.



9.5 The Head Severity Index (HSI) measures the risk of head injury in a car crash. It is calculated by:

$$HSI = \int_0^t [a(t)]^{2.5}\, dt$$

where $a(t)$ is the normalized acceleration (acceleration in m/s^2 divided by 9.81 m/s^2) and t is time in seconds during a crash. The acceleration of a dummy head measured during a crash test is given in the following table.

t (ms)	0	5	10	15	20	25	30	35	40	45	50	55	60
a (m/s^2)	0	3	8	20	33	42	40	48	60	12	8	4	3

Determine the HSI.
(*a*) Use the composite trapezoidal method.
(*b*) Use the composite Simpson's 1/3 method.
(*c*) Use the composite Simpson's 3/8 method.

9.6 Evaluate the integral

$$I = \int_0^{\pi} \sin^2 x\, dx$$

using the following methods:
(*a*) Simpson's 1/3 method. Divide the whole interval into six subintervals.
(*b*) Simpson's 3/8 method. Divide the whole interval into six subintervals.

The exact value of the integrals is $I = \pi/2$. Compare the results and discuss the reasons for the differences.

9.7 Evaluate the integral

$$I = \int_0^{2.4} \frac{2x}{1+x^2}\, dx$$

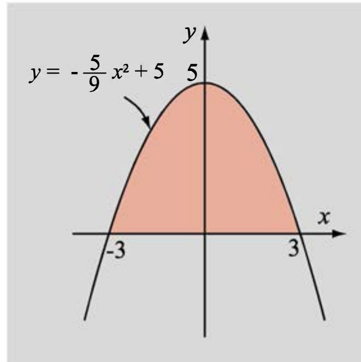
using the following methods:
(*a*) Simpson's 1/3 method. Divide the whole interval into six subintervals.
(*b*) Simpson's 3/8 method. Divide the whole interval into six subintervals.

The exact value of the integral is $I = \ln\frac{169}{25}$. Compare the results and discuss the reasons for the differences.

9.8 Evaluate the integral in Problem 9.7 using
(*a*) three-point Gauss quadrature;
(*b*) four-point Gauss quadrature.

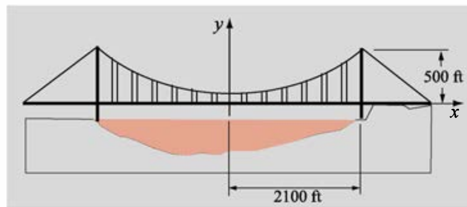
9.9 The area of the shaded region shown in the figure can be calculated by:

$$A = \int_{-3}^{3}\left(-\frac{5}{9}x^2+5\right)\, dx = 20$$



Evaluate the integral using the following methods:
(*a*) Simpson's 1/3 method. Divide the whole interval into four subintervals.
(*b*) Simpson's 3/8 method. Divide the whole interval into nine subintervals.
(*c*) Three-point Gauss quadrature.
Compare the results and discuss the reasons for the differences.

9.10 The central span of the Golden Gate bridge is 4200 ft long and the towers' height from the roadway is 500 ft. The shape of the main suspension cables can be approximately modeled by the equation:



$$f(x) = C\left(\frac{e^{x/C}+e^{-x/C}}{2}-1\right) \quad \text{for} \quad -2100 \le x \le 2100 \text{ ft}$$

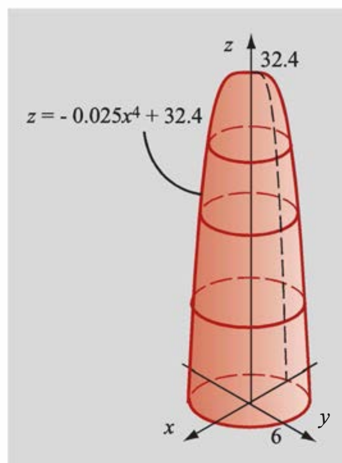
where $C = 4491$.

By using the equation $L = \int_a^b \sqrt{1+[f'(x)]^2}\, dx$, determine the length of the main suspension cables with the following integration methods:
(*a*) Simpson's 1/3 method. Divide the whole interval into eight subintervals.
(*b*) Simpson's 3/8 method. Divide the whole interval into nine subintervals.
(*c*) Three-point Gauss quadrature.

9.11 A silo structure is made by revolving the curve $z = -0.025x^4 + 32.4$ from $x = 0$ m to $x = 6$ m about the z-axis, as shown in the figure to the right.

The surface area, *S*, that is obtained by revolving a curve $z = f(x)$ in the domain from a to b around the z-axis can be calculated by:

$$S = 2\pi\int_a^b x\sqrt{1+[f'(x)]^2}\, dx$$



Calculate the surface area of the silo with the following integration methods:
(*a*) Simpson's 1/3 method. Divide the whole interval into four subintervals.
(*b*) Simpson's 3/8 method. Divide the whole interval into six subintervals.
(*c*) Three-point Gauss quadrature method.

9.12 Determine the volume of the silo in Problem 9.11. The volume, V, that is obtained by revolving a curve $x = f(z)$ in the domain from a to b around the z-axis can be calculated by:

$$V = \pi\int_a^b [f(z)]^2\,dz$$

Calculate the volume of the silo with the following integration methods:
(*a*) Simpson's 1/3 method. Divide the whole interval into four subintervals.
(*b*) Simpson's 3/8 method. Divide the whole interval into six subintervals.
(*c*) Three-point Gauss quadrature.

9.13 In the standard Simpson's 1/3 method (Eq. (9.16)), the points used for the integration are the endpoints of the domain, a and b, and the middle point $(a+b)/2$. Derive a new formula for Simpson's 1/3 method in which the points used for the integration are $x = a$, $x = b$, and $x = (a+b)/3$.

9.14 The value of π can be calculated from the integral $\pi = \frac{1}{2}\int_{-1}^{1}\frac{4}{1+x^2}dx$.
(*a*) Approximate π using the composite trapezoidal method with six subintervals.
(*b*) Approximate π using the composite Simpson's 1/3 method with six subintervals.

9.15 Evaluate the integral $\int_0^2 \cos\left(\frac{\pi x^2}{2}\right)dx$ using second-level Romberg integration. Use $n = 1$ in the first estimate with the trapezoidal method.

9.16 Evaluate the error integral defined as: $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ at $x = 2$ by using four-point Gauss quadrature.

9.17 Evaluate the integral

$$\int_0^3 (-0.5x^5 + 3.9x^4 - 8.1x^3 + 2.7x^2 + 5.9x + 1.5)\,dx$$

using four-point Gauss quadrature.

9.18 Show that the truncation error for the composite trapezoidal method is of the order of h^2, where h is the step size (width of subinterval).

Problems to be programmed in MATLAB
Solve the following problems using MATLAB environment. Do not use MATLAB's built-in functions for integration.

9.19 Write a user-defined MATLAB function for integration with the composite trapezoidal method of a function $f(x)$ that is given in a set of n discrete points. The points don't have to be spaced equally. For the function name and arguments use `I=IntPointsTrap(x,y)`, where the input arguments `x` and `y` are vectors with the values of x and the corresponding values of $f(x)$, respectively. The output argument `I` is the value of the integral. To carry out the integration use Eq. (9.12).
(*a*) Use `IntPointsTrap` to solve Problem 9.2.
(*b*) Use `IntPointsTrap` to solve Problem 9.3.

9.20 Write a user-defined MATLAB function for integration with the composite Simpson's method of a function $f(x)$ that is given in a set of n discrete points that are spaced equally. For the function name and arguments use `I=SimpsonPoints(x,y)`, where the input arguments `x` and `y` are vectors with the values of x and the corresponding values of $f(x)$, respectively. The output argument `I` is the value of the integral. If the number of intervals in the data points is divisible by 3, the integration is done with the composite Simpson's 3/8 method. If the number of intervals in the data points is one more than a number divisible by 3, the integration in the first interval is done with the trapezoidal method and the integration over the rest of the intervals is done with the composite Simpson's 3/8 method. If the number of intervals in the data points is two more than a number divisible by 3, then the integration over the first two intervals is done with Simpsons's 1/3 method and the integration over the rest of the intervals is done with the composite Simpson's 3/8 method.
(*a*) Use `SimpsonPoints` to solve Problem 9.1.
(*b*) Use `SimpsonPoints` to solve Problem 9.5.

9.21 Write a user-defined MATLAB function that uses the composite trapezoidal method for integration of a function $f(x)$ that is given in analytical form (equation). For the function name and arguments use `I=Compzoidal(Fun,a,b)`. `Fun` is a name for the function that is being integrated. It is a dummy name for the function that is imported into `Compzoidal`. The actual function that is integrated should be written as an anonymous or a user-defined function that calculates, using element-by-element operations, the values of $f(x)$ for given values of x. It is entered as a function handle when `Compzoidal` is used. `a` and `b` are the limits of integration, and `I` is the value of the integral. The function `Compzoidal` calculates the value of the integral in iterations. In the first iteration the interval $[a, b]$ is divided into two subintervals. In every iteration that follows, the number of subintervals is doubled. The iterations stop when the difference in the value of the integral between two successive iterations is smaller than 0.1%. Use `Compzoidal` to solve Problems 9.6 and 9.7.

9.22 Write a user-defined MATLAB function that uses the composite Simpson's 1/3 method for integration of a function $f(x)$ that is given in analytical form (equation). For the function name and arguments use `I=Simpson13(Fun,a,b)`. `Fun` is a name for the function that is being integrated. It is a dummy name for the function that is imported into `Simpson13`. The actual function that is integrated should be written as an anonymous or a user-defined function that calculates, using element-by-element operations, the values of $f(x)$ for given values of x. It is entered as a function handle when `Simpson13` is used. `a` and `b` are the limits of integration, and `I` is the value of the integral. The function `Simpson13` calculates the value of the integral in iterations. In the first iteration the interval $[a, b]$ is divided into two subintervals. In every iteration that follows, the number of subintervals is doubled. The iterations stop when the difference in the value of the integral between two successive iterations is smaller than 0.1%. Use `Simpson13` to solve Problems 9.6 and 9.7.

9.23 Write a user-defined MATLAB function that uses the composite Simpson's 3/8 method for integration of a function $f(x)$ that is given in analytical form (equation). For the function name and arguments use `I=Simpsons38(Fun,a,b)`. `Fun` is a name for the function that is being integrated. It is a dummy name for the function that is imported into `Simpson38`. The actual function that is integrated should be written as an anonymous or a user-defined function that calculates, using element-by-element operations, the values of $f(x)$ for given values of x. It is entered as a function handle when `Simpson38` is used. `a` and `b` are the limits of integration, and `I` is the value of the integral. The integration function calculates the value of the integral in iterations. In the first iteration the interval $[a, b]$ is divided into three subintervals. In every iteration that follows, the number of subintervals is doubled. The iterations stop when the difference in the value of the integral between two successive iterations is smaller than 0.1%. Use `Simpsons38` to solve Problems 9.6 and 9.7.

9.24 Write a user-defined MATLAB function for integration of a function $f(x)$ in the domain $[-1, 1]$ $(\int_{-1}^{1} f(x)dx)$ with four-point Gauss quadrature. For the function name and arguments use `I=GaussQuad4(Fun)`, where `Fun` is a name for the function that is being integrated. It is a dummy name for the function that is imported into `GaussQuad4`. The actual function that is integrated should be written as an anonymous or a user-defined function that calculates the value of $f(x)$ for a given value of x. It is entered as a function handle when `GaussQuad4` is used. The output argument `I` is the value of the integral. Use `GaussQuad4` to solve Problem 9.9.

9.25 Write a user-defined MATLAB function for integration of a function $f(x)$ in the domain $[a, b]$ $(\int_{a}^{b} f(x)dx)$ with five-point Gauss quadrature. For function name and arguments use `I=GaussQuad5ab(Fun,a,b)`, where `Fun` is a name for the function that is being integrated. It is a dummy name for the function that is imported into `GaussQuad5ab`. The actual function that is integrated should be written as an anonymous or a user-defined function that calculates the value of $f(x)$ for a given value of x. It is entered as a function handle when `GaussQuad5ab` is used. The output argument `I` is the value of the integral. Use `GaussQuad5ab` to evaluate the integral in Example 9-2.

9.26 The error function $erf(x)$ (also called the Gauss error function), which is used in various disciplines (e.g., statistics, material science), is defined as:

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$

Write a user-defined MATLAB function that calculates the error function. For function name and arguments use `ef=ErrorFun(x)`. Use the user-defined function `Simpson38` that was written in Problem 9.23 for the integration inside `ErrorFun`.

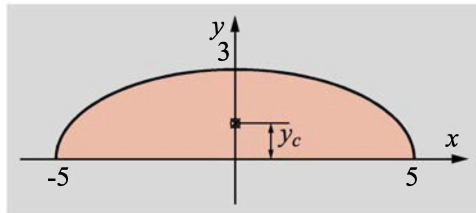
(*a*) Use `ErrorFun` to make a plot of the error function for $0 \le x \le 2$. The spacing between points on the plot should be 0.02.

(*b*) Use `ErrorFun` to make a two-column table with values of the error function. The first column displays values of x from 0 to 2 with spacing of 0.2, and the second column displays the corresponding values of the error function.

Problems in math, science, and engineering

Solve the following problems using MATLAB environment. As stated, use the MATLAB programs that are presented in the chapter, programs developed in previously solved problems, or MATLAB's built-in functions.

9.27 The centroid of the half-ellipse-shaped cross-sectional area shown is given by:



$$y_c = \frac{4}{25\pi}\int_{-5}^{5} y\sqrt{9 - y^2}\,dy$$

Calculate y_c.

(*a*) Use the user-defined function `Simpsons38` written in Problem 9.23.

(*b*) Use one of MATLAB's built-in integration functions.

9.28 The moment of inertia, I_x, about the x axis of the half-ellipse-shaped cross-sectional area shown in Problem 9.27 is given by:

$$I_x = \frac{10}{3}\int_{-5}^{5} y^2\sqrt{9 - y^2}\,dy$$

Calculate I_x.

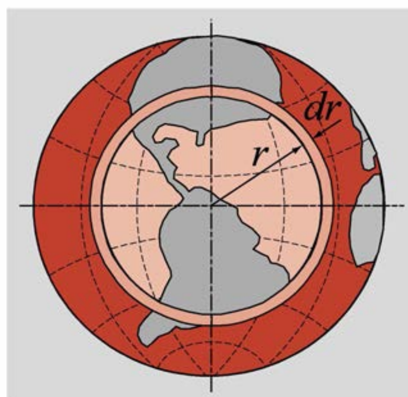
(*a*) Use the user-defined function `Simpsons38` written in Problem 9.23.

(*b*) Use one of MATLAB's built-in integration functions.

9.29 The density, ρ, of the Earth varies with the radius, r. The following table gives the approximate density at different radii:

r (km)	0	800	1200	1400	2000	3000	3400	3600	4000	5000	5500	6370
ρ (kg/m^3)	13000	12900	12700	12000	11650	10600	9900	5500	5300	4750	4500	3300

The mass of the Earth can be calculated by:



$$m = \int_0^{6370} \rho 4\pi r^2 dr$$

Determine the mass of the earth by using the data in the table.

(*a*) Use the user-defined function `IntPointsTrap` that was written in Problem 9.19.

(*b*) Use MATLAB's built-in function `trapz`.

(*c*) Use MATLAB's built-in `interp1` function (with the `spline` option for the interpolation method) to generate a new interpolated data set from the data that is given in the table. For spacing divide the domain `[0, 6370]` into 50 equal subintervals (use the `linspace` command). Calculate the mass of the Earth by integrating the interpolated data set with MATLAB's built-in function `trapz`.

9.30 A pretzel is made by a robot that is programed to place the dough according to the curve given by the following parametric equations:

$$x = (2.5 - 0.3t^2)\cos(t) \qquad y = (3.3 - 0.4t^2)\sin(t)$$

where $-4 \le t \le 3$. The length of a parametric curve is given by the integral:

$$\int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt$$

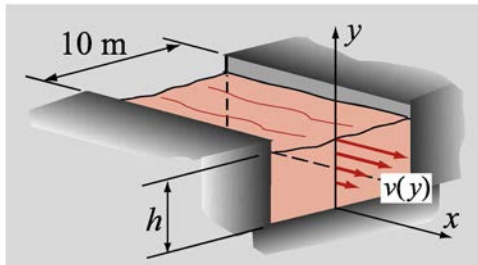
Determine the length of the pretzel. For the integration use:

(*a*) The user-defined function `SimpsonPoints` that was written in Problem 9.20.

(*b*) MATLAB's built-in function `trapz`.

9.31 The flow rate Q (volume of water per second) in the channel shown can be calculated by:

$$Q = 10\int_0^h v(y)dy$$



where $v(y)$ is the water speed, and $h = 5$ m is the overall height of the water. The water speed at different heights are given in the table.

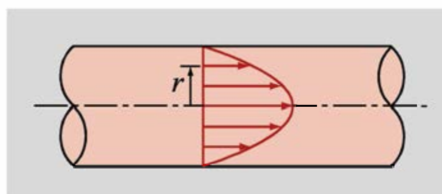
(*a*) Use the user-defined function `IntPointsTrap` that was written in Problem 9.19.

(*b*) Use MATLAB's built-in function `trapz`.

y (m)	0	0.3	0.5	1	1.5	2	2.5	3	4	5
v (m/s)	0	0.4	0.5	0.56	0.6	0.63	0.66	0.68	0.71	0.74

9.32 Measurements of the velocity distribution of a fluid flowing in a pipe (laminar flow) are given in the table. The flow rate Q (volume of fluid per second) in the pipe can be calculated by:

$$Q = \int_0^r 2\pi v r dr$$



Use the data in the table to evaluate Q.

(*a*) Use the user-defined function `IntPointsTrap` that was written in Problem 9.19.

(*b*) Use the user-defined function `SimpsonPoints` that was written in Problem 9.20.

(*c*) Use MATLAB's built-in function `trapz`.

r (in)	0.0	0.25	0.5	0.75	1	1.25	1.5	1.75	2.0
v (in/s)	38.0	37.6	36.2	33.6	29.7	24.5	17.8	9.6	0

9.33 The value of π can be approximated by calculating the integral:

$$\int_0^1 \frac{4}{1+x^2}\, dx$$

Write a MATLAB program in a script file that calculates the value of the integral numerically with the composite trapezoidal method and the composite Simpson's 1/3 method in eight iterations. In the first iteration the interval $[0, 1]$ is divided into two subintervals. In every iteration that follows, the number of subintervals is doubled. (In the last iteration the domain is divided into 256 subintervals.) In every iteration, the program calculates the error in the value of π (for each of the methods) which is defined by the absolute value of the difference between the value calculated by the numerical integration and the value of π in MATLAB (`pi`). Show the results in a figure (log scale on both axes) that displays the error versus the number of subintervals for each method. (Two plots in the same figure.)

9.34 The surface of steel can be hardened by increasing the concentration of carbon. This is done in a process called carburizing in which the surface of the steel is exposed to a high concentration of carbon at elevated temperature. This causes the carbon to diffuse into the steel. At constant temperature, the relationship between the concentration of carbon C_x at a distance x (in m) from the surface, and time t (in seconds), is given by:

$$\frac{C_x - C_o}{C_s - C_0} = 1 - erf\left(\frac{x}{2\sqrt{Dt}}\right) \quad (erf \text{ is the error function})$$

where C_0 is the initial uniform carbon concentration, C_s is the carbon concentration that the steel is exposed to, and D is the diffusion coefficient. Consider the case where $C_0 = 0.2\%$, $C_s = 1.2\%$, and $D = 1.4 \times 10^{-11}\ \text{m}^2/\text{s}$.

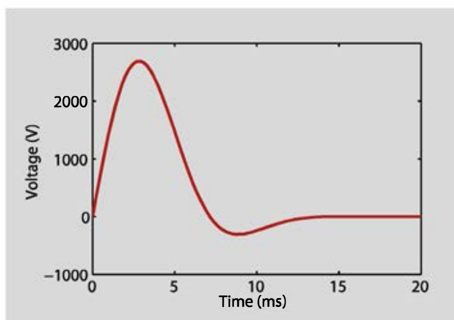
(*a*) For $t = 3$ h, calculate and plot the concentration of carbon as a function of x for $0 \le x \le 1.5$ mm.
(*b*) For $x = 0.4$ mm, calculate and plot the concentration of carbon as a function of t for $0 \le t \le 10$ h.
Solve the problem by using the user-defined function `ErrorFun` that was written in Problem 9.26.

9.35 The figure shows the output pulse from an MDS defibrillator. The voltage as a function time is given by:

$$v(t) = 3500\sin(140\pi t)e^{-63\pi t}\ \text{V}$$

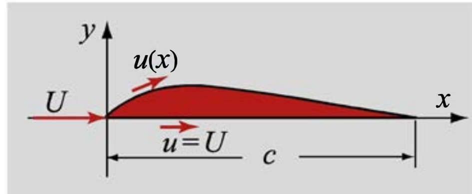
The energy, E, delivered by this pulse can be calculated by:

$$E = \int_0^t \frac{[v(t)]^2}{R}\, dt \ \text{Joules.}$$



where R is the impedance of the patient. For $R = 50\Omega$, determine the time when the pulse has to be switched off if 250 J of energy is to be delivered.
(*a*) Use the composite Simpson's 3/8 method (user-defined function `Simpsons38` that was written in Problem 9.23).
(b) Use one of MATLAB's built-in functions.

9.36 For the airfoil shown in the figure, the lift force, F_L, can be calculated from measurements of the air speed u along the surface by integration:



$$F_L = \frac{1}{2}\rho U^2 L\int_{x=0}^{x=c}\left[\left(\frac{u}{U}\right)^2 - 1\right]dx$$

where L is the length of the wing, ρ is the density, and u/U is measured versus x as given below:

x (m)	0	0.0125	0.025	0.0375	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
u/U	0	0.969	1.241	1.279	1.279	1.286	1.305	1.314	1.310	1.201	1.072	0.948	0.857	0.804

Determine the lift force for a wing $L = 3$ m long and chord length $c = 0.5$ m, if the speed $U = 160$ km/h and $\rho = 1.225$ kg/m^3.

(*a*) Use the user-defined function `IntPointsTrap` that was written in Problem 9.19.

(*b*) Use MATLAB's built-in function `trapz`.

9.37 The force per unit length f that is exerted by the wind on the 24 ft tall sail as a function of it height z is given by:

$$f = 160\frac{z}{z+4}e^{-z/8} \quad \text{lb/ft}$$

The total force F on the sail is calculated by:

$$F = \int_0^{24} f\,dz$$

Determine the total force.

(*a*) Use the user-defined function `Simpsons38` that was written in Problem 9.23.

(*b*) Use one of MATLAB's built-in integration functions.

9.38 Solve Problem 9.37 by with the five-point Gauss quadrature method, by using the user-defined function `GaussQuad5ab` that was written in Problem 9.26.

9.39 Evaluate the following infinite integral numerically:

$$I = \int_0^{\infty}\frac{1}{x^2+1}dx$$

Write a MATLAB program in a script file in which the value of the integral is calculated for different values of the upper limit. Start by using 1 for the upper limit, then 2, then 4, and so on (multiply the previous value by 2). Calculate the relative difference between the last two evaluations ($(I_i - I_{i-1})/I_{i-1}$), and stop when the relative difference between the last two iterations is smaller than 0.00001. Each time the integral should be evaluated with the composite Simpson's 1/3 method. The value of the integral is calculated using iterations. In the first iteration the interval $[0, b]$ is divided into two subintervals. In every iteration that follows, the number of subintervals is doubled. The iterations stop when the difference in the value of the integral between two successive iterations is smaller than 0.01%. If Problem 9.22 was solved, use the user-defined function `Simpson13` for the integration. Compare the numerical value with the exact value of $\pi/2$.

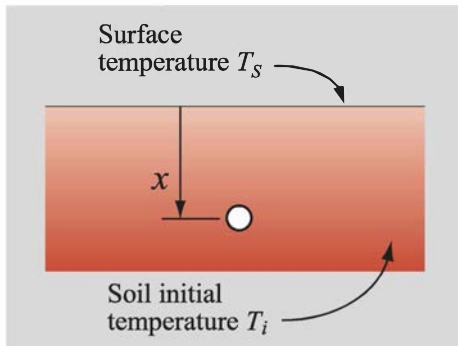
9.40 A thermocouple is used to measure the temperature of a flowing gas in a duct. The time-dependence of the temperature of the spherical junction of a thermocouple is given by the implicit integral equation:

$$t = -\int_{T_i}^{T} \frac{\rho V C}{A_S[h(T - T_\infty) + \varepsilon\sigma_{SB}(T^4 - T_{surr}^4)]}\,dT$$

where T is the temperature of the thermocouple junction at time t, ρ is the density of the junction material, V is the volume of the spherical junction, C is the heat capacity of the junction, A_S is the surface area of the junction, h is the convection heat transfer coefficient, T_∞ is the temperature of the flowing gas, ε is the emissivity of the junction material, σ_{SB} is the Stefan–Boltzmann constant, and T_{surr} is the temperature of the surrounding duct wall.

For $T_i = 298\,\text{K}$, $\varepsilon = 0.9$, $\rho = 8500\,\text{kg/m}^3$, $C = 400$ J/kg/K, $T_\infty = 473$ K, $h = 400$ W/m^2/K, $V = 5.0 \times 10^{-10}\,\text{m}^3$, $A_S = 1.0 \times 10^{-6}\,\text{m}^2$, $\sigma_{SB} = 5.67 \times 10^{-8}\,\text{W/(m}^2\text{K}^4)$, and $T_{surr} = 673$ K, determine the time it takes for the thermocouple junction temperature to increase to 490 K, using Romberg integration.

9.41 In the design of underground pipes, there is a need to estimate the temperature of the ground. The temperature of the ground at various depths can be estimated by modeling the ground as a semi-infinite solid initially at constant temperature. The temperature at depth, x, and time, t, can be calculated from the expression:

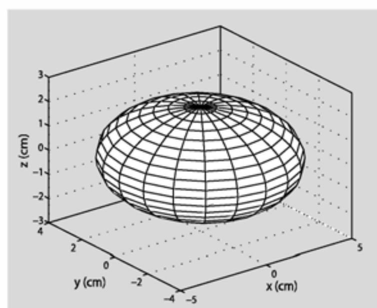


$$\frac{T(x,t) - T_S}{T_i - T_S} = erf\left(\frac{x}{2\sqrt{\alpha t}}\right) = \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{\alpha t}}} e^{-u^2}du$$

where T_S is the surface temperature, T_i is the initial soil temperature, and $\alpha = 0.138 \times 10^{-6}$ m^2/s is the thermal diffusivity of the soil. Answer the following questions taking $T_S = -15\ ^\circ\text{C}$ and $T_i = 12\,^\circ\text{C}$.

(*a*) Find the temperature at a depth $x = 1$ m after 30 days ($t = 2.592 \times 10^6$ s).
(*b*) Write a MATLAB program in a script file that generates a plot that shows the temperature as a function of time at a depth of $x = 0.5$ m for 40 days. Use increments of 1 day.
(*c*) Write a MATLAB program that generates a three-dimensional plot (T vs. x and t) showing the temperature as a function of depth and time for $0 \le x \le 3$ m and $0 \le t \le 2.592 \times 10^7$ s.

9.42 In imaging and treatment of breast cancers, an ellipsoidal shape may be used to represent certain tumors so that changes in their surface areas may be quantified and monitored during treatment. The surface area of an ellipsoid is given by:



$$S = 8ab\int_0^{\pi/2}\int_0^{\pi/2} \sin\theta\sqrt{1 - p\sin^2\theta}\ \,d\theta d\phi$$

where $p = \delta\sin^2\phi + \varepsilon\cos^2\phi$, $\delta = 1 - \frac{c^2}{a^2}$, $\varepsilon = 1 - \frac{c^2}{b^2}$, and $2a$, $2b$, and $2c$ are the major dimensions of the ellipsoid along the x, y, and z axes, respectively. For $2a = 9.5$ cm, $2b = 8$ cm, and $2c = 4.2$ cm, calculate the surface area of this ellipsoidal tumor using the trapezoidal method.