26. Find the sum of the series $2 + 6x + 2x^2 + 20x^3 + \cdots$, $|x| < 1$

27. Using power series representation of $\frac{e^x - 1}{x}$, show that $\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$

28. Find a series of powers of x that converges to $\tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

29. Use power series to find the value of $\int_0^{\frac{1}{2}} \frac{dx}{1+x^4}$ to three places of decimal

30. Estimate $\int_0^1 x^2 e^{-x^2}\, dx$ to three places of decimal.

Chapter 9

FIRST-ORDER DIFFERENTIAL EQUATIONS

Mathematical models for real world phenomena often take the form of equations relating various quantities and their rates of change (derivatives). For example, the motion of a particle involves the distances covered in time t and velocity v and/or acceleration a. Now the rate of change $\frac{ds}{dt}$ of s with respect to t is the velocity v and rate of change $\frac{dv}{dt}$ of velocity with respect to t is the acceleration a. A particle moving in a straight line has an equation of motion as $s = f(t)$, where t is in seconds and s is in meters. Its velocity satisfies the equation

$$v = \frac{ds}{dt} = 4t^2 + 5t - 3.$$

This leads us to the definition of a **differential equation (D.E.)**.

D.E. AND THEIR CLASSIFICATIONS

(9.1) Definition. An equation involving one dependent variable and its derivatives with respect to one or more independent variables, is called **differential equation**. For example

(i) $\frac{dy}{dx} + y\cos x = \sin x$

(ii) $\dfrac{d^2y}{dx^2} + xy\left(\dfrac{dy}{dx}\right)^2 = 0$

(iii) $\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2} = \dfrac{d^2y}{dx^2}$

(iv) $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = nx$

(v) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$

are differential equations

(9.2) Definition. A differential equation, in which ordinary derivatives of the dependent variable with respect to a single independent variable occur, is called an **ordinary differential equation (O.D.E.)** Equations **(i)**, **(ii)** and **(iii)** above are examples of ordinary differential equation.

(9.3) Definition. A differential equation involving partial derivatives of the dependent variable with respect to more than one independent variable is called a **partial differential equation**. Equations **(iv)** and **(v)** given above are partial differential equations.

(9.4) Definition. The **order** of a differential equation is the order of the highest derivative that occurs in the equation.

(9.5) Definition. The **degree** of a differential equation is the greatest exponent of the highest order derivative that appears in the equation. (The dependent variable and its derivatives should be expressed in a form free from radicals and fractions).

The differential equations given in **(9.1)** have the following orders and degrees

(i) order 1, degree 1

(ii) order 2, degree 1

(iii) order 2, degree 2, exponent of $\dfrac{d^2y}{dx^2}$ is 2 after removing the radical by squaring both sides of the equation.

(iv) order, 1, degree 1

(v) order 2, degree 1.

We shall study ordinary differential equations only.

Recall that a function $T : U \to V$, where U, V, are vector spaces over the same field F, is called **linear** if, for $\alpha, \beta \in F, x, y \in U$,

$$T(\alpha x + \beta y) = \alpha\, T(x) + \beta\, T(y).$$

(9.6) Definition. (Linear Differential Equations). An ordinary differential equation

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}\right) = 0$$

is said to be linear if F is a linear function of the variables

$$x, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}$$

A similar definition applies to partial differential equations.

Thus the general linear ordinary differential equation of order n is

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\,y = F(x)$$

where $a_0(x)$ is not identically zero.

The equation $\dfrac{d^3y}{dx^3} + 2e^x\dfrac{d^2y}{dx^2} + y\dfrac{dy}{dx} = x^3$ $\dfrac{d}{dx}\cdot y$ $\dfrac{d}{dx}\cdot\dfrac{dy}{dx}$

is not linear because of the term $y\dfrac{dy}{dx}$.

It should be carefully noted that in a linear ordinary differential equation. $\dfrac{d^2y}{dx^2}$

(i) the dependent variable y and its derivatives are all of degree one.

(ii) no products of y or any of its derivatives appear

(iii) no transcedental function of y and / or its derivatives occur.

A differential equation which is not linear is called a nonlinear differential equation.

Differential equations occur in the mathematical formulation of many problems in science and engineering. Some such problems are.

(i) Determining the motion of projectile, rocket, satellite or planet.

(ii) Finding the charge or current in an electric circuit.

(iii) Study of chemical reactions.

(iv) Determination of curves with given geometrical properties.

(9.7) Definition. A **solution** (or **integral**) of a differential equation is a relation between the variables, not containing derivatives, such that this relation and the derivatives obtained from it satisfy the given differential equation identically. For example,

The equation

$$\frac{dy}{dx} = -\lambda y \quad \text{has a solution}$$

$$y = c\,e^{-\lambda x}, \quad \text{where } c \text{ is an arbitrary constant.}$$

The equation

$$\frac{d^2y}{dx^2} + y = 0 \qquad \text{has solutions}$$

$$y = A\cos x, \quad y = B\sin x \text{ and } y = A\cos x + B\sin x,$$

where A and B are arbitrary constants.

A solution of a differential equation which contains as many arbitrary constants as the order of the equation is called **general solution** (or **integral**) of the differential equation. A solution obtained from the general solution by giving particular values to the constants is called a **particular solution** (or **integral**). The graph of a particular integral is called an **integral curve** of the differential equation.

FORMATION OF DIFFERENTIAL EQUATIONS

(9.8) Given a relation

$$f(x, y, c_1, c_2, \ldots, c_n) = 0 \qquad (1)$$

between variables x, y and containing n constants $c_1, c_2, \ldots, c_n$, it is always possible to form a differential equation of order n such that the given relation (1) is the general solution of the equation. This is done by differentiating (1) n times thereby obtaining n equations and then eliminating the n constants from the original relation and n derived equations. The method is illustrated by means of examples.

Example 1. The equation

$$y = x + a \qquad (1)$$

represents a family of parallel straight lines for different values of a. Elimination of one constant 'a' requires two equations. The second equation is obtained by differentiating (1).

Thus $\frac{dy}{dx} = 1$ is the differential equation of the relation (1), with a eliminated.

Example 2. Form the differential equation by eliminating the two constants A and B from the relation

$$y = A\sin x + B\cos x. \qquad (1)$$

Solution. It is clear that three equations are required to eliminate two unknowns A and B. We obtain two other needed equations by successive differentiation of (1). Thus from (1), we have

$$\frac{dy}{dx} = A\cos x - B\sin x \qquad (2)$$

$$\frac{d^2y}{dx^2} = -A\sin x - B\cos x = -y, \text{ using (1)} \qquad (3)$$

So $$\frac{d^2y}{dx^2} + y = 0 \qquad (4)$$

is the required differential equation and (1) is its general solution.

Example 3. Find the differential equation of all parabolas whose axes are parallel to the y-axis.

Solution. General equation of a parabola whose axis is parallel to the y-axis is

$$y = ax^2 + bx + c \qquad (1)$$

In order to obtain its differential equation, we have to eliminate a, b, c from (1). For that we need three more equations. Differentiating (1) successively, we have

$$\frac{dy}{dx} = 2ax + b,$$

$$\frac{d^2y}{dx^2} = 2a,$$

$$\frac{d^3y}{dx^3} = 0$$

The last equation does not contain any of the constants a, b and c. Thus

$$\frac{d^3y}{dx^3} = 0$$

is the differential equation of all parabolas whose axes are parallel to the y-axis.

INITIAL AND BOUNDARY CONDITIONS

We have observed that general solution of a differential equation contains the same number of arbitrary constants as is the order of the equation. Sometimes we need to find the solutions of differential equations subject to supplementary conditions. Two types of conditions will be often encountered.

(9.9) Definition. (Initial Conditions). It is often required to find the solution of a differential equation subject to certain conditions. If the conditions relate to one value of the independent variable such as $y = y_0$ at $x = x_0$ (written as $y(x_0) = y_0$) and $\frac{dy}{dx} = y'(x_0)$ at $x = x_0$, where x_0 belongs to some interval $]\,\alpha, \beta\,[$ then they are called **initial conditions** (or **one-point boundary conditions**) and x_0 is called the **initial point**. An **initial value problem** consists of a differential equation (of any order) together with a collection of initial conditions that must be satisfied by the solution of the differential equation and derivatives at the initial point.

(9.10) Definition. (Boundary Conditions). The problem of finding the solution of a differential equation such that all the associated constraints relate to two different values of the independent variable is called a **two-point boundary value problem** (or simply a **boundary value problem**). The associated supplementary **boundary conditions** are called **two-point boundary conditions**.

Example 4. Solve $$\frac{dy}{dx} = 2x \qquad (1)$$

such that $y(1) = 4$.

Solution. This is an initial value problem. We note that $y = x^2 + c$, c being arbitrary constant, is the general solution of (1). Since $y(1) = 4$, we have

$$y(1) = 1^2 + c.$$

Therefore, $4 = 1 + c$ or $c = 3$.

Thus $y = x^2 + 3$ is the solution of the initial value problem (1).

Note that the general solution represents a family of parabolas for different values of c. The solution $y = x^2 + 3$ is a particular member of the family that passes through (1, 4).

Example 5. Solve: $$\frac{d^2y}{dx^2} + y = 0 \qquad (1)$$

subject to the conditions

$$y(\pi/4) = \sqrt{2},\ y'(\pi/4) = \frac{1}{\sqrt{2}}.$$

Solution. Since both the conditions relate to one value of x, namely $x = \pi/4$, this is an initial value problem. We have already noted in Example 2 that

$$y = A\sin x + B\cos x \qquad (2)$$

is the general solution of (1). Differentiating (2) w.r.t. x, we get

$$y' = A\cos x - B\sin x. \qquad (3)$$

By the given conditions, we have respectively from (2) and (3)

$$y(\pi/4) = \sqrt{2} = A\sin\pi/4 + B\cos\pi/4 = \frac{A}{\sqrt{2}} + \frac{B}{\sqrt{2}}$$

$$y'(\pi/4) = \frac{1}{\sqrt{2}} = A\cos\pi/4 - B\sin\pi/4 = \frac{A}{\sqrt{2}} - \frac{B}{\sqrt{2}}.$$

Hence $A + B = 2$

$A - B = 1$.

$\frac{3}{2} + B = 2$, $B = 2 - \frac{3}{2}$, $B = \frac{4-3}{2} = \frac{1}{2}$

Solving for A and B, we find that

$$A = \frac{3}{2},\quad B = \frac{1}{2}.$$

With these values of A and B, the particular solution of (1) is

$$y = \frac{3}{2}\sin x + \frac{1}{2}\cos x.$$

Example 6. Verify that $y = c_1\cos x$ and $y = c_2\sin x$ are solutions of $\frac{d^2y}{dx^2} + y = 0$. Find a particular solution of the equation satisfying the boundary conditions

$$y(0) = 1,\ y(\pi/2) = 2$$

Solution. We have

$$y = c_1\cos x$$

Differentiating twice, we get

$$\frac{dy}{dx} = -c_1\sin x$$

$$\frac{d^2y}{dx^2} = c_1\cos x = -y$$

or $$\frac{d^2y}{dx^2} + y = 0.$$

Thus $y = c_1\cos x$ is a solution of

$$\frac{d^2y}{dx^2} + y = 0. \qquad (1)$$

Similarly, it can be checked that $y = c_2\sin x$ is also a solution of (1).

General solution of (1) is

$$y = c_1\cos x + c_2\sin x. \qquad (2)$$

Applying the boundary conditions, we obtain from (2)

$$y(0) = 1 = c_1 + 0$$

and $$y\left(\frac{\pi}{2}\right) = 2 = 0 + c_2.$$

Thus $c_1 = 1$ and $c_2 = 2$. Hence the particular solution of (1) satisfying the given conditions is

$$y = \cos x + 2\sin x.$$

Example 7. Solve the boundary value problem

$$\frac{d^2y}{dx^2} + y = 0,\quad y(0) = 1,\quad y(\pi) = 5.$$

Solution. Applying the boundary conditions to the general solution

$$y = A\sin x + B\cos x$$

of the given equation, we have

$$y(0) = 1 = B$$

$$y(\pi) = 5 = -B.$$

Thus we obtain two values of B and we are unable to determine any definite value of A. Hence the boundary value problem has no solution.

It follows that a boundary value problem need not always have a solution.

EXERCISE 9.1

1. Classify each of the following equations as ordinary or partial differential equations, state the order and degree of each equation and determine whether the equation is linear or nonlinear.

(i) $\dfrac{d^3y}{dx^3} + 4\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 3y = \cos x$

(ii) $x^2\,dy + y^2\,dx = 0$

(iii) $\dfrac{\partial^3 u}{\partial x^3} + \dfrac{\partial^2 u}{\partial y^2} + \left(\dfrac{\partial u}{\partial z}\right)^2 + ux^3 + uy^2 + uz = 0$

(iv) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + u\dfrac{\partial u}{\partial x} + u\dfrac{\partial u}{\partial y} + u = 0$

(v) $\left(\dfrac{dy}{dx}\right)^2 = \left(\dfrac{d^2y}{dx^2} + y\right)^{\frac{3}{2}}$.

2. Form the differential equation of which the given function is a solution:

(i) $y = x + 3e^{-x}$

(ii) $y = (x^3 + c)\,e^{-3x}$, c being an arbitrary constant

(iii) $ax + \ln|y| = y + b$

(iv) $y = ae^x + b\ln|x| + cx + d$

(v) $x^2 + y^2 + 2gx + 2fy + c = 0$, f, g and c being arbitrary constants.

(vi) $u = f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$; $(x, y, z) \neq (0, 0, 0)$

(vii) $u = f(x - ay) + g(x + ay)$, where f and g are twice differentiable functions.

3. Find the differential equation of all

(i) circles of radius a

(ii) circles that pass through the origin

(iii) ellipses in standard form

(iv) parabolas, each of which has a latus rectum $4a$ and whose axes are parallel to the x-axis.

(v) hyperbolas in standard form

(vi) coincs whose axes coincide with the axes of coordinates.

Solve the following initial value problems

(i) $\dfrac{dy}{dx} = -\dfrac{x}{y}$, $\quad y(3) = 4$,

given that the differential equation has $x^2 + y^2 = c^2$ as the general solution

(ii) $\dfrac{dy}{dx} + y = 2xe^{-x}$, $\quad y(-1) = e + 3$,

given that the differential equation has $y = (x^2 + c)e^{-x}$ as the general solution

(iii) $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 12y = 0$, $\quad y(0) = -2,\ y'(0) = 6$

where $y = Ae^{4x} + Be^{-3x}$ is the general solution of the given differential equation.

(iv) $x\dfrac{dy}{dx} + 2y = 4x^2$, $\quad y(1) = 2$,

given that $y = x^2 + \dfrac{c}{x^2}$ is the general solution of the differential equation.

(v) $x^3\dfrac{d^3y}{dx^3} - 3x^2\dfrac{d^2y}{dx^2} + 6x\dfrac{dy}{dx} - 6y = 0$, $\quad y(2) = 0,\ y'(2) = 2,\ y''(2) = 6$,

given that $y = c_1x + c_2x^2 + c_3x^3$ is the general solution of the given differential equation.

Solve the boundary value problems:

(i) $\dfrac{d^2y}{dx^2} + y = 0$, $\quad y(0) = 1,\ y'\left(\dfrac{\pi}{2}\right) = +1$

given that $y = c_1\sin x + c_2\cos x$ is the general solution of the given equation.

(ii) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 0$, $\quad y(0) = 0,\ y(1) = 1$,

where $y = c_1e^x + c_2e^{2x}$ is the general solution of the given equation.

In the previous section we classified the differential equations and also saw how a differential equation can be formed by eliminating constants from a given functional equation. The problem of finding general solution (integral) of a given differential equation will now be considered. The solution of any differential equation may or may not exist. Even if the integral of a given equation exists, it may not be easy to find. We shall only discuss methods of solutions of special types of differential equations. The study of existence of solutions is beyond the scope of this book.

SEPARABLE EQUATIONS

(9.11) Definition. A differential equation of the type

$$F(x)\,G(y)\,dx + f(x)\,g(y)\,dy = 0 \quad (1)$$

is called an equation with **separable variables** or simply a separable equation. Equation (1) may be written as

$$\frac{F(x)}{f(x)}\,dx + \frac{g(y)}{G(y)}\,dy = 0$$

which can be easily integrated.

Example 8. Solve: $\dfrac{dy}{dx} = \dfrac{x^2}{y}$ (1)

Solution. Equation (1) can be written as

$$y\,dy = x^2\,dx.$$

Integrating both the sides, we get

$$\frac{y^2}{2} = \frac{x^3}{3} + c_1$$

or $\quad 3y^2 = 2x^3 + c$

which is the required solution of (1).

Example 9. Solve: $\dfrac{dy}{dx} = \dfrac{1}{x\tan y}$.

Solution. The given equation is separable and can be written as

$$\frac{dx}{x} = \tan y\,dy.$$

Integrating, we have

$$\ln|x| = -\ln|\cos y| + c$$

or $\quad \ln|x| + \ln|\cos y| = c$

i.e., $\quad \ln|x\cos y| = c$

or $\quad |x\cos y| = e^c = a$ (say)

or $\quad x\cos y = a \quad$ because $a > 0$.

Example 10. Solve: $x\sin y\,dx + (x^2+1)\cos y\,dy = 0$. (1)

Solution. Dividing (1) by $(x^2+1)\sin y$, we have

$$\frac{x}{x^2+1}\,dx + \cot y\,dy = 0$$

which is a separable equation. Therefore,

$$\int\frac{x\,dx}{x^2+1} + \int\cot y\,dy = c_1 = \ln|c_2|$$

i.e., $\quad \frac{1}{2}\ln(x^2+1) + \ln|\sin y| = \ln|c_2|$

or $\quad \ln(x^2+1) + 2\ln|\sin y| = 2\ln|c_2|$

Now $\quad 2\ln|\sin y| = \ln|\sin y|^2 = \ln(\sin y)^2 = \ln\sin^2 y$

And $\quad 2\ln|c_2| = \ln|c_2|^2 = \ln c_2^2 = \ln c$, where $c_2^2 = c > 0$

Hence $\quad \ln(x^2+1) + \ln\sin^2 y = \ln c$.

From this it follows that

$$(x^2+1)\sin^2 y = c \quad \text{is the required solution.}$$

Example 11. Solve the initial value problem

$$\frac{dy}{dx} = \frac{2x}{y + x^2 y}, \qquad y(0) = -2$$

Solution. We have

$$\frac{dy}{dx} = \frac{2x}{y(1+x^2)}$$

or $\quad y\,dy = \dfrac{2x}{1+x^2}\,dx$

Integrating, we obtain

$$\frac{y^2}{2} = \ln(1+x^2) + \ln c, \quad c \text{ being a constant.}$$

$$= \ln(c(1+x^2))$$

or $\quad y^2 = \ln(c^2(1+x^2)^2) \quad (1)$

Now setting $y(0) = -2$ into (1), we have

$$4 = \ln c^2 \quad \text{or} \quad c^2 = e^4.$$

So (1) becomes

$$y^2 = \ln(e^4(1+x^2)^2)$$

which is the required solution.

EXERCISE 9.2

Solve (Problems 1–15):

1. $\frac{dy}{dx} = \frac{x^2}{y(1+x^3)}$
2. $\frac{dy}{dx} + y^2 \sin x = 0$
3. $\frac{dy}{dx} = 1 + x + y^2 + xy^2$
4. $(xy + 2x + y + 2)\,dx + (x^2 + 2x)\,dy = 0$
5. $\frac{dy}{dx} = 2x^2 + y - x^2y + xy - 2x - 2$
6. $\csc y\,dx + \sec x\,dy = 0$
7. $y(1+x)\,dx + x(1+y)\,dy = 0$
8. $y\sqrt{1+x^2}\,dx + x\sqrt{1+y^2}\,dy = 0$
9. $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$
10. $(e^x + 1)\,y\,dy = (y+1)\,e^x\,dx$
11. $\frac{dy}{dx} = \frac{y^3 + 2y}{x^2 + 3x}$
12. $(\sin x + \cos x)\,dy + (\cos x - \sin x)\,dx = 0$
13. $e^x\left(1 + \frac{dy}{dx}\right) = xe^{-y}$
14. $xe^{x^2+y}\,dx = y\,dy$
15. $(2x \cos y)\,dx + x^2(\sec y - \sin y)\,dy = 0.$

Solve the initial value problems:

16. $2(y-1)\,dy = (3x^2 + 4x + 2)\,dx$, $\quad y(0) = -1$
17. $(3x+8)(y^2+4)\,dx - 4y(x^2+5x+6)\,dy = 0$, $\quad y(1) = 2$
18. $(1+2y^2)\,dy = y\cos x\,dx$, $\quad y(0) = 1$
19. $8\cos^2 y\,dx + \csc^2 x\,dy = 0$, $\quad y\left(\frac{\pi}{12}\right) = \frac{\pi}{4}$
20. $\frac{dy}{dx} = \frac{x(x^2+1)}{4y^3}$, $\quad y(0) = -\frac{1}{\sqrt{2}}$.

HOMOGENEOUS EQUATIONS

(9.12) Definition. A function $f(x, y)$ is called **homogeneous of degree** n if

$$f(tx, ty) = t^n f(x, y),$$

where t is a nonzero real number. Thus $\sqrt{xy}$, $\frac{x^{10}+y^{10}}{x^2+y^2}$ and $\sin\left(\frac{x}{y}\right)$ are homogeneous function of degree 1, 8 and 0 respectively. (Check!)

A first order differential equation

$$\frac{dy}{dx} = f(x, y) \quad (1)$$

is said to be **homogeneous** if f is a homogeneous function of any degree. If (1) is written in the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

then it is called homogeneous if $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree.

The equation

$$\frac{dy}{dx} = \ln x - \ln y + \frac{x+y}{x-y} = \ln\frac{1}{y/x} + \frac{1+y/x}{1-y/x}$$

is homogeneous, but

$$\frac{dy}{dx} = \frac{y^3 + 2xy}{x^2} = y\left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right)$$

is not homogeneous.

(9.13) Theorem. A homogeneous equation

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right) \quad (1)$$

can be transformed into a separable equation (in the variables v and x) by the substitution $y = vx$.

Proof. Put $y = vx$ into (1). Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and (1) becomes

$$v + x\frac{dv}{dx} = g(v)$$

or

$$v - g(v) + x\frac{dv}{dx} = 0$$

or

$$[v - g(v)]\,dx + x\,dv = 0.$$

This equation is separable and can be solved as in the previous section.

Example 12. Solve: $\frac{dy}{dx} = \frac{x^3 + y^3}{x^2y + xy^2}$

Solution. We have

$$\frac{dy}{dx} = \frac{1 + (y/x)^3}{(y/x) + (y/x)^2} \quad (1)$$

so that the equation is homogenous. Setting $y = vx$ into (1), we obtain

$$v + x\frac{dv}{dx} = \frac{1+v^3}{v+v^2} = \frac{1+v^3}{v(1+v)} = \frac{v^2-v+1}{v}$$

or $$x\frac{dv}{dx} = \frac{v^2-v+1}{v} - v = \frac{1-v}{v}$$

or $$\frac{v}{1-v}\,dv = \frac{dx}{x}$$

or $$\left(-1+\frac{1}{1-v}\right)dv = \frac{dx}{x}$$

Integrating, we get

$$-v - \ln|1-v| = \ln|x| + \ln|c|$$

or $$\ln|cx(1-v| + v = 0.$$

Replacing v by y/x in the above equation, we have

$$\ln|c(x-y)| + \frac{y}{x} = 0$$

which is the required solution.

Example 13. Solve the initial value problem

$$(x^2+3y^2)\,dx - 2xy\,dy = 0, \qquad y(2) = 6.$$

Solution. Here

$$\frac{dy}{dx} = \frac{x^2+3y^2}{2xy} = \frac{3(y/x)^2+1}{2(y/x)} \qquad (1)$$

which shows that the equation is homogeneous. Putting $y = vx$ into (1), we have

$$v + x\frac{dv}{dx} = \frac{3v^2+1}{2v}$$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

or $$x\frac{dv}{dx} = \frac{3v^2+1}{2v} - v = \frac{v^2+1}{2v}$$

or $$\frac{2v}{1+v^2}\,dv = \frac{dx}{x}.$$

On integrating, we get

$$\ln(1+v^2) = \ln|x| + \ln|c|$$

or $$1+v^2 = |cx|$$

Replacing v by y/x, we obtain

$$1+\frac{y^2}{x^2} = |cx|$$

or $$x^2+y^2 = |cx|x^2.$$

If $x \geq 0$, we can write this as

$$x^2+y^2 = cx^3$$

or $$y^2 = cx^3 - x^2$$

or $$y = \pm\sqrt{cx^3-x^2}$$

Applying the initial condition, we get

$$y(2) = 6 = \pm\sqrt{8c-4}$$

or $$8c - 4 = 36 \quad \text{i.e.,} \quad c = 5.$$

Hence $y = \sqrt{5x^3-x^2}$ is the required solution. We take the plus sign in the radical since $y(2)$ is positive.

D.E. REDUCIBLE TO HOMOGENEOUS FORM

(1.14) The differential equation

$$(a_1x+b_1y+c_1)\,dx + (a_2x+b_2y+c_2) = 0$$

not homogeneous. But it can be reduced to a homogeneous form as illustrated below:

Case I. If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then make the transformations

$$x = X+h, \; y = Y+k.$$

The given equation becomes

$$(a_1X+b_1Y+a_1h+b_1k+c_1)\,dX + (a_2X+b_2Y+a_2h+b_2k+c_2)\,dY = 0. \qquad (1)$$

Let h and k be the solution of the system of equations

$$\left.\begin{aligned} a_1h+b_1k+c_1 &= 0 \\ a_2h+b_2k+c_2 &= 0 \end{aligned}\right\}$$

Then for these values of h and k, (1) reduces to the homogeneous equation

$$(a_1X+b_1Y)\,dX + (a_2X+b_2Y)\,dY = 0$$

in the variables X and Y.

Case II. If $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, then put

$z = a_1x + b_1y$ and the given equation will reduce to a separable equation in the variables x and z.

Example 14. Solve $\dfrac{dy}{dx} = \dfrac{2y - x + 5}{2x - y - 4}$. (1)

Solution. Putting

$$x = X + h, \quad y = Y + k$$

the given equation becomes

$$\frac{dY}{dX} = \frac{2Y - X + 2k - h + 5}{2X - Y + 2h - k - 4} \tag{2}$$

The solution of the system of equations

$$\left.\begin{aligned} -h + 2k + 5 &= 0 \\ 2h - k - 4 &= 0 \end{aligned}\right\}$$

is $\quad h = 1, \quad k = -2.$

For these values of h and k, (2) reduces to

$$\frac{dY}{dX} = \frac{2Y - X}{2X - Y} \tag{3}$$

which is homogeneous.

Putting $Y = vX$ into (3), we have

$$v + X\frac{dv}{dX} = \frac{2v - 1}{2 - v}$$

or $\quad X\dfrac{dv}{dX} = \dfrac{2v - 1}{2 - v} - v = \dfrac{v^2 - 1}{2 - v}$ which is separable. Therefore,

$$\left(\frac{2 - v}{v^2 - 1}\right) dv = \frac{dX}{X}.$$

Integrating, we obtain

$$\ln\left|\frac{v - 1}{v + 1}\right| - \frac{1}{2}\ln|v^2 - 1| = \ln|X| + \ln|c_0|$$

or $\quad \ln\left|\dfrac{v-1}{v+1}\right|^2 = \ln|X|^2 + \ln|c_0|^2 + \ln|v^2 - 1|$

$$= \ln\{|c_0 X|^2 . |v^2 - 1|\}$$

or $\quad \left|\dfrac{v-1}{v+1}\right|^2 = c|X|^2 . |v - 1| . |v + 1|$, where $c = c_0^2$

or $\quad |v - 1| = c|X|^2 . |v + 1|^3.$

Replacing v by Y/X, this becomes

$$\left|\frac{Y - X}{X}\right| = c|X|^2 \left|\frac{Y + X}{X}\right|^3$$

or $\quad |Y - X| = c|Y + X|^3$ (4)

But $X = x - 1$, $Y = y + 2$. Hence (4) takes the form

$$|y - x + 3| = c|y + x + 1|^3$$

which is the required solution.

Example 15. Solve: $(2x + y + 1)\,dx + (4x + 2y - 1)\,dy = 0$ (1)

Solution. Since $\dfrac{a_1}{a_2} = \dfrac{2}{4} = \dfrac{b_1}{b_2} = \dfrac{1}{2}$, we put $2x + y = z$. Then (1) becomes

$$(z + 1)\,dx + (2z - 1)(dz - 2\,dx) = 0$$

or $\quad (z + 1 - 4z + 2)\,dx + (2z - 1)\,dz = 0$

i.e., $\quad 3(1 - z)\,dx + (2z - 1)\,dz = 0$ which is separable.

Dividing by $1 - z$, we have

$$3\,dx + \frac{2z - 1}{1 - z}\,dz = 0$$

or $\quad 3\,dx + \left(-2 + \dfrac{1}{1 - z}\right) dz = 0.$

Integrating, we obtain

$$3x - 2z - \ln|1 - z| = c_0$$

Replacing z by $2x + y$, it reduces to

$$3x - 2(2x + y) - \ln|1 - 2x - y| = c_0$$

or $\quad -x - 2y - \ln|2x + y - 1| = c_0$

or $\quad \ln|2x + y - 1| + x + 2y = c,$ where $c = -c_0$

is the required solution.

EXERCISE 9.3

Solve **(Problems 1 – 10)**:

1. $(x - y)\,dx + (x + y)\,dy = 0$ 2. $(y^2 + 2xy)\,dx + x^2\,dy = 0$
3. $(x^2 - 3y^2)\,dx + 2xy\,dy = 0$
4. $3x\cos(y/x)\,dy = [2x\sin(y/x) + 3y\cos(y/x)]\,dx$
5. $(x^2 + xy + y^2)\,dx - x^2\,dy = 0$ 6. $y\,dy + x\,dx = \sqrt{x^2 + y^2}\,dx$

7. $\dfrac{dy}{dx} = \dfrac{4y-3x}{2x-y}$

8. $x \sin\left(\dfrac{y}{x}\right) dy = \left[y \sin\left(\dfrac{y}{x}\right) - x\right] dx$

9. $\left(x^3 + y^2\sqrt{x^2+y^2}\right) dx - xy\sqrt{x^2+y^2}\, dy = 0$

10. $\left(\sqrt{x+y}+\sqrt{x-y}\right) dx - \left(\sqrt{x+y}-\sqrt{x-y}\right) dy = 0.$

Solve the initial value problems (**Problems 11–14**):

11. $\dfrac{dy}{dx} = \dfrac{x+y}{x}$, $\quad y(1) = 1$

12. $\left(y+\sqrt{x^2+y^2}\right) dx - x\,dy = 0,$ $\quad y(1) = 0$

13. $(2x-5y)\,dx + (4x-y)\,dy = 0,$ $\quad y(1) = 4$

14. $(3x^2+9xy+5y^2)\,dx - (6x^2+4xy)\,dy = 0,$ $\quad y(2) = -6.$

Solve:

15. $\dfrac{dy}{dx} = \dfrac{x+3y-5}{x-y-1}$

16. $\dfrac{dy}{dx} = -\dfrac{4x+3y+15}{2x+y+7}$

17. $(3y-7x-3)\,dx + (7y-3x-7)\,dy = 0$

18. $\dfrac{dy}{dx} = \dfrac{3x-4y-2}{3x-4y-3}$

19. $\dfrac{dy}{dx} = \dfrac{y-x+1}{y-x+5}$

20. $\dfrac{dy}{dx} = \dfrac{x-2y+5}{2x+y-1}$

EXACT EQUATIONS

(9.15) Definition. The expression

$$M(x, y)\,dx + N(x, y)\,dy \qquad (1)$$

is called an **exact differential** if there exists a continuously differentiable function $f(x, y)$ of two real variables x and y such that the expression equals the total differential df. We know from calculus that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Thus, if (1) is exact then

$$M(x, y) = \frac{\partial f}{\partial x} = f_x \quad \text{and} \quad N(x, y) = \frac{\partial f}{\partial y} = f_y.$$

If (1) is an exact differential then the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is called an **exact equation**.

(9.16) Theorem. The differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0 \qquad (1)$$

is an exact equation if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

where the functions $M(x, y)$ and $N(x, y)$ have continuous first order partial derivatives.

Proof. Suppose that the equation (1) is exact so that $M\,dx + N\,dy$ is an exact differential. By definition, there exists a function $f(x, y)$ such that

$$M(x, y) = \frac{\partial f}{\partial x} = f_x \quad \text{and} \quad N(x, y) = \frac{\partial f}{\partial y} = f_y.$$

Then $\quad M_y = \dfrac{\partial M}{\partial y} = f_{xy} = \dfrac{\partial^2 f}{\partial y\,\partial x}$

and $\quad N_x = \dfrac{\partial N}{\partial x} = f_{yx} = \dfrac{\partial^2 f}{\partial x\,\partial y}$

Since M and N possess continuous first order partial derivatives, we have $f_{xy} = f_{yx}$ and, therefore

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{as desired.}$$

The proof of the converse is omitted since it is beyond our scope.

(9.17) Solution of an Exact Equation.

If $\quad M(x, y)\,dx + N(x, y)\,dy = 0 \qquad (1)$

is an exact equation, then there exists a function $f(x, y)$ such that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$
$$= M(x, y)\,dx + N(x, y)\,dy.$$

Therefore, $\dfrac{\partial f}{\partial x} = M$ and $\dfrac{\partial f}{\partial y} = N.$

Integrating $\dfrac{\partial f}{\partial x} = M$ with respect to x, we have

$$f(x, y) = \int M\,dx + h(y). \qquad (2)$$

The constant of integration $h(y)$ is an arbitrary function of y since it must vanish under differentiation w.r.t. x.

Differentiating (2) w.r.t. y, we get

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\int M\,dx\right) + h'(y)$$

i.e.,
$$\frac{\partial f}{\partial y} = N = \frac{\partial}{\partial y}\left(\int M\,dx\right) + h'(y)$$

or
$$h'(y) = N - \frac{\partial}{\partial y}\left(\int M\,dx\right)$$

Integrating the above equation w.r.t. y, we obtain h and hence $f(x, y) = c$ is the required solution of (1).

Example 16. Solve: $(3x^2y + 2)\,dx + (x^3 + y)\,dy = 0$ (1)

Solution. Here $M = 3x^2y + 2$ and $N = x^3 + y$

$$\frac{\partial M}{\partial y} = 3x^2 \quad , \quad \frac{\partial N}{\partial x} = 3x^2.$$

Thus $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ and so the equation is exact.

To find the solution of (1), we note that the left hand side of the equation is an exact differential. Therefore, there exists a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = 3x^2y + 2 \qquad (2)$$

and
$$\frac{\partial f}{\partial y} = x^3 + y \qquad (3)$$

Integrating (2) w.r.t. x, we have

$$f(x, y) = x^3y + 2x + h(y),$$

where $h(y)$ is the constant of integration. Differentiating the above equation w.r.t. y and using (3), we obtain

$$\frac{\partial f}{\partial y} = x^3 + h'(y) = x^3 + y$$

or
$$h'(y) = y.$$

Integrating, we have

$$h(y) = \frac{y^2}{2}.$$

Thus $f(x, y) = x^3y + 2x + \dfrac{y^2}{2}.$

Hence the general solution of (1) is

$$x^3y + 2x + \frac{y^2}{2} = c.$$

Alternative Method:

Integrating (2) and (3) w.r.t. x and y respectively, we have

$$f(x, y) = x^3y + 2x + h(y)$$

and
$$f(x, y) = x^3y + \frac{y^2}{2} + g(x)$$

Thus $h(y) = \dfrac{y^2}{2}$ and $g(x) = 2x$

The general solution is

$$x^3y + 2x + \frac{y^2}{2} = c.$$

Example 17. Solve the initial value problem

$$(2y \sin x \cos x + y^2 \sin x)\,dx + (\sin^2 x - 2y \cos x)\,dy = 0, \qquad y(0) = 3$$

Solution. Here

$$M = 2y \sin x \cos x + y^2 \sin x$$

and
$$N = \sin^2 x - 2y \cos x$$

$$\frac{\partial M}{\partial y} = 2 \sin x \cos x + 2y \sin x$$

$$\frac{\partial N}{\partial x} = 2 \sin x \cos x + 2y \sin x$$

Thus $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

showing that the given equation is exact.

Hence there exists a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = 2y \sin x \cos x + y^2 \sin x \qquad (1)$$

$$\frac{\partial f}{\partial y} = \sin^2 x - 2y \cos x \qquad (2)$$

Integrating (1) w.r.t. x, we have

$$f(x, y) = y \sin^2 x - y^2 \cos x + h(y).$$

Differentiating this equation w.r.t. y and using (2), we get

$$\sin^2 x - 2y \cos x + h'(y) = \sin^2 x - 2y \cos x$$

i.e., $h'(y) = 0$ and so $h(y) = c_1.$

Hence the general solution of the given equation is

$$f(x, y) = c_2$$

i.e., $y \sin^2 x - y^2 \cos x + c_1 = c_2$

or $y \sin^2 x - y^2 \cos x = c_2 - c_1 = c$.

Applying the initial condition that when $x = 0$, $y = 3$, we have

$$-9 = c$$

Hence $y^2 \cos x - y \sin^2 x = 9$

is the required solution

EXERCISE 9.4

Solve (Problems 1–10):

1. $(3x^2 + 4xy)\,dx + (2x^2 + 2y)\,dy = 0$
2. $(2xy + y - \tan y)\,dx + (x^2 - x\tan^2 y + \sec^2 y)\,dy = 0$
3. $\dfrac{x+y}{y-1}dx - \dfrac{1}{2}\left(\dfrac{x+1}{y-1}\right)^2 dy = 0$ 4. $\dfrac{dy}{dx} = -\dfrac{ax + hy}{hx + by}$
5. $(1 + \ln xy)\,dx + \left(1 + \dfrac{x}{y}\right)dy = 0$ 6. $\dfrac{y\,dx + x\,dy}{1 - x^2y^2} + x\,dx = 0$
7. $(6xy + 2y^2 - 5)\,dx + (3x^2 + 4xy - 6)\,dy = 0$
8. $(y\sec^2 x + \sec x \tan x)\,dx + (\tan x + 2y)\,dy = 0$
9. $(y\cos x + 2xe^y)\,dx + (\sin x + x^2e^y - 1)\,dy = 0$
10. $(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x)\,dx + (xe^{xy}\cos 2x - 3)\,dy = 0$.

Solve the initial value problems:

11. $(2xy - 3)\,dx + (x^2 + 4y)\,dy = 0,$ $y(1) = 2$
12. $(2x\cos y + 3x^2y)\,dx + (x^3 - x^2\sin y - y)\,dy = 0,$ $y(0) = 2$
13. $(3x^2y^2 - y^3 + 2x)\,dx + (2x^3y - 3xy^2 + 1)\,dy = 0,$ $y(-2) = 1$
14. $\dfrac{3-y}{x^2}dx + \dfrac{y^2 - 2x}{xy^2}dy = 0,$ $y(-1) = 2$
15. $(4x^3e^{x+y} + x^4e^{x+y} + 2x)\,dx + (x^4e^{x+y} + 2y)\,dy = 0,$ $y(0) = 1.$

INTEGRATING FACTORS

(9.18) Definition. *If the differential equation*

$$M(x, y)\,dx + N(x, y)\,dy = 0 \qquad (1)$$

is not exact but when it is multiplied by a function $\mu(x, y)$ and the resulting equation

$$\mu(x, y)M(x, y)\,dx + \mu(x, y)N(x, y)\,dy = 0$$

is exact, then $\mu(x, y)$ is called an **integrating factor (I.F.)** of the differential equation (1). The number of integrating factors of an equation may be infinite.

We list below (without proofs) some rules to find the integrating factors of equations of special types.

Rule I. If $M(x, y)\,dx + N(x, y)\,dy = 0$

is not exact and $\dfrac{M_y - N_x}{N} = P$, where P is a function of x only, then (1) has an integrating factor $\mu(x)$ which also depends on x. $\mu(x)$ is solution of the differential equation (1)

$$\frac{d\mu}{dx} = P\mu$$

i.e., $$\mu(x) = \exp\int P\,dx$$

Note that $M_y = \dfrac{\partial M}{\partial y}$, $N_x = \dfrac{\partial N}{\partial x}$

Rule II. If $\dfrac{N_x - M_y}{M} = Q$, where Q is a function of y only, then the differential equation $M\,dx + N\,dy = 0$ has an integrating factor

$$\mu(y) = \exp\int Q\,dy$$

→ only on "x" dependent
→ on "y" dependent

Rule III. If $M\,dx + N\,dy = 0$ (1)

is homogeneous and $xM + yN \neq 0$, then

$\dfrac{1}{xM + yN}$ is an I.F. of (1).

Rule IV. If $M\,dx + N\,dy = 0$ is of the form

$$yf(xy)\,dx + xg(xy)\,dy = 0$$

and $xM - yN \neq 0$, then

$\dfrac{1}{xM - yN}$ is an I.F. of (1).

The following differential formulas are useful in the calculation of certain exact equations:

(i) $d\left(\dfrac{y}{x}\right) = \dfrac{x\,dy - y\,dx}{x^2}$

(ii) $d\left(\dfrac{x}{y}\right) = \dfrac{y\,dx - x\,dy}{y^2}$

(iii) $d(xy) = x\,dy + y\,dx$

(iv) $d(x^2+y^2) \quad = \quad 2(x\,dx + y\,dy)$

(v) $d\left(\ln \dfrac{x}{y}\right) \quad = \quad \dfrac{y\,dx - x\,dy}{xy}$

(vi) $d\left(\arctan \dfrac{x}{y}\right) \quad = \quad \dfrac{y\,dx - x\,dy}{x^2+y^2}$

Example 18. Solve: $y\,dx + (x^2y - x)\,dy = 0$.

Solution. The equation is not exact. Rearranging the equation, we have

$$y\,dx - x\,dy + x^2y\,dy \quad = \quad 0$$

or
$$\frac{y\,dx - x\,dy}{x^2} + y\,dy \quad = \quad 0.$$

Now it is an exact equation and may be written as

$$-d\left(\frac{y}{x}\right) + y\,dy \quad = \quad 0.$$

Integrating, we have

$$-\frac{y}{x} + \frac{y^2}{2} \quad = \quad c$$

or $\quad xy^2 - 2y \quad = \quad cx \quad$ is the general solution.

Example 19. $(x^2 - 2x + 2y^2)\,dx + 2xy\,dy = 0$. (1)

Solution. Here $\dfrac{M_y - N_x}{N} = \dfrac{4y - 2y}{2xy} = \dfrac{1}{x}$.

Therefore, I.F. $\mu(x, y)$ is the solution of

$$\frac{d\mu}{dx} \quad = \quad \frac{\mu}{x}$$

or $\quad \mu \quad = \quad x \quad$ is an I.F.

Multiplying the equation by x, we have

$$(x^3 - 2x^2 + 2xy^2)\,dx + 2x^2y\,dy = 0.$$

This equation is exact. The reader can easily find that its solution is

$$\frac{x^4}{4} - \frac{2x^3}{3} + x^2y^2 \quad = \quad c_0$$

or $\quad 3x^4 - 8x^3 + 12x^2y^2 = c \quad$ is the solution of (1).

ample 20. Solve: $dx + \left(\dfrac{x}{y} - \sin y\right) dy = 0$.

tion. Here, by Rule II,

$$\frac{N_x - M_y}{M} = \frac{\frac{1}{y} - 0}{1} = \frac{1}{y}$$

function of y only. Therefore,

$$\mu(y) \quad = \quad \exp \int \frac{dy}{y} \quad = \quad e^{\ln y} \quad = \quad y$$

I.F. Multiplying the equation by y, we have

$$y\,dx + (x - y\sin y)\,dy \quad = \quad 0$$

or
$$y\,dx + x\,dy - y\sin y\,dy \quad = \quad 0$$

or
$$d(xy) - y\sin y\,dy \quad = \quad 0.$$

Integrating, we get

$$xy + y\cos y - \sin y \quad = \quad c$$

which is the required solution.

ample 21. Solve: $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$. (1)

tion. The equation is homogeneous but not exact. We have

$$xM + yN \; = \; x^3y - 2x^2y^2 - x^3y + 3x^2y^2 = x^2y^2 \neq 0$$

So, using Rule III,

$\dfrac{1}{xM + yN} = \dfrac{1}{x^2y^2}$ is an I.F. Therefore, multiplying (1) by $\dfrac{1}{x^2y^2}$, we obtain

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy \quad = \quad 0.$$

This equation is exact. Integrating, we get

$$\frac{x}{y} - 2\ln|x| + 3\ln|y| = \; c$$

the required solution.

Note that, here the term $\dfrac{x}{y}$, involving both x and y, has been taken only once.

ample 22. Solve: $y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0$. (1)

tion. The equation is of the form

$$yf(xy)\,dx + xg(xy)\,dy \quad = 0.$$

Now, $\quad xM - yN \; = \; x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3 = 3x^3y^3 \neq 0.$

Therefore, $\dfrac{1}{x^3y^3}$ is an I.F.

Multiplying (1) by $\frac{1}{x^3y^3}$, we have

$$\left(\frac{1}{x^2y}+\frac{2}{x}\right)dx+\left(\frac{1}{xy^2}-\frac{1}{y}\right)dy = 0$$

This is an exact equation. Integrating, we get

$$\frac{-1}{xy}+2\ln|x|-\ln|y| = c$$

as the required solution.

EXERCISE 9.5

Solve (by finding an I.F.):

1. $(x^9y^3+y)\,dx-x\,dy = 0$
2. $x\,dy-y\,dx = (x^2+y^2)\,dx$
3. $(x^2+x-y)\,dx+x\,dy = 0$
4. $dy+\frac{y-\sin x}{x}dx = 0$
5. $y(2xy+e^x)\,dx-e^x\,dy = 0$
6. $(y^4+2y)\,dx+(xy^3+2y^4-4x)\,dy = 0$
7. $(x^2+y^2+2x)\,dx+2y\,dy = 0$
8. $(x^2+y^2)\,dx-2xy\,dy = 0$
9. $(3y+4xy^2)\,dx+(2x+3x^2y)\,dy = 0$
10. $(4x+3y^2)\,dx+2xy\,dy = 0$
11. $(y-xy^2)\,dx+(x+x^2y^2)\,dy = 0$
12. $(3x^2y^4+2xy)\,dx+(2x^3y^3-x^2)\,dy = 0$
13. $y-x\frac{dy}{dx} = x+y\frac{dy}{dx}$
14. $\frac{dy}{dx} = e^{2x}+y-1$
15. $(y^2+xy)\,dx-x^2\,dy = 0$
16. $(3xy+y^2)\,dx+(x^2+xy)\,dy = 0$
17. $(3x^2y+2xy+y^3)\,dx+(x^2+y^2)\,dy = 0$
18. $y\,dx+(2xy-e^{-2y})\,dy = 0$
19. $e^x\,dx+(e^x\cot y+2y\csc y)\,dy = 0$
20. $(x+2)\sin y\,dx+x\cos y\,dy = 0$

LINEAR EQUATIONS

(9.19) Definition. A first order ordinary differential equation is **linear** in the dependent variable y and the independent variable x if it is or can be written in the form

$$\frac{dy}{dx}+P(x)y = Q(x), \tag{1}$$

where P and Q are functions of x.

(9.20) Solution of a Linear Equation. The linear equation

$$\frac{dy}{dx}+P(x)y = Q(x)$$

can be written as

$$[P(x)y-Q(x)]\,dx+dy = 0 \tag{2}$$

which is of the form $M\,dx+N\,dy=0$, where

$$M = P(x)y-Q(x) \quad\text{and}\quad N = 1.$$

Now

$$\frac{\partial M}{\partial y} = P(x) \quad\text{and}\quad \frac{\partial N}{\partial x} = 0$$

Thus (2) is not exact unless $P(x)=0$ in which case (1) is separable. However, an integrating factor (depending only on x) of (2) may be easily found. Let $\mu(x)$ be an I.F. of (2). Then multiplying (2) by $\mu(x)$, we get

$$[\mu(x)P(x)y-\mu(x)Q(x)]\,dx+\mu(x)\,dy = 0. \tag{3}$$

Now (3) is an exact equation if and only if

$$\frac{\partial}{\partial y}[\mu(x)P(x)y-\mu(x)Q(x)] = \frac{\partial}{\partial x}[\mu(x)]$$

This condition reduces to

$$\mu(x)P(x) = \frac{d}{dx}[\mu(x)]$$

i.e.,

$$\mu P(x) = \frac{d\mu}{dx}$$

or

$$\frac{d\mu}{\mu} = P(x)\,dx,$$

Integrating, we obtain

$$\ln|\mu| = \int P(x)\,dx$$

or

$$\mu = \exp\left[\int P(x)\,dx\right] > 0.$$

Thus $e^{\int P\,dx}$ $\left[\text{or } \exp\int P(x)\,dx\right]$ is an I.F. of the linear equation (1). Multiplying (1) by $e^{\int P\,dx}$, we have

$$e^{\int P(x)\,dx}\frac{dy}{dx}+e^{\int P(x)\,dx}P(x)y = Q(x)\,e^{\int P(x)\,dx}$$

or

$$\frac{d}{dx}\left[y\,e^{\int P(x)\,dx}\right] = Q(x)\,e^{\int P(x)\,dx}$$

Integrating this we obtain the solution of (1) in the form

$$y\,e^{\int P(x)\,dx} = \int\left[Q(x)\,e^{\int P(x)\,dx}\right]dx+c.$$

Example 23. Solve: $(x-1)^3 \dfrac{dy}{dx} + 4(x-1)^2 y = x+1$

Solution. We write the equation in the standard form

$$\frac{dy}{dx} + \frac{4}{x-1} y = \frac{x+1}{(x-1)^3} \qquad (1)$$

Here $P(x) = \dfrac{4}{x-1}$. Therefore, an I.F. of (1) is

$$\exp\left[\int \frac{4dx}{x-1}\right] = \exp[\ln(x-1)^4] = (x-1)^4.$$

Multiplying (1) by this I.F., we get

$$(x-1)^4 \frac{dy}{dx} + 4(x-1)^3 y = x^2 - 1$$

or

$$\frac{d}{dx}[y(x-1)^4] = x^2 - 1$$

Integrating, we obtain

$$y(x-1)^4 = \frac{x^3}{3} - x + c$$

which is the required solution.

Example 24. Solve: $(x + 2y^3)\dfrac{dy}{dx} = y.$

Solution. We have

$$\frac{dy}{dx} = \frac{y}{x+2y^3}$$

This equation is clearly not linear in y. But in the first order differential equation the roles of x and y are interchangeable in the sense that either variable may be regarded as dependent variable. Let us regard x as dependent variable and y as independent variable. The equation may be written as

$$\frac{dx}{dy} - \frac{1}{y}x = 2y^2 \qquad (1)$$

which is linear in x with

$$\text{I.F.} = \exp\left[\int\left(-\frac{1}{y}\right)dy\right] = \exp\left[\ln\frac{1}{y}\right] = \frac{1}{y}.$$

Multiplying (1) by $\dfrac{1}{y}$, we get

$$\frac{1}{y}\frac{dx}{dy} - \frac{1}{y^2}x = 2y \quad \text{or} \quad \frac{d}{dy}\left(\frac{x}{y}\right) = 2y.$$

Integrating, we have

$$\frac{x}{y} = y^2 + c \quad \text{or} \quad x = y(y^2 + c)$$

the required solution.

Example 25. Solve the initial value problem

$$\frac{dr}{d\theta} + r\tan\theta = \cos^2\theta, \qquad r\left(\frac{\pi}{4}\right) = 1.$$

Solution. The equation is linear in r with

$$\text{I.F.} = \exp\left[\int \tan\theta\, d\theta\right] = \exp[\ln\sec\theta] = \sec\theta,$$

(taking $\sec\theta$ positive)

Multiplying the given equation by $\sec\theta$, we have

$$\sec\theta\frac{dr}{d\theta} + r\sec\theta\tan\theta = \cos\theta$$

$$\frac{d}{d\theta}[r\sec\theta] = \cos\theta$$

Integrating, we obtain

$$r\sec\theta = \sin\theta + c$$

Applying the initial condition, we have

$$1 \,.\, \sec\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) + c$$

or

$$c = \sqrt{2} - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

Therefore,

$$r\sec\theta = \sin\theta + \frac{1}{\sqrt{2}}$$

or

$$r = \sin\theta\cos\theta + \frac{\cos\theta}{\sqrt{2}}$$

or

$$2r = \sin 2\theta + \sqrt{2}\cos\theta \quad \text{is the required solution.}$$

THE BERNOULLI EQUATION[1]

(9.21) Definition. An equation of the form

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n \qquad (1)$$

is called the **Bernoulli differential equation**. This equation is linear if $n = 0$ or 1. If n is not zero or 1, then (1) is reducible to a linear equation. Dividing by y^n, (1) becomes

$$y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = Q(x) \qquad (2)$$

In (2), put $v = y^{1-n}$ then it reduces to

$$\frac{dv}{dx} + (1-n)\,P(x)\,v = (1-n)\,Q(x)$$

1 a
b

2 a, b

which is linear in v.

Note. Consider the equation

$$f'(y)\frac{dy}{dx} + P(x)f(y) = Q(x)$$

Letting $v = f(y)$, this equation becomes

$$\frac{dv}{dx} + P(x)\,v = Q(x)$$

which is linear in v.

Example 26. Solve $\dfrac{dy}{dx} + \dfrac{xy}{1-x^2} = xy^{\frac{1}{2}}$. (1)

Solution. Dividing by $y^{\frac{1}{2}}$, (1) becomes

$$y^{-\frac{1}{2}}\frac{dy}{dx} + \frac{x}{1-x^2}\,y^{\frac{1}{2}} = x. \qquad (2)$$

Put $\qquad y^{\frac{1}{2}} = v$

or $\qquad \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = \frac{dv}{dx}$

Then (2) reduces to

$$\frac{dv}{dx} + \frac{x}{2(1-x^2)}\,v = \frac{x}{2} \qquad (3)$$

1. After the name of Swiss mathematician Jacques Bernoulli (1654 – 1705) who studied it in 1669 C.E.

This is linear in v.

$$\text{I.F.} = \exp\left[\int \frac{x}{2(1-x^2)}\,dx\right] = \exp\left[\frac{-1}{4}\ln(1-x^2)\right] = (1-x^2)^{-\frac{1}{4}}$$

Multiplying (3) by $(1-x^2)^{-\frac{1}{4}}$, we get

$$(1-x^2)^{-\frac{1}{4}}\frac{dv}{dx} + \frac{x}{2(1-x^2)^{5/4}}\,v = \frac{x}{2(1-x^2)^{1/4}}$$

or $$\frac{d}{dx}\left[(1-x^2)^{-\frac{1}{4}}v\right] = \frac{-1}{4}\left[-2x(1-x^2)^{-\frac{1}{4}}\right]$$

Integrating, we have

$$v(1-x^2)^{-\frac{1}{4}} = \frac{-1}{4}\,\frac{(1-x^2)^{3/4}}{3/4} + c$$

or $$v = c(1-x)^{-\frac{1}{4}} - \frac{1-x^2}{3}$$

or $$y^{\frac{1}{2}} = c(1-x^2)^{-\frac{1}{4}} - \frac{1-x^2}{3}$$

is the required solution of (1)

EXERCISE 9.6

Solve (Problems 1–15)

1. $\dfrac{dy}{dx} + \left(\dfrac{2x+1}{x}\right)y = e^{-2x}$
2. $\dfrac{dy}{dx} + \dfrac{3y}{x} = 6x^2$
3. $\dfrac{dy}{dx} + \dfrac{y}{x\ln x} = \dfrac{3x^2}{\ln x}$
4. $\dfrac{dy}{dx} + 3y = 3x^2 e^{-3x}$
5. $\cos^3 x\dfrac{dy}{dx} + y\cos x = \sin x$
6. $x\dfrac{dy}{dx} + (1 - x\cot x)\,y = x$
7. $(x+1)\dfrac{dy}{dx} - ny = e^x(x+1)^{n+1}$
8. $(x^2+1)\dfrac{dy}{dx} + 2xy = 4x^2$
9. $x\dfrac{dy}{dx} + 2y = \sin x$
10. $(1+x^2)\dfrac{dy}{dx} + 4xy = \dfrac{1}{(1+x^2)^2}$
11. $\dfrac{dy}{dx} = \dfrac{1}{e^y - x}$
12. $(x+2y^3)\dfrac{dy}{dx} = y$
13. $x\dfrac{dy}{dx} + y = y^2\ln x$
14. $\dfrac{dy}{dx} + y = xy^3$
15. $x\dfrac{dy}{dx} - 2x^2y = y\ln y$.

Solve the initial value problems:

16. $(x^2+1)\frac{dy}{dx} + 4xy = x,$ $\quad y(2) = 1$

17. $e^x[y - 3(e^x+1)^2]\,dx + (e^x+1)\,dy = 0,$ $\quad y(0) = 4$

18. $\frac{dy}{dx} + \frac{y}{2x} = \frac{x}{y^3},$ $\quad y(1) = 2$

19. $x(2+x)\frac{dy}{dx} + 2(1+x)y = 1 + 3x^2,$ $\quad y(-1) = 1$

20. $x\frac{dy}{dx} + 3y = x^3y^2,$ $\quad y(1) = 2$

ORTHOGONAL TRAJECTORIES

(9.22) (Rectangular Coordinates) Definition. It has been observed that the general solution of a first order differential equation contains one arbitrary constant. When this constant is assigned different values, one obtains a one-parameter family of curves. Each of these curves represents a particular solution of the given differential equation.

On the other hand, given a one-parameter family of curves

$$f(x, y, c) = 0, \tag{1}$$

c being parameter, then each member of the family is a particular solution of some differential equation. In fact, this differential equation is obtained by elimination of the parameter c between (1) and the relation obtained by differentiating (1) w.r.t. x.

Let $f(x, y, c) = 0$ and $F(x, y, k) = 0$ be two families of curves with parameters c and k. If each curve in either family is intersected orthogonally by every curve in the other family, then each family is said to be **orthogonal trajectory** of the other. Recall that two curves are said to be orthogonal (intersect orthogonally) if their tangents at the point of intersection are perpendicular to each other.

For example, the families of curves given by

$$x^2 + y^2 = c^2 \quad \Rightarrow \quad f(x, y, c) \equiv x^2 + y^2 - c^2 = 0$$

and

$$y = kx \quad \Rightarrow \quad F(x, y, k) \equiv y - kx = 0$$

are orthogonal as illustrated graphically below:

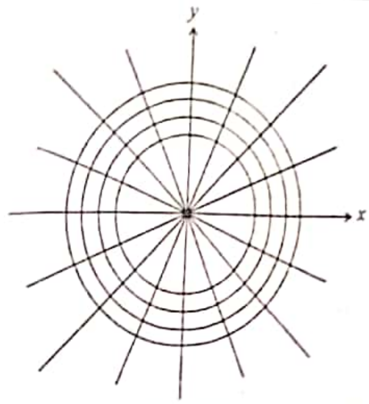


Figure 9.1

Let the given family of curves be (1). By eliminating c, we find the differential equation of this family. Suppose the differential equation of (1) is

$$\frac{dy}{dx} = F(x, y)$$

The differential equation of the orthogonal family is

$$\frac{dy}{dx} = \frac{-1}{F(x, y)} \tag{2}$$

The solution of (2) is the required family of orthogonal trajectories of (1).

Example 27. Find the orthogonal trajectories of the family of circles

$$x^2 + y^2 = c^2 \tag{1}$$

Solution. Differentiating (1) w.r.t. x, we have

$$\frac{dy}{dx} = \frac{-x}{y}.$$

The differential equation of the orthogonal trajectories of (1) is

$$\frac{dy}{dx} = \frac{-1}{-x/y} = \frac{y}{x}. \tag{2}$$

We now solve the differential equation (2). This equation is separable,

$$\frac{dy}{y} = \frac{dx}{x}$$

Integrating, we get

$$y = kx \tag{3}$$

which is the required equation of the orthogonal trajectories of (1). This equation represents a family of straight line through the origin. The orthogonal trajectories are shown in Figure 9.1 above.

Example 28. Find the orthogonal trajectories of the family of curves

$$y = ce^{-x/4}$$

Solution. Differentiating (1) w.r.t. x, we obtain (1)

$$\frac{dy}{dx} = -\frac{c}{4}e^{-x/4} = -\frac{1}{4}y$$

which is the differential equation of (1). Differential equation of the orthogonal trajectories is

$$\frac{dy}{dx} = \frac{4}{y}. \qquad (2)$$

Solving (2), we get

$$y^2 = 8x + k$$

as an equation of the orthogonal trajectories of (1).

(9.23) Polar Coordinates

Let $f(r, \theta, c) = 0$ be the equation of a family of curves. The differential equation of this family can be obtained by elimination of c. Suppose the differential equation of this family is

$$P\,dr + Q\,d\theta = 0$$

where P and Q are functions of r and θ. (1)

We know from calculus that if ϕ is the angle between the radius vector and the tangent to a curve of the given family at any point (r, θ), then

$$\tan\phi = r\frac{d\theta}{dr}.$$

If ϕ_1 is the angle between the radius vector and the tangent to an orthogonal trajectory at (r, θ), then

$$\phi_1 = \frac{\pi}{2} + \phi$$

or $\quad \tan\phi_1 = -\cot\phi$

i.e., $\quad \tan\phi_1 \tan\phi = -1$

for the two curves to be orthogonal.

From (1), we have

$$\frac{d\theta}{dr} = -\frac{P}{Q}$$

or $$r\frac{d\theta}{dr} = -\frac{Pr}{Q}$$

Hence the differential equation of the orthogonal trajectories is

$$r\frac{d\theta}{dr} = \frac{Q}{Pr}. \qquad (2)$$

Solution of (2) is the required family of orthogonal trajectories of the family $f(r, \theta, c) = 0$.

Example 29. Find the orthogonal trajectories of the family of cardioids

$$r = a(1 + \cos\theta). \qquad (1)$$

Solution. Differentiating (1) w.r.t. r, we have

$$1 = a(-\sin\theta)\frac{d\theta}{dr}$$

or $$\frac{d\theta}{dr} = \frac{-1}{a\sin\theta} = -\frac{1+\cos\theta}{r\sin\theta}$$

or $$r\frac{d\theta}{dr} = \frac{-(1+\cos\theta)}{\sin\theta}$$

is the differential equation of (1).

Differential equation of the orthogonal trajectories is

$$r\frac{d\theta}{dr} = \frac{\sin\theta}{1+\cos\theta}. \qquad (2)$$

Separating variables in (2), we get

$$\frac{dr}{r} = \frac{1+\cos\theta}{\sin\theta}d\theta = \csc\theta\, d\theta + \cot\theta\, d\theta.$$

Hence $\quad \ln|r| = -\ln|\csc\theta - \cot\theta| + \ln|\sin\theta| + \ln|b|$

or $\quad r = b\sin\theta(\csc\theta - \cot\theta) = b(1 - \cos\theta)$

is an equation of the orthogonal trajectories of (1).

EXERCISE 9.7

Find an equation of orthogonal trajectories of the curve of each of the following families (**Problems 1–16**):

1. $x^2 - y^2 = c$
2. $x = cy^2$
3. $x^2 + y^2 = cx$
4. $y = e^{cx}$
5. $y = x - 1 + ce^{-x}$
6. $xy = c$
7. $x = \frac{y^2}{4} + \frac{c}{y^2}$
8. $y = (x - c)^2$

9. $y^2 = x^2 + cx$ 10. $x^2 + y^2 = 1 + 2cy$

11. $r = a(1 + \sin\theta)$ 12. $r^2 = a \sin 2\theta$

13. $r^n = a^n \cos n\theta$ 14. $r = \dfrac{a}{2 + \cos\theta}$

15. $r = a \sin n\theta$ 16. $r^n \cos n\theta = a^n$

17. A family of curves whose family of orthogonal trajectories is the same as the given family is called **self-orthogonal**. Show that

$$y^2 = 4cx + 4c^2$$

is self-orthogonal.

18. Prove that the family of confocal coincs

$$\frac{x^2}{c^2} + \frac{y^2}{c^2 - 1} = 1$$

is self-orthogonal.

In this section we shall consider first order differential equations with degree more than one. In what follows, $\dfrac{dy}{dx}$ will be denoted by p.

We have already studied various methods of finding the solution of some special types of the nonlinear first order and first degree differential equations. Such equations were separable, exact, homogenous and so on. We shall briefly discuss techniques to find solutions of special types of first order nonlinear differential equations of higher degree.

EQUATIONS SOLVABLE FOR p

Example 30. Solve: $x^2p^2 + xp - y^2 - y = 0$ (1)

Solution. We factorize the left hand member of (1) to obtain

$$(x^2p^2 - y^2) + (xp - y) = (xp - y)(xp + y + 1) = 0.$$

Therefore, either

$$xp - y = 0 \quad (2)$$

or

$$xp + y + 1 = 0 \quad (3)$$

(2) gives $x\dfrac{dy}{dx} = y$ or $\dfrac{dy}{dx} = \dfrac{dx}{x}$

or

$$y = cx. \quad (4)$$

From (3), we have $xp = -(y + 1)$

or

$$x\frac{dy}{dx} = -(y+1) \quad \text{or} \quad \frac{dy}{y+1} = \frac{-dx}{x}$$

which yields $x(y + 1) = c$ (5)

Combining (4) and (5), the required solution of (1) is

$$(y - cx)(xy + x - c) = 0$$

Example 31. Solve:

$$xp^3 - (x^2 + x + y)p^2 + (x^2 + xy + y)p - xy = 0 \quad (1)$$

Solution. By inspection, we find that $p - 1$ is a factor of left hand member of (1). Thus, the given equation is

$$(p - 1)[xp^2 - (x^2 + y)p + xy] = 0$$

or

$$(p - 1)(xp - y)(p - x) = 0.$$

Therefore, either

$$p - 1 = 0 \quad (2)$$

or

$$xp - y = 0 \quad (3)$$

or

$$p - x = 0. \quad (4)$$

From (2), we have $\dfrac{dy}{dx} = 1$ or $y = x + c$ (5)

From (3), we get $x\dfrac{dy}{dx} = y$ or $y = cx$ (6)

From (4), we obtain $\dfrac{dy}{dx} = x$

or

$$y = \frac{x^2}{2} + c \quad \text{or} \quad x^2 + 2(y - c) = 0. \quad (7)$$

The general solution of (1) is obtained by combining (5), (6) and (7). Thus

$$(y - x - c)(y - cx)(x^2 - 2y + 2c) = 0$$

is the general solution of (1).

EQUATIONS SOLVABLE FOR y

Example 32. Solve: $y = p^2x + p$. (1)

Solution. Differentiating (1) w.r.t. x, we have

$$\frac{dy}{dx} = p = p^2 + 2xp\frac{dp}{dx} + \frac{dp}{dx}$$

or $$\frac{dp}{dx}(2px+1) + p^2 - p = 0$$

or $$\frac{dp}{dx} = \frac{-p(p-1)}{2px+1}$$

or $$\frac{dx}{dp} = \frac{-2px-1}{p(p-1)}$$

i.e., $$\frac{dx}{dp} + \frac{2x}{p-1} = \frac{-1}{p(p-1)} \qquad (2)$$

which is linear in x.

$$\text{I.F.} = \exp\left[\int \frac{2}{p-1}dp\right] = \exp\left[\ln(p-1)^2\right] \Rightarrow (p-1)^2.$$

Multiplying (2) by $(p-1)^2$ and integrating, we get

$$x(p-1)^2 = c - p + \ln p \qquad (3)$$

$$x = \frac{c-p+\ln p}{(p-1)^2}.$$

Substituting this value of x into (1), we have

$$y = p^2 \cdot \frac{c-p+\ln p}{(p-1)^2} + p$$

or $$y(p-1)^2 = p^2(c-p+\ln p) + p \qquad (4)$$

Thus, (3) and (4) constitute the required solution of (1) with p as a parameter.

Example 33. Solve: $y + px = p^2x^4$. (1)

Solution. From (1), we get

$$y = p^2x^4 - px.$$

Differentiating the above equation w.r.t. x, we have

$$\frac{dy}{dx} = p = 4x^3p^2 + 2px^4\frac{dp}{dx} - p - x\frac{dp}{dx}$$

or $$2p - 4x^3p^2 - 2x^4p\frac{dp}{dx} + x\frac{dp}{dx} = 0$$

or $$2p(1-2px^3) + x(1-2px^3)\frac{dp}{dx} = 0.$$

or $$(1-2px^2)\left(2p + x\frac{dp}{dx}\right) = 0$$

Hence, either $1 - 2px^3 = 0$ (2)

or $2p + x\dfrac{dp}{dx} = 0.$ (3)

The equation (3) gives

$$\frac{dp}{p} + \frac{2dx}{x} = 0$$

or $$\ln p + 2\ln x = \ln c$$

or $$px^2 = c \quad \text{or} \quad p = \frac{c}{x^2}.$$

Substituting this value of p into (1), we obtain

$$y = c^2 - \frac{c}{x}$$

or $xy - c^2x + c = 0$ is the required solution.

We have yet to deal with relation (2). If we eliminate p from (1) and (2), we get

$$y = -\frac{1}{4x^2}. \qquad (3)$$

It is easy to check that (3) is also a solution of (1) and this solution does not involve any constant.

EQUATIONS SOLVABLE FOR x

Example 34. Solve: $xp = 1 + p^2$. (1)

Solution. We have

$$x = \frac{1}{p} + p \qquad (2)$$

Differentiating (1) w.r.t. y, we have

$$\frac{dx}{dy} = \frac{1}{p} = -\frac{1}{p^2}\frac{dp}{dy} + \frac{dp}{dx}$$

or
$$\frac{1}{p} = \left(1 - \frac{1}{p^2}\right)\frac{dp}{dy}$$

or
$$1 = \left(p - \frac{1}{p}\right)\frac{dp}{dy}$$

which is a separable equation. Therefore,

$$dy = \left(p - \frac{1}{p}\right)dp.$$

Integrating, we get $\quad c + y = \dfrac{p^2}{2} - \ln p$

i.e., $\quad 2y = p^2 - 2\ln p - 2c \qquad (3)$

Thus, (2) and (3) constitute the solution of (1).

CLAIRAUT'S EQUATION[1]

(9.24) The equation

$$y = xp + f(p), \qquad (1)$$

is known as **Clairaut's equation**. Differentiating both sides of (1), w.r.t. x, we get

$$p = p + xp' + f'(p)\,.\,p'.$$

Cancelling like terms, we have

$$xp' + p'f'(p) = 0$$

or
$$p'\,[x + f'(p)] = 0.$$

Since one of the factors must be zero, two different solutions arise.

(i) If $p' = 0$, then $p = c$ and substitution of this value into (1) yields the general solution

$$y = cx + f(c).$$

(ii) If $x + f'(p) = 0$, then $x = -f'(p)$ and (1) can be rewritten as

$$y = -pf'(p) + f(p)$$

1. Named after the French astronomer and mathematician Alexis Claude Clairaut (1713 – 1765)

Thus x and y are both expressed as functions of p and we obtain the parametric equations

$$\left.\begin{aligned} x &= -f'(p) \\ y &= f(p) - pf'(p) \end{aligned}\right\} \qquad (2)$$

of a curve, representing a solution of (1). The solution (2) is called the **singular solution**. This solution is not deducible from the general solution. p may be eliminated between the two equations in (2) to get a relation in x and y involving no constant.

Example 35. Find the general solution and singular solution of

$$y = xp + \frac{1}{4}p^4. \qquad (1)$$

Solution. The general solution of the equation is

$$y = cx + \frac{1}{4}c^4.$$

Differentiating (1) w.r.t. x, we have

$$p = p + xp' + p^3p'$$

or
$$x = -p^3. \qquad (2)$$

We eliminate p from (1) and (2) and get

$$y = x(-x)^{1/3} + \frac{1}{4}(-x)^{4/3} = -\frac{3}{4}x(-x)^{1/3}$$

or $\quad 64y^3 + 27x^4 = 0\quad$ is the singular solution.

Example 36. Solve: $x^2(y - px) = yp^2$.

Solution. This is not Clairaut's equation. Let us write

$$x^2 = u \quad \text{and} \quad y^2 = v.$$

Then, $\quad 2x\,dx = du \quad$ and $\quad 2y\,dy = dv.$

Hence $\quad \dfrac{y}{x}\dfrac{dy}{dx} = \dfrac{dv}{du}$

$$p = \frac{dy}{dx} = \frac{x}{y}\frac{dv}{du}.$$

Substituting this value of p into (1), we get

$$x^2\left(y - \frac{x^2}{y}\frac{dv}{du}\right) = \frac{x^2}{y}\left(\frac{dv}{du}\right)^2$$

or $$y^2 - x^2\frac{dv}{du} = \left(\frac{dv}{du}\right)^2$$

or $$v - u\frac{dv}{du} = \left(\frac{dv}{du}\right)^2$$

or $$v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2. \qquad (2)$$

This is Clairaut's equation. Hence its general solution is

$$v = cu + c^2$$

or $y^2 = cx^2 + c$ is the general solution of (1).

Differentiating (2) w.r.t. u, we get

$$\frac{dv}{du} = u\frac{d^2v}{du^2} + \frac{dv}{du} + 2\,\frac{dv}{du}\,\frac{d^2v}{du^2}$$

or $$\frac{d^2v}{du^2}\left(u + 2\,\frac{dv}{du}\right) = 0.$$

If $\dfrac{d^2v}{du^2} = 0$, then $v = cu + c^2$, which is the general solution.

If $u + 2\dfrac{dv}{du} = 0$, then

$$u = -2\frac{dv}{du} \qquad (3)$$

and $$v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2 \qquad (2)$$

give the singular solution.

Eliminating $\dfrac{dv}{du}$ from (3) and (2), we obtain

$$v = -\frac{u^2}{4}$$

or $u^2 + 4v = 0$ is the singular solution.

Replacing u, v by x^2 and y^2 respectively, we have

(i) $y^2 = cx^2 + c^2$ as the general solution.

(ii) $x^4 + 4y^2 = 0$ as the singular solution.

EXERCISE 9.8

Solve (Problems 1–25):

1. $p^2 + p - 6 = 0$
2. $x^2p^2 + xyp - 6y^2 = 0$
3. $p^2y + (x - y)p - x = 0$
4. $p^3 - (x^2 + xy + y^2)p + xy^2 + x^2y = 0$
5. $xp^2 + (y - 1 - x^2)p - x(y - 1) = 0$
6. $xyp^2 + (x + y)p + 1 = 0$
7. $p^2 - (x^2y + 3)p + 3x^2y = 0$
8. $yp^2 + (x - y^2)p - xy = 0$
9. $(y + x)^2p^2 + (2y^2 + xy - x^2)p + y(y - x) = 0$
10. $xy(x^2 + y^2)(p^2 - 1) = p(x^4 + x^2y^2 + y^4)$
11. $xp^2 - 3yp + 9x^2 = 0$
12. $p^2 + x^3p - 2x^2y = 0$
13. $p^2 + 4x^5p - 12x^4y = 0$
14. $x^8p^2 + 3xp + 9y = 0$
15. $p^2 + 3xp - y = 0$
16. $y = px + x^3p^2$
17. $xp^2 - 2yp + ax = 0$
18. $p = \tan\left(x - \dfrac{p}{1 + p^2}\right)$
19. $p^3 - 4xyp + 8y^2 = 0$
20. $ap^2 + py - x = 0$
21. $e^{4x}(p - 1) + e^{2y}p^2 = 0$
22. $yp^2 - 2xp + y = 0$
23. $p^2\cos^2 y + p\sin x\cos x\cos y - \sin y\cos^2 x = 0$
24. $(px - y)(py + x) = 2p$
25. $y^2(y - xp) = x^4p^2$.

Find the general solution and the singular solution of each of the following differential equations:

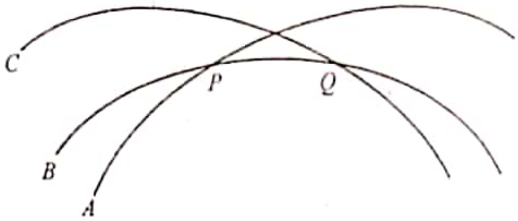
26. $y = xp - \ln p$
27. $y = xp - e^p$
28. $y = xp + a\sqrt{1 + p^2}$
29. $y = xp - \sqrt{p}$
30. $y = xp + p^3$.

ENVELOPE

(9.25) Definition. Let $f(x, y, c) = 0$ be a one-parameter family of curves. Suppose all members of the family of curves are drawn for various values of the parameter c arranged in order of magnitude. The two curves that correspond to two consecutive values of c will be designated as **neighbouring curves**. The locus of the ultimate points of intersection of neighbouring curves is called the **envelope** of the family $f(x, y, c) = 0$.

Let A, B, C represent three neighbouring intersecting members of the family. Let P be the point of intersection of A and B and Q be the point of intersection of B and C. By definition P and Q are points on the envelope. Thus the curve B and the envelope have two contiguous points common, and therefore, have ultimately a common tangent. Hence

B and the envelope touch each other. In the same way, we may show that the envelope touches any other member of the family. Thus a curve E which, at each of its points touches someone of the curves of the family is the envelope of the family.



For the family of circles with centres on the x-axis and radius 1, that is, the family of circles with equation

$$(x-1)^2 + y^2 = 1,$$

the pair of lines $y = \pm 1$ is the envelope. It is easy to see that the lines $y = \pm 1$ touch each member of the family of the circles.

(9.26) Equation of the Envelope. To find an equation of the envelope of the family $f(x, y, c) = 0$, consider two neighbouring curves

$$\left.\begin{aligned} f(x, y, c) &= 0 \\ \text{and} \qquad f(x, y, c+h) &= 0 \end{aligned}\right\} \tag{1}$$

Find the intersection of these two curves and let $h \to 0$. The point of intersection then must approach the point of contact of the curve $f(x, y, c) = 0$ with the envelope. At the point of intersection the equation

$$\frac{f(x, y, c+h) - f(x, y, c)}{h} = 0 \tag{2}$$

is true as well as (1). Letting $h \to 0$, we have from (2)

$$f_c(x, y, c) = 0 \tag{3}$$

$$\text{and from (1)} \quad f(x, y, c) = 0 \tag{4}$$

If we eliminate c from (3) and (4), we obtain an equation of the envelope. The eliminant is called the **c-discriminant** of the family $f(x, y, c) = 0$.

The c-discriminant may contain loci other than the envelope.

Example 37. The family of parabolas

$$(x-c)^2 - 2y = 0$$

has the x-axis as envelope.

SINGULAR SOLUTIONS

(9.27) Let

$$f(x, y, p) = 0 \tag{1}$$

be a nonlinear first order differential equation in which the left hand member is a polynomial in p. The general solution of this differential equation will be a one-parameter family

$$f(x, y, c) = 0. \tag{2}$$

The envelope E of (2) is a curve which, at each of its points, touches some one member of the family (2). At a point of contact P of the envelope and a member of (2), the values of x, y, p are the same. But the values of x, y, p for the curve at P satisfy (1). Hence the values of x, y, p at every point of the envelope also satisfy (1). Thus the envelope of family (2) is a solution of the differential equation (1).

Example 38. Consider $p^2 - xp + y = 0$.

This is Clairaut's equation and its general solution (replacing simply p by c) is

$$y = cx + c^2. \tag{1}$$

Now we find the envelope of the family (1).

$$f(x, y, c) \equiv y - cx + c^2 = 0 \tag{2}$$

$$f_c \equiv \frac{\partial f}{\partial c} = -x + 2c = 0. \tag{3}$$

Substituting the value of c from (3) into (2), we have

$$y - x\left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)^2 = 0$$

i.e., $$y - \frac{x^2}{2} + \frac{x^2}{4} = 0$$

or $$4y = x^2. \tag{4}$$

(4) is an equation of the envelope of the family (1).

We check whether (4) is a solution of the given differential equation. Differentiating (4) w.r.t. x, we get

$$4\frac{dy}{dx} = 2x \quad \text{or} \quad 2p = x. \tag{5}$$

Substituting from (4) and (5) into $p^2 - px + y$, we have

$$\frac{x^2}{4} - \frac{x^2}{2} + \frac{x^2}{4} = 0$$

Thus (4) is a solution of the given differential equation. This solution does not involve any constant and it cannot be derived from the general solution by giving particular values to c. Such a solution is called a **singular solution (S.S.)**.

(9.28) Definition. A solution of a differential equation $f(x, y, p) = 0$ is called a **singular solution (S.S.)** if

(i) it is not derived from the general solution by giving any particular value to the arbitrary constants

(ii) at each of its points, it is tangent to some member of the one-parameter family of curves represented by the general solution.

We have seen that the envelope, if any, of the one-parameter family of curves represented by the general solution of a differential equation is a singular solution. Thus **the c-discriminant equation may contain singular solution, if any**.

The singular solution may also be obtained from the differential equation directly without finding the general solution. Since at ultimate point of intersection of neighbouring curves the p's for the intersecting curves become equal, and thus the locus of the points where p's have equal roots will include the envelope. If we eliminate p from

$$f(x, y, p) = 0 \quad \text{and} \quad \frac{\partial f}{\partial p} = 0$$

the resulting equation is called the **p-discriminant**. The p-discrminant represents the locus for each point of which $f(x, y, p) = 0$ has equal roots.

If envelope of the general solution of $f(x, y, p) = 0$ exists, it will be contained in the p-discriminant. **Thus the p-discriminant equation may contain (i) singular solution (ii) solutions that are not singular and (iii) such loci which are not solutions at all.**

Example 39. Solve: $xp^2 - 2yp + 4x = 0$ (1)

Solution. $2yp = xp^2 + 4x$ or $2y = xp + \dfrac{4x}{p}$.

Differentiating w.r.t. x, we get

$$2p = x\frac{dp}{dx} + p + \frac{4}{p} - \frac{4x}{p^2}\frac{dp}{dx}$$

or $$p - \frac{4}{p} = \left(x - \frac{4x}{p^2}\right)\frac{dp}{dx}$$

or $$\frac{p^2 - 4}{p} = \frac{x\,(p^2 - 4)}{p^2}\,\frac{dp}{dx}$$

or $$\frac{dp}{p} = \frac{dx}{x}$$

or $$p = cx. \qquad (2)$$

Eliminating p from (1) and (2), we obtain

$$x(c^2x^2) - 2ycx + 4x = 0$$

or $$c^2x^2 - 2yc + 4 = 0 \qquad (3)$$

is the general solution.

From (3), we obtain the c-discriminant as

$$4y^2 - 16x^2 = 0 \quad \text{or} \quad y^2 = 4x^2.$$

The p-discriminant is [from (1)]

$$4y^2 = 16x^2 \quad \text{or} \quad y^2 = 4x^2$$

Since the c-discrminant and p-discriminant are the same viz., $y^2 = 4x^2$ and it satisfies the given differential equation, it is the singular solution.

Example 40. Solve: $x^2p^2 + yp\,(2x + y) + y^2 = 0$ (1)

by making the substitutions $y = u$, $xy = v$ and find the singular solutions.

Solution. $y = u, \qquad x = \dfrac{v}{y} = \dfrac{v}{u}$.

Thus $$\frac{dy}{du} = 1, \qquad \frac{dx}{du} = \frac{u\dfrac{dv}{du} - v}{u^2}$$

$$p = \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{u^2}{u\dfrac{dv}{du} - v}.$$

Substituting into (1), we have

$$\frac{v^2}{u^2}\left[\frac{u^2}{u\dfrac{dv}{du} - v}\right]^2 + (2v + u^2)\frac{u^2}{u\dfrac{dv}{du} - v} + u^2 = 0$$

or $$v^2 + (2v + u^2)\left(u\frac{dv}{du} - v\right) + \left(u\frac{dv}{du} - v\right)^2 = 0$$

or $$v^2 + 2uv\frac{dv}{du} - 2v^2 + u^3\frac{dv}{du} - u^2v + u^2\left(\frac{dv}{du}\right)^2 - 2uv\frac{dv}{du} + v^2 = 0$$

or $$v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2$$

which is a Clairaut's equation. Its solution is

$$v = cu + c^2$$

i.e., $$xy = cy + c^2 \qquad (2)$$

is the general solution of (1).

From (2), the c-discrminant is

$$y^2 + 4xy = 0 \quad \text{i.e.,} \quad y(y+4x) = 0$$

Therefore, $y = 0, \quad y + 4x = 0.$ (3)

Clearly, $y = 0$ satisfies (1) and so it is a singular solution.

We also check whether $y + 4x = 0$ is a solution of (1).

Differentiating (3), we have $p = -4$. Substituting $p = -4$ into the left hand member of (1) and using (3), we obtain

$$16x^2 - 4x(-4)(-2x) + (-4x)^2 = 32x^2 - 32x^2 = 0$$

Thus $y + 4x = 0$ is also a singular solution.

Example 41. Solve

$$(x^2-1)p^2 - 2xyp - x^2 = 0 \qquad (1)$$

and find the singular solution, if any.

Solution. Solving the equation for y, we have

$$2y = \frac{(x^2-1)p^2 - x^2}{xp} = xp - \frac{p}{x} - \frac{x}{p}.$$

Differentiating w.r.t. x, we get

$$2p = xp' + p - \frac{xp' - p}{x^2} - \frac{p - xp'}{p^2}, \text{ where } p' = \frac{dp}{dx}$$

or $$p = p'\left(x - \frac{1}{x} + \frac{x}{p^2}\right) + \frac{p}{x^2} - \frac{1}{p}$$

or $$p\left(1 - \frac{1}{x^2} + \frac{1}{p^2}\right) = xp'\left(1 - \frac{1}{x^2} + \frac{1}{p^2}\right)$$

Therefore, $\dfrac{dp}{p} = \dfrac{dx}{x}$. or $p = cx.$

Substituting this value of p into (1), we get

$$(x^2-1)c^2x^2 - 2x^2yc - x = 0$$

or $$(x^2-1)c^2 - 2yc - 1 = 0 \qquad (2)$$

is the general solution of (1).

From (2), the c-discriminant is

$$4y^2 + 4(x^2-1) = 0$$

i.e., $$x^2 + y^2 = 1$$

which is also envelope of the family (2).

We also note that the p-discriminant is

$$4x^2y^2 + 4x^2(x^2-1) = 0$$

i.e., $$x^2 + y^2 = 1.$$

Since the envelope and the p-discriminant are same, the singular solution is

$$x^2 + y^2 = 1.$$

Example 42. Find the singular solution of

$$x^3p^2 + x^2yp + a^3 = 0 \qquad (1)$$

Solution. The p-discriminant of (1) is

$$x^4y^2 - 4x^3a^3 = 0 \quad \text{or} \quad x^3(y^2x - 4a^3) = 0. \qquad (2)$$

The parts of (2) that satisfy (1) are singular solutions of the given equation. From (2), we have

$$x = 0 \quad \text{or} \quad y^2x - 4a^3 = 0 \qquad (3)$$

We rewrite (1) as

$$x^3 + x^2y\frac{dx}{dy} + a^3\left(\frac{dx}{dy}\right)^2 = 0. \qquad (4)$$

Clearly, $x = 0$ satisfies (4).

Thus $x = 0$ is a singular solution of (1).

Differentiating $y^2x - 4a^3 = 0$ w.r.t. x, we have

$$2xyp + y^2 = 0 \quad \text{or} \quad y(2xp + y) = 0$$

i.e., $$p = -\frac{y}{2x}.$$

Putting this value of p into the left hand member of (1) and using (3), we obtain

$$x\left(-\frac{y}{2x}\right)^2 + x^2y\left(-\frac{y}{2x}\right) + a^3 = -\frac{xy^2}{4} + a^3 = -(xy^2 - 4a^3) = 0$$

Thus $xy^2 - 4a^3 = 0$ and its derivatives satisfy (1) and so it is a solution of (1).

Hence $x = 0$ and $xy^2 - 4a^3 = 0$

are singular solutions of (1).

THE RICATTI EQUATION[1]

(9.29) We have already studied first order linear differential equation

$$y' + P(x)\,y = R(x). \quad (1)$$

If we add the term $Q(x)\,y^2$ to the left hand member of (1), we obtain a nonlinear equation

$$y' + P(x)\,y + Q(x)\,y^2 = R(x) \quad (2)$$

(2) is called the **Ricatti** equation.

In many cases, the solution of (2) cannot be expressed in terms of **elementary functions**. However, the Ricatti equation

$$y' + Py + Qy^2 = R \quad (1)$$

can be reduced to a linear equation by the substitution $y = y_1 + \frac{1}{u}$, where y_1 is a particular solution of (1) and u is an unknown nonzero function of x.

Proof. Let $y = y_1 + \frac{1}{u}$ be as given.

Differentiating w.r.t. x, we have

$$y' = \frac{dy}{dx} = \frac{dy_1}{dx} - \frac{1}{u^2}\frac{du}{dx}.$$

Substituting for y and y' into (1), we get

$$\frac{dy_1}{dx} - \frac{1}{u^2}\frac{du}{dx} + P\left(y_1 + \frac{1}{u}\right) + Q\left(y_1^2 + \frac{2y_1}{u} + \frac{1}{u^2}\right) = R$$

or
$$\frac{dy_1}{dx} + Py_1 + Qy_1^2 - R - \frac{1}{u^2}\left(\frac{du}{dx} - Pu - 2Qy_1u - Q\right) = 0 \quad (2)$$

Since y_1 is a solution of (1), we have

$$\frac{dy_1}{dx} + Py_1 + Qy_1^2 - R = 0$$

and so (2) reduces to

$$\frac{du}{dx} - (P + 2Qy_1)\,u = Q \quad (3)$$

which is a linear equation.

Thus if a particular solution of (1) is known then its general solution can be found.

1. After the name of Italian mathematician Count Jacobo Fransecco Ricatti (1676 – 1754).

Example 43. Solve: $\frac{dy}{dx} - y^2 = -1, \quad y(0) = 3;$

given that $y_1 = 1$ is a particular solution of the given equation.

Solution. Here $P = 0,\ Q = -1,\ R = -1.$

Writing $y = 1 + \frac{1}{u}$, the given equation reduces to

$$\frac{du}{dx} - (0 + 2(-1)(1))\,u = -1$$

i.e.,
$$\frac{du}{dx} + 2u = -1, \quad (1)$$

which is a linear equation.

I.F. of (2) is $e^{\int 2\,dx} = e^{2x}$.

Multiplying (1) by the I.F., we get

$$\frac{du}{dx}e^{2x} + 2ue^{2x} = -e^{2x}$$

or
$$\frac{d}{dx}(ue^{2x}) = -e^{2x}$$

Integrating, we obtain

$$ue^{2x} = -\int e^{2x}\,dx + c = \frac{-e^{2x}}{2} + c$$

or
$$u = -\frac{1}{2} + \frac{c}{e^{2x}} \quad (2)$$

From $y = 1 + \frac{1}{u}$, we get by the initial condition,

$$y(0) = 3 = 1 + \frac{1}{u(0)} \quad \text{or} \quad u(0) = \frac{1}{2}.$$

Hence from (2), we have

$$u(0) = \frac{1}{2} = -\frac{1}{2} + c \quad \text{or} \quad c = 1$$

Thus
$$u = -\frac{1}{2} + \frac{1}{e^{2x}} = \frac{2 - e^{2x}}{2e^{2x}}$$

Required solution is

$$y = 1 + \frac{1}{u} = 1 + \frac{2e^{2x}}{2 - e^{2x}} = \frac{2 + e^{2x}}{2 - e^{2x}}$$

Example 44. Solve $\dfrac{dy}{dx} - \dfrac{y}{x} - x^3y^2 = -x^5$, by finding a particular solution. (1)

Solution. It is a Ractti equation with

$$P = -\frac{1}{x}, \quad Q = -x^3 \quad \text{and} \quad R = -x^5$$

An obvious solution of (1) is $y_1 = x$. Substituting $y = x + \dfrac{1}{u}$ into (1), we have

$$\frac{du}{dx} - \left[-\frac{1}{x} + 2(-x^3)x\right]u = -x^3$$

or
$$\frac{du}{dx} + \left(\frac{1}{x} + 2x^4\right)u = -x^3, \tag{2}$$

which is a linear equation.

Its
$$\text{I.F.} = e^{\int\left(\frac{1}{x}+2x^4\right)dx}$$
$$= e^{\ln x}\, e^{\frac{2}{5}x^5} = xe^{\frac{2}{5}x^5}$$

Multiplying (2) by the I.F. and integrating, we have

$$uxe^{\frac{2}{5}x^5} = -\int x^4 \cdot e^{\frac{2}{5}x^5}dx + c' = -\frac{1}{2}e^{\frac{2}{5}x^5} + c'$$

or
$$u = \frac{c' - \frac{1}{2}e^{\frac{2}{5}x^5}}{xe^{\frac{2}{5}x^5}}$$

Hence the general solution of (1) is

$$y = x + \frac{xe^{\frac{2}{5}x^5}}{c' - \frac{1}{2}e^{\frac{2}{5}x^5}} = \frac{x\left(c' + \frac{1}{2}e^{\frac{2}{5}x^5}\right)}{c' - \frac{1}{2}e^{\frac{2}{5}x^5}}$$

or
$$\frac{y}{x} = \frac{c' + \frac{1}{2}e^{\frac{2}{5}x^5}}{c' - \frac{1}{2}e^{\frac{2}{5}x^5}}$$

or
$$\frac{y-x}{y+x} = \frac{e^{\frac{2}{5}x^5}}{2c'} = ce^{\frac{2}{5}x^5}$$

is the required solution.

EXERCISE 9.9

Solve and find the singular solution, if any. (Problems 1–10)

1. $y = px - p^2$
2. [illegible]
3. $xp^2 - (x-y)p + 1 - y = 0$
4. $4p^2 = 9x$
5. $4xp^2 = (3x-1)^2$
6. $p^2 + 2px^3 - 4x^2y = 0$
7. $6p^2y^2 - 3px - y = 0$
8. [illegible]
9. $xp^4 - 2yp^3 + 12x^3 = 0$
10. [illegible]

Investigate each of the following for singular solutions by finding the p-discriminant (Problems 11–16):

11. [illegible]
12. $4x(x-1)(x-2)p^2 - (3x^2 - 6x + 2)^2 = 0$
13. $p^4 - 2p^2 + 1 - y^4 = 0$
14. $p^3 - 4xyp + 8y^2 = 0$
15. $4p^3 + 3xp - y = 0$
16. $p^3 - 2x^2p - 4xy = 0$

Solve the following Ricatti equations:

17. $\dfrac{dy}{dx} - y - \dfrac{2}{x^3}y^2 = -x^2$, given that $y_1 = x^2$ is a particular solution.
18. $\dfrac{dy}{dx} + \dfrac{3}{x}y - y^2 = \dfrac{1}{x^2}$, given that $y_1 = \dfrac{1}{x}$ is a particular solution.
19. $\dfrac{dy}{dx} - 4y - y^2 = 4$
20. $\dfrac{dy}{dx} = 7 - 6y - y^2$
21. $\dfrac{dy}{dx} + (\cot x)y - y^2 = -\csc^2 x$

Mathematics takes us into the region of absolute necessity, for which not only the actual world but also every possible world must conform.

BERTRAND RUSSEL
(1872-1970 C.E.)
English Philosopher, Mathematician and Author

Mathematics is the standard of objective truth for all intellectual endeavours.

HERMANN WEYL
(1885-1955 C.E.)
German Mathematician

Chapter **10**

DIFFERENTIAL EQUATIONS OF HIGHER ORDER

In Chapter 9, we studied methods of solving special types of linear and nonlinear differential equations of the first order. In this chapter, we shall consider systematic methods for the solution of certain classes of differential ~~equations~~ of order more than one.

$\frac{dy}{dx} + P(x)\,y = Q(x)$

LINEAR DIFFERENTIAL EQUATIONS

(10.1) Definition. A linear differential equation of order n in the dependent variable y and the independent variable x is of the form

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\,y = F(x), \quad (1)$$

(St. form)

where $a_0(x)$, $a_1(x)$, $\cdots$, $a_{n-1}(x)$, $a_n(x)$ and $F(x)$ are functions of the independent variable x only and $a_0(x)$ is not identically zero. Using primes, (1) is also written as

$$a_0(x)\,y^{(n)} + a_1(x)\,y^{(n-1)} + \cdots + a_{n-1}(x)\,y' + a_n(x)\,y = F(x).$$

We shall first study equations of the type

$$a_0\,y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = F(x) \quad (2)$$

where $a_0, a_1, \cdots, a_{n-1}, a_n$ are real constants.

Equation (1) is with variable coefficients while (2) is with constant coefficients.

In order to solve (2), we shall first consider the equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0. \tag{3}$$

The coefficient of $y^{(n)}$ may be made 1 by dividing throughout by a_0. The differential equation (3) is called **homogeneous** linear differential equation of order n. [The use of the word **homogeneous** here is quite different from the one already mentioned in 9.12]. If $F(x)$ is not identically zero then (2) is called **nonhomogeneous** and (3) is called the **associated homogeneous** equation of (2).

(10.2) Definition. If $y_1(x), y_2(x), \cdots, y_m(x)$ are m functions of an independent variable x and $c_1, c_2, \cdots, c_m$ are constants, then the expression

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_m y_m(x)$$

is called a **linear combination** of $y_1(x), y_2(x), \cdots, y_m(x)$. We usually write $y_1, y_2, \cdots, y_m$ instead of $y_1(x), y_2(x), \cdots, y_m(x)$ when it is clear from the context that $y_1, y_2, \cdots, y_m$ are functions of x.

As in vector spaces, the m nonzero functions $y_1, y_2, \cdots, y_m$ are called **linearly dependent** if and only if, there exist constants $c_1, c_2, \cdots, c_m$, at least one of which is nonzero, such that

$$c_1 y_1 + c_2 y_2 + \cdots + c_m y_m = 0.$$

The functions $y_1, y_2, \cdots, y_m$ are called **linear independent** if and only if, they are not linearly dependent, i.e., if and only if

$$c_1 y_1 + c_2 y_2 + \cdots + c_m y_m = 0$$

implies $\quad c_1 = c_2 = \cdots = c_m = 0.$

(10.3) Before investigating a solution of (2) of **(10.1)**, we state the following facts:

(i) Every homogeneous linear nth-order differential equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0 \tag{3}$$

has n linearly independent solutions $y_1, y_2, \cdots, y_n$.

(ii) If $y_1, y_2, \cdots, y_m$ are n linearly independent solutions of (3), then any linear combination

$$y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n \quad \text{of} \quad y_1, y_2, \cdots, y_n$$

is the general solution of (3); $c_1, c_2, \cdots, c_m$ being arbitrary constants.

(iii) Let y_p be any **particular solution** of (2) of **(10.1)**, i.e., y_p does not contain any constant, then $y_c + y_p$ is the **general solution** of (2).

(ii) and (iii) can be easily checked by actual substitutions into (3) and (2) of (10.1). Proof of (i) is beyond the scope of this book.

Thus to find the general solution of (2) of **(10.1)**, we have to find a linearly independent set of n solutions of (3) so as to determine y_c and a particular solution y_p of (2) and then obtain

$$y = y_c + y_p \tag{4}$$

the general solution of (2). In the general solution (4), y_c is called the **Complementary Function (C.F.)** and y_p is called the **Particular Integral (P.I.)** of (2).

These statements are also true for the equation (1) of **(10.1)** with variable coefficients.

HOMOGENEOUS LINEAR EQUATIONS

(10.4) Consider the equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \tag{1}$$

where $a_0, a_1, \cdots, a_{n-1}, a_n$ are real constants. To find a solution of (1) we shall try a judicious guess. The differential equation (1) requires a function y with the property that if y and its successive derivatives are each multiplied by constants a_j, $j = n, n-1, \cdots, 1, 0$, and the resulting products are then added, the sum should equal zero. This can only happen if the function is such that its various derivatives are constant multiples of itself. The exponential function $y = e^{mx}$, m being a constant, has such properties. Here we have

$$\frac{dy}{dx} = me^{mx}, \quad \frac{d^2 y}{dx^2} = m^2 e^{mx}, \quad \cdots, \quad \frac{d^n y}{dx^n} = m^n e^{mx}.$$

Substituting into (1) we have

$$a_0 m^n e^{mx} + a_1 m^{n-1} e^{mx} + \cdots + a_{n-1} m e^{mx} + a_n e^{mx} = 0$$

or

$$e^{mx} (a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n) = 0$$

Since $e^{mx} \neq 0$, we have

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0 \tag{2}$$

Thus, $y = e^{mx}$ is a solution of (1) if and only if m is a solution of (2). Equation (2) is called the **characteristics (or auxiliary) equation** of the given differential equation (1). Observe that (2) can be obtained from (1) by merely replacing the kth derivative in (1) by m^k $(k = n, n-1, \cdots, 1)$. Three cases arise according as the roots of (2) are.

(I) real and distinct

(II) real and repeated

(III) complex.

Case I. Distinct Real Roots

Let $m_1, m_2, \cdots, m_n$

be n distinct real roots of (2). Then $e^{m_1 x}, e^{m_2 x}, \cdots, e^{m_n x}$ are n distinct solutions of (1). These n solutions are linearly independent. Hence the general solution of (2) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x},$$

where $c_1, c_2, \cdots, c_n$ are arbitrary constants.

Case II. Repeated Real Roots

In equation (1), writing $D \equiv \frac{d}{dx}$, we have

$$(a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n) y = 0$$

or $\quad [f(D)] y = 0,$

where $\quad f(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n.$

Note that if we write D for m in the characteristic equation (2) then it is the same as $f(D) = 0$. If $m_1, m_2, \cdots, m_n$ are the roots of $f(D) = 0$, then (1) may be written as

$$(D - m_1)(D - m_2) \cdots (D - m_n) y = 0.$$

Let the root m_1 be repeated twice say $m_2 = m_1$.

The part of the general solution of (1) corresponding to the twice repeated root m_1 of (2) is solution of

$$(D - m_1)^2 y = 0$$

i.e., of $\quad (D - m_1)(D - m_1) y = 0 \qquad (3)$

Let $\quad (D - m_1) y = V.$

Then (3) becomes

$$(D - m_1) V = 0$$

or $\quad \frac{dV}{dx} - m_1 V = 0. \qquad (4)$

Separating the variables, we have

$$\frac{dV}{V} = m_1 \, dx.$$

Therefore, $\quad \ln V = m_1 x + k, \quad$ where k is a constant

or $\quad V = c_2 e^{m_1 x}, \quad$ where c_2 is a constant.

Replacing V in (4), we obtain

$$(D - m_1) y = c_2 e^{m_1 x}$$

or $\quad \frac{dy}{dx} - m_1 y = c_2 e^{m_1 x} \qquad (5)$

which is a linear equation of order one.

Its $\quad$ I.F. $= \exp\left(\int (-m_1)\, dx\right) = e^{-m_1 x}$

Multiplying (5) by $e^{-m_1 x}$, we get

$$\frac{d}{dx}\left(y e^{-m_1 x}\right) = c_2$$

or $\quad y e^{-m_1 x} = c_2 x + c_1$

or $\quad y = (c_1 + c_2 x) e^{m_1 x}$

is the part of the general solution corresponding to the twice repeated root m_1. The general solution of (1) is

$$y = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

In the same manner, if the characteristic equation (2) has the triple repeated root m_1, the corresponding part of the general solution of (1) is the solution of

$$(D - m_1)^3 y = 0.$$

Proceeding as before, we can easily find

$$y = (c_1 + c_2 x + c_3 x^2) e^{m_1 x}$$

as the part of the general solution corresponding to this triple repeated root m_1.

If the characteristic equation (2) has the real root m_1 occurring k times then the part of the general solution of (1) corresponding to the k-fold repeated root m_1 is

$$y = (c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) e^{m_1 x}.$$

Case III. Complex Roots Real part of

Suppose the characteristic equation has the complex number $a + ib$ as a non-repeated root. Since coefficients of (1) are real, the conjugate complex number $a - ib$ is also a non-repeated root. The corresponding part of the general solution is

$$y = k_1 e^{(a+ib)x} + k_2 e^{(a-ib)x}, \text{ where } k_1 \text{ and } k_2 \text{ are arbitrary constants}$$

i.e.,

$$\begin{aligned} y &= e^{ax}\left[k_1 e^{ibx} + k_2 e^{-ibx})\right] \\ &= e^{ax}\left[k_1(\cos bx + i\sin bx) + k_2(\cos bx - i\sin bx)\right] \\ &= e^{ax}\left[(k_1 + k_2)\cos bx + i(k_1 - k_2)\sin bx\right] \\ &= e^{ax}\left[c_1 \sin bx + c_2 \cos bx\right] \end{aligned}$$

where $c_1 = i(k_1 - k_2)$, $c_2 = k_1 + k_2$ are two arbitray constants.

If $a + ib$ and $a - ib$ are conjugate complex roots, each repeated k times, then the corresponding part of the general solution of (1) may be written as

$$y = \left[(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1})\sin bx + (c_{k+1} + c_{k+2} x + \cdots + c_{2k} x^{k-1})\cos bx\right]$$

In the examples given below we use these concepts to solve homogeneous linear differential equations.

Example 1. Solve: $(D^2 + 4D + 3)y = 0$

Solution. The characteristic equation is

$$D^2 + 4D + 3 = 0$$

with roots $D = -1, -3$.

Hence the general solution of the given equation is

$$y = c_1 e^{-x} + c_2 e^{-3x}.$$

Synthetic division

$$\begin{array}{c|cccc} 1 & 1 & -5 & 7 & -3 \\ & & 1 & -4 & +3 \\ \hline & 1 & -4 & +3 & 0 \end{array}$$

$D-1=0$

$D^2 - 4D + 3 = 0$

Example 2. Solve: $(D^3 - 5D^2 + 7D - 3)y = 0$

Solution. The characteristic equation is

$$D^3 - 5D^2 + 7D - 3 = 0. \quad (1)$$

By inspection, $D = 1$ is a solution of this equation. The other two roots can be found from the quadratic factor of

$$(D - 1)(D^2 - 4D + 3) = 0.$$

Hence all roots of (1) are $D = 1, 1, 3$.

The general solution is

$$y = (c_1 + c_2 x)e^x + c_3 e^{3x}$$

Example 3. Solve: $(D^3 - D^2 + D - 1)y = 0$

Solution. The characteristic equation is

$$D^3 - D^2 + D - 1 = 0$$

By inspection, $D = 1$ is a root of this equation.

Now,

$$D^3 - D^2 + D - 1 = (D - 1)(D^2 + 1) = 0$$

Hence the other two roots are

$$D = \pm i = a + ib \text{ with } a = 0, b = 1$$

The general solution is

$$y = c_1 e^x + e^{0x}(c_2 \sin x + c_3 \cos x)$$

or

$$y = c_1 e^x + c_2 \sin x + c_3 \cos x$$

Example 4. Solve: $(D^2 + D - 12)y = 0$, where $y(2) = 2$, $y'(2) = 0$

Solution. The characteristic equation is

$$D^2 + D - 12 = 0$$

with roots $D = 3, -4$.

Hence the general solution of the given equation is

$$y = c_1 e^{3x} + c_2 e^{-4x} \quad (1)$$

We now determine the coefficients (constants) c_1, c_2 in (1) from the initial conditions as follows.

Since the initial conditions are given at $x = 2$, we rewrite the general solution in the form

$$y = k_1 e^{3(x-2)} + k_2 e^{-4(x-2)} \quad (2)$$

where $k_1 = c_1 e^6$, $k_2 = c_2 e^{-8}$.

Applying initial condition, (i.e. replacing c_1, c_2 by the new constants $k_1 e^6$, $k_2 e^{-8}$ respectively in (1), we get

$$y(2) = 2 = k_1 + k_2 \quad (3)$$

Differentiating (2) w.r.t. x, we have

$$y' = \frac{dy}{dx} = 3k_1 e^{3(x-2)} - 4k_2 e^{-4(x-2)}$$

Therefore, $y'(2) = 0 = 3k_1 - 4k_2$. (4)

Solving (3) and (4), we obtain

$$k_1 = \frac{8}{7}, \quad k_2 = \frac{6}{7}$$

Hence the solution (2) of the differential equation satisfying the given conditions is

$$y = \frac{8}{7}\, e^{3(x-2)} + \frac{6}{7}\, e^{-4(x-2)}$$

Example 5. Solve $(D^2 + 4D + 5) = 0, \quad y(0) = 1, \quad y'(0) = 0.$

Solution. The characteristic equation is

$$D^2 + 4D + 5 = 0$$

which has roots $D = -2 \pm i$.

The general solution is

$$y = e^{-2x}(c_1 \sin x + c_2 \cos x) \qquad (1)$$

Applying the given conditions, we have, from (1)

$$y(0) = 1 = c_2.$$

Differentiating (1) w.r.t. x, we get

$$y' = -2e^{-2x}(c_1 \sin x + c_2 \cos x) + e^{-2x}(c_1 \cos x - c_2 \sin x)$$

Therefore, $y'(0) = 0 = -2c_2 + c_1$, giving $c_1 = 2$.

Substituting the values of c_1 and c_2 into (1), the required solution is

$$y = e^{-2x}(2 \sin x + \cos x).$$

Example 6. Solve $(D^3 - 3D^2 + 4)\, y = 0.$

$$y(0) = 1, \; y'(0) = -8, \; y''(0) = -4.$$

Solution. The characteristic equation is

$$D^3 - 3D^2 + 4 = 0$$

or $\quad (D + 1)(D^2 - 4D + 4) = 0.$

Therefore, $\quad D = -1, 2, 2.$

The general solution is

$$y = c_1 e^{-x} + e^{2x}(c_2 + c_3 x) \qquad (1)$$

Now, from (1) $\quad y' = -c_1 e^{-x} + 2e^{2x}(c_2 + c_3 x) + c_3 e^{2x}$

and $\quad y'' = c_1 e^{-x} + 4e^{2x}(c_2 + c_3 x) + 4c_3 e^{2x}$

Applying initial conditions, we get from the above three equation

$$y(0) = 1 = c_1 + c_2 \qquad (2)$$

$$y'(0) = -8 = -c_1 + 2c_2 + c_3 \qquad (3)$$

$$y''(0) = -4 = c_1 + 4c_2 + 4c_3 \qquad (4)$$

Multiplying (3) by -4 and adding to (4), we have

$$28 = 5c_1 - 4c_2 \qquad (5)$$

Multiplying (2) by 4 and adding to (5), we obtain

$$9c_1 = 32 \quad \text{or} \quad c_1 = \frac{32}{9}$$

Therefore, $\quad c_2 = 1 - \frac{32}{9} = -\frac{23}{9}$ and $c_3 = \frac{6}{9}$

The required solution is

$$y = \frac{32}{9}\, e^{-x} + e^{2x}\left(\frac{-23}{9} + \frac{6}{9}x\right) = \frac{1}{9}\left(32\, e^{-x} - 23\, e^{2x} + 6xe^{2x}\right)$$

complete

EXERCISE 10.1

Solve: (Problems 1 – 15)

1. $(9D^2 - 12D + 4)\, y = 0$
2. $(75D^2 + 50D + 12)\, y = 0$
3. $(D^3 - 4D^2 + D + 6)\, y = 0$
4. $(D^3 + D^2 + D + 1)\, y = 0$
5. $(D^3 - 6D^2 + 12D - 8)\, y = 0$
6. $(D^3 - 6D^2 + 3D + 10)\, y = 0$
7. $(D^3 - 27)\, y = 0$
8. $(4D^4 - 4D^3 - 3D^2 + 4D - 1)\, y = 0$
9. $(D^4 + 2D^3 - 2D^2 - 6D + 5)\, y = 0$
10. $(D^4 - 5D^3 + 6D^2 + 4D - 8)\, y = 0$
11. $(D^4 - 4D^3 - 7D^2 + 22D + 24)\, y = 0$
12. $(D^4 + 4)\, y = 0$
13. $(D^4 - D^3 - 3D^2 + D + 2)\, y = 0$
14. $(16D^6 + 8D^4 + D^2)\, y = 0$
15. $(D^4 + 6D^3 + 15D^2 + 20D + 12)\, y = 0$

Solve the following equations with the given conditions:

16. $(D^2 + 8D - 9)\, y = 0;$ $\quad y(1) = 1, \; y'(1) = 0$
17. $(D^2 + 6D + 9)\, y = 0;$ $\quad y(0) = 2, \; y'(0) = -3$
18. $(D^2 + 6D + 13)\, y = 0;$ $\quad y(0) = 3, \; y'(0) = -1$
19. $(D^3 - 6D^2 + 11D - 6)\, y = 0;$ $\quad y(0) = 0 = y'(0), \; y''(0) = 2$
20. $(D^4 - D^3)\, y = 0;$ $\quad y(0) = y'(0) = 1, \; y''(1) = 3e, \; y'''(1) = e$

DIFFERENTIAL OPERATORS

(10.5) Let $T : V \to V$, where V is a vector space over a field F, be a **linear transformation** (or an **operator**) from V to V.

Linear transformations S, T, U defined on V have the following properties.

(i) $S + T = T + S$ where $(S + T)\,v = S(v) + T(v)$, $v \in V$,

(ii) $(S + T) + U = S + (T + U)$.

(iii) $S(T + U) = ST + SU$

(iv) $(ST)\,U = S(TU)$

(v) $ST \neq TS$ in general.

Here $ST : V \to V$ is a linear transformation defined by:

$$(ST)(v) = S(Tv) \quad \text{for all } v \in V.$$

Let X be the vector space of all real (or complex) valued real (or complex) functions possessing a kth order derivative for every $k = 1, 2, 3, \cdots$. For each $g \in X$, $y = g(x) \in R$ (or $\in C$). D^k which associates with each $y = g(x)$, its kth derivative, is a linear operator on X. We write $D^k y$ as $\dfrac{d^k y}{d^k x}$. Since the sum and product of the linear operators and scalar multiple of a linear operator are also linear operators, the linear combination

$$f(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n I \tag{1}$$

of $D^n, D^{n-1}, \cdots, D, I$ is also a linear operator. Here, I is the identity linear operator defined by $I(y) = y$ for all $y = g(x)$ and $a_0, a_1, \cdots, a_n$ are scalars. The image of a $y = g(x)$ under $f(D)$ is written as

$$\begin{aligned} f(D)y &= a_0 D^n y + a_1 D^{n-1} y + \cdots + a_{n-1} Dy + a_n y \\ &= a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y. \end{aligned}$$

The equation $\quad f(D)y = 0$

is then a **linear differential equation of order n.**

The operators $D, D^2, \cdots, D^n$ and $f(D)$, given by (1), are called **differential operators**.

We have written D for $\dfrac{d}{dx}$, D^2 for $\dfrac{d^2}{dx^2}$ and so on D^n for $\dfrac{d^n}{dx^n}$. Thus

$$Dy = \frac{dy}{dx}, \quad D^2 y = \frac{d^2 y}{dx^2}, \quad \text{and } D^n y = \frac{d^n y}{dx^n}.$$

The expression

$$A = f(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n I$$

is called **a differential operator of order n.** This may be thought of as an operator which, when applied to any function y, results in

$$Ay = f(D)\,y = a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n Iy.$$

NONHOMOGENEOUS LINEAR EQUATIONS

In this section we discuss the solution of nonhomogeneous linear equation (2) defined in (10.1).

(10.6) Solution of the equation

$$f(D)\,y = F(x) \tag{1}$$

where $\quad f(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n.$

We have already remarked in (10.3) that the general solution of (1) consists of two parts, namely

(i) Complementary Function (C.F.)

(ii) Particular Integral (P.I.)

The C.F. is the solution of the homogeneous equation $f(D)\,y = 0$ and we have described different cases of its solution in (10.4).

To find the P.I. of (1), we write

$$y = \frac{1}{f(D)} F(x)$$

and try to evaluate $y = \dfrac{1}{f(D)} F(x)$.

A real number m is said to be a zero of $f(D)$ if $f(m) = 0$.

Suppose $f(D)$ has n distinct zeros $m_1, m_2, \cdots, m_n$.

Then $\quad \dfrac{1}{f(D)} F(x) = \dfrac{1}{(D - m_1)(D - m_2) \ldots (D - m_n)} F(x).$

Suppose $\quad \dfrac{1}{D-m_n}\,F(x) = y^*.$

Then $\quad (D-m_n)\,y^* = F(x)$

or $\quad \dfrac{dy}{dx} - m_n y^* = F(x)$

which is a linear equation of the first order. Its

$$\text{I.F.} = \exp\left(\int(-m_n)\right)dx = \exp(-m_n x).$$

Hence $\quad \dfrac{d}{dx}\left(ye^{-m_n x}\right) = F(x)\,e^{-m_n x}$

or $\quad y^* = e^{m_n x}\int F(x)\,e^{-m_n x}\,dx,$

omitting the constant of integration, since we are looking for a particular solution.

Therefore,

$$\frac{1}{f(D)}\,F(x) = \frac{1}{(D-m_1)\ldots(D-m_{n-1})}\,e^{m_n x}\int e^{-m_n x}F(x)\,dx.$$

Next we evaluate $\quad \dfrac{1}{D-m_{n-1}}\,e^{m_n x}\int e^{-m_n x}F(x)\,dx$

as before and continue the process. This is the required P.I. of the equation (1).

Alternatively, we may also write

$$\begin{aligned}\frac{1}{f(D)}\,F(x) &= \frac{1}{(D-m_1)(D-m_2)\ldots(D-m_n)}\,F(x)\\ &= \left[\frac{A_1}{D-m_1}+\frac{A_2}{D-m_2}+\ldots+\frac{A_n}{D-m_n}\right]F(x)\end{aligned}$$

after resolving $\dfrac{1}{f(D)}$ into partial fractions

$$\begin{aligned}&= A_1e^{m_1x}\int e^{-m_1x}F(x)\,dx + A_2e^{m_2x}\int e^{-m_2x}F(x)\,dx + \cdots\\ &\qquad + A_ne^{m_nx}\int e^{-m_nx}F(x)\,dx\end{aligned}$$

which is the required P.I. of the equation (1).

Note: Mostly we follow the alternative method.

Example 7. Solve $\quad (D^3-D)y = e^x$

Solution. The characteristic equation is

$$D^3 - D = 0 \text{ with roots } 0, 1, -1$$

$$\text{C.F.} = y_c = c_1 + c_2' e^x + c_3 e^{-x}$$

$$\text{P.I.} = y_p = \frac{e^x}{D(D+1)(D-1)}$$

Now $\quad \dfrac{1}{D-1}\,e^x = e^x\int e^{-x}\,e^x\,dx = xe^x$, by the method described in (**10.6**)

Thus $\quad \dfrac{1}{D(D+1)(D-1)}\,e^x = \dfrac{1}{D(D+1)}\,xe^x$

Again,

$$\frac{1}{D+1}\,(xe^x) = e^{-x}\int e^x\,xe^x\,dx = \frac{xe^x}{2} - \frac{e^x}{4}$$

Hence

$$\begin{aligned}\frac{1}{D(D+1)}(xe^x) &= \frac{1}{D}\left(\frac{xe^x}{2}-\frac{e^x}{4}\right) = \int\frac{xe^x}{2}\,dx - \int\frac{e^x}{4}\,dx\\ &= \frac{xe^x}{2} - \frac{3}{4}\,e^x\end{aligned}$$

Therefore, the general solution of the equation is

$$\begin{aligned}y = y_c + y_p &= c_1 + c_2'\,e^x + c_3e^{-x} + \frac{xe^x}{2} - \frac{3}{4}\,e^x\\ &= c_1 + \left(c_2' - \frac{3}{4}\right)e^x + c_3e^{-x} + \frac{x\,e^x}{2}\\ &= c_1 + c_2\,e^x + c_3\,e^{-x} + \frac{x\,e^x}{2}\end{aligned}$$

Alternative Method, Consider $f(D)^{-1}$ as the inverse of the operator $f(D)$, so that

$$\begin{aligned}y_p &= \frac{1}{D(D+1)(D-1)}\,e^x\\ &= \left(-\frac{1}{D}+\frac{1}{2}\,\frac{1}{1+D}+\frac{1}{2}\,\frac{1}{D-1}\right)e^x\\ &= -\int e^x\,dx + \frac{1}{2}\,e^{-x}\int e^x\,.\,e^x\,dx + \frac{1}{2}\,e^x\int e^{-x}e^x\,dx\\ &= -e^x + \frac{1}{4}\,e^x + \frac{1}{2}\,xe^x\\ &= \frac{xe^x}{2}\end{aligned}$$

, since e^x already occurs in the C.F. and so the terms involving e^x are omitted

WORKING RULES FOR FINDING P.I.

The method of evaluating $\frac{1}{f(D)} F(x)$ explained in (10.6) involves successive integrations and is quite unwieldy. We list below short methods for finding the P.I. when $F(x)$ is function of a particular type.

(10.7) To Evaluate $\frac{1}{f(D)} e^{ax}$, where

$$f(D) = a_0D^n + a_1D^{n-1} + \cdots + a_{n-1} D + a_n$$

We know that

$$D^k e^{ax} = a^k e^{ax}, \text{ where } k \text{ is a positive integer.}$$

$$\begin{aligned} \text{Hence } f(D) e^{ax} &= (a_0D^n + a_1D^{n-1} + \cdots + a_{n-1} D + a_n) e^{ax} \\ &= a_0D^n e^{ax} + a_1D^{n-1} e^{ax} + \cdots + a_{n-1} De^{ax} + a_n e^{ax} \\ &= a_0a^n e^{ax} + a_1a^{n-1} e^{ax} + \cdots + a_{n-1} a e^{ax} + a_n e^{ax} \\ &= f(a) e^{ax}. \end{aligned}$$

Applying $\frac{1}{f(D)}$ to both sides of (1), we obtain

$$\frac{f(D)}{f(D)} e^{ax} = \frac{1}{f(D)} f(a) e^{ax}$$

or
$$e^{ax} = f(a)\frac{1}{f(D)} e^{ax}$$

i.e.,
$$\frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}, \qquad \text{if } f(a) \neq 0$$

If $f(a) = 0$, then a is a root of $f(D) = 0$.

Let a be a k-fold root of $f(D) = 0$ so that $(D-a)^k$ is a factor of $f(D)$. Then

$$f(D) = (D-a)^k \phi(D), \text{ where } \phi(a) \neq 0.$$

We first check the effect of $(D-a)^k$ on $e^{ax} p(x)$, for a polynomial $p(x)$ in x.

We have
$$(D-a)(e^{ax} p(x)) = D(e^{ax} p(x)) - a e^{ax} p(x) = e^{ax} D p(x)$$
$$(D-a)^2(e^{ax} p(x)) = (D-a)(e^{ax} Dp(x)) = e^{ax} D^2p(x).$$

Continuing in this way, we get

$$(D-a)^k(e^{ax} p(x)) = e^{ax} D^k p(x) \qquad (3)$$

Setting $p(x) = x^k$ in (3), we are led to

$$(D-a)^k (x^k e^{ax}) = e^{ax} D^k x^k = k!\, e^{ax}$$

or
$$\phi(D) (D-a)^k (x^k e^{ax}) = \phi(D)\, k!\, e^{ax}$$
$$= k!\, \phi(D) e^{ax} = k!\, \phi(a) e^{ax}, \quad \text{by (1)}$$

Operating on both sides by $\frac{1}{\phi(D)(D-a)^k}$, we obtain

$$x^k e^{ax} = \frac{1}{\phi(D)(D-a)^k} k!\, \phi(a) e^{ax}$$

or
$$\frac{1}{\phi(D)(D-a)^k} e^{ax} = \frac{x^k e^{ax}}{k!\, \phi(a)}. \qquad (5)$$

(10.8) Principle of Superposition. Let

$$f(D) y = (a_0D^n + a_1D^{n-1} + \cdots + a_{n-1} D + a_n) y = F(x) \qquad (1)$$

be a linear differential equation of order n. If $F(x) = F_1(x) + F_2(x)$, then particular integral of (1) is the sum of particular integrals of

$$F(D) y = F_1(x) \qquad (2)$$

and
$$F(D) y = F_2(x). \qquad (3)$$

Proof. Let y_1 and y_2 be particular integrals of (2) and (3) respectively.

Then
$$f(D) y_1 = F_1(x)$$

and
$$f(D) y_2 = F_2(x)$$

Suppose $y = y_1 + y_2$. Setting this value of y into (1), we obtain

$$\begin{aligned} f(D) y &= f(D)(y_1 + y_2) \\ &= f(D) y_1 + f(D) y_2 \\ &= F_1(x) + F_2(x) = F(x) \end{aligned}$$

showing that y is a solution of (1).

Example 8. Solve: $(D^3 + 1) y = 1 + e^{-x} + e^{2x}$.

Solution. The characteristic equation is

$$D^3 + 1 = 0 \text{ with roots } -1, \frac{1 \pm i\sqrt{3}}{2}.$$

Therefore, C.F. is

$$y_c = c_1e^{-x} + e^{x/2}\left(c_2 \sin\frac{\sqrt{3}}{2}x + c_3 \cos\frac{\sqrt{3}}{2}x\right).$$

The P.I. is given by

$$y_p = \frac{1}{D^3+1}(1+e^{-x}+e^{2x})$$

$$= \frac{1}{D^3+1}e^{0x} + \frac{1}{D^3+1}e^{2x} + \frac{1}{(D+1)(D^2-D+1)}e^{-x}$$

$$= 1+\frac{e^{2x}}{9} + \frac{1}{3(D+1)}e^{-x}, \quad \text{using } \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax},$$

$$\text{for } \frac{1}{D^2-D+1}e^{-x} \text{ with } D=-1 \quad \text{when } f(-1)\neq 0$$

$$= 1+\frac{e^{2x}}{9} + \frac{1}{3}xe^{-x}, \text{ by using (4) of (10.7).}$$

Hence the general solution of the equation is

$$y = y_c + y_p$$

$$= c_1e^{-x} + e^{x/2}\left(c_2 \sin\frac{\sqrt{3}}{2}x + c_3 \cos\frac{\sqrt{3}}{2}x\right) + 1 + \frac{e^{2x}}{9} + \frac{1}{3}xe^{-x}.$$

(10.9) To find the P.I. when $F(x) = \sin ax$ or $\cos ax$.

Here we have to evaluate

$$\frac{1}{f(D)}\sin ax \quad \text{and} \quad \frac{1}{f(D)}\cos ax.$$

From Euler's Theorem, we have

$$e^{iax} = \cos ax + i\sin ax$$

Thus $\frac{1}{f(D)}\sin ax$ and $\frac{1}{f(D)}\cos ax$ are respectively imaginary and real parts of $\frac{1}{f(D)}e^{iax}$.

If $f(D)$ contains only even powers of D, say $f(D) = f(D^2)$, it is easy to see that

$$\frac{1}{f(D^2)}\begin{cases}\sin ax\\ \cos ax\end{cases} = \frac{1}{f(-a^2)}\begin{cases}\sin ax\\ \cos ax\end{cases}, \quad \text{provided } -a^2 \text{ is not a zero of } f(D^2).$$

Example 9. Solve: $(D^2-5D+6)y = \sin 3x$

Solution. The roots of the characteristic equation

$$D^2-5D+6 = 0 \text{ are 2 and 3.}$$

$$\text{C.F.} = y_c = c_1e^{2x}+c_2e^{3x}$$

$$\text{P.I.} = y_p = \frac{1}{D^2-5D+6}\sin 3x$$

which is imaginary part of $\frac{1}{(D-2)(D-3)}e^{3ix}$

Now $\frac{1}{(D-2)(D-3)}e^{3ix} = \frac{e^{3ix}}{(3i-2)(3i-3)}$, by using $\frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)}$ and $f(a)\neq 0$

$$= \frac{-1+5i}{78}e^{3ix}$$

$$= \left(-\frac{1}{78}+\frac{5}{78}i\right)(\cos 3x + i\sin 3x)$$

Its imaginary part is $-\frac{1}{78}\sin 3x + \frac{5}{78}\cos 3x = y_p$.

Hence the required general solution is

$$y = c_1e^{2x}+c_2e^{3x} - \frac{1}{78}\sin 3x + \frac{5}{78}\cos 3x.$$

(10.10) Theorem. If a is not a zero of $f(D)$, then

$$\frac{1}{f(D-a)}e^{ax}F(x) = e^{ax}\frac{1}{f(D)}F(x).$$

This replacing of D by $D+a$ is known as **exponential shift**.

Proof. We have already shown in (10.7) that

$$(D-a)^k(e^{ax}p(x)) = e^{ax}D^kp(x)$$

Using linearity of differential operators, we conclude that, when $f(D)$ is a polynomial (with constant coefficients), then

$$f(D-a)(e^{ax}p(x)) = e^{ax}f(D)p(x) \qquad (1)$$

Suppose $f(D)p(x) = F(x)$;

Then $$p(x) = \frac{1}{f(D)}F(x). \qquad (2)$$

From (1) and (2), we have

$$f(D-a)\, e^{ax} \frac{1}{f(D)} F(x) = e^{ax} F(x)$$

Operating on both sides by $\frac{1}{f(D-a)}$, we get

$$e^{ax} \frac{1}{f(D)} F(x) = \frac{1}{f(D-a)} e^{ax} F(x)$$

or
$$\frac{1}{f(D-a)} e^{ax} F(x) = e^{ax} \frac{1}{f(D)} F(x),$$

provided that a is not a zero of $f(D)$.

Example 10. Solve: $(D^3 + D^2 - 4D - 4)\, y = e^{2x} \cos 3x$.

Solution. The characteristic equation is

$$D^3 + D^2 - 4D - 4 = 0.$$

By inspection $D = 2$ is a root of this equation. The other roots are $D = -2, -1$

$$\text{C.F.} = y_c = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{2x}$$

$$\text{P.I.} = y_p = \frac{1}{(D-2)(D+2)(D+1)} e^{2x} \cos 3x$$

$$= e^{2x} \frac{1}{D(D+4)(D+3)} \cos 3x, \text{ by the exponential shift}$$

$$= e^{2x} \frac{1}{(D^3 + 7D^2 + 12D)} \cos 3x$$

$$= e^{2x} \frac{1}{-9D - 63 + 12D} \cos 3x, \qquad \text{putting } D^2 = -3^2$$

$$= e^{2x} \frac{1}{3(D-21)} \cos 3x$$

$$= e^{2x} \frac{(D+21)(\cos 3x)}{3(D^2 - 441)}$$

$$= e^{2x} \frac{(D+21)}{3(-9-441)} \cos 3x$$

$$= e^{2x} \frac{-1}{3 \times 450} [-3 \sin 3x + 21 \cos 3x]$$

$$= \frac{e^{2x}}{450} (\sin 3x - 7 \cos 3x).$$

Alternative Method. Here, for $\frac{1}{D(D+4)(D+3)} \cos 3x$, we proceed as follow

$$\frac{\cos 3x}{(D^3 + 7D^2 + 12D)} = \text{Re} \frac{1}{(D^3 + 7D^2 + 12D)} e^{3ix}, \text{ by (10.9)}$$

$$= \text{Re} \frac{e^{3ix}}{(-27i - 63 + 36i)}, \text{ by (2) of (10.7)}$$

$$= \text{Re} \frac{e^{3ix}}{9(i-7)}$$

$$= \text{Re} \frac{(i+7)\, e^{3ix}}{-450}$$

$$= \text{Re} \frac{(i+7)(\cos 3x + i \sin 3x)}{-450}$$

$$= -\frac{7}{450} \cos 3x + \frac{1}{450} \sin 3x$$

Therefore,

$$y_p = \frac{e^{2x}}{450} (\sin 3x - 7 \cos 3x) \quad \text{as before}$$

The general solution is

$$y = y_c + y_p$$

$$= c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{2x} + \frac{e^{2x}}{450} (\sin 3x - 7 \cos 3x).$$

(10.11) Series Expansion of Polynomials or Expressions involving Operators.

This method is useful when $F(x)$ is a polynomial in x.

Let $f(D)\, y = F(x)$ be such that $F(x)$ is a polynomial in x. To evaluate the particular integral

$$y_p = \frac{1}{f(D)} F(x)$$

It is often useful [illegible] in a series in D by the Binomial Theorem for negative exponent [illegible]

[illegible] $\cdots) F(x)$

The derivatives on the right will vanish after certain stage since

$$D^n x^r = 0 \quad \text{if} \quad n > r.$$

The following binomial expansions will be useful in this connection:

(i) $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

(ii) $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$

Example 11. Solve $(D^3 - 2D + 1)y = 2x^3 - 3x^2 + 4x + 5 \quad (= p(x))$

Solution. The characteristic equation is

$$D^3 - 2D + 1 = 0$$

or $$(D-1)(D^2 + D - 1) = 0$$

$$D = 1, \frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}$$

$$y_c = c_1 e^x + c_2 e^{\frac{-1+\sqrt{5}}{2}x} + c_3 e^{\frac{-1-\sqrt{5}}{2}x}$$

Next,

$$y_p = \frac{1}{1 - 2D + D^3}(2x^3 - 3x^2 + 4x + 5)$$

Now $$\frac{1}{1-2D+D^3} = [1 - (2D - D^3)]^{-1}$$

$$= 1 + (2D - D^3) + (2D - D^3)^2 + (2D - D^3)^3 + \cdots$$

$$= 1 + 2D - D^3 + 4D^2 + 8D^3,$$

neglecting D^4 and higher powers of D in view of the degree 3 of the polynomial $p(x)$ in x. Therefore

$$y_p = (1 + 2D + 4D^2 + 7D^3)(2x^3 - 3x^2 + 4x + 5)$$

$$= 2x^3 - 3x^2 + 4x + 5 + 2(6x^2 - 6x + 4) + 4(12x - 6) + 7(12)$$

$$= 2x^3 + 9x^2 + 40x + 73.$$

The general solution is

$$y = y_c + y_p$$

$$= c_1 e^x + c_2 e^{\frac{-1+\sqrt{5}}{2}x} + c_3 e^{\frac{-1-\sqrt{5}}{2}x} + 2x^3 + 9x^2 + 40x + 73.$$

EXERCISE 10.2

Find the general solution of each of the following (Problems 1-15)

1. $(D^2 + 3D - 4)y = 15e^x$
2. $(D^2 - 3D + 2)y = e^x + e^{2x}$
3. $(D^2 - 2D - 3)y = 2e^x - 10\sin x$
4. $(D^4 - 2D^3 + D^2)y = x^3 + 3x + 1$
5. $(D^3 - D^2 + D - 1)y = 4\sin x$
6. $(D^3 - 2D^2 - 3D + 10)y = 40\cos x$
7. $(D^2 + 4)y = 4\sin^2 x$ 2(1-Cos2x)
8. $(D^3 + D)y = 2x^2 + 4\sin x$
9. $(D^4 + D^2)y = 3x^2 + 6\sin x - 2\cos x$
10. $(D^2 - 2D + 4)y = e^x\cos x$
11. $(D^3 - D^2 + 3D + 5)y = e^x\sin 2x$
12. $(D^3 - 7D - 6)y = e^{2x}(1 + x)$
13. $(D^2 - 7D + 12)y = e^{2x}(x^3 - 5x^2)$
14. $(D^4 + 8D^2 - 9)y = 9x^3 + 5\cos 2x$
15. $(D^3 + 3D^2 - 4)y = \sinh x - \cos^2 x.$

Solve the initial value problems:

16. $y'' - 8y' + 15y = 9xe^{2x}$, $\quad y(0) = 5,\ y'(0) = 10$
17. $y'' - 4y' + 13y = 8\sin 3x$, $\quad y(0) = 1,\ y'(0) = 2$
18. $y'' - 4y = 2 - 8x$, $\quad y(0) = 0,\ y'(0) = 5$
19. $y'' + y = x\sin x$, $\quad y(0) = 1,\ y'(0) = 2$
20. $y''' + 3y'' + 7y' + 5y = 16e^{-x}\cos 2x$, $\quad y(0) = 2,\ y'(0) = -4,\ y''(0) = -2$

THE METHOD OF UNDETERMINED COEFFICIENTS (U.C.)

(10.12) We have studied some special cases in which particular integral can be evaluated by the inverse operator. Now we consider the method of undetermined coefficients which can prove simpler in finding the particular integral of the equation $f(D)y = F(x)$ when $F(x)$ is

(i) an exponential function (e^{ax})

(ii) a polynomial $(b_0x^n + b_1x^{n-1} + \cdots + b_n)$

(iii) sinusoidal function ($\sin ax$ or $\cos ax$)

(iv) the more general case in which $F(x)$ is sum of a product of terms of the above types, such as

$$F(x) = e^{ax}(b_0x^n + b_1x^{n-1} + \cdots + b_n)\begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix}.$$

The P.I. y_p will be constructed according to the following table:

$F(x)$ is of the form	Take y_p as
1. a	$A x^k$
2. $a x^n$ (n is a +ive integer)	$x^k (A_0 x^n + A_1 x^{n-1} + \cdots + A_{n-1} x + A_n)$
3. $a x^n e^{rx}$ (n is a +ive integer)	$x^k (A_0 x^n + A_1 x^{n-1} + \cdots + A_{n-1} x + A_n) e^{rx}$
4. $C x^n \cos ax$ 5. $C x^n \sin ax$	$x^k (A \cos \alpha x + B \sin \alpha x)$
6. $C x^n e^{rx} \cos \alpha x$ 7. $C x^n e^{rx} \sin \alpha x$	$x^k [(A_0 x^n + A_1 x^{n-1} + \cdots + A_{n-1} x + A_n) e^{rx} \cos \alpha x$ $+ (B_0 x^n + B_1 x^{n-1} + \cdots + B_{n-1} x + B_n) e^{rx} \sin \alpha x)]$

In x^k, k is the smallest nonnegative integer which will ensure that no term in y_p is already in the C.F.

If $F(x)$ is sum of several terms, write y_p for each term individually and then add up all of them.

The y_p and its derivatives will be substituted into the equation $f(D)\, y = F(x)$ and coefficients of like terms on the left hand and right hand sides will be equated to determine the U.C. $A_0, A_1, \cdots, A_n, B_0, B_1, \cdots, B_n$.

The method is illustrated by examples as follows.

Example 12. Solve: $y'' - 3y' + 2y = 2x^3 - 9x^2 + 6x$ (1)

Solution. C.F. is easily found as

$$y_c = c_1 e^x + c_2 e^{2x}$$

For a particular solution, we assume

$$y_p = x^k (Ax^3 + Bx^2 + Cx + D).$$

Since no term of the C.F. is present in y_p, we take $k = 0$ and so

$$y_p = Ax^3 + Bx^2 + Cx + D$$

$$y_p' = 3Ax^2 + 2Bx + C$$

$$y_p'' = 6Ax + 2B.$$

Substituting for y_p, y_p' and y_p'' into (1), we have

$$(6Ax + 2B) - 3(3Ax^2 + 2Bx + C) + 2(Ax^3 + Bx^2 + Cx + D) = 2x^3 - 9x^2 + 6x$$

or

$$2Ax^3 + x^2(2B - 9A) + x(6A - 6B + 2C) + 2B - 3C + 2D = 2x^3 - 9x^2 + 6x.$$

Equating coefficients of like terms, we obtain

Coeff. of x^3: $2A = 2$ or $A = 1$

Coeff. of x^2: $2B - 9A = -9$ or $2B = 9A - 9$ giving $B = 0$

Coeff. of x: $6A - 6B + 2C = 6$ or $C = 0$

Coeff. of x^0: $2B - 3C + 2D = 0$ or $D = 0$

So $y_p = x^3$

The required general solution is

$$y = c_1 e^x + c_2 e^{2x} + x^3.$$

Example 13. Solve: $y'' - 3y' + 2y = x^2 e^x$ (1)

Solution. The C.F. is

$$y_c = c_1 e^x + c_2 e^{2x}$$

To construct a particular solution, we use (3) of the table in **(10.12)**.

$$y_p = x^k (Ax^2 + Bx + C) e^x$$

Since e^x is already in C.F., we take $k = 1$ so that no term of C.F. is in y_p. Hence the modified P.I. is

$$y_p = (Ax^3 + Bx^2 + Cx) e^x$$

$$y_p' = (Ax^3 + Bx^2 + Cx) e^x + (3Ax^2 + 2Bx + C) e^x$$

$$y_p'' = (Ax^3 + Bx^2 + Cx) e^x + 2(3Ax^2 + 2Bx + C) e^x + (6Ax + 2B) e^x$$

Substituting for y_p, y_p' and y_p'' into (1), we have

$$(Ax^3 + Bx^2 + Cx) e^x + 2(3Ax^2 + 2Bx + C) e^x + (6Ax + 2B) e^x$$
$$- 3(Ax^3 + Bx^2 + Cx) e^x - 3(3Ax^2 + 2Bx + C) e^x$$
$$+ 2(Ax^3 + Bx^2 + Cx) e^x = x^3 e^x$$

or

$$[-3Ax^2 + (6A - 2B) x + C] e^x = x^3 e^x.$$

Equating coefficients of like terms, we obtain

Coeff. of x^3: $-3A = 1$ or $A = -\frac{1}{3}$

Coeff. of x: $6A - 2B = 0$ or $B = -1$

Coeff. of x^0: $C = 0$

so $y_p = -\frac{1}{3} x^3 e^x - x^2 e^x$

The required general solution is

$$y = c_1 e^x + c_2 e^{2x} - \frac{1}{3} x^3 e^x - x^2 e^x$$

Example 14. Solve: $y'' + 4y = xe^x + x \sin 2x$ (1)

Solution. The C.F., as readily found above, is

$$y_c = c_1 \sin 2x + c_2 \cos 2x.$$

For the P.I. of (1), we find the P.I. of

$y'' + 4y = x e^x$ (2)

and $y'' + 4y = x \sin 2x$ (3)

separately. Their sum will be the P.I. of (1) (by the **Principle of Superposition**).

For a particular solution of (2), we have

$$y_p = x^k (Ax + B) e^x.$$

Since no term of C.F. is in the y_p, we must take $k = 0$. Therefore,

$$y_p = (Ax + B) e^x$$

$$y_p' = (Ax + B) e^x + Ae^x$$

$$y_p'' = (Ax + B) e^x + 2Ae^x.$$

Substituting for y_p, y_p' and y_p'' into (2), we have

$$(Ax + B) e^x + 2Ae^x + 4(Ax + B) e^x = xe^x$$

or $5Ax e^x + (2A + 5B) e^x = xe^x$

Equating coefficients of like terms, we obtain

Coeff. of $x e^x$: $5A = 1$ or $A = \frac{1}{5}$

Coeff. of e^x: $2A + 5B = 0$ or $B = -\frac{2}{25}$

So P.I. of (2) is $y_p = \frac{1}{5} xe^x - \frac{2}{25} e^x.$ (4)

For a particular solution of (3), we assume

$$y_p = x^k \left[(Cx + D) \sin 2x + (Ex + F) \cos 2x\right]$$

Since $\sin 2x$ and $\cos 2x$ are already in the C.F., we must take $k = 1$, so that the P.I. is

$$y_p = (Cx^2 + Dx) \sin 2x + (Ex^2 + Fx) \cos 2x$$

$$y_p' = (Cx^2 + Dx)(2 \cos 2x) + (2Cx + D) \sin 2x + (Ex^2 + Fx)(-2 \sin 2x) + (2Ex + F) \cos 2x$$

$$y_p'' = (Cx^2 + Dx)(-4 \sin 2x) + (2Cx + D)(2 \cos 2x) + (2Cx + D)(2 \cos 2x) + 2C \sin 2x + (Ex^2 + Fx)(-4 \cos 2x) + (2Ex + F)(-2 \sin 2x) + (2Ex + F)(-2 \sin 2x) + 2E \cos 2x$$

Substituting into (3), we have

$$(Cx^2 + Dx)(-4 \sin 2x) + 4(2Cx + D) \cos 2x + 2C \sin 2x + (Ex + Fx)(-4 \cos 2x) - 4(2Ex + F) \sin 2x + 2E \cos 2x + 4(Cx^2 + Dx) \sin 2x + 4(Ex^2 + Fx) \cos 2x = x \sin 2x$$

Equating the coefficients of like terms, we get

Coeff. of $x \sin 2x$: $-8E = 1$ or $E = -\frac{1}{8}$

Coeff. of $x \cos 2x$: $8C = 0$ or $C = 0$

Coeff. of $\sin 2x$: $2C - 4F = 0$ or $F = 0$

Coeff. of $\cos 2x$: $2E + 4D = 0$ or $D = \frac{1}{16}$

Hence P.I. of (3) is

$$y_p = \frac{1}{16} x \sin 2x - \frac{1}{8} x^2 \cos 2x \quad (5)$$

Sum of (4) and (5) is the required P.I. of (1), i.e.,

$$y_p = \frac{1}{5} xe^x - \frac{2}{25} e^x + \frac{1}{16} x \sin 2x - \frac{1}{8} x^2 \cos 2x$$

is the particular solution of (1)

Hence the general solution of (1) is

$$y = c_1 \sin 2x + c_2 \cos 2x + \frac{1}{5} xe^x - \frac{2}{25} e^x + \frac{1}{16} x \sin 2x - \frac{1}{8} x^2 \cos 2x$$

EXERCISE 10.3

Solve by the method of U.C. (Problems 1–9)

1. $y'' - 4y' + 4y = e^{2x}$
2. $y'' + 2y' + 5y = 6\sin 2x + 7\cos 2x$
3. $2y'' + 3y' + y = x^2 + 3\sin x$
4. $y'' + 2y' + y = e^x \cos x$
5. $y'' + y = 12\cos^2 x$
6. $y'' - 3y' + 2y = 2x^2 + 2x\,e^x$
7. $y''' + y' = 2x^2 + 4\sin x$
8. $y''' + y'' + 3y' - 5y = 5\sin 2x + 10x^2 + 3x + 7$
9. $y^{(4)} + 8y'' + 16y = \sin x$
10. Write the general form of the P.I. (without evaluating the U.C.) for
    (i) $y'' + 2y' + 2y = 4e^{-x}x^2\sin x + 3e^{-x} + 2e^{-x}\cos x$
    (ii) $y'' + 3y' + 2y = e^x(x^2+1)\sin 2x + 4e^x + 3e^{-x}\cos x$

THE CAUCHY-EULER[1] EQUATION

(10.13) An equation of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}x\frac{dy}{dx} + a_n y = F(x) \qquad (1)$$

is called **Cauchy-Euler** (or **equidimensional**) equation $a_0, a_1, \cdots, a_{n-1}, a_n$ are real constants.

The equation can be reduced to a linear differential equation with constant coefficients by the transformation

$$x = e^t \quad \text{or} \quad t = \ln x.$$

Then

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} + \frac{dy}{dt}\left(-\frac{1}{x^2}\right) = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

1. Leonhard Euler (1701 – 1783). A Swiss mathematician.

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}\right]$$

$$= \frac{1}{x^2}\frac{d}{dx}\left(\frac{d^2y}{dt^2}\right) + \frac{d^2y}{dt^2}\frac{d}{dx}\left(\frac{1}{x^2}\right) - \frac{1}{x^2}\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\frac{d}{dx}\left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{x^2}\frac{d}{dt}\left(\frac{d^2y}{dt^2}\right)\frac{dt}{dx} - \frac{2}{x^3}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} + \frac{2}{x^3}\frac{dy}{dt}$$

$$= \frac{1}{x^3}\frac{d^3y}{dt^3} - \frac{3}{x^3}\frac{d^2y}{dt^2} + \frac{2}{x^3}\frac{dy}{dt}$$

$$= \frac{1}{x^3}\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right)$$

If we write $D = \dfrac{d}{dx}$ and $\Delta = \dfrac{d}{dt}$, then

$$xD = \Delta$$
$$x^2D^2 = \Delta^2 - \Delta = \Delta(\Delta - 1)$$
$$x^3D^3 = \Delta^3 - 3\Delta^2 + 2\Delta = \Delta(\Delta-1)(\Delta-2)$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$x^nD^n = \Delta(\Delta-1)(\Delta-2)\cdots(\Delta-n+1)$$

Substituting these values of $xD, x^2D^2, \cdots$ into (1), we obtain an equation of nth order with constant coefficients having t as the independent variable. This equation can be solved by the previous methods.

Example 15. Solve: $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = x^3.$ (1)

Solution. Let $x = e^t$. Then we have

$$t = \ln x$$
$$xD = \Delta$$
$$x^2D^2 = \Delta(\Delta - 1).$$

Substituting into (1), we get

$$[\Delta(\Delta-1) - 2\Delta + 2]\,y = e^{3t}$$

or $$[\Delta^2 - 3\Delta + 2]\,y = e^{3t}. \qquad (2)$$

The characteristic equation has roots 2 and 1. Therefore, C.F. of (2) is

$$y_c = c_1 e^t + c_2 e^{2t}$$

P.I. of equation (2) is

$$y_p = \frac{1}{(\Delta - 1)(\Delta - 2)} e^{3t} = \frac{1}{2} e^{3t}$$

The general solution of (2) is

$$y = c_1 e^t + c_2 e^{2t} + \frac{1}{2} e^{3t}$$

Replacing t by $\ln x$ (or e^t by x), we have

$$y = c_1 x + c_2 x^2 + \frac{1}{2} x^3$$

as the general solution of (1).

Example 16. Solve:

$$x^4 \frac{d^4y}{dx^4} + 6x^3 \frac{d^3y}{dx^3} + 9x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = (1 + \ln x)^2 \qquad (1)$$

Solution. We let $x = e^t$ so that $t = \ln x$. By (10.13), we have

$$\begin{aligned} xD &= \Delta \\ x^2D^2 &= \Delta(\Delta - 1) = \Delta^2 - \Delta \\ x^3D^3 &= \Delta(\Delta - 1)(\Delta - 2) = \Delta^3 - 3\Delta^2 + 2\Delta \\ x^4D^4 &= \Delta(\Delta - 1)(\Delta - 2)(\Delta - 3) = \Delta^4 - 6\Delta^3 + 11\Delta^2 - 6\Delta. \end{aligned}$$

Substituting into (1), we get

$$[\Delta^4 - 6\Delta^3 + 11\Delta^2 - 6\Delta + 6(\Delta^3 - 3\Delta^2 + 2\Delta) + 9(\Delta^2 - \Delta) + 3\Delta + 1]\, y = (1 + t)^2$$

or

$$(\Delta^4 + 2\Delta^2 + 1)\, y = 1 + 2t + t^2 \qquad (2)$$

The roots of the characteristic equation of (2) are $\pm i, \pm i$.

Therefore, C.F. of (2) is

$$y_c = (c_1 + c_2 t) \sin t + (c_3 + c_4 t) \cos t$$

To find a particular integral of (2), we use the method of U.C. Let the particular integral of (2) be

$$\begin{aligned} y_p &= At^2 + Bt + C \\ y_p' &= 2At + B \\ y_p'' &= 2A. \end{aligned}$$

Substituting into (2), we obtain

$$\begin{aligned} 4A + At^2 + Bt + C &= 1 + 2t + t^2 \\ At^2 + Bt + (4A + C) &= t^2 + 2t + 1 \end{aligned}$$

Equating coefficients of like terms, we have

$$A = 1, \quad B = 2, \quad 4A + C = 1 \quad \text{or} \quad C = -3.$$

Therefore, a particular integral of (2) is

$$y_p = t^2 + 2t - 3$$

Hence the general solution of (2) is

$$y = (c_2 + c_2 t) \sin t + (c^3 + c_4 t) \cos t + t^2 + 2t - 3 \qquad (3)$$

Replacing t by $\ln x$ in (3), we get

$$y = (c_1 + c_2 \ln x) \sin(\ln x) + (c_3 + c_4 \ln x) \cos(\ln x) + (\ln x)^2 + 2 \ln x - 3$$

as the general solution of (1).

EXERCISE 10.4

Solve

1. $x^2 \frac{d^2y}{dx^2} + 7x \frac{dy}{dx} + 5y = x^5$

2. $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 5y = x^2 \sin(\ln x)$

3. $x^2 \frac{d^2y}{dx^2} - (2m - 1)x \frac{dy}{dx} + (m^2 + n^2)y = n^2 x^m \ln x$

4. $4x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 3y = \sin \ln(-x), \quad x < 0$

5. $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10x + \frac{10}{x}$

6. $x^4 \frac{d^3y}{dx^3} + 2x^3 \frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 1$

7. $x^3 \frac{d^3y}{dx^3} + 4x^2 \frac{d^2y}{dx^2} - 5x \frac{dy}{dx} - 15y = x^4$

8. $(x+1)^2\dfrac{d^2y}{dx^2} + (x+1)\dfrac{dy}{dx} + y = 4[\cos\ln(x+1)]^2$

9. $(2x+1)^2\dfrac{d^2y}{dx^2} - 6(2x+1)\dfrac{dy}{dx} + 16y = 8(2x+1)^2$

10. $x^2y'' + 2xy' - 6y = 10x^2$, $\quad y(1) = 1,\ y'(1) = -6$

11. $x^2y'' - 2xy' + 2y = x\ln x$, $\quad y(1) = 1,\ y'(1) = 0$

12. $x^3y''' + 2x^2y'' + xy' - y = 15\cos(2\ln x)$, $\quad y(1) = 2,\ y'(1) = -3,\ y''(1) = 0$

REDUCTION OF ORDER

(10.14) If one solution of the second order linear equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \tag{1}$$

(where P, Q are not necessarily constants and may be functions of x) is known, then we can use it to find the general solution of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = F(x) \tag{2}$$

The procedure, due to D' Alembert, is known as the method of **reduction of order.**

Suppose it is known that $y = y_1$ is a solution of (1). We assume that

$$y = vy_1 \tag{3}$$

is a solution of (2), where v is a function of x to be determined. From (3), we get

$$\frac{dy}{dx} = v\frac{dy_1}{dx} + y_1\frac{dv}{dx},$$

$$\frac{d^2y}{dx^2} = v\frac{d^2y_1}{dx^2} + 2\frac{dv}{dx}\frac{dy_1}{dx} + y_1\frac{d^2v}{dx^2}$$

Substituting into (2), we have

$$v\frac{d^2y_1}{dx^2} + 2\frac{dv}{dx}\frac{dy_1}{dx} + y_1\frac{d^2v}{dx^2} + Pv\frac{dy_1}{dx} + Py_1\frac{dv}{dx} + Qvy_1 = F(x)$$

or

$$y_1\frac{d^2v}{dx^2} + \left(2\frac{dy_1}{dx} + Py_1\right)\frac{dv}{dx} + \left(\frac{d^2y_1}{dx^2} + P\frac{dy_1}{dx} + Qy_1\right)v = F(x) \tag{4}$$

Since $y = y_1$ is a solution of (1)

$$\frac{d^2y_1}{dx^2} + P\frac{dy_1}{dx} + Qy_1 = 0$$

and, therefore, equation (4) reduces to

$$y_1\frac{d^2v}{dx^2} + \left(2\frac{dy_1}{dx} + Py_1\right)\frac{dv}{dx} = F(x) \tag{5}$$

Setting $\dfrac{dv}{dx} = u$ in (5), we obtain

$$y_1\frac{du}{dx} + \left(2\frac{dy_1}{dx} + Py_1\right)u = F(x)$$

which is a linear equation of the first order in u and can be solved for u. From $\dfrac{dv}{dx} = u$, we determine v and hence the solution $y = vy_1$. The method is illustrated by means of examples.

Example 17. Solve: $\dfrac{d^2y}{dx^2} + y = \csc x$ (1)

Solution. The C.F. of (1) is

$$y_c = c_1\sin x + c_2\cos x$$

We may take any special value of y_c as y_1. Let us take y_1 to be the value of y_c when $c_1 = 1$ and $c_2 = 0$. Then assume that

$$y = v\sin x \tag{2}$$

is a solution of (1). From (2), we get

$$\frac{dy}{dx} = \sin x\frac{dv}{dx} + v\cos x$$

$$\frac{d^2y}{dx^2} = \sin x\frac{d^2v}{dx^2} + 2\cos x\frac{dv}{dx} - v\sin x.$$

Substituting into (1), we obtain

$$\sin x\frac{d^2v}{dx^2} + 2\cos x\frac{dv}{dx} - v\sin x + v\sin x = \csc x$$

or

$$\frac{d^2v}{dx^2} + 2\cot x\frac{dv}{dx} = \csc^2 x.$$

Setting $u = \dfrac{dv}{dx}$, the above equation becomes

$$\frac{du}{dx} + 2\cot x\, u = \csc^2 x \qquad (3)$$

which is linear equation of first order. An integrating factor of (3) is

$$\exp\left[\int 2\cot x\, dx\right] = \exp\left[\ln \sin^2 x\right] = \sin^2 x$$

Multiplying (3) by $\sin^2 x$ and integrating, we have

$$u \sin^2 x = x,$$

neglecting constant of integration since we seek only a particular solution

or $\quad u = x\csc^2 x$

or $\quad \dfrac{dv}{dx} = x\csc^2 x$

or $\quad v = \int x\csc^2 x\, dx = -x\cot x + \ln|\sin x|$, (on integrating by parts)

Hence a particular solution of (1) is

$$y_p = v\sin x = -x\cos x + \sin x\left(\ln|\sin x|\right)$$

The general solution of (1) is

$$y = c_1 \sin x + c_2 \cos x - x\cos x + \sin x\left(\ln|\sin x|\right)$$

Example 18. Solve $(x+2)\dfrac{d^2y}{dx^2} - (2x+5)\dfrac{dy}{dx} + 2y = (x+1)\,e^x$ (1)

Solution. We note that $y = e^{2x}$ makes the left hand side of (1) zero. Therefore, we put

$$y = v e^{2x}$$

$$\frac{dy}{dx} = \left(\frac{dv}{dx} + 2v\right)e^{2x}$$

$$\frac{d^2y}{dx^2} = \left(\frac{d^2v}{dx^2} + 4\frac{dv}{dx} + 4v\right)e^{2x}$$

Substituting into (1), we obtain

$$(x+2)\left(\frac{d^2v}{dx^2} + 4\frac{dv}{dx} + 4v\right)e^{2x} - (2x+5)\left(\frac{dv}{dx} + 2v\right)e^{2x} + 2v\,e^{2x} = (x+1)e^x$$

or
$$(x+2)\frac{d^2v}{dx^2}\,e^{2x} + \left[4(x+2) - (2x+5)\right]\frac{dv}{dx}\,e^{2x} + \left[4(x+2) - 2(2x+5) + 2\right]v e^{2x} = (x+1)e^x$$

or
$$(x+2)\frac{d^2v}{dx^2} + (2x+3)\frac{dv}{dx} = (x+1)\,e^{-x}$$

Writing $u = \dfrac{dv}{dx}$, the above equation becomes

$$(x+2)\frac{du}{dx} + (2x+3)\,u = (x+1)\,e^{-x}$$

or
$$\frac{du}{dx} + \frac{2x+3}{x+2}\,u = \frac{x+1}{x+2}\,e^{-x} \qquad (2)$$

which is linear equation of first order. An integrating factor of (2) is

$$\exp\left[\int \frac{2x+3}{x+2}\,dx\right] = \exp\left[\int\left(2 - \frac{1}{x+2}\right)dx\right]$$

$$= \exp\left[2x - \ln(x+2)\right] = \frac{e^{2x}}{x+2}$$

Multiplying (2) by $\dfrac{e^{2x}}{x+2}$, we have

$$\frac{d}{dx}\left(u\,\frac{e^{2x}}{x+2}\right) = \frac{x+1}{(x+2)^2}\,e^x$$

or
$$u\,\frac{e^{2x}}{x+2} = \int \frac{x+1}{(x+2)^2}\,e^x\,dx + c_1$$

$$= \int \frac{e^x}{x+2}\,dx - \int \frac{e^x}{(x+2)^2}\,dx + c_1$$

$$= \frac{e^x}{x+2} + c_1$$

or $\quad u = e^{-x} + c_1(x+2)\,e^{-2x}$

Therefore, $\quad \dfrac{dv}{dx} = e^{-x} + c_1(x+2)\,e^{-2x}$

Integrating, we get

$$v = -e^{-x} - \frac{1}{4}\,c_1(2x+5)\,e^{-2x} + c_2$$

Hence $\quad y = ve^{2x} = -e^x - \dfrac{1}{4}\,c_1(2x+5) + c_2 e^{2x}$

is the general solution of (1).

Note. It is easy to see that $y'' + Py' + Qy = 0$ is satisfied by $y = e^{\pm x}$ if $1 \pm P + Q = 0$ and by $y = x$ if $P + Qx = 0$. These can be used to find a solution of $y'' + Py' + Q = 0$ by inspection.

EXERCISE 10.5

Solve:

1. $\frac{d^2y}{dx^2} + y = \sec^3 x$

2. $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$

3. $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$ (**Legendre's**[1] equation of order one)

4. $(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 1$

5. $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 8x^3$

6. $x^2\frac{d^2y}{dx^2} - (x^2+2x)\frac{dy}{dx} + (x+2)y = x^3e^x$

7. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \frac{1}{(1+e^x)^2}$

8. $x\frac{d^2y}{dx^2} - (2x+1)\frac{dy}{dx} + (x+1)y = (x^2+x-1)e^x$

9. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = (1+x+x^2+\dots+x^{25})e^{2x}$, given that $y = e^{2x}$ is a solution of the associated homogeneous equation

10. $\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 3y = 2\sec x$, given that $y = \sin x$ is a solution of the associated homogeneous equation

11. $x\frac{d^2y}{dx^2} - 2(x+1)\frac{dy}{dx} + (x+2)y = (x-1)e^{2x}$

12. $(1-x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x(1-x^2)^{3/2}$. Ans 1

1 Legendre, Adrien Marie (1752 – 1833). A French mathematician

THE WRONSKIAN[1]

(10.15) Definition. If y_1, y_2 are two differentiable functions of x on $I = [a, b]$ then their **Wronskian**, denoted by $W = W[y_1, y_2]$, is defined by

$$W = W[y_1, y_2] = y_1y'_2 - y'_1y_2 = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$$

Similarly, the Wronskian of three differentiable functions y_1, y_2, y_3 on $I = [a, b]$ is defined by

$$W = W[y_1, y_2, y_3] = \begin{vmatrix} y_1 & y_2 & y_3 \\ y'_1 & y'_2 & y'_3 \\ y''_1 & y''_2 & y''_3 \end{vmatrix}.$$

The definition can be extended in a similar manner for the Wronskian of n differentiable functions on $I = [a, b]$

(10.16) Theorem. Let y_1, y_2 be two solutions of the homogeneous linear equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0, \tag{1}$$

where P, Q are functions of x and are continuous on $I = [a, b]$. Then their Wronskian $W = W[y_1, y_2]$ is either identically zero or is never zero on I.

Proof. We have

$$W = y_1y'_2 - y'_1y_2$$

Differentiating w.r.t. x, we get

$$W' = y_1y''_2 + y_1'y_2' - y_1'y_2' - y''_1y_2$$
$$= y_1y''_2 - y''_1y_2$$

Since y_1, y_2 are solutions of (1), we have

$$y''_1 + Py_1' + Qy_1 = 0 \tag{2}$$

and

$$y_2'' + Py_2' + Qy_2 = 0 \tag{3}$$

Multiply (2) by y_2 and (3) by y_1 to have an equivalent system

$$y''_1y_2 + Py'_1y_2 + Qy_1y_2 = 0 \tag{4}$$
$$y_1y''_2 + Py'_2y_1 + Qy_1y_2 = 0 \tag{5}$$

1 Named after the Polish mathematician Jozef Maria Hoene Wronski (1778 – 1853).

Subtracting (4) from (5), we obtain

$$(y_1 y_2'' - y_2 y_1'') + P(y_1 y_2' - y_1' y_2) = 0$$

or
$$\frac{dW}{dx} + PW = 0$$

The general solution of the above equation is readily found as

$$W = c \exp\left(-\int P\,dx\right)$$

Since the exponential function is never zero, W is identically zero if $c = 0$ or is never zero when $c \neq 0$.

(10.17) Theorem. If y_1, y_2 are two solutions of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0,$$

then their Wronskian $W = W[y_1, y_2] = 0$ if and only if y_1, y_2 are linearly dependent.

Proof. If one of the two solutions is identitically zero, then the theorem is obviously true.

Assume that y_1, y_2 are both nonzero and let y_1, y_2 be linearly dependent. Then $y_1 = cy_2$, where c is a constant.

i.e.,
$$\frac{y_1}{y_2} = c$$

or
$$\frac{d}{dx}\left(\frac{y_1}{y_2}\right) = \frac{y_2 y_1' - y_1 y_2'}{y_2^2} = \frac{dc}{dx} = 0$$

i.e.,
$$y_1 y'_2 - y'_1 y_2 = 0$$

$\Rightarrow$
$$W[y_1, y_2] = 0$$

Conversely, if $W[y_1, y_2] = 0$, then $\frac{d}{dx}\left(\frac{y_1}{y_2}\right) = \frac{W}{y_2^2} = 0$.

$\Rightarrow$
$$\frac{y_1}{y_2} = c, \text{ where } c \text{ is a constant}$$

Thus $y_1 = c\,y_2$ and so y_1, y_2 are linearly dependent.

(10.18) Corollary. $W[y_1, y_2] \neq 0$ if and only if y_1, y_2 are linearly independent.

Note: Theorem 10.17 and Corollary 10.18 can only be applied to check the linear dependence or linear independence of differentiable functions which are solutions of a linear homogeneous O.D.E. Arbitrary differentiable functions exist with their Wronskian zero but they are linearly independent.

Consider the functions $y_1 = x^3$, $y_2 = |x^3|$ on $]-\infty, \infty[$. Here $W(y_1, y_2) = 0$ but y_1 and y_2 are linearly independent. (Verify!).

VARIATION OF PARAMETERS

(10.19) We found the solution of the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = F(x), \tag{1}$$

where P, Q are functions of x, in the previous section by reduction of order of (1). The solution of (1) can be determined by a procedure known as the **method of variation of parameters**[1]. This method can be applied even to equations of higher order.

Suppose that linearly independent solutions of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \tag{2}$$

are given by $y = y_1(x)$ and $y = y_2(x)$. Then the complementary function of (1) is

$$y_c = c_1 y_1 + c_2 y_2$$

where c_1 and c_2 are arbitrary constants. We replace the arbitrary constants c_1 and c_2 by unknown functions $u_1(x)$ and $u_2(x)$ and require that

$$y_p = u_1 y_1 + u_2 y_2 \tag{3}$$

be a particular solution of (1). In order to determine the two functions u_1 and u_2 we need two conditions. One condition is that (3) must satisfy (1). A second condition can be imposed arbitrarily.

Differentiating (3) w.r.t. x, we have

$$y_p' = (u_1' y_1 + u_2' y_2) + u_1 y_1' + u_2 y_2'$$

If we differentiate the above equation, y_p'' will contain u_1'' and u_2''. To avoid second derivatives of u_1 and u_2, we put

$$u_1' y_1 + u_2' y_2 = 0. \tag{4}$$

With this condition, we have

$$y_p' = u_1 y_1' + u_2 y_2'$$

so that

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''.$$

---

1. Discovered by the French mathematician J.L. Lagrange (1736 – 1813)

Scanned with CamScanner

Substituting for y_p, y_p', y_p'' into equation (1), we get

$$\left(u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''\right) + P\left(u_1 y_1' + u_2 y_2'\right) + Q\left(u_1 y_1 + u_2 y_2\right) = F(x)$$

or $$u_1\left(y_1'' + P y_1' + Q y_1\right) + u_2\left(y_2'' + P y_2' + Q y_2\right) + u_1' y_1' + u_2' y_2' = F(x)$$

Expressions within the parentheses are zero since y_1 and y_2 are solutions of (2). Hence

$$u_1' y_1' + u_2' y_2' = F(x). \qquad (5)$$

Taking (4) and (5) together, we have two equations in the two unknowns u_1' and u_2':

$$u_1' y_1 + u_2' y = 0$$

and $$u_1' y_1' + u_2' y_2' = F(x).$$

Solving these, we have

$$\left.\begin{aligned} u_1' &= \frac{-y_2 F(x)}{y_1 y_2' - y_1' y_2} \\ u_2' &= \frac{y_1 F(x)}{y_1 y_2' - y_1' y_2} \end{aligned}\right\} \qquad (6)$$

In (6), the Wronskian of y_1, y_2, namely, $W(y_1, y_2) = y_1 y_2' - y_1' y_2 \neq 0$, since y_1, y_2 are linearly independent solutions of (2).

Integrating (6), we find u_1 and u_2 as

$$\left.\begin{aligned} u_1 &= \int \frac{-y_2 F(x)}{y_1 y_2' - y_1' y_2}\,dx = \int \frac{-y_2 F(x)}{W}\,dx \\ u_2 &= \int \frac{y_1 F(x)}{y_1 y_2' - y_1' y_2}\,dx = \int \frac{y_1 F(x)}{W}\,dx \end{aligned}\right\} \qquad (7)$$

where W is the Wronskian of y_1, y_2.

Thus y_p is completely determined.

In numerical problems, instead of performing the complete process, formulas (7) will be directly applied to evaluate u_1 and u_2.

Example 19. Find the general solution of

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x e^x \ln x \qquad (1)$$

Solution. The C.F. of (1) is

$$y_c = c_1 e^x + c_2 x e^x$$

Let $$y_p = u_1 e^x + u_2 x e^x.$$

Here $$y_1 = e^x, \quad y_2 = x e^x, \quad F(x) = x e^x \ln x$$

$$W = W(y_1, y_2) = y_1 y_2' - y_1' y_2 = e^x\left(e^x + x e^x\right) - x e^x \cdot e^x = e^{2x}.$$

By the formulas (7) of (10.19), we have

$$u_1 = \int \frac{-y_2 F(x)}{W}\,dx = \int \frac{-x e^x \cdot x e^x \ln x}{e^{2x}}\,dx$$

and $$u_2 = \int \frac{y_1 F(x)}{W}\,dx = \int \frac{e^x \cdot x e^x \ln x}{e^{2x}}\,dx$$

i.e., $$u_1 = -\int x^2 \ln x\,dx = -(\ln x)\frac{x^3}{3} + \int \frac{x^2}{3}\,dx = \frac{x^3}{9} - \frac{x^3}{3}\ln x$$

$$u_2 = \int x \ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx = -\frac{x^2}{4} + \frac{x^2}{2}\ln x.$$

$$y_p = \left(\frac{x^3}{9} - \frac{x^3}{3}\ln x\right)e^x + \left(-\frac{x^2}{4} + \frac{x^2}{2}\ln x\right)x e^x$$

$$= e^x\left(\frac{x^3}{9} - \frac{x^3}{3}\ln x - \frac{x^3}{4} + \frac{x^3}{2}\ln x\right) = e^x\left(-\frac{5x^3}{36} + \frac{x^3}{6}\ln x\right).$$

The general solution of (1) is

$$y = y_c + y_p = c_1 e^x + c_2 x e^x + e^x\left(-\frac{5x^3}{36} + \frac{x^3}{6}\ln x\right).$$

Example 20. Find the general solution of

$$x^2\frac{d^2y}{dx^2} - x(x+2)\frac{dy}{dx} + (x+2)\,y = x^3 \qquad (1)$$

given that $y = x e^x$ is a solution of the associated homogeneous equation

$$x^2\frac{d^2y}{dx^2} - x(x+2)\frac{dy}{dx} + (x+2)\,y = 0 \qquad (2)$$

Solution. The given equation in the standard form is

$$\frac{d^2y}{dx^2} - \frac{x+2}{x}\frac{dy}{dx} + \frac{x+2}{x^2}\,y = x$$

Since $P + Qx = -\frac{x+2}{x} + x\frac{x+2}{x^2} = 0$, $y = x$ is also a solution of (2). The two solutions, $y = x$ and $y = x e^x$ are linearly independent. The complementary function of (1) is

$$y_c = c_1 x + c_2 x e^x.$$

We assume that

$$y_p = u_1 x + u_2 x e^x$$

is a particular solution of (1) (3)

Here $y_1 = x$, $y_2 = x e^x$, $F(x) = x$

$$W = W(y_1, y_2) = y_1 y_2' - y_1' y_2 = x(x e^x + e^x) - x e^x = x^2 e^x$$

By the formulas (7) of (10.19), we have

$$u_1 = \int \frac{-y_2 F(x)}{W}\, dx \quad \text{and} \quad u_2 = \int \frac{y_1 F(x)}{W}\, dx$$

i.e., $$u_1 = -\int \frac{x^2 e^x}{x^2 e^x}\, dx = -x \quad \text{and} \quad u_2 = \int \frac{x^2}{x^2 e^x}\, dx = -e^{-x}$$

Substituting for u_1, u_2 into (3), a particular solution of (1) is

$$y_p = -x^2 - x$$

The general solution of (1) is

$$y = y_c + y_p = c_1 x + c_2 x e^x - x^2 - x$$

Example 21. Explain how the method of variation of parameters can be applied to find a particular solution of a nonhomogeneous linear third order differential equation whose complementary function is known. Apply the method to find a particular solution of

$$\frac{d^3 y}{dx^3} + \frac{dy}{dx} = \cos x.$$

Solution. Suppose the C.F. of a linear third order differential equation

$$\frac{d^3 y}{dx^3} + P\frac{d^2 y}{dx^2} + Q\frac{dy}{dx} + Ry = F(x) \tag{1}$$

is known to be

$y_c = c_1 y_1 + c_2 y_2 + c_3 y_3$; y_1, y_2, y_3 being linearly independent solutions of the associated homogeneous equation of (1)

We assume that

$$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3, \text{ where } u_1, u_2, u_3 \text{ are functions of } x.$$

$$y_p' = u_1 y_1' + u_2 y_2' + u_3 y_3' + u_1' y_1 + u_2' y_2 + u_3' y_3$$

We set $$u_1' y_1 + u_2' y_2 + u_3' y_3 = 0 \tag{2}$$

Then $$y_p' = u_1 y_1' + u_2 y_2' + u_3 y_3'$$

$$y_p'' = u_1 y_1'' + u_2 y_2'' + u_3 y_3'' + u_1' y_1' + u_2' y_2' + u_3' y_3'$$

Again set $$u_1' y_1' + u_2' y_2' + u_3' y_3' = 0 \tag{3}$$

Then $$y_p'' = u_1 y_1'' + u_2 y_2'' + u_3 y_3''$$

$$y_p''' = u_1 y_1''' + u_2 y_2''' + u_3 y_3''' + u_1' y_1'' + u_2' y_2'' + u_3' y_3''$$

Substituting for y_p, y_p', y_p'', y_p''' into equation (1), we have

$$(u_1 y_1''' + u_2 y_2''' + u_3 y_3''' + u_1' y_1'' + u_2' y_2'' + u_3' y_3'') + P(u_1 y_1'' + u_2 y_2'' + u_3 y_3'')$$
$$+ Q(u_1 y_1' + u_2 y_2' + u_3 y_3') + R(u_1 y_1 + u_2 y_2 + u_3 y_3) = F(x)$$

or $$u_1 (y_1''' + P y_1'' + Q y_1' + R y_1) + u_2 (y_2''' + P y_2'' + Q y_2' + R y_2)$$
$$+ u_3 (y_3''' + P y_3'' + Q y_3' + R y_3) + u_1' y_1'' + u_2' y_2'' + u_3' y_3'' = F(x)$$

i.e., $$u_1 \cdot 0 + u_2 \cdot 0 + u_3 \cdot 0 + u_1' y_1'' + u_2' y_2'' + u_3' y_3'' = F(x) \tag{4}$$

From the system of equations (2), (3) and (4), we find u_1', u_2' and u_3' and on integration we get u_1, u_2 and u_3.

Now we apply this method to find P.I. of

$$\frac{d^3 y}{dx^3} + \frac{dy}{dx} = \csc x.$$

The C.F. of the equation is obtained by solving the equation $D(D^2 + 1) = 0$. Thus

$$y_c = c_1 + c_2 \cos x + c_3 \sin x \text{ with } y_1 = 1, y_2 = \cos x, y_3 = \sin x.$$

For the particular integral, let

$$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3.$$
$$= u_1 + u_2 \cos x + u_3 \sin x$$

Substituting for y_1, y_2, y_3 and their derivatives into equations (2), (3) and (4) above, we have the following system of linear equations in u_1', u_2', u_3'.

$$u_1' + u_2' \cos x + u_3' \sin x = 0$$
$$u_1' \cdot 0 - u_2' \sin x + u_3' \cos x = 0$$
$$u_1' \cdot 0 - u_2' \cos x - u_3' \sin x = \csc x.$$

We solve this system by the Guassian elimination method

Augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & \cos x & \sin x & 0 \\ 0 & -\sin x & \cos x & 0 \\ 0 & -\cos x & -\sin x & \csc x \end{array}\right]$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 0 & \csc x \\ 0 & -\sin x & \cos x & 0 \\ 0 & -\cos x & -\sin x & \csc x \end{array}\right] \quad \text{by } R_1 + R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 0 & \csc x \\ 0 & -\sin x \cos x & \cos^2 x & 0 \\ 0 & -\cos x \sin x & -\sin^2 x & 1 \end{array}\right] \quad \text{by } (\cos x) R_2 \text{ and } (\sin x) R_3$$

$$\underset{\sim}{R}\left[\begin{array}{ccc|c} 1 & 0 & 0 & \csc x \\ 0 & -\sin x \cos x & \cos^2 x & 0 \\ 0 & 0 & -1 & 1 \end{array}\right] \quad \text{by } R_3 - R_2$$

Therefore,

$$-u'_3 = 1 \quad \text{or} \quad u'_3 = -1, \qquad \text{(from } R_3\text{)}$$

$$(-\sin x \cos x)\, u'_2 + (\cos^2 x)\, u'_3 = 0, \qquad \text{(from } R_2\text{)}$$

or $$u'_2 = -\cot x.$$

$$u'_1 = \csc x, \qquad \text{(from } R_1\text{)}$$

Hence $$u_1 = \int \csc x\, dx = \ln|\csc x - \cot x|$$

$$u_2 = \int \frac{-\cos x}{\sin x}\, dx = -\ln|\sin x|$$

and $$u_3 = \int -dx = -x.$$

$$y_p = u_1 + u_2 \cos x + u_3 \sin x$$

$$= \ln|\csc x - \cot x| - (\cos x)\ln|\sin x| - x \sin x$$

is the required particular solution.

EXERCISE 10.6

Find a particular solution of each of the following (**Problems 1–9**).

1. $\dfrac{d^2y}{dx^2} + 4y = \sec 2x$

2. $\dfrac{d^2y}{dx^2} + y = \tan x \sec x$

3. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = (1 + e^{-x})^{-1}$

4. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = e^{-2x} \sec x$

5. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = \dfrac{e^{2x}}{1+x}$

6. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = e^x \arcsin x$

7. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 5y = e^x \tan 2x$

8. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = e^{-x} \ln x$

9. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 2e^{-x} \tan^2 x.$

10. Find the general solution of
$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^3 e^x,$$
given that $y_1 = x^2$ is a solution of the associated homogeneous equation.

11. Find the general solution of
$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = \frac{1}{1+x},$$
given tiat $y_1 = \dfrac{1}{x}$ is a solution of the associated homogeneous equation.

12. Find te general solution of
$$x(x-2)\frac{d^2y}{dx^2} - (x^2-2)\frac{dy}{dx} + 2(x-1)\,y = 3x^2(x-2)^2 e^x,$$
given that $y_1 = e^x$ is a solution of the associated homogeneous equation.

13. Find the general solution of
$$(\sin^2 x)\frac{d^2y}{dx^2} - (\sin 2x)\frac{dy}{dx} + (1 + \cos^2 x)\,y = \sin^3 x,$$
given that $y_1 = \sin x$ and $y_2 = x \sin x$ are linearly independent solutions of the associated homogeneous equation.

14. Use the method of Example 21 to find a particular solution of

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = \frac{2e^x}{x^2}$$

15. By the method of variation of parameters, find a particular solution of

$$\frac{d^3y}{dx^3} - 2\frac{dy}{dx} - 4y = e^{-x}\tan x.$$

ONE VARIABLE ABSENT (NONLINEAR D.E.)

(10.20) Consider an equation in which the dependent variable y is missing i.e., an equation of the form

$$f\left(\frac{d^ny}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots, \frac{dy}{dx}, x\right) = 0 \qquad (1)$$

If we set $p = \frac{dy}{dx}$, then (1) takes the form

$$f\left(\frac{d^{n-1}p}{dx^{n-1}}, \frac{d^{n-2}p}{dx^{n-2}}, \ldots, p, x\right) = 0$$

so that the order of (1) has been reduced by one.

If the derivative of lowest order occurring in the equation (1) is $\frac{d^ry}{dx^r}$, then put $\frac{d^ry}{dx^r} = p$.

Similarly, consider the equation

$$f\left(\frac{d^ny}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots, \frac{dy}{dx}, y\right) = 0 \qquad (2)$$

in which x is absent. We write

$$p = \frac{dy}{dx}$$

so that $$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(p\frac{dp}{dy}\right) = \frac{d}{dy}\left(p\frac{dp}{dy}\right)\frac{dy}{dx} = p\left(\frac{dp}{dy}\right)^2 + p^2\frac{d^2p}{dy^2}.$$

Substituting for $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \cdots$ the equation (2) will be transformed into an equation in p and y of order $n-1$. This equation in then solved for p and y. Such an equation may reduce to more than one equation in p and y. The solution of each of these equations must satisfy (2) in order that it is a solution of (2).

Example 22. Solve. $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + ax = 0 \qquad (1)$

Solution. Put $\frac{dy}{dx} = p$ and so $\frac{d^2y}{dx^2} = \frac{dp}{dx}$.

Then (1) becomes

$$(1+x^2)\frac{dp}{dx} + xp + ax = 0$$

or $$\frac{dp}{a+p} = -\frac{x}{1+x^2}\,dx.$$

Integrating, we have

$$\ln|a+p| = \ln(1+x^2)^{-\frac{1}{2}} + \ln|c_1|$$

$$a+p = \frac{c_1}{\sqrt{1+x^2}}$$

Therefore,

$$a + \frac{dy}{dx} = \frac{c_1}{\sqrt{1+x^2}}$$

or $$dy = \left(\frac{c_1}{\sqrt{1+x^2}} - a\right)dx$$

or $$y = c_1\sinh^{-1}x - ax + c_2$$

is the general solution of (1).

Example 23. Solve: $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx} \qquad (1)$

Solution. Here x is absent. We set

$$\frac{dy}{dx} = p \quad \text{and so} \quad \frac{d^2y}{dx^2} = p\frac{dp}{dy}.$$

Hence (1) is exact. Its first integral is

$$R_0 \frac{dy}{dx} + R_1 y = \int (2\cos x - 2x)\, dx + c_1$$

where $\quad R_0 = P_0 = x^2 + 1$

$$R_1 = P_1 - P_0' = 4x - 2x = 2x.$$

Thus, the first integral becomes

$$(x^2+1)\frac{dy}{dx} + 2xy = 2\sin x - x^2 + c_1 \qquad (2)$$

This is again exact, since

$$P_0 = x^2 + 1, \quad P_1 = 2x \quad \text{and} \quad P_1 - P_0' = 2x - 2x = 0.$$

First integral of (2) is

$$R_0\, y = \int (2\sin x - x^2 + c_1)\, dx + c_2$$

where $\quad R_0 = P_0 = x^2 + 1.$

Thus $\quad (x^2+1)\, y = -2\cos x - \dfrac{x^3}{3} + c_1 x + c_2$

is the general solution of (1).

Example 25. Solve

$$(2x-1)\frac{d^3y}{dx^3} + (4+x)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0. \qquad (1)$$

Solution. Here $n = 3$

$$P_0 = 2x - 1, \quad P_1 = 4 + x, \; P_2 = 2, \; P_3 = 0$$

$$P_3 - P_2' + P_1'' - P_0''' = 0 - 0 + 0 - 0 = 0.$$

Hence (1) is exact. Its first integral is

$$R_0 \frac{d^2y}{dx^2} + R_1 \frac{dy}{dx} + R_2 y = c_1, \qquad (2)$$

where $\quad R_0 = P_0 = 2x - 1$

$$R_1 = P_1 - P_0' = 4 + x - 2 = 2 + x$$

$$R_2 = P_2 - P_1' + P_0'' = 2 - 1 = 1$$

Hence (2) becomes

$$(2x-1)\frac{d^2y}{dx^2} + (2+x)\frac{dy}{dx} + y = c_1. \qquad (3)$$

In (3), we have

$$P_0 = 2x - 1, \quad P_1 = 2 + x, \quad P_2 = 1$$

and

$$P_2 - P_1' + P_0'' = 1 - 1 + 0 = 0.$$

Therefore (3) is exact. First integral of (3) is

$$R_0 \frac{dy}{dx} + R_1 y = \int c_1\, dx + c_2, \qquad (4)$$

where $\quad R_0 = P_0 = 2x - 1$

$$R_1 = P_1 - P_0' = 2 + x - 2 = x$$

Therefore (4) becomes

$$(2x-1)\frac{dy}{dx} + xy = c_1 x + c_2 \qquad (5)$$

In (5), we have

$$P_0 = 2x - 1, \quad P_1 = x$$

and $\quad P_1 - P_0' = x - 2 \neq 0$

Thus, (5) is not exact. We can write it as

$$\frac{dy}{dx} + \frac{x}{2x-1}\, y = \frac{c_1 x + c_2}{2x-1}$$

which is a linear equation. Its

$$\text{I.F.} = \exp\left(\int \frac{x\,dx}{2x-1}\right) = \exp\left(\int \left(\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2x-1}\right) dx\right)$$

$$= \exp\left(\frac{1}{2}x + \frac{1}{4}\ln(2x-1)\right) = e^{\frac{1}{2}x}\,.\,(2x-1)^{\frac{1}{4}}.$$

So $\quad \dfrac{d}{dx}\left[y e^{\frac{1}{2}x} \cdot (2x-1)^{\frac{1}{4}}\right] = \dfrac{e^{\frac{1}{2}x}(2x-1)^{\frac{1}{4}}(c_1 x + c_2)}{(2x-1)}$

or $\quad y e^{\frac{1}{2}x} \cdot (2x-1)^{\frac{1}{4}} = \displaystyle\int \frac{e^{\frac{1}{2}x}(c_1 x + c_2)}{(2x-1)^{\frac{3}{4}}}\, dx.$

TWO SPECIAL TYPES

(10.22) In this section we consider differential equations of the following types:

(i) $\dfrac{d^n y}{dx^n} = f(x)$ (ii) $\dfrac{d^2 y}{dx^2} = f(y)$

The differential equation in (i) is exact and its solution can be found by successive integration.

The differential equation in (ii) is not exact. In order to solve it, multiply both sides of (ii) by $2\dfrac{dy}{dx}$ to obtain

$$2\frac{dy}{dx}\frac{d^2y}{dx^2} = 2\frac{dy}{dx}f(y) = 2f(y)\frac{dy}{dx}$$

or $$\frac{d}{dx}\left(\frac{dy}{dx}\right)^2 = 2\frac{dy}{dx}f(y)$$

Integrating, we get

$$\left(\frac{dy}{dx}\right)^2 = 2\int f(y)\,dy + c$$

or $$\frac{dy}{dx} = \left[2\int f(y)\,dy + c\right]^{1/2}$$

which is a separable equation and can be solved.

Example 26. Solve: $x^4\dfrac{d^4y}{dx^4} + 1 = 0.$ (1)

Solution. $$\frac{d^4y}{dx^4} = -\frac{1}{x^4} = -x^{-4}.$$

Integrating successively, we have

$$\frac{d^3y}{dx^3} = -\frac{x^{-3}}{-3} + c_1$$

$$\frac{d^2y}{dx^2} = -\frac{x^{-3+1}}{3\cdot 2} + c_1x + c_2$$

$$\frac{dy}{dx} = -\frac{x^{-2+1}}{3\cdot 2\,(-1)} + c_1\frac{x^2}{2} + c_2x + c_3$$

$$y = \frac{1}{6}\ln|x| + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3x + c_4$$

is the general solution of (1).

Example 27. Solve: $\dfrac{d^2y}{dx^2} = a^2y.$ (1)

Solution. Multiplying both the sides of (1) by $2\dfrac{dy}{dx}$, we have

$$2\frac{dy}{dx}\frac{d^2y}{dx^2} = 2a^2y\frac{dy}{dx}$$

Integrating, we get

$$\left(\frac{dy}{dx}\right)^2 = a^2y^2 + c = a^2(y^2 + c_1)$$

$$\frac{dy}{dx} = a\sqrt{y^2 + c_1}$$

or $$\frac{dy}{\sqrt{y^2 + c_1}} = a\,dx$$

or $$\ln\left(y + \sqrt{y^2 + c_1}\right) = ax + c_2$$

is the general solution of the given equation.

EXERCISE 10.7

Solve:

1. $2\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 + 4 = 0$

2. $2x\dfrac{d^3y}{dx^3}\dfrac{d^2y}{dx^2} = \left(\dfrac{d^2y}{dx^2}\right)^2 - a^2$

3. $x\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^3 - \dfrac{dy}{dx} = 0$

4. $x\dfrac{d^3y}{dx^3} - 2\dfrac{d^2y}{dx^2} = 12x^3,$ $y(1) = 0,\ y'(1) = 1,\ y''(1) = 0$

5. $2y\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 = 1$

6. $y\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 = 4y^2\ln x,$ $y(1) = e,\ y'(1) = 2e$

7. $(1 + y^2)\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^3 + \dfrac{dy}{dx} = 0$

8. $y\dfrac{d^2y}{dx^2} + 4y^2 - \dfrac{1}{2}\left(\dfrac{dy}{dx}\right)^2 = 0,$ $y(0) = 1,\ y'(0) = \sqrt{8}$

9. $(2x^2 + 3x)\dfrac{d^2y}{dx^2} + (6x + 3)\dfrac{dy}{dx} + 2y = (x + 1)\, e^x$

10. $\sin x \dfrac{d^2y}{dx^2} - \cos x \dfrac{dy}{dx} + 2y \sin x = 0$

11. $(x + \sin x)\dfrac{d^3y}{dx^3} + 3(1 + \cos x)\dfrac{d^2y}{dx^2} - 3 \sin x \dfrac{dy}{dx} - y \cos x = -\sin x$

12. $(e^x + 2x)\dfrac{d^4y}{dx^4} + 4(e^x + 2)\dfrac{d^3y}{dx^3} + 6e^x \dfrac{d^2y}{dx^2} + 4e^x \dfrac{dy}{dx} + e^x\, y = \dfrac{1}{x^5}$

13. $x^5 \dfrac{d^2y}{dx^2} + 3x^3 \dfrac{dy}{dx} + (3 - 6x)\, x^2 y = x^4 + 2x - 5$

14. (i) $\dfrac{d^3y}{dx^3} = \ln x$ (ii) $\dfrac{d^2y}{dx^2} = x^2 \sin x$

15. (i) $\dfrac{d^2y}{dx^2} = -\cot y \csc^2 y;\ y(0) = \dfrac{\pi}{2},\ y'(0) = 1$ (ii) $\dfrac{d^2y}{dx^2} = -\dfrac{a^2}{y^2}$

LINEAR SYSTEMS OF D.E.

(10.23) The differential equations studied in previous sections involved one independent variable x and a dependent variable y. In many problems in applied mathematics, there occur several dependent variables which are functions of a single independent variable. The mathematical formulation of such problems gives rise to a system of simultaneous differential equations. In this section, we briefly discuss such a system. We shall restrict our discussion to three dependent variables x, y and z which are all functions of an independent variable t. The theory can be easily extended to n dependent variables of a single independent variable.

Let x, y, z be functions of t and let D denote $\dfrac{d}{dt}$. The system of three equations in three unknowns x, y, z such as

$$\left.\begin{aligned} L_{11}(D)\,x + L_{12}(D)\,y + L_{13}(D)\,z &= f_1(t) \\ L_{21}(D)\,x + L_{22}(D)\,y + L_{23}(D)\,z &= f_2(t) \\ L_{31}(D)\,x + L_{32}(D)\,y + L_{33}(D)\,z &= f_3(t) \end{aligned}\right\} \qquad (1)$$

where $L_{ij}(D)$, $(1 \le i, j \le 3)$ are linear operators, is called a [1]**linear system of differential equations**. The system (1) is said to be a **linear system with constant coefficients** if $L_{ij}(D)$ are polynomials in D with coefficients not involving the independent variable. For example,

$$\begin{aligned} (D^2 + D)\,x \qquad - D^2y + 3Dz &= 0 \\ (D^2 - 1)\,x + (2D + 3)\,y + Dz &= t \\ (D + 1)^2\,x \qquad + 5\,Dy - z &= t^2 \end{aligned}$$

is a linear system of differential equations with constant coefficients.

Here we shall study only linear system of differential equations with constant coefficients.

(10.24) Linear Systems with Constant Coefficients. Let

$$\left.\begin{aligned} L_{11}(D)\,x + L_{12}(D)\,y + L_{13}(D)\,z &= f_1(t) \\ L_{21}(D)\,x + L_{22}(D)\,y + L_{23}(D)\,z &= f_2(t) \\ L_{31}(D)\,x + L_{32}(D)\,y + L_{33}(D)\,z &= f_3(t) \end{aligned}\right\} \qquad (1)$$

be a linear system of differential equations with constant coefficients, i.e. $L_{ij}(D)$ are polynomials in D, $1 \le i, j \le 3$. A set of functions $x = f(t)$, $y = g(t)$, $z = h(t)$ is called a **solution** of the system (1), if each of the three equations of the system is satisfied on substitution of these values of the variables x, y, z and their derivatives with respect to t.

The solution of (1) can be found only if

$$\det [\, L \,] \ne 0,$$

where $[\, L \,] = [L_{ij}(D)]$ is the matrix of coefficients in (1).

As in the case of linear algebraic systems discussed in Chapter 4, the following elementary operations can be performed on a linear differential system with constant coefficients to produce another equivalent system:

(1) Interchanging two equations.

(2) Multiplying an equation by a nonzero constant.

(3) Operating on both sides of an equation by a polynomial operator and adding (subtracting) the result to (from) another equation.

These elementary operations will be used to find the general solution of the system (1).

Before solving a linear system, it should be changed into operator notation.

1. Recall that $L \equiv a_0(t)\, D^n + a_1(t)\, D^{n-1} + \dots + a_{n-1}(t)\, D + a_n(t)$ is an nth order linear operator, $a_0(t) \ne 0$.

Example 28. Solve the system

$$\frac{dx}{dt} + \frac{dy}{dt} + y - x = e^{2t}$$

$$\frac{d^2x}{dt^2} + \frac{dy}{dt} = 3e^{2t}$$

Solution. Using the operator notation with $D = \frac{d}{dt}$, we can write the given system as

$$(D-1)x + (D+1)y = e^{2t} \quad (1)$$

$$D^2x + Dy = 3e^{2t} \quad (2)$$

We shall eliminate one of the dependent variables from (1) and (2) to get an equation in one dependent variable.

Operating on (1) by D and on (2) by $D+1$, we have

$$D(D-1)x + D(D+1)y = De^{2t} = 2e^{2t} \quad (3)$$

$$(D+1)D^2x + (D+1)Dy = (D+1)3e^{2t} = 9e^{2t} \quad (4)$$

Subtracting (3) from (4), we get

$$(D^3 + D^2 - D^2 + D)x + 0 = 7e^{2t}$$

or $$(D^3 + D)x = 7e^{2t}$$

Its solution is easily found as

$$x = c_1 + c_2 \sin t + c_3 \cos t + \frac{7}{10}e^{2t} \quad (5)$$

Substituting for x from (5) into (2), we obtain

$$Dy = c_2 \sin t + c_3 \cos t + \frac{1}{5}e^{2t}$$

Integrating, we have

$$y = -c_2 \cos t + c_3 \sin t + \frac{1}{10}e^{2t} + c_4 \quad (6)$$

To verify, substitute x from (5) and y from (6) into both (1) and (2) and obtain after simplification

$$c_4 - c_1 = 0, \quad 0 = 0$$

Thus $c_4 = c_1$ and (5) and (6) constitute the required general solution of the given system provided that we replace c_4 in (6) by c_1.

Note. It is proved in advanced books that the number of independent constants in the general solution of a linear system of differential equations is the same as the degree in D of the determinant of the operational coefficients of the equations. If the determinant of operational coefficients is identically zero the system may have no solution or it may have solutions with any number of constants.

We note that the coefficient determinant of the system of Example 28 is

$$\begin{vmatrix} D-1 & D+1 \\ D^2 & D \end{vmatrix}$$

which is of degree three in D. Hence the number of independent constants in the general solution should be three as we have already obtained.

Example 29. Solve

$$\frac{d^2x}{dt^2} + \frac{dy}{dt} - x + y = 1$$

$$\frac{d^2y}{dt^2} + \frac{dx}{dt} - x + y = 0$$

Solution. Writing $D = \frac{d}{dt}$, the given system in operator notation is

$$(D^2 - 1)x + (D+1)y = 1 \quad (1)$$

$$(D-1)x + (D^2+1)y = 0 \quad (2)$$

Operating on (1) by (D^2+1) and on (2) by $(D+1)$, we have

$$(D^2+1)(D^2-1)x + (D^2+1)(D+1)y = 1 \quad (3)$$

$$(D+1)(D-1)x + (D+1)(D^2+1)y = 0 \quad (4)$$

Subtracting (4) from (3), we get

$$(D^4 - 1 - D^2 + 1)x = 1$$

or $$D^2(D^2-1)x = 1.$$

Its solution is

$$x = c_1 + c_2t + c_3e^t + c_4e^{-t} - \frac{t^2}{2}. \quad (A)$$

Again applying the operator $(D-1)$ on (1) and (D^2-1) on (2), we obtain

$$(D-1)(D^2-1)x + (D-1)(D+1)y = (D-1)1 = -1 \quad (5)$$

$$(D^2-1)(D-1)x + (D^2-1)(D^2+1)y = 0. \quad (6)$$

Subtracting (5) from (6), we have

$$(D^4 - 1 - D^2 + 1)\,y = 1$$

or

$$D^2(D^2 - 1)\,y = 1$$

Its general solution is

$$y = k_1 + k_2 t + k_3 e^t + k_4 e^{-t} - \frac{t^2}{2} \qquad \text{(B)}$$

Thus the solution of the system must be of the forms given by (A) and (B) for some choice of constants $c_1, c_2, c_3, c_4, k_1, k_2, k_3, k_4$. The determinant of operational coefficients for the system is

$$\begin{vmatrix} D^2 - 1 & D + 1 \\ D - 1 & D^2 + 1 \end{vmatrix}$$

which is of degree 4 in D. Hence the number of independent constants in the general solution should be four.

Substituting for x from (A) and for y from (B) into the second equation of the system, we get

$$\left[c_2 + c_3 e^t - c_4 e^{-t} - t\right] + \left[- c_1 - c_2 t - c_3 e^t - c_4 e^{-t} + \frac{t^2}{2}\right]$$

$$+ \left[k_3 e^t + k_4 e^{-t} - 1\right] + \left[k_1 + k_2 t + k_3 e^t + k_4 e^{-t} - \frac{t^2}{2}\right] = 0$$

or

$$(c_2 - c_1 - 1 + k_1) + (-1 - c_2 + k_2)\,t + 2k_3 e^t + 2(-c_4 + k_4)\,e^{-t} = 0$$

In order that the pair (A) and (B) satisfy the second equation of the system, we must have

$$\begin{aligned} c_2 - c_1 - 1 + k_1 &= 0 \quad \text{or} \quad k_1 = 1 + c_1 - c_2 \\ -1 - c_2 + k_2 &= 0 \quad \text{or} \quad k_2 = 1 + c_2 \\ k_3 &= 0 \quad \text{or} \quad k_3 = 0 \\ - c_4 + k_4 &= 0 \quad \text{or} \quad k_4 = c_4 \end{aligned}$$

Hence

$$x = c_1 + c_2 t + c_3 e^t + c_4 e^{-t} - \frac{t^2}{2}$$

$$y = (1 + c_1 - c_2) + (c_2 + 1)\,t + c_4 e^{-t} - \frac{t^2}{2}$$

is the general solution of the given system.

EXERCISE 10.8

Find the general solution of each of the following linear systems:

1. $\dfrac{dx}{dt} = y$, $\quad \dfrac{dy}{dt} = -4x + 4y$

2. $\dfrac{dx}{dt} = x + y$, $\quad \dfrac{dy}{dt} = 4x - 2y$

3. $2\dfrac{dx}{dt} + \dfrac{dy}{dt} - x - y = e^{-t}$, $\quad \dfrac{dx}{dt} + \dfrac{dy}{dt} + 2x + y = e^t$

4. $2\dfrac{dx}{dt} - 2\dfrac{dy}{dt} - 3x = t$, $\quad 2\dfrac{dx}{dt} + 2\dfrac{dy}{dt} + 3x + 8y = 2$

5. $\dfrac{d^2x}{dt^2} + \dfrac{dy}{dt} + 6x = 0$, $\quad \dfrac{dx}{dt} + \dfrac{dy}{dt} + 2x - 2y = 2$

6. $\dfrac{dx}{dt} - 2x - y = 3e^{2t}$, $\quad \dfrac{dy}{dt} + 4x - 2y = te^{2t}$

7. $\dfrac{dx}{dt} + \dfrac{dy}{dt} + 5x + 4y = e^{-t}$, $\quad \dfrac{dx}{dt} + \dfrac{dy}{dt} + 2x + y = 3$

8. $2\dfrac{dx}{dt} + \dfrac{dy}{dt} + 3x + 4y = -\cos t$, $\quad \dfrac{dx}{dt} + \dfrac{dy}{dt} + x + 2y = 2\sin t$

9. $\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2x + 6y = 2e^t$, $\quad 2\dfrac{dx}{dt} + 3\dfrac{dy}{dt} + 3x + 8y = -1$

10. $\dfrac{dx}{dt} - x - y = 0, \quad x(0) = 1$; $\quad \dfrac{dy}{dt} - y = 0, \quad y(0) = 0$

11. $\dfrac{dx}{dt} - x - 2y = t - 1, \quad x(0) = 0$; $\quad \dfrac{dy}{dt} - 3x - 2y = -5t - 2, \quad y(0) = 4$

12. $\dfrac{dx}{dt} - 3y = 0$, $\quad \dfrac{dy}{dt} + z = 0$, $\quad \dfrac{dz}{dt} + y = 0$

13. $\dfrac{d^2x}{dt^2} + \dfrac{dy}{dt} - y = 0$, $\quad 2\dfrac{dx}{dt} + \dfrac{dz}{dt} - x - z = 0$, $\quad \dfrac{dx}{dt} + \dfrac{dy}{dt} + 3x - 4y + 3z = 0$

There are many differential equations whose solutions cannot be obtained in terms of **elementary functions** by the methods already discussed. However, their solutions can be obtained in the form of power series. The procedure is similar to the method of undetermined coefficients for polynomials but the number of coefficients will not be finite in this case.

We begin our study with examples of first order differential equations.

POWER SERIES SOLUTIONS OF FIRST ORDER D.E.

Example 30. Consider the equation

$$y' = -2y \tag{1}$$

We assume that (1) has a power series solution

$$y = \sum_{n=0}^{\infty} c_n x^n$$

The series can be differentiated term by term so that

$$y' = \sum_{n=0}^{\infty} nc_n x^{n-1} = \sum_{n=1}^{\infty} nc_n x^{n-1}$$

Substituting for y and y' into (1), we have

$$\sum_{n=1}^{\infty} nc_n x^{n-1} = -2\sum_{n=0}^{\infty} c_n x^n \tag{2}$$

In (2), we reindex the series on the left by setting $n-1=k$ so that

$$\sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k$$

$$= \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n, \text{ since } k \text{ is dummy index.}$$

Now (2) becomes

$$\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n = -2\sum_{n=0}^{\infty} c_n x^n.$$

Equating coefficients of like terms, we have the recursion formula

$$(n+1)c_{n+1} = -2c_n \tag{3}$$

Setting $n = 0, 1, 2, 3, \cdots$ into (3), we get the successive coefficients in terms of c_0 as below

$$c_1 = -2c_0$$

$$c_2 = -\frac{2}{2}c_1 = 2c_0$$

$$c_3 = -\frac{2}{3}c_2 = -\frac{2\cdot 2}{3}c_0$$

The solution of the given equation becomes

$$y = c_0 - 2c_0 x + 2c_0 x^2 - \frac{2\cdot 2}{3}c_0 x^3 + \cdots$$

$$= c_0\left(1 + (-2x) + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \cdots\right). \tag{4}$$

The power series (4) can be easily seen as a convergent series with interval of convergence $]-\infty, \infty[$. Thus (4) is indeed a solution of (1).

If we solve (1) directly, we find that

$$y = c_0 e^{-2x}$$

a solution of (1) which is clearly the same as (4) with e^{-2x} expressed as a series.

Note: When an exact solution of an equation can be easily found, series solution is not beneficial.

Example 31. Find a series solution of the initial value problem

$$(1+x)y' - my = 0, \qquad y(0) = 1,$$

m being any real number.

Solution. We assume a series solution as

$$y = \sum_{n=0}^{\infty} c_n x^n. \tag{1}$$

Differentiation of both sides of (1) gives

$$y' = \sum_{n=0}^{\infty} nc_n x^{n-1}.$$

Substituting the values of y and y' into the given equation, we have

$$(1+x)\sum_{n=0}^{\infty} nc_n x^{n-1} - m\sum_{n=0}^{\infty} c_n x^n = 0$$

or $$\sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} nc_n x^n - m\sum_{n=0}^{\infty} c_n x^n = 0$$

or $$\sum_{n=0}^{\infty} (n+1)c_{n+1}x^n + \sum_{n=0}^{\infty} (n-m)c_n x^n = 0.$$

Equating coefficients of x^n, we get

$$(n+1)c_{n+1} = (m-n)c_n$$

Since $y(0)=1$, we obtain $c_0 = 1$ from (1).

Substituting $n = 0, 1, 2, 3, \cdots$, into (2), we have

$$c_1 = mc_0 = m$$

$$c_2 = \frac{m-1}{2}c_1 = \frac{m(m-1)}{2!}$$

$$c_3 = \frac{m-2}{3}c_2 = \frac{m(m-1)(m-2)}{3!}$$

$$\vdots \qquad \vdots \qquad \vdots$$

Inserting the values of $c_0, c_1, c_2, \cdots$ into (1), we obtain

$$y = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots$$

It may be easily checked that this series converges for $|x| < 1$. Thus (3) is a series solution of the given initial value problem.

If we solve the given equation directly, we have by separation of variables

$$\frac{y'}{y} = \frac{m}{1+x}.$$

Integrating, we get

$$\ln y = m\ln(1+x) + \ln c$$

or $$y = c(1+x)^m.$$

Applying the initial condition, we have

$$c = 1. \quad \text{Thus} \quad y = (1+x)^m$$

is solution of the given problem.

Comparing (3) and (4), we obtain

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots$$
$$+ \frac{m(m-1)(m-2)\cdots(m-r+1)}{r!}x^r + \cdots$$

which is the Binomial Theorem for exponent m, m real.

If m is a positive integer, then all the terms $c_r x^r$ for $r > m$ are zero and the series terminates at x^m. The right hand side of (3) becomes a polynomial of degree m which converges for all values of x.

EXERCISE 10.9

Apply the power series method to solve the following differential equations **(Problems 1 – 9)**

1. $y' = y\left(1 + \frac{1}{x}\right)$
2. $(x^2+x)y' = (2x+1)y$
3. $y' - ky = 0$
4. $y' + y - 1 = 0$
5. $x(1-x)y' = y$
6. $xy' = (2x^2+1)y$
7. $(1-x^2)y' = y$
8. $xy' - (x+2)y = -2x^2 - 2x$
9. $(x+1)y' - (2x+3)y = 0.$
10. Express $\arcsin x$ in the form of a power series $\sum_{n=0}^{\infty} c_n x^n$ by solving

$$y' = \frac{1}{\sqrt{1-x^2}}, \qquad y(0) = 0$$

in two different ways. Hence deduce the formula

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3\cdot 2^3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5\cdot 2^5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{7\cdot 2^7} + \cdots$$

SECOND-ORDER LINEAR EQUATIONS

Before we take up the problem of solving second order linear differential equations with variable coefficients by the power series method, we need some definitions.

(10.25) Definition. A real valued function $f: R \to R$ is said to be analytic at a point x_0 if it has a power series representation

$$f(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^n$$

that is convergent in some **neighbourhood**[1] of x_0. For example the functions e^x, $\sin x$, $\cos x$ and polynomials in x are all analytic at all points of R. The function represented by the sum of a power series is analytic at all points inside its interval of convergence. The powers series

$$1 + x + x^2 + \cdots$$

has the sum function $\frac{1}{1-x}$, $(|x| < 1)$ and it is analytic at all points within its interval of convergence i.e. at all points of $]-1, 1[$.

(10.26) Definition. Let

$$a_0(x)\,y'' + a_1(x)\,y' + a_2(x)\,y = 0 \tag{1}$$

be a linear homogeneous differential equation with variable coefficients $a_0(x)$, $a_1(x)$ and $a_2(x)$ that are continuous functions of x in some interval $]\alpha, \beta[$. If $a_0(x) \neq 0$ in its domain, we divide by $a_0(x)$ so that the equation (1) becomes

$$y'' + \frac{a_1(x)}{a_0(x)}\,y' + \frac{a_2(x)}{a_0(x)}\,y = 0$$

or $\quad y'' + P(x)\,y' + Q(x)\,y = 0$

where $\quad P(x) = \frac{a_1(x)}{a_0(x)},\; Q(x) = \frac{a_2(x)}{a_0(x)} \qquad (2)$

(2) is called the **standard** (or **normalized**) form of (1).

(10.27) Definition. Let $\quad y'' + P(x)\,y' + Q(x)\,y = 0 \qquad (1)$

be in the standard form. If the coefficients $P(x)$ and $Q(x)$ are analytic at a point $x = x_0$, then the point $x = x_0$ is called an **ordinary point** of the differential equation (1). Any point, that is not an ordinary point of (1), is called a **singular point**.

Thus for a point $x = x_0$ to be an ordinary point of (1), $P(x)$ and $Q(x)$ must be represented by power series in $x - x_0$ that are covergent in some neighbourhood of x_0.

If $P(x)$ and $Q(x)$ are algebraic functions then they are analytic at all points except where their denominators vanish. In this case all points, except where the denominators of $P(x)$ and $Q(x)$ vanish, are ordinary points of the differential equation.

---

1. A neighbourhood of x_0 is any open interval $]-a, a[$ containing x_0.

Example 32. Find the ordinary points of

$$(x^2 - 1)\,y'' + 3xy' + (x + 1)\,y = 0 \tag{1}$$

Solution. Writing (1) in the standard form, we have

$$y'' + \frac{3x}{x^2-1}\,y' + \frac{x+1}{x^2-1}\,y = 0$$

Here $\quad P(x) = \frac{3x}{x^2-1}$, $Q(x) = \frac{x+1}{x^2-1} = \frac{1}{x-1}$

$P(x)$ and $Q(x)$ are algebraic functions whose denominators vanish at $x = 1$, $x = -1$. Except for these points, all points on the real line R, are ordinary points of (1).

The points $x = \pm 1$ are singular points of (1).

Example 33. Find ordinary points of

$$y'' + \cos x\,y' + e^x\,y = 0 \tag{1}$$

Solution. Here $\quad P(x) = \cos x, \qquad Q(x) = e^x$

and these are transcendental functions. Since $\cos x$ and e^x have power series representations

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

and the series converge for all x, every point x of R is an ordinary point of (1). There are no singular points of (1).

SERIES SOLUTIONS AROUND ORDINARY POINTS

A sufficient condition for the existence of a power series solution of a linear differential equation

$$y'' + P(x)\,y' + Q(x)\,y = 0 \tag{1}$$

around an ordinary point is stated in the following theorem whose proof is omitted.

(10.28) Theorem. If $x = x_0$ is an ordinary point of (1) then it is always possible to find its two independent power series solutions of the form $y = \sum_{n=0}^{\infty} c_n\,(x - x_0)^n$. The power series will converge for $|x - x_0| < R$, where R is a positive number.

(10.29) The following procedure will be adopted to find a power series solution of

$$y'' + P(x)\,y' + Q(x) = 0 \qquad (1)$$

around an ordinary point $x = x_0$ of (1)

I. If $P(x)$ and $Q(x)$ are algebraic functions then multiply (1) by the L.C.M. of denominators of $P(x)$ and $Q(x)$ so that (1) becomes

$$a_0(x)\,y'' + a_1(x)\,y' + a_2(x)\,y = 0$$

and $a_0(x)$, $a_1(x)$ and $a_2(x)$ are polynomials in x.

If $P(x)$ and $Q(x)$ involve **transcendental functions**[1], replace them by their power series about the ordinary point $x = x_0$.

II. Assume $y = \sum\limits_{n=0}^{\infty} c_n (x-x_0)^n$ is a power series solution of (1)

III. Differentiate $y = \sum\limits_{n=0}^{\infty} c_n (x-x_0)^n$ twice and substitute into (1) for y, y' and y''.

IV. Equate coefficients of like powers of x to have a recursion relation from which coefficients c_n will be obtained step by step.

The method is illustrated by the following examples.

Example 34. Find the power series solution of

$$y'' - xy' - y = 0 \qquad (1)$$

around the ordinary point $x = 0$.

Solution. Let

$$y = \sum_{n=0}^{\infty} c_n x^n$$

be a solution of (1). Then

$$y' = \sum_{n=0}^{\infty} n c_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1)\, c_n x^{n-2}.$$

Substituting into (1) for y, y' and y'', we obtain

$$\sum_{n=0}^{\infty} n(n-1)\, c_n x^{n-2} - x \sum_{n=0}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

1. Functions which are not algebraic functions.

$$\sum_{n=2}^{\infty} n(n-1)\, c_n x^{n-2} - \sum_{n=0}^{\infty} (n+1)\, c_n x^n = 0 \qquad (2)$$

We reindex the first sum by setting $n-2 = k$ so that

$$\sum_{n=2}^{\infty} n(n-1)\, c_n x^{n-2} = \sum_{k=0}^{\infty} (k+2)(k+1)\, c_{k+2} x^k$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)\, c_{n+2} x^n, \text{ since } k \text{ is dummy variable.}$$

Now (2) becomes

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)\, c_{n+2} - (n+1)\, c_n\right] x^n = 0$$

Equating the coefficients of x^n, we get, for all natural numbers n,

$$(n+2)\, c_{n+2} - c_n = 0 \quad \text{or} \quad c_{n+2} = \frac{c_n}{n+2} \qquad (3)$$

(3) is the recursion formula from which the coefficients c_n will be obtained step by step.

Putting $n = 0$ and 1, we note that

$$c_2 = \frac{c_0}{2} \quad \text{and} \quad c_3 = \frac{c_1}{3}$$

Thus the coefficients c_n can be expressed in terms of c_0 and c_1 according as n is even or odd. For even and odd n, (3) may be written as

$$c_{2n+2} = \frac{c_{2n}}{2n+2} \qquad (4)$$

$$\text{and} \quad c_{2n+3} = \frac{c_{2n+1}}{2n+3} \qquad (5)$$

Setting $n = 0, 1, 2, 3 \cdots$, into (4) and (5), we have

$$c_2 = \frac{c_0}{2}$$

$$c_4 = \frac{c_2}{4} = \frac{c_0}{2^3} = \frac{c_0}{2^2 \cdot 2!}$$

$$c_6 = \frac{c_4}{2 \cdot 3} = \frac{c_0}{2^4 \cdot 3} = \frac{c_0}{2^3 \cdot 3!}$$

$$c_8 = \frac{c_6}{8} = \frac{c_0}{2^3 \cdot 3!} = \frac{c_0}{2^4 \cdot 4!}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

and $$c_3 = \frac{c_1}{3}$$

$$c_5 = \frac{c_3}{5} = \frac{c_1}{3 \cdot 5}$$

$$c_7 = \frac{c_5}{7} = \frac{c_1}{3 \cdot 5 \cdot 7}$$

$$c_9 = \frac{c_7}{9} = \frac{c_1}{3 \cdot 5 \cdot 7 \cdot 9}$$

$$\vdots \quad \vdots \quad \vdots$$

Thus $$u(x) = c_0 + \frac{c_0}{2} x^2 + \frac{c_0}{2^2 \cdot 2!} x^4 + \frac{c_0}{2^3 \cdot 3!} x^6 + \cdots$$

and $$v(x) = c_1 x + \frac{c_1}{3} x^3 + \frac{c_1}{3 \cdot 5} x^5 + \frac{c_1}{3 \cdot 5 \cdot 7} x^7 + \cdots$$

are two linearly independent solutions of (1).

The general power series solution is

$$y = u(x) + v(x)$$

$$= c_0\left(1 + \frac{x^2}{2} + \frac{x^4}{2^2 \cdot 2!} + \frac{x^6}{2^3 \cdot 3!} + \frac{x^8}{2^4 \cdot 4!} + \cdots\right) + c_1\left(x + \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} + \frac{x^7}{3 \cdot 5 \cdot 7} + \cdots\right)$$

$$= c_0 e^{\frac{x^2}{2}} + c_1\left(x + \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} + \frac{x^7}{3 \cdot 5 \cdot 7} + \cdots\right).$$

Example 35. Find the series solution of Airy's[1] equation

$$y'' - xy = 0. \qquad (1)$$

Solution. Here $x = 0$ is an ordinary point of (1).

Let $y = \sum_{n=0}^{\infty} c_n x^n$ be a series solution of (1).

Then $y' = \sum_{n=0}^{\infty} n c_n x^{n-1}$ and $y'' = \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}$.

1. Named after the English astronomer Sir George Biddel Airy (1801 – 1892).

Substituting for y and y'' into (1), we have

$$\sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

or $$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=3}^{\infty} c_{n-3} x^{n-2} = 0$$

or $$2c_2 + \sum_{n=3}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=3}^{\infty} c_{n-3} x^{n-2} = 0$$

Equating coefficients of like terms, we get

$$2c_2 = 0$$

and $$n(n-1) c_n - c_{n-3} = 0, \qquad n \geq 3$$

or $$c_n = \frac{c_{n-3}}{n(n-1)}. \qquad (2)$$

Setting $n = 3, 4, 5, 6, \cdots$ into (2), we have

$$c_3 = \frac{c_0}{6}$$

$$c_4 = \frac{c_1}{3 \cdot 4}$$

$$c_5 = \frac{c_2}{5 \cdot 4} = 0 = c_8 = c_{11} = \cdots = c_{3n+2}$$

$$c_6 = \frac{c_3}{6 \cdot 5} = \frac{c_0}{6 \cdot 5 \cdot 3 \cdot 2}$$

$$c_7 = \frac{c_4}{7 \cdot 6} = \frac{c_1}{7 \cdot 6 \cdot 4 \cdot 3}.$$

Thus

$c_2, c_5, c_8, \cdots, c_{3n+2}, \cdots$ are all zero,

$c_3, c_6, c_9, \cdots, c_{3n}, \cdots$ are all dependent on c_0.

$c_4, c_7, c_{10}, \cdots, c_{3n+1}, \cdots$ are all dependent on c_1.

$c_n = \dfrac{c_{n-3}}{n(n-1)}$ implies that $c_{3n} = \dfrac{c_{3n-3}}{3n(3n-1)}$

Therefore, $c_3 = \dfrac{c_0}{3\cdot 2} = \dfrac{c_0}{6}$

$$c_6 = \frac{c_3}{6\cdot 5} = \frac{c_0}{6\cdot 5\cdot 3\cdot 2} = \frac{c_0}{180}$$

$$c_9 = \frac{c_6}{9\cdot 8} = \frac{c_0}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2} = \frac{c_0}{12960}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$c_{3n+1} = \frac{c_{3n-2}}{3n(3n+1)} \quad \text{and so}$$

$$c_4 = \frac{c_1}{3\cdot 4} = \frac{c_1}{12}$$

$$c_7 = \frac{c_4}{6\cdot 7} = \frac{c_1}{3\cdot 4\cdot 6\cdot 7} = \frac{c_1}{504}$$

$$c_{10} = \frac{c_7}{9\cdot 10} = \frac{c_1}{504\times 90} = \frac{c_1}{45360}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

Two independent solutions are

$$u(x) = c_0\left[\frac{1}{6}x^3 + \frac{1}{180}x^6 + \frac{1}{12960}x^9 + \cdots\right]$$

and $$v(x) = c_1\left[x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \frac{1}{45360}x^{10} + \cdots\right]$$

So the general power series solution is

$$y = u(x) + v(x)$$

Example 36. Find the series solution of

$$(x^2 - 2x)\,y'' + 5(x-1)\,y' + 3y = 0 \qquad (1)$$

around the ordinary point $x = 1$

Solution. Let $y = \sum_{n=0}^{\infty} c_n (x-1)^n$ be a solution of (1). Then

$$y' = \sum_{n=0}^{\infty} n\,c_n (x-1)^{n-1},$$

$$y'' = \sum_{n=0}^{\infty} n(n-1)\,c_n (x-1)^{n-2}$$

Substituting these values of y, y' and y'' into (1), we have

$$(x^2-2x)\sum_{n=0}^{\infty} n(n-1)\,c_n (x-1)^{n-2} + 5(x-1)\sum_{n=0}^{\infty} n\,c_n (x-1)^{n-1} + 3\sum_{n=0}^{\infty} c_n (x-1)^n = 0$$

or $$[(x-1)^2 - 1]\sum_{n=0}^{\infty} n(n-1)\,c_n (x-1)^{n-2} + 5(x-1)\sum_{n=0}^{\infty} nc_n (x-1)^{n-1}$$

$$+\,3\sum_{n=0}^{\infty} c_n (x-1)^n = 0$$

or $$\sum_{n=0}^{\infty} n(n-1)\,c_n (x-1)^n + 5\sum_{n=0}^{\infty} nc_n (x-1)^n + 3\sum_{n=0}^{\infty} c_n (x-1)^n$$

$$-\sum_{n=2}^{\infty} n(n-1)\,c_n (x-1)^{n-2} = 0$$

or $$\sum_{n=0}^{\infty} [n(n-1) + 5n + 3]\,c_n (x-1)^n - \sum_{n=0}^{\infty} (n+2)(n+1)\,c_{n+2} (x-1)^n = 0$$

Equating the coefficients of $(x-1)^n$, we have

$$(n^2 + 4n + 3)\,c_n = (n+2)(n+1)\,c_{n+2}$$

or $$c_{n+2} = \frac{n+3}{n+2}\,c_n \qquad (2)$$

is the recursion relation for finding the coefficients. From (2), we get

$$c_{2n} = \frac{2n+1}{2n}\,c_{2n-2} \qquad (3)$$

by taking $n + 2 = 2k$ so that $n = 2k - 2$ and replacing k by n in the resulting expression involving k.

and $$c_{2n+1} = \frac{2n+2}{2n+1}\,c_{2n-1} \qquad (4)$$

Setting $n = 1, 2, 3, \cdots$ into (3) and (4), we obtain

$$c_2 = \frac{3}{2}\,c_0$$

$$c_4 = \frac{5}{4}\,c_2 = \frac{5\cdot 3}{4\cdot 2}\,c_0$$

$$c_6 = \frac{7}{6}\, c_4 = \frac{7\cdot5\cdot3}{6\cdot4\cdot2}\, c_0$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$c_{2n} = \frac{3\cdot5\cdot7\cdots(2n+1)}{2\cdot4\cdot6\cdots 2n}\, c_0$$

and $c_3 = \frac{4}{3}\, c_1$

$$c_5 = \frac{6}{5}\, c_3 = \frac{6\cdot4}{5\cdot3}\, c_1$$

$$c_7 = \frac{8}{7}\, c_5 = \frac{8\cdot6\cdot4}{7\cdot5\cdot3}\, c_1$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$c_{2n+1} = \frac{4\cdot6\cdot8\cdots(2n+2)}{3\cdot5\cdot7\cdots(2n+1)}\, c_1$$

Required solution is

$$y = c_0 + c_0\left[\frac{3}{2}(x-1)^2 + \frac{3\cdot5}{2\cdot4}(x-1)^4 + \cdots\right] + c_1(x-1) + c_1\left[\frac{4}{3}(x-1)^3 + \frac{6\cdot5}{5\cdot3}(x-1)^5 + \cdots\right]$$

$$= c_0 + c_0 \sum_{n=1}^{\infty} \frac{3\cdot5\cdot7\cdots(2n+1)}{2\cdot4\cdot6\cdots 2n}(x-1)^{2n} + c_1(x-1) + c_1 \sum_{n=1}^{\infty} \frac{4\cdot6\cdot8\cdots(2n+2)}{3\cdot5\cdot7\cdots 2n+1}(x-1)^{2n+1}$$

Example 37. Find the series solution of

$$xy'' + y\sin x = 0$$

around the ordinary point $x = 0$.

Solution. The given equation in the standard form is

$$y'' + \frac{\sin x}{x}\, y = 0 \qquad (1)$$

Here $Q(x) = \frac{\sin x}{x}$

$$= \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{y^5}{5!} - \cdots\right) = 1 - \frac{x^2}{6} + \frac{x^4}{120} - \cdots$$

and this series converges for all x. Thus every value of x is an ordinary point and in particular $x = 0$ is an ordinary point.

Let $y = \sum_{n=0}^{\infty} c_n x^n$ be a solution of (1)

Then, on substitution of the values of y and y'' into (1), we have

$$\sum_{n=2}^{\infty} n(n-1)\, c_n x^{n-2} + \left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{720} + \cdots\right)(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots) = 0$$

or
$$2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \cdots + c_0 + c_1x + \left(c_2 - \frac{c_0}{6}\right)x^2 + \left(-\frac{c_1}{6} + c_3\right)x^3 + \left(c_4 + \frac{c_0}{120} - \frac{c_2}{6}\right)x^4 + \cdots = 0$$

Equating coefficients of like terms, we get

$$2c_2 + c_0 = 0 \quad \text{or} \quad c_2 = -\frac{1}{2}\, c_0$$

$$6c_3 + c_1 = 0 \quad \text{or} \quad c_3 = -\frac{1}{6}\, c_1$$

$$12c_4 + c_2 - \frac{c_0}{6} = 0$$

or $$12c_4 = -c_2 + \frac{c_0}{6} = \frac{1}{2}\, c_0 + \frac{c_0}{6} = \frac{2}{3}\, c_0 \quad \text{or} \quad c_4 = \frac{1}{18}\, c_0$$

$$20c_5 - \frac{c_1}{6} + c_3 = 0$$

or $$20c_5 = \frac{c_1}{6} - c_3 = \frac{c_1}{6} + \frac{c_1}{6} = \frac{c_1}{3} \quad \text{or} \quad c_5 = \frac{c_1}{60}$$

$$30c_6 + c_4 + \frac{c_0}{120} - \frac{c_2}{6} = 0$$

or $$30c_6 = -c_4 - \frac{c_0}{120} + \frac{c_2}{6} = -\frac{1}{18}\, c_0 - \frac{1}{120}\, c_0 - \frac{1}{12}\, c_0 = -\frac{53}{360}\, c_0.$$

Required solution is

$$y = c_0 - \frac{1}{2} c_0 x^2 + \frac{1}{18} c_0 x^4 - \frac{53}{360} c_0 x^6 + \cdots$$

$$+ c_1 x - \frac{1}{6} c_1 x^3 + \frac{1}{60} c_1 x^5 + \cdots$$

$$= c_0\left(1 - \frac{1}{2}x^2 + \frac{1}{18}x^4 - \frac{53}{360}x^6 + \cdots\right) + c_1\left(x - \frac{1}{6}x^3 + \frac{1}{60}x^5 + \cdots\right)$$

Example 38. Find the power series solution of the initial value problem

$$x(2-x)\,y'' - 6(x-1)\,y' - 4y = 0, \quad y(1) = 1, \quad y'(1) = 0$$

Solution. Since the initial conditions are given at $x = 1$, we seek a power series solution of the form $y = \sum_{n=0}^{\infty} c_n (x-1)^n$.

Substituting into the given equation for y, y' and y'', we have

$$[1-(x-1)^2] \sum_{n=0}^{\infty} n(n-1)\,c_n (x-1)^{n-2} - 6(x-1) \sum_{n=0}^{\infty} nc_n (x-1)^{n-1}$$

$$-4 \sum_{n=0}^{\infty} c_n (x-1)^n = 0$$

or
$$\sum_{n=2}^{\infty} n(n-1)\,c_n (x-1)^{n-2} - \sum_{n=2}^{\infty} n(n-1)\,c_n (x-1)^n - 6 \sum_{n=1}^{\infty} nc_n (x-1)^n$$

$$-4 \sum_{n=0}^{\infty} c_n (x-1)^n = 0$$

or
$$\sum_{n=0}^{\infty} (n+1)(n+2)\,c_{n+2} (x-1)^n - \sum_{n=2}^{\infty} n(n-1)\,c_n (x-1)^n - 6 \sum_{n=1}^{\infty} nc_n (x-1)^n$$

$$-4 \sum_{n=0}^{\infty} c_n (x-1)^n = 0$$

or
$$2c_2 + 6c_3(x-1) + \sum_{n=2}^{\infty} (n+1)(n+2)\,c_{n+2} (x-1)^n - \sum_{n=2}^{\infty} n(n-1)\,c_n (x-1)^n$$

$$-6c_1(x-1) - 6\sum_{n=2}^{\infty} nc_n (x-1)^n - 4c_0 - 4c_1(x-1) - 4\sum_{n=2}^{\infty} c_n (x-1)^n = 0$$

or
$$(2c_2 - 4c_0) + (6c_3 - 10c_1)(x-1) + \sum_{n=2}^{\infty} \Big[(n+2)(n+1)\,c_{n+2} - n(n-1)\,c_n$$

$$-6nc_n - 4c_n\Big](x-1)^n = 0$$

Equating coefficients of like powers of $x - 1$, we have

Coeff. of $(x-1)^0$: $2c_2 - 4c_0 = 0$ or $c_2 = 2c_0 = 2$,

since from initial condition, $y(1) = c_0 = 1$

Coeff. of $(x-1)$: $6c_3 - 10c_1 = 0$ or $c_3 = \frac{10}{6} c_1 = 0$, since $c_1 = y'(1) = 0$

Coeff. of $(x-1)^n$: $(n+2)(n+1)c_{n+2} + (-n^2 + n - 6n - 4)\,c_n = 0$, $n \geq 2$

i.e., $$c_{n+2} = \frac{n^2 + 5n + 4}{(n+2)(n+1)}\, c_n = \frac{n+4}{n+2}\, c_n$$

Therefore, $c_4 = \frac{6}{4} c_2 = 3$

$$c_5 = \frac{7}{6} c_3 = 0 = c_7 = \cdots = c_{2n+1}$$

$$c_{2n+2} = \frac{2n+4}{2n+2}\, c_{2n} = \frac{n+2}{n+1}\, c_{2n} \qquad n = 0, 1, 2, \cdots$$

$$c_6 = \frac{4}{3} c_4 = \frac{4}{3} \cdot 3 = 4$$

$$c_8 = \frac{5}{4} c_6 = 5$$

$$c_{10} = \frac{6}{5} c_8 = 6$$

$$\vdots \quad \vdots \quad \vdots$$

$$c_{2n} = \frac{n+1}{n}\, c_{2n-2} = n + 1.$$

Required solution is

$$y = 1 + 2(x-1)^2 + 3(x-1)^4 + 4(x-1)^6 + \cdots + (n+1)(x-1)^{2n} + \cdots$$

$$= \sum_{n=0}^{\infty} (n + 1\,(x-1)^{2n}.$$

EXERCISE 10.10

Find the series solution of each of the following differential equations around the indicated ordinary point (**Problems 1 – 14**)

1. $(x^2 - 1)\,y'' + 4xy' + 2y = 0$, around $x = 0$
2. $y'' - x^2 y = 0$, around $x = 0$
3. $y'' - x^3 y = 0$, around $x = 0$
4. $y'' + \dfrac{3x}{1+x^2}\,y' + \dfrac{1}{1+x^2}\,y = 0$, around $x = 0$
5. $y'' + xy' + (x^2 + 2)\,y = 0$, around $x = 0$
6. $y'' - 4xy' - 4y = 4 + 6x$, around $x = 0$
7. $y'' - 2x^2 y' + 4xy = x^2 + 2x + 4$, around $x = 0$
8. $(1 - x^2)\,y'' - 2xy' + m(m+1)\,y = 0$, (Legendre's equation) around $x = 0$
9. $y'' + (\alpha + \beta \cos 2x)\,y = 0$, (Mathieu's equation) around $x = 0$
10. $y'' + x^3 y' + 3x^2 y = e^x$, around $x = 0$
11. $y'' + (x^2 - 1)\,y = 0$, around $x = 1$
12. $y'' + (x - 3)\,y' + y = 0$, around $x = 2$
13. $y'' + xy' + (\ln x)\,y = 0$, around $x = 1$
14. $y'' + (\sin x)\,y = 0$, around $x = \frac{\pi}{2}$

Solve the initial value problems using power series method:

15. $y'' + xy' + 2y = 0$, $y(0) = 4$, $y'(0) = -1$
16. $y'' + e^x y = 0$, $y(0) = 1$, $y'(0) = -1$
17. $(x^2 + 1)\,y'' + xy' + 2xy = 0$, $y(0) = 2$, $y'(0) = 3$
18. $y'' + (x^2 + 2x + 1)\,y' - (4 + 4x)\,y = 0$, $y(-1) = 0$, $y'(-1) = 1$
19. $y'' + xy = e^{2x}$, $y(0) = 0$, $y'(0) = 1$
20. $xy'' + y' + 2y = 0$, $y(1) = 2$, $y'(1) = 4$.

APPLICATIONS OF D.E.

Differential equations can be used in the solutions of many problems in physical, biological and social sciences. In this section, we shall study such problems which can be formulated in terms of first and second order differential equations. The solution of the differential equation will lead to the solution of the given problem.

The following examples illustrate the techniques.

Example 39. A stone weighting 8 lbs falls from rest toward the earth from a great height. As it falls it is acted upon by air resistance that is numerically $\frac{1}{2}\,v$ (in pounds) where v is the velocity (in feet per second). Find the velocity and distance fallen at time t.

Solution. We use Newton's second law $F = ma$ to formulate mathematical model of this problem.

The forces acting on the body are:

(i) F_1, the weight of the stone acting downward and hence is positive.

(ii) F_2, the air resistance, numerically equal to $\frac{1}{2}\,v$, which acts upward and is therefore negative.

Newton's second law becomes

$$m\frac{dv}{dt} = F_1 + F_2$$

i.e., $$\frac{1}{4}\frac{dv}{dt} = 9 - \frac{1}{2}v, \quad \text{taking } g = 32.$$

or $$\frac{dv}{dt} = 32 - 2v$$

or $$\frac{dv}{dt} + 2v = 32$$

which is a linear equation. We now solve this differential equation.

Here

$$\text{I.F.} = e^{\int 2dt} = e^{2t}$$

and so, $$\frac{d}{dt}(ve^{2t}) = 32\,e^{2t}$$

Integrating, we get

$$ve^{2t} = 16\,e^{2t} + c \quad \text{or} \quad v = 16 + ce^{-2t}.$$

Now $v(0) = 0$.

Hence $0 = 16 + c$ or $c = -16$

Therefore, $v = 16\{1 - e^{-2t}\}$ (1)

is the velocity after time t.

(1) may be written as

$$\frac{dx}{dt} = 16(1 - e^{-2t}). \qquad (2)$$

From (2), we have,

$$dx = 16(1 - e^{-2t})\,dt$$

or $$x = 16t + 8e^{-2t} + k.$$

Applying the initial condition $x(0) = 0$, we find $k = -8$.

Hence

$$x = 16t + 8e^{-2t} - 8$$

is the distance fallen after time t.

Example 40. A body of constant mass is projected upward from the earth's surface with an initial velocity v_0. Assuming there is no air resistance, but taking into consideration the variation of the earth's gravitational field with altitude, find the smallest initial velocity for which the body will not return to the earth (This is the so called **escape velocity**).

Solution. The general expression for the weight $w(x)$ of a body of mass m is obtained from Newton's inverse-square law of gravitational attraction. If R is the radius of the earth and x is the altitude above sea level, then $w(x) = \dfrac{k}{(R+x)^2}$, k being constant. At $x = 0$, $w = mg$, hence $k = mgR^2$ and $w(x) = \dfrac{mgR^2}{(R+x)^2}$. The only force acting on the body is its weight which acts downward. Thus the equation of motion is

$$m\frac{dv}{dt} = \frac{-mgR^2}{(R+x)^2}$$

or $$v\frac{dv}{dt} = \frac{-mgR^2}{(R+x)^2}.$$

Separating the variables and integrating, we have

$$\frac{1}{2}v^2 = \frac{gR^2}{R+x} + c.$$

But $v = v_0$ at $t = 0$ i.e., at $x = 0$.

Therefore, $c = \dfrac{1}{2}v_0^2 - gR$

and so $$v^2 = v_0^2 - 2gR + \frac{2gR^2}{R+x}. \qquad (1)$$

The escape velocity is found by requiring that v given by (1) remains positive for all (positive) value of x. Thus we must have

$$v_0^2 \geq 2gR.$$

Hence the escape velocity is

$$v_0 = \sqrt{2gR} = 6.9 \text{ miles/sec. taking } R = 4000 \text{ miles.}$$

Example 41. The radioactive isotopes thorium 234 disintegrates at a rate proportional to the amount present. It is found that in one week 17.06 % of this material has disintegrated. Find an expression for the amount of material at any time. Also determine how long will it take for one half of this material to disintegrate?

Solution. Let y be the amount of thorium 234 present at any time t (t in days). Let $y = y_0$ at $t = 0$. We have

$$\frac{dy}{dt} = -ky \qquad (1)$$

$$y(0) = y_0, \quad y(7) = 0.8204\, y_0,$$

k being constant of proportionality.

Solution of (1) is

$$y = ce^{-kt}$$

At $t = 0$, $c = y_0$.

Therefore, $y = y_0 e^{-kt}$.

Now $y(7) = 0.8204\, y_0 = y_0 e^{-7k}$

or $e^{-7k} = 0.8204$

or $-7k = \ln 0.8204$

or $k = \dfrac{-\ln 0.8204}{7} = -0.02828.$

Hence, $y = y_0 e^{-0.02828\,t}$ (2)

gives the value of y at any time.

Now we want to find t when $y = 0.5\, y_0$. This is obtained from (2) by

$$0.5\, y_0 = y_0 e^{-0.02828\,t}$$

or $e^{0.02828\,t} = 0.5$

or $-0.02828\,t = \ln 0.5$

$= -\ln 2 = -0.6951$

or $t = \dfrac{0.66951}{0.02828} = 24.5$ days.

Example 42. A tank contains x_0 kg of salt dissolved in 200 litres of water. Starting at time $t = 0$ water containing 1/2 kg of salt per litre enters the tank at the rate of 4 litres/min and the well-stirred solution leaves the tank at the same rate. Find the concentration of salt in the tank at any time $t > 0$.

Solution. Let x denote the amount of salt in the tank at time t. Then the rate of change of the salt in the tank at time t is equal to the rate at which salt enters the tank minus the rate at which it leaves the tank.

The rate at which salt enters the tank is

$$\left(\frac{1}{2}\text{ kg/litre}\right)(4\text{ litres/min}) = 2\text{ kg/min}$$

The rate at which salt leaves the tank is

$$\left(\frac{x}{200}\text{ kg/litre}\right)(4\text{ litres/min}) = \frac{x}{50}\text{ kg/min}$$

Hence we have

$$\frac{dx}{dt} = 2 - \frac{x}{50}$$

or
$$\frac{dx}{dt} + \frac{x}{50} = 2$$

$$\text{I.F.} = e^{\int \frac{1}{50}dt} = e^{0.02t}$$

Therefore,

$$\frac{d}{dt}(xe^{0.02t}) = 2e^{0.02t}$$

Integrating, we get

$$xe^{0.02t} = \int 2e^{0.02t}\,dt + c$$

$$= \frac{2}{0.02}e^{0.02t} + c$$

or
$$x = 100 + c\,e^{-0.02t}$$

When $t = 0$, $x = x_0$. Thus

$$x_0 = 100 + c \quad \text{i.e.,} \quad c = x_0 - 100.$$

Hence
$$x = 100 + (x_0 - 100)\,e^{-0.02t}$$

$$= x_0 e^{-0.02t} + 100\left(1 - e^{-0.02t}\right).$$

The concentration $c(t)$ of salt in the tank is given by

$$c(t) = \frac{x}{200} = \frac{x_0 e^{-0.02t}}{200} + \frac{1}{2}(1 - e^{-0.02t})\text{ kg/litre.}$$

Example 43. A litre of ice cream at a temperature of $-15°C$ is removed from the deep freezer and placed in a room where the temperature is $20°C$. If after 15 minutes the temperature of the ice cream is $-10°C$, how long will it take the ice cream to reach a temperature of $0°C$?

Solution. Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between the temperatures of the object and its surroundings. Thus if $\theta(t)$ is temperature of the ice cream at time t and T is the temperature of the room, then

$$\frac{d\theta}{dt} = -k(\theta - T), \tag{1}$$

where $k > 0$ is constant of proportionality.

From (1), we have

$$\frac{d\theta}{\theta - T} = -k\,dt.$$

Integrating, we get

$$\ln(\theta - T) = -kt + \ln c$$

or
$$\theta - T = ce^{-kt}.$$

At $t = 0$, let $\theta = \theta(0)$, then

$$\theta(0) - T = c.$$

Therefore,

$$\theta - T = (\theta(0) - T)\,e^{-kt}. \tag{2}$$

To determine k we need one more condition. At $t = t_1$, let $\theta = \theta(t_1)$. Then from (2),

$$\theta(t_1) = T + (\theta(0) - T)\,e^{-kt_1}$$

or
$$\frac{\theta(t_1) - T}{\theta(0) - T} = e^{-kt_1}$$

or
$$k = -\frac{1}{t_1}\ln\frac{\theta(t_1) - T}{\theta(0) - T}$$

At $t = 0$, $\theta(0) = -15$.

At $t = t_1 = 20$, $\theta(20) = -10$.

Therefore,

$$k = -\frac{1}{20}\ln\frac{-10 - 20}{-15 - 20}$$

$$= -\frac{1}{20}\ln\frac{6}{7} \quad \text{which is the value of the constant of proportionality.}$$

We need t such that $\theta(t) = 0$.

From (2), we have

$$0 = T + (\theta(0) - T)\,e^{-kt}$$

or $$0 = 20 + (-35)\,e^{-kt}, \text{ since } T = 20,\ \theta(0) = -15$$

or $$e^{-kt} = \frac{20}{35}$$

or $$-kt = \ln\frac{4}{7}$$

i.e., $$t = -\frac{1}{k}\ln\frac{4}{7} = \frac{20}{\ln\frac{6}{7}}\ln\frac{4}{7}$$

$$= \frac{20(\ln 4 - \ln 7)}{\ln 6 - \ln 7} = \frac{20(1.3863 - 1.9459)}{1.7928 - 1.9459}$$

$$= \frac{20(-0.5596)}{-0.1541} = 72.63 \text{ min.}$$

Thus the ice cream reaches a temperature of 0°C after 72.63 min. of its removal from the deep freezer.

Example 44. A car, moving at a certain velocity and with constant acceleration, is applied brakes to make it stop. The car stops 10 seconds after the brakes are applied and travels 300 meters during this time. Find the law of motion of the car during this 10 second interval. Also find the distance covered, the speed and acceleration of the car at the moment the brakes are applied.

Solution. Suppose that the car is represented by a particle moving in a straight line at the time brakes are applied. We have the following initial conditions.

$$s = 0 \text{ at } t = 0,\quad s = 300 \text{ at } t = 10,\quad v = 0 \text{ at } t = 10$$

Now the particle (car) is moving at a constant acceleration. So

$\frac{dv}{dt} = a$ = constant. This is a first order separable differential equation. Hence

$$v = at + c_1 \tag{1}$$

Also $\frac{ds}{dt} = v = at + c_1$ which, again, is a first order separable differential equation. The solution of this differential equation is

$$s = \frac{1}{2}at^2 + c_1 t + c_2 \tag{2}$$

Now $s = 0$ at $t = 0$. So $c_2 = 0$. Next, at $t = 10$, $v = 0$, so, from (1), $c_1 = -10a$. Hence (2) becomes

$$s = \frac{1}{2}at^2 - 10at. \tag{3}$$

At $s = 300$, $t = 10$, so from equation (3), we have

$$300 = \frac{1}{2}a \times 100 - 10a \times 10$$

$$= -50a$$

So the initial acceleration is $a = -6$ m/sec^2

Putting this value of a into (3), we get

$$s = 60t - 3t^2 \tag{4}$$

as the law of motion.

Initial speed of the car $= v = -10a = 60$ m/sec.

The distance covered by the car before coming to rest is (with $t = 10$ seconds)

$$s = 600 - 300 \text{ m} = 300 \text{ m}$$

Example 45. A projectile is fired from a platform 5 m above the ground, with an initial velocity of 250 m/sec. The only force affecting the projectile, during its flight, is taken to be the gravity which is equivalent to a force having a downward acceleration of 9.8 m/sec^2. Find the equation which gives the projectile's height above the ground as a function of the time t with $t = 0$ when the projectile is fired. Find also the height of the projectile from the ground after 5 seconds. After how many seconds the velocity will be zero and what will be height at that time?

Solution. Let s denote the height of the projectile after t seconds of its flight, v its velocity at time t and a its acceleration. The quantities a, v and s are related by the following derivatives

$$\frac{ds}{dt} = v,\quad \frac{dv}{dt} = a,\quad \frac{d^2s}{dt^2} = a$$

Since the gravitational acceleration acts in the direction of decreasing s, we have the following differential equation

$$\frac{dv}{dt} = -9.8$$

or $$\frac{d^2s}{dt^2} = -9.8 \tag{1}$$

with initial conditions

$$v = \frac{ds}{dt} = 0,\quad s = 5,\quad t = 0.$$

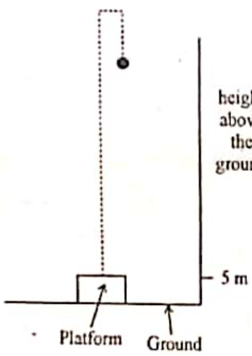


Figure 10.1

The second order differential equation (1) can to be integrated twice to give v and s as :

$$v = \frac{ds}{dt} = -9.8\,t + c_1 \qquad (2)$$

$$s = -9.8\,\frac{t^2}{2} + c_1 t + c_2 \qquad (3)$$

As $v = 250$ m/sec when $t = 0$, $c_1 = 250$ from (2). Also, at $t = 0$, $s = 5$, so that, from (3), we have

$$5 = s = 0 + c_2 \quad \text{so that} \quad c_2 = 5$$

Hence (3) becomes

$$s = -44.9\,t^2 + 250\,t + 5 \qquad (4)$$

To find the height of the projectile after 5 seconds, we have, from (4), when $t = 0$,

$$s = (-4.9 \times 25 + 250 \times 5 + 5)\ \text{m}$$

$$= (-122.5 + 1250 + 5)\ \text{m} = 1132.5\ \text{m}$$

Also from (2), the velocity after 5 seconds is

$$v = (-9.8 \times 5 + 250)\ \text{m/sec}$$

$$= (-49.0 + 250)\ \text{m/sec} = 201\ \text{m/sec}.$$

To find the time taken for the velocity to be zero, we use equation (2) with $c_1 = 250$. This equation gives

$$0 = -9.8\,t + 250$$

so that $$t = \frac{250}{9.8} = 25.51 \text{ seconds}.$$

At $t = 25.5$, the distance covered is (from (4))

$$s = (-4.9) \times (25.5)^2 + 250 \times (25.5) + 5 = 3193\ \text{m}.$$

Thus the height is 3193 m when $v = 0$.

(10.30) The Harmonic Oscillator Equation. In the real world, there are a number of quantities that oscillate or vibrate in a uniform manner, repeating themselves periodically in definite intervals of time. Some of the examples are alternating electric currents, sound waves, light waves, radio waves, electromagnetic waves, pendulums, mass-spring systems, human heartbeat, periodic variation of the population of a plant or animal species.

The simples mathematical model for such quantities is

$$y = A\cos(\omega t - \phi) \qquad (1)$$

where y is the oscillating quantity, t is time, usually measured in seconds, A is a positive constant, called **amplitude** of the oscillation, ω is a positive constant called the **angular frequency** of the oscillation and ϕ is a constant called the **phase angle**.

For each equation of the type (1), the constant v given by

$$v = \frac{w}{2p}$$

is called the **frequency** of the oscillation. If t is measured in seconds, then v represents the number of complete oscillations, called **cycles** per second. One cycle per second is called **hertz** (or Hz). The quantity T defined by

$$T = \frac{1}{v}$$

is called the **period of the oscillation.**

A quantity y oscillating in accordance with the equation (1) is said to be undergoing a **simple harmonic oscillation.** Also, any physical device which produces a quantity that undergoes simple harmonic oscillation is called a **harmonic oscillator.**

The **harmonic oscillator equation**

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \qquad (2)$$

is obtained from (1) by differentiating the equation (1) twice. Hence (1) is a solution of the differential equation (2).

The differential equation (2) is a second order linear differential equation and can be solved by the usual techniques of solving such equations.

Example 46. (Mass-Spring System). Suppose that a mass m is suspended by a perfectly elastic spring with spring constant k. The mass of the spring and the air friction are neglected. A vertical axis y is set up so that when the mass and the spring are hanging in equilibrium, the mass is lifted to the position with y-coordinate $y = A_0$ and is released at time $t = 0$ with the initial velocity $v_0 = 0$. Discuss the motion of the particle of mass m and find.

(i) the equation of motion and (ii) the frequency of the oscillation.

Solution. The mass moves up and down between A_0 and $-A_0$. Suppose that, at time t, the velocity of m is given by

$$v = \frac{dy}{dt}$$

and its acceleration is

$$a = \frac{dv}{dt} = \frac{d^2y}{dt^2}.$$

By Hooke's law, the unbalanced force F applied on a mass m by the spring is given by $F = -ky$

So, by Newton's second law of motion, we have

$$F = ma = -ky,$$

so that $$a + \frac{k}{m}y = 0$$

i.e., $$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0 \qquad (1)$$

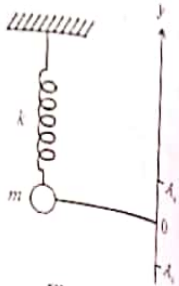


Figure 10.2

is the differential equation of the vibrating mass

Let $\frac{k}{m} = \omega^2$ so that $\omega = \sqrt{\frac{k}{m}}$ Then (1) can be written as

$$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

The general solution of this differential equation is

$$y = A_1 \cos \omega t + B_1 \sin \omega t = A \cos(\omega t - \phi).$$

When $t = 0, y(0) = A_0, A = \sqrt{A_0^2 + 0^2} = A_0, \phi = 0$ so that the equation of motion is

$$y = A_0 \cos \omega t = A_0 \cos \sqrt{\frac{k}{m}}\, t$$

The frequency of the oscillation is

$$v = \frac{w}{2p} = \frac{1}{2p}\sqrt{\frac{k}{m}}.$$

EXERCISE 10.11

1. A ball weighting $\frac{3}{4}$ lb is thrown vertically upward from a point 6 ft above the surface of the earth with an initial velocity of 20 ft/sec. As it rises it is acted upon by air resistance that is numerically equal to $\frac{1}{64}$ v, where v is the velocity in feet per sec. How high will the ball rise?

2. A bullet weighing 1 oz. is fired vertically downward from a stationary helicopter with a muzzle velocity of 1200 ft/sec. The air resistance (in pounds) is numerically equal to $10^{-4} v^2$, where v is the velocity (in feet per second). Find the velocity of the bullet as a function of the time.

3. A parachutist weighting 192 lbs (including equipment) falls from rest toward the earth. Before the parachute opens, the air resistance equals $\frac{3}{4}$ v. The parachute opens 10 sec. after the fall begins and the air resistance is then $\frac{3}{4}v^2$, where v is the velocity in feet per second. Find the velocity of the parachutist (i) when the parachute opens (ii) after the parachute opens.

4. The population of Pakistan increases at a rate proportional to the number of inhabitants present at any time t. If the population doubles in 40 years, in how many years will it triple?

5. Assume that the population of the earth changes at a rate proportional to the current population. It is estimated that at time $t = 0$ (1650 C.E.) the earth's population was 600 million, at $t = 300$(1950 C.E.) its population was 2.8 billion. Find an expression giving the population of the earth at any time. Assuming that the maximum population the earth can support is 25 billion, when will this limit be reached?

6. A newly built fish farm is stocked with 400 fish at time $t = 0$ (month). Thereafter the population increases at the rate of $\sqrt{P}$ per month when there are P fish in the farm. What is fish population at time t?

7. A 1000/litre tank contains a mixture of water and chlorine. Fresh water is pumped in at a rate of 6 litres per second. The fluid is well stirred and pumped out at a rate of 8 litres per second. If the initial concentration of chlorine is 0.02 grams per litre, find the amount of chlorine in the tank as a function of t.

8. A tank contains 200 litres of toxic solution containing 20 kg of pollutant. Fresh water is poured into the tank at the rate of 8 litres/min and the well stirred mixture leaves the tank at the same rate. Find an expression for the amount of pollutant in the tank at any time t.

9. A coffee cup has a temperature of 100°C when freshly poured and one minute later has cooled to 95°C in a room at 25°C. Determine when the coffee reaches a temperature of 70°C?

10. A steel bar at a temperature of 110°C is moved to a room where the constant temperature is 10°C. After one hour, the temperature of the bar is 60°C. How much time is required for it to reach a temperature of 30°C?

11. The marginal cost for producing x units of a product is given by

$$\frac{dC}{dt} = \frac{360}{\sqrt{x}} \text{ rupees per unit of the product.}$$

Find the cost C of manufacturing x units of the product if the fixed cost is Rs. 48,000 when $x = 1600$ units.

12. Suppose that a certain disease in Pakistan spreads in such a way that the rate of change of the infected people varies as the number of infected people. If the number of cases of the disease in a given year is reduced by 25 %, in how many years the present 75000 infected cases will reduce to 3000 cases only?

13. Bacteria grown in a certain culture increase at a rate proportional to the number of bacteria present. If there are 4000 bacteria present initially and if the number of bacteria triples in half an hour, how many bacteria are present after t hours ? How many are present after 2 hours ? After how many hours bacteria will grow to one million ?

14. It is observed that the rate at which a solid substance dissolves in fluid varies directly as the product of the amount of undissolved solid present in the solvent and the difference between the saturation concentration and the instantaneous concentration of the substance. If 20 kg of soluble substance is put into a tank containing 120 kg of solvent and, at the end of 12 minutes the concentration is observed to be 1 part in 30, find the amount of soluble substance in the solution at any time t if the saturation concentration is 1 part of soluble substance in 3 parts of solvent.

15. The temperature of a machine, when it is first shut down after operating, is 220°C and temperature of the surrounding air is 30°C. After 20 minutes, the temperature of the machine is 160°C. Find a function that gives the temperature of the machine at any time t and then find the temperature of the machine 30 minutes after it is shut down.

16. A mass of 10 kg attached to a spring stretches it 0.5 m from its natural length. The system is set in motion by displacing the mass 0.1 m above the equilibrium position. Find the spring constant k and solve the initial value problem.

17. Establish the mass-spring equation

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F(t)$$

by taking into account damping due to air resistance on a mass m by a force proportional to the velocity and also a time dependent force $F(t)$ acting on the mass, where k is the spring constant and c is the damping coefficient.

18. A mass of 4 kg attached to a spring stretches it 0.2 m from its natural length. The damping coefficient is 10 kg/sec. and the system is set in motion from equilibrium with a downward initial velocity of 2 m/sec. Find the differential equation and the initial conditions satisfied by the system.

10

Chapter 11

THE LAPLACE TRANSFORM

The **Laplace transform**[1] is an efficient technique for solving linear differential equations with constant coefficients. We shall study its basic properties and will apply them to solve initial value problems. As the name suggests, Laplace Transform is an operator which transforms a function f of the variable t into a function F of the variable s.

It will be seen later that the Laplace transform of a function is a convergent improper integral. A necessary pre-requisite for the study of Laplace transform is familiarity with the convergence of improper integrals.

THE LAPLACE TRANSFORM

(11.1) Definition. **(Piecewise Continents Function)** A real-valued function f defined on an interval $[a, b]$ is said to be **piecewise continuous** in $[a, b]$ if there exists a partition

$$P = \{ a = x_0, x_1, x_2, \cdots, x_{n-1}, x_n = b \}$$

of $[a, b]$ such that f is continuous in the interior of each subinterval $[x_i, x_{i+1}]$ and has finite one-sided limits $\lim\limits_{x \to x_i^+} f(x)$ and $\lim\limits_{x \to x_{i+1}^-} f(x)$ at the end points of each subinterval $(i = 0, 1, 2, \cdots, n-1)$. The one-sided limits $\lim\limits_{x \to x_i^+} f(x)$ and $\lim\limits_{x \to x_{i+1}^-} f(x)$ are finite but unequal so that f has a finite number of **jumps (ordinary) discontinuities at the points of**

1. Named for the eminent French astronomer, mathematician and physicist Pierre Simon Marquis de Laplace (1749 – 1827).

subdivision of $[a, b]$. Clearly, a continuous function is piecewise continuous and a piecewise continuous function is bounded. It is also known from calculus that if f is piecewise continuous on $[a, b]$, then $\int_a^b f(x)\,dx$ exists.

(11.2) Definition. Let f be a real-valued piecewise continuous function defined on $[0, \infty[$. The **Laplace transform** of f, denoted by $\mathscr{L}(f)$, is the function F defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt \qquad (1)$$

provided the improper integral in (1) converges.

The domain of F is the set of all real numbers s for which the above integral converges.

Note that the operation transforms the given function f of the variable t into a new function F of the variable s and is written symbolically $F(s) = \mathscr{L}\{f(t)\}$ or simply $F = \mathscr{L}(f)$.

(11.3) Definition. If $F = \mathscr{L}\{f\}$ as in **11.2**, then the original function f is called the **inverse Laplace transform** of F and is denoted by $f = \mathscr{L}^{-1}\{F\}$. Clearly,

$$\mathscr{L}^{-1}\{F\} = \mathscr{L}^{-1}\{\mathscr{L}(f)\} = f.$$

Thus, if $\mathscr{L}\{f(t)\} = F(s)$ then $\mathscr{L}^{-1}\{F(s)\} = f(t)$.

The inverse Laplace transforms will be useful for solving initial value problems.

Example 1. Let $f(t) = 1$ on $[0, \infty[$. Then

$$\mathscr{L}\{f\} = \mathscr{L}\{1\} = \int_0^\infty e^{-st}\,dt = \lim_{h\to\infty}\int_0^h e^{-st}\,dt$$

$$= \lim_{h\to\infty}\left[-\frac{e^{-st}}{s}\right]_0^h = \lim_{h\to\infty}\left[-\frac{e^{-hs}}{s} + \frac{1}{s}\right]$$

$$= \frac{1}{s} = F(s), \quad \text{provided } s > 0.$$

Example 2. Let $f(t) = t^n$, n being a positive integer. Evaluate $\mathscr{L}\{f(t)\}$.

Solution. Here $\mathscr{L}\{f(t)\} = \mathscr{L}\{t^n\} = \int_0^\infty e^{-st}\,t^n\,dt$

Integrating by parts, taking t^n as first function, we have

$$\mathscr{L}\{t^n\} = \left[t^n \cdot \frac{e^{-st}}{-s}\right]_0^\infty + \int_0^\infty nt^{n-1} \cdot \frac{e^{-st}}{s}\,dt$$

$$= \left[\frac{t^n}{-se^{st}}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt = \frac{n}{s} \cdot \mathscr{L}\{t^{n-1}\},$$

since successive application of L' Hospital's rule to $\lim_{t\to\infty}\frac{t^n}{e^{st}}$ gives its value zero.

Hence $\quad \mathscr{L}\{t^n\} = \frac{n}{s}\,\mathscr{L}\{t^{n-1}\}$

$$= \frac{n}{s} \cdot \frac{n-1}{s}\,\mathscr{L}\{t^{n-2}\}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$= \frac{n}{s} \cdot \frac{n-1}{s} \cdot \ldots \cdot \frac{1}{s}\,\mathscr{L}\{1\}.$$

$$= \frac{n!}{s^n} \cdot \frac{1}{s}, \quad \text{as } \mathscr{L}\{1\} = \frac{1}{s} \text{ by Example 1.}$$

$$= \frac{n!}{s^{n+1}} = F(s).$$

Example 3. Compute $\mathscr{L}\{e^{at}\}$, where a is a constant and $s \neq a$.

Solution. Here $f(t) = e^{at}$

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{e^{at}\} = \int_0^\infty e^{-st}e^{at}\,dt$$

$$= \int_0^\infty e^{(a-s)t}\,dt = \lim_{h\to\infty}\int_0^h e^{(a-s)t}\,dt$$

$$= \lim_{h\to\infty}\left[\frac{e^{(a-s)t}}{a-s}\right]_0^h$$

$$= \lim_{h\to\infty}\left[\frac{e^{(a-s)h}}{a-s} - \frac{1}{a-s}\right]$$

$$= \begin{cases} \dfrac{1}{s-a} & \text{if } s > a \\ \infty & \text{if } s = a \end{cases}$$

Here $e^{-(s-a)h} \to 0$ as $h \to \infty$ and $s > a$, while $e^{-(s-a)h} \to \infty$ as $h \to \infty$ and $s < a$.

When $a = s$, $f(t) = e^{st}$ and

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st}\, e^{st}\, dt = \int_0^\infty dt = \{t\}_0^\infty = \infty$$

Therefore, $\mathscr{L}\{e^{at}\} = \dfrac{1}{s-a}$, $s > a$

Example 4. Find the Laplace transforms of

(i) $\cos at$ (ii) $\sin at$

Solution. By definition

$$\mathscr{L}\{\cos at\} = \int_0^\infty e^{-st} \cos at\, dt$$

and $\mathscr{L}\{\sin at\} = \displaystyle\int_0^\infty e^{-st} \sin at\, dt.$

Therefore, for $i = \sqrt{-1}$,

$$\begin{aligned}
\mathscr{L}\{\cos at\} + i\,\mathscr{L}\{\sin at\} &= \int_0^\infty e^{-st} \cos at\, dt + i\int_0^\infty e^{-st} \sin at\, dt \\
&= \int_0^\infty e^{-st} (\cos at + i \sin at)\, dt \\
&= \int_0^\infty e^{-st} e^{iat}\, dt \\
&= \lim_{h\to\infty} \left[\frac{e^{(ia-s)t}}{ia-s}\right]_0^h \\
&= \lim_{h\to\infty} \left[\frac{e^{(ia-s)h}}{ia-s} - \frac{1}{ia-s}\right] \\
&= \begin{cases} \dfrac{1}{s-ia} & \text{if } s > 0 \\ \text{undefined} & \text{if } s < 0 \end{cases} \\
&= \frac{s+ia}{s^2+a^2} \quad \text{if } s > 0
\end{aligned}$$

Here $\displaystyle\lim_{h\to\infty} \frac{e^{(ia-s)h}}{ia-s} = 0$ for $s > 0$ and is undefined for $s < 0$.

Equating real and imaginary parts, we get

(i) $\mathscr{L}\{\cos at\} = \dfrac{s}{s^2+a^2}$, $s > 0$

(ii) $\mathscr{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$, $s > 0$

Note. $\mathscr{L}\{\cos at\}$ and $\mathscr{L}\{\sin at\}$ could also be evaluated directly by definition

$$\mathscr{L}\{\cos at\} = \int_0^\infty e^{-st} \cos at\, dt$$

Integrate twice by parts and the result will follow.

Example 5. Consider the function f defined by $f(t) = \dfrac{1}{t}$. For the Laplace transform of $\dfrac{1}{t}$, we first check the convergence of $\displaystyle\int_0^\infty \frac{e^{-st}}{t}\, dt.$

$$\int_0^\infty \frac{e^{-st}}{t}\, dt = \int_0^1 \frac{e^{-st}}{t}\, dt + \int_1^\infty \frac{e^{-st}}{t}\, dt.$$

For $0 \le t \le 1$, we have $e^{-st} \ge e^{-s}$ if $s > 0$.

Therefore, $\displaystyle\int_0^\infty \frac{e^{-st}}{t}\, dt \ge \int_0^1 \frac{e^{-s}}{t}\, dt + \int_1^\infty \frac{e^{-st}}{t}\, dt.$

But
$$\begin{aligned}
\int_0^1 \frac{e^{-s}}{t}\, dt &= \frac{1}{e^s} \lim_{h\to 0} [\ln t]_h^1 \\
&= \frac{1}{e^s} \lim_{h\to 0} (\ln 1 - \ln h) \\
&= \infty, \text{ since } \ln h \to -\infty \text{ as } h \to 0.
\end{aligned}$$

Hence $\displaystyle\int_0^1 \frac{e^{-st}}{t}\, dt$ also diverges to ∞. Consequently, $\displaystyle\int_0^\infty \frac{e^{-st}}{t}\, dt$ diverges and so by definition, $\mathscr{L}\left\{\dfrac{1}{t}\right\}$ does not exist.

It is obvious from this example that Laplace transforms exist only for certain class of functions. We shall later state and prove a theorem that guarantees the existence of Laplace transform of functions satisfying certain conditions. For that we need

(11.4) Definition. A function f defined $[0, \infty[$ is said to be of **exponential order** a as $t \to \infty$ if there exist real constants a, $M > 0$ and $T > 0$ such that

$$|f(t)| \le Me^{at} \qquad \text{for } t \ge T.$$

(11.5) Theorem. Let f be a piecewise continuous function defined on $[0, \infty[$. If f is of exponential order a as $t \to \infty$ then $\mathcal{L}\{f(t)\}$ exists for all $s > a$.

Proof. Since f is of exponential order a, there exist positive real numbers M and T such that

$$|f(t)| \le Me^{at} \qquad , \qquad t \ge T \tag{1}$$

The theorem will be proved if we show that $\int_0^\infty e^{-st} f(t)\, dt$ converges.

Now

$$\int_0^\infty e^{-st} f(t)\, dt = \int_0^T e^{-st} f(t)\, dt + \int_T^\infty e^{-st} f(t)\, dt \tag{2}$$

Since f is piecewise continuous, the first integral on the right of (2) exists. Thus the convergence of $\int_0^\infty e^{-st} f(t)\, dt$ depends on the convergence of $\int_T^\infty e^{-st} f(t)\, dt$. But

$$\begin{aligned}
\left|\int_T^\infty e^{-st} f(t)\, dt\right| &\le \int_T^\infty \left|e^{-st} f(t)\right| dt \\
&\le \int_T^\infty e^{-st} Me^{at}\, dt, \qquad \text{by (1)} \\
&= M\int_T^\infty e^{(a-s)t}\, dt \\
&= M \lim_{h \to \infty} \left[\frac{e^{(a-s)t}}{a-s}\right]_T^h \\
&= \lim_{h \to \infty} M\left[\frac{e^{(a-s)h}}{a-s} - \frac{e^{(a-s)T}}{a-s}\right] \\
&= \begin{cases} M\left(0 - \dfrac{e^{(a-s)T}}{a-s}\right) & \text{if } a-s<0 \\ \infty & \text{if } a-s \ge 0 \end{cases}
\end{aligned}$$

Here $\lim_{h \to \infty} \dfrac{e^{(a-s)h}}{a-s} = 0$ if $a-s<0$ and $\lim_{h \to \infty} \dfrac{e^{(a-s)h}}{a-s} = \infty$ if $a-s \ge 0$.

Thus $\mathcal{L}\{f(t)\}$ exists for all $s > a$.

Note. The conditions stated in Theorem 11.5 are sufficient but not necessary. There exist functions which do not satisfy the hypothesis of **(11.5)** but still possess Laplace transforms.

Example 6. Consider the function f defined by

$$f(t) = t^{-\frac{1}{2}}$$

Clearly, f is not defined at $t = 0$, but it will be shown that $\mathcal{L}\left\{t^{-\frac{1}{2}}\right\}$ exists. By definition, we have

$$\mathcal{L}\left\{t^{-\frac{1}{2}}\right\} = \int_0^\infty e^{-st} t^{-\frac{1}{2}}\, dt \tag{1}$$

Let $st = x$. Then $s\,dt = dx$ or $dt = \frac{1}{s}\, dx$. So

$$t^{-\frac{1}{2}} = \left(\frac{x}{s}\right)^{-\frac{1}{2}} = \sqrt{\frac{s}{x}}$$

Substituting these values of t and dt into (1), we have

$$\begin{aligned}
\mathcal{L}\left\{t^{-\frac{1}{2}}\right\} &= \frac{1}{\sqrt{s}} \int_0^\infty e^{-x} x^{-\frac{1}{2}}\, dx \\
&= \frac{1}{\sqrt{s}}\, \Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{\pi}{s}} \qquad \text{(from calculus)}
\end{aligned}$$

Here $\Gamma(t)$ (**the gamma function of** t) is defined by

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\, dx \qquad \text{with} \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Thus $\mathcal{L}\left\{t^{-\frac{1}{2}}\right\}$ exists.

PROPERTIES OF THE LAPLACE TRANSFORM

(11.6) Theorem. (The Linearity Property).

Let $f(t) = a\, g(t) + b\, h(t)$, where a, b are constants and $\mathcal{L}\{g(t)\}$ and $\mathcal{L}\{h(t)\}$ exist. Then $\mathcal{L}\{f(t)\}$ exists and

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{a\, g(t) + b\, h(t)\} = a\, \mathcal{L}\{g(t)\} + b\, \mathcal{L}\{h(t)\}.$$

Proof. By definition,

$$\mathscr{L}\{a\,g(t) + b\,h(t)\} = \int_0^\infty e^{-st}\{a\,g(t) + b\,h(t)\}\,dt$$

$$= a\int_0^\infty e^{-st} g(t)\,dt + b\int_0^\infty e^{-st} h(t)\,dt$$

$$= a\,\mathscr{L}\{g(t)\} + b\,\mathscr{L}\{h(t)\}$$

(11.7) Theorem. (The Differentiation Formula). Let f be continuous on $[\,0, \infty\,[$ and of exponential order a. Let f' be piecewise continuous on every finite closed interval $0 \le t \le b$. Then $\mathscr{L}\{f'(t)\}$ exists for $s > a$ and

$$\mathscr{L}\{f'(t)\} = s\,\mathscr{L}\{f(t)\} - f(0)$$

Proof. By Theorem 11.5, $\mathscr{L}\{f'(t)\}$ exists.

Let $F(s) = \mathscr{L}\{f(t)\}$.

Then, by definition,

$$\mathscr{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= 0 - f(0) + s\,\mathscr{L}\{f(t)\} = s\,\mathscr{L}\{f(t)\} - f(0) = s\,F(s) - f(0)$$

Here $|e^{-st} f(t)| \le Me^{(a-s)t} \to 0$ as $t \to \infty$ and $s > a$.

(11.8) Corollary. If $F(s) = \mathscr{L}\{f(t)\}$, then

$$\mathscr{L}\{f''(t)\} = s^2 F(s) - s\,f(0) - f'(0)$$

Proof.

$$\mathscr{L}\{f''(t)\} = s\,\mathscr{L}\{f'(t)\} - f'(0), \quad \text{by (11.7)}$$

$$= s\,[s\,\mathscr{L}\{f(t)\} - f(0)] - f'(0)$$

$$= s^2\,\mathscr{L}\{f(t)\} - s\,f(0) - f'(0)$$

$$= s^2 F(s) - s\,f(0) - f'(0).$$

(11.9) Corollary. $\mathscr{L}\{f^{(n)}(t)\} = s^n\,\mathscr{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{(n-1)}(0)$

Proof.

$$\mathscr{L}\{f^{(n)}(t)\} = s\,\mathscr{L}\{f^{(n-1)}(t)\} - f^{(n-1)}(0)$$

$$= s\,[s\,f^{(n-2)}(t)\} - f^{(n-2)}(0)] - f^{(n-1)}(0)$$

$$= s^2\,\mathscr{L}\{f^{(n-2)}(t)\} - s\,f^{(n-2)}(0) - f^{(n-1)}(0)$$

$$= s^2\,[s\,\mathscr{L}\{f^{(n-3)}(t)\} - f^{(n-3)}(0)] - s\,f^{(n-2)}(0) - f^{(n-1)}(0)$$

$$= s^3\,\mathscr{L}\{f^{(n-3)}(t)\} - s^2 f^{(n-3)}(0) - s\,f^{(n-2)}(0) - f^{(n-1)}(0).$$

Continuing in this way, we get the required result.

(11.10) Theorem. (First Shifting or Translation Property).

Let $F(s) = \mathscr{L}\{f(t)\}$ exist for $s > b$. For any constant a such that $s > a + b$, $\mathscr{L}\{e^{at} f(t)\}$ exists and $\mathscr{L}\{e^{at} f(t)\} = F(s-a)$.

Proof. By definition, $F(s) = \int_0^\infty e^{-st} f(t)\,dt$

So $$\mathscr{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt = \int_0^\infty e^{-(s-a)t} f(t)\,dt \quad \text{exists}$$

provided $(s-a) > b$ i.e. $s > a + b$, and

$$\mathscr{L}\{e^{at} f(t)\} = F(s-a).$$

Example 7. Compute $\mathscr{L}\{\sinh at\}$ and $\mathscr{L}\{\cosh at\}$.

Solution. Here $\sinh at = \dfrac{e^{at} - e^{-at}}{2}$.

We have

$$\mathscr{L}\{\sinh at\} = \mathscr{L}\left\{\frac{e^{at} - e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\,\mathscr{L}\{e^{at}\} - \frac{1}{2}\,\mathscr{L}\{e^{-at}\}, \text{ by (11.6)}$$

$$= \frac{1}{2}\cdot\frac{1}{s-a} - \frac{1}{2}\cdot\frac{1}{s+a} = \frac{a}{s^2 - a^2}.$$

Similarly, $\mathscr{L}\{\cosh at\} = \dfrac{s}{s^2 - a^2}$.

Example 8. Compute $\mathscr{L}\{\cos^2 at\}$

Solution. Let
$$f(t) = \cos^2 at$$
$$f'(t) = -2a\cos at \sin at = -a \sin 2at$$

By (11.7), $\quad \mathscr{L}\{f'(t)\} = s\,\mathscr{L}\{f(t)\} - f(0)$

or $\quad s\,\mathscr{L}\{f(t)\} = \mathscr{L}\{f'(t)\} + f(0)$

i.e., $\quad s\,\mathscr{L}\{\cos^2 at\} = -a\,\mathscr{L}\{\sin 2at\} + f(0)$

$$= -a \cdot \frac{2a}{s^2+4a^2} + 1, \text{ as in Example 4}$$
$$= \frac{s^2+2a^2}{s^2+4a^2}$$

Therefore, $\quad \mathscr{L}\{f(t)\} = \mathscr{L}\{\cos^2 at\} = \dfrac{s^2+2a^2}{s(s^2+4a^2)}$

Alternative Method:
$$\cos^2 at = \frac{\cos 2at + 1}{2}$$

So, by the Linearity Property of the transform, we have
$$\mathscr{L}\{\cos^2 at\} = \frac{1}{2}\left[\mathscr{L}\{\cos 2at\} + \mathscr{L}\{1\}\right]$$
$$= \frac{1}{2}\left\{\frac{s}{s^2+4a^2} + \frac{1}{s}\right\} = \frac{s^2+2a^2}{s(s^2+4a^2)}$$

Example 9. Evaluate $\mathscr{L}\{e^{3t}(t^3 + \sin 2t)\}$

Solution.
$$\mathscr{L}\{e^{3t}(t^3+\sin 2t)\} = \mathscr{L}\{e^{3t}t^3 + e^{3t}\sin 2t\}$$
$$= \mathscr{L}\{e^{3t}t^3\} + \mathscr{L}\{e^{3t}\sin 2t\}, \text{ by (11.6)}$$

Now $\quad \mathscr{L}\{t^3\} = \dfrac{3!}{s^4}$

$$\mathscr{L}\{\sin 2t\} = \frac{2}{s^2+4}$$

By (11.10), $\quad \mathscr{L}\{e^{3t}t^3\} = \dfrac{3!}{(s-3)^4}$ and $\mathscr{L}\{e^{3t}\sin 2t\} = \dfrac{2}{(s-3)^2+4}$

Therefore,
$$\mathscr{L}\{e^{3t}(t^3+\sin 2t)\} = \frac{3!}{(s-3)^4} + \frac{2}{(s-3)^2+4}.$$

Example 10. Compute $\mathscr{L}\{te^{at}\cos bt\}$.

Solution. Consider $\quad te^{at}e^{ibt} = te^{(a+ib)t}$

Let $f(t) = t$, then $\mathscr{L}\{f(t)\} = \mathscr{L}\{t\} = \dfrac{1}{s^2}$.

By (11.10), $\quad \mathscr{L}\{te^{(a+ib)t}\} = \dfrac{1}{[s-(a+ib)]^2} = \dfrac{1}{[(s-a)-ib]^2}$

$$= \frac{[(s-a)+ib]^2}{[\{(s-a)-ib\}\{(s-a)+ib\}]^2}$$
$$= \frac{(s-a)^2 - b^2 + 2ib(s-a)}{[(s-a)^2+b^2]^2}$$

Equating real parts, we have
$$\mathscr{L}\{t\,e^{at}\cos bt\} = \frac{(s-a)^2-b^2}{[(s-a)^2+b^2]^2}$$

(11.11) Theorem. Suppose $\mathscr{L}\{f(t)\} = F(s)$ exists for $s > a$. Then
$$\mathscr{L}\{t f(t)\} = -F'(s).$$

Proof. Consider
$$\frac{d}{ds}\mathscr{L}\{f(t)\} = \frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$
$$= \int_0^\infty \frac{d}{ds} e^{-st} f(t)\,dt = \int_0^\infty -t\,e^{-st} f(t)\,dt = -\mathscr{L}\{t f(t)\}$$

Thus $\quad F'(s) = -\mathscr{L}\{t f(t)\}$

or $\quad \mathscr{L}\{t f(t)\} = -F'(s)$, as desired.

Using the above result repeatedly, we have
$$\mathscr{L}\{t^n f(t)\} = -\frac{d}{ds}\left[\mathscr{L}\{t^{n-1} f(t)\}\right]$$
$$= (-1)^2 \frac{d^2}{ds^2}\left[\mathscr{L}\{t^{n-2} f(t)\}\right]$$
$$\vdots \quad \vdots \quad \vdots$$
$$= (-1)^n \frac{d^n}{ds^n}\left[\mathscr{L}\{f(t)\}\right].$$

Example 11. Compute $\mathscr{L}\{t^3 e^{-t}\}$.

Solution.

$$f(t) = e^{-t}$$

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{e^{-t}\} = \frac{1}{s+1}$$

By (**11.11**), $$\mathscr{L}\{t^3 e^{-t}\} = (-1)^3 \frac{d^3}{ds^3}\,[\mathscr{L}\{e^{-t}\}] = -\frac{d^3}{ds^3}\left(\frac{1}{s+1}\right)$$

$$= -\frac{(-1)^3\, 3!}{(s+1)^4} = \frac{6}{(s+1)^4}$$

(11.12) Theorem. If $\mathscr{L}\{f(t)\} = F(s)$ then

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du, \text{ provided } \lim_{t \to 0^+} \frac{f(t)}{t} \text{ exists.}$$

Proof. Let $\dfrac{f(t)}{t} = g(t)$. Then $f(t) = t\,g(t)$.

$$F(s) = \mathscr{L}\{f(t)\} = \mathscr{L}\{t\,g(t)\}$$

$$= -\frac{d}{ds}\,(\mathscr{L}\{g(t)\}), \text{ by } (\mathbf{11.11})$$

Integrating, we have

$$\mathscr{L}\{g(t)\} = -\int_\infty^s F(u)\,du = \int_s^\infty F(u)\,du$$

(11.13) Theorem. If f is piecewise continuous and is of exponential order a, then

$$\mathscr{L}\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\,\mathscr{L}\{f(t)\}$$

Proof. The integral

$$g(t) = \int_0^t f(u)\,du$$

is a continuous function of t. Since $f(t)$ is of exponential order a, $|f(t)| = M e^{at}$. Therefore,

$$|g(t)| = \left|\int_0^t f(u)\,du\right| = M\int_0^t e^{au}\,du = \frac{M}{a}\,\{e^{at} - 1\}.$$

By the Fundamental Theorem of Integral Calculus, $g'(t) = f(t)$ except at points where f is discontinuous. Hence $g'(t)$ is piecewise continuous. By (**11.7**), we have

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{g'(t)\} = s\,\mathscr{L}\{g(t)\} - g(0), \quad s > a$$

$$= s\,\mathscr{L}\{g(t)\}, \text{ since } g(0) = 0$$

Thus $$\mathscr{L}\{g(t)\} = \frac{1}{s}\,\mathscr{L}\{f(t)\}$$

Example 12. Compute $\mathscr{L}\left\{\dfrac{\sin t}{t}\right\}$

Solution. Let $f(t) = \sin t$. Then

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{\sin t\} = \frac{1}{s^2+1} = F(s)$$

Set $g(t) = \dfrac{\sin t}{t}$. By (**11.12**), we have

$$\mathscr{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty F(u)\,du$$

$$= \int_s^\infty \frac{1}{1+u^2}\,du = [\arctan u]_s^\infty = \frac{\pi}{2} - \arctan s.$$

Example 13. Evaluate $\mathscr{L}\left\{\displaystyle\int_0^t \frac{1-\cosh au}{u}\,du\right\}$

Solution. Let $f(t) = 1 - \cosh at$. Then

$$\mathscr{L}\{f(t)\} = \frac{1}{s} - \frac{s}{s^2 - a^2} = F(s), \text{ as in Example 7.}$$

Set $g(t) = \dfrac{1-\cosh at}{t}$. Then by (**11.12**), we get

$$\mathscr{L}\left\{\frac{1-\cosh at}{t}\right\} = \int_s^\infty \left(\frac{1}{u} - \frac{u}{u^2-a^2}\right) du$$

$$= \left[\ln u - \frac{1}{2}\ln(u^2 - a^2)\right]_s^\infty$$

$$= \lim_{u \to \infty} \left[\frac{1}{2}\ln u^2 - \frac{1}{2}\ln(u^2 - a^2)\right] + \frac{1}{2}\ln(s^2 - a^2) - \ln s$$

$$= \frac{1}{2}\lim_{u \to \infty} \ln\left(\frac{u^2}{u^2 - a^2}\right) + \frac{1}{2}\ln\left(\frac{s^2 - a^2}{s^2}\right)$$

$$= \frac{1}{2}\ln \lim_{u \to \infty}\left(\frac{u^2}{u^2 - a^2}\right) + \frac{1}{2}\ln\left(\frac{s^2 - a^2}{s^2}\right)$$

$$= \frac{1}{2}\ln 1 + \frac{1}{2}\ln\left(\frac{s^2 - a^2}{s^2}\right) = \frac{1}{2}\ln\left(\frac{s^2 - a^2}{s^2}\right)$$

By **(11.13)**, we have

$$\mathscr{L}\left\{\int_0^t \frac{1 - \cosh au}{u}\, du\right\} = \frac{1}{s}\,\mathscr{L}\left\{\frac{1 - \cosh at}{t}\right\} = \frac{1}{2s}\ln\left(\frac{s^2 - a^2}{s^2}\right)$$

(11.14) Definition. **(Unit Step Function).** Let $a \geq 0$. The function u_a defined on $]0, \infty[$ by

$$u_a(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$$

is called the **unit step function**. If $a = 0$ then

$$u_0(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0 \end{cases}$$

Since u_a is defined on $]0, \infty[$, we have $u_0(t) = 1$ for $t > 0$.

Clearly, the unit step function is of exponential order.

(11.15) Theorem. Let u_a be the unit step function. Then

$$\mathscr{L}\{u_a(t)\} = \frac{e^{-as}}{s}.$$

Proof. By definition,

$$\mathscr{L}\{u_a(t)\} = \int_0^\infty e^{-st}\, u_a(t)\, dt$$

$$= \int_0^a 0\, e^{-st}\, dt + \int_a^\infty e^{-st}\, dt = \int_a^\infty e^{-st}\, dt$$

$$= \lim_{h \to \infty}\left[\frac{e^{-st}}{-s}\right]_a^h = \lim_{h \to \infty}\left[\frac{-e^{-hs}}{s} + \frac{e^{-as}}{s}\right]$$

2nd shifting thm

$$= \frac{e^{-as}}{s},\ s > 0, \text{ because } \lim_{h \to \infty}\frac{e^{-hs}}{s} = 0$$

(11.16) Theorem. Let f be a function of exponential order a and $\mathscr{L}\{f(t)\} = F(s)$. For the function

$$u_a(t) f(t - a) = \begin{cases} 0 & \text{if } 0 < t < a \\ f(t - a) & \text{if } t > a, \end{cases}$$

$$\mathscr{L}\{u_a(t) f(t - a)\} = e^{-as} F(s)$$

Proof.
$$\boxed{\mathscr{L}\{u_a(t) f(t - a)\}} = \int_0^\infty e^{-st}\, u_a(t) f(t - a)\, dt$$

$$= \int_0^a e^{-st}\, 0\, dt + \int_a^\infty e^{-st} f(t - a)\, dt$$

$$= \int_a^\infty e^{-st} f(t - a)\, dt \qquad (1)$$

Putting $t - a = \tau$ into (1), we get

$$\mathscr{L}\{u_a(t) f(t - a)\} = e^{-as}\int_0^\infty e^{-s\tau} f(\tau)\, d\tau = e^{-as} F(s)$$

This is known as the **Second Translation Property.**

Example 14. Find the Laplace transform of

$$f(t) = \begin{cases} 0 & \text{if } 0 < t < \frac{\pi}{2} \\ \cos t & \text{if } t > \frac{\pi}{2} \end{cases}$$

Solution. We have to express $\cos t$ in terms of $t - \frac{\pi}{2}$ so as to apply **(11.16)**.

As $\cos t = -\sin\left(t - \frac{\pi}{2}\right)$, let

$$g(t) = \begin{cases} 0 & \text{if } 0 < t < \frac{\pi}{2} \\ \sin\left(t - \frac{\pi}{2}\right) & \text{if } t > \frac{\pi}{2} \end{cases}$$

Then $\quad f(t) = -u_{\pi/2}(t)\,g(t)$

$$\mathscr{L}\{f(t)\} = -\mathscr{L}\left\{u_{\pi/2}(t)\sin\left(t-\frac{\pi}{2}\right)\right\}$$

$$= -e^{\left(-\frac{\pi}{2}\right)s}\ \mathscr{L}\{\sin t\} \text{ by } \mathbf{(11.16)}$$

$$= -e^{-\frac{\pi}{2}s}\ \frac{1}{s^2+1}$$

Alternative Method

By definition,

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^{\pi/2} e^{-st} f(t)\,dt + \int_{\pi/2}^\infty e^{-st} f(t)\,dt$$

$$= 0 + \int_{\pi/2}^\infty e^{-st}\cos t\,dt$$

$$= \left[\frac{e^{-st}}{-s}\cos t\right]_{\pi/2}^\infty + \frac{1}{s}\int_{\pi/2}^\infty e^{-st}(-\sin t)\,dt$$

$$= \frac{-1}{s}\int_{\pi/2}^\infty e^{-st}\sin t\,dt$$

$$= \frac{1}{s^2}\left[e^{-st}\sin t\right]_{\frac{\pi}{2}}^\infty - \frac{1}{s^2}\int_{\pi/2}^\infty e^{-st}\cos t\,dt$$

$$= -\frac{e^{-\frac{\pi}{2}s}}{s^2} - \frac{1}{s^2}F(s)$$

Therefore, $\quad \left(1+\frac{1}{s^2}\right)F(s) = -\frac{e^{-\frac{\pi}{2}s}}{s^2}$

or $\quad F(s) = -\frac{e^{-\frac{\pi}{2}s}}{s^2+1}$

TABLE OF SOME LAPLACE TRANSFORMS

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$	$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
1. 1	$\frac{1}{s}$, $s>0$	2. t	$\frac{1}{s^2}$
3. t^n	$\frac{n!}{s^{n+1}}$, $s>0$	4. t^a, $\alpha > -1$	$\frac{\Gamma(a+1)}{s^{a+1}}$
5. $\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$	6. e^{at}	$\frac{1}{s-a}$, $s>a$
7. te^{at}	$\frac{1}{(s-a)^2}$	8. $\sin at$	$\frac{a}{s^2+a^2}$, $s>0$
9. $\cos at$	$\frac{s}{s^2+a^2}$, $s>0$	10. $\sinh at$	$\frac{a}{s^2-a^2}$, $s>\lvert a\rvert$
11. $\cosh at$	$\frac{s}{s^2-a^2}$, $s>\lvert a\rvert$	12. $t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$, $s>a$
13. $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$, $s>a$	14. $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$, $s>a$
15. $t\sin at$	$\frac{2as}{(s^2+a^2)^2}$, $s>0$	16. $t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$, $s>0$
17. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$, $c>0$	18. $\int_0^t f(u)\,du$	$\frac{1}{s}F(s)$
19. $t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}F(s)$	20. $\frac{f(t)}{t}$	$\int_s^\infty F(u)\,du$
21. $u_a(t)$	$\frac{e^{-as}}{s}$	22. $u_a(t)f(t-a)$	$e^{-as}F(s)$
23. $\sin at - at\cos at$	$\frac{2a^3}{(s^2+a^2)^2}$	24. $f'(t)$	$sF(s)-f(0)$
25. $1-\cos at$	$\frac{a^2}{s(s^2+a^2)}$	26. $at-\sin at$	$\frac{a^3}{s^2(s^2+a^2)}$
27. $\sinh at - \sin at$	$\frac{2a^3}{s^4-a^4}$	28. $\cosh at - \cos at$	$\frac{2a^2 s}{s^4-a^4}$

EXERCISE 11.1

Compute the Laplace transform of each of the following (**Problems 1 – 28**):

1. $t^2 + 6t - 17$
2. e^{3t+5}
3. $\sin(7t + 4)$
4. $\cos(at + b)$
5. $\cosh(5t - 3)$
6. $(t^3 - 1)\,e^{-2t}$
7. $e^{-t}\sin 2t$
8. $e^{3t}\cosh 4t$
9. $\cos t \cos 2t$
10. $\sin^3 t$
11. $te^{-3t}\sin at$
12. $\sinh^2 at$
13. $\cosh at \sin at$
14. $\sinh at \cos at$
15. $\cosh at \cos bt$
16. $[\,t\,]$, the bracket function
17. t^{α}, $\alpha > -1$. Hence find $\mathscr{L}\{t^{5/2}\}$
18. $t^2 \sin at$
19. $t^2 \cos at$
20. $t \sin^2 at$
21. $t^2 \cos^2 2t$
22. $\dfrac{\sin at}{t}$
23. $\dfrac{1 - \cos at}{t}$
24. $\displaystyle\int_0^t \frac{\sin au}{u}\,du$
25. $\displaystyle\int_0^t \frac{1 - \cos au}{u}\,du$
26. $\dfrac{\sinh at}{t}$
27. $\ln t$
28. $f(t) = \begin{cases} 0 & \text{if } t < 3 \\ (t-3)^3 & \text{if } t > 3 \end{cases}$
29. If $\mathscr{L}\{f(t)\} = F(s)$ for $s > a$, show that $\mathscr{L}\{f(ct)\} = \dfrac{1}{c}\,F\left(\dfrac{s}{c}\right)$, $c > 0$ and $s > ca$
30. Compute $\mathscr{L}\{\sin\sqrt{t}\}$. Deduce $\mathscr{L}\left\{\dfrac{\cos\sqrt{t}}{\sqrt{t}}\right\}$.

INVERSE LAPLACE TRANSFORM

Recall Definition 11.3 that if $F(s)$ is the Laplace transform of a function $f(t)$, then $f(t)$ is called the **inverse Laplace transform** of $F(s)$ and is denoted by $\mathscr{L}^{-1}\{F(s)\} = f(t)$. In this section, we shall compute inverse Laplace transforms of certain functions. This will be done with the aid of Table of Laplace Transforms of elementary functions and properties of the inverse transform which follow from the theorems and properties of the Laplace transform.

eg

(11.17) Theorem. (Linearity Property). If $\mathscr{L}^{-1}\{F_1(s)\} = f_1(t)$ and

$$\mathscr{L}^{-1}\{F_2(s)\} = f_2(t), \text{ then for any constants } a,\, b$$

$$\begin{aligned} \mathscr{L}^{-1}\{a\,F_1(s) + b\,F_2(s)\} &= a\,\mathscr{L}^{-1}\{F_1(s)\} + b\,\mathscr{L}^{-1}\{F_2(s)\} \\ &= a\,f_1(t) + b\,f_2(t) \end{aligned}$$

It is a direct consequence of Theorem 11.6.

(11.18) Theorem. If $\mathscr{L}^{-1}\{F(s)\} = f(t)$, then .

(i) $\mathscr{L}^{-1}\{F(s-a)\} = e^{at} f(t)$

(ii) $\mathscr{L}^{-1}\{F(cs)\} = \dfrac{1}{c}\, f\left(\dfrac{t}{c}\right)$, $c > 0$

(iii) $\mathscr{L}^{-1}\{F^{(n)}(s)\} = (-1)^n\, t^n f(t)$

(iv) $\mathscr{L}^{-1}\left\{\dfrac{F(s)}{s}\right\} = \displaystyle\int_0^t f(u)\,du$

(v) $\mathscr{L}^{-1}\{e^{-as} F(s)\} = u_a(t)\, f(t-a)$, where $u_a(t)$ is the unit step function.

These results follow from the relevant theorems on Laplace transform.

Example 15. Compute $\mathscr{L}^{-1}\left\{\dfrac{5s}{s^2+5}\right\}$.

Solution. Here $\dfrac{5s}{s^2+5} = 5 \cdot \dfrac{s}{s^2 + (\sqrt{5})^2}$.

So, from the Table of Laplace Transforms, we have

$$\mathscr{L}^{-1}\left\{\frac{5s}{s^2+5}\right\} = 5\,\mathscr{L}^{-1}\left\{\frac{s}{s^2 + (\sqrt{5})^2}\right\} = 5\cos\sqrt{5}\; t.$$

Example 16. Find $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2+2s-15}\right\}$

Solution. Here

$$\frac{1}{s^2+2s-15} = \frac{1}{(s+5)(s-3)} = \frac{A}{s+5} + \frac{B}{s-3}$$

$$= \frac{1}{8}\left[\frac{1}{s-3} - \frac{1}{s+5}\right]$$

Therefore, $$\mathscr{L}^{-1}\left\{\frac{1}{s^2+2s-15}\right\} = \frac{1}{8}\,\mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\} - \frac{1}{8}\,\mathscr{L}^{-1}\left\{\frac{1}{s+5}\right\}$$

$$= \frac{1}{8}\,e^{3t} - \frac{1}{8}\,e^{-5t}$$

Example 17. Compute $\mathscr{L}^{-1}\left\{\dfrac{3s+17}{s^2+8s+25}\right\}$ ✓

Solution. Here

$$\frac{3s+17}{s^2+8s+25} = \frac{3(s+4)+5}{(s+4)^2+3^2}$$

So $$\mathscr{L}^{-1}\left\{\frac{3s+17}{s^2+8s+25}\right\} = 3\,\mathscr{L}^{-1}\left\{\frac{s+4}{(s+4)^2+3^2}\right\} + 5\,\mathscr{L}^{-1}\left\{\frac{1}{(s+4)^2+3^2}\right\}$$

$$= 3e^{-4t}\,\mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \frac{5}{3}\,e^{-4t}\,\mathscr{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$= 3e^{-4t}\cos 3t + \frac{5}{3}\,e^{-4t}\sin 3t.$$

Example 18. Evaluate $\mathscr{L}^{-1}\left\{\dfrac{s^2}{(s^2+a^2)^2}\right\}$ ✓

Solution. Here $$\frac{s^2}{(s^2+a^2)^2} = \frac{1}{s^2+a^2} - \frac{a^2}{(s^2+a^2)^2}$$

So $$\mathscr{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} - a^2\,\mathscr{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$$

$$= \frac{1}{a}\,\sin at - a^2\,\mathscr{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} \qquad (1)$$

Now $$\mathscr{L}\{t\sin at\} = -\frac{d}{ds}\,\mathscr{L}\{\sin at\}$$

$$= -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$$

$$= \frac{2as}{(s^2+a^2)^2}$$

Therefore, $$2a\,\mathscr{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = t\sin at$$

By the integration property (11.18) (iv), we get

$$2a\,\mathscr{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\cdot\frac{1}{s}\right\} = 2a\,\mathscr{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$$

$$= \int_0^t u\sin au\,du$$

$$= \left[\frac{-u\cos au}{a}\right]_0^t - \frac{1}{a}\int_0^t \cos au\,du$$

$$= -\frac{t\cos at}{a} - \frac{1}{a^2}\,\sin at$$

Substituting into (1), we have

$$\mathscr{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = \frac{1}{a}\,\sin at + \frac{t\cos at}{2} - \frac{\sin at}{2a}$$

$$= \frac{1}{2a}\,(\sin at + at\cos at).$$

Example 19. Compute $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2}\right\}$

Solution. Here $$\mathscr{L}\{u_2(t)\} = \frac{e^{-2s}}{s}.$$

Therefore, $$\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} = \int_0^t u_2(\tau)\,d\tau$$

$$= (t-2)\,u_2(t).$$

Again, $$\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s^3}\right\} = \int_0^t (t-2)\,u_2(\tau)\,d\tau$$

$$= u_2(t)\,\frac{(t-2)^2}{2}.$$

Alternative Method:

$$\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s^3}\right\} = u_2(t)f(t-2) \quad , \quad \text{where } f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{t^2}{2}$$

$$= u_2(t)\,\frac{(t-2)^2}{2}$$

Example 20. Find $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2(s^2-a^2)}\right\}$

Solution. Here $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2-a^2}\right\} = \dfrac{1}{a}\sinh at.$

Using the integration property **(11.18)(iv)**, we get

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s^2-a^2)}\right\} = \frac{1}{a}\int_0^t \sinh au\,du$$

$$= \frac{1}{a^2}\left[\cosh au\right]_0^t = \frac{1}{a^2}\{\cosh at - 1\}.$$

Applying the same property again, we obtain

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2(s^2-a^2)}\right\} = \frac{1}{a^2}\int_0^t (\cosh au - 1)\,du$$

$$= \frac{1}{a^3}\sinh at - \frac{1}{a^2}t$$

Alternative Method:

Here
$$\frac{1}{s^2(s^2-a^2)} = \frac{1}{a^2}\left(\frac{1}{s^2-a^2} - \frac{1}{s^2}\right) \checkmark$$

Therefore,
$$\mathscr{L}^{-1}\left\{\frac{1}{s^2(s^2-a^2)}\right\} = \frac{1}{a^2}\mathscr{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\} - \frac{1}{a^2}\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= \frac{1}{a^3}\sinh at - \frac{1}{a^2}t \quad .$$

(11.19) Definition. (**Convolution**). Let $f(t)$ and $g(t)$ be piecewise continuous functions on $[\,0, \infty\,[$ The **convolution** of f and g, denoted by $f \bullet g$, is defined by

$$(f \bullet g)t = \int_0^t f(t-u)\,g(u)\,du$$

It can be easily verified that

(i) $f \bullet g = g \bullet f$

(ii) $f \bullet (g+h) = f \bullet g + f \bullet h$

(iii) $f \bullet 0 = 0$

The following result is useful in evaluating the Laplace transform and its inverse.

(11.20) Theorem. Let $f(t)$ and $g(t)$ be piecewise continuous on $[\,0, \infty\,[$ and of exponential order and $\mathscr{L}(f) = F(s)$, $\mathscr{L}(g) = G(s)$ Then

(i) $\mathscr{L}(f \bullet g)(s) = F(s)\,G(s)$

(ii) $\mathscr{L}^{-1}\{F(s)\,G(s)\}(t) = (f \bullet g)t$

$$= \int_0^t f(t-u)\,g(u)\,du$$

The proof of this theorem is omitted

Example 21. Let $f(t) = t$ and $g(t) = \cos t$ Compute $f \bullet g$ and verify

Solution. $(f \bullet g)\,t = \displaystyle\int_0^t (t-u)\cos u\,du$

$$= \int_0^t t\cos u\,du - \int_0^t u\cos u\,du$$

$$= t\left[\sin u\right]_0^t - \left\{\left[u\sin u\right]_0^t - \int_0^t \sin u\,du\right\}$$

$$= t\sin t - t\sin t + \left[-\cos u\right]_0^t$$

$$= 1 - \cos t$$

$$\mathscr{L}(f) = \frac{1}{s^2} = F(s),\ \mathscr{L}(g) = \frac{s}{1-s^2} = G(s)$$

(i)
$$\mathscr{L}(f \bullet g) = \int_0^\infty e^{-st}\left\{\int_0^t (t-u)\cos u\,du\right\}dt$$

$$= \int_0^\infty e^{-st}(1-\cos t)\,dt$$

$$= \int_0^\infty e^{-st}\,dt - \int_0^\infty e^{-st}\cos t\,dt$$

$$= \frac{1}{s} - \frac{s}{s^2+1} = \frac{1}{s(s^2+1)} = \frac{1}{s^2} \cdot \frac{s}{s^2+1}$$

$$= F(s)\,G(s)$$

(ii) $$\mathscr{L}^{-1}\{F(s)\,G(s)\} = \mathscr{L}^{-1}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$$

$$= \mathscr{L}^{-1}\left(\frac{1}{s}\right) - \mathscr{L}^{-1}\left(\frac{s}{s^2+1}\right)$$

$$= 1 - \cos t = (f * g).$$

Example 22. Compute $\mathscr{L}^{-1}\left\{\dfrac{2a^2}{(s^2+a^2)^2}\right\}$.

Solution. We have $\mathscr{L}^{-1}\left\{\dfrac{a}{s^2+a^2}\right\} = \sin at.$

By **(11.20) (ii)**, we get

$$\mathscr{L}^{-1}\left\{\frac{2a^2}{(s^2+a^2)^2}\right\} = 2(\sin at * \sin at)$$

$$= 2\int_0^t \sin a(t-u)\sin au\,du$$

$$= 2\int_0^t \{\sin at\cos au - \cos at\sin au\}\sin au\,du$$

$$= \sin at\int_0^t 2\cos au\sin au\,du - 2\cos at\int_0^t \sin^2 au\,du$$

$$= \sin at\int_0^t \sin 2au\,du - \cos at\int_0^t 2\sin^2 au\,du$$

$$= \sin at\left[-\frac{\cos 2at}{2a}\right]_0^t - \cos at\int_0^t (1-\cos 2au)\,du$$

$$= -\frac{\sin at(\cos 2at - 1)}{2a} - \cos at\left[t - \frac{\sin 2at}{2a}\right]$$

$$= \frac{1}{2a}(\sin 2at\cos at - \cos 2at\sin at) - t\cos at + \frac{1}{2a}\sin at$$

$$= \frac{1}{2a}\sin(2at - at) + \frac{1}{2a}\sin at - t\cos at$$

$$= \frac{1}{a}\sin at - t\cos at.$$

EXERCISE 11.2

Compute the inverse Laplace transform of each of the following (**Problems 1 – 20**).

1. $\dfrac{s-2}{s^2-2}$
2. $\dfrac{3s+1}{s^2-6s+18}$
3. $\dfrac{9s-67}{s^2-16s+\cancel{49}35}$
4. $\dfrac{as+b}{s^2+2cs+d}$, $d > c^2 > 0$
5. $\dfrac{s}{(s+a)^2+b^2}$
6. $\dfrac{1}{(s^2+a^2)(s^2+b^2)}$
7. $\dfrac{1}{(s-1)(s^2+4)}$
8. $\dfrac{7s+5}{(3s-8)^2}$
9. $\dfrac{5s+3}{(s+7)^3}$
10. $\dfrac{2s-3}{2s^3+3s^2-2s-3}$
11. $\dfrac{2s^3+6s^2+21s+52}{s(s+2)(s^2+4s+13)}$
12. $\dfrac{1}{(s^2+4)(s^2+6s-5)}$
13. $\dfrac{s^3+3s^2-s-3}{(s^2+2s+5)^2}$
14. $\arctan\dfrac{a}{s}$
15. $\ln\dfrac{s^2+1}{(s-1)^2}$
16. $\ln\dfrac{s^2+a^2}{s^2+b^2}$
17. $\dfrac{e^{-3s}}{s^2(s^2+9)}$
18. $e^{-\pi s}\dfrac{s}{s^2-4s+5}$
19. $e^{-2s}\cdot\dfrac{s+6}{s^3-5s^2+6s}$
20. $e^{-3s}\cdot\dfrac{3s-7}{s^2-10s+26}$

In each of **Problems 21 – 23**, use the Convolution Theorem 11.20 to evaluate the inverse Laplace transform:

21. $\dfrac{1}{s^2(s+5)}$
22. $\dfrac{s}{(s+1)(s^2+4)}$
23. $\dfrac{1}{(s^2+1)(s^2+4s+5)}$
24. Show that $\mathscr{L}^{-1}\left\{\dfrac{s^3}{s^4+4a^4}\right\} = \cosh at\cos at$
25. Show that $\mathscr{L}^{-1}\left\{\dfrac{s}{s^4+4a^4}\right\} = \dfrac{1}{2a^2}\sinh at\sin at.$

SOLUTIONS OF INITIAL VALUE PROBLEMS

(11.21) In this section we shall use the powerful tool of Laplace transform to solve constant coefficients linear initial value problems. It will be seen that by the methods of Laplace transform, a differential equation can be converted into an algebraic equation. The independent variable will be t instead of x and the dependent variable will remain y as before in the differential equations to be considered here.

If $y(t)$ is a solution of a differential equation, then the Laplace transform of $y(t)$ will be denoted by $\mathscr{L}\{y(t)\} = Y(s)$ in conformity with our earlier notation. The differentiation formula (**Theorem 11.7**) and its corollaries are the most important properties to be employed in the solutions of the problems.

The following procedure will be adopted:

I. Given an initial value problem, take Laplace transform of both sides. Use Theorem 11.7 and initial conditions to convert the differential equation into an algebraic equation in $Y(s)$.

II. Solve the algebraic equation for $Y(s)$.

III. The inverse transform $\mathscr{L}^{-1}\{Y(s)\} = y(t)$ is the required solution of the given problem.

The method is applicable to both homogeneous and nonhomogeneous equations. However, for nonhomogeneous equations, the function $f(t)$ on the right hand side of an equation in standard form must possess Laplace transform.

This method is now illustrated by examples.

Example 23. Use the Laplace transform to solve

$$\frac{dy}{dt} + y = -e^t, \qquad (1)$$

$$y(0) = 2.$$

Solution. Taking Laplace transform of both sides of (1), we have

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} + \mathscr{L}\{y(t)\} = -\mathscr{L}\{e^t\}. \qquad (2)$$

Now $\mathscr{L}\left\{\frac{dy}{dt}\right\} = s\,Y(s) - y(0)$, by (11.7)

$= s\,Y(s) - 2$, on applying the initial condition.

Substituting into (2), we get

$$s\,Y(s) - 2 + Y(s) = -\frac{1}{s-1}$$

or $$(s+1)\,Y(s) = 2 - \frac{1}{s-1} = \frac{2s-3}{s-1}$$

or $$Y(s) = \frac{2s-3}{(s-1)(s+1)} = \frac{5}{2}\cdot\frac{1}{s+1} - \frac{1}{2}\cdot\frac{1}{s-1}$$

Applying $\mathscr{L}^{-1}$ on both sides, we have

$$\mathscr{L}^{-1}\{Y(s)\} = y(t) = \frac{5}{2}\,\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{2}\,\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= \frac{5}{2}e^{-t} - \frac{1}{2}e^t$$

is the required solution.

Example 24. Solve $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = t\,e^t$, (1)

$$y(0) = 0, \quad y'(0) = 0$$

by the method of Laplace transform.

Solution. Taking Laplace transform of both sides of (1), we have

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} - 2\,\mathscr{L}\left\{\frac{dy}{dt}\right\} + \mathscr{L}\{y(t)\} = \mathscr{L}\{t\,e^t\}$$

or $$s^2 Y(s) - s\,y(0) - y'(0) - 2s\,Y(s) - 2y(0) + Y(s) = \frac{1}{(s-1)^2}$$

or $$(s^2 - 2s + 1)\,Y(s) = \frac{1}{(s-1)^2}$$

i.e., $$Y(s) = \frac{1}{(s-1)^4}.$$

Therefore, $$\mathscr{L}^{-1}\{Y(s)\} = y(t) = \mathscr{L}^{-1}\left\{\frac{1}{(s-1)^4}\right\} = \frac{1}{6}t^3 e^t$$

is the required solution.

Example 25. Solve $\frac{d^2y}{dt^2} + y = 2u_4(t)\sin\pi t$, (1)

$$y(0) = 1, \; y'(0) = 0.$$

Solution. Taking Laplace transform of both sides of (1), we get

$$s^2 Y(s) - s \cdot 1 - 0 + Y(s) = 2e^{-4s}\,\frac{\pi}{s^2+\pi^2}\,, \quad \sin \pi t = \sin \pi(t-4).$$

or

$$Y(s) = \frac{s + \dfrac{2\pi e^{-4s}}{s^2+\pi^2}}{s^2+1}$$

$$= \frac{s}{s^2+1} + 2\pi e^{-4s}\left(\frac{1}{(s^2+1)(s^2+\pi^2)}\right)$$

$$= \frac{s}{s^2+1} + 2\pi e^{-4s}\left[\frac{1}{\pi^2-1}\cdot\frac{1}{s^2+1} - \frac{1}{\pi^2-1}\cdot\frac{1}{s^2+\pi^2}\right]$$

$$\mathcal{L}^{-1}\{Y(s)\} = y(t) = \cos t + \frac{2\pi}{\pi^2-1}\,u_4(t)\sin(t-4) - \frac{2}{\pi^2-1}\,u_4(t)\sin \pi(t-4)$$

$$= \cos t + \frac{2}{\pi^2-1}\,u_4(t)\left[\pi \sin(t-4) - \sin \pi t\right]$$

is the required solution.

Example 26. Solve

$$\frac{d^3y}{dt^3} - 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} - 6y = e^{4t}, \qquad (1)$$

$$y(0) = y'(0) = y''(0) = 0.$$

Solution. Taking Laplace transform of both sides of (1), we have

$$s^3 Y(s) - s^2\,y(0) - s\,y'(0) - y''(0) - 6s^2\,Y(s) + 6s\,y(0) + 6y'(0)$$

$$+ 11s\,Y(s) - 11\,y(0) - 6Y(s) = \frac{1}{s-4}$$

or

$$(s^3 - 6s^2 + 11s - 6)\,Y(s) = \frac{1}{s-4}$$

or

$$Y(s) = \frac{1}{(s^3-6s^2+11s-6)(s-4)}$$

$$= \frac{1}{(s-1)(s-2)(s-3)(s-4)}$$

$$= \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3} + \frac{D}{s-4}$$

$$= -\frac{1}{6}\,\frac{1}{s-1} + \frac{1}{2}\,\frac{1}{s-2} - \frac{1}{2}\cdot\frac{1}{s-3} + \frac{1}{6}\cdot\frac{1}{s-4}$$

Therefore,

$$\mathcal{L}^{-1}\{Y(s)\} = y(t) = -\frac{1}{6}\,e^{t} + \frac{1}{2}\,e^{2t} - \frac{1}{2}\,e^{3t} + \frac{1}{6}\,e^{4t}$$

is the solution of (1).

Example 27. Solve $\dfrac{d^4y}{dt^4} - y = u_1(t) - u_2(t),$ (1)

$$y(0) = y'(0) = y''(0) = y'''(0) = 0.$$

Solution. Taking Laplace transform of (1), we have

$$s^4\,Y(s) - s^3\,y(0) - s^2\,y'(0) - s\,y''(0) - y'''(0) - Y(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$$

or

$$(s^4 - 1)\,Y(s) = \frac{e^{-s} - e^{-2s}}{s}$$

or

$$Y(s) = (e^{-s} - e^{-2s})\,\frac{1}{s(s-1)(s+1)(s^2+1)}$$

Now,

$$\frac{1}{s(s-1)(s+1)(s^2+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1} + \frac{Ds+E}{s^2+1}$$

or

$$1 = A(s^4-1) + Bs(s+1)(s^2+1) + Cs(s-1)(s^2+1) + (Ds+E)\,s(s^2-1) \qquad (2)$$

Setting $s = 0, 1, -1$ into (2), we find

$$A = -1, \qquad B = \frac{1}{4}, \qquad C = \frac{1}{4}.$$

Equate coefficients of like terms in (2).

Coeff. of s^4: $0 = A + B + C + D$ or $D = \frac{1}{2}$

Coeff. of s^3: $0 = B - C + E$ or $E = 0.$

Therefore,

$$Y(s) = (e^{-s} - e^{-2s})\left\{\frac{-1}{s} + \frac{1}{4(s-1)} + \frac{1}{4(s+1)} + \frac{1}{2}\,\frac{s}{s^2+1}\right\}$$

Taking inverse Laplace transform, we have

$$\mathcal{L}^{-1}\{Y(s)\} = y(t) = u_1(t)\left[-1 + \frac{1}{4}e^{t-1} + \frac{1}{4}e^{-(t-1)} + \frac{1}{2}\cos(t-1)\right]$$

$$- u_2(t)\left[-1 + \frac{1}{4}e^{t-2} + \frac{1}{4}e^{-(t-2)} + \frac{1}{2}\cos(t-2)\right]$$

as the desired solution.

Example 28. Solve $t\frac{d^2y}{dt^2} - t\frac{dy}{dt} - y = 0,$ (1)

$$y(0) = 0, \quad y'(0) = 3.$$

Solution. We have

$t = -\frac{d}{ds}$

$$\mathcal{L}\left\{t\frac{d^2y}{dt^2}\right\} = -\frac{d}{ds}\,\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\}$$

$$= -\frac{d}{ds}\,\{s^2\,Y(s) - s\,y(0) - y'(0)\}$$

$$= -s^2\,Y'(s) - 2\,s\,Y(s) + y(0)$$

$$= -s^2\,Y'(s) - 2s\,Y(s). \quad (2)$$

$$\mathcal{L}\left\{t\frac{dy}{dt}\right\} = -\frac{d}{ds}\,\{s\,Y(s) - y(0)\}$$

$$= -s\,Y'(s) - Y(s). \quad (3)$$

Taking Laplace transform of (1) and using (2) and (3), we get

$$-s^2\,Y'(s) - 2s\,Y(s) + s\,Y'(s) + Y(s) - Y(s) = 0$$

or $\quad (s^2 - s)\,Y'(s) + 2s\,Y(s) = 0$

i.e., $\quad Y'(s) + \frac{2s}{s^2 - s}\,Y(s) = 0$

or $\quad \frac{Y'(s)}{Y(s)} = \frac{-2}{s-1}$

Integrating, we have

$$\ln|Y(s)| = -2\ln|s-1| + \ln c$$

or $\quad Y(s) = \frac{c}{(s-1)^2}$

Therefore, $\mathcal{L}^{-1}\{\,Y(s)\,\} = y(t) = c\,t\,e^t$ (4)

Differentiate (4) w.r.t t and apply the initial conditions.

$$y'(t) = c\,e^t + c\,t\,e^t$$

$$3 = y'(0) = \left[c(t+1)\,e^t\right]_{t=0} = c.$$

Thus the solution is $y(t) = 3t\,e^t$.

Example 29. Find the solution $(x(t), y(t))$ of the system

$$\frac{dx}{dt} - x + y = 2e^t, \qquad x(0) = 0$$

$$\frac{dy}{dt} + x - y = e^t, \qquad y(0) = 0$$

Solution. Taking Laplace transform, we have

$$s\,X(s) - x(0) - X(s) + Y(s) = \frac{2}{s-1}$$

$$s\,Y(s) - y(0) + X(s) - Y(s) = \frac{1}{s-1}$$

or $\quad (s-1)\,X(s) + Y(s) = \frac{2}{s-1}$ (1)

and $\quad X(s) + (s-1)\,Y(s) = \frac{1}{s-1}$ (2)

Multiply (2) by $s-1$ and subtract (1) from the resulting equation to have

$$[(s-1)^2 - 1]\,Y(s) = 1 - \frac{2}{s-1} = \frac{s-3}{s-1}$$

or $\quad Y(s) = \frac{s-3}{(s-1)\,s(s-2)}$

$$= \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$$

$$= -\frac{3}{2}\cdot\frac{1}{s} + \frac{2}{s-1} - \frac{1}{2}\cdot\frac{1}{s-2}$$

Therefore, $\mathcal{L}^{-1}\{\,Y(s)\,\} = y(t) = -\frac{3}{2} + 2e^t - \frac{1}{2}\,e^{2t}.$

Differentiating w.r.t. t, we obtain

$$\frac{dy}{dt} = 2e^t - e^{2t}.$$

Substituting for y and $\frac{dy}{dt}$ into second equation of the system, we get

$$x = e^t + y - \frac{dy}{dt} = e^t - \frac{3}{2} + 2e^t - \frac{1}{2}\,e^{2t} - 2e^t + e^{2t}$$

$$= -\frac{3}{2} + e^t + \frac{1}{2}\,e^{2t}.$$

Solution of the system is

$$x(t) = -\frac{3}{2} + e^t + \frac{1}{2}\,e^{2t},$$

$$y(t) = -\frac{3}{2} + 2e^t - \frac{1}{2}\,e^{2t}.$$