Advanced Coding Theory and Cryptography

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Chapter 1

An introduction to Gröbner bases

Theorem 2.1.10 (Hilbert's Basis Theorem)

Proof. We proceed by induction on the number of variables. Let $I \subset A[X]$ be an ideal not finitely generated, we may assume it can be constructed by an infinite sequence $(f_i)_{i \in \mathbb{N}}$ of independent polynomials of minimal degree. "Independent" means that $f_i \in I \setminus J_i$ where we set $J_i :=< f_0, \dots, f_{i-1} >$. Now let $a_i := lc(f_i)$ be the leading coefficient of f_i and consider $J := a_0, a_1, \dots \subset A$. We know that J can be a basis for an ideal in A but since A is a Noetherian ring we have that there exists a finite basis for such ideal, say $J =< a_1, \dots, a_N >$. We claim that $I =< f_1, \dots, f_N >=: I'$.
Suppose by contrary that this is not true then take a polynomial $f_{N+1} \in I$, we want to show that it is a linear combination of elements of I', so first of all let's look at the leading coefficient:

$$a_{N+1} = u_1 a_1 + u_2 a_2 + \cdots + u_N a_N$$

this is true since A is Noetherian ring. Consider

$$g := \sum_{i=1}^{N} u_i f_i x^{deg(f_{N+1}) - deg(f_i)} \in I'$$

it has the same degree and same leading coefficient as f_{N+1}. Now $f_{N+1} - g \notin I'$ and has degree strictly less than f_{N+1} contraddicting its minimality. Therefore $f_{N+1} - g$ must be 0 and $f_{N+1} \in I'$.

The induction follows since we can consider $A[X_1, \dots, X_m] = A'[X_m]$ where $A' := A[X_1, \dots, X_{m-1}]$ which we know is a Noetherian ring. □

Lemma 2.1.13 (Dickson's Lemma)

Proof. We proceed by induction on the number of variables, by first proving the case with one variable. So we are considering $\mathcal{M} = \{X_1^\alpha | \alpha \in \mathbb{N}\}$, and $T \subset \mathcal{M}$ a semigroup ideal. Since every $t_i \in T$ is of the form $t = X_1^{\alpha_i}$ we consider $\beta = \min\{\alpha_i | X_1^{\alpha_i} \in T\}$. We claim that $T =< X_1^\beta >$. Indeed let $t_j \in T$ then it is of the form $t_j = X_1^{\alpha_j}$ so $\frac{t_j}{t_i} = X_1^{\alpha_j - \beta}$ is well defined where $\alpha_j - \beta > 0$ by minimality of β. We can take $\gamma = \alpha_j - \beta$ hence:

$$t_j = X_1^\beta \cdot X_1^\gamma \in < X_1^\beta >= T$$

We prove the more general case so let be $m \in \mathbb{N}$ arbitrary and assume the lemma proved for $m-1$.
Let $T \subset \mathcal{M} = \{X_1^{a_1} \cdots X_m^{a_m} \mid (a_1, \ldots, a_m) \in \mathbb{N}^m\}$. Consider also the projection map $\pi(X_1^{a_1} \cdots X_m^{a_m}) = X_1^{a_1} \cdots X_{m-1}^{a_{m-1}}$. By induction hypothesis $\pi(T)$ is a finitely generated semigroup ideal so we can find a basis, say $\pi(T) =< t_1, \ldots, t_k >$. Now let:

$$A_i := \min\{a_m \mid X_m^{a_m} | t, t \in T, \pi(t) = t_i\} \quad \forall i = 1, \ldots, k$$

and furthermore

$$A := \min\{a_m \mid X_m^{a_m} \in T\}$$

We claim that $T =< t_1 X_m^{A_1}, \ldots, t_k X_m^{A_k}, X_m^A >$ which is a finite set.
So pick an arbitrary $t \in T$ so $t = \pi(t) X_m^{a_{m_t}}$ for some $a_{m_t} \in \mathbb{N}$, we know that $\exists t_i$ such that $\pi(t) = s \cdot t_i$, therefore $t = s \cdot t_i \cdot X_m^{a_{m_t}}$ and by minimality of A_i we obtain that for:

$$t = s \cdot t_i \cdot X_m^{a_{m_t}} = s \cdot t_i \cdot X_m^{A_i} \cdot X_m^\gamma$$

for $\gamma = a_{m_t} - A_i$. Now $\forall t \in T$ we have proved that $t \in < t_i \cdot X_m^{A_i} >$ which is contained in $< t_1 X_m^{A_1}, \ldots, t_k X_m^{A_k}, X_m^A >$ □

Theorem 2.1.14

Proof. $\Rightarrow$ Let $f \in I$ then we can write:

$$f = \sum_{i=1}^k f_i \cdot p_i = f_1 \cdot p_1 + f_2 \cdot p_2 + \ldots + f_k \cdot p_k, \ f_i \in \mathcal{P}$$

So evaluating $f(A)$ means to evaluate every p_i so:

$$f(A) = f_1 \cdot p_1(A) + f_2 \cdot p_2(A) + \ldots + f_k \cdot p_k(A) =$$

$$= f_1 \cdot 0 + f_2 \cdot 0 + \cdots + f_k \cdot 0 = 0$$

$\Leftarrow$ Trivial by setting $f = p_i \ \forall i = 1, \ldots, k$

□

Theorem 2.1.17

Proof. We already know that I and J are finitely generated so by keeping in mind that $I \subset J$ we can let:

$$I =< p_1, \dots, p_k > \quad and \quad J =< p_1, \dots, p_h >, \; h \geq k$$

Now pick $A \in \mathcal{V}_{\mathbb{F}}(J)$ arbitrary, for every $g \in I$ we have that $g \in J$ therefore $g(A) = 0$ which means that $A \in \mathcal{V}_{\mathbb{F}}(I)$ for every A. Therefore $\mathcal{V}_{\mathbb{F}}(J) \subset \mathcal{V}_{\mathbb{F}}(I)$ □

Proposition 2.2.6

Proof. Assume that

$$f = h_1 g_{i_1} + h_2 g_{i_2} + \dots + h_s g_{i_s} + r_1 = k_1 g_{j_1} + k_2 g_{j_2} + \dots + k_t g_{j_t} + r_2$$

with $g_{i_l}, g_{j_l} \in \mathcal{G}$ and $h_l, k_l, r_1, r_2 \in \mathcal{P}$. We obtain that neither r_1 nor r_2 are divisible by any $lm(g), g \in \mathcal{G}$. Therefore we can write:

$$0 = f - f = (h_1 g_{i_1} + h_2 g_{i_2} + \dots + h_s g_{i_s} + r_1) - (k_1 g_{j_1} + k_2 g_{j_2} + \dots + k_t g_{j_t} + r_2)$$

Hence:

$$r_2 - r_1 = (h_1 g_{i_1} + h_2 g_{i_2} + \dots + h_s g_{i_s}) - (k_1 g_{j_1} + k_2 g_{j_2} + \dots + k_t g_{j_t})$$

Now the LHS belongs to the ideal by definition, i.e. $\exists g \in \mathcal{G}$ such that $lm(g) | lm(r_2 - r_1)$ but $lm(r_2 - r_1)$ is $lm(r_2)$ or $lm(r_1)$, so the only way to be divisible is to be 0. □

Corollary 2.2.9

Proof. $\Rightarrow$ $\mathcal{V}(I) = \emptyset$ means that there exists $f \in I$ that has no roots in $\overline{\mathbb{K}}^m$, but this is possible only for a polynomial of degree 0, i.e. a constant, say c in the base field of K. $c = X^0 * c = 1 * c$ therefore $1 \in I$.

$\Leftarrow$ For $f = 1 \in I$ we have no roots, therefore $\mathcal{V}(I) = \emptyset$.

□

Lemma 2.2.13

Proof. Since $\gcd(lm(p_1), lm(p_2)) = 1$ we can write the S-polynomial as follows:

$$S(p_1, p_2) = p_1 lt(p_2) - p_2 lt(p_1)$$

We assume $lc(p_i) = 1, i = 1, 2$ therefore $lt(p_i) = lm(p_i)$ for reading simplicity. Furthermore we write $p_i = lm(p_i) + r_i$ hence:

$$p_1 lt(p_2) - p_2 lt(p_1) = lm(p_2)(lm(p_1) + r_1) - lm(p_1)(lm(p_2) + r_2) =$$

$$= lm(p_2)r_1 - lm(p_1)r_2 = r_1(p_2 - r_2) - r_2(p_1 - r_1) =$$

$$= r_1p_2 - r_2p_1$$

Now since $lm(r_1) < lm(p_1)$, $lm(r_2) < lm(p_2)$ and $\gcd(lm(p_1), lm(p_2)) = 1$ we have that $lm(S)$ is $lm(r_1p_2)$ or $lm(r_2p_1)$ but not both.
Assume $lm(S) = lm(r_1p_2)$ therefore $lm(S)$ is divisible by $lm(p_2)$ by a factor of $lm(r_1)$. Therefore in the division algorithm:

$$S \xrightarrow{p_2} r_1p_2 - r_2p_1 - lm(r_1)p_2 =$$

$$= (r_1 - lm(r_1))p_2 - r_2p_1$$

Which has the same form as the starting point, therefore we can repeat the algorithm til we obtain 0. $\square$

Proposition 2.2.14

Proof. Set $J_i := lm(g) \mid g \in G_i$, we want to prove is that $G_{i+1} \supsetneq G_i$ implies that $J_{i+1} \supsetneq J_i$. By construction of the algorithm we have that $G_{i+1} = G_i \cup \{r\}$ hence $J_{i+1} = J_i \cup \{lm(r)\}$ because $lm(g) \nmid lm(r)$ for any $g \in G_i$. As we know J is a semigroup ideal of $\mathcal{P}$. But $\mathcal{P}$ is Noetherian which means that we do not have infinite ideal chains, or in other words J is finitely generated. So the algorithm stops. $\square$

Chapter 2

Gröbner bases and 0-dim ideals

Theorem 3.1.4

Proof. To check that I is 0-dimensional we prove that its variety contains a finite number of points. Let $E := < X_i^q - X_i \mid 1 \leq i \leq m >$ whose variety is exactly the vector space $\mathbb{F}_q^m$. Now let $J := I \setminus E$, it is easy to see that $\mathcal{V}(I) = \mathcal{V}(E) \cap \mathcal{V}(J) \subseteq \mathbb{F}_q^m$ hence $\#\mathcal{V}(I) \leq \#\mathbb{F}_q^m = q^m$ which is finite, thus I is 0-dimensional.

To prove that I is radical it is sufficient to show that $\sqrt{I} \subseteq I$ since the other way around is trivial by definition of radical ideal. Given a polynomial $f \in \sqrt{I}$ this belongs to I if and only if $\exists n \in \mathbb{N}$ such that $f^n \in I$ or in other words $f^n \equiv 0 \ rem \ I$. To begin with notice that $f^q \equiv f \ rem \ I$, indeed take:

$$f := a_1 X_1^{\alpha(1,1)} \cdots X_m^{\alpha(m,1)} + \cdots + a_n X_1^{\alpha(1,n)} \cdots X_m^{\alpha(m,n)}$$

Where $\alpha_{(i,j)} \in \mathbb{N}$ and $a_j \in \mathbb{F}$. Now by rising to the power of q e obtain:

$$\begin{gathered}
f^q = (a_1 X_1^{\alpha(1,1)} \cdots X_m^{\alpha(m,1)} + \cdots + a_n X_1^{\alpha(1,n)} \cdots X_m^{\alpha(m,n)})^q = \\
= (a_1 X_1^{\alpha(1,1)} \cdots X_m^{\alpha(m,1)})^q + \cdots + (a_n X_1^{\alpha(1,n)} \cdots X_m^{\alpha(m,n)})^q = \\
= a_1 (X_1^q)^{\alpha(1,1)} \cdots (X_m^q)^{\alpha(m,1)} + \cdots + a_n (X_1^q)^{\alpha(1,n)} \cdots (X_m^q)^{\alpha(m,n)} = \\
= a_1 X_1^{\alpha(1,1)} \cdots X_m^{\alpha(m,1)} + \cdots + a_n X_1^{\alpha(1,n)} \cdots X_m^{\alpha(m,n)} \\
= f \mod I
\end{gathered}$$

Therefore given $f \in \sqrt{I}$ then $f^n \in I \iff f^n \equiv 0 \mod I$ we can have two cases for n, i.e. $n < q$ and $n \geq q$ but we know that $f^n \equiv f^{n \mod q} \mod I$ so we can consider only the case $n < q$. So we can state the result as follows:

$$f \in \sqrt{I} \Rightarrow f^n \in I \Rightarrow f^n \cdot f^{q-n} \in I \iff f^q \in I \iff f \in I$$

We thus get that $I = \sqrt{I}$. $\square$

Lemma 3.1.9

Proof. Let $T^* := \{X_1^{z_1}, \ldots, X_m^{z_m}, \} \subset T$, it is easy to see that $\Delta(T^*)$ forms an m-dimensional rectangle in the space of monomials, therefore we can compute its volume as follows:

$$\#\Delta(T^*) = \prod_{j=1}^{m} z_j$$

Now the remaining part of T forms an m-dimensional polyhedron which is contained in $\Delta(T)$ and has volume:

$$\prod_{j=1}^{m} (z_j - i_j)$$

so to compute the actual value of $\#\Delta(T)$ one must subtract such volume from $\#\Delta(T^*)$ obtaining:

$$\#\Delta(T) = \prod_{j=1}^{m} z_j - \prod_{j=1}^{m} (z_j - i_j)$$

$\square$

Theorem 3.2.1

Proof. We have $S := \{P_1, \ldots, P_k\}$ and want a Gröbner basis $\mathcal{G}'$ of $I' := \mathcal{I}(S)$. If $S = \{A\}$ with $A := (a_1, \ldots, a_m)$ then $\mathcal{I}(S) =< (X_1 - a_1), \ldots, (X_m - a_m) >$, notice that the leading monomials in the generating basis are relatively coprime therefore $\mathcal{S}(g_i, g_j) = 0 \ \forall \ i \neq j$ therefore it is also a Gröbner basis. What we want to prove in the general case is that given $f \in I$ there exist $g \in \mathcal{G}'$ such that $lm(g) \mid lm(f)$.

So let $f \in \mathcal{I}(S \cup \{B\})$, $B \in \mathbb{K}^m$ this means that $f(B) = 0 \ \forall \ P_i \in S \cup \{B\}$. It is easy to see that $f \in \mathcal{I}(S)$ so given $\mathcal{G}$ a Gröbner basis of $\mathcal{I}(S)$ we get that exist $g \in \mathcal{G}$ such that $lm(g) \mid lm(f)$. We distinguish three cases here:

1. If $g(B) = 0$ then $g \in \mathcal{G}'$ and this case is trivial.

2. Suppose $g(B) \neq 0$ and $lm(g) \succ lm(g_*)$. in this case:

   $$g' := g - \frac{g(B)}{g_*(B)} \cdot g_*$$

   Now $g'(B) = 0$ and the leading monomial is left unchanged so $lm(g') \mid lm(f)$ and so $g' \in \mathcal{G}'$.

3. Suppose $g = g_*$ then $g(B) \neq 0$. We claim that there exist $g_* \cdot (x_i - b_i)$, $0 \leq i \leq m$ such that $lm(g_* \cdot (x_i - b_i)) \mid lm(f)$. Obviously for every i it holds that $(g_* \cdot (x_i - b_i))(B) = 0$. We see that $lm(g_* \cdot (x_i - b_i)) = x_i \cdot lm(g_*)$, if our claim is false then it must be $lm(g_*) = lm(f)$ (the reasoning is as follows: if $lm(g_*) \mid lm(f)$ there must exist x_i such that $x_i \cdot lm(g) \mid lm(f)$ otherwise $lm(g_*) = lm(f)$) therefore keeping in mind that $f \in \mathcal{I}(S)$ we have that:

$$f = g_* + h_1 g_1 + \cdots + h_l \cdot g_l$$

with $g_l \in \mathcal{G}$ and $lm(g_i) \prec lm(g_*)$ therefore evaluating in B we obtain:

$$0 = f(B) = g_*(B) + h_1(B)g_1(B) + \cdots + h_l(B) \cdot g_l(B) = g_*(B) \neq 0$$

which is a contraddiction. So our claim is true and $g(x_i - b_i) \in \mathcal{G}'$ allowing $\mathcal{G}'$ to be a Gröbner basis.

□

Proposition 3.4.2

Proof. Recall that $N(I)$ is the set of monomials that are not leading monomials of elements of $I \subseteq \mathbb{F}[X_1, \ldots, X_m]$. Let $\mathcal{G}$ be a Gröbner basis of I. We want to prove that the elements of the set $\{M + I \mid M \in N(I)\}$ are linearly independent and they span all R.

It is easy to prove that they are linealry independent over F since they differ each other by at least a variable (e.g. X_1 and X_1X_2) or a degree in at least one variable (e.g. X_1X_2 and $X_1X_2^2$).

To prove that they span all R let $f \in \mathbb{F}[X_1, \ldots, X_m]$ with $f \neq 0$, it belongs to a nonzero residue class in the quotient algebra $[f] \in R$ whose representative has leading monomial $lm(f \mod I) \in N(I)$ as otherwise there will exist $g \in \mathcal{G}$ such that $lm(g) \mid lm(f \mod I)$. This extends to all the other monomials $M_i \in Supp(f \mod I)$ simply because $M_i \prec lm(f \mod I)$. □

Chapter 3

Affine Variety Codes

Theorem 5.1.1

Proof. Write $\mathbb{F}^* = \mathbb{F}_q^* = \{P_1, \ldots, P_n\}$ where $n = q - 1$. Consider the generator matrix of RS_k:

$$G = \begin{pmatrix} 1_{|P_1} & \cdots & 1_{|P_n} \\ \vdots & \ddots & \vdots \\ X^{k-1}_{|P_1} & \cdots & X^{k-1}_{|P_n} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ P_1 & P_2 & \cdots & P_n \\ \vdots & \vdots & \ddots & \vdots \\ P_1^{k-1} & P_2^{k-1} & \cdots & P_n^{k-1} \end{pmatrix}$$

Notice that a polynomial evaluation (codeword) in $\mathbb{F}[x]$ is precise combination of rows of G Suppose first that there are two polynomials giving the same codeword:

$$c_1 = (f_1(P_1), \ldots, f_1(P_n)) = (f_2(P_1), \ldots, f_2(P_n)) = c_2$$

Since $deg(f_1), deg(f_2) < k \leq n$, $f_1 - f_2$ is a polynomial of degree less than n which has n zeroes that is impossible unless $f_1 = f_2 \Rightarrow c_1 = c_2$. In other words there is no row in G that is a linear combination of the others. Hence $dim(G) = \#rows(G) = k$.

For the distance we prove both $\geq$ and $\leq$:

Notice that the weight of a codeword:

$$(f_1(P_1), \ldots, f_1(P_n)) = (f_2(P_1), \ldots, f_2(P_n))$$

is the number of points of $\mathbb{F}^*$ that are nonzeroes of f. Therefore let f be a polynomial with as many zeroes as possible, i.e. generating a minimum weight codeword. f has at most $k-1$ zeroes in $\mathbb{F}^*$ hence c can have at most $k-1$ zero coordinates which means that the code has distance $d = w_H(f) \geq n - k + 1$.

On the other hand consider the polynomial:

$$f = \prod_{i=1}^{k-1} (x - P_i)$$

it has degree $k-1$ and $k-1$ solutions therefore the codeword it generates has exactly weight $n-k+1$. So the bound is thight. $\square$

Theorem 5.2.1

Proof. Here we write $\mathbb{F}^m = \{P_1, \ldots, P_n\}$ with $n = q^m$ Let $c \in RM_s \setminus \{0\}$ then again:

$$c = (f(P_1), \ldots, f(P_n))$$

for some $f \in \mathbb{F}[X_1, \ldots, X_m]$ and let $lm(f) = X_1^{i_1} \cdots X_m^{i_m}$. By definitionof the code $deg(f) \leq s \leq m(q-1) < q^m$ so f can have at most $deg(f)$ zeroes and thank to this we can say that $c = 0 \iff f = 0$.

Obviously $i_1 + \cdots + i_m \leq s$ and $0 \leq i_1, \ldots, i_m \leq q-1$ since every coordinate of $P_{i,j}$ (the $j-th$ coordinate of P_i) is a value of $\mathbb{F}$ so it respects $P_{i,j}^q = P_{i,j}$.
Set $I :=< f > + < X_1^q - X_1, \ldots, X_m^q - X_m >$, it is 0-dimensional and radical by theorem 3.1.4. The zeroes of f over $\mathbb{F}^m$ are:

$$\mathcal{V}_{\mathbb{F}^{>}}(I) = N(I) \subseteq \Delta(I) =< X_1^q, \ldots, X_m^q, X_1^{i_1} \cdots X_m^{i_m} >$$

Therefore we can compute a lower bound for the weight of c that is:

$$w_H(c) = n - N(I) \geq n - \#\Delta(I)$$

$$= q^m - (q^m - \prod_{j=1}^{m}(q - i_j)) = \prod_{j=1}^{m}(q - i_j)$$

$$= (q - i_1) \cdots (q - i_m) =: L$$

Now we need to minimize L we want as many $(q - i_h) = 1$ as possible, i.e. $i_h = q - 1$, but since $s = a(q-1) + b$, we can do this only for a factors, so take:

$$i_1 = \cdots = i_a = q - 1 \quad and \quad i_{a+1} = b$$

Then $i_1 + \cdots + i_m = a(q-1) + b$ and so we get:

$$L = (q - (q-1))^a \cdot (q - b) \cdot q^{m-a-1} = (q - b) \cdot q^{m-a-1}$$

To show that this bound is thight we find a polynomial that generates a codeword of weight exactly L. Write $\mathbb{F} = \{\alpha_1, \ldots, \alpha_q\}$ and consider the following polynomial:

$$g := (\prod_{l=1}^{q-1}(\prod_{i=1}^{a}(X_i - \alpha_l)))(\prod_{t=1}^{b}(X_{a+1} - \alpha_t))$$

So $lm(g) = X_1^{q-1} \cdots X_a^{q-1} X_{a+1}^b$, has degree $deg(g) = a(q-1) + b$ and has exactly $(q-b)q^{m-a-1}$ non zeroes. $\square$

Exercise 5.8.2

Proof. Let's check that g has actually the claimed number of nonzeroes, we have $q-1$ values that given to X_1 make g vanish, which means that there are $(q-1)q^{m-1}$ vectors in $\mathbb{F}^m$ that are zeroes because they make a factor of g containing X_1 vanish. We do the same for X_2 but without considering vectors already taken for X_1, i.e. we can take $(q-1)q^{m-2}$ vectors. We go on like this for $a+1$ variables obtaining:

$$\overbrace{(q-1)q^{m-1}+(q-1)q^{m-2}+\cdots+(q-1)q^{m-a}}^{R}+\overbrace{bq^{m-a-1}}^{Q}$$

Consider the two parts R and Q separately:

$$R=(q-1)(q^{m-a})\frac{(q^a-1)}{q-1}=(q^m-q^{m-a})$$

$$R+Q=(q^m-q^{m-a})+bq^{m-a-1}$$

Therefore the number of nonzeroes are:

$$q^m-(R+Q)=q^m-(q^m-q^{m-a})-bq^{m-a-1}$$
$$=q^{m-a}-bq^{m-a-1}=qq^{m-a-1}-bq^{m-a-1}$$
$$=(q-b)q^{m-a-1}$$

□

Lemma 5.3.1

Proof. Write again $\mathbb{F}^m=\{P_1,\dots,P_n\}$ where $n=q^m$, and the simple field $\mathbb{F}=\{\alpha_1,\dots,\alpha_q\}$. We can simply take the following polynomial:

$$f:=\prod_{j=1}^{m}\left(\prod_{t=1}^{i_j}(X_j-\alpha_t)\right)$$

See that $lm(f)=X_1^{i_1}\cdots X_m^{i_m}$, and now we check the number of nonzeroes by counting the number of zeroes as before. Call $I_q=<X_1^q-X_1,\dots,X_m^q-X_m,f>$ and recall that for a 0-dimensional ideal $\#N(I)=\#\mathcal{V}(I)$ hence the number of zeroes of f is:

$$\#N(I_q)=i_1q^{m-1}+i_2q^{m-2}(q-i_1)+\cdots+i_m\prod_{j=1}^{m-1}(q-i_j)$$

$$=q^m-\prod_{j=1}^{m}(q-i_j)$$

Hence the weight of a the codeword c generated by f will be:

$$w_H(c) = q^m - \#N(I_q) = \prod_{j=1}^{m}(q - i_j)$$

$\square$

Theorem 5.3.2

Proof. We first fix a monomial ordering $\prec$ and then take a nonzero codeword $c \in Hyp_q(s,m) - \{0\}$. Using the same notation as in Lemma 5.3.1 we have that:

$$c = (f(P_1), \dots, f(P_n))$$

with $f \in \mathbb{F}[X_1, \dots, X_m]$ non zero and having $lm(f) = X_1^{i_1} \cdots X_m^{i_m}$. Let's count the number of nonzeroes of f, call $I_q =< X_1^q - X_1, \dots, X_m^q - X_m, f >$:

$$w_H(c) = q^m - \#N(I_q) \geq q^m - \#\Delta(I_q) = \prod_{j=1}^{m}(q - i_j) \geq q^m - s$$

Notice that the last inequality comes from the definition of hyperbolic code. We can now apply Lemma 5.3.1 to find a polynomial with that leading monomial and $q^m - s$ nonzero points. So the bound is tight. $\square$

Lemma 5.4.2

Proof. In order to minimize the value $\prod_{l=1}^{m}(q - i_l)$ we try to have as many small factors (i.e. $(q - i_l) = 1$) as possible. To do this we take $i_1 = s - 1$ and $i_2 = 1$ and $i_3 = \cdots = i_m = 0$. Hence the product becomes:

$$\prod_{l=1}^{m}(q - i_l) = q^m - \overline{s}_1 q^{m-1} + \overline{s}_2 q^{m-2} - \cdots (-1)^m \overline{s}_m$$

Where $\overline{s}_k$ for $1 \leq k \leq m$ is the k-th symmetric polynomial in the variables $\{i_1, \dots, i_m\}$. Notice that for $k \geq 3$ every term of s_k is made up by three variables, which means that at least one of them must be 0. Notice furthermore that for the same reason:

$$\overline{s}_1 = i_1 + i_2 = s \quad and \quad \overline{s}_2 = i_1 \cdot i_2 = s - 1$$

Therefore what survives of the polynomial is:

$$\prod_{l=1}^{m}(q - i_l) = q^m - \overline{s}_1 q^{m-1} + \overline{s}_2 q^{m-2}$$

$$= q^m - s q^{m-1} + (s-1) q^{m-2}$$

$\square$

Lemma 5.4.3

Proof. We try to minimize the value $(s-i_1)\prod_{l=2}^{m}(q-i_l)$. Since $s \leq q-1$ we procede by taking $i_2 = q-1$ now by the relation $i_1 + \cdots + i_m = q$ we have 1 more to spend. To choose on which i_l we spend it consider the following argument for $a, b \in \mathbb{N}, a < b$:

$$(a-1)b = ab - b < ab - a = a(b-1)$$

Therefore by setting $a = s$ and $b = q$ the obvious choice will be $i_1 = 1$. Thus we get:

$$(s-1)\prod_{l=3}^{m}(q-i_l) = (s-1)(q^{m-3} - \overline{s}_1 q^{m-4} + \cdots (-1)^{m-2}\overline{s}_{m-2})$$

Now by the same argument we had in Lemma 5.4.2 we see that no $\overline{s}_k$ survives since all the $i_l = 0$ for $l \geq 3$. Hence the minimum value is $(s-1)q^{m-2}$. □

Lemma 5.7.1

Proof. Assume $u \cdot v \neq 0$ then $\sum_{i=1}^{n} u_i v_i \neq 0$ therefore at least one factor $a_i b_i$ survives. Therefore in the worst case we will obtain:

$$u * v = (0, \ldots, 0, u_i v_i, 0 \ldots, 0) \neq \mathbf{0}$$

□

Exercise 5.8.15

Proof. Let $z \in \mathbb{F}^n$ with $u \cdot (v * z) \neq 0$ then we write:

$$0 \neq u \cdot (v_1 z_1, \ldots, u_n z_n) = \sum_{i=1}^{n} u_i v_1 z_1 = z \cdot (u * v)$$

Therefore neither $(u * v)$ nor z can be 0 □

Lemma 5.7.2

Proof. Let f, g be polynomials, then we write:

$$ev(f \cdot g) = ((fg)(P_1), \ldots, (fg)(P_n))$$

but we already know that $(fg)(A) = f(A)g(A)$ (provable by expanding f, g in sum of monomials) so:

$$ev(f \cdot g) = (f(P_1)g(P_1), \ldots, f(P_n)g(P_n)) = ev(f) * ev(g)$$

□

Lemma 5.7.3(STUCK)

Proof. Consider a vector space $E \lhd \mathbb{F}^n$ with $\dim(E) = k$ and a vector basis of E:

$$\mathcal{B} := \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k,1} & b_{k,2} & \cdots & b_{k,n} \end{pmatrix}$$

We are going to consider the column space of $\mathcal{B}$. To start with assume that $w_H(c) = k - 1$ and w.l.o.g. assume that only the first $k - 1$ coordinates of c are different from 0, we perform moltiplication only on $\mathcal{B}$, so consider:

$$\mathcal{B} * c := \begin{pmatrix} b_{1,1} * c_1 & b_{1,2} * c_2 & \cdots & b_{1,n} * c_n \\ b_{2,1} * c_1 & b_{2,2} * c_2 & \cdots & b_{2,n} * c_n \\ \vdots & \vdots & \ddots & \vdots \\ b_{k,1} * c_1 & b_{k,2} * c_2 & \cdots & b_{k,n} * c_n \end{pmatrix}$$

What we obtain is that the columns between the k-th and the $n-th$ of $\mathcal{B} * c$ must be 0 $\square$

Exercise 5.8.14

Proof. Let $s \in N_{\prec_w}(I)$ such that $\mathbf{c} \cdot s \neq 0$. If $s \in \square_{\prec_w} L$ then there exists a polynomial $f \in L$ such that $\mathbf{c} \cdot ev(f) = \sum\limits_{v_i \in Supp(f)} c_i v_i = 0$. But:

$$0 = \mathbf{c} \cdot ev(f) = \mathbf{c} \cdot ev(s) + \mathbf{c} \cdot ev(\lambda) = \mathbf{c} \cdot ev(s) + 0 \neq 0$$

where lambda is the remainin part of f, (i.e. $\lambda = f - lt(f)$). The last equality holds by minimality of s. We got a contraddiction and therefore the thesis. $\square$

Chapter 4

Order Domain Codes

Proposition 6.1.6

Proof. What we want to prove is that for any $f \in R_q$ such that $Supp(f) \in N_{\prec_w}(I)$ and $lm(f) = p$ holds that:

$$lm(fh\ rem\ \mathcal{G}) = lm(ph\ rem\ \mathcal{G})$$

The fact $w(ph) = w(p) + w(h) \in w(N_{\prec_w}(I))$ means that $ph \in N_{\prec_w}(I)$ so that $ph = ph\ rem\ \mathcal{G}$. Hence we can write thanks to Lemma 6.1.2:

$$w(ph\ rem\ \mathcal{G}) = w(ph) = w(lm(fh)) = w(lm(fh\ rem\ \mathcal{G}))$$

But for the second order domain conditions two monomials have the same weight if and only if they are the same monomial. The second part follows the same reasoning. □

Exercise 6.5.3

Proof. Let $\Gamma = w(N(I))$ and prove the three properties that characterize a semigroup.

1. Set $e = w(1) = 0$ then for any $m \in \Gamma$ let $\alpha = w(m)$ and so:

   $$\alpha + e = w(m \cdot 1) = w(m) = \alpha$$

2. Let $m, n \in \Gamma$ with $\alpha = w(m)$ and $\beta = w(m)$. It could be that $m \cdot n \notin N(I)$ but by Lemma 6.1.2 we can write:

   $$\alpha + \beta = w(m \cdot n) = w(m \cdot n\ rem\ \mathcal{G}) \in \Gamma$$

3. Let $m_1, m_2, m_3 \in N(I)$ with $w(m_1) = \alpha, w(m_2) = \beta$ and $w(m_3) = \gamma$. So then:

   $$\begin{aligned} \alpha + (\beta + \gamma) &= w(m_1 \cdot (m_2 \cdot m_3)) \\ &= w((m_1 \cdot m_2) \cdot m_3) = (\alpha + \beta) + \gamma \end{aligned}$$

   Here I intentionally omitted $rem\ \mathcal{G}$ for the sake of reading simplicity.

$\square$

Theorem 6.1.7

Proof. To begin with we translate theorem 5.6.9 which aims at finding an upper bound for the cardinality of the set $N_{\prec}(I_q + < f >)$. To do this we compute the cardinality of the set:

$$\Omega_p = \{s \in N(I_q) \mid \exists h \in N(I_q) \ s.t. \ (p, h) \ is \ OWB, lm(ph \ rem \ \mathcal{G}) = s\}$$

for each $p \in \square_{\prec} L$ and take the minimum.
So with the notation $\square = \square_{\prec} L$ consider now the set:

$$\min_{p \in \square}\{\delta(p)\} = \min_{p \in \square} \#\{s \in N(I_q) \mid \exists h \in N(I) \ s.t. \ w(p) + w(h) = w(s)\}$$

Proposition 6.1.6 shows that the belonging conditions of this last set is the same of requiring that (p, h) is OWB. Moreover we also proved that $N(I_q)$ is a semigroup the two sets are equal. To translate theorem 5.7.4 we have to consider another kind of set. In such theorem we counted the number of OWB pairs that give rise to a monomial $s \in N(I_q) \setminus \square_{\prec} L$, we then built a polynomial space and measured its dimension. So we are considering:

$$\min_{s \in N(I_q) \setminus \square_{\prec} L} = \{p \in N(I_q) \mid \exists h \in N(I_q) \ s.t. \ (p, h) \ is \ OWB, lm(ph \ rem \ \mathcal{G}) = s\}$$

and the set:

$$\min_{h \in N(I_q)}\{\mu(w(h))\}$$

Now procede as above and apply Proposition 6.1.6 when required. $\square$

Theorem 6.2.5

Proof. W.2) Consider $f = F + I$ and $g = G + I$ then:

$$\rho(f + g) = \max\{w(m) \mid m \in Supp(F + G)\}$$

$$= \max\{w(lm(F)), w(lm(G))\}$$

$$= \max\{\rho(F), \rho(G))\}$$

but thanks to Lemma 6.1.2 the weights are left unchanged during reduction by a Gröbner basis therefore:

$$\rho(f + g) = \max\{\rho(F), \rho(G)\} = \max\{\rho(f), \rho(g)\}$$

W.4) Let again $f = F + I$ and $g = G + I$. If $\rho(f) = \rho(g)$ then $\rho(f) = w(lm(F)) = w(lm(G)) = \rho(g)$ but since ρ is bijective it must be that $lm(f) = lm(g)$. Now we distinguish two cases:

$f = g$ Take $a = 1$ then $\rho(f - ag) = \rho(0) = -\infty \prec_w \rho(g)$

$f \neq g$ In this case we can write $f = c_1 lm(g) + \lambda$ and similarly $g = c_2 lm(g) + \gamma$. In this case we set $a = \frac{c_1}{c_2}$ (observe taht a exists and is different from 0 because we are in a field) so that:

$$\rho(f - ag) = \rho(\lambda - \frac{c_1}{c_2}\gamma) \prec_w \rho(g)$$

(Obviously by maximality of $lm(g)$).

$\square$