ME 441 Final exam
Fall 2023
Due 12/17 by the end of the day

- **General rules:**
- **Read the following lines and start right away!**
- You may only discuss exam related material with the instructor and not any other person.
- Information in the lecture notes, textbook and descriptions and hints provided in the exam are all you need to answer the questions and it is best to keep focus on these resources to avoid confusion. But you may use any other resource (that excludes any person other than the instructor) if you wish, **after providing proper citation**.
- Using a resource without citation **is violation of the academic honesty policy**.
- Write clearly and legibly on a white paper so that your thoughts could be followed easily in the grading process.
- If you use a software to find your answers, make sure you provide thorough information about your steps. Final form of the expressions found by software need to be transferred to your exam papers and integrated with your work. A printout of your code should be added as appendix.
- If derivation is not asked for, you don't have to derive! Feel free to use final form of equations, shape functions, etc. as long as we covered those equations in class or they are written in the book.

Academic honesty policy: We enforce academic honesty policy strictly. Violating exam rules could result in referring your case to the board of academic. Make sure you write **honor pledge statement**: "*I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own*." on your exam cover page. Grading will NOT be possible without having a signed honor pledge on your cover page.

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ME 441: FEM Final Exam

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December 12, 2023

Problem 1

Problem Definition.

Consider the Gauss Quadrature rule in a square domain defined by $(\xi, \eta) = (\pm 1, \pm 1)$. The Gauss Quadrature rule of order 1, 2, and 3 provides exact integrals for polynomials integrands of order 1,3, and 5 respectively. Now consider Gauss Quadrature rule in a triangular domain defined by $(r, s) = (0, 1)$. The Gauss Quadrature rule of order 1, 2, and 3 for this domain provides exact integrals for polynomials integrands of order 1,2, and 3.
Explain why increasing the order of exactness in Gauss Quadrature rule in triangular domain (r, s) is much slower compared to square domains defined by (ξ, η). An alternate question statement is why even orders do not appear in Gauss Quadrature rule in square domains (ξ, η).

Answer.

Gauss Quadrature (GQ) in square domains is evaluated for $(\xi, \eta) = (\pm 1, \pm 1)$, while in triangular domains, such a GQ is evaluated in a domain defined by $(r, s) = (0, 1)$. In addition, GQ integration in square domains is faster because it can utilize the fact that odd terms in the polynomial function do not have any contributions to the integral when the bound is defined as $\xi = [-1, 1]$ (similarly for η); that is, we can increase the degree of the polynomial by 1 and still not have an increased variable/equation effect in the system of interest. In other words, we can use one extra degree of polynomial integration with the Gauss Quadrature scheme in square domains without having to worry about the state of the number of variables (*i.e.,* increasing) in the system. Let's take an example of 1st order polynomial using 1 GP (Gauss Point):

$$\int_{-1}^{1} (a_0 + a_1 x)\, dx \approx w_1 f(q_1) \rightarrow 2a_0 \approx w_1(a_0 + a_1 q_1)$$

Note that, the term $a_1 x$ (*i.e., x*) has no contribution to the exact integral above (left side of the equation) as it gets canceled out after the bounds are applied. Now, factoring the coefficients of a_0 and a_1 yields

$$a_0(2 - w_1) - a_1(q_1 w_1) = 0$$

Here, to equate L.H.S. and R.H.S., both $(2-w_1)$ and $(q_1 w_1)$, individually, have to be 0.

$$2 - w_1 = 0 \rightarrow w_1 = 2 \tag{1}$$

$$q_1 w_1 = 0 \rightarrow q_1 = 0 \quad (w_1 \neq 0) \tag{2}$$

Hence, we get GP = 0 and weight = 2 for 1 GP with 1 order of polynomial.

1 GP with polynomial degree 2 (keeping GP fixed and increasing polynomial order):

$$\int_{-1}^{1} (a_0 + a_1 x + a_2 x^2)\, dx \approx w_1 f(q_1) \rightarrow 2a_0 + \frac{2a_2}{3} \approx w_1(a_0 + a_1 q_1 + a_2 q_1^2)$$

$$a_0(2 - w_1) - a_1(q_1 w_1) + a_2(\frac{2}{3} - w_1 q_1^2) = 0$$

$$2 - w_1 = 0 \rightarrow w_1 = 2 \tag{3}$$

$$q_1 w_1 = 0 \rightarrow q_1 = 0 \quad (w_1 \neq 0) \tag{4}$$

$$\frac{2}{3} - w_1 q_1^2 = 0 \rightarrow q_1 = \pm\frac{1}{\sqrt{3}} \quad (w_1 = 2) \tag{5}$$

We see more than one solution for the same GP ($q_1 = 0$ and $q_1 = \pm\frac{1}{\sqrt{3}}$) from increasing the order of polynomial with keeping GP fixed at 1. Hence, GP of order 1 is exact up to the polynomial order of 1 only. So, increasing terms in the polynomial will give us extra equations for which the system will have more equations than variables (no unique solution).

2 GP with polynomial degree of 2:

Similarly, for 2 GP with polynomial degree of 2,

$$2 - w_1 - w_2 = 0 \rightarrow w_1, w_2 = 1 \tag{6}$$

$$q_1 w_1 + q_2 w_2 = 0 \rightarrow q_1 + q_2 = 0 \tag{7}$$

$$\frac{2}{3} - w_1 q_1^2 - w_2 q_2^2 = 0 \rightarrow q_1^2 + q_2^2 = \frac{2}{3} \tag{8}$$

Here, (7) and (8) can be solved to get q_1, q_2. (2 unknowns, 2 equations).

For degree 3,

$$2 - w_1 - w_2 = 0 \rightarrow w_1, w_2 = 1 \tag{9}$$

$$q_1 w_1 + q_2 w_2 = 0 \rightarrow q_1 + q_2 = 0 \tag{10}$$

$$\frac{2}{3} - w_1 q_1^2 - w_2 q_2^2 = 0 \rightarrow q_1^2 + q_2^2 = \frac{2}{3} \tag{11}$$

$$w_1 q_1^3 + w_2 q_2^3 = 0 \rightarrow q_1^3 + q_2^3 = 0 \tag{12}$$

Here, (10) and (12) represent the same equation (equivalent equations). **Thus, for n order of Gauss Quadrature, we can have exact integral for polynomials of order up to $2n-1$ for square domains.**

Conversely, within $[0, 1]$, which is used in the triangular domains, both odd and even terms in the polynomial contribute to the exact integral. **Thus, for** n

order of Gauss Quadrature, we can have exact integral for polynomials of order up to n for triangular domains. A generalized formula to derive GQ points and weights for triangular regions (Shivaram, 2013; Hussain and Karim, 2020) is given by:

$$\int_0^1 \int_0^{1-x} (a_0+a_1x+a_2y)\,dydx \approx w_1 f(q_1,p_1) \rightarrow \frac{a_0}{2}+\frac{a_1}{6}+\frac{a_2}{6} \approx w_1(a_0+a_1q_1+a_2p_1)$$

Note that, the terms a_1x, a_1y (i.e., x, y) have a contribution to the exact integral above (left side of the equation) as they do not get canceled out after the bounds are applied, unlike in square domains.

$$a_0(\frac{1}{2}-w_1)+a_1(\frac{1}{6}-q_1w_1)+a_2(\frac{1}{6}-p_1w_1)=0$$

$$\frac{1}{2}-w_1=0 \rightarrow w_1=\frac{1}{2}, (\text{also}, w_2=\frac{1}{2}) \tag{13}$$

$$q_1w_1=\frac{1}{6} \rightarrow q_1=\frac{1}{3} \quad (w_1=\frac{1}{2}) \tag{14}$$

$$p_1w_1=\frac{1}{6} \rightarrow p_1=\frac{1}{3} \quad (w_1=\frac{1}{2}) \tag{15}$$

Now, if we increase the polynomial degree by 1, we will have a_0, a_1, a_2, a_3, a_4, but we will not have any equivalent equations, hence more equations than unknowns (w_1, q_1, p_1). In square domains, this increased order will have no effects on the integral and hence, GQ can leverage this to get integrals without experiencing much change to the system of equations.

So, it can be said that increasing polynomial terms increases equations but for cancellation of odd terms (between $[-1, 1]$) in square domains accounts for that increase in equation and hence makes two equations equivalent. This is why GQ integration in square domains is faster than that in triangular domains. Henceforth, for square domains, the Gauss Quadrature rule of order 1, 2, and 3 (n) provides exact integrals for polynomials integrands of order 1, 3, and 5 $(2n-1)$, respectively. However, such a Gauss Quadrature rule of order 1, 2, and 3 (n) for triangular domains provides exact integrals for polynomials integrands of order 1, 2, and 3 (n), respectively.

Problem 2

Problem Definition.

Define the meaning of "Reduced Integration" and "Full-integration". Point to one problem that can arise by using "Reduced Integration". Why would we possibly want to avoid "Full-integration"?

Answer.

The main difference between full and reduced order integration is that for Q4 elements, we use 2×2 Gauss points (order 2) for the former and only 1 (order 1) for the latter. That is, we use 1 order less than the required full integration in a reduced order integration scheme.

Pros and Cons of Reduced Integration

As far as advantages are concerned, such reduced order integration can result in numerical integration that is $4\times$ faster than the full integration. But the

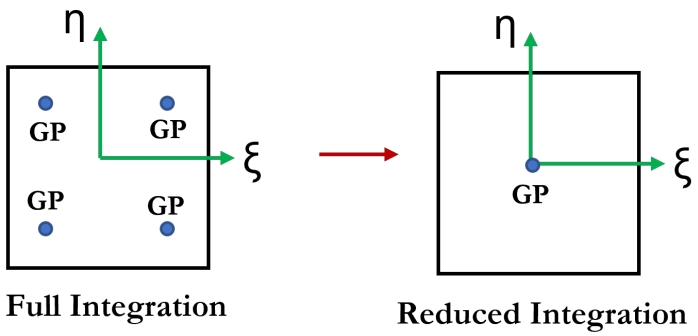

Figure 1: Reduced and full integration schemes

caveat is that we lose accuracy and accumulation of errors can produce unwanted results. Also, reduced-order integration, albeit faster than full-order integration, can produce weird-looking results, as shown in Fig. 2.

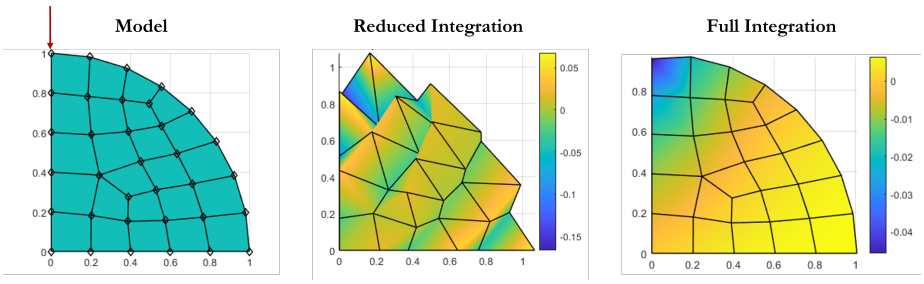

Figure 2: Accuracy in predicting field variable in RI and FI

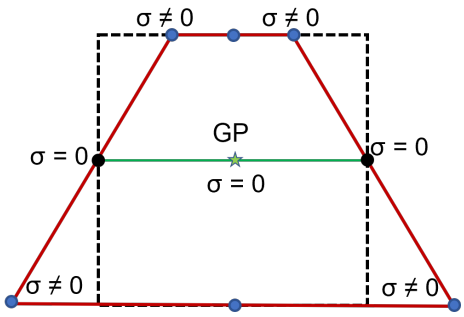

Figure 3: Hourglassing

Another major issue with such a reduced order scheme is that the emergence of 0 strain at the Gauss point for which the model predicts no stress (0^{th} order) in the system, even though all other points experience finite stresses. This phenomenon is known as *Hourglassing*, where the deformation mode results in 0^{th} order stress at the Gauss point. In FEM, the typical algorithm for calculating nodal stresses depends on the calculation of stresses at the Gauss points at first. Then, the algorithm extrapolates GP stresses to the nodes of the element to get stress distribution. Note that stresses at the nodes are not calculated directly (but extrapolation from GP is used) because such direct calculations are known to produce more inaccurate results than the extrapolation approach. So, if GP experiences no stress, the algorithm will not be able to extrapolate stresses to the nodes (hourglassing mode of deformation). Hence, the model fails to estimate stress at the nodes because it has 0 stress at GP. An example of hourglassing mode is shown in Fig. 3:

Reason to avoid full integration:

One reason to avoid full integration is to limit the consumption of computational resources, as such full-order schemes require almost 4-fold more resources to compute the integral.

Problem 3

Problem Definition.

Explain the significance of a) Jacobian matrix and b) determinant of the Jacobian matrix (or in short, Jacobian). What can go wrong if Jacobian becomes zero?

Answer.

Jacobian Matrix

Jacobian matrix is used to map between iso-parametric (natural) and real spaces. It contains information regarding the derivatives of real coordinates (x, y) with respect to iso-parametric coordinates (ξ, η). Jacobian matrix, defined as

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix},$$

is used in FEM to denote the relationship between gradients of displacements in (x, y) and (ξ, η) spaces. It is critical in mapping to iso-parametric space so that we can take advantage of the Gauss Quadrature in the (ξ, η) space and then, map back (inverse Jacobian matrix) to real space as needed. Another way to interpret the Jacobian matrix is that this is a scaling matrix that allows us to take any size and shape of Q4 and turn it into a square of side length = 2 in the iso-space. That being said, if the Jacobian matrix is a unit matrix, then such spaces (real and iso) coincide with each other, meaning there is a one-to-one scaling/mapping between the spaces.

Jacobian (Determinant of Jacobian Matrix)

The determinant of a Jacobian matrix is significant for understanding how much scaling is done by the Jacobian matrix. In other words, Jacobian refers to how distorted has the element become, compared to its ideal shape. "The Jacobian

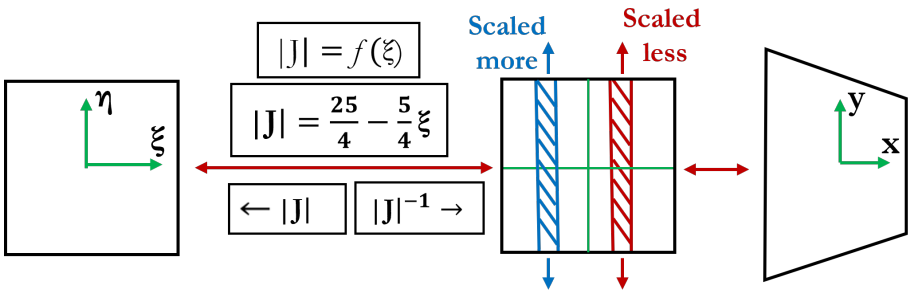

Figure 4: Scaling by $|J|$

at a point gives the best linear approximation of the distorted parallelogram near that point, and the Jacobian determinant gives the ratio of the area of the approximating parallelogram to that of the original square." (Wikipedia contributors, 2023)

What happens if $|J|$ becomes 0.

(I) $|J| = 1$: perfect, no distortion
(II) $|J| < 1$: distortion is present
(III) $|J| \to 0$: element is distorted significantly
(IV) $|J| = 0$: elemental distortion is so colossal that $|J|^{-1}$ does not exist anymore and thus, we cannot map back to (x, y) coordinates (since inverse of $|J|$ does not exist anymore). An alternate argument is that we need $|J|$ to integrate over infinitesimal areas to determine the stiffness matrix. If $|J| = 0$, then the stiffness matrix ($[K]$), defined as

$$[K] = \int_{-1}^{1} [B]^T [E][B] t |J| \, d\xi d\eta \ ,$$

does not exist as well.

Problem 4

Problem Definition.

Explain why we need to choose plane strain or plain stress assumption when solving elasticity problems in 2D. What considerations or criteria are used in making this choice?

Answer.

In 3D problems, the constitutive matrix ($[C]$) is 6×6 which includes normal and shear stress components. However, for 2D plane problems, we can ignore one dimension (*e.g.*, z) as a method of simplification to reduce the effort of computation to a great extent (simplification reduces $[C]$ to 3×3). In doing such a simplification technique, we can either choose plane stress or plane strain since making one of them 0 will not make the other 0; therefore, we consider taking either plane stress or plane strain for such simplification purposes. If, hypothetically, making one of them 0 would make the other one 0, then we could have used a simple generalized reduced version of the constitutive matrix. Even if we make $\sigma_z = 0$ in plane-stress assumption, $\epsilon_z \neq 0$ for which we need to account for that and that is done by choosing an appropriate constitutive matrix.

Plane Stress.

In plane-stress assumption, $\sigma_z = 0$ for which the strain in the third direction (ϵ_z) is not 0 and is a function of strains in the other two directions.

$$\epsilon_z = -\nu(\epsilon_x + \epsilon_y) + \frac{\sigma_z}{E}$$
$$\epsilon_z = -\nu(\epsilon_x + \epsilon_y)$$

If there is no load acting in the third direction, then we can condition the problem with a plane stress assumption. The constitutive matrix ($[C]$) in this case is defined as

$$[C] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix}.$$

Plane Strain.

In plane-strain assumption, $\epsilon_z = 0$ for which the stress in the third direction (σ_z) is not 0 and is a function of stresses in the other two directions.

$$\epsilon_z = -\nu(\epsilon_x + \epsilon_y) + \frac{\sigma_z}{E}$$

$$0 = -\nu(\epsilon_x + \epsilon_y) + \frac{\sigma_z}{E}$$

$$\frac{\sigma_z}{E} = \nu(\frac{\sigma_x}{E} + \frac{\sigma_y}{E})$$

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

If there is no displacement (or constrained) in the third direction or if the length in the third direction is long compared to the other ones ($\epsilon_z = \frac{\Delta l_z}{l_{zo}} \rightarrow \epsilon_z \approx 0$, as $l_{zo} \uparrow\uparrow$), then we can condition the problem with a plane strain assumption. The constitutive matrix ($[C]$) is:

$$[C] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix}$$

Problem 5

Problem Definition.

Problem 5 (5 pts) – Consider the definite integral I = $\int_{-1}^{1}\int_{0}^{2}\frac{8+2x^2}{3+y}dxdy$. The exact solution to this integral is I = 64 ln(2)/3. Use order 1 to 3 of Gauss quadrature rule to calculate this integral numerically and compare the accuracy of each order with the results from exact integration method. Is increasing the order of Gauss quadrature method beyond the 3rd order likely to give exact results? Explain why?
Hint: Note the bounds of the integral

Answer.

Given,

$$\int_{-1}^{1}\int_{0}^{2}\frac{8+2x^2}{3+y}dxdy$$

Exact Integral

Exact integral of the integrand given is: $I_{exact} = \frac{64\,ln(2)}{3} = 14.79$

Gauss Quadrature Integral

Here, the inner integral ($\int_0^2$ .... dx) does not have Gauss Quadrature bounds (*i.e.,* $[-1, 1]$). So, we need to use a change-of-variable technique to go from $[0, 2]$ to $[-1, 1]$. To do that, we can take $t = x - 1$, which will give us

$$x = 0 \rightarrow t = -1$$
$$x = 2 \rightarrow t = 1$$

$$I_{GQ} = \int_{-1}^{1}\int_{-1}^{1}\frac{8+2(t+1)^2}{3+y}dtdy$$

Gauss Quadrature (1st Order: 1 point)

$$I_{GQ1} = \frac{8+2(0+1)^2}{3+0}4$$
$$I_{GQ1} = \frac{40}{3}$$
$$I_{GQ1} = 13.33$$

$$\begin{aligned}\text{Accuracy} &= (1 - |\frac{I_{\text{exact}} - I_{GQ1}}{I_{\text{exact}}}|) \times 100 \\ &= 90.13\%\end{aligned}$$

Gauss Quadrature (2nd Order: 2×2 points):

$$I_{GQ2} = \frac{8 + 2(\frac{1}{\sqrt{3}} + 1)^2}{3 + \frac{1}{\sqrt{3}}}1 + \frac{8 + 2(-\frac{1}{\sqrt{3}} + 1)^2}{3 + \frac{1}{\sqrt{3}}}1 + \frac{8 + 2(\frac{1}{\sqrt{3}} + 1)^2}{3 - \frac{1}{\sqrt{3}}}1 + \frac{8 + 2(-\frac{1}{\sqrt{3}} + 1)^2}{3 - \frac{1}{\sqrt{3}}}1$$

$$I_{GQ2} = \frac{138 + 2\sqrt{3}}{39} + \frac{150 - 34\sqrt{3}}{39} + \frac{150 + 34\sqrt{3}}{39} + \frac{138 - 2\sqrt{3}}{39}$$

$$I_{GQ2} = 14.77$$

$$\begin{aligned}\text{Accuracy} &= (1 - |\frac{I_{\text{exact}} - I_{GQ2}}{I_{\text{exact}}}|) \times 100 \\ &= 99.86\%\end{aligned}$$

Gauss Quadrature (3rd Order: 3×3 points):

$$I_{GQ3} = \frac{8 + 2(0 + 1)^2}{3 + 0}\frac{8}{9}\times\frac{8}{9} + \frac{8 + 2(0 + 1)^2}{3 + \frac{\sqrt{3}}{\sqrt{5}}}\frac{8}{9}\times\frac{5}{9} + \frac{8 + 2(0 + 1)^2}{3 - \frac{\sqrt{3}}{\sqrt{5}}}\frac{8}{9}\times\frac{5}{9} + \frac{8 + 2(\frac{\sqrt{3}}{\sqrt{5}} + 1)^2}{3 + 0}\frac{5}{9}\times\frac{8}{9}$$
$$+ \frac{8 + 2(-\frac{\sqrt{3}}{\sqrt{5}} + 1)^2}{3 + 0}\frac{5}{9} \times \frac{8}{9} + \frac{8 + 2(\frac{\sqrt{3}}{\sqrt{5}} + 1)^2}{3 + \frac{\sqrt{3}}{\sqrt{5}}}\frac{5}{9} \times \frac{5}{9} + \frac{8 + 2(-\frac{\sqrt{3}}{\sqrt{5}} + 1)^2}{3 + \frac{\sqrt{3}}{\sqrt{5}}}\frac{5}{9} \times \frac{5}{9}$$
$$+ \frac{8 + 2(\frac{\sqrt{3}}{\sqrt{5}} + 1)^2}{3 - \frac{\sqrt{3}}{\sqrt{5}}}\frac{5}{9} \times \frac{5}{9} + \frac{8 + 2(-\frac{\sqrt{3}}{\sqrt{5}} + 1)^2}{3 - \frac{\sqrt{3}}{\sqrt{5}}}\frac{5}{9} \times \frac{5}{9}$$

$$I_{GQ3} = 2.634 + 1.308 + 2.219 + 2.354 + 1.333 + 1.169 + 0.663 + 1.983 + 1.124$$

$$I_{GQ3} = 14.79$$

$$\begin{aligned}\text{Accuracy} &= (1 - |\frac{I_{\text{exact}} - I_{GQ3}}{I_{\text{exact}}}|) \times 100 \\ &= 100\%\end{aligned}$$

It is evident from the accuracy calculations that 3rd-order GQ can reproduce the exact solution. Going beyond 3rd order will not produce any more improved results as accuracy cannot be greater than 100%. It will be either the same as 3rd order or will be worse. Either way, it will be computationally more intensive as more points—16 to be exact in 4th order—are required to evaluate the integral, compared to 9 points in 3rd order. Hence, 3rd-order GQ provides an exact solution to the posed problem, and going beyond that will not give a better result than what is already obtained from the 3rd order.

Problem 6

Problem Definition.

Problem 6 (10 pts) – In isoparametric elements, we can map the geometry of the element (meaning position x and y) using the shape functions of the element. Consider a quadratic (6-noded, with 6 shape functions) isoparametric triangle that has straight sides and mid nodes at the center of each side. Show that mapping of a triangular geometry using a quadratic triangle produces the same coordinates for the corners of the triangular region as a linear triangle (3-noded. Consider the coordinates of the corners to be arbitrary (x_i, y_i) where i=1,2,3.

Answer.

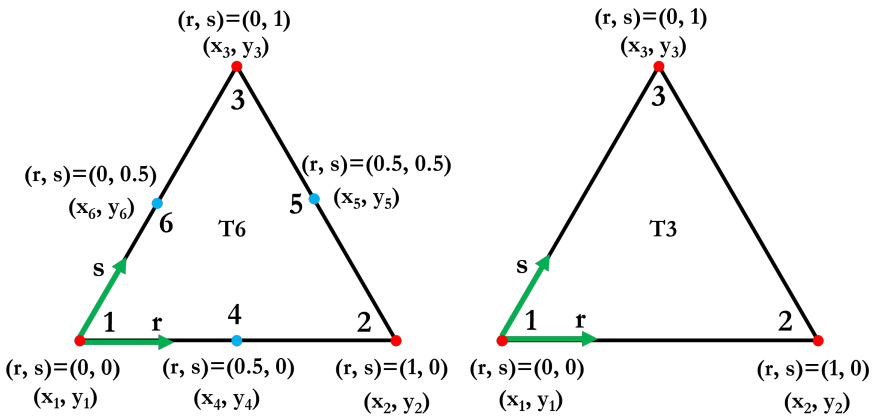

Figure 5: Nodes and nodal coordinates in T6 and T3

T6 (Quadratic Triangle):

$$[N] = [x][A]^{-1}$$

$$\begin{bmatrix} N_1 & N_4 & N_2 & N_5 & N_3 & N_6 \end{bmatrix}_{1\times 6} = \begin{bmatrix} 1 & r & s & rs & r^2 & s^2 \end{bmatrix}_{1\times 6} \begin{bmatrix} 1 & r_1 & s_1 & r_1 s_1 & r_1^2 & s_1^2 \\ 1 & r_4 & s_4 & r_4 s_4 & r_4^2 & s_4^2 \\ 1 & r_2 & s_2 & r_2 s_2 & r_2^2 & s_2^2 \\ 1 & r_5 & s_5 & r_5 s_5 & r_5^2 & s_5^2 \\ 1 & r_3 & s_3 & r_3 s_3 & r_3^2 & s_3^2 \\ 1 & r_6 & s_6 & r_6 s_6 & r_6^2 & s_6^2 \end{bmatrix}^{-1}_{6\times 6}$$

$$= \begin{bmatrix} 1 & r & s & rs & r^2 & s^2 \end{bmatrix}_{1\times 6} \begin{bmatrix} 1 & 0 & 0 & 0\times 0 & 0^2 & 0^2 \\ 1 & 0.5 & 0 & 0.5\times 0 & 0.5^2 & 0^2 \\ 1 & 1 & 0 & 1\times 0 & 1^2 & 0^2 \\ 1 & 0.5 & 0.5 & 0.5\times 0.5 & 0.5^2 & 0.5^2 \\ 1 & 0 & 1 & 0\times 1 & 0^2 & 1^2 \\ 1 & 0 & 0.5 & 0\times 0.5 & 0^2 & 0.5^2 \end{bmatrix}^{-1}_{6\times 6}$$

$$\begin{bmatrix} N_1 & N_4 & N_2 & N_5 & N_3 & N_6 \end{bmatrix}_{1\times 6}$$

$$=$$

$$\begin{bmatrix} 2r^2+4rs-3r+2s^2-3s+1 & 4r-4rs-4r^2 & 2r^2-r & 4rs & 2s^2-s & 4s-4rs-4s^2 \end{bmatrix}_{1\times 6}$$

Now, we can use interpolation to get points in (x, y) coordinates:

$$x(r,s) = N_1x_1 + N_4x_4 + N_2x_2 + N_5x_5 + N_3x_3 + N_6x_6$$
$$y(r,s) = N_1y_1 + N_4y_4 + N_2y_2 + N_5y_5 + N_3y_3 + N_6y_6$$

$$x(r,s) = x_1\left(2\,r^2 + 4\,r\,s - 3\,r + 2\,s^2 - 3\,s + 1\right) - x_6\left(4\,r\,s - 4\,s + 4\,s^2\right) - x_4\left(4\,r\,s - 4\,r + 4\,r^2\right) - x_2\left(r - 2\,r^2\right) - x_3\left(s - 2\,s^2\right) + 4\,r\,s\,x_5$$

Evaluate $x(r, s)$ at the known nodal positions of T6. For $(r, s) = (0, 0)$: $x = x_1$. Similarly, for $(0.5, 0), (1, 0), (0.5, 0.5), (0, 1), (0, 0.5)$ positions at (r, s) space, we get $x = x_4, x_2, x_5, x_3, x_6$, respectively (this is expected). Similarly, for y coordinates:

$$y(r,s) = y_1\left(2\,r^2 + 4\,r\,s - 3\,r + 2\,s^2 - 3\,s + 1\right) - y_6\left(4\,r\,s - 4\,s + 4\,s^2\right) - y_4\left(4\,r\,s - 4\,r + 4\,r^2\right) - y_2\left(r - 2\,r^2\right) - y_3\left(s - 2\,s^2\right) + 4\,r\,s\,y_5$$

For $(0.5, 0), (1, 0), (0.5, 0.5), (0, 1), (0, 0.5)$ positions at (r, s) space, we get $y = y_1, y_4, y_2, y_5, y_3, y_6$ respectively. Here, the corner points are $(r, s) = (0, 0), (1, 0), (0, 1)$ and their corresponding mapping results in $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ coordinates.

T3 (Linear Triangle):

Now, in T3, there are only 3 nodes (at 3 corners of the triangle). So to prove that the iso-parametric mapping for any triangular element produces the same corner coordinates for both T3 and T6, we need to prove that $(r, s) = (0, 0), (1, 0), and (0, 1)$ in T6 coincide with the same (x, y) values as for T3.

$$\begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}_{1\times 3} = \begin{bmatrix} 1 & r & s \end{bmatrix}_{1\times 3} \begin{bmatrix} 1 & r_1 & s_1 \\ 1 & r_2 & s_2 \\ 1 & r_3 & s_3 \end{bmatrix}^{-1}_{3\times 3}$$

$$\begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}_{1\times 3} = \begin{bmatrix} 1 & r & s \end{bmatrix}_{1\times 3} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1}_{3\times 3}$$

$$\begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}_{1\times 3} = \begin{bmatrix} 1-r-s & r & s \end{bmatrix}_{1\times 3}$$

$$x(r,s) = N_1 x_1 + N_2 x_2 + N_3 x_3$$
$$y(r,s) = N_1 y_1 + N_2 y_2 + N_3 y_3$$

$$x(r,s) = r\,x_2 - x_1\,(r+s-1) + s\,x_3$$
$$y(r,s) = r\,y_2 - y_1\,(r+s-1) + s\,y_3$$

Evaluate $x(r,s)$ at the known nodal positions of T3. For $(0,0),(1,0),(0,1)$ positions at (r,s) space, we get $x = x_1, x_2, x_3$, respectively (this is expected). Similarly, for y coordinates: $y = y_1, y_2, y_3$ respectively. But, from the figure, we can see that in T6 with 6 nodes, the corner nodes are defined as the 1st, 3rd, and 5th node. Here, the corner points are $(r,s) = (0,0),(1,0),(0,1)$ and their corresponding mapping results in $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ coordinates. **So, we can say that mapping of a triangular geometry using a quadratic triangle produces the same coordinates for the corners of the triangular region as a linear triangle (3-noded).**

Problem 7

Problem Definition.

Problem 7 (20 pts) Consider the differential equation below, with the listed boundary conditions.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{1+x^2}; \qquad u(0) = u(1) = 0$$

a) To obtain an approximate solution, suggest the lowest order but sensible choice of polynomial form as a trial solution. Apply the boundary conditions to your trial function before proceeding to the next step.
b) Try your approximation function directly into the differential equation and find relationship(s) for unknown coefficient(s) in the polynomial.
c) Try method of weighted residuals with least square formulation and find the unknown coefficient(s).
d) Comment on results obtained in b and c. Which method makes more sense?
e) Explain why method of weighted residuals can be applied to any differential equation but exact solution techniques work on only a few differential equation expressions.

Answer.

Given,

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{1}{1+x^2} \\ u(0) = u(1) &= 0 \end{aligned} \tag{16}$$

(a) Strong Form

For the aforementioned differential equation, we consider a trial polynomial solution, $u = a_0 + a_1 x + a_2 x^2$. We need at least an order of 2 for the trial polynomial since the differential equation (of order 2) must be continuous and differentiable up to order 2. Since this is a trial solution, we need first to apply the boundary conditions (BCs), that is $u(0) = u(1) = 0$.

$$\begin{aligned} 0 &= a_0 + a_1 0 + a_2 0^2 \text{ (1st BC)} \\ \rightarrow a_0 &= 0 \end{aligned}$$

$$\begin{aligned} 0 &= a_1 1 + a_2 1^2 \text{ (2nd BC)} \\ \rightarrow a_1 &+ a_2 = 0 \end{aligned}$$

So, we have $a_0 = 0$ and a relationship between the other two coefficients, $a_1 + a_2 = 0$. Hence, the trial solution becomes:

$$u = a_1 x - a_1 x^2 \tag{17}$$

(b) Strong Form cont'd

Now, taking two derivatives of (17) will yield:

$$\frac{\partial u}{\partial x} = a_1 - 2a_1 x$$
$$\frac{\partial^2 u}{\partial x^2} = -2a_1$$

Now, equating (16) with $-2a_1$,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{1+x^2} = -2a_1$$
$$a_1 = \frac{-1}{2(1+x^2)}$$

Therefore the trial solution becomes (from (17)),

$$u = \frac{-x}{2(1+x^2)} + \frac{x^2}{2(1+x^2)} \tag{18}$$
$$u = \frac{x^2 - x}{2(1+x^2)} \tag{19}$$

This does not make sense since a_1 is supposed to be a constant but here, it is a function of x. Hence, the strong form of a differential equation is not the best in terms of providing a solution for all types of differential equations.

(c) Weak Form

We take a weak form of the trial solution for the differential equation and use weighted residual (least-squares method) to get the coefficients of the trial solution. We can start with the first-order polynomial as a trial solution but then, $\frac{\partial^2 u}{\partial x}$ will not exist and no residual can be calculated. So, we can start with the second or third order where the second derivative will exist (hence the residual is not 0). The third-order trial function is more accurate than the second-order one and we show both of them in the following steps. The steps are as follows:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{1+x^2} \tag{20}$$

Step 1: Take an approximate trial solution suited for the least-square method and apply boundary conditions:

$$\tilde{u}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{21}$$
$$\tilde{u}(0) = 0 = a_0 \tag{22}$$
$$\tilde{u}(1) = 0 = a_1 + a_2 + a_3 \tag{23}$$
$$a_1 = -a_2 - a_3 \tag{24}$$

Thus, we have $a_0 = 0$ and $a_1 = -a_2 - a_3$.

Step 2: Put a_0, a_1, a_2, a_3 in the $\tilde{u}(x)$ equation and take derivative twice.

$$\tilde{u} = -a_2 x - a_3 x + a_2 x^2 + a_3 x^3 \tag{25}$$
$$\tilde{u} = a_2(x^2 - x) + a_3(x^3 - x) \tag{26}$$

$$\tilde{u}' = (2x - 1)a_2 + (3x^2 - 1)a_3 \tag{27}$$
$$\tilde{u}'' = 2a_2 + 6xa_3 \tag{28}$$

Step 3: The difference between exact and trial function is known as residual (R). So, from (28) and (20), we can write:

$$R(x) = \frac{1}{1 + x^2} - (2a_2 + 6xa_3) \tag{29}$$
$$R(x) = \frac{1}{1 + x^2} - 2a_2 - 6xa_3 \tag{30}$$

Step 4: Take the derivative of R with respect to a_2 and a_3 to minimize S, where $S = \int_{x_1}^{x_2} R(x)^2 dx$ (*i.e.*, $\frac{\partial S}{\partial a_i} = 0$, where $i = 1, 2, ...$ no of coefficients). Therefore, from (30),

$$\frac{\partial R(x)}{\partial a_2} = -2 \tag{31}$$
$$\frac{\partial R(x)}{\partial a_3} = -6x \tag{32}$$

Step 5: Minimize S with respect to the coefficients to ensure residual is minimized. So,

$$S = \int_0^1 R(x)^2 dx \tag{33}$$
$$\frac{\partial S}{\partial a_2} = \int_0^1 2R(x) \frac{\partial R(x)}{\partial a_2}\, dx = 0 \tag{34}$$
$$0 = \int_0^1 R(x) \frac{\partial R(x)}{\partial a_2}\, dx \tag{35}$$
$$\int_0^1 (\frac{1}{1 + x^2} - 2a_2 - 6xa_3)(-2)\, dx = 0 \tag{36}$$
$$[4a_2 x + 6a_3 x^2 - 2\arctan(x)]_0^1 = 0 \tag{37}$$
$$4a_2 + 6a_3 - 1.57 = 0 \tag{38}$$
$$4a_2 + 6a_3 = 1.57 \tag{39}$$

Similarly, for a_3,

$$0 = \int_0^1 R(x) \frac{\partial R(x)}{\partial a_3} \, dx \tag{40}$$

$$0 = \int_0^1 (\frac{1}{1+x^2} - 2a_2 - 6xa_3)(-6x) \, dx \tag{41}$$

$$\int_0^1 (12a_2 x + 36a_3 x^2) \, dx - \int_1^2 (\frac{6dz}{2z}) \, dz = 0 \text{ (let, } 1 + x^2 = z) \tag{42}$$

$$[6a_2 x^2 + 12a_3 x^3]_0^1 - 3(ln(z))_1^2 = 0 \tag{43}$$

$$6a_2 + 12a_3 - 2.079 = 0 \tag{44}$$

Step 6: Solving (39) and (44) will yield values for a_2 and a_3.

$$\begin{aligned} 4a_2 + 6a_3 &= 1.57 \\ 6a_2 + 12a_3 &= 2.08 \\ a_2 = 0.53, \; a_3 &= -0.0917 \end{aligned}$$

Step 7: Plug a_2, a_3 into (24) to get a_1.

$$\begin{aligned} a_1 &= -0.4383 \\ a_2 &= 0.53 \\ a_3 &= -0.0917 \end{aligned}$$

Therefore, the trial solution is

$$\tilde{u}(x) = -0.4383x + 0.53x^2 - 0.0917x^3$$

$$\tilde{u}(x) = -0.44x + 0.53x^2 - 0.09x^3$$

Sanity check (plugging BCs back):

$$\begin{aligned} \tilde{u}(0) &= 0 \\ \tilde{u}(1) &= 0 \end{aligned}$$

Solving it using 2nd order polynomial.

Step 1:

$$\begin{aligned} \tilde{u}(x) &= a_0 + a_1 x + a_2 x^2 \\ \tilde{u}(0) &= 0 = a_0 \\ \tilde{u}(1) &= 0 = a_1 + a_2 \\ a_1 &= -a_2 \end{aligned}$$

Step 2:

$$\tilde{u}' = a_1 + 2xa_2$$
$$\tilde{u}'' = 2a_2$$

Step 3:

$$R(x) = \frac{1}{1+x^2} - 2a_2$$

Step 4:

$$\frac{\partial R(x)}{\partial a_2} = -2$$

Step 5:

$$\int_0^1 (\frac{1}{1+x^2} - 2a_2)(-2)\, dx = 0$$
$$a_2 = 0.393$$

Steps 6-7:

$$a_1 = -0.393$$
$$a_2 = 0.393$$

$$\tilde{u}(x) = -0.393x + 0.393x^2$$

The exact solution is:

$$u = xtan^{-1}(x) - \frac{1}{2}ln(1+x^2) - 0.438x$$

(d) Strong vs. Weak Results

Coefficients obtained in (b) are functions of x (*i.e.,* $a_1, a_2 = f(x)$), which seems counter-intuitive since we know that, from how the trial solution was initiated, the coefficients are supposed to be constants, not functions of x. And, results obtained from the weak form prove that such coefficients are, in fact, constants. So, weak-form results make more sense and are better at approximating than strong ones (in an average sense). Even a 2nd- or 3rd-order polynomial weak form is better at approximation than strong forms (as seen in Fig. 6). A 3rd-order weak solution approximates the exact solution the best among the solutions considered in this example. Note that the term 'weak' does not mean the solution is weak, rather it means that the continuity condition imposed by the exact solution (strong form) has been weakened and relaxed so as to we at least have an approximation for any kind of differential equation. The figure below shows both strong and weak forms of the solution to the given differential equation.

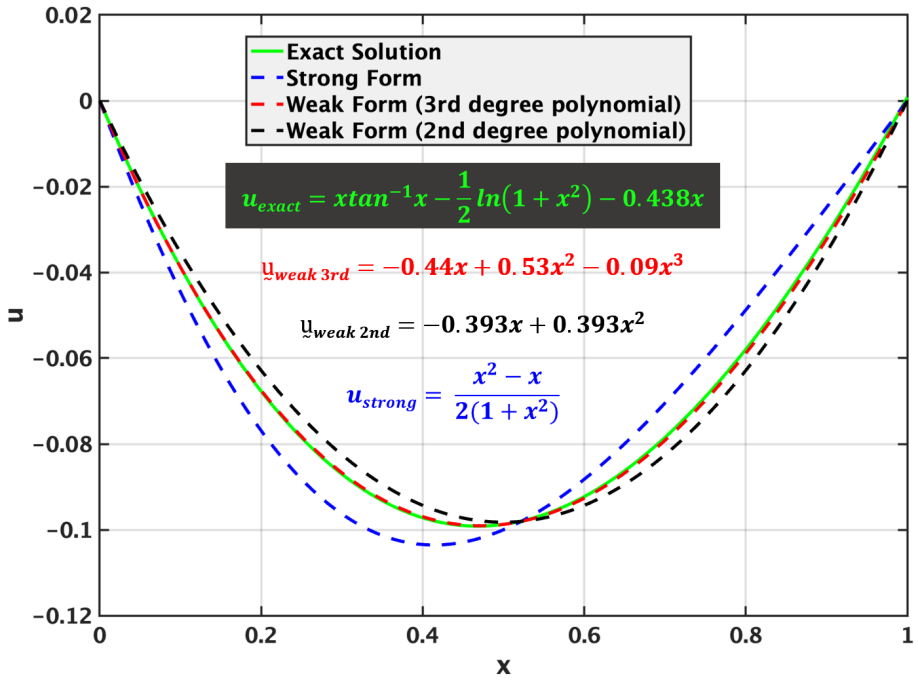

Figure 6: Exact vs Strong vs Weak Form

(e) Why weak form is better as a generalized solution.

The method of weighted residual (weak form) works well with any differential equation but the exact solution techniques (*e.g.,* strong form) work well with a handful of differential equations. The reasons are as follows:

- This is because the strong form requires the solution to be continuous and differentiable at least up to the order of the differential equation (*e.g.,* if the differential equation is of order 2, then the solution needs to be differentiable up to order 2). This imposes constraints on the solution and makes it difficult to come up with a continuous, differentiable solution. On the other hand, the weak form reduces such continuity conditions by converting the differential into an integral functional that is easier to evaluate. Hence, the weak form can approximate a solution to any differential equation.

- In addition, the strong form requires the error to be 0 at any point of the domain (thus not approximation-friendly). However, the weak form requires the integral to be 0 over the domain (*i.e.,* allow for some error to remove continuity constraints to at least have an approximate solution).

- The weak-form solution has the boundary conditions incorporated in the actual equation (integral functional) and does not need to satisfy boundary conditions at every step of the solution. On the contrary, strong forms need to meet the boundary-condition (BC) criteria at any point (*i.e.,* they have

BCs expressed separately). This is why strong forms are not very handy for solving differential equations (in a general sense).

- Next, the weak form can be turned into a matrix form that is easier to solve since working with matrices is flexible.
- Interestingly, various weights can be applied (which will lead to different methods—Galerkin, Least-Squares, etc.—to the residual (*i.e.,* differential equation - assumed solution), and then, the coefficients can be solved by solving a system of linear equations, which is very cool.

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Wikipedia contributors (2023). Jacobian matrix and determinant — Wikipedia, the free encyclopedia. `https://en.wikipedia.org/w/index.php?title=Jacobian_matrix_and_determinant&oldid=1185327400`. [Online; accessed 12-December-2023].

MATLAB Code for Problem 6

```
% ========================================================
%%% ME 441: FEM, Final Exam Problem 6
%%% Afnan Mostafa
%%% 12/12/2023
% ========================================================

%% %%%%%%%%%%%%%%%%%%%% Clearing space %%%%%%%%%%%%%%%%%

clear
clc
close all
rng('shuffle')
tic

%% %%%%%%%%%%%%% T3 and T6 %%%%%%%%%%%%%%%%%%%%

syms r s

syms r1 r2 r3 r4 r5 r6 s1 s2 s3 s4 s5 s6 x1 x2 x3 x4 x5 x6 y1 y2 y3 y4
    y5 y6

%% %%%%%%%%%%%%%%%%% T3 %%%%%%%%%%%%%%%%%%%%%%

xMat_t3 = [1 r s];
AMat_t3 = [
    1 r1 s1;
    1 r2 s2;
    1 r3 s3];

N_t3 = xMat_t3*(inv(AMat_t3)); %  [N] = [x][A]^-1
N_eval_t3 = subs(N_t3,[r1,r2,r3,s1,s2,s3],[0,1,0,0,0,1]);

X_t3 = N_eval_t3(1)*x1+N_eval_t3(2)*x2+N_eval_t3(3)*x3;
```

```
33 Y_t3 = N_eval_t3(1)*y1+N_eval_t3(2)*y2+N_eval_t3(3)*y3;
34
35 corner1_t3_x = subs(X_t3,[r,s],[0,0]);
36 corner2_t3_x = subs(X_t3,[r,s],[1,0]);
37 corner3_t3_x = subs(X_t3,[r,s],[0,1]);
38 corner1_t3_y = subs(Y_t3,[r,s],[0,0]);
39 corner2_t3_y = subs(Y_t3,[r,s],[1,0]);
40 corner3_t3_y = subs(Y_t3,[r,s],[0,1]);
41
42 %% %%%%%%%%%%%% T6 %%%%%%%%%%%%%%%%%%%%%%%%%
43
44 xMat_t6 = [1 r s r*s r^2 s^2];
45 AMat_t6 = [
46     1 r1 s1 r1*s1 r1^2 s1^2;
47     1 r4 s4 r4*s4 r4^2 s4^2;
48     1 r2 s2 r2*s2 r2^2 s2^2;
49     1 r5 s5 r5*s5 r5^2 s5^2;
50     1 r3 s3 r3*s3 r3^2 s3^2;
51     1 r6 s6 r6*s6 r6^2 s6^2];
52
53 N_t6 = xMat_t6*(inv(AMat_t6));
54
55 N_eval_t6 = subs(N_t6,[r1,r2,r3,r4,r5,r6,s1,s2,s3,s4,s5,s6
       ],[0,1,0,0.5,0.5,0,0,0,1,0,0.5,0.5]);
56
57 X_t6 = N_eval_t6(1)*x1+N_eval_t6(2)*x4+N_eval_t6(3)*x2+N_eval_t6(4)*x5+
       N_eval_t6(5)*x3+N_eval_t6(6)*x6;
58 Y_t6 = N_eval_t6(1)*y1+N_eval_t6(2)*y4+N_eval_t6(3)*y2+N_eval_t6(4)*y5+
       N_eval_t6(5)*y3+N_eval_t6(6)*y6;
59
60 %% %%%%%%%% get corner coords %%%%%%%%%%%%%%%%%%
61
62 corner1_t6_x = subs(X_t6,[r,s],[0,0]);
63 corner2_t6_x = subs(X_t6,[r,s],[1,0]);
64 corner3_t6_x = subs(X_t6,[r,s],[0,1]);
65 corner1_t6_y = subs(Y_t6,[r,s],[0,0]);
66 corner2_t6_y = subs(Y_t6,[r,s],[1,0]);
67 corner3_t6_y = subs(Y_t6,[r,s],[0,1]);
68
69 %% %%%%%%%%%% check the corner coords %%%%%%%%%%%%%
70
71 sprintf('(x, y) coords of corner 1 in T3, (%s, %s)', corner1_t3_x,
       corner1_t3_y)
72 sprintf('(x, y) coords of corner 1 in T6, (%s, %s)', corner1_t6_x,
       corner1_t6_y)
73 sprintf('(x, y) coords of corner 2 in T3, (%s, %s)', corner2_t3_x,
       corner2_t3_y)
74 sprintf('(x, y) coords of corner 2 in T6, (%s, %s)', corner2_t6_x,
       corner2_t6_y)
75 sprintf('(x, y) coords of corner 3 in T3, (%s, %s)', corner3_t3_x,
       corner3_t3_y)
76 sprintf('(x, y) coords of corner 3 in T6, (%s, %s)', corner3_t6_x,
       corner3_t6_y)
77
78 toc
79
80 % ========================END OF SCRIPT=========================
```