ME 441 Exam1 (v1)
Fall 2023
Due 10/30 by midnight

- **General rules:**
- **Read the following lines and start right away!**
- You may only discuss exam related material with the instructor and not any other person.
- Information in the lecture notes, textbook and descriptions and hints provided in the exam are all you need to answer the questions and it is best to keep focus on these resources to avoid confusion. But you may use any other resource (that excludes any person other than the instructor) if you wish, **after providing proper citation**.
- Using a resource without citation **is violation of the academic honesty policy**.
- Write clearly and legibly on a white paper so that your thoughts could be followed easily in the grading process.
- If you use a software to find your answers, make sure you provide thorough information about your steps. Final form of the expressions found by software need to be transferred to your exam papers and integrated with your work. A printout of your code should be added as appendix.
- If derivation is not asked for, you don't have to derive! Feel free to use final form of equations, shape functions, etc. as long as we covered those equations in class or they are written in the book.

Academic honesty policy: We enforce academic honesty policy strictly. Violating exam rules could result in referring your case to the board of academic. Make sure you write **honor pledge statement**: "*I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own*." on your exam cover page. Grading will NOT be possible without having a signed honor pledge on your cover page.

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own

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FEM

ME 441

Mid Term

Afnan Mostafa

Q1.

Reasons FEM results vary depending on the modeling setup:

1 FEM results largely depend on the shape functions and these shape functions are dependent on how many nodes per element are considered for our initial definition of the setup.

Ex. CST has constant derivative of shape functions and the shape functions themselves are planes as compared to LST wherein the shape functions are 2nd order. LST is much more accurate in terms of explaining gradient fields however it is expensive (computationally). So, to accommodate constant strain in CST, one needs to use more and more mesh to make it look like not constant anymore. CST with coarse mesh will produce incorrect approximations.

2 FEM results can vary if the number of nodes is increased/decreased during the modeling setup (changing the field variable, ϕ, values/ nodal values).

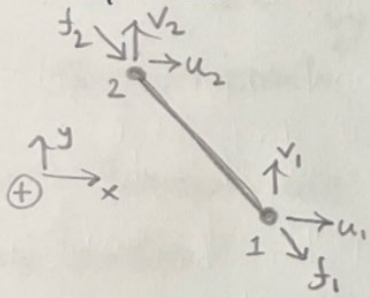
3 Finally, the type of connecting element (Q8, Q4, LST, CST) will result in different FEM results.

better

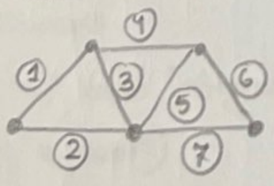
Q2.

Trusses

1. Can't take moments, only axial loads (made up with bars)



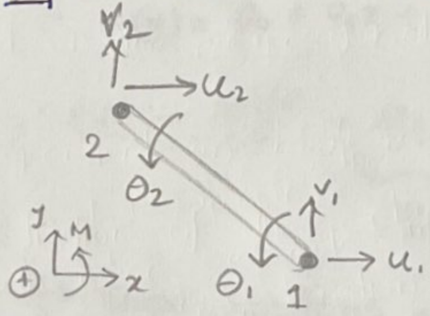
2. Trusses consist of multiple bars that are pinned at the connecting grid points.



Truss

Beams

1. Can take moments, shear, axial loads



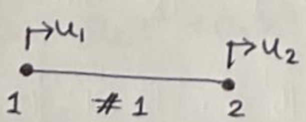
2. Beams are basically solid shaft (it can be standalone).

Cantilever Beam

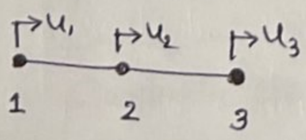
Q3

1D elements can be represented with multi-noded FEM representations.

For example,



(1D 2-noded element) /bar



(1D 3-noded elements) /bar

1

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Here,

For 2-noded element,

$$\phi(x) = a_0 + a_1 x$$

Let ϕ be u

$$\therefore u(x) = a_0 + a_1 x$$

$$\therefore \varepsilon(x) = \frac{\partial u(x)}{\partial x} = \underbrace{a_1}_{\text{Constant strain}}$$

→ Shape functions will be of 1st order.

$$\therefore [B] = \underbrace{[\partial][N]}_{\text{Slope of plane}} \rightarrow \text{Constant}$$

$$[K] = \underbrace{[B]^T [E][B]}_{\text{Constant}} \int_v dv$$

No need to take integral (PVWVE theorem) } good but need to use fine mesh to accommodate inaccurate model description.

For 3-noded element,

$$\phi(x) = a_0 + a_1 x + a_2 x^2$$

Let ϕ be u

$$\therefore u(x) = a_0 + a_1 x + a_2 x^2$$

$$\therefore \varepsilon(x) = \frac{\partial u(x)}{\partial x} = \underbrace{a_1 + 2a_2 x}_{\text{linear strain}}$$

→ Shape functions will be of ~~linear~~ 2nd order.

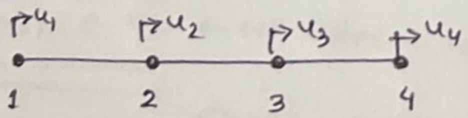
$$\therefore [B] = [\partial][N] \rightarrow \underbrace{\text{1st order}}_{\text{derivative field}}$$

$$[K] = \int_v [B]^T [E][B] \, dv$$

↳ still need to take integrals (doable)

Hence,

4-noded bars will have 3rd order shape functions.



#1

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$\varepsilon(x) = \frac{\partial u(x)}{\partial x} = \underbrace{a_1 + 2a_2 x + 3a_3 x^2}_{\text{Quadratic strain}}$$

$$[B] = [\partial]\underbrace{[N]}_{\text{3rd order}} \longrightarrow \text{2nd order}$$

Hence, derivative fields are 2nd order and we need to do a lot of integrals.

Hence, more than 3 nodes eleme are not standard for engineering.

Q5

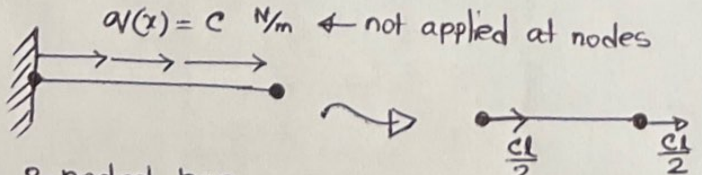
Assembly in FEM:

Assembly refers to the method of extending an elemental stiffness matrix, [kele], to a global DOF stiffness matrix, $[k_{ele}]_{gl}$ and then, adding all $[k_{ele\,1}]_{gl}$, $[k_{ele\,2}]_{gl}$ . . . $[k_{ele\,n}]_{gl}$ to house a structural stiffness matrix, $[k_{str}]$.

Algorithm:

1. Find the elements in stiffness matrices
2. Expand them to global DOF
3. Add them together.

Q4. ~~Q6~~. **Consistent Loads**: A set of equivalent nodal forces that produce the same work that forces in non-nodal positions produce.



2-noded bar

$$[k]\{d\} = \underbrace{\{f_v\}}_{\text{Consistent body force vector}} + \underbrace{\{f_s\}}_{\text{consistent surface force vector}}$$

→ True for any element

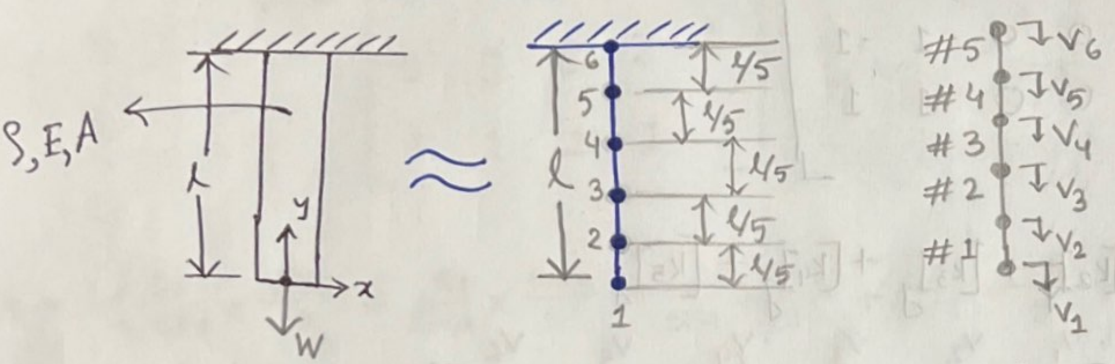
→ True for any interpolation function

why? → Hence, if we have consistent loads, we can do calculations for any [N], any shape/size, any element.

This above relation is true for any size or space.

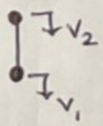
Q6.

a) 5 2-noded bar elements:



For each 2-noded bar, we need to determine local [K] first.

#1:



$$[K_1] = \frac{EA}{l/5} \begin{array}{cc} v_1 & v_2 \end{array} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

#2:

$$[k_2] = \frac{EA}{l/5} \begin{array}{cc} v_2 & v_3 \end{array} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

#3:

$$[k_3] = \frac{EA}{l/5} \begin{array}{cc} v_3 & v_4 \end{array} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

#4:

$$[k_4] = \frac{EA}{l/5} \begin{array}{cc} v_4 & v_5 \end{array} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

#5:

$$[k_5] = \frac{EA}{l/5} \begin{array}{cc} v_5 & v_6 \end{array} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Expanding to global DOF:

$$[k_1]_g = \frac{5EA}{l} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$[k_2]_g = \frac{5EA}{l} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$[k_3]_g = \frac{5EA}{l} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$[k_4]_g = \frac{5EA}{l} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$[k_5]_g = \frac{5EA}{l} \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

$$\therefore [K_{global}] = [k_1]_g + [k_2]_g + [k_3]_g + [k_4]_g + [k_5]_g$$

$$[K_{global}] = \frac{5EA}{l} \begin{array}{cc} \begin{matrix} V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \end{matrix} & \\ \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} & \begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{matrix} \end{array}_{6\times6}$$

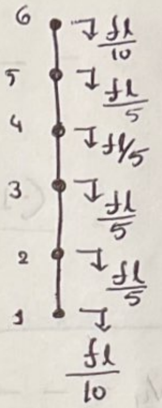
we know, $V_6 = 0$

$$\therefore [K_{global}]_{BC} = \frac{5EA}{l} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & \cancel{1}2 \end{bmatrix}_{5\times5}$$

$$\therefore \{f\} = [K_{global}]_{BC} \{V\}$$

$$\{V\} = \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{Bmatrix} \qquad \{f\} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{Bmatrix} = \frac{l}{5EA} \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 4 & 3 & 2 & 1 \\ 3 & 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} -\frac{\rho A l g}{10} \\ -\left(\frac{\rho A l g}{10} + \frac{\rho A l g}{10}\right) \\ -\left(\frac{\rho A l g}{10} + \frac{\rho A l g}{10}\right) \\ -\left(\frac{\rho A l g}{10} + \frac{\rho A l g}{10}\right) \\ -\left(\frac{\rho A l g}{10} + \frac{\rho A l g}{10}\right) \end{Bmatrix}$$



$$\therefore V_1 = -\left(\frac{5l}{5EA} \times \frac{\rho A l g}{10} + \frac{4l}{5EA} \times \frac{\rho A l g}{5} + \frac{3l}{5EA} \times \frac{\rho A l g}{5} + \frac{2l}{5EA} \times \frac{\rho A l g}{5} + \frac{l}{5EA} \times \frac{\rho A l g}{5}\right)$$

$$= -\frac{\rho l^2 g}{10E} - \frac{4\rho l^2 g}{25E} - \frac{3\rho l^2 g}{25E} - \frac{2\rho l^2 g}{25E} - \frac{\rho l^2 g}{25E}$$

$$= \frac{-5\rho l^2 g - 8\rho l^2 g - 6\rho l^2 g - 4\rho l^2 g - 2\rho l^2 g}{50E}$$

$$= \frac{-25\rho l^2 g}{50E}$$

$$= -\frac{\rho l^2 g}{2E} \qquad \boxed{\therefore V_1 = -\frac{\rho l^2 g}{2E}}$$

$$V_2 = -\frac{2\rho l^2 g}{25E} - \frac{4\rho l^2 g}{25E} - \frac{3\rho l^2 g}{25E} - \frac{2\rho l^2 g}{25E} - \frac{\rho l^2 g}{25E}$$

$$= \frac{(-2-4-3-2-1)\rho l^2 g}{25E}$$

$$\boxed{\therefore V_2 = -\frac{12\rho l^2 g}{25E}}$$

$$V_3 = -\frac{3\rho l^2 g}{50E} - \frac{3\rho l^2 g}{25E} - \frac{3\rho l^2 g}{25E} - \frac{2\rho l^2 g}{25E} - \frac{\rho l^2 g}{25E}$$

$$= \frac{(-3-6-6-4-2)\rho l^2 g}{50E}$$

$$\boxed{\therefore V_3 = -\frac{21\rho l^2 g}{50E}}$$

$$V_4 = -\frac{\rho l^2 g}{25E} - \frac{2\rho l^2 g}{25E} - \frac{2\rho l^2 g}{25E} - \frac{2\rho l^2 g}{25E} - \frac{\rho l^2 g}{25E}$$

$$= \frac{(-1-2-2-2-1)\rho l^2 g}{25E}$$

$$\boxed{\therefore V_4 = -\frac{8\rho l^2 g}{25E}}$$

$$V_5 = -\frac{\rho l^2 g}{50E} - \frac{\cancel{2}\rho l^2 g}{25E} - \frac{\rho l^2 g}{25E} - \frac{\rho l^2 g}{25E} - \frac{\rho l^2 g}{25E}$$

$$= \frac{(-1-2-2-2-2)\rho l^2 g}{50E}$$

$$\boxed{\therefore V_5 = -\frac{9\rho l^2 g}{50E}}$$

$$\therefore \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{Bmatrix} = \begin{Bmatrix} -\frac{\rho l^2 g}{2E} \\ -\frac{12\rho l^2 g}{25E} \\ -\frac{21\rho l^2 g}{50E} \\ -\frac{8\rho l^2 g}{25E} \\ -\frac{9\rho l^2 g}{50E} \\ 0 \end{Bmatrix}$$

Exact solⁿ:

$$V = \frac{\rho g(y^2 - l^2)}{2E}$$

$$\therefore V_{1e} = \frac{\rho g(0^2 - l^2)}{2E} = -\frac{\rho g l^2}{2E}$$

$$V_{2e}\left(y = \frac{l}{5}\right) = \frac{\rho g\left(\frac{l^2}{25} - l^2\right)}{2E} = -\frac{12\rho l^2 g}{25E}$$

$$V_{3e}\left(y = \frac{2l}{5}\right) = \frac{\rho g\left(\frac{4l^2}{25} - l^2\right)}{2E} = -\frac{21\rho l^2 g}{50E}$$

$$V_{4e}\left(y = \frac{3l}{5}\right) = \frac{\rho g\left(\frac{9l^2}{25} - l^2\right)}{2E} = -\frac{8\rho l^2 g}{25E}$$

$$V_{5e}\left(y = \frac{4l}{5}\right) = \frac{\rho g\left(\frac{16l^2}{25} - l^2\right)}{2E} = -\frac{9\rho l^2 g}{50E}$$

$V_{1e} = V_1$

$V_{2e} = V_2$

$V_{3e} = V_3$

$V_{4e} = V_4$

$V_{5e} = V_5$

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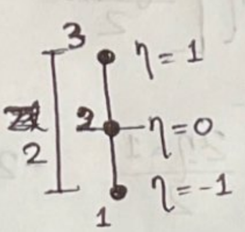
$\boxed{b}$

Using isoparametric Formulation:



$$\phi(\xi) = a_0 + a_1 \xi + a_2 \xi^2$$

It's a vertical bar, hence in (ξ, η) space,

$$\phi(\eta) = a_0 + a_1 \eta + a_2 \eta^2$$

$$\therefore [N] = [1 \quad \eta \quad \eta^2] \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

$$= [1 \quad \eta \quad \eta^2] \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-\eta}{2} + \frac{\eta^2}{2} & 1 + 0 - \eta^2 & \frac{\eta + \eta^2}{2} \end{bmatrix}$$

$$[N] = \begin{bmatrix} \underbrace{\frac{\eta(\eta - 1)}{2}}_{N_1} & \underbrace{1 - \eta^2}_{N_2} & \underbrace{\frac{\eta(\eta+1)}{2}}_{N_3} \end{bmatrix}$$

$$[B] = [\partial][N]$$

$$\frac{\partial}{\partial y}$$

$$\therefore \frac{\partial y}{\partial \eta} = \frac{(y_3 - y_1)}{2} = \frac{l}{2}$$

$$\therefore \frac{\partial \eta}{\partial y} = \frac{2}{l} = J$$

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$$\therefore [B] = \frac{\partial}{\partial y}[N] = J\frac{\partial}{\partial \eta}\begin{bmatrix} \frac{\eta(\eta-1)}{2} & 1-\eta^2 & \frac{\eta(\eta+1)}{2} \end{bmatrix}$$

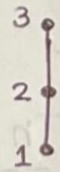
$$= J\begin{bmatrix} \frac{2\eta-1}{2} & -2\eta & \frac{2\eta+1}{2} \end{bmatrix}$$

$$[k] = \int_{y_1}^{y_2} [B]^T [E][B] A\,dy$$

$$= \int_{-1}^{1} J\begin{bmatrix} \frac{2\eta-1}{2} \\ -2\eta \\ \frac{2\eta+1}{2} \end{bmatrix} E\; J\begin{bmatrix} \frac{2\eta-1}{2} & -2\eta & \frac{2\eta+1}{2} \end{bmatrix} A\,d\eta\,\frac{l}{2}$$

$$= \frac{4}{l^2} EA \frac{l}{2} \times \int_{-1}^{1}\begin{bmatrix} \frac{(2\eta-1)^2}{4} & -\frac{(2\eta-1)2\eta}{2} & \frac{(2\eta-1)(2\eta+1)}{4} \\ -\frac{(2\eta-1)2\eta}{2} & 4\eta^2 & -\frac{(2\eta+1)2\eta}{2} \\ \frac{(2\eta+1)(2\eta-1)}{4} & -\frac{(2\eta+1)2\eta}{2} & \frac{(2\eta+1)^2}{4} \end{bmatrix} d\eta$$

$$= \frac{\cancel{2}EA}{2\cancel{4}l}\int_{-1}^{1}\begin{bmatrix} (2\eta-1)^2 & -4\eta(2\eta-1) & (2\eta-1)(2\eta+1) \\ -4\eta(2\eta-1) & 16\eta^2 & -4\eta(2\eta+1) \\ (2\eta+1)(2\eta-1) & -4\eta(2\eta+1) & (2\eta+1)^2 \end{bmatrix} d\eta \leftarrow$$



$$= \cancel{\frac{EA}{2l}}\ \Big|\ \cancel{\tfrac{4}{3}\eta^2 - 4\eta + 1} \quad \cancel{4\eta - 8\eta^2} \quad 4\eta^2$$

After integrating,

$$[k] = \frac{EA}{2l} \begin{bmatrix} \frac{4}{3}\eta^3 - 2\eta^2 + \eta & -\frac{8}{3}\eta^3 + 2\eta^2 & \frac{4}{3}\eta^3 - \eta \\ -\frac{8}{3}\eta^3 + 2\eta^2 & \frac{16\eta^3}{3} & -\frac{8}{3}\eta^3 - 2\eta^2 \\ \frac{4}{3}\eta^3 - \eta & -\frac{8}{3}\eta^3 - 2\eta^2 & \frac{4}{3}\eta^3 + 2\eta^2 + \eta \end{bmatrix}_{-1}^{1}$$

After putting limits,

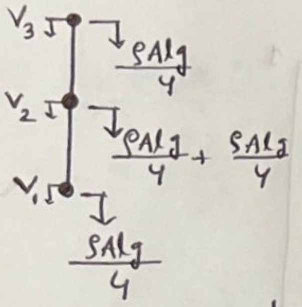
$$[k] = \frac{EA}{2l} \begin{bmatrix} 4.67 & -5.33 & 0.67 \\ -5.33 & 10.67 & -5.33 \\ 0.67 & -5.33 & 4.67 \end{bmatrix}$$

$$\begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix} = \frac{2l}{EA} \begin{bmatrix} 4.67 & -5.33 & 0.67 \\ -5.33 & 10.67 & -5.33 \\ 0.67 & -5.33 & 4.67 \end{bmatrix}^{-1} \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix}$$

Applying BC, $V_3 = 0$

$$\therefore \begin{Bmatrix} V_1 \\ V_2 \end{Bmatrix} = \frac{2l}{EA} \begin{bmatrix} 4.67 & -5.33 \\ -5.33 & 10.67 \end{bmatrix}^{-1} \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

$$\begin{Bmatrix} V_1 \\ V_2 \end{Bmatrix} = \frac{2l}{EA} \begin{bmatrix} 0.49 & 0.25 \\ 0.25 & 0.22 \end{bmatrix} \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$



$$\therefore \begin{Bmatrix} V_1 \\ V_2 \end{Bmatrix} = \frac{2l}{EA}\begin{bmatrix} 0.49 & 0.25 \\ 0.25 & 0.22 \end{bmatrix}\begin{Bmatrix} -\frac{\rho A l g}{4} \\ -\frac{\rho A l g}{2} \end{Bmatrix}$$

$$\therefore V_1 = -\frac{2\times 0.49 l}{EA}\times\frac{\rho A l g}{4} - \frac{2\times 0.25 l}{EA}\times\frac{\rho A l g}{2}$$

$$= -\frac{0.98\rho l^2 g}{4E} - \frac{0.5\rho l^2 g}{2E}$$

$$= \frac{-(0.98+1)\rho l^2 g}{4E}$$

$$\boxed{\therefore V_1 = -0.495\frac{\rho l^2 g}{E}}$$

$$V_2 = -\frac{2\times 0.25 l}{EA}\times\frac{\rho A l g}{4} - \frac{2\times 0.22 l}{EA}\times\frac{\rho A l g}{2}$$

$$= -\frac{0.5\rho l^2 g}{4E} - \frac{0.44\rho l^2 g}{2E}$$

$$= \frac{-(0.5+0.88)\rho l^2 g}{4E}$$

$$\boxed{\therefore V_2 = -0.345\frac{\rho l^2 g}{E}}$$

$$\begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} -0.495\frac{\rho l^2 g}{E} \\ -0.345\frac{\rho l^2 g}{E} \\ 0 \end{Bmatrix}$$

Exact Solution:

$$V_{1e} = \frac{\rho g (y^2 - l^2)}{2E} = \frac{-\rho g (l^2)}{2E} = -0.5 \frac{\rho g l^2}{E}$$

$$V_{2e}\left(y = \frac{l}{2}\right) = \frac{\rho g\left(\frac{l^2}{4} - l^2\right)}{2E} = \frac{-3\rho l^2 g}{8E} = -0.375 \frac{\rho l^2 g}{E}$$

[redacted] $\boxed{d}$

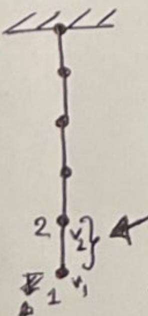
In this case, the displacements obtained from 3-noded bar solution are not exactly same as those of 2-noded bars solution. This is because of the increased number of nodes in 2-noded bar elements. The more nodes we use to produce an FEM result, the better the solution will be (ignoring the computational efforts).

2-noded Solution : 6 nodes $\rightarrow$ better approximation

3-noded solution : 3 nodes $\rightarrow$ not bad but worse that 2-noded.

[redacted] $\boxed{c}$ For non-nodal values, we use shape functions to get them using MATLAB (code is attached to Appendix).

2-noded bars:



$$\begin{cases} N_{1,1} = \dfrac{y_2 - y}{y_2 - y_1} \\ N_{2,1} = \dfrac{y - y_1}{y_2 - y_1} \end{cases}$$

for each bar:

$$V(y) = N_{1,1} V_1 + N_{2,1} V_2$$

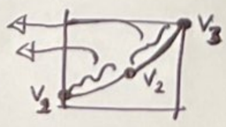
$V_1, V_2 \rightarrow$ nodal values.

3-noded bar:

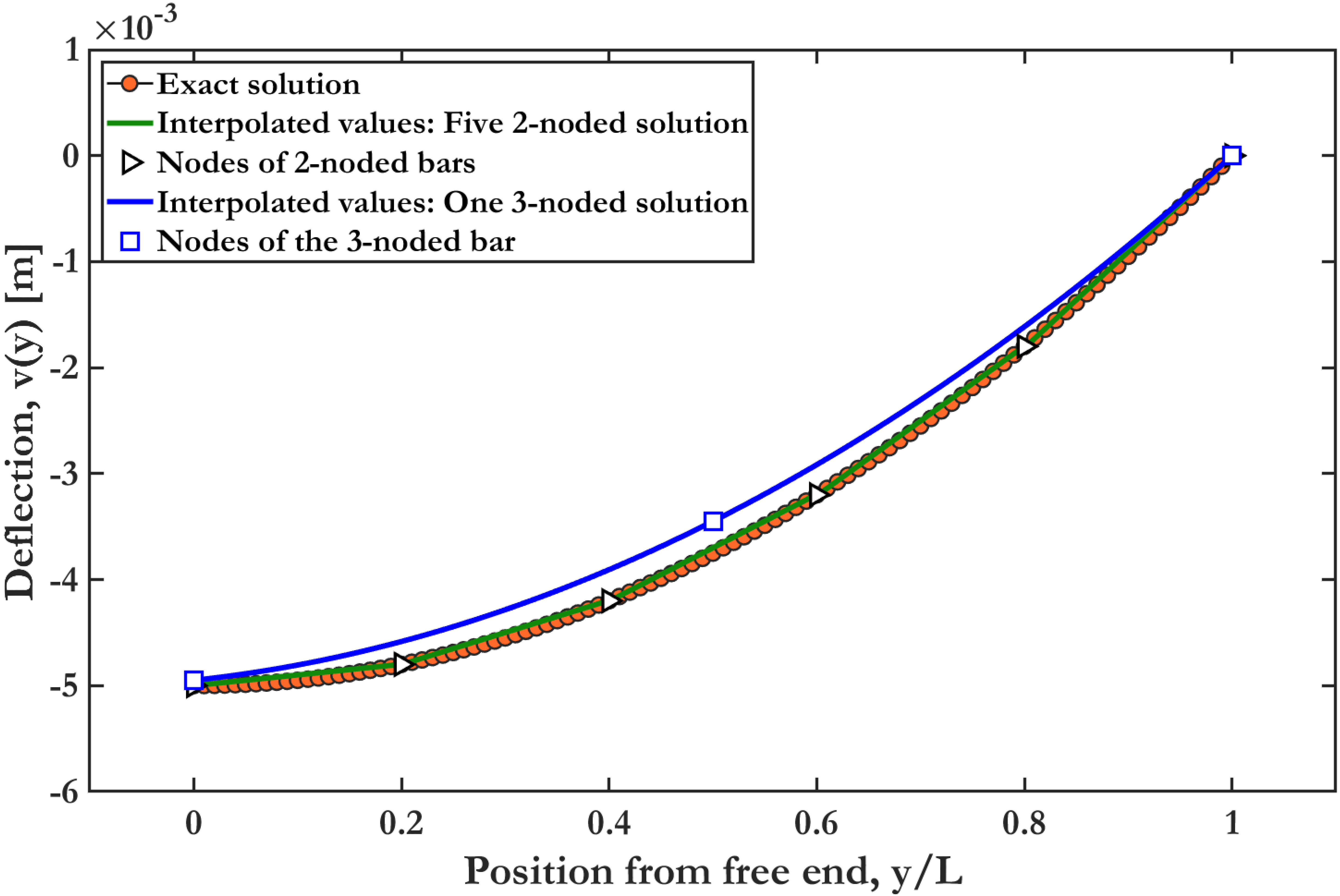
Shape functions, $[N] = \left[\underbrace{\frac{\eta(\eta-1)}{2}}_{N_1} \quad \underbrace{1-\eta^2}_{N_2} \quad \underbrace{\frac{\eta(\eta+1)}{2}}_{N_3} \right]$

for 1 3-noded bar:

$$\therefore V(y) = N_1 V_1 + N_2 V_2 + N_3 V_3$$

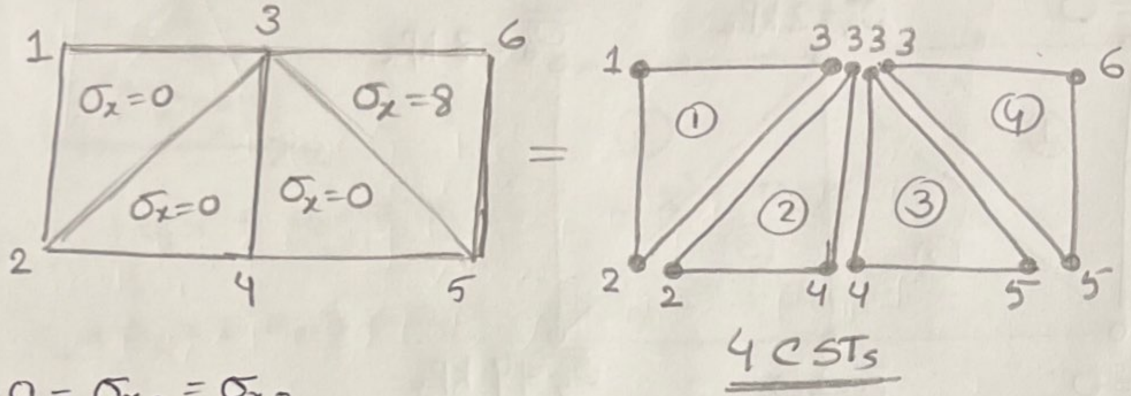


Scanned with CamScanner



×10^-3
1
0
-1
-2
-3
-4
-5
-6
Exact solution
Interpolated values: Five 2-noded solution
Nodes of 2-noded bars
Interpolated values: One 3-noded solution
Nodes of the 3-noded bar
Deflection, v(y) [m]
0
0.2
0.4
0.6
0.8
1
Position from free end, y/L

Q7



$\sigma_{x,1} = 0 = \sigma_{x,2} = \sigma_{x,3}$

$\sigma_{x,4} = 8$ kPa

There are total 6 nodes.

Average stresses:

Node 1 : 1 element (CST = 1)

$\therefore \overline{\sigma_1} = \frac{0}{1} = 0$ kPa

Node 2 : 2 CSTs → 1 + 2

$\overline{\sigma_2} = \frac{0+0}{2} = 0$ kPa

Node 3 : 4 CSTs → 1 + 2 + 3 + 4

$\therefore \overline{\sigma_3} = \frac{0+0+0+8}{4} = 2$ kPa

Node 4 : 2 CSTs → 2 + 3

$\overline{\sigma_4} = \frac{0+0}{2} = 0$ kPa

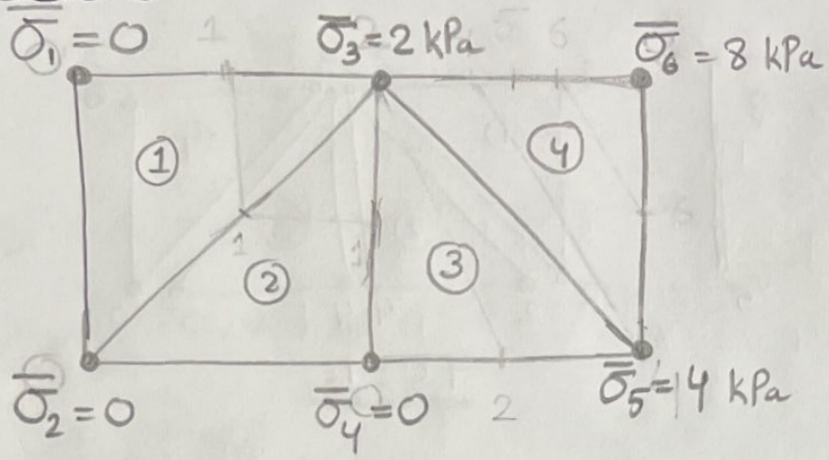
Node 5 : 2 CSTs → 3 + 4

$\overline{\sigma_5} = \frac{0+8}{2} = 4$ kPa

Node 6 : 1 CST → 4

$\therefore \overline{\sigma_6} = \frac{8}{1} = 8$ kPa

$\bar{\sigma}$ at nodal points:



For 1,

$$u(x,y) = a_0 + a_1 x + a_2 y \qquad V(x,y) = a_3 + a_4 x + a_5 y$$

$$\varepsilon_x(x,y) = \frac{\partial u(x,y)}{\partial x} = a_1 \quad \therefore \sigma_x = E\varepsilon_x \quad \therefore \sigma_x \rightarrow \text{Constant}$$

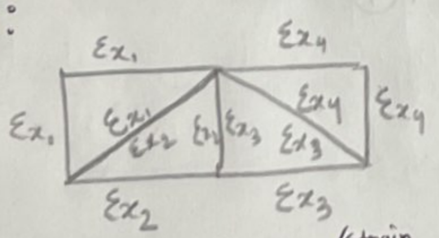
$$\boxed{\sigma_x = Ea_1} = A$$

but to avoid sudden stress jumps, we do average & interpolate.

To interpolate between the nodes, we can use the ~~the~~ concept of the derivative of shape functions. $\frac{\partial}{\partial x}[N] \Rightarrow$ derivative of a plane $\Rightarrow$ Constant

$[N] \rightarrow$ Plane $\qquad \frac{\partial}{\partial x}[N] \rightarrow$ constant

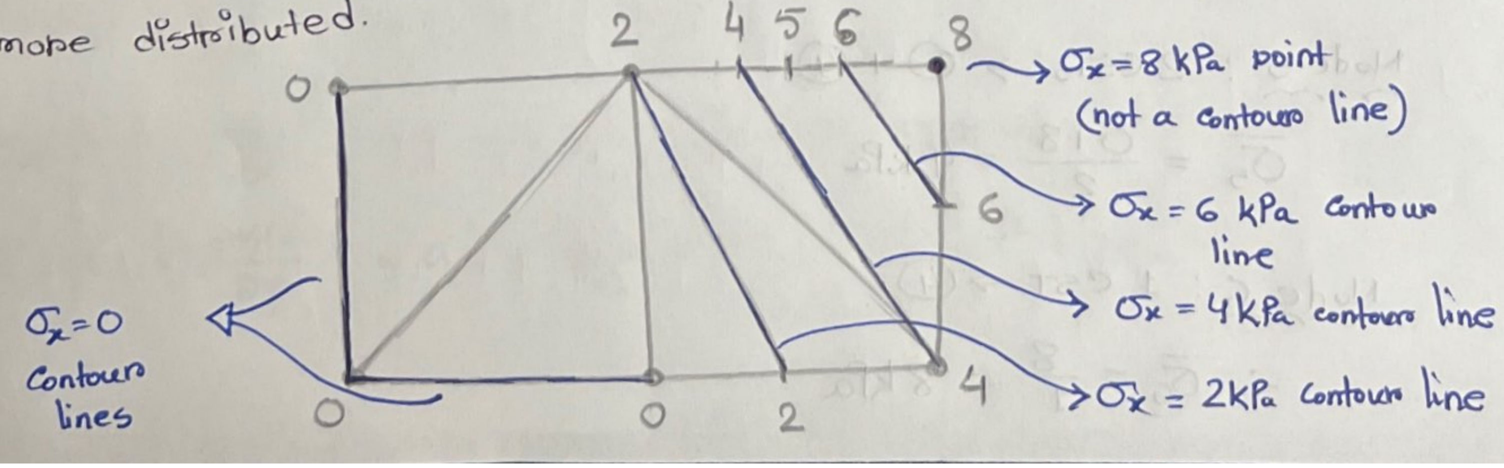
$\therefore$ ε field:



$\leftarrow$ ~~linear~~ & constant strain fields

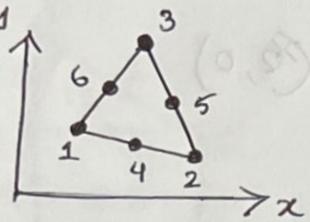
After interpolation, stress/strain field will not be constant any more but more distributed.

equally spaced due to $\sigma = Ea_1$



$\boxed{Q8}$

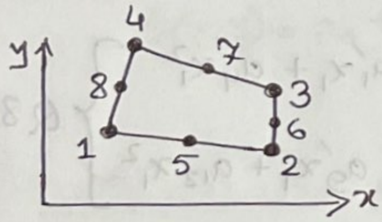
LST:

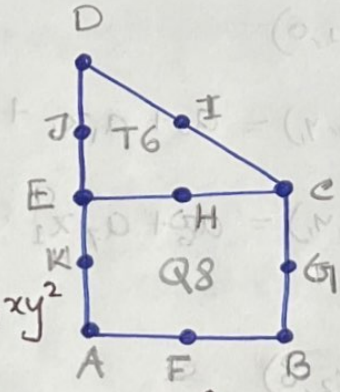


$$u(x,y) = a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2$$

$$V(x,y) = a_6 + a_7 x + a_8 y + a_9 xy + a_{10} x^2 + a_{11} y^2$$

Q8:

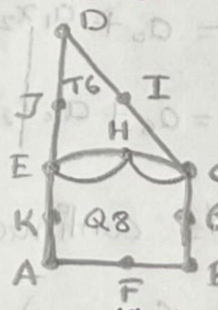




$$u_1(x,y) = a_0' + a_1' x + a_2' y + a_3' xy + a_4' x^2 + a_5' y^2 + a_6' x^2 y + a_7' xy^2$$

$$v_1(x,y) = a_8' + a_9' x + a_{10}' y + a_{11}' xy + a_{12}' x^2 + a_{13}' y^2 + a_{14}' x^2 y + a_{15}' xy^2$$

Nodes: Q8 → ABCEFGHK LST → CDEHIJ



Displacement field

Shared nodes by Q8 & T6 : EHC

At the nodal points, both Q8 and T6 must have the same displacement value but the intermediate field variable values will not be similar due to the ~~gr~~ difference in gradients.

$$\left.\begin{aligned} \varepsilon_x(x,y) &= \frac{\partial u(x,y)}{\partial x} = a_1 + a_3 y + 2a_4 x \\ \varepsilon_y(x,y) &= \frac{\partial v(x,y)}{\partial y} = a_8 + a_9 x + 2a_{11} y \end{aligned}\right\} \text{LST}$$

$$\left.\begin{aligned} \varepsilon_{x'}(x,y) &= \frac{\partial u_1(x,y)}{\partial x} = a_1' + a_3' y + 2a_4' x + 2a_6' xy + a_7' y^2 \\ \varepsilon_{y'}(x,y) &= \frac{\partial v_1(x,y)}{\partial y} = a_{10}' + a_{11}' x + 2a_{13}' y + a_{14}' x^2 + 2a_{15}' xy \end{aligned}\right\} \text{Q8}$$

Scanned with CamScanner

Now,

Let's consider $E(0,0)$, $H(x_1,0)$ & $C(x_2,0)$

E:

$\therefore u(x,y) = a_0 + \cancel{a_7}$ & $u'(x,y) = a_0'$

$V(x,y) = a_6$ $\quad$ $V'(x,y) = a_8'$

H: $(x_1,0)$

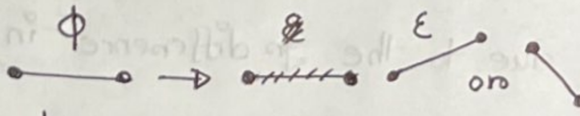
$$\text{LST}\begin{cases} u(x_1,y_1) = a_0 + a_1 x_1 + a_4 x_1^2 \approx u'(x_1,y_1) = a_0' + a_1' x_1 + a_4' x_1^2 \\ v(x_1,y_1) = a_6 + a_7 x_1 + a_{10} x_1^2 \approx V'(x_1,y_1) = a_8' + a_9' x_1 + a_{12}' x_1^2 \end{cases} \Big\} Q8$$

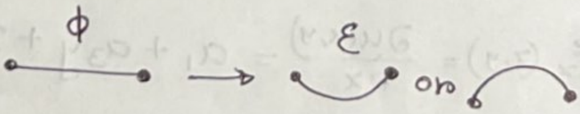
C: $(x_2,0)$

$$\text{LST}\begin{cases} u(x_2,y_2) = a_0 + a_1 x_2 + a_4 x_2^2 \approx u'(x_2,y_2) = a_0' + a_1' x_2 + a_4' x_2^2 \\ v(x_2,y_2) = a_6 + a_7 x_2 + a_{10} x_2^2 \approx V'(x_2,y_2) = a_8' + a_9' x_2 + a_{12}' x_2^2 \end{cases} \Big\} Q8$$

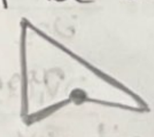
$\therefore$ Field is continuous at the nodal positions and they have the same ϕ (u or v). E, H, C: $\left.\begin{matrix} u = u' \\ v = v' \end{matrix}\right\}$ $\boxed{C^\circ \text{ continuous}}$

Strain:

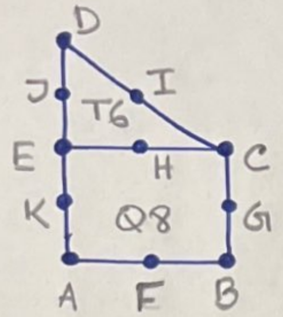
LST → liner strain field between nodes: 

Q8 → Quadratic strain field between nodes: 

For displacement purposes Q8 & T6 can be used together without any issue. However, for stress analysis, there will be a gap/discontinuity between their strain fields. Thus, finer mesh is needed for such analysis. Hence, the shared boundary is not C^1 continuous but C°.



Q8
Contd



ϕ:

(only considering 1 field variable, $\phi \approx u$)

LST (T6):

$$\phi(x,y) = a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2 \Big\} \text{fully polynomial}$$

Q8:

$$\phi_1(x,y) = a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2 + \underbrace{a_6 x^2 y + a_7 x y^2}_{\text{extra terms}}$$

$$\phi_{,x} = \frac{\partial \phi}{\partial x} = a_1 + a_3 y + 2a_4 x \qquad \text{quadratic strain}$$

$$\phi_{1,x} = \frac{\partial \phi_1}{\partial x} = \underbrace{a_1 + a_3 y + 2a_4 x}_{\text{linear strain}} + \underbrace{2a_6 xy + a_7 y^2}_{\text{extra terms}}$$

$\phi(x,y)$

$$\phi_{,xx} = \frac{\partial^2 \phi}{\partial x^2} = 2a_4$$

$$\phi_{1,xx} = \frac{\partial^2 \phi_1}{\partial x^2} = 2a_4 + 2a_6 y$$

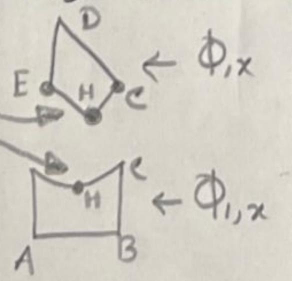
$\phi(x,y)$ is continuous at the shared boundary albeit there is a gap between the elements.

C^0 is achieved at the nodal points (shared) of the boundary.

However, the $\phi_{,x}$ & $\phi_{1,x}$ are different along the edge since $\phi_{,x}$ is linear in x & y and $\phi_{1,x}$ has linear and quadratic strain terms

ϕ is C^0 continuous

ϕ_1 is C^1 continuous



$\boxed{\text{The shared boundary between LST \& Q8 is } C^0 \text{ continuous}}$

Ans

Appendix for Question 6(a-d)


```
%% Afnan Mostafa
%% 10/28/2023
%% code for ME 441 Midterm Q6, Fall '23

%% ~~~ clear space ~~~ %%
clear
clc
close all
rng('shuffle')

%% %%%%%%%%%%%%%%%%%% parameters (selected from previous HW) %%%%%%%%%%%%%
%
E = 10^6;
rho = 10^3;
g = 10;
l=1;
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%% %%%%%%%%%%%%%%%%%%% Analytical solution %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
n = 100;
dy = l/n;
y = 0:dy:l;
v_exact = (rho*g*(y.^2 - l^2))./(2*E);
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%% %%%%%%%%%%%%%%%%%%% Five 2-noded elements %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
nodal_val = [-1/2 -12/25 -21/50 -8/25 -9/50 0]; %% from analysis
nodal_val = nodal_val.*((rho*g*l^2)./E);
elements = 5;
start_point = 0;
seg_len = l/elements;
arr_V = [];
% iterate through each elements to get non-nodal values
for i=1:elements
    start_point = (i-1)*seg_len;
    end_point = start_point + seg_len;
    yvalues = linspace(start_point,end_point,100);
    bar_N1 = (end_point - yvalues)./seg_len;
    bar_N2 = (yvalues - start_point)./seg_len;
    disp_bars = bar_N1.*nodal_val(i)+bar_N2.*nodal_val(i+1);
    arr_V = [arr_V disp_bars];
end
y_val = linspace(0,l,length(arr_V));
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

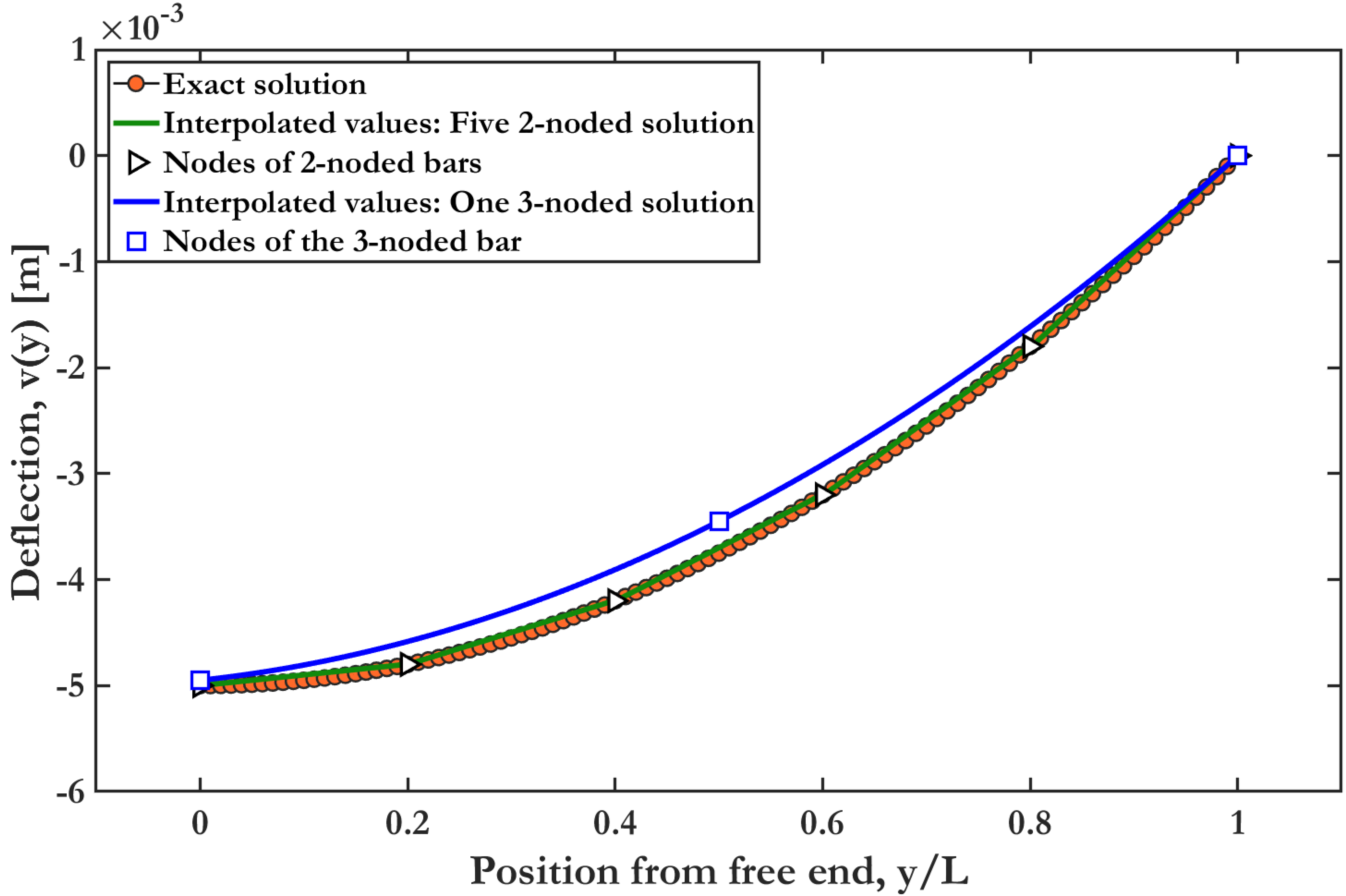
```
%% %%%%%%%%%%%%%%%%%%% One 3-noded element %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
elements_3 = 2;
step = l/elements_3;
y2 = 0:step:l;
three_nod_val = [-0.495 -0.345 0].*((rho*g*l^2)./E);   %% from analysis
% casi, eta space shape functions and displacement
eta = linspace(-1,1,100);
N1 = eta.*(eta-1)./2;
N2 = 1-(eta).^2;
N3 = eta.*(eta+1)./2;
disp_3nodes = N1.*three_nod_val(1)+N2.*three_nod_val(2)+N3.*three_nod_val(3);
y_val_3nodes = 0.5.*(eta+1);
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%% %%%%%%%%%%%%%%%%%%%% plotting exact + two FEM solutions %%%%%%%%%%%%%%%%
%
p1 = plot(y,v_exact,'om-','DisplayName','Exact Solution','LineWidth',1.5,...
    'MarkerSize',12,...
    'MarkerEdgeColor','k',...
    'MarkerFaceColor', 'm' );
hold on
box on
p2 = plot(y_val, arr_V,'-k','DisplayName',...
    'Interpolated values for 2-noded bars','LineWidth',3,...
    'MarkerSize',6,...
    'MarkerEdgeColor','k',...
    'MarkerFaceColor', 'none' );
p3 = plot(y_val_3nodes, disp_3nodes,'-r','DisplayName',...
    'Interpolated values for 3-noded bar','LineWidth',3,...
    'MarkerSize',6,...
    'MarkerEdgeColor','r',...
    'MarkerFaceColor', 'none' );
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%% %%%%%%%%%%%%%%%%%%%% plotting nodes %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
p21 = plot(linspace(0,1,6), nodal_val,'ok','DisplayName',...
    'nodes of 2-noded bars','LineWidth',2,...
    'MarkerSize',12,...
    'MarkerEdgeColor','k',...
    'MarkerFaceColor', 'w' );
p31 = plot(linspace(0,1,3), three_nod_val,'sb','DisplayName',...
    'nodes of the 3-noded bar','LineWidth',2,...
    'MarkerSize',14,...
    'MarkerEdgeColor','r',...
```

```
    'MarkerFaceColor', 'w' );
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%% %%%%%%%%%%%%%%%%%%%% plotting features %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
set(gca,'FontName','Garamond','FontSize',24,'FontWeight','bold',...
    'LineWidth',2,'XMinorTick','off',...
    'YMinorTick','off','GridAlpha',0.07,...
    'GridLineStyle','--','LineWidth',2);
xlabel('Position from free end, y/L',...
    'FontName','Garamond','FontSize',28);legend('Numerical', 'Exact');
ylabel('Deflection, v(y) [m]',...
    'FontName','Garamond','FontSize',28)
legend([p1,p2,p21,p3,p31],{'Exact solution', ...
    'Interpolated values: Five 2-noded solution','Nodes of 2-noded bars',...
    'Interpolated values: One 3-noded solution','Nodes of the 3-noded bar'},...
    'Location', 'Northwest');
set(gcf,'units','points','position',[100,100,1024,700])
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```



×10^-3
1
0
-1
-2
-3
-4
-5
-6
0
0.2
0.4
0.6
0.8
1
Exact solution
Interpolated values: Five 2-noded solution
Nodes of 2-noded bars
Interpolated values: One 3-noded solution
Nodes of the 3-noded bar
Deflection, v(y) [m]
Position from free end, y/L